

Understanding the small object argument

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Abstract

The small object argument is a transfinite construction which, starting from a set of maps in a category, generates a weak factorisation system on that category. As useful as it is, the small object argument has some problematic aspects: it possesses no universal property; it does not converge; and it does not seem to be related to other transfinite constructions occurring in categorical algebra. In this paper, we give an “algebraic” refinement of the small object argument, cast in terms of Grandis and Tholen’s natural weak factorisation systems, which rectifies each of these three deficiencies.

1 Introduction

The concept of *factorisation system* provides us with a way of viewing a category \mathcal{C} as a compositional product of two subcategories \mathcal{L} and \mathcal{R} . The two key ingredients are an axiom of *factorisation*, which affirms that any map of \mathcal{C} may be written as a map of \mathcal{L} followed by a map of \mathcal{R} , and an axiom of *orthogonality*, which assures us that this decomposition is unique up to unique isomorphism. From these two basic axioms a very rich theory can be developed, and a very useful one, since most categories arising in mathematical practice will admit at least a few different factorisation systems.

However, in those mathematical areas where the primary objects of study are themselves higher-dimensional entities – most notably, topology and higher dimensional category theory – the notion of factorisation system is frequently too strong,

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since we would like factorisations be unique, not up to isomorphism, but up to something weaker. Thus in a 2-category, we might want uniqueness up-to-equivalence; or in a category of topological spaces, uniqueness up-to-homotopy.

The usual way of achieving this is to pass from factorisation systems to *weak factorisation systems*. The modifier “weak” has the familiar effect of turning an assertion of unique existence into an assertion of mere existence, here in respect to the diagonal fill-ins which are guaranteed to us by the axiom of orthogonality.

From this, we would not necessarily expect the factorisations in a weak factorisation system (henceforth w.f.s.) to be unique up to anything at all: but remarkably, each weak factorisation system generates its own notion of “equivalence” which respect to which its factorisations *are* unique. The framework within which this is most readily expressed is that of Quillen’s *model categories* [20], which consist in a clever interaction of two w.f.s.’s on a category: but we can make do with a single w.f.s., and for the purposes of this paper, we will.

Whilst in many respects, the theory of w.f.s.’s is similar to the theory of factorisation systems (which we will henceforth call *strong* factorisation systems to avoid ambiguity), there are some puzzling aspects to it: and notable amongst these is the manner in which one typically constructs a w.f.s.

In the case of strong factorisation systems, there is a very elegant theory which, given a sufficiently well-behaved category \mathcal{C} , can generate a strong factorisation system from any set of maps $J \subset \mathcal{C}^2$. The \mathcal{R} -maps will be the maps which are *right orthogonal* to each of the maps in J (in a sense which we recall more precisely in Section 2); and the \mathcal{L} -maps, those which are left orthogonal to each of the maps in \mathcal{R} . The key difficulty is how we should build the factorisations, and for this we are able bring to bear a well-established body of knowledge concerning transfinite constructions in categories, on which the definitive word is [17].

There is a corresponding theory for weak factorisation systems. Again, we suppose ourselves given a well-behaved \mathcal{C} and a set of maps J , but this time we take for \mathcal{R} the class of maps *weakly* right orthogonal to J , and for \mathcal{L} , the class of maps *weakly* left orthogonal to \mathcal{R} . To obtain a weak factorisation system, we must also have factorisation of maps: and for this, we apply a construction known as the *small object argument*, introduced by Quillen [20], and first given in its full generality by Bousfield [6].

The problem lies in divining the precise nature of the small object argument. It is certainly some kind of transfinite construction: but it is a transfinite construction which does not converge, has no universal property, and does not seem to be an instance of any other known transfinite construction.

In this paper, we present a modification of the small object argument which rectifies each of these deficiencies: it is guaranteed to converge; the factorisations it provides are freely generated by the set J , in a suitable sense; and it may be

construed as an instance of a familiar free monoid construction.

To make this possible, we must adopt a rather different perspective on weak factorisation systems. The definition of a w.f.s. specifies classes of maps \mathcal{L} and \mathcal{R} together with axioms which affirm properties: that *there exist* factorisations, or that *there exist* certain diagonal fill-ins. But a key tenet of category theory is that anything we specify in terms of properties should have an equally valid expression in terms of structure: and in the case of w.f.s.'s, a suitable “algebraic” reformulation is given by Tholen and Grandis’ notion of *natural weak factorisation system* [13].

The extra algebraicity provided by natural w.f.s.’s allows us a clearer view of what is actually going on in the small object argument. We now have a *functor* from the category of natural w.f.s.’s on \mathcal{C} into **CAT** which sends each natural w.f.s. to its category of \mathcal{L} -maps; and we can factor this functor through **CAT**/ \mathcal{C}^2 . We may view the resultant functor $\mathbf{NWFS}(\mathcal{C}) \rightarrow \mathbf{CAT}/\mathcal{C}^2$ as being the “semantics” side of a syntax/semantics adjunction: for which the syntax side is precisely our refinement of the small object argument.

Although all our arguments will be cast in terms of natural w.f.s.’s, we will see that there are ramifications for plain w.f.s.’s as well, since our refined version of the small object argument can equally well be applied there, giving rise to factorisations which are less redundant than the original argument, and in many cases can be easily calculated by hand.

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2 Notions of factorisation system

In this section, we describe in detail the various sorts of factorisation system mentioned in the Introduction.

2.1 Most familiar is the notion of *strong factorisation system* $(\mathcal{L}, \mathcal{R})$ on a category \mathcal{C} , introduced by Freyd and Kelly in [9]. This is given by two classes of maps \mathcal{L} and \mathcal{R} in \mathcal{C} which are each closed under composition with isomorphisms, and which satisfy the axioms of

(factorisation) Every map $e: X \rightarrow Y$ in \mathcal{C} can be written as $e = gf$, where $f \in \mathcal{L}$ and $g \in \mathcal{R}$; and

(orthogonality) $f \perp g$ for all $f \in \mathcal{L}$ and $g \in \mathcal{R}$, where $f \perp g$ means that for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{k} & D \end{array} \quad (2.1)$$

in \mathcal{C} , there is a unique map $j: B \rightarrow C$ such that $gj = k$ and $jf = h$.

Instead of writing $f \perp g$, we may also say that f is *left orthogonal* to g or that g is *right orthogonal* to f ; moreover, given a class \mathcal{A} of maps in \mathcal{C} , we write

$$\perp \mathcal{A} = \{ g \in \mathcal{C}^2 \mid f \perp g \text{ for all } f \in \mathcal{A} \} \quad \text{and} \quad \mathcal{A}^\perp = \{ f \in \mathcal{C}^2 \mid f \perp g \text{ for all } g \in \mathcal{A} \};$$

and this sets up a Galois connection on the collection of all classes of maps in \mathcal{C} . In a strong factorisation system, we have $\mathcal{R} = \mathcal{L}^\perp$ and $\mathcal{L} = \perp \mathcal{R}$, so that the classes \mathcal{L} and \mathcal{R} determine each other.

2.2 We arrive at the notion of a *weak factorisation system* [6] by making two alterations to the above definition. One is minor: we require that \mathcal{L} and \mathcal{R} are closed under retracts in the arrow category \mathcal{C}^2 , rather than merely closed under isomorphism. The other is more far-reaching: we replace the orthogonality condition with

(weak orthogonality) $f \pitchfork g$ for all $f \in \mathcal{L}$ and $g \in \mathcal{R}$, where $f \pitchfork g$ means that for every commutative square as in (2.1), there exists a (not necessarily unique) fill-in $j: B \rightarrow C$ such that $gj = k$ and $jf = h$.

We now have a Galois connection $\pitchfork(\cdot) \dashv (\cdot)^\pitchfork$; and again, the classes \mathcal{L} and \mathcal{R} of a w.f.s. determine each other by the equations $\mathcal{L} = \pitchfork \mathcal{R}$ and $\mathcal{R} = \mathcal{L}^\pitchfork$. However, the classes \mathcal{L} and \mathcal{R} need not determine the factorisation of a map, even up to isomorphism, as the following examples show:

2.3 Examples:

- (i) (Epi, Mono) is a strong factorisation system on **Set**; but (Mono, Epi) is a weak factorisation system. For the latter, there are two natural choices of factorisation for a map $f: X \rightarrow Y$: the *graph* factorisation which goes via $X \times Y$; and the *cograph* factorisation, which goes through $X + Y$.
- (ii) There is a weak factorisation system on **Cat** given by (injective equivalences, isofibrations). A *injective equivalence* is a functor which is both injective on objects and an equivalence of categories; whilst an *isofibration* is a functor along which all isomorphisms have liftings.

- (iii) There is a weak factorisation system (anodyne extensions, Kan fibrations) on $\mathbf{SSet} = [\Delta^{\text{op}}, \mathbf{Set}]$, the category of simplicial sets. The Kan fibrations are easy to describe: they are precisely the maps which are weakly right orthogonal to the set of *horn inclusions* $\Lambda^k[n] \rightarrow \Delta[n]$. The anodyne extensions are the class of maps weakly left orthogonal to all Kan fibrations; more explicitly, they are obtained by closing the set of horn inclusions under countable composition, cobase change, coproduct and retract.

2.4 As we mentioned in the Introduction, Grandis and Tholen’s *natural weak factorisation systems* [13] provide an algebraisation of the notion of weak factorisation system. In order to motivate the definition, we first give a similar algebraisation of the notion of strong factorisation system.

So suppose that we are given a strong factorisation system $(\mathcal{L}, \mathcal{R})$ on a category \mathcal{C} , together with for each map of \mathcal{C} a choice of factorisation:

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda_f} Kf \xrightarrow{\rho_f} Y,$$

where $\lambda_f \in \mathcal{L}$ and $\rho_f \in \mathcal{R}$. It follows from the orthogonality property that this assignation may be extended in a unique way to a *functorial factorisation*: which is to say, a functor $F: \mathcal{C}^{\mathbf{2}} \rightarrow \mathcal{C}^{\mathbf{3}}$ (where $\mathbf{2}$ and $\mathbf{3}$ are the ordinals $(0 \leq 1)$ and $(0 \leq 1 \leq 2)$ respectively) which splits the “face map” $d_1: \mathcal{C}^{\mathbf{3}} \rightarrow \mathcal{C}^{\mathbf{2}}$ given by

$$d_1(X \xrightarrow{f} Y \xrightarrow{g} Z) = (X \xrightarrow{gf} Z).$$

2.5 This face map is induced by the functor $\delta_1: \mathbf{2} \rightarrow \mathbf{3}$ picking out the unique arrow $0 \rightarrow 2$: we have $d_1 = \mathcal{C}^{\delta_1}$. There are two other functors $\delta_0, \delta_2: \mathbf{2} \rightarrow \mathbf{3}$, and homming these into \mathcal{C} induces two further face maps $d_0, d_2: \mathcal{C}^{\mathbf{3}} \rightarrow \mathcal{C}^{\mathbf{2}}$, with

$$d_0(X \xrightarrow{f} Y \xrightarrow{g} Z) = (Y \xrightarrow{g} Z) \quad \text{and} \quad d_2(X \xrightarrow{f} Y \xrightarrow{g} Z) = (X \xrightarrow{f} Y).$$

Postcomposing our functorial factorisation $F: \mathcal{C}^{\mathbf{2}} \rightarrow \mathcal{C}^{\mathbf{3}}$ with these induces functors $L, R: \mathcal{C}^{\mathbf{2}} \rightarrow \mathcal{C}^{\mathbf{2}}$, which send an object f of $\mathcal{C}^{\mathbf{2}}$ to λ_f and ρ_f respectively.

2.6 There is further structure in $\mathbf{Cat}(\mathbf{2}, \mathbf{3})$ which we can make use of: we have natural transformations $\gamma_{2,1}: \delta_2 \Rightarrow \delta_1$ and $\gamma_{1,0}: \delta_1 \Rightarrow \delta_0$, and by homming these into \mathcal{C} , we induce natural transformations $c_{2,1}: d_2 \Rightarrow d_1$ and $c_{1,0}: d_1 \Rightarrow d_0$. Postcomposing $F: \mathcal{C}^{\mathbf{2}} \rightarrow \mathcal{C}^{\mathbf{3}}$ with these now gives us natural transformations $\Phi: L \Rightarrow \text{id}_{\mathcal{C}^{\mathbf{2}}}$ and

$\Lambda: \text{id}_{\mathcal{C}^2} \Rightarrow R$ with components

$$\Phi_f = \begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ \lambda_f \downarrow & & \downarrow f \\ Kf & \xrightarrow{\rho_f} & Y \end{array} \quad \text{and} \quad \Lambda_f = \begin{array}{ccc} X & \xrightarrow{\lambda_f} & Kf \\ f \downarrow & & \downarrow \rho_f \\ Y & \xrightarrow{\text{id}_Y} & Y. \end{array}$$

2.7 Now, because $F: \mathcal{C}^2 \rightarrow \mathcal{C}^3$ arose from a strong factorisation system, the corresponding $\Lambda: \text{id}_{\mathcal{C}^2} \Rightarrow R$ will provide the unit for a reflection of \mathcal{C}^2 into the full subcategory of \mathcal{C}^2 spanned by the \mathcal{R} -maps. To see this, consider a morphism

$$\begin{array}{ccc} X & \xrightarrow{h} & W \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{k} & Z \end{array}$$

from f to g in \mathcal{C}^2 , with g an \mathcal{R} -map. Then applying orthogonality to the square

$$\begin{array}{ccc} X & \xrightarrow{h} & W \\ \lambda_f \downarrow & & \downarrow g \\ Kf & \xrightarrow{k \cdot \rho_f} & Z \end{array}$$

we obtain a map $j: Kf \rightarrow W$ making both triangles commute; and so the map $(h, k): f \rightarrow g$ factors uniquely through Λ_f as

$$f \xrightarrow{\Lambda_f} \rho_f \xrightarrow{(j, k)} g.$$

Thus the subcategory spanned by the \mathcal{R} -maps is a full, replete, reflective subcategory of \mathcal{C}^2 , via the reflector $\Lambda: \text{id}_{\mathcal{C}^2} \Rightarrow R$, and so (R, Λ) extends uniquely to an idempotent monad $\mathbf{R} = (R, \Lambda, \Pi)$ whose category of \mathbf{R} -algebras may be identified with this subcategory. Dually, the pair (L, Φ) may be extended uniquely to an idempotent comonad $\mathbf{L} = (L, \Phi, \Sigma)$ whose category of coalgebras is isomorphic to the full subcategory of \mathcal{C}^2 spanned by the \mathcal{L} -maps. Thus we have proved:

2.8 Proposition: *There is a bijective correspondence between strong factorisation systems $(\mathcal{L}, \mathcal{R})$ on a category \mathcal{C} for which a choice of factorisation for every map has been made, and functorial factorisations $F: \mathcal{C}^2 \rightarrow \mathcal{C}^3$ for which the corresponding pointed endofunctor (R, Λ) underlies an idempotent monad and the corresponding copointed endofunctor (L, Φ) underlies an idempotent comonad.*

The notion of natural weak factorisation system now arises by generalising the situation of this Proposition in a very obvious way: by dropping the requirement of idempotency.

2.9 Definition: [13] A *natural weak factorisation system* on a category \mathcal{C} is given by a functorial factorisation $F: \mathcal{C}^2 \rightarrow \mathcal{C}^3$, together with an extension of the corresponding pointed endofunctor (R, Λ) to a monad $\mathbf{R} = (R, \Lambda, \Pi)$; and an extension of the corresponding copointed endofunctor (L, Φ) to a comonad $\mathbf{L} = (L, \Phi, \Sigma)$.

Observe that we can reconstruct F from \mathbf{L} and \mathbf{R} , and thus we may speak simply of a natural weak factorisation system (\mathbf{L}, \mathbf{R}) .

2.10 Examples:

- (i) There is a natural w.f.s. on **Set** whose underlying functorial factorisation is the graph factorisation of Examples 2.3(i):

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\langle id, f \rangle} X \times Y \xrightarrow{\pi_2} Y.$$

Dually, there is a natural w.f.s. on **Set** which factors f through $X + Y$. These examples generalise to any category with products or coproducts, as the case may be.

- (ii) There is a natural w.f.s. on **Cat** whose underlying functorial factorisation is given by

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \quad \mapsto \quad \mathcal{C} \xrightarrow{\lambda_F} \mathcal{D} \downarrow F \xrightarrow{\rho_F} \mathcal{D},$$

where $\mathcal{D} \downarrow F$ is the *comma category* whose objects are triples $(c, d, f: d \rightarrow Fc)$; λ_F is the functor sending c in \mathcal{C} to $(id: Fc \rightarrow Fc)$ in $\mathcal{D} \downarrow F$; and ρ_F is the functor sending $(f: d \rightarrow Fc)$ in $\mathcal{D} \downarrow F$ to d . There are variations on this theme: we can replace $\mathcal{D} \downarrow F$ with the dual comma category $F \downarrow \mathcal{D}$; or with the *iso-comma category* $\mathcal{D} \downarrow_{\cong} F$, which is the full subcategory of $\mathcal{D} \downarrow F$ whose objects are the invertible arrows. These examples generalise to any 2-category with comma objects.

- (iii) By Proposition 2.8, any strong factorisation system on \mathcal{C} gives rise to a natural weak factorisation system on \mathcal{C} .

2.11 It is not immediately clear that a natural w.f.s. deserves the name of weak factorisation system. To show that this is so, we must exhibit suitable analogues of the axioms of factorisation and weak orthogonality; for which we must first identify

what the \mathcal{L} -maps and \mathcal{R} -maps are. Now, for a strong factorisation system, we can reconstruct the \mathcal{L} - and \mathcal{R} -maps from the associated comonad L and monad R as their respective coalgebras and algebras; and thus it is natural to define:

2.12 Definition: Let (L, R) be a natural w.f.s. on \mathcal{C} . We write **L-Map** for the category of L -coalgebras, and call its objects *L-maps*; and write **R-Map** for the category of R -algebras and call its objects *R-maps*.

Note that being an L - or R -map is structure on, and not a property of, a map of \mathcal{C} .

2.13 Examples:

- (i) For the natural w.f.s. on **Set** which factors $f: X \rightarrow Y$ through $X+Y$, an R -map structure on $g: C \rightarrow D$ is a splitting for g : that is, a morphism $g^*: Y \rightarrow X$ with $gg^* = \text{id}_Y$. An L -map structure on $f: A \rightarrow B$ exists just when f is a monomorphism, and in this case is uniquely determined: thus the comonad L is “property-like”, though not idempotent.
- (ii) For the natural w.f.s. on **Cat** which factors $F: \mathcal{C} \rightarrow \mathcal{D}$ through $\mathcal{D} \downarrow F$, an R -map is a *split fibration*: that is, a Grothendieck fibration with chosen liftings that compose up strictly. An L -map is, roughly speaking, an inclusion of a reflective subcategory: more precisely, an L -map structure on a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is given by specifying a functor $F^*: \mathcal{D} \rightarrow \mathcal{C}$ and a natural transformation $\eta: 1_{\mathcal{D}} \Rightarrow FF^*$ satisfying $F^*F = 1_{\mathcal{D}}$, $F^*\eta = \text{id}_{F^*}$ and $\eta F = \text{id}_F$. For the n.w.f.s. which factors through $F \downarrow \mathcal{D}$ instead, the R -algebras are split opfibrations and the L -coalgebras, inclusions of coreflective subcategories; whilst if we factor through $\mathcal{D} \downarrow_{\cong} F$, then R -algebras are split isofibrations, and L -coalgebras are retract equivalences.
- (iii) If we view a strong factorisation system $(\mathcal{L}, \mathcal{R})$ on \mathcal{C} as a natural w.f.s., then the L -maps and R -maps reduce to \mathcal{L} -maps and \mathcal{R} -maps. In this particular case, being an L - or R -map returns to being a mere property; and this is because the comonad L and monad R are idempotent.

Further details on these examples may be found in [13].

2.14 With this definition of L -map and R -map, it is now clear that any natural w.f.s. (L, R) admits an axiom of factorisation: given a map $f: C \rightarrow D$, we obtain an L -map structure on $\lambda_f: C \rightarrow Kf$ by applying the cofree functor $\mathcal{C}^2 \rightarrow \mathbf{L-Map}$, and an R -map structure on $\rho_f: Kf \rightarrow D$ by applying the free functor $\mathcal{C}^2 \rightarrow \mathbf{R-Map}$.

2.15 More interestingly, we also have an axiom of weak orthogonality. To see this, suppose that we are given a square like (2.1) together with an L-coalgebra structure on f and an R-algebra structure on g . Thus we have a coaction morphism $e: f \rightarrow Lf$ and an action morphism $m: Rg \rightarrow g$, which the (co)algebra axioms force to be of the following forms:

$$e = \begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ f \downarrow & & \downarrow \lambda_f \\ B & \xrightarrow{s} & Kf \end{array} \quad \text{and} \quad m = \begin{array}{ccc} Kf & \xrightarrow{p} & C \\ \rho_g \downarrow & & \downarrow g \\ D & \xrightarrow{\text{id}_D} & D. \end{array}$$

Furthermore, we may view the square (2.1) as a map $(h, k): f \rightarrow g$ in \mathcal{C}^2 ; and so applying the functorial factorisation of (L, R) yields an arrow $K(h, k): Kf \rightarrow Kg$ in \mathcal{C} . We now obtain a diagonal fill-in for (2.1) as the composite

$$B \xrightarrow{s} Kf \xrightarrow{K(h, f)} Kg \xrightarrow{p} C. \quad (2.2)$$

Note that this fill-in is canonically determined by the L-map structure on f and the R-map structure on g . Indeed, it is reasonable to view an L-map structure as encoding a coherent choice of lifting opposite every R-map, and vice versa.

2.16 Example: Let us see how we obtain diagonal fill-ins for the natural w.f.s. on **Cat** which factors $F: \mathcal{C} \rightarrow \mathcal{D}$ through $\mathcal{D} \downarrow F$. We suppose ourselves given a square of functors

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{H} & \mathcal{C} \\ F \downarrow & & \downarrow G \\ \mathcal{B} & \xrightarrow{K} & \mathcal{D}, \end{array}$$

with F an L-coalgebra and G an R-algebra. The L-coalgebra structure on F provides us with a functor $F^*: \mathcal{B} \rightarrow \mathcal{A}$ and a natural transformation $\eta: 1 \Rightarrow FF^*$. Thus we can define a functor $HF^*: \mathcal{B} \rightarrow \mathcal{C}$ and a natural transformation

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{HF^*} & \mathcal{C} \\ K \searrow & \xrightarrow{\alpha} & \swarrow G \\ & \mathcal{D}; & \end{array}$$

indeed, we have $GHF^* = KFF^*$, and so can take $\alpha = K\eta: K \Rightarrow KFF^*$. Now using the R-algebra structure on G , we may factorise this 2-cell as:

$$\begin{array}{ccc}
 & HF^* & \\
 \mathcal{B} & \begin{array}{c} \xrightarrow{\quad} \\ \Uparrow \bar{\alpha} \\ \xrightarrow{\quad} \end{array} & \mathcal{C} \\
 & J & \\
 & = & \\
 & \begin{array}{c} \swarrow K \\ \searrow G \end{array} & \\
 & \mathcal{D}, &
 \end{array}$$

where J is given by reindexing HF^* along α . It is not hard to see that this functor $J: \mathcal{C} \rightarrow \mathcal{D}$ is precisely the fill-in specified by equation (2.2) above.

2.17 Remark: It follows from the observations of §2.14 and §2.15 that any natural w.f.s. (L, R) on a category \mathcal{C} has an underlying plain w.f.s. For if we define \mathcal{L} to be the class of arrows in \mathcal{C} which admit some L-coalgebra structure and \mathcal{R} to be the class of arrows admitting some R-algebra structure, then $(\mathcal{L}, \mathcal{R})$ will satisfy all the axioms required of a w.f.s., except possibly for closure under retracts. So we take $\bar{\mathcal{L}}$ and $\bar{\mathcal{R}}$ to be the respective retract-closures of \mathcal{L} and \mathcal{R} ; and now the pair $(\bar{\mathcal{L}}, \bar{\mathcal{R}})$ gives a w.f.s. on \mathcal{C} .

2.18 It turns to be very useful to strengthen the notion of natural w.f.s. slightly. For this, we consider the natural transformations $\Pi: RR \Rightarrow R$ and $\Sigma: L \Rightarrow LL$ associated to a natural w.f.s. (L, R) . We may denote their respective components at $f \in \mathcal{C}^2$ by

$$\Pi_f = \begin{array}{ccc} K\rho_f & \xrightarrow{\pi_f} & Kf \\ \rho_f \downarrow & & \downarrow \rho_f \\ B & \xrightarrow{\text{id}_B} & B \end{array} \quad \text{and} \quad \Sigma_f = \begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \lambda_f \downarrow & & \downarrow \lambda_f \\ Kf & \xrightarrow{\sigma_f} & K\lambda_f \end{array};$$

again, the arrows written as identities are forced to be so by the (co)monad axioms. Now, these maps σ_f and π_f provide us with the components of a natural transformation $\Delta: LR \Rightarrow RL$ whose component at f is given by:

$$\Delta_f = \begin{array}{ccc} Kf & \xrightarrow{\sigma_f} & K\lambda_f \\ \lambda_f \downarrow & & \downarrow \rho_{\lambda_f} \\ K\rho_f & \xrightarrow{\pi_f} & Kf. \end{array}$$

(That this square commutes is a consequence of the (co)monad axioms). We will say that a natural w.f.s. *satisfies the distributivity axiom* if this natural transformation

$\Delta: LR \Rightarrow RL$ defines a distributive law of L over R in the sense of [4]. Note that this is a property of a natural w.f.s., rather than extra structure on it.

2.19 Example: We may check that each of the natural w.f.s.'s given so far satisfies the distributivity axiom.

2.20 There are important results about n.w.f.s.'s that are true only if we restrict to those for which the distributivity axiom holds. Two such results are Theorem 4.14 and Theorem A.1 below; and there is another which allows us to characterise R -maps purely in terms of lifting properties against the L -maps, and vice versa. In order that these results should be valid, we henceforth modify the definition of natural w.f.s. to include the requirement that the distributivity axiom should hold.

3 Free and algebraically-free natural w.f.s.'s

3.1 Our goal is to use the theory of natural w.f.s.'s to give a categorically coherent reformulation of the small object argument. As we stated in the Introduction, this argument provides the means by which, starting from a set of maps J , one may produce a w.f.s. *cofibrantly generated* by J : that is, a w.f.s. $(\mathcal{L}, \mathcal{R})$ for which $\mathcal{R} = J^{\text{th}}$.

3.2 Examples: All the weak factorisation systems of Examples 2.3 are cofibrantly generated:

- For the w.f.s. (Mono, Epi) on **Set**, a suitable J is given by the set containing the single map $!: 0 \rightarrow 1$.
- For the (injective equivalences, isofibrations) w.f.s. on **Cat**, a suitable J is given by the single map $\lrcorner b \lrcorner : 1 \rightarrow \mathbf{Iso}$, where **Iso** is the indiscrete category on the set $\{a, b\}$.
- For the w.f.s. (anodyne extensions, Kan fibrations) on **SSet**, a suitable J is given by the set of horn inclusions $\{\Lambda_n^k \rightarrow \Delta_n\}$.

To give our reformulation of the small object argument, we will need to provide a notion of “cofibrantly generated” natural w.f.s. However, a careful analysis reveals two candidates for this notion. In this section, we study these candidates and their relationship to each other.

3.3 We begin by forming the entities that we have met so far into categories. Suppose we are given functorial factorisations F and $F': \mathcal{C}^2 \rightarrow \mathcal{C}^3$ on \mathcal{C} . We define

a *morphism of functorial factorisations* $\alpha: F \rightarrow F'$ to be a natural transformation $\alpha: F \Rightarrow F'$ which upon whiskering with $d_1: \mathcal{C}^3 \rightarrow \mathcal{C}^2$ becomes the identity transformation $\text{id}_{\mathcal{C}^2} \Rightarrow \text{id}_{\mathcal{C}^2}$. To give such a morphism is to give a family of maps $\alpha_f: Kf \rightarrow K'f$, natural in f , and making diagrams of the following form commute:

$$\begin{array}{ccc}
 & A & \\
 \lambda_f \swarrow & & \searrow \lambda'_f \\
 Kf & \xrightarrow{\alpha_f} & K'f \\
 \rho_f \searrow & & \swarrow \rho'_f \\
 & B &
 \end{array}$$

Suppose now that F and F' underlie natural w.f.s.'s (L, R) and (L', R') on \mathcal{C} , and consider a morphism of functorial factorisations $\alpha: F \rightarrow F'$. By whiskering the natural transformation $\alpha: F \Rightarrow F'$ with the other two face maps $d_0, d_2: \mathcal{C}^3 \rightarrow \mathcal{C}^2$, we induce natural transformations $\alpha_l: L \Rightarrow L'$ and $\alpha_r: R \Rightarrow R'$; and we will say that $\alpha: F \rightarrow F'$ is a *morphism of natural w.f.s.'s* just when α_l is a comonad morphism and α_r a monad morphism.

3.4 Let us write $\mathbf{NWFS}(\mathcal{C})$ for the category of n.w.f.s.'s on \mathcal{C} . We may define a “semantics” functor $\mathcal{G}: \mathbf{NWFS}(\mathcal{C}) \rightarrow \mathbf{CAT}/\mathcal{C}^2$, which sends a n.w.f.s. (L, R) to its category of L-coalgebras $\mathbf{L-Map}$, equipped with the forgetful functor into \mathcal{C}^2 ; and sends a morphism $\alpha: (L, R) \rightarrow (L', R')$ of n.w.f.s.'s to the morphism

$$\begin{array}{ccc}
 \mathbf{L-Map} & \xrightarrow{(\alpha_l)_*} & \mathbf{L'-Map} \\
 & \searrow U_L & \swarrow U_{L'} \\
 & \mathcal{C}^2 &
 \end{array}$$

of $\mathbf{CAT}/\mathcal{C}^2$. Here $(\alpha_l)_*$ is the functor which sends an L-coalgebra $x: X \rightarrow LX$ to the L'-coalgebra

$$X \xrightarrow{x} LX \xrightarrow{(\alpha_l)x} L'X.$$

Our first candidate for the notion of “cofibrantly generated” n.w.f.s. is now:

3.5 Definition: Let $I: \mathcal{J} \rightarrow \mathcal{C}^2$ be an object of $\mathbf{CAT}/\mathcal{C}^2$, with \mathcal{J} small; and let (L, R) be a n.w.f.s. on \mathcal{C} . We will say that (L, R) is *free on \mathcal{J}^1* if we can provide a

¹Here we commit the usual abuse of notation in denoting a category $I: \mathcal{J} \rightarrow \mathcal{C}^2$ over \mathcal{C}^2 merely by its domain category \mathcal{J} .

morphism

$$\begin{array}{ccc}
 \mathcal{J} & \xrightarrow{\eta} & \mathbf{L}\text{-Map} \\
 & \searrow I & \swarrow U_L \\
 & & \mathcal{C}^2
 \end{array}$$

of $\mathbf{CAT}/\mathcal{C}^2$ which exhibits (L, R) as a reflection of I along \mathcal{G} : which is to say that, for any n.w.f.s. (L', R') on \mathcal{C} and functor $F: \mathcal{J} \rightarrow \mathbf{L}'\text{-Map}$ over \mathcal{C}^2 , there is a unique morphism of n.w.f.s.'s $\alpha: (L, R) \rightarrow (L', R')$ for which $F = (\alpha_L)_* \circ \eta$.

3.6 Remark: There is a dual semantics functor $\mathcal{H}: \mathbf{NWFS}(\mathcal{C}) \rightarrow (\mathbf{CAT}/\mathcal{C}^2)^{\text{op}}$, which sends a n.w.f.s. to its category of R-algebras: and a corresponding notion of an n.w.f.s. being *cofree* on \mathcal{J} . However, being cofree is significantly less common than being free, primarily because the conditions under which we will construct free n.w.f.s.'s – typically, local presentability or local boundedness – are much more prevalent than their duals.

3.7 Whilst Definition 3.5 is natural from a categorical perspective, it has an obvious drawback: it provides no analogue of the equation $\mathcal{R} = J^\natural$ which a cofibrantly generated w.f.s. satisfies. Definition 3.9, our second candidate for the notion of “cofibrantly generated” n.w.f.s., will rectify this. Before we can give it, we will need a preliminary result.

3.8 Proposition: *Let \mathcal{C} be a category. Then the Galois connection ${}^\natural(-) \dashv (-)^\natural$ induced by the notion of weak orthogonality may be lifted to an adjunction*

$$\mathbf{CAT}/\mathcal{C}^2 \begin{array}{c} \xrightarrow{{}^\natural(-)} \\ \perp \\ \xleftarrow{(-)^\natural} \end{array} (\mathbf{CAT}/\mathcal{C}^2)^{\text{op}} .$$

Proof. First we give the functor $(-)^{\natural}: (\mathbf{CAT}/\mathcal{C}^2)^{\text{op}} \rightarrow \mathbf{CAT}/\mathcal{C}^2$. On objects, this sends a category $U: \mathcal{A} \rightarrow \mathcal{C}^2$ over \mathcal{C}^2 to the following category \mathcal{A}^{\natural} over \mathcal{C}^2 . Its objects are pairs (g, ϕ) , where g is a morphism of \mathcal{C} and ϕ is a coherent choice of lifting against every element of \mathcal{A} : which is to say, a mapping which to each object $a \in \mathcal{A}$ and square

$$\begin{array}{ccc}
 A & \xrightarrow{h} & C \\
 Ua \downarrow & & \downarrow g \\
 B & \xrightarrow{k} & D
 \end{array} \tag{3.1}$$

in \mathcal{C} , assigns a morphism $\phi(a, h, k): B \rightarrow C$ making both triangles commute, and subject to the following naturality condition: if we are given a morphism $\sigma: a \rightarrow a'$ of \mathcal{A} whose image under U is the morphism

$$\begin{array}{ccc} A & \xrightarrow{s} & A' \\ Ua \downarrow & & \downarrow Ua' \\ B & \xrightarrow{t} & B', \end{array}$$

of \mathcal{C}^2 , then we have $\phi(a, hs, kt) = \phi(a', h, k) \circ t$. A morphism of \mathcal{A}^{th} from (g, ϕ) to (g', ϕ') is a morphism $(u, v): g \rightarrow g'$ of \mathcal{C}^2 which respects the choice of liftings in ϕ and ϕ' , in the sense that the equation $u \circ \phi(a, h, k) = \phi'(a, uh, vk)$ holds for all suitable a, h and k . The functor exhibiting \mathcal{A}^{th} as a category over \mathcal{C}^2 is the evident forgetful functor.

This defines $(-)^{\text{th}}$ on objects of $\mathbf{CAT}/\mathcal{C}^2$; and to extend this definition to morphisms, we consider a further category \mathcal{B} over \mathcal{C}^2 and a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ over \mathcal{C}^2 : from which we obtain a map $F^{\text{th}}: \mathcal{B}^{\text{th}} \rightarrow \mathcal{A}^{\text{th}}$ over \mathcal{C}^2 by sending the object $(c, \phi(-, *, ?))$ of \mathcal{B}^{th} to the object $(c, \phi(F(-), *, ?))$ of \mathcal{A}^{th} .

We define the functor ${}^{\text{th}}(-)$ in the same way as $(-)^{\text{th}}$, but with Ua and g swapped around in equation (3.1). It remains only to exhibit the adjointness ${}^{\text{th}}(-) \dashv (-)^{\text{th}}$: for which it is easy to see that, given categories $U: \mathcal{A} \rightarrow \mathcal{C}^2$ and $V: \mathcal{B} \rightarrow \mathcal{C}^2$ over \mathcal{C}^2 , we may identify both

$$\text{functors } \mathcal{A} \rightarrow {}^{\text{th}}\mathcal{B} \quad \text{and} \quad \text{functors } \mathcal{B} \rightarrow \mathcal{A}^{\text{th}}$$

over \mathcal{C}^2 with “ $(\mathcal{A}, \mathcal{B})$ -lifting operations”: that is, functions ψ which given an object $a \in \mathcal{A}$, an object $b \in \mathcal{B}$ and a commuting square

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ Ua \downarrow & & \downarrow Vb \\ B & \xrightarrow{k} & D, \end{array}$$

provide a morphism $\psi(a, b, h, k): B \rightarrow C$ making both triangles commute; and subject to the obvious naturality condition with respect to morphisms of both \mathcal{A} and \mathcal{B} . \square

In particular, we see from §2.15 that any any n.w.f.s. (L, R) comes equipped with a privileged $(L\text{-Map}, R\text{-Map})$ -lifting operation: which by the above proof, we may view as a privileged morphism lift: $R\text{-Map} \rightarrow L\text{-Map}^{\text{th}}$ over \mathcal{C}^2 .

3.9 Definition: Let $I: \mathcal{J} \rightarrow \mathcal{C}^2$ be a category over \mathcal{C}^2 , and (L, R) a n.w.f.s. on \mathcal{C} . We say that (L, R) is *algebraically-free* on \mathcal{J} if we can provide a morphism $\eta: \mathcal{J} \rightarrow \mathbf{L-Map}$ over \mathcal{C} for which the functor

$$\mathbf{R-Map} \xrightarrow{\text{lift}} \mathbf{L-Map}^{\text{th}} \xrightarrow{\eta^{\text{th}}} \mathcal{J}^{\text{th}} \quad (3.2)$$

is an isomorphism of categories.

3.10 Remark: The terminology we have chosen deliberately recalls the distinction which is made in [17] between the *free* and the *algebraically-free* monad generated by a pointed endofunctor. We will partially justify this in Section 5, by showing that algebraic-freeness in our sense can be seen as a special case of algebraic-freeness in the sense of [17]; and in the Appendix, where we prove the implication “algebraically-free \Rightarrow free” for n.w.f.s.’s.

However, there are some results of [17] which the author has been unable to find an analogue of: in particular, he has been unable to produce either positive or negative results about the implication “free \Rightarrow algebraically-free”. The corresponding implication does not hold in the theory of monads; and whilst it seems unlikely that it should hold here either, a proof of this fact has been elusive. Despite this, we will be able to show in Section 5 that any free n.w.f.s. which we come across *in mathematical practice* will be algebraically-free.

3.11 Examples: The natural w.f.s. on **Set** which factors $f: X \rightarrow Y$ through $X + Y$ is algebraically-free: we let \mathcal{J} be the terminal category and let $I: \mathcal{J} \rightarrow \mathbf{Set}^2$ pick out the object $!: 0 \rightarrow 1$. It is now easy to see that the category \mathcal{J}^{th} consists precisely of the R-algebras: morphisms $g: C \rightarrow D$ equipped with a splitting $g^*: D \rightarrow C$.

However, none of the other natural w.f.s.’s described in Examples 2.10 are free or algebraically-free: and this despite being close relatives of plain w.f.s.’s which are cofibrantly generated. The problem for these examples is that, although an R-map structure affirms the existence of certain liftings, it also asserts certain coherence conditions between those liftings, which cannot be expressed in the language of orthogonality.

A fair intuition is that the (algebraically)-free natural w.f.s.’s are the natural w.f.s.’s which may be specified by a “signature” \mathcal{J} of lifting properties; but subject to no “equations” between these liftings.

We may relate the notion of algebraically-free n.w.f.s. quite directly to that of cofibrantly generated w.f.s., if we assume the axiom of choice in our metatheory:

3.12 Proposition*: *Let \mathcal{C} be a category and J a set of maps in \mathcal{C} ; and let \mathcal{J} denote the set J , viewed as a discrete subcategory of \mathcal{C}^2 . If the algebraically-free n.w.f.s. (\mathbf{L}, \mathbf{R}) on $\mathcal{J} \hookrightarrow \mathcal{C}^2$ exists, then its underlying plain w.f.s. $(\bar{\mathcal{L}}, \bar{\mathcal{R}})$ is the w.f.s. cofibrantly generated by J .*

Proof. Recall from §2.17 that the class of maps \mathcal{R} consists of those maps in \mathcal{C} admitting some \mathbf{R} -algebra structure; and that $\bar{\mathcal{R}}$ consists of all retracts of maps in \mathcal{R} . We are required to show that $\bar{\mathcal{R}} = J^\natural$; and since J^\natural is easily seen to be closed under retracts, it will suffice to show that $\mathcal{R} = J^\natural$.

Now, since (\mathbf{L}, \mathbf{R}) is algebraically-free on \mathcal{J} , we have $\mathbf{R}\text{-Map} \cong \mathcal{J}^\natural$ over \mathcal{C}^2 ; and so a morphism $f \in \mathcal{C}^2$ will admit an \mathbf{R} -algebra structure, and thus lie in \mathcal{R} , just when it can be lifted through the forgetful functor $\mathcal{J}^\natural \rightarrow \mathcal{C}^2$. But an object of \mathcal{J}^\natural consists of a map of \mathcal{C} equipped with a choice of lifting against every map in the set J , subject to no further coherence conditions; and so, if we allow ourselves the axiom of choice, f will admit a lifting through \mathcal{J}^\natural just when $f \in J^\natural$. Thus we have that $\mathcal{R} = J^\natural$ as desired. \square

4 Constructing free natural w.f.s.'s

4.1 We now ready to give our analogue of the small object argument, which will be a general apparatus by means of which we can construct free, and even algebraically-free, n.w.f.s.'s on a category \mathcal{C} .

For our argument to work, we will at least require \mathcal{C} to be cocomplete: but in order to guarantee the convergence of certain transfinite sequences we construct, we must impose some further “smallness” property on \mathcal{C} .

4.2 Given a regular cardinal α , we say that $X \in \mathcal{C}$ is α -presentable if the representable functor $\mathcal{C}(X, -): \mathcal{C} \rightarrow \mathbf{Set}$ preserves α -filtered colimits. The first smallness property we may consider on \mathcal{C} is that:

(*) For every $X \in \mathcal{C}$, there is an α_X for which X is α_X -presentable.

This is certainly the case for any category \mathcal{C} which is *locally presentable* in the sense of [10]. However, it does not obtain in categories such as the category of topological spaces, the category of Hausdorff topological spaces, or the category of topological groups: and since we would like our argument to be valid in such contexts, we will require a more general notion of smallness.

4.3 Recall that a strong factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} is said to be *proper* if every \mathcal{E} -map is an epimorphism and every \mathcal{M} -map a monomorphism; and is said to

be *well-copowered* if every object of \mathcal{C} possesses, up-to-isomorphism, a mere set of \mathcal{E} -quotients. We say that an object X is α -*bounded* with respect to a proper $(\mathcal{E}, \mathcal{M})$ if $\mathcal{C}(X, -)$ preserves α -filtered unions of \mathcal{M} -subobjects (in the sense of sending them to α -filtered unions of sets). The second smallness property we consider on \mathcal{C} supposes some proper, well-copowered $(\mathcal{E}, \mathcal{M})$, and says that:

(\dagger) For every $X \in \mathcal{C}$, there is an α_X for which X is α_X -bounded with respect to $(\mathcal{E}, \mathcal{M})$.

Top, **Haus** and **TopGrp** all satisfy (\dagger), with $\mathcal{M} =$ the subspace inclusions in the first two cases, and $\mathcal{M} =$ the inclusion of subgroups which are also subspaces in the third.

We may now state the main result of the paper.

4.4 Theorem: *Let \mathcal{C} be a cocomplete category satisfying either $(*)$ or (\dagger) , and let $I: \mathcal{J} \rightarrow \mathcal{C}^2$ be a category over \mathcal{C}^2 with \mathcal{J} small. Then the free n.w.f.s. on \mathcal{J} exists, and is algebraically-free on \mathcal{J} .*

In this section, we will prove freeness: in the next, algebraic-freeness.

4.5 We begin by factorising the semantics functor \mathcal{G} through a pair of intermediate categories. The first is the category $\mathbf{LNWFS}(\mathcal{C})$ of “left halves of n.w.f.s.’s”. Its objects (F, L) are functorial factorisations F on \mathcal{C} together with an extension of the corresponding (L, Φ) to a comonad L ; and its morphisms are maps of functorial factorisations which respect the comonad structure. There is an obvious functor $\mathcal{G}_1: \mathbf{NWFS}(\mathcal{C}) \rightarrow \mathbf{LNWFS}(\mathcal{C})$ sending (L, R) to (F, L) .

The second category we consider is $\mathbf{Cmd}(\mathcal{C}^2)$, the category of comonads on \mathcal{C}^2 . We have a functor $\mathcal{G}_2: \mathbf{LNWFS}(\mathcal{C}) \rightarrow \mathbf{Cmd}(\mathcal{C}^2)$, which sends (F, L) to L ; and we have the semantics functor $\mathcal{G}_3: \mathbf{Cmd}(\mathcal{C}^2) \rightarrow \mathbf{CAT}/\mathcal{C}^2$ which sends a comonad to its category of coalgebras, and a comonad morphism $\gamma: C \rightarrow C'$ to $\gamma_*: \mathbf{C-Coalg} \rightarrow \mathbf{C'-Coalg}$. We now have that

$$\mathcal{G} = \mathbf{NWFS}(\mathcal{C}) \xrightarrow{\mathcal{G}_1} \mathbf{LNWFS}(\mathcal{C}) \xrightarrow{\mathcal{G}_2} \mathbf{Cmd}(\mathcal{C}^2) \xrightarrow{\mathcal{G}_3} \mathbf{CAT}/\mathcal{C}^2,$$

so that we may give a reflection along \mathcal{G} by giving a reflection along each functor \mathcal{G}_1 , \mathcal{G}_2 and \mathcal{G}_3 in turn. For \mathcal{G}_3 , we have the following well-known result, which was first stated at this level of generality by Dubuc [7]; but see also [2].

4.6 Proposition: *Let \mathcal{C} be cocomplete, and let $U: \mathcal{A} \rightarrow \mathcal{C}^2$ be a small category over \mathcal{C}^2 . Then \mathcal{A} admits a reflection along $\mathcal{G}_3: \mathbf{Cmd}(\mathcal{C}^2) \rightarrow \mathbf{CAT}/\mathcal{C}^2$.*

Proof. Because \mathcal{A} is small and \mathcal{C}^2 cocomplete (since \mathcal{C} is), we can form the left Kan extension of U along itself:

$$\begin{array}{ccc} & \mathcal{A} & \\ U \swarrow & \rightleftarrows_{\theta} & \searrow U \\ \mathcal{C}^2 & \xrightarrow{\text{Lan}_U(U)} & \mathcal{C}^2, \end{array}$$

whose defining property is that θ should provide the unit for a representation

$$[\mathcal{C}^2, \mathcal{C}^2](\text{Lan}_U(U), -) \cong [\mathcal{A}, \mathcal{C}^2](U, (-) \circ U).$$

In particular, corresponding to the identity transformation $\text{id}_U: U \Rightarrow U$, we have a natural transformation $\epsilon: \text{Lan}_U(U) \Rightarrow \text{id}_{\mathcal{C}^2}$; whilst corresponding to the composite natural transformation

$$U \xrightarrow{\theta} \text{Lan}_U(U) \circ U \xrightarrow{\text{Lan}_U(U) \circ \theta} \text{Lan}_U(U) \circ \text{Lan}_U(U) \circ U$$

we have a natural transformation $\Delta: \text{Lan}_U(U) \Rightarrow \text{Lan}_U(U) \circ \text{Lan}_U(U)$. It is now easy to check that ϵ and Δ make $\text{Lan}_U(U)$ into a comonad on \mathcal{C}^2 , the so-called *density comonad* of U . This has the property that comonad morphisms $(\text{Lan}_U(U), \epsilon, \Delta) \rightarrow \mathbb{T}$ are in bijection with left coactions of \mathbb{T} on U , which in turn are in bijection with liftings of $U: \mathcal{A} \rightarrow \mathcal{C}^2$ through the category of \mathbb{T} -coalgebras: and this is precisely the universal property for $\text{Lan}_U(U)$ to be a reflection of U along \mathcal{G}_3 . \square

Next, we consider reflections along $\mathcal{G}_2: \mathbf{LNWFS}(\mathcal{C}) \rightarrow \mathbf{Cmd}(\mathcal{C}^2)$. These exist under very mild hypotheses indeed:

4.7 Proposition: *If \mathcal{C} has pushouts, then $\mathcal{G}_2: \mathbf{LNWFS}(\mathcal{C}) \rightarrow \mathbf{Cmd}(\mathcal{C}^2)$ has a left adjoint.*

Proof. Let us say that an endofunctor $F: \mathcal{C}^2 \rightarrow \mathcal{C}^2$ *preserves domains* if $\text{dom} \circ F = \text{dom}$. Given two such endofunctors F and F' , we will say that a natural transformation α between them *preserves domains* if $\text{dom} \circ \alpha = \text{id}_{\text{dom}}$. Finally, we will say that a comonad (T, ϵ, Δ) on \mathcal{C}^2 *preserves domains* if T , ϵ and Δ all preserve domains.

It is now a simple but instructive exercise to show that $\mathbf{LNWFS}(\mathcal{C})$ is isomorphic to the full subcategory of $\mathbf{Cmd}(\mathcal{C}^2)$ whose objects are the domain-preserving comonads. Thus the Proposition will follow if we can show this subcategory to be reflective.

To do this, we first observe that there is a strong factorisation system on \mathcal{C}^2 whose left class \mathcal{P} consists of the pushout squares, and whose right class consists \mathcal{D}

of the squares whose domain component is an isomorphism. In fact, if we make a choice of pushouts in \mathcal{C} , then we obtain a functorial factorisation of every map into a pushout square followed by a square whose domain component is an *identity*.

We can lift the factorisation system $(\mathcal{P}, \mathcal{D})$ to one of the same name on $[\mathcal{C}^2, \mathcal{C}^2]$; and the accompanying functorial factorisation lifts too, allowing us to factor every map of $[\mathcal{C}^2, \mathcal{C}^2]$ as a map whose components are pushouts, followed by one whose domain components are identities.

Suppose now that we are given a comonad $S = (S, \epsilon, \Delta)$ on \mathcal{C}^2 : we construct its reflection into domain-preserving comonads as follows. We start by factorising the counit of S as

$$\epsilon = S \xrightarrow{\phi} \hat{S} \xrightarrow{\hat{\epsilon}} \text{id}_{\mathcal{C}^2},$$

where the components of ϕ are pushout squares, and the domain components of $\hat{\epsilon}$ are identities. From this latter fact, we deduce that both \hat{S} and $\hat{\epsilon}$ preserve domains. We now consider the following diagram:

$$\begin{array}{ccc} S & \xrightarrow{\Delta} & SS & \xrightarrow{\phi\phi} & \hat{S}\hat{S} \\ \phi \downarrow & & & & \downarrow \hat{\epsilon}\hat{S} \\ \hat{S} & \xrightarrow{\text{id}_{\hat{S}}} & \hat{S} & & \hat{S}. \end{array}$$

Since ϕ is in \mathcal{P} , and $\hat{\epsilon}\hat{S}$ in \mathcal{D} , we obtain by orthogonality a unique diagonal fill-in $\hat{\Delta}: \hat{S} \Rightarrow \hat{S}\hat{S}$. Since both $\hat{\epsilon}\hat{S}$ and $\text{id}_{\hat{S}}$ are domain-preserving, we deduce that $\hat{\Delta}$ is too.

A little calculus with the unique diagonal fill-in property and the comonad axioms for (S, ϵ, Δ) now yields the comonad axioms for $\hat{S} = (\hat{S}, \hat{\epsilon}, \hat{\Delta})$; and it is immediate that $\phi: S \Rightarrow \hat{S}$ then satisfies the necessary axioms for it to lift to a comonad morphism $\phi: S \rightarrow \hat{S}$.

We claim that this ϕ provides the desired reflection of S into domain-preserving comonads. Indeed, suppose we are given another domain-preserving comonad $T = (T, e, D)$, and a morphism of comonads $\psi: S \rightarrow T$. Then we have the following commutative square:

$$\begin{array}{ccc} S & \xrightarrow{\psi} & T \\ \phi \downarrow & & \downarrow e \\ \hat{S} & \xrightarrow{\hat{\epsilon}} & \text{id}_{\mathcal{C}^2}. \end{array}$$

The map ϕ is in \mathcal{P} , and e is in \mathcal{D} : so by orthogonality, we induce a unique natural transformation $\hat{\psi}: \hat{S} \Rightarrow T$. The comonad morphism axioms for $\hat{\psi}$ now follow from the axioms for ψ and uniqueness of diagonal fill-ins. \square

4.8 We have thus reduced the problem of constructing free n.w.f.s.'s to the problem of constructing reflections along $\mathcal{G}_1: \mathbf{NWFS}(\mathcal{C}) \rightarrow \mathbf{LNWFS}(\mathcal{C})$. The key to constructing these will be to exhibit a monoidal structure on $\mathbf{LNWFS}(\mathcal{C})$ whose corresponding category of monoids is isomorphic to $\mathbf{NWFS}(\mathcal{C})$.

We will deduce the existence of this monoidal structure from a more general result characterising natural w.f.s.'s on \mathcal{C} as *bialgebra* objects in the category of functorial factorisations on \mathcal{C} . Now, usually when one considers bialgebra objects in a category, it is with reference to a symmetric or braided monoidal structure on that category: but here we will need something slightly more general.

4.9 By a *two-fold monoidal category* [3], we mean a category \mathcal{V} equipped with two monoidal structures $(\otimes, I, \alpha, \lambda, \rho)$ and $(\odot, \perp, \alpha', \lambda', \rho')$ in such a way that the functors $\odot: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ and $\perp: 1 \rightarrow \mathcal{V}$, together with the natural transformations α', λ' and ρ' , are lax monoidal with respect to the (\otimes, I) monoidal structure.

Of course, being lax monoidal is not merely a property of a functor, but extra structure on it: and in this case, the extra structure amounts to giving maps

$$m: \perp \otimes \perp \rightarrow \perp, \quad c: I \rightarrow I \odot I \quad \text{and} \quad j: I \rightarrow \perp$$

making (\perp, j, m) into a \otimes -monoid and (I, j, c) into a \odot -comonoid; together with a natural family of maps

$$z_{A,B,C,D}: (A \odot B) \otimes (C \odot D) \rightarrow (A \otimes C) \odot (B \otimes D)$$

obeying six coherence laws. It follows that \otimes and I are oplax monoidal with respect to the (\odot, \perp) monoidal structure; and in fact, we may take this as an alternative definition of two-fold monoidal category.

4.10 Examples:

- Any braided or symmetric monoidal category is two-fold monoidal, with the two monoidal structures coinciding; the maps $z_{A,B,C,D}$ are built from braidings/symmetries and associativity isomorphisms: c.f. [16].
- If \mathcal{V} is a cocomplete symmetric monoidal category, then the functor category $[X \times X, \mathcal{V}]$ has a two-fold monoidal structure. The first monoidal structure (\otimes, I) is given by matrix multiplication, whilst the second structure (\odot, \perp) is given pointwise.
- Similarly, if \mathcal{V} is a cocomplete symmetric monoidal category, then the functor category $[\mathbb{N}, \mathcal{V}]$ has a two-fold monoidal structure on it. The first monoidal structure (\otimes, I) is the *substitution tensor product*, with unit given by $I(n) = 0$

for $n \neq 1$ and $I(1) = I$; and binary tensor given by

$$(F \otimes G)(n) = \sum_{\substack{m, k_1, \dots, k_m \\ k_1 + \dots + k_m = n}} F(m) \otimes G(k_1) \otimes \dots \otimes G(k_m).$$

The second monoidal structure (\odot, \perp) is again given pointwise.

Further examples and applications to topology may be found in [3, 8].

4.11 A two-fold monoidal category $(\mathcal{V}, \otimes, I, \odot, \perp)$ provides a suitable environment to define a notion of bialgebra. Indeed, because the \odot -monoidal structure is lax monoidal with respect to the \otimes -structure, it lifts to the category $\mathbf{Mon}_{\otimes}(\mathcal{V})$ of \otimes -monoids in \mathcal{V} . Thus we define the category of (\otimes, \odot) -bialgebras to be

$$\mathbf{Bialg}_{\otimes, \odot}(\mathcal{V}) := \mathbf{Comon}_{\odot}(\mathbf{Mon}_{\otimes}(\mathcal{V})).$$

Now, because the \otimes -monoidal structure is also oplax monoidal with respect to the \odot -monoidal structure, it lifts to the category of \odot -comonoids in \mathcal{V} ; and thus we obtain an alternative definition of bialgebra by setting

$$\mathbf{Bialg}'_{\otimes, \odot}(\mathcal{V}) := \mathbf{Mon}_{\otimes}(\mathbf{Comon}_{\odot}(\mathcal{V})).$$

However, it is not hard to see that these two constructions yield isomorphic results. Indeed, to give an object of either $\mathbf{Bialg}(\mathcal{V})$ or $\mathbf{Bialg}'(\mathcal{V})$ is to give an object A of \mathcal{V} ; maps $\eta: I \rightarrow A$ and $\mu: A \otimes A \rightarrow A$ making it into a \otimes -monoid; and maps $\epsilon: A \rightarrow \perp$ and $\delta: A \rightarrow A \odot A$ making it into a \odot -comonoid; all subject to the commutativity of the following four diagrams:

$$\begin{array}{ccc}
\begin{array}{ccc} I & \xrightarrow{\eta} & A \\ c \downarrow & & \downarrow \Delta \\ I \odot I & \xrightarrow{\eta \odot \eta} & A \odot A \end{array} & \begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ \epsilon \otimes \epsilon \downarrow & & \downarrow \epsilon \\ \perp \otimes \perp & \xrightarrow{m} & \perp \end{array} & \begin{array}{ccc} & A & \\ \eta \nearrow & & \searrow \epsilon \\ I & \xrightarrow{j} & \perp \end{array} \\
& & & (4.1)
\end{array}$$

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\mu} & A \\
\Delta \otimes \Delta \downarrow & & \downarrow \Delta \\
(A \odot A) \otimes (A \odot A) & \xrightarrow{z_{A,A,A,A}} & (A \otimes A) \odot (A \otimes A) \xrightarrow{\mu \odot \mu} A \odot A
\end{array}$$

Likewise, to give a morphism of either $\mathbf{Bialg}(\mathcal{V})$ or $\mathbf{Bialg}'(\mathcal{V})$ is to give a map $f: A \rightarrow B$ of \mathcal{V} which is both a monoid morphism and a comonoid morphism. We

may summarise this by saying that, in the following diamond of forgetful functors

$$\begin{array}{ccc}
 & \mathbf{Bialg}_{\otimes, \odot}(\mathcal{V}) & \\
 \swarrow & & \searrow \\
 \mathbf{Comon}_{\odot}(\mathcal{V}) & & \mathbf{Mon}_{\otimes}(\mathcal{V}), \\
 \searrow & & \swarrow \\
 & \mathcal{V} &
 \end{array} \tag{4.2}$$

each west-pointing arrow forgets monoid structure, and each east-pointing arrow forgets comonoid structure.

4.12 Examples:

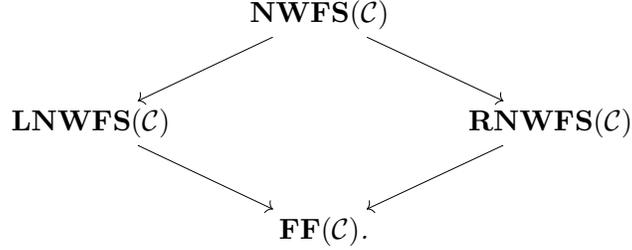
- If view a braided or symmetric monoidal category as a two-fold monoidal category, then a bialgebra in our sense is precisely a bialgebra in the usual sense.
- In the two-fold monoidal category $[X \times X, \mathcal{V}]$, a \otimes -monoid is a \mathcal{V} -category with object set X ; an \odot -comonoid is an $X \times X$ -indexed family of comonoids in \mathcal{V} ; and a (\otimes, \odot) -bialgebra is a *comonoidal* \mathcal{V} -category with object set X : which we may view either as a comonoid in $\mathcal{V}\text{-Cat}$, or as a \mathcal{V} -category whose homsets are comonoids and whose unit and composition maps are comonoid morphisms.
- A bialgebra in the two-fold monoidal category $[\mathbb{N}, \mathcal{V}]$ is what is sometimes called a *Hopf operad*: namely, an operad in \mathcal{V} whose objects of n -ary operations are comonoids; and whose substitution maps are morphisms of comonoids.

Bialgebras in two-fold monoidal categories play a central role in recent work [19] of François Lamarche.

4.13 Let us write $\mathbf{FF}(\mathcal{C})$ for the category of functorial factorisations on \mathcal{C} , and let us write $\mathbf{RNWFS}(\mathcal{C})$ for the category dual to $\mathbf{LNWFS}(\mathcal{C})$: so its objects are pairs (F, R) of a functorial factorisation F on \mathcal{C} together with an extension of the corresponding (R, Λ) to a monad.

4.14 Theorem: *There is a two-fold monoidal structure on $\mathbf{FF}(\mathcal{C})$ such that the diamond of forgetful functors (4.2) is, up-to-isomorphism, the diamond of forgetful*

functors



Proof. We begin by exhibiting two strict monoidal structures on $\mathbf{FF}(\mathcal{C})$. We do this by describing two different categories which are both isomorphic to $\mathbf{FF}(\mathcal{C})$, and which both admit obvious strict monoidal structures: then by transport of structure, we induce the required monoidal structures on $\mathbf{FF}(\mathcal{C})$.

The first category we consider is the category of domain-preserving copointed endofunctors and copointed endofunctor maps on \mathcal{C}^2 . It is easy to see that this category is isomorphic to $\mathbf{FF}(\mathcal{C})$; and that it has a strict monoidal structure (\odot, \perp) on it, with unit

$$\perp = (\text{id}_{\text{id}_{\mathcal{C}^2}} : \text{id}_{\mathcal{C}^2} \Rightarrow \text{id}_{\mathcal{C}^2})$$

and tensor product

$$(\Phi : L \Rightarrow \text{id}_{\mathcal{C}^2}) \odot (\Phi' : L' \Rightarrow \text{id}_{\mathcal{C}^2}) = (\Phi\Phi' : LL' \Rightarrow \text{id}_{\mathcal{C}^2}).$$

When we transport this along the isomorphism with $\mathbf{FF}(\mathcal{C})$, we obtain the following monoidal structure. The unit \perp is the functorial factorisation

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{f} Y \xrightarrow{\text{id}_Y} Y$$

and the tensor product $F' \odot F$ of two functorial factorisations $F, F' : \mathcal{C}^2 \rightarrow \mathcal{C}^3$ is given by

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda'_{\lambda_f}} K' \lambda_f \xrightarrow{\rho_f \circ \rho'_{\lambda_f}} Y.$$

Furthermore, to give a \odot -comonoid structure on some $F \in \mathbf{FF}(\mathcal{C})$ is to give a comonoid structure on the corresponding copointed (L, Φ) ; but this is precisely to extend it to a comonad on \mathcal{C}^2 . Thus we may identify $\mathbf{Comon}_{\odot}(\mathbf{FF}(\mathcal{C}))$ with $\mathbf{LWFS}(\mathcal{C})$.

The second category we consider is the category of codomain-preserving pointed endofunctors on \mathcal{C}^2 . Again, this is isomorphic to $\mathbf{FF}(\mathcal{C})$, and again, it has a strict monoidal structure given by composition. When we transport this back to $\mathbf{FF}(\mathcal{C})$, we obtain the monoidal structure whose unit I is the functorial factorisation

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\text{id}_X} X \xrightarrow{f} Y;$$

and whose tensor product $F' \otimes F$ is the functorial factorisation

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda'_{\rho_f} \circ \lambda_f} K' \rho_f \xrightarrow{\rho'_{\rho_f}} Y.$$

To make $F \in \mathbf{FF}(\mathcal{C})$ into a monoid with respect to this monoidal structure is now to give an extension of the corresponding (R, Λ) to a monad; and so we have $\mathbf{Mon}_{\otimes}(\mathbf{FF}(\mathcal{C})) \cong \mathbf{RNWFS}(\mathcal{C})$ as required.

We next show that these two monoidal structures on $\mathbf{FF}(\mathcal{C})$ can be made into a two-fold monoidal structure. Since I is initial and \perp terminal in $\mathbf{FF}(\mathcal{C})$, for this we need only give the family of interchange maps $z_{A,B,C,D}: (A \odot B) \otimes (C \odot D) \rightarrow (A \otimes C) \odot (B \otimes D)$: and this we do explicitly. The factorisation $(A \odot B) \otimes (C \odot D)$ sends a map $f: X \rightarrow Y$ to

$$X \xrightarrow{\lambda^A(\lambda^B(\rho_f^{C \odot D})) \circ \lambda^C(\lambda_f^D)} K^A(\lambda^B(\rho_f^{C \odot D})) \xrightarrow{\rho^B(\rho_f^{C \odot D}) \circ \rho^A(\lambda^B(\rho_f^{C \odot D}))} Y,$$

where $\rho_f^{C \odot D}$ abbreviates the map $\rho_f^D \circ \rho^C(\lambda_f^D)$; whilst $(A \otimes C) \odot (B \otimes D)$ sends f to

$$X \xrightarrow{\lambda^A(\rho^C(\lambda_f^{B \otimes D})) \circ \lambda^C(\lambda_f^{B \otimes D})} K^A(\rho^C(\lambda_f^{B \otimes D})) \xrightarrow{\rho^B(\rho_f^D) \circ \rho^A(\rho^C(\lambda_f^{B \otimes D}))} Y,$$

where $\lambda_f^{B \otimes D}$ abbreviates the map $\lambda^B(\rho_f^D) \circ \lambda_f^D$. To give $z_{A,B,C,D}$ we must therefore give suitable maps $K^A(\lambda^B(\rho_f^{C \odot D})) \rightarrow K^A(\rho^C(\lambda_f^{B \otimes D}))$. For this, we consider the following square:

$$\begin{array}{ccc} K^C(\lambda_f^D) & \xrightarrow{K^C(\text{id}_X, \lambda^B(\rho_f^D))} & K^C(\lambda_f^{B \otimes D}) \\ \lambda^B(\rho_f^{C \odot D}) \downarrow & & \downarrow \rho^C(\lambda_f^{B \otimes D}) \\ K^B(\rho_f^{C \odot D}) & \xrightarrow{K^B(\rho^C(\lambda_f^D), \text{id}_Y)} & K^B(\rho_f^D). \end{array}$$

This square commutes, with both sides equal to

$$K^C(\lambda_f^D) \xrightarrow{\rho^C(\lambda_f^D)} K^D f \xrightarrow{\lambda^B(\rho_f^D)} K^B(\rho_f^D),$$

and so we may view it as a morphism $\lambda^B(\rho_f^{C \odot D}) \rightarrow \rho^C(\lambda_f^{B \otimes D})$ in \mathcal{C}^2 : applying K^A to which yields the required map $K^A(\lambda^B(\rho_f^{C \odot D})) \rightarrow K^A(\rho^C(\lambda_f^{B \otimes D}))$ in \mathcal{C} . The (extensive) remaining details are left to the reader.

Thus we have a two-fold monoidal structure (\otimes, \odot) on $\mathbf{FF}(\mathcal{C})$: and to complete the proof, we must show that the corresponding bialgebras are precisely n.w.f.s.'s on \mathcal{C} . But to equip a functorial factorisation with both a \otimes -monoid and an \odot -comonoid structure is to give extensions of the corresponding (R, Λ) to a monad \mathbf{R} , and the corresponding (L, Φ) to a comonad \mathbf{L} ; and it is now a short calculation to show that the bialgebra axioms (4.1) will hold just when the distributivity axiom holds for (\mathbf{L}, \mathbf{R}) . \square

4.15 This Theorem implies that an object $X \in \mathbf{LNWFS}(\mathcal{C})$ will admit a reflection along the functor $\mathcal{G}_1: \mathbf{NWFS}(\mathcal{C}) \rightarrow \mathbf{LNWFS}(\mathcal{C})$ just when the free \otimes -monoid on X exists. But since the unit I of the monoidal structure on $\mathbf{LNWFS}(\mathcal{C})$ is also an initial object, to construct the free monoid on X is equally well to construct the free monoid on the pointed object $!: I \rightarrow X$. In order to do this, we may employ a standard transfinite construction: which we now describe.

4.16 If \mathbf{On} denotes the category of all small ordinals, then a *transfinite sequence* in a category \mathcal{V} is a functor $X: \mathbf{On} \rightarrow \mathcal{V}$, whose value at an ordinal α we denote by X_α , and whose value at the unique morphism $\alpha \rightarrow \beta$ (for $\alpha \leq \beta$) we denote by $X_{\alpha, \beta}: X_\alpha \rightarrow X_\beta$. We say that a transfinite sequence $X: \mathbf{On} \rightarrow \mathcal{V}$ *converges* at an ordinal γ if the maps $X_{\alpha, \beta}$ are isomorphisms for all $\beta \geq \alpha \geq \gamma$.

Let \mathcal{V} now be a cocomplete monoidal category. Given a pointed object $t: I \rightarrow T$ in \mathcal{V} , we may form a transfinite sequence $X: \mathbf{On} \rightarrow \mathcal{V}$ which we call the *free monoid sequence* for (T, t) . We build this sequence, together with a family of maps $\sigma_\alpha: T \otimes X_\alpha \rightarrow X_{\alpha+}$, by the following transfinite induction:

- $X_0 = I$, $X_1 = T$, $X_{0,1} = t$, and $\sigma_0 = \rho_T: T \otimes I \rightarrow T$;
- For a successor ordinal $\beta = \alpha^+$, we give X_β and $\sigma_\beta: T \otimes X_\beta \rightarrow X_{\beta+}$ by the following coequaliser diagram:

$$\begin{array}{ccccc}
 & & \sigma_\alpha & \rightarrow & X_\beta & \xrightarrow{t \otimes X_\beta} & T \otimes X_\beta & \xrightarrow{\sigma_\beta} & X_{\beta+}, \\
 & \nearrow & & & & & \nearrow & & \\
 T \otimes X_\alpha & & & & & & & & \\
 & \searrow & & & & & \searrow & & \\
 & & T \otimes t \otimes X_\alpha & \rightarrow & T \otimes T \otimes X_\alpha & \xrightarrow{T \otimes \sigma_\alpha} & T \otimes X_\beta & &
 \end{array}$$

and give $X_{\beta, \beta+}$ by the composite $\sigma_\beta \circ (t \otimes X_\beta)$;

- For a non-zero limit ordinal γ , we give X_γ by $\text{colim}_{\alpha < \gamma} X_\alpha$, with connecting maps $X_{\alpha, \gamma}$ given by the injections into the colimit. We give $X_{\gamma+}$ and σ_γ by

the following coequaliser diagram:

$$\begin{array}{ccc}
 & \text{colim } X_{\alpha^+} = X_{\gamma} & \\
 \text{colim } \sigma_{\alpha} \nearrow & & \searrow t \otimes X_{\gamma} \\
 \text{colim}(T \otimes X_{\alpha}) & \xrightarrow{\text{can}} & T \otimes \text{colim } X_{\alpha} = T \otimes X_{\gamma} \xrightarrow{\sigma_{\gamma}} X_{\gamma^+}
 \end{array}$$

where “can” is the map induced by the cocone $T \otimes X_{\alpha} \rightarrow T \otimes \text{colim } X_{\alpha}$. We give X_{γ, γ^+} by the composite $\sigma_{\gamma} \circ (t \otimes X_{\gamma})$.

The following is now Theorem 23.3 of [17].

4.17 Proposition: *Let \mathcal{V} be a cocomplete monoidal category in which each functor $(-) \otimes X: \mathcal{V} \rightarrow \mathcal{V}$ preserves connected colimits; and let $t: I \rightarrow T$ be a pointed object of \mathcal{V} . If the free monoid sequence for (T, t) converges at stage γ , then X_{γ} is the free monoid on (T, t) , with the universal map given by $X_{1, \gamma}: T \rightarrow X_{\gamma}$.*

In fact, this result is a mild generalisation of [17], since we require $(-) \otimes X$ to preserve only connected colimits, rather than all colimits; but it is trivial to check that this does not affect the argument in any way.

In order to apply this result, we observe that:

4.18 Proposition: *If \mathcal{C} is a cocomplete category, then $\mathbf{LNWFS}(\mathcal{C})$ is also cocomplete; and moreover, each functor $(-) \otimes X: \mathbf{LNWFS}(\mathcal{C}) \rightarrow \mathbf{LNWFS}(\mathcal{C})$ preserves connected colimits.*

Proof. We first note that the category $\mathbf{FF}(\mathcal{C})$ may be obtained by taking the category $[\mathcal{C}^2, \mathcal{C}]$, slicing this over the object $\text{cod}: \mathcal{C}^2 \rightarrow \mathcal{C}$; and then coslicing this under the object $v: \text{dom} \Rightarrow \text{cod}$ given by $v_f = f$ for all $f \in \mathcal{C}^2$. Consequently, $\mathbf{FF}(\mathcal{C})$ will be cocomplete whenever \mathcal{C} is. But by Theorem 4.14, the functor $U: \mathbf{LNWFS}(\mathcal{C}) \rightarrow \mathbf{FF}(\mathcal{C})$ is a forgetful functor from a category of comonoids, and as such creates colimits, so that $\mathbf{LNWFS}(\mathcal{C})$ is also cocomplete.

In order to see that each functor $(-) \otimes X: \mathbf{LNWFS}(\mathcal{C}) \rightarrow \mathbf{LNWFS}(\mathcal{C})$ preserves connected colimits, we consider the composite

$$V := \mathbf{LNWFS}(\mathcal{C}) \xrightarrow{U} \mathbf{FF}(\mathcal{C}) \xrightarrow{d_0 \circ (-)} [\mathcal{C}^2, \mathcal{C}],$$

where we recall that postcomposing with d_0 sends a functorial factorisation $F: \mathcal{C}^2 \rightarrow \mathcal{C}^3$ to the corresponding endofunctor $R: \mathcal{C}^2 \rightarrow \mathcal{C}^2$. It is easy to see that $d_0 \circ (-)$ creates connected colimits; and since U creates all colimits, we conclude that V creates connected colimits.

Now observe that V sends the monoidal structure on $\mathbf{LNWFS}(\mathcal{C})$ to the compositional monoidal structure on $[\mathcal{C}^2, \mathcal{C}^2]$, so that we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{LNWFS}(\mathcal{C}) & \xrightarrow{(-)\otimes X} & \mathbf{LNWFS}(\mathcal{C}) \\ V \downarrow & & \downarrow V \\ [\mathcal{C}^2, \mathcal{C}^2] & \xrightarrow{(-)\circ VX} & [\mathcal{C}^2, \mathcal{C}^2]. \end{array}$$

We wish to show that $(-)\otimes X$ preserves connected colimits: but because V creates them, it suffices to show that the composite around the top preserves connected colimits; and this follows from the fact that both functors V and $(-)\circ VX$ around the bottom preserve connected colimits. \square

Thus the free monoid on $X \in \mathbf{LNWFS}(\mathcal{C})$ will exist whenever the free monoid sequence for $!: I \rightarrow X$ converges. Sufficient conditions for convergence are given by Theorem 15.6 of [17], which when adapted to the present situation becomes:

4.19 Proposition: *Let \mathcal{V} be a cocomplete monoidal category, and let $t: I \rightarrow T$ be a pointed object of \mathcal{V} . If the functor $T \otimes (-): \mathcal{V} \rightarrow \mathcal{V}$ preserves either λ -filtered colimits; or λ -indexed unions of \mathcal{M} -subobjects for some proper, well-copowered $(\mathcal{E}, \mathcal{M})$ on \mathcal{V} , then the free monoid sequence for (T, t) converges.*

4.20 There is a problem if we apply this result with $\mathcal{V} = \mathbf{LNWFS}(\mathcal{C})$, since the second of the two smallness criteria requires a proper, well-copowered $(\mathcal{E}, \mathcal{M})$ on \mathcal{V} ; and even if we have such an $(\mathcal{E}, \mathcal{M})$ on the category \mathcal{C} , we will not, in general, be able to lift it to $\mathbf{LNWFS}(\mathcal{C})$. In order to resolve this problem, we consider again the composite

$$V := \mathbf{LNWFS}(\mathcal{C}) \xrightarrow{U} \mathbf{FF}(\mathcal{C}) \xrightarrow{d_0 \circ (-)} [\mathcal{C}^2, \mathcal{C}^2].$$

We saw above that this preserves both connected colimits and monoidal structure; and so takes the free monoid sequence on $!: I \rightarrow X$ in $\mathbf{LNWFS}(\mathcal{C})$ to the *free monad sequence* on the underlying pointed endofunctor $\Lambda: \text{id}_{\mathcal{C}^2} \Rightarrow R$ of X . Moreover, V reflects isomorphisms: hence the convergence of the latter sequence guarantees the convergence of the former.

Thus, it will suffice to apply Proposition 4.19 for $\mathcal{V} = [\mathcal{C}^2, \mathcal{C}^2]$, which avoids the problem described above, since any proper, well-copowered $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} can be lifted without trouble to $[\mathcal{C}^2, \mathcal{C}^2]$. In fact, it will suffice to lift to \mathcal{C}^2 , since when we instantiate Proposition 4.19 at $\mathcal{V} = [\mathcal{C}^2, \mathcal{C}^2]$, the requirement that $T \otimes (-): [\mathcal{C}^2, \mathcal{C}^2] \rightarrow [\mathcal{C}^2, \mathcal{C}^2]$ should preserve λ -filtered colimits or unions may be safely reduced to the requirement that $T: \mathcal{C}^2 \rightarrow \mathcal{C}^2$ should preserve the same.

We may summarise this argument as follows:

4.21 Proposition: *Let there be given a cocomplete category \mathcal{C} ; and let $(F, L) \in \mathbf{LNWFS}(\mathcal{C})$. If the functor $R = d_0 \circ F: \mathcal{C}^2 \rightarrow \mathcal{C}^2$ preserves either λ -filtered colimits; or λ -indexed unions of \mathcal{M} -subobjects for some proper, well-copowered $(\mathcal{E}, \mathcal{M})$ on \mathcal{C}^2 , then the free monoid sequence for $!: I \rightarrow (F, L)$ converges: and in particular, the reflection of (F, L) along $\mathcal{G}_1: \mathbf{NWFS}(\mathcal{C}) \rightarrow \mathbf{LNWFS}(\mathcal{C})$ exists.*

We are now ready to prove the first part of our main theorem:

4.22 Proposition: *Let \mathcal{C} be a cocomplete category satisfying one of the smallness conditions $(*)$ or (\dagger) , and let $I: \mathcal{J} \rightarrow \mathcal{C}^2$ be a category over \mathcal{C}^2 with \mathcal{J} small. Then the free n.w.f.s. on \mathcal{J} exists.*

Proof. By Proposition 4.6 and Proposition 4.7, we may find an object $(F, L) \in \mathbf{LNWFS}(\mathcal{C})$ which is a reflection of \mathcal{J} along $\mathcal{G}_3\mathcal{G}_2: \mathbf{LNWFS}(\mathcal{C}) \rightarrow \mathbf{CAT}/\mathcal{C}^2$. We now wish to apply Proposition 4.21 to (F, L) : so for a \mathcal{C} satisfying $(*)$, we will show that $R = d_0 \circ F: \mathcal{C}^2 \rightarrow \mathcal{C}^2$ preserves λ -filtered colimits for some λ ; whilst for a \mathcal{C} satisfying (\dagger) , we will show that R preserves λ -indexed unions of \mathcal{M} -subobjects for the induced factorisation system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C}^2 . Since the proof is the same in both cases, we restrict our attention to the former.

We begin by considering the following diagram:

$$\begin{array}{ccc} & & \mathcal{C}^2 \\ & & \nearrow^{d_0} \\ \mathcal{C}^2 & \xrightarrow{F} & \mathcal{C}^3 \\ & & \searrow_{d_2} \\ & & \mathcal{C}^2 \end{array}$$

The upper composite is $R: \mathcal{C}^2 \rightarrow \mathcal{C}^2$, which we are to show preserves λ -filtered colimits; but since d_0 and d_2 preserve and reflect connected colimits, we may equally well show that the lower composite $L = d_2 \circ F$ preserves λ -filtered colimits.

Now, from Proposition 4.6 and Proposition 4.7, the functor L has the following explicit description. First we form the left Kan extension of $I: \mathcal{J} \rightarrow \mathcal{C}^2$ along itself to obtain a functor $M: \mathcal{C}^2 \rightarrow \mathcal{C}^2$. We may describe this by the usual coend formula

$$M(f) = \int^{j \in \mathcal{J}} \mathcal{C}^2(I(j), f) \cdot I(j).$$

We now consider the counit transformation $\epsilon: M \Rightarrow \text{id}_{\mathcal{C}^2}$, whose component at f is the map

$$\epsilon_f: \int^{j \in \mathcal{J}} \mathcal{C}^2(I(j), f) \cdot I(j) \rightarrow f$$

corresponding to the identity transformation $\mathcal{C}^2(I(-), f) \Rightarrow \mathcal{C}^2(I(-), f)$; and we factor this transformation ϵ as

$$M \xrightarrow{\xi} L \xrightarrow{\Phi} \text{id}_{\mathcal{C}^2},$$

where each component of ξ is a pushout; and each component of Φ is the identity in its domain.

Let us first show that L preserves any colimit which M does. Suppose that $A: \mathcal{I} \rightarrow \mathcal{C}^2$ is a small diagram whose colimit is preserved by K , and consider the following diagram:

$$\begin{array}{ccccc} \text{colim}_i M A_i & \xrightarrow{\text{colim}_i \xi_{A_i}} & \text{colim}_i L A_i & \xrightarrow{\text{colim}_i \Phi_{A_i}} & \text{colim}_i A_i \\ \text{can}_M \downarrow & & & & \downarrow = \\ M \text{ colim}_i A_i & \xrightarrow{\xi_{\text{colim}_i A_i}} & L \text{ colim}_i A_i & \xrightarrow{\Phi_{\text{colim}_i A_i}} & \text{colim}_i A_i. \end{array} \quad (4.3)$$

The class \mathcal{P} of morphisms in \mathcal{C}^2 which are pushout squares is the left class of a strong factorisation system, and hence stable under colimit: and thus not only $\xi_{\text{colim}_i A_i}$, but also $\text{colim}_i \xi_{A_i}$, is in \mathcal{P} . Likewise, the class \mathcal{D} of morphisms in \mathcal{C}^2 which are domain-isomorphisms is also the left class of a strong factorisation system on \mathcal{C}^2 , whose corresponding right class is the class of codomain-isomorphisms. Hence \mathcal{D} is also stable under colimit; and so both $\Phi_{\text{colim}_i A_i}$ and $\text{colim}_i \Phi_{A_i}$ are in \mathcal{D} .

The orthogonality property for $(\mathcal{P}, \mathcal{D})$ now implies that there is a unique map $\phi: \text{colim}_i L A_i \rightarrow L \text{ colim}_i A_i$ rendering (4.3) commutative; and moreover, that ϕ is invertible, since can_M is. But the canonical morphism $\text{can}_L: \text{colim}_i L A_i \rightarrow L \text{ colim}_i A_i$ makes (4.3) commute; and so we deduce that $\text{can}_L = \phi$ is invertible as required.

Thus L preserves any colimit which M does: so we will be done if we can find some λ for which M preserves λ -filtered colimits. Now, for each $j \in \mathcal{J}$, we have the morphism $I(j): X \rightarrow Y$ of \mathcal{C}^2 : and by condition (*), we can find a λ_j for which both X and Y are λ_j -presentable; from which it follows that $I(j)$ is λ_j -presentable in \mathcal{C}^2 . Thus, if we take λ to be a regular cardinal larger than each λ_j , then each $I(j)$ is λ -presentable in \mathcal{C}^2 .

We now show that K preserves λ -filtered colimits. Indeed, suppose that $A: \mathcal{I} \rightarrow \mathcal{C}^2$ is a λ -filtered diagram in \mathcal{C}^2 ; then we have that

$$\begin{aligned} M(\text{colim}_i A_i) &= \int^{j \in \mathcal{J}} \mathcal{C}^2(I(j), \text{colim}_i A_i) \cdot I(j) \\ &\cong \int^{j \in \mathcal{J}} (\text{colim}_i \mathcal{C}^2(I(j), A_i)) \cdot I(j) \quad (\text{as } I(j) \text{ is } \lambda\text{-presentable}) \\ &\cong \text{colim}_i \int^{j \in \mathcal{J}} \mathcal{C}^2(I(j), A_i) \cdot I(j) \quad (\text{as colimits commute with colimits}) \\ &= \text{colim}_i M(A_i), \end{aligned}$$

as desired. □

5 Constructively-free implies algebraically-free

In this Section, we prove that all free n.w.f.s.'s obtained by the procedure of the previous Section are algebraically-free. In order to do this, we will need to establish a link between our notion of algebraically-free n.w.f.s., and [17]'s notion of algebraically-free monad. We begin, therefore, by recalling the latter.

5.1 Let $\sigma: \text{id} \Rightarrow S$ be a pointed endofunctor on some category \mathcal{V} . An *S-algebra* is an object $X \in \mathcal{V}$ together with a morphism $x: SX \rightarrow X$ satisfying $x.\sigma = \text{id}_X$; and an *S-algebra morphism* $(X, x) \rightarrow (Y, y)$ is a morphism $f: X \rightarrow Y$ of \mathcal{V} for which $f.x = y.Sf$. We write $S\text{-Alg}$ for the category of *S-algebras* and *S-algebra morphisms*. A *morphism of pointed endofunctors* $(S, \sigma) \Rightarrow (T, \tau)$ is a natural transformation $\alpha: S \Rightarrow T$ satisfying $\tau = \alpha.\sigma$; and any such morphism induces a functor $\alpha^*: T\text{-Alg} \rightarrow S\text{-Alg}$ sending (X, x) to $(X, x.\alpha_X)$.

If we are given a monad $\mathbb{T} = (T, \eta, \mu)$ on \mathcal{V} , we can consider its category $\mathbb{T}\text{-Alg}$ of algebras *qua* monad; or we can consider its category $T\text{-Alg}$ of algebras *qua* pointed endofunctor. Evidently, every \mathbb{T} -algebra is a T -algebra, and so we have an inclusion functor $\text{inc}: \mathbb{T}\text{-Alg} \rightarrow T\text{-Alg}$.

Now let (S, σ) be a pointed endofunctor on \mathcal{V} . We say that a monad $\mathbb{T} = (T, \eta, \mu)$ is *algebraically-free* on (S, σ) if we can provide a morphism of pointed endofunctors $\alpha: (S, \sigma) \Rightarrow (T, \eta)$ such that the composite

$$\mathbb{T}\text{-Alg} \xrightarrow{\text{inc}} T\text{-Alg} \xrightarrow{\alpha^*} S\text{-Alg}$$

is an isomorphism of categories.

The main result we will need about algebraically-free monads is the following, which is Theorem 22.3 of [17]:

5.2 Proposition: *Let \mathcal{V} be a cocomplete category, and let (S, σ) be a pointed endofunctor of \mathcal{V} . If the free monad sequence $X: \mathbf{On} \rightarrow [\mathcal{V}, \mathcal{V}]$ for (S, σ) converges at stage γ , then the morphism $X_{1,\gamma}: S \Rightarrow X_\gamma$ exhibiting X_γ as the free monad on S also exhibits it as the algebraically-free monad on S .*

5.3 We are now ready to prove the second part of our main Theorem. We suppose given a cocomplete \mathcal{C} , so that any small $I: \mathcal{J} \rightarrow \mathcal{C}^2$ over \mathcal{C}^2 has a reflection (F', L') along $\mathcal{G}_3\mathcal{G}_2$; and we now say that the free n.w.f.s. on such a \mathcal{J} *exists constructively* just when the free monoid sequence for (F', L') converges.

5.4 Proposition: *Let \mathcal{C} be a cocomplete category, and let $I: \mathcal{J} \rightarrow \mathcal{C}^2$ be a small category over \mathcal{C}^2 . If the free n.w.f.s. on \mathcal{J} exists constructively, then it is algebraically-free on \mathcal{J} .*

Proof. Let us write (L, R) for the free n.w.f.s. on \mathcal{J} , and (F', L') for the reflection of \mathcal{J} along $\mathcal{G}_3\mathcal{G}_2$. By constructive existence, we obtain (L, R) as the convergent value X_γ of the free monoid sequence on (F', L') ; and so if $\eta: \mathcal{J} \rightarrow \mathbf{L-Map}$ exhibits (L, R) as the free n.w.f.s. on \mathcal{J} , then the corresponding morphism $\alpha: (F', L') \rightarrow (F, L)$ of $\mathbf{LNWFS}(\mathcal{C})$ is the map $X_{1,\gamma}$ of this free monoid sequence. Now, applying the functor

$$V := \mathbf{LNWFS}(\mathcal{C}) \xrightarrow{U} \mathbf{FF}(\mathcal{C}) \xrightarrow{d_0 \circ (-)} [\mathcal{C}^2, \mathcal{C}^2]$$

to this free monoid sequence yields the free monad sequence for the pointed endofunctor $\Lambda': \text{id}_{\mathcal{C}^2} \Rightarrow R'$: and the convergence of the former guarantees the convergence of the latter. Thus by Proposition 5.2, we deduce that the map of pointed endofunctors $\alpha_r: (R', \Lambda') \rightarrow (R, \Lambda)$, obtained by applying V to α , exhibits R as the algebraically-free monad on (R', Λ') .

We now consider the following diagram:

$$\begin{array}{ccccc} \mathbf{R-Map} & \xrightarrow{\text{lift}} & \mathbf{L-Map}^{\text{th}} & \xrightarrow{\eta^{\text{th}}} & \mathcal{J}^{\text{th}} \\ \text{id} \downarrow & & \downarrow G & & \downarrow H \\ \mathbf{R-Alg} & \xrightarrow{\text{inc}} & \mathbf{R-Alg} & \xrightarrow{(\alpha_r)^*} & \mathbf{R'-Alg}. \end{array} \quad (5.1)$$

By algebraic-freeness of R , the composite along the bottom is an isomorphism; and we would like to deduce that the composite along the top is an isomorphism. To do this, it suffices to find isomorphisms G and H as indicated which make both squares commute.

We begin by constructing G . Recall that an object of $\mathbf{L-Map}^{\text{th}}$ is a pair (g, ϕ) consisting of a morphism $g: C \rightarrow D$ and a mapping ϕ which to each object $a \in \mathbf{L-Map}$ and square

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ U_L(a) \downarrow & & \downarrow g \\ B & \xrightarrow{k} & D \end{array}$$

in \mathcal{C} , assigns a fill-in $\phi(a, h, k): B \rightarrow C$ which is natural with respect to morphisms of $\mathbf{L-Map}$. Now, to give such a ϕ is equally well to give a natural transformation

$$\phi: \mathcal{C}^2(U_L(-), g) \Rightarrow \mathcal{C}^2(U_L(-), \text{id}_C): (\mathbf{L-Map})^{\text{op}} \rightarrow \mathbf{Set}$$

which is a section of the natural transformation $\mathcal{C}^2(U_L(-), \text{id}_C) \Rightarrow \mathcal{C}^2(U_L(-), g)$ induced by postcomposition with $(\text{id}_C, g): \text{id}_C \rightarrow g$. But $U_L: \mathbf{L-Map} \rightarrow \mathcal{C}^2$ has a

right adjoint given by the cofree functor $C_L : \mathcal{C}^2 \rightarrow \mathbf{L-Map}$; and thus we have an isomorphism

$$\mathcal{C}^2(U_L(-), g) \cong \mathbf{L-Map}(-, C_L(g)).$$

So $\mathcal{C}^2(U_L(-), g)$ is represented by $C_L(g)$; and thus by the Yoneda Lemma, ϕ is uniquely determined by where it sends the counit map $U_L C_L g \rightarrow g$; which is to say, by the fill-in it provides for the square

$$\begin{array}{ccc} C & \xrightarrow{\text{id}} & C \\ \lambda_g \downarrow & & \downarrow g \\ Kf & \xrightarrow{\rho_g} & D \end{array}$$

in \mathcal{C} . But to provide a fill-in for this square is precisely to make g into an algebra for the pointed endofunctor (R, Λ) . Thus we have an isomorphism between objects of $\mathbf{L-Map}^{\text{th}}$ and objects of $R\text{-Alg}$; and it is now straightforward to extend this to the required isomorphism of categories $G : \mathbf{L-Map}^{\text{th}} \rightarrow R\text{-Alg}$, and to verify that this G makes the left-hand square of (5.1) commute.

We now complete the proof by constructing the isomorphism $H : \mathcal{J}^{\text{th}} \rightarrow R'\text{-Alg}$. Proceeding as above, we see that to give an object of \mathcal{J}^{th} is to give a morphism $g : C \rightarrow D$ of \mathcal{C} together with a natural transformation $\phi : \mathcal{C}^2(I(-), g) \Rightarrow \mathcal{C}^2(I(-), \text{id}_C)$ which is a section of the natural transformation $\mathcal{C}^2(I(-), \text{id}_C) \Rightarrow \mathcal{C}^2(I(-), g)$ induced by postcomposing with $(\text{id}_C, g) : \text{id}_C \rightarrow g$. Now, if we write

$$Mg := \int^{j \in \mathcal{J}} \mathcal{C}^2(I(j), g) \cdot I(j)$$

and ϵ_g for the counit map $Mg \rightarrow g$ as before, then to give ϕ is equivalently to give a morphism $k : Mg \rightarrow \text{id}_C$ satisfying $\epsilon_g = (\text{id}_C, g) \circ k$. Furthermore, we obtain $L'g$ from Mg by factorising ϵ_g as

$$\epsilon_g = Mg \xrightarrow{\xi_g} L'g \xrightarrow{\Phi'_g} g,$$

where ξ_g is a pushout square, and Φ'_g is the identity in its domain; and so given such a map $k : Mg \rightarrow \text{id}_C$, applying unique diagonalisation to the diagram

$$\begin{array}{ccc} Mg & \xrightarrow{k} & \text{id}_C \\ \xi_g \downarrow & & \downarrow (\text{id}_C, g) \\ L'g & \xrightarrow{\Phi'_g} & g \end{array}$$

shows that k is induced by a unique morphism $m: L'g \rightarrow \text{id}_C$. But to give such a morphism is to give a diagonal fill-in for the square

$$\begin{array}{ccc} C & \xrightarrow{\text{id}} & C \\ \lambda'_g \downarrow & & \downarrow g \\ K'f & \xrightarrow{\rho'_g} & D \end{array}$$

in \mathcal{C} ; which in turn is to make g into an algebra for the pointed endofunctor (R', Λ') . The remaining details are again straightforward. \square

6 Comparison with the small object argument

Since we have advertised the argument of Theorem 4.4 as an adaptation of the small object argument, it behooves us to investigate the relationship between the two. To do this, we combine our main Theorem with Proposition 3.12 to deduce:

6.1 Proposition: *Let \mathcal{C} be a cocomplete category satisfying either of the smallness conditions (*) or (†); and let J be a set of maps in \mathcal{C} . Then the w.f.s. $(\mathcal{L}, \mathcal{R})$ cofibrantly generated by J exists.*

6.2 Since the two classes of maps \mathcal{L} and \mathcal{R} of this w.f.s. are entirely determined by the equations $\mathcal{L} = {}^{\#}\mathcal{R}$ and $\mathcal{R} = J^{\#}$, the content of this Proposition is that we may find an $(\mathcal{L}, \mathcal{R})$ -factorisation for every map of \mathcal{C} . This is also the content of the small object argument, and so we may compare the two by comparing the choices of factorisation which they provide. For a detailed account of the small object argument, we refer the reader to [6] or [15].

6.3 Suppose we are given a category \mathcal{C} and a set of maps J as in the Proposition; and let $g: C \rightarrow D$ be a morphism of \mathcal{C} that we wish to factorise. The first step in both the small object argument and our argument turns out to be the same. In the small object argument, we form the set S whose elements are squares

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{k} & D \end{array}$$

such that $f \in J$. We then form the coproduct

$$\begin{array}{ccc} \sum_{x \in S} A_x & \xrightarrow{[h_x]_{x \in S}} & C \\ \sum_{x \in S} f_x \downarrow & & \downarrow g \\ \sum_{x \in S} B_x & \xrightarrow{[k_x]_{x \in S}} & D \end{array} \quad (6.1)$$

and define an object $K'g$ and morphisms λ'_g and ρ'_g by factorising this square as

$$\begin{array}{ccccc} \sum_{x \in S} A_x & \xrightarrow{[h_x]_{x \in S}} & C & \xrightarrow{\text{id}_C} & C \\ \sum_{x \in S} f_x \downarrow & & \downarrow \lambda'_g & & \downarrow g \\ \sum_{x \in S} B_x & \xrightarrow{\xi_g} & K'g & \xrightarrow{\rho'_g} & D, \end{array} \quad (6.2)$$

where the left-hand square is a pushout.

On the other hand, suppose we view J as a discrete subcategory \mathcal{J} of \mathcal{C}^2 ; and write $I: \mathcal{J} \hookrightarrow \mathcal{C}^2$ for the inclusion functor. Then we may view (6.1) as the morphism

$$\epsilon_g: \int^{f \in \mathcal{J}} \mathcal{C}^2(I f, g) \cdot I f \rightarrow g,$$

of \mathcal{C}^2 ; which is to say, the component at g of the counit transformation

$$\epsilon: \text{Lan}_I(I) \Rightarrow \text{id}_{\mathcal{C}^2}: \mathcal{C}^2 \rightarrow \mathcal{C}^2.$$

We may then view (6.2) as the component at g of the factorisation of ϵ into a map which is componentwise a pushout, followed by a map whose domain components are identities. Thus the assignment $g \mapsto (\lambda'_g, \rho'_g)$ obtained from the small object argument is just the underlying factorisation of the reflection of $I: \mathcal{J} \hookrightarrow \mathcal{C}^2$ along $\mathcal{G}_3 \mathcal{G}_2$.

6.4 At this point, the two arguments under consideration diverge from each other. The small object argument is the more naive of the two: it simply iterates the above procedure, each time replacing the map g with the map ρ'_g . This gives rise to the countable sequence

$$\begin{array}{ccccccc} C & \xrightarrow{\lambda'_g} & K'g & \xrightarrow{\lambda'_{\rho'_g}} & K'\rho'_g & \xrightarrow{\lambda'_{\rho'_{\rho'_g}}} & \dots \\ \downarrow g & & \downarrow \rho'_g & & \downarrow \rho'_{\rho'_g} & & \\ D & \xrightarrow{\text{id}_D} & D & \xrightarrow{\text{id}_D} & D & \xrightarrow{\text{id}_D} & \dots \end{array}$$

which we extend transfinitely by taking colimits at limit ordinals. However, as pointed out in [1], this sequence *almost never converges*. Instead, the small object argument requires one to choose an arbitrary ordinal at which to stop: or rather, an ordinal which is large enough to ensure that the right part of the corresponding factorisation lies in J^{th} .

6.5 Our argument produces a different transfinite sequence, whose first few terms are:

$$\begin{array}{ccccccc} C & \xrightarrow{\lambda'_g} & K'g & \xrightarrow{\lambda''_g} & K''g & \xrightarrow{\lambda'''_g} & \cdots \\ \downarrow g & & \downarrow \rho'_g & & \downarrow \rho''_g & & \\ D & \xrightarrow{\text{id}_D} & D & \xrightarrow{\text{id}_D} & D & \xrightarrow{\text{id}_D} & \cdots \end{array};$$

here, $K''g$ is the coequaliser

$$K'g \begin{array}{c} \xrightarrow{\lambda'_{\rho'_g}} \\ \xrightarrow{K'(\lambda'_g, \text{id}_D)} \end{array} K'\rho'_g \longrightarrow K''g,$$

and in general, the term at stage α in this sequence will be a quotient of the corresponding term at stage α in the small object argument.

We may understand this quotienting process as follows. The small object argument provides a way of taking a map $g: C \rightarrow D$, and recursively adding elements to its domain which witness the required lifting properties against the set J . This process must be recursive, since the process of adding witnesses can create new instances of the lifting properties: which in turn will require new witnesses to be added, and so on.

However, the small object argument is badly behaved: at each stage it adds new witnesses for *all* instances of the required lifting properties – including those instances for which witnesses were added at a previous stage of the induction. The effect of the quotienting process which our argument carries out is to collapse these superfluous new witnesses back onto their predecessors.

7 Applications

We end the paper with two simple applications of Theorem 4.4.

7.1 In Examples 2.3, we saw that the set J of horn inclusions generates a plain w.f.s. (anodyne extensions, Kan fibrations) on \mathbf{SSet} . If we view the set J as a discrete subcategory $\mathcal{J} \hookrightarrow \mathbf{SSet}^2$, then it also generates a natural w.f.s. (\mathbf{L}, \mathbf{R}) .

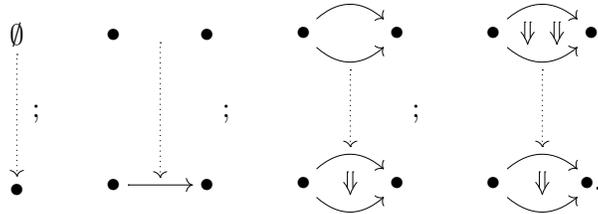
By restricting the monad $R: \mathbf{SSet}^2 \rightarrow \mathbf{SSet}^2$ of this natural w.f.s. to the slice over the terminal object, we obtain a monad $T: \mathbf{SSet} \rightarrow \mathbf{SSet}$, whose category of algebras is the category \mathbf{AlgKan} of “algebraic Kan complexes”: simplicial sets equipped with a chosen filler for every horn, subject to no further coherence conditions. Since \mathbf{AlgKan} is finitarily monadic over \mathbf{SSet} , it is locally finitely presentable, and hence provides a rich categorical base for further constructions.

Observe that the morphisms of \mathbf{AlgKan} are maps of simplicial sets which *strictly* preserve the chosen fillers. Whilst these maps are of some theoretical importance, we are more likely to be interested in the category \mathbf{AlgKan}_ψ whose objects are the same, but whose morphisms are arbitrary maps of simplicial sets. We may obtain this category by considering the adjunction

$$\mathbf{AlgKan} \begin{array}{c} \xrightarrow{U} \\ \dashv \top \\ \xleftarrow{F} \end{array} \mathbf{SSet}.$$

This generates a comonad FU on \mathbf{AlgKan} ; and the corresponding co-Kleisli category is precisely \mathbf{AlgKan}_ψ . In particular, we deduce that the inclusion functor $\mathbf{AlgKan} \hookrightarrow \mathbf{AlgKan}_\psi$ has a left adjoint. It is a corresponding result which forms the cornerstone of two-dimensional monad theory [5, Theorem 3.13].

7.2 For an example even more in the spirit of [5], we consider the category $\mathcal{C} = \mathbf{2-Cat}$ and the set of maps J given as follows:



These maps generate a plain w.f.s. which is one half of the model structure on $\mathbf{2-Cat}$ described by Lack in [18]. Our purpose here will be to consider the corresponding natural w.f.s. (L, R) generated by these maps, where as usual we view J as a discrete subcategory $\mathcal{J} \hookrightarrow \mathcal{C}^2$.

In particular, if we take the comonad L for this natural w.f.s. and restrict it to the coslice under the initial object, we obtain a comonad $Q: \mathbf{2-Cat} \rightarrow \mathbf{2-Cat}$. We can describe Q quite explicitly. Given a 2-category \mathcal{K} , we first form the free 2-category FUK on the underlying 1-graph of \mathcal{K} . Then we take the counit 2-functor $\epsilon_{\mathcal{K}}: FUK \rightarrow \mathcal{K}$ and factorise it as

$$\epsilon_{\mathcal{K}} = FUK \xrightarrow{\xi_{\mathcal{K}}} Q\mathcal{K} \xrightarrow{\phi_{\mathcal{K}}} \mathcal{K}$$

where $\xi_{\mathcal{K}}$ is bijective on objects and 1-cells, and $\phi_{\mathcal{K}}$ is locally fully faithful. The resultant $Q\mathcal{K}$ is precisely the “homomorphism classifier” of \mathcal{K} : it is characterised by an isomorphism, natural in \mathcal{L} , between

$$2\text{-functors } Q\mathcal{K} \rightarrow \mathcal{L} \quad \text{and} \quad \text{pseudofunctors } \mathcal{K} \rightarrow \mathcal{L}.$$

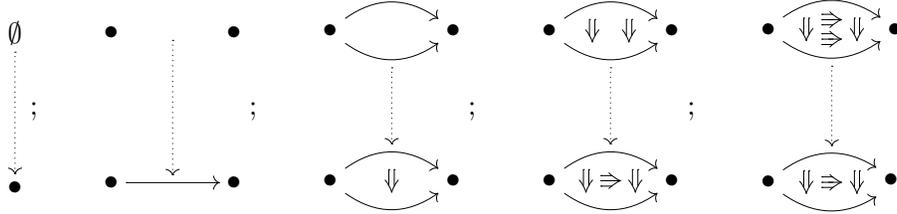
It follows from this characterisation that the co-Kleisli category of Q is the category $\mathbf{2-Cat}_\psi$ of 2-categories and pseudofunctors between them.

Observe that in this example, we at no point had to define what a “pseudofunctor” was: it emerged simply from applying our apparatus for a well-chosen set of maps J . Of course, since we already knew what pseudofunctors were, we did not gain much from this; however, it suggests that for a more complex \mathcal{C} , we may be able to define a suitable notion of “pseudomorphism” simply by applying the above argument for a suitable set of maps J .

As an example of this, let us consider the category \mathbf{Tricat} of tricategories and (strict) structure-preserving maps between them, and see how this argument allows us to derive the notion of trihomomorphism. By “tricategory”, we will mean [14]’s algebraic definition of tricategory, so that \mathbf{Tricat} is finitarily monadic over the category \mathbf{GSet}_3 of 3-dimensional globular sets; and in particular is locally finitely presentable. Let us write

$$\mathbf{Tricat} \begin{array}{c} \xrightarrow{U} \\ \dashv \! \! \dashv \\ \xleftarrow{F} \end{array} \mathbf{GSet}_3$$

for the free/forgetful adjunction. We define a set J of morphisms in \mathbf{Tricat} by taking the following set of maps in \mathbf{GSet}_3 :



and applying the free functor F to each of them. We now proceed as before: we consider this set J as a discrete subcategory $\mathcal{J} \hookrightarrow \mathbf{Tricat}^2$ and let (L, R) be the n.w.f.s. generated by \mathcal{J} ; and then let Q be the comonad on \mathbf{Tricat} given by the restriction of L to the coslice under the initial object.

We now *define* a trihomomorphism $\mathcal{S} \rightarrow \mathcal{T}$ to be a strict morphism $Q\mathcal{S} \rightarrow \mathcal{T}$, and define the category \mathbf{Tricat}_ψ of tricategories and trihomomorphisms to be the co-Kleisli category of Q . The notion of trihomomorphism we obtain in this way cannot be the one we are used to from [12], since the latter does not admit a strictly associative composition: see [11]. Nonetheless, we can show that our new

notion of trihomomorphism is equivalent to the old one, in that we can exhibit a biequivalence between a suitably defined 2-category of these new trihomomorphisms and a corresponding bicategory of the usual ones.

The full details of this will be worked out in a forthcoming paper; but for now, let us merely say that this method should immediately extend to (sufficiently algebraic) weak n -categories and even weak ω -categories, thereby allowing us to give a notion of “weak morphism of ω -categories” which admits a strictly associative composition.

A Algebraically-free implies free

The purpose of this Appendix is to sketch a proof of the following result:

A.1 Theorem: *Let (L, R) be a n.w.f.s. on \mathcal{C} which is algebraically-free on $I: \mathcal{J} \rightarrow \mathcal{C}^2$. Then (L, R) is free on \mathcal{J} .*

Proof. We first define a monoidal structure on the category $\mathbf{CAT}/\mathcal{C}^2$. Given $U: \mathcal{A} \rightarrow \mathcal{C}^2$ and $V: \mathcal{B} \rightarrow \mathcal{C}^2$, their tensor product $W: \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}^2$ is obtained by first taking the pullback

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{B} & \longrightarrow & \mathcal{A} \\ \downarrow \lrcorner & & \downarrow U \\ \mathcal{B} & \xrightarrow{V} & \mathcal{C}^2 \\ & & \downarrow \text{dom} \\ & & \mathcal{C}, \end{array}$$

and then defining the projection W by $W(a, b) = Ua \circ Vb$. The unit for this tensor product is the object $(s_0: \mathcal{C} \rightarrow \mathcal{C}^2)$, where s_0 is the functor induced by homming the unique map $\sigma_0: \mathbf{2} \rightarrow \mathbf{1}$ into \mathcal{C} ; thus $s_0(c) = \text{id}_c: c \rightarrow c$.

Next, we show that, for any n.w.f.s. (L, R) on \mathcal{C} , the object $(U_L: \mathbf{L-Map} \rightarrow \mathcal{C}^2)$ is a monoid with respect to this monoidal structure: the key point being that, given L-map structures on $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we may define an L-map structure on $gf: X \rightarrow Z$. Indeed, if these two L-map structures are provided by morphisms $s: Y \rightarrow Kf$ and $t: Z \rightarrow Kg$ (as in §2.15), then the L-map structure on the composite gf is given by:

$$Z \xrightarrow{t} Kg \xrightarrow{K(s, \text{id}_Z)} K(g \circ \rho_f) \xrightarrow{K(K(1, g), 1)} K\rho_{gf} \xrightarrow{\pi_{gf}} K(gf).$$

The remaining details are routine; and by dualising, we see that $U_R: \mathbf{R-Map} \rightarrow \mathcal{C}^2$ is also a monoid in $\mathbf{CAT}/\mathcal{C}^2$.

We may now show that, if $\alpha: (L, R) \rightarrow (L', R')$ is a map of n.w.f.s.’s, then the induced functors $(\alpha_l)_*: \mathbf{L-Map} \rightarrow \mathbf{L'-Map}$ and $(\alpha_r)^*: \mathbf{R'-Map} \rightarrow \mathbf{R-Map}$ are maps

of monoids; so that the semantics functors \mathcal{G} and \mathcal{H} may be lifted to functors

$$\hat{\mathcal{G}}: \mathbf{NWFS}(\mathcal{C}) \rightarrow \mathbf{Mon}(\mathbf{CAT}/\mathcal{C}^2)$$

and $\hat{\mathcal{H}}: \mathbf{NWFS}(\mathcal{C}) \rightarrow (\mathbf{Mon}(\mathbf{CAT}/\mathcal{C}^2))^{\text{op}}$.

We now arrive at a crucial juncture in the proof: we show that $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$ are fully faithful. In the case of $\hat{\mathcal{G}}$, for example, we consider n.w.f.s.'s (\mathbf{L}, \mathbf{R}) and $(\mathbf{L}', \mathbf{R}')$ on \mathcal{C} , and a map of monoids $F: \mathbf{L}\text{-Map} \rightarrow \mathbf{L}'\text{-Map}$ over \mathcal{C}^2 ; and must show that there is a unique morphism $\alpha: (\mathbf{L}, \mathbf{R}) \rightarrow (\mathbf{L}', \mathbf{R}')$ for which $F = (\alpha_l)_*$. To do this, we consider squares of the following form:

$$\begin{array}{ccc} A & \xrightarrow{\lambda'_f} & K'f \\ \lambda_f \downarrow & & \downarrow \rho'_f \\ Kf & \xrightarrow{\rho_f} & B. \end{array}$$

We can make ρ'_f into an \mathbf{R}' -map, since it is the free \mathbf{R}' -map on f . Similarly, we can make λ_f into an \mathbf{L} -map; and by applying the functor $F: \mathbf{L}\text{-Map} \rightarrow \mathbf{L}'\text{-Map}$, we may make it into an \mathbf{L}' -map. Now we apply the lifting operation associated with $(\mathbf{L}', \mathbf{R}')$ to obtain a morphism $\alpha_f: Kf \rightarrow K'f$. These maps α_f provide the components of a morphism between the underlying functorial factorisations of (\mathbf{L}, \mathbf{R}) and $(\mathbf{L}', \mathbf{R}')$: it remains only to check that the comonad and monad structures are preserved. This is just a matter of checking details, but makes essential use of two facts: that F is a map of monoids; and that the distributivity axiom holds in (\mathbf{L}, \mathbf{R}) and $(\mathbf{L}', \mathbf{R}')$.

Next, we prove that for any category $U: \mathcal{A} \rightarrow \mathcal{C}^2$ over \mathcal{C}^2 , the category $\mathcal{A}^{\text{th}} \rightarrow \mathcal{C}^2$ is a monoid in $\mathbf{CAT}/\mathcal{C}^2$. The key point is to show that, whenever we equip morphisms $f: C \rightarrow D$ and $g: D \rightarrow E$ of \mathcal{C} with coherent choices of liftings against the elements of \mathcal{A} , we induce a corresponding equipment on the composite gf . Indeed, given $a \in \mathcal{A}$ and a square

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ \downarrow Ua & & \downarrow f \\ & & D \\ & & \downarrow g \\ B & \xrightarrow{k} & E, \end{array}$$

we define $\phi_{gf}(a, h, k): B \rightarrow C$ as follows. First we form $j := \phi_g(a, fh, k): B \rightarrow D$; and now we take $\phi_{gf}(a, h, k) := \phi_f(a, h, j): B \rightarrow C$.

We may now check that if $F: \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of $\mathbf{CAT}/\mathcal{C}^2$, then the morphism $F^\natural: \mathcal{B}^\natural \rightarrow \mathcal{A}^\natural$ respects the monoid structures on \mathcal{A}^\natural and \mathcal{B}^\natural , so that the functors $(-)^\natural$, and dually ${}^\natural(-)$, lift to functors

$$\begin{aligned} & (-)^\natural: (\mathbf{CAT}/\mathcal{C}^2)^{\text{op}} \rightarrow \mathbf{Mon}(\mathbf{CAT}/\mathcal{C}^2) \\ \text{and} \quad & {}^\natural(-): \mathbf{CAT}/\mathcal{C}^2 \rightarrow (\mathbf{Mon}(\mathbf{CAT}/\mathcal{C}^2))^{\text{op}}. \end{aligned}$$

Finally, we may show that for any n.w.f.s. (L, R) on \mathcal{C} , the canonical operation of lifting $\text{lift}: \mathbf{R-Map} \rightarrow \mathbf{L-Map}^\natural$ is a monoid morphism in $\mathbf{CAT}/\mathcal{C}^2$. Again, this is simply a matter of checking details.

We now have all the material we need to prove the Theorem. We suppose ourselves given a n.w.f.s. (L, R) which is algebraically-free on $I: \mathcal{J} \rightarrow \mathcal{C}^2$ via the morphism $\eta: \mathcal{J} \rightarrow \mathbf{L-Map}$: and are required to show that (L, R) is free on \mathcal{J} . So consider a further n.w.f.s. (L', R') on \mathcal{C} , and a morphism $F: \mathcal{J} \rightarrow \mathbf{L'-Map}$ over \mathcal{C}^2 . We can form the following diagram of functors over \mathcal{C}^2 :

$$\begin{array}{ccc} \mathbf{R-Map} & \xrightarrow{\text{lift}} & \mathbf{L-Map}^\natural & \xrightarrow{\eta^\natural} & \mathcal{J}^\natural. \\ & & & \nearrow F^\natural & \\ \mathbf{R'-Map} & \xrightarrow{\text{lift}} & \mathbf{L'-Map}^\natural & & \end{array}$$

By algebraic-freeness, the composite along the top is invertible, and so we obtain from this diagram a functor $\mathbf{R'-Map} \rightarrow \mathbf{R-Map}$. But every map in the diagram is a map of monoids, and hence the induced functor $\mathbf{R'-Map} \rightarrow \mathbf{R-Map}$ is too; and so is induced by a unique morphism of n.w.f.s.'s $\alpha: (L, R) \rightarrow (L', R')$.

It requires a little more work to show $(\alpha_l)_* \circ \eta = F$, and that α is the unique morphism of n.w.f.s.'s with this property. The two essential facts that we need are that, for any n.w.f.s. (L, R) , the canonical morphism $\mathbf{L-Map} \rightarrow {}^\natural(\mathbf{R-Map})$ is a monomorphism; and that, for any morphism of n.w.f.s.'s $\alpha: (L, R) \rightarrow (L', R')$, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{R-Map} & \xrightarrow{\text{lift}} & \mathbf{L-Map}^\natural \\ (\alpha_r)_* \downarrow & & \downarrow ((\alpha_l)_*)^\natural \\ \mathbf{R'-Map} & \xrightarrow{\text{lift}} & \mathbf{L'-Map}^\natural. \end{array}$$

We leave these details to the reader. □

References

- [1] J. Adámek, H. Herrlich, J. Rosický, and W. Tholen. On a generalized small-object argument for the injective subcategory problem. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 43(2):83–106, 2002.
- [2] H. Applegate and M. Tierney. Categories with models. In *Seminar on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 156–244. Springer, Berlin, 1969.
- [3] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, and R. Vogt. Iterated monoidal categories. *Advances in Mathematics*, 176(2):277–349, 2003.
- [4] J. Beck. Distributive laws. In *Seminar on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 119–140. Springer, Berlin, 1969.
- [5] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *Journal of Pure and Applied Algebra*, 59(1):1–41, July 1989.
- [6] A. K. Bousfield. Constructions of factorization systems in categories. *Journal of Pure and Applied Algebra*, 9(2-3):207–220, 1977.
- [7] E. J. Dubuc. *Kan extensions in enriched category theory*. Lecture Notes in Mathematics, Vol. 145. Springer-Verlag, Berlin, 1970.
- [8] S. Forcey. Enrichment over iterated monoidal categories. *Algebraic & Geometric Topology*, 4:95–119 (electronic), 2004.
- [9] P. J. Freyd and G. M. Kelly. Categories of continuous functors. I. *Journal of Pure and Applied Algebra*, 2:169–191, 1972.
- [10] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 221.
- [11] R. Garner and N. Gurski. The low-dimensional structures that tricategories form. arXiv preprint 0711.1761.
- [12] R. Gordon, A. J. Power, and R. Street. Coherence for tricategories. *Memoirs of the American Mathematical Society*, 117(558), 1995.
- [13] M. Grandis and W. Tholen. Natural weak factorisation systems. *Archivum Mathematicum*, 42:397–408, 2006.
- [14] N. Gurski. *An algebraic theory of tricategories*. PhD thesis, University of Chicago, 2006.

- [15] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [16] A. Joyal and R. Street. Braided tensor categories. *Advances in Mathematics*, 102(1):20–78, 1993.
- [17] G. M. Kelly. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bulletin of the Australian Mathematical Society*, 22(1):1–83, 1980.
- [18] S. Lack. A Quillen model structure for 2-categories. *K-Theory*, 26(2):171–205, June 2002.
- [19] F. Lamarche. Exploring the gap between linear and classical logic. *Theory and Applications of Categories*, 18(17):473–535, 2007.
- [20] D. G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.