Chapter IV. Generalized Equivariant Cohomology

In this chapter we show how to construct generalized equivariant cohomology theories, using G-spectra. We then show how any generalized theory is connected by a spectral sequence to the "classical" theory of Chapter I.

1. Equivariant cohomology via G-spectra

We work with the category of spaces with base points in this section. Let $Y$ be a G-spectrum. Then for any G-space $X$ we have homomorphisms

$$n_k: [[S^{k-n}X; Y_k]] \xrightarrow{\phi} [[S^{k-n+1}X; SY_k]] \xrightarrow{\epsilon_{k+1}} [[S^{k-n+1}X; Y_{k+1}]].$$

Thus, with these maps, the groups $[[S^{k-n}X; Y_k]]$ form a direct system and we define

$$(1.1) \tilde{H}_n^G(X; Y) = \lim_k [[S^{k-n}X; Y_k]] = \lim_k [S^kX; Y_{m+k}].$$

Note that if $X$ is locally compact then this is the same as

$$(1.2) \pi_n(E(X,Y)) = \lim_k \pi_{k-n}(E(X,Y_k)).$$

Note that $[[S^kX; Y_{n+k}]] \simeq [[X; \Omega^kY_{n+k}]].$ If $A \subset X$ is invariant under $G$, then for any G-space $W$ there is the exact sequence

$$[[X \cup C_A; W]] \rightarrow [[X; W]] \rightarrow [[A; W]]$$

of (2.1) in Chapter III. If $(X, A)$ is a pair of G-complexes, then $X \cup C_A$ has the same equivariant homotopy type as does $X/A$. Thus, taking $W = \Omega^kY_{n+k}$, and passing to the limit over $k$, we obtain the exact sequence

$$(1.3) \tilde{H}_n^G(X/A; Y) \rightarrow \tilde{H}_n^G(X; Y) \rightarrow \tilde{H}_n^G(A; Y)$$

on the category $G_0$ of G-complexes with base point.
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Using the natural homeomorphism $S^{k-n}X \approx S^{k-(n+1)}SX$ we obtain a natural isomorphism $s_k : \{ [S^{k-n}X; Y_k] \} \rightarrow \{ [S^{k-(n+1)}SX; Y_k] \}$. These commute with the $\eta_k$ and hence define a natural isomorphism

$$S^* : H^*_G(X; Y) \rightarrow H^*_G(SX, Y).$$

We have shown that $H^*_G(X; Y)$ defines an equivariant cohomology theory on $G_0$.

2. Exact couples

In this section we provide some background from the theory of exact couples. Let

$$\begin{array}{ccc}
D & \xrightarrow{i} & D \\
\downarrow{k} & & \downarrow{j} \\
E & & \\
\end{array}$$

be an exact couple where $E$ and $D$ are bigraded, $k$ has total degree 1 and $i$ and $j$ have total degree 0. Note that $(jk)^2 = 0$ and let $H(E)$ be the homology of $E$ with respect to the differential $jk$. The derived couple of (2.1) is

$$\begin{array}{ccc}
iD & \xrightarrow{i'} & iD \\
\downarrow{k'} & & \downarrow{j'} \\
H(E) & & \\
\end{array}$$

where $i' = i \mid iD$, $j'$ is induced by $ji^{-1}$ and $k'$ is induced by $k$. Let $D_1 = D$ and $E_1 = E$. Iterating the above procedure we obtain the (r-1)st derived couple
where \( E_r = H(E_{r-1}) \) and \( D_r = iD_{r-1} = i^{r-1}D \).

We shall now assume that
\[
\begin{align*}
\deg i &= (-1,1) \\
\deg j &= (0,0) \\
\deg k &= (1,0)
\end{align*}
\]
and it is then easy to check that
\[
\begin{align*}
\deg i_r &= (-1,1) \\
\deg j_r &= (r-1,1-r) \\
\deg k_r &= (1,0)
\end{align*}
\]

We let \( d_r = j_r k_r \) which has degree \((r,1-r)\). The system \( \{E_r^{p,q}\} \) together with the differentials \( d_r \) then form a spectral sequence.

We shall now assume that, for some integer \( N \),
\[
\begin{align*}
E_r^{p,q} &= 0 \quad \text{for } p < 0 \text{ and for } p > N \\
D_r^{p,q} &= 0 \quad \text{for } p < 0.
\end{align*}
\]

From the exact sequence
\[
\ldots \to D_r^{p,q} \xrightarrow{i} E_r^{p,q} \xrightarrow{k} D_r^{p+1,q} \xrightarrow{i} D_r^{p,q+1} \xrightarrow{j} E_r^{p,q+1} \xrightarrow{k} D_r^{p+1,q+1} \xrightarrow{i} E_r^{p,q+2} \xrightarrow{k} D_r^{p+1,q+2} \xrightarrow{i} E_r^{p,q+3} \xrightarrow{k} D_r^{p+1,q+3} \xrightarrow{i} \ldots
\]
we see that
\[
i: D_r^{p+1,q} \xrightarrow{\cong} D_r^{p,q+1} \quad \text{for } p > N.
\]

For \( n = p+q \) we let \( J^n \) be a group which is isomorphic to \( D_r^{p,q+1} \) for \( p > N \) and let \( q^{p,q+1}: J^n \to D_r^{p,q+1} \) be some isomorphism chosen so that
commutes. Following $\varnothing$ by iterates of $i$ we have homomorphisms $\varnothing^{p,q+1}: J^n \rightarrow D^{p,q+1}$ defined for all $p$ (with $n = p+q$) such that (2.5) commutes.

If $r > N$ we see that $d_r = 0$, since $E^p_r, q = 0$ for $p < 0$ and for $p > N$. Thus

$$E^p_r, q \simeq E^p_{r+1} \simeq \ldots$$

for $r > N$ and we let $E^p_r, q$ denote the common value. The $(r-1)$st derived couple has the form

$$\ldots i^{r-1} D^{p,r}_p, q \rightarrow E^p_r, q \rightarrow i^{r-1} D^{p+1,r}_p, q \rightarrow i^{r-1} D^{p+2,r}_p, q \rightarrow \ldots$$

Now $i^{r-1} D^{p,r}_p, q \subset D^{p-r+1,r}_p, q_r - 1 = 0$ for $r$ sufficiently large and $i^{r-1} D^{p+1,r}_p, q_r - 1 = \text{Im} \varnothing^{p+1,r}_p, q \subset D^{p+1,r}_p, q$ for $r$ sufficiently large.

Thus, for $r$ large, this exact sequence has the form

$$0 \rightarrow E^p_r, q \rightarrow \text{Im} \varnothing^{p+1,r}_p, q \rightarrow \text{Im} \varnothing^{p+1,r}_p, q \rightarrow 0.$$

That is, we have an exact sequence

$$0 \rightarrow E^p_r, q \rightarrow \frac{J^{p,q}}{\ker \varnothing^{p+1,r}_p, q} \rightarrow \frac{J^{p,q}}{\ker \varnothing^{p,r}_p, q+1} \rightarrow 0.$$

Put

$$J^{p,q} = \ker \{ \varnothing^{p,q+1} : J^{p,q} \rightarrow D^{p,q+1} \}$$

so that (2.7) provides the isomorphism

$$E^p_r, q \simeq J^{p,q}/J^{p+1,q-1}. $$

Thus we have that the spectral sequence $E^p_r, q$ converges to the graded group associated with the (finite) filtration
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... \supset J^p,q \supset J^{p+1},q-1 \supset ...

of \( J^{p+q} = \mathbb{D}^{M+1},p+q-M \) for \( M \geq N \).

3. The spectral sequence of a filtered G-complex

Let \( K \) be a G-complex and let \( \{ K_r \} \) be a sequence of G-subcomplexes such that

\[
\begin{align*}
K_r & \subset K_{r+1} \\
K_{-1} & = \emptyset \\
K_N & = K
\end{align*}
\]

(3.1)

where \( N \) is some given integer.

Let \( \{ \mathcal{H}^*, \delta^* \} \) be any equivariant cohomology theory and put

\[
\begin{align*}
E^{p,q} & = \mathcal{H}^{p+q}(K_p, K_{p-1}) \\
D^{p,q} & = \mathcal{H}^{p+q-1}(K_{p-1})
\end{align*}
\]

(3.2)

Then the exact cohomology sequence of the pair \( (K_p, K_{p-1}) \) provides an exact couple

\[
\begin{array}{ccc}
D & \rightarrow & D \\
& \searrow & \\
& E \leftarrow &
\end{array}
\]

as in section 2.

The differential \( d_1 \) is the composition

\[
E_1^{p,q} = \mathcal{H}^{p+q}(K_p, K_{p-1}) \rightarrow \mathcal{H}^{p+q}(K_p) \rightarrow \mathcal{H}^{p+q+1}(K_{p+1}, K_p) = E_1^{p+1,q}
\]

(3.3)

And the spectral sequence converges to the graded group associated with the filtration

\[
J^{p,q} = \ker(\mathcal{H}^{p+q}(K) \rightarrow \mathcal{H}^{p+q}(K_{p-1}))
\]

of \( J^{p+q} = \mathcal{H}^{p+q}(K) \).
4. The main spectral sequence

Let \( \{ \mathcal{H}^*, s^* \} \) be any equivariant cohomology theory and let \( K \) be a \( G \)-complex of dimension \( N < \infty \). If \( K \) is not finite then we shall assume that \( \mathcal{H}^* \) also satisfies the axiom:

\[
(A) \text{ If } S \text{ is a discrete } G \text{-set with orbits } S_\alpha \text{ then } \prod i_\alpha^*: \mathcal{H}^n(S) \to \prod \mathcal{H}^n(S_\alpha) \text{ is an isomorphism, where } i_\alpha: S_\alpha \to S \text{ is the inclusion.}
\]

Letting \( K_p = K^p \), the \( p \)-skeleton of \( K \), the preceding section provides a spectral sequence with

\[
E_1^{p,q} = \mathcal{H}^{p+q}(K^p, K^{p-1}) \cong \mathcal{H}^{p+q}(K^p/K^{p-1}).
\]

Now

\[
K^p/K^{p-1} \cong S^p C_p^+
\]

the \( p \)-th reduced suspension of the discrete \( G \)-set \( C_p^+ \) where \( C_p \) stands for the set of all \( p \)-cells of \( K \). Thus

\[
E_1^{p,q} \cong \mathcal{H}^{p+q}(S^p C_p^+) \cong \mathcal{H}^q(C_p^+) \cong \mathcal{H}^q(C_p).
\]

Now let \( h^q G \) denote the coefficient system of Chapter I, section 4, example (1). That is

\[
h^q(G/H) = \mathcal{H}^q(G/H) = \mathcal{H}^q((G/H)^+).
\]

We shall define an isomorphism

\[(h.1) \quad \alpha: \mathcal{H}^q(C_p^+) \cong C^p_G(K; h^q) \]

as follows:

For \( \sigma \in C_p \) let

\[
i_\sigma: (G/G_\sigma)^+ \to C_p^+
\]

be the equivariant map defined by \( i_\sigma(gG_\sigma) = g \sigma \in C_p \). Also let

\[
j_\sigma: C_p^+ \to (G/G_\sigma)^+
\]
be defined by \( j_\sigma(g\sigma) = gG_\sigma \) and \( j_\sigma(\tau) = \text{base point if } \tau \text{ is not in the orbit of } \sigma \). Note that

\[
\begin{align*}
  i_g \sigma &= i_\sigma \hat{g} \\
  j_g \sigma &= \hat{g}^{-1} j_\sigma \\
  j_\sigma i_\sigma &= 1 \\
  j_\tau i_\sigma &= 0 \text{ (the base point) if } \tau \notin G(\sigma),
\end{align*}
\]

(4.2)

where \( \hat{g} = R_g : G/G_\sigma = G/gG_\sigma g^{-1} \to G/G_\sigma \). Also note that \( i_\sigma j_\sigma \) is the identity on \( G(\sigma) \) and collapses everything else to the base point.

We have the induced maps

\[
\begin{align*}
  i_\sigma^* &: \hat{\mathcal{H}}^q(C_p) \to \hat{\mathcal{H}}^q(C^*_p) = h^q(G/G_\sigma) \\
  j_\sigma^* &: \hat{\mathcal{H}}^q(C^*_p) \to \hat{\mathcal{H}}^q(C_p).
\end{align*}
\]

Define, for \( \lambda \in \hat{\mathcal{H}}^q(C_p) \) and \( \sigma \in C_p \),

(4.3) \( \alpha(\lambda)(\sigma) = i_\sigma^*(\lambda) \).

To check that \( \alpha(\lambda) \) is equivariant we compute

\[
\alpha(\lambda)(g\sigma) = i_{g\sigma}^*(\lambda) = (i_{\sigma}\hat{g})^*(\lambda) = \hat{g}^*i_\sigma^*(\lambda) = \hat{g}^*(\alpha(\lambda)(\sigma))
\]

as was to be shown. (See Chapter I, sections 5 and 6.)

We must check that \( \alpha \) is an isomorphism. We shall show that its inverse is given by the map

\[
\beta: C^P_G(K,h^q) \to \hat{\mathcal{H}}^q(C^*_p)
\]

defined as follows: Let \( f \in C^P_G(K,h^q) \). Note that

\[
 j_{g\sigma}^*(f(g\sigma)) = (\hat{g}^{-1} j_\sigma)^*(\hat{g}^*(f(\sigma))) = j_\sigma^*(f(\sigma)).
\]

Let \( T \subset C_p \) be a system of representatives of the orbits of \( G \) on the set \( C_p \) and define
(4.4) \[ \beta(f) = \prod_{\sigma \in T} j_\sigma^*(f(\sigma)). \]

Now we compute
\[
\alpha(\beta(f))(\sigma) = i_\sigma^*(\beta(f)) = i_\sigma^*(\prod_{\tau \in T} j_\tau^*(f(\tau)))
= i_\sigma^*(f(\sigma)) = (j_i \circ i_\sigma)^* f(\sigma) = f(\sigma)
\]
so that \( \alpha \beta = 1. \) Also
\[
\beta(\alpha(\lambda))(\sigma) = \prod_{\sigma \in T} j_\sigma^*(\alpha(\lambda)(\sigma))
= \prod_{\sigma \in T} (i_\sigma^*(\lambda)) = \prod_{\sigma \in T} (i_\sigma j_i)^*(\lambda) = \lambda
\]
so that \( \beta \alpha = 1. \) Thus \( \alpha \) is an isomorphism as was to be shown.

Now we claim that under the isomorphism
\[ E^{p,q}_1 \cong \wedge^q \pi^*(C^p) \cong C_0^q(K; h^q) \]
the differential \( d_1 \) becomes, up to sign, the coboundary.

We first remark that, up to sign, \( d_1 : E^{p,q}_1 \to E^{p+1,q}_1 \) may be identified with the homomorphism
\[ \tilde{\psi}^{p+1}_q(K^p/K^{p-1}) \cong \tilde{\psi}^{p+1}_q(S(K^p/K^{p-1})) \cong \tilde{\psi}^{p+1}_q(K^{p+1}/k^p) \]
where \( \tilde{\psi}^q : K^{p+1}/k^p \to S(K^p/K^{p-1}) \) is an equivariant map defined as follows: If \( \sigma \) is a \((p+1)\)-cell and \( f_\sigma : S^p \to K^p \) is a characteristic map (chosen equivariantly) we follow \( f_\sigma \) by collapsing \( K^p \) and suspending \( S^{p+1} \to S(K^p/K^{p-1}) \) (unreduced on the left, reduced on the right). Then the cell \( \sigma/\partial \subset K^{p+1}/k^p \) is identified with \( S^{p+1} \) in a canonical way (taking the base point into the north pole of \( S^{p+1} \)). The resulting maps \( \sigma/\partial \to S(K^p/K^{p-1}) \) are put together to form the map \( \tilde{\psi}^q : K^{p+1}/k^p \to S(K^p/K^{p-1}) \). The verification of this relies on the fact that in the Puppe sequence for the inclusion \( i : K^p/K^{p-1} \to K^{p+1}/k^{p-1} \) the map \( C_i \to S(K^p/K^{p-1}) \) may be identified with \( \tilde{\psi}^{p+1}_{p+1} \). The details will be left to the reader.
Now $KP^{p+1}/K^p \cong SP^{p+1}C_{p+1}$ and $S(KP^p/K^{p-1}) \cong SP^{p+1}C_p$ so that the map $\psi_p$ is described by the induced maps $\sigma/\delta \subset SP^{p+1}_p \rightarrow SP^{p+1}_p = \bigvee_{\tau} S(\tau/\hat{\tau}) + S(\tau/\hat{\tau})$ (where $\sigma, \tau \in C_{p+1}^p$). It is easy to see that, in fact, this map has degree $[\tau: \sigma]$ (see Chapter I, section 1).

Thus $d_1$ is induced, up to sign, by

$$\eta_p^*: \tilde{\mathcal{H}}^{q+1}(SC_p^+) \rightarrow \tilde{\mathcal{H}}^{q+1}(SC_{p+1}^+)$$

where $\eta_p: SC_{p+1}^+ = \bigvee_{\sigma} S_\sigma + \bigvee_{\tau} S_\tau = SC_p^+$ is an equivariant map such that the induced map $S_\sigma + S_\tau$ has degree $[\tau: \sigma]$. (Here we use $S_\sigma$ to stand for a copy of the circle indexed by the cell $\sigma$.)

We claim that the following diagram commutes

$$\tilde{\mathcal{H}}^{q+1}(SC_p^+) \xrightarrow{\eta_p^*} \tilde{\mathcal{H}}^{q+1}(SC_{p+1}^+)$$

where we use $S$ to denote the suspension isomorphism. The proof is straightforward but will involve some cumbersome details.

First, suppose $\sigma$ is a $(p+1)$-cell and $\tau$ is a $p$-cell of $K$ with $K(\tau) \subset K(\sigma)$. Then let $\Theta^\tau_\sigma$ denote the equivariant map $G/G^\tau_\sigma \rightarrow G/G^\tau_\sigma$ induced by inclusion $G^\tau_\sigma \subset G^\tau_\sigma$. Using $[\tau: \sigma]$ to denote maps of degree $[\tau: \sigma]$ we note that the diagram

$$\begin{array}{c}
S(G/G^\tau_\sigma)^+ \xrightarrow{\delta^\tau_\sigma} SC^+_p \xrightarrow{\eta_p} SC^+_p \\
\bigvee_{\tau} S(G/G^\tau_\sigma)^+ \xrightarrow{\bigvee_{\tau} \delta^\tau_\sigma} \bigvee_{\tau} S(G/G^\tau_\sigma)^+ \\
\bigvee_{\tau \in T} S(G/G^\tau_\sigma)^+ \xrightarrow{\bigvee_{\tau \in T} \delta^\tau_\sigma} \bigvee_{\tau \in T} S(G/G^\tau_\sigma)^+ \\
\end{array}$$

is commutative. The proof involves some detailed calculations.
of equivariant maps commutes, where $T$ is the set of all $p$-cells $\tau$ with $K(\tau) \subseteq K(\sigma)$.

The induced diagram in cohomology is

$$
\begin{align*}
\tilde{\mathcal{H}}(S(G/G_\sigma)^+)^* & \xrightarrow{(S_{\sigma})^*} \tilde{\mathcal{H}}(S_{p+1}^*) & \xrightarrow{\eta^*} \tilde{\mathcal{H}}(S_{p}^*) \\
\sum_{\tau} \tilde{\mathcal{H}}(S(G/G_\phi)^+) & \xrightarrow{\sum(S_{\phi}^*)^*} \sum_{\tau} \tilde{\mathcal{H}}(S(G/G_\tau)^+) \\
\sum_{\tau \in T} \tilde{\mathcal{H}}(S(G/G_\sigma)^+) & \xrightarrow{\sum(S_{\phi}^*)^*} \sum_{\tau \in T} \tilde{\mathcal{H}}(S(G/G_\tau)^+)
\end{align*}
$$

Since $(S \phi)^* = S \cdot \phi^* \cdot S^{-1}$ we obtain from this diagram that

$$
(4.6) \quad \sum_{\tau} \tilde{\mathcal{H}}(S(G/G_\phi)^+) = S\left[ \sum_{\tau} \tilde{\mathcal{H}}(S(G/G_\tau)^+) \right] S^{-1}.
$$

Now let us verify that (4.5) commutes. Let $\lambda \in \tilde{\mathcal{H}}(S_{p+1}^*)$ and, as usual, let $\sigma$ be a $(p+1)$-cell of $K$. Then

$$
(4.7) \quad a(S^{-1}(\eta_p^*(\lambda)))(\sigma) = i^*_\sigma(S^{-1}(\eta_p^*(\lambda)))
$$

$$
= S^{-1}(\eta_p^*(\lambda)) = \sum_{\tau} \tilde{\mathcal{H}}(S_{\phi}^*)^* (S^{-1}(\lambda)).
$$

(The last equality comes from (4.6).) On the other hand

$$
\delta^p(aS^{-1}(\lambda))(\sigma) = \sum_{\tau} \tilde{\mathcal{H}}(S_{\phi}^*)^* (a(S^{-1}(\lambda))(\tau))
$$

directly from the definition of $\delta^p$. This may be further simplified to

$$
\sum_{\tau} \tilde{\mathcal{H}}(S_{\phi}^*)^* i^*_\tau(S^{-1}(\lambda)),
$$

the same as in (4.7). This shows that (4.5) commutes and hence, finally, that $d_1 : E^1_{p,q} \to E^{p+1, q}_1$ becomes the coboundary under our isomorphism with $C^*_G(K; h^q)$. Thus we have

$$
(4.8) \quad E^p_{2,q} \simeq H^p_G(K; h^q).
$$
As noted before, the spectral sequence converges (when \( \dim K < \infty \)) to the graded group associated with some filtration of \( \mathcal{H}^{p+q}(K) \).

5. The "classical" uniqueness theorem

Suppose that \( \mathcal{H}^* \) is an equivariant cohomology theory satisfying the dimension axiom (4) of section 2, Chapter I. Let \( h \in \mathcal{C}_G \) denote the "coefficients" of this theory. That is \( h(G/H) = \mathcal{H}^0(G/H) \), and so on. Let \( K \) be a finite dimensional \( G \)-complex. If \( K \) is infinite we assume that (A) of the last section is satisfied.

In this case the spectral sequence of the last section degenerates for \( r \geq 2 \). In fact

\[
E_2^{p,q} = \begin{cases} 
H_G^p(K;h); & q = 0 \\
0 & q \neq 0 
\end{cases}
\]

It follows that, in fact,

\[
(5.1) \quad \mathcal{H}^p(K) \approx H_G^p(K;h)
\]

and naturality is not hard to verify. Thus this is the only equivariant classical cohomology theory having coefficients \( h \).

The reader should note that, for general \( h \in \mathcal{C}_G \), \( h \) is indeed the coefficient system of the cohomology theory \( H^*_G(K;h) \). That is, there is a natural isomorphism

\[
h(G/H) \approx H_G^0(G/H;h).
\]