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Hopf Rings, Dieudonné Modules, and $E_*\Omega^2S^3$

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This paper is dedicated J. Michael Boardman

ABSTRACT. The category of graded, bicommutative Hopf algebras over the prime field with p elements is an abelian category which is equivalent, by work of Schoeller, to a category of graded modules, known as Dieudonné modules. Graded ring objects in Hopf algebras are called Hopf rings, and they arise in the study of unstable cohomology operations for extraordinary cohomology theories. The central point of this paper is that Hopf rings can be studied by looking at the associated ring object in Dieudonné modules. They can also be computed there, and because of the relationship between Brown-Gitler spectra and Dieudonné modules, calculating the Hopf ring for a homology theory E_* comes down to computing $E_*\Omega^2S^3$ – which Ravenel has done for $E = BP$. From this one recovers the work of Hopkins, Hunton, and Turner on the Hopf rings of Landweber exact cohomology theories.

The are two major algebraic difficulties encountered in this approach. The first is to decide what a ring object is in the category of Dieudonné modules, as there is no obvious symmetric monoidal pairing associated to a tensor product of modules. The second is to show that Hopf rings pass to rings in Dieudonné modules. This involves studying universal examples, and here we pick up an idea suggested by Bousfield: torsion-free Hopf algebras over the p-adic integers with some additional structure, such as a self-Hopf-algebra map that reduces to the Verschiebung, can be easily classified.

An abelian category $\mathcal{A}$ with a set of small projective generators is equivalent to a category $\mathcal{M}$ of modules over some ring R. In addition, if $\mathcal{A}$ and $\mathcal{M}$ are symmetric monoidal categories and the equivalence of categories $\mathcal{A} \to \mathcal{M}$ respects the monoidal structure, one can study the ring objects in $\mathcal{A}$ by studying the ring objects in $\mathcal{M}$. The purpose of this paper is to develop this observation in the case where $\mathcal{A}$ is the category $\mathcal{HA}_+$ of graded, bicommutative Hopf algebras over the prime field $\mathbb{F}_p$. The graded ring objects in $\mathcal{HA}^+$ are called Hopf rings and they arise naturally when studying unstable cohomology operations for some cohomology theory E^* (see [**2, 11, 21**]).

To state some results, fix a prime $p > 2$. (A slight rewording gives the results at $p = 2$). We will restrict attention to the sort of Hopf algebra that arises in algebraic topology; namely, to Hopf algebras that are skew-commutative and so

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that the degree 0 part H_0 of H is the group algebra of an abelian group. We will call the category of such Hopf algebras $\mathcal{HA}_\pm$. If X is a pointed space, then $H_*(\Omega^2 X;\mathbb{F}_p) = H_*\Omega^2 X \in \mathcal{HA}_\pm$. Schoeller [**23**] has essentially proved that there is an equivalence of categories

$$D_* : \mathcal{HA}_\pm \to \mathcal{D}\pm$$

where $\mathcal{D}_\pm$ is the category of graded Dieudonné modules. An object $M \in \mathcal{D}_\pm$ is a non-negatively graded abelian group M so that M_{2n+1} is an $\mathbb{F}_p$ vector space and there are homomorphisms

$$F : M_{2n} \to M_{2pn} \text{ and } V : M_{2n} \to M_{2n/p}$$

so that $FV = VF = p$ and $V = 1$ if $n = 0$. If p does not divide n, we set $V = 0$. It follows that if $2n = 2p^k s$, $(p, s) = 1$, then $p^{k+1}M_{2n} = 0$. If $H \in \mathcal{HA}_\pm$, then the action of F and V on D_*H reflect the Frobenius and Verschiebung, respectively, of H.

The category $\mathcal{HA}_\pm$ is a symmetric monoidal category. As with the tensor product of abelian groups, the symmetric monoidal pairing arises by considering bilinear maps. An example of a bilinear map in $\mathcal{HA}_\pm$ is supplied by considering a ring spectrum E. The functor $X \mapsto E^n X$ is representable in the homotopy category of spaces; indeed, if $E(n) = \Omega^\infty\Sigma^n E$, then for all CW complexes X one has

$$[X, E(n)] \cong E^n X.$$

The cup-product pairing $E^n X \times E^m X \to E^{n+m} X$ is induced by a map of spaces

$$E(n) \wedge E(m) \to E(n + m)$$

and the resulting map of coalgebras

$$H_*E(n) \otimes H_*E(m) \to H_*E(n + m)$$

is a bilinear map of Hopf algebras. One can axiomatize this situation (see §5 or [**21**]) and, following Hunton and Turner [**13**], we prove in §7 that given H and K in $\mathcal{HA}_\pm$, there is a universal bilinear map

$$H \otimes K \to H \boxtimes K.$$

The pairing $\boxtimes : \mathcal{HA}_\pm \times \mathcal{HA}_\pm \to \mathcal{HA}_\pm$ is symmetric monoidal; the unit is the group ring $\mathbb{F}_p[\mathbb{Z}]$.

Next one would like to calculate $D_*(H \boxtimes K)$. If $\phi : H_1 \otimes H_2 \to K$ is a bilinear pairing of Hopf algebras, one obtains a bilinear pairing of Dieudonné modules

$$D_*\phi : D_*H_1 \times D_*H_2 \to D_*K$$

in the sense that $D_*\phi$ is a bilinear map of graded abelian groups, and

$$VD\phi_*(x, y) = D\phi_*(Vx, Vy)$$

and

$$D\phi_*(Fx, y) = F(D\phi_*(x, Vy)) \qquad D\phi_*(x, Fy) = F(D\phi_*(Vx, y)).$$

If $M, N \in \mathcal{D}$ there is a universal such bilinear pairing

$$M \times N \to M \boxtimes_{\mathcal{D}} N$$

which is easy to write down (see Equation 7.6). Then $\boxtimes_{\mathcal{D}}$ is a symmetric monoidal pairing and one of the main results is $D_*(H \boxtimes K) \cong D_*H \boxtimes_{\mathcal{D}} D_*K$. See Theorem 7.7. This leads to effective computations.

We then employ this to study Hopf rings. If E is a ring spectrum, then $H_*\mathbf{E} = \{H_*E(n)\}$ is a $\mathbb{Z}$-graded ring object in $\mathcal{HA}_\pm$ and hence $D_*H_*\mathbf{E}$ is a $\mathbb{Z}$-graded ring object in $\mathcal{D}_\pm$. This last means that given

$$x \in D_mH_*E(j) \qquad\qquad y \in D_nH_*E(k)$$

there is a product $x \circ y \in D_{m+n}H_*E(j+k)$ so that

$$V(x\circ y) = Vx \circ Vy, \quad (Fx)\circ y = F(x \circ Vy), \quad x \circ Fy = F((Vx)\circ y).$$

Such an object will be called a Dieudonné ring. If the ring spectrum is homotopy commutative, then this product satisfies the following skew commutativity formula:

$$x \circ y = (-1)^{nm+jk} y \circ x.$$

Since

$$D_0H_*E(k) = \pi_0E(k) = E^{-k},$$

$D_*H_*\mathbf{E}$ is actually an E^* algebra; so $D_*H_*\mathbf{E}$ is an E^*-Dieudonné algebra.

To compute this object, we use the fact the functor on spectra

$$X \mapsto D_nH_*\Omega^\infty X$$

which assigns to a space X the degree n part of of $D_*H_*\Omega^\infty X$ is actually part of a homology theory if $n \not\equiv \pm 1 \mod (2p)$. In fact, by [**9**], if $B(n)$ is the nth Brown-Gitler spectrum, there is a natural surjection

$$B(n)_nX \to D_nH_*\Omega^\infty X$$

which is an isomorphism if $n \not\equiv \pm 1 \mod (2p)$. Thus, for a ring spectrum E one obtains a surjection

$$E_kB(n) \to D_nH_*E(n-k)$$

and if $\mathcal{B} = \{B(n)\}_{n\geq 0}$ is the graded Brown-Gitler spectrum, one gets a degree-shearing surjection

$$E_*\mathcal{B} \to D_*H_*\mathbf{E}$$

of bigraded groups. In fact this can be made into a morphism of E_* Dieudonné algebras (see §10). In many cases, the kernel of this map can be analyzed. Furthermore, the Snaith splitting of Ω^2S^3 and the analysis of the summands done by a variety of authors ([**3, 5, 12, 16**]) shows that there is a filtration on $E_*\Omega^2S^3$ so that the associated graded object is $E_*\mathcal{B}$. Since Ravenel [**19**] has effectively calculated $BP_*\Omega^2S^3$, one can deduce a great deal about $D_*H_*\mathbf{E}$ for Landweber exact theories. In particular one can recover the Hopkins-Hunton-Turner results [**11, 13**] which, in this form, have a pleasing statement – one that, in fact, succinctly encodes the Ravenel-Wilson relation of complex oriented theories. (See Theorem 10.3.)

This paper is divided into three sections. The first two are devoted to the algebra of Hopf algebras, their associated Dieudonné modules, and the appropriate bilinear pairings. We spend most of our energy discussing graded commutative (as opposed to skew commutative) Hopf algebras, moving on to skew commutative Hopf algebras and the topological applications cited above only in the third section. Because of the splitting principle for skew-commutative Hopf algebras (see [**18**] and Proposition 9.1) the passage from commutative to skew-commutative is easy. As above, we denote graded bicommutative Hopf algebras by $\mathcal{HA}^+$.

In order to come to grips with some of the algebra involved we spend a great deal of time working with universal examples. The projective generators of $\mathcal{HA}^+$ include the Hopf algebras

$$H(n) = \mathbb{F}_p[x_0, x_1, \dots, x_k]$$

where $n = 2p^k s$, $(p,s) = 1$, $\deg(x_i) = 2p^i s$, all with Witt vector diagonal. This is the reduction module p of a Hopf algebra over the p-adic integers $\mathbb{Z}_p$

$$CW_s(k) = \mathbb{Z}_p[x_0, x_1, \dots, x_k]$$

with the unique diagonal so that the Witt polynomials

$$w_i = x_0^{p^i} + px_1^{p^{i-1}} + \cdots + p^i x_i$$

are primitive. (The CW stands for co-Witt.) The $\mathbb{Z}_p$-Hopf algebra $CW_s(k)$ comes equipped with a lift of the Verschiebung; that is, there is a degree lowering Hopf algebra map

$$\psi : CW_s(k) \longrightarrow CW_s(k)$$

which reduces to the Verschiebung $\xi : H(n) \to H(n)$. Picking up a thread suggested by Bousfield [**1**], it turns out that torsion-free, graded, connected Hopf algebras over $\mathbb{Z}_p$ with a lift of the Verschiebung are completely classified by their indecomposables. Furthermore, if H is such a Hopf algebra, then $D_*(\mathbb{F}_p \otimes_{\mathbb{Z}_p} H)$ can be simply computed in terms of QH. (See Theorem 4.8.) This and other related topics occupy the first four sections. This is the first part of the paper.

The second part of the paper is devoted to bilinear pairings, developing the formulas cited above, and proving the isomorphism

$$D_*H \boxtimes_{\mathcal{D}} \mathcal{D}_*K \cong D_*(H \boxtimes K).$$

There is a table of contents at the end of this introduction and a glossary of symbols before the references.

This project has it roots in a conversation with Bill Dwyer, who noted that the work of Moore and Smith [**18**] shows that the functor

$$Z \mapsto H_*\Omega Z$$

has excellent exactness properties when Z is a loop space. Furthermore, Dwyer suggested, this fact could be used to study Hopf rings. I knew these exactness properties as the statement that, on spectra, $X \mapsto D_*H_*\Omega^\infty X$ was part of a homology theory.

Much of what is done here can be greatly generalized. The restriction to the prime field is dictated by homotopy theory, not by algebra, and all of the algebraic results pass to any perfect field of characteristic p. The internal grading on the Hopf algebras can probably also be dropped, although some care must be taken to deal with those Hopf algebras that are neither group rings nor connected in the sense of [**6**].

Contents

Part I: Classifying Hopf Algebras

1. The Dwork Lemma and algebras with a lift of the Frobenius

This preliminary section introduces the basic tool we will use for constructing morphims of Hopf algebras. Fix a prime p.

Let $x_0, x_1, \dots$ be a sequence of indeterminants and let $\mathbb{Z}_p[x_0, x_1, x_2, \dots]$ be the free graded commutative algebra over the p-adic integers $\mathbb{Z}_p$ in the indeterminants x_i. Let

$$w_n = w_n(x) = w_n(x_0, \dots, x_n) = x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^n x_n$$

be the nth Witt polynomial. Here and below, $x = (x_0, x_1, \dots)$.

The following is known as the Dwork Lemma.

Lemma 1.1. *Let A be a commutative torsion-free $\mathbb{Z}_p$ algebra. Suppose A has a ring endomorphism*

$$\varphi : A \to A$$

so that $\varphi(x) \equiv x^p \mod p$. Then, given a sequence of elements $g_n \in A$, $n \geq 0$, so that

$$g_n \equiv \varphi g_{n-1} \mod p^n,$$

there are unique elements q_n, $n \geq 0$, so that

$$w_n(q) = w_n(q_0, q_1, \dots, q_n) = g_n.$$

Proof. Note that $w_n(x_0, \dots, x_n) = w_{n-1}(x_0^p, \dots, x_{n-1}^p) + p^n x_n$. Thus q_n is determined by the formula

$$p^n q_n = g_n - w_{n-1}(q_0^p, \dots, q_{n-1}^q) \tag{1.1}$$

provided the right hand side of this equation is divisible by p^n. But

$$\begin{aligned} w_{n-1}(q_0^p, \ldots, q_{n-1}^p) &\equiv \varphi w_{n-1}(q_0, \ldots, q_{n-1}) \mod p^n \\ &= \varphi g_{n-1} \\ &\equiv g_n \mod p^n. \end{aligned}$$

$\square$

REMARK 1.2. There is an obvious graded analog of this result. One requires that the indeterminants x_i have degree $p^i m$ for some positive integer m. Then the Witt polynomial w_i also has degree $p^i m$, as do the solutions q_i.

Algebras A over $\mathbb{Z}_p$ equipped with algebra endomorphisms $\varphi : A \to A$ so that $\varphi(x) \equiv x^p \mod p$ will be said to have a lift of Frobenius. If $A = \mathbb{Z}_p[\mathbf{T}]$ is the free commutative graded algebra on a graded set of generators $\mathbf{T}$, then one can define $\varphi : A \to A$ by setting $\varphi(t) = t^p$ for $t \in \mathbf{T}$.

For the next result, let $y = (y_0, y_1, \ldots)$ be another set of indeterminants.

COROLLARY 1.3. *There exist unique polynomials*

$$a_i(x, y) \in \mathbb{Z}_p[x_0, x_1, \ldots, y_0, y_1, \ldots]$$

so that

$$w_n(a_0, a_1, \ldots) = w_n(x) + w_n(y).$$

This is immediate from the Dwork Lemma. Note that induction and the equation 1.1 imply that a_n is a polynomial in $x_0, \ldots, x_n, y_0, \ldots, y_n$.

We use this result to define a diagonal

$$\Delta : \mathbb{Z}_p[x_0, x_1, \ldots] \to \mathbb{Z}_p[x_0, x_1, \ldots] \otimes \mathbb{Z}_p[x_0, x_1, \ldots] \tag{1.2}$$

by

$$\Delta(x_i) = a_i(x \otimes 1, 1 \otimes x).$$

Here $x \otimes 1 = (x_0 \otimes 1, x_1 \otimes 1, \ldots)$ and similarly for $1 \otimes x$.

LEMMA 1.4. *With this diagonal $\mathbb{Z}_p[x_0, x_1, \ldots]$ becomes a bicommutative Hopf algebra over $\mathbb{Z}_p$. The Witt polynomials $w_n(x)$ are primitive.*

PROOF. That Δ is coassociative and cocommutative follows from the uniqueness clause of Lemma 1.1. For example, Δ is cocommutative because $a_i(x, y) = a_i(y, x)$, which in turn follows from the uniqueness of the a_i and the equation

$$\begin{aligned} w_n(a_0(x, y), a_1(x, y), \ldots) &= w_n(x) + w_n(y) = w_n(y) + w_n(x) \\ &= w_n(a_0(y, x), a_1(y, x), \ldots). \end{aligned}$$

The Witt polynomials are primitive by construction. Put another way,

$$\begin{aligned} \Delta w_n(x_0, x_1, \ldots) &= w_n(\Delta x_0, \Delta x_1, \ldots) \\ &= w_n(a_0(x \otimes 1, 1 \otimes x), a_1(x \otimes 1, 1 \otimes x), \ldots) \\ &= w_n(x \otimes 1) + w_n(1 \otimes x) \\ &= w_n(x) \otimes 1 + 1 \otimes w_n(x). \end{aligned}$$

$\square$

REMARK 1.5. 1. Corollary 1.3 and Lemma 1.4 can, together, be rephrased as follows: there is a unique Hopf algebra structure on $\mathbb{Z}_p[x_0, x_1, \ldots]$ so that the Witt polynomials $w_n(x)$ are primitive.

2. Since $a_n(x, y)$ is a polynomial in $x_0, \ldots, x_n, y_0, \ldots, y_n$, the diagonal on the Hopf algebra $\mathbb{Z}_p[x_0, x_1, \ldots]$ restricts to a diagonal on $\mathbb{Z}_p[x_0, \ldots, x_n]$ making the latter a sub-Hopf algebra of $\mathbb{Z}_p[x_0, x_1, \ldots]$.

3. Witt vectors can be defined as follows. If $\mathcal{A}$ is the category of $\mathbb{Z}_p$ algebras and $k \in \mathcal{A}$, then the Witt vectors on k is the set

$$W(k) = \mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}_p[x_0, x_1, \ldots], k)$$

with group structure induced from the Hopf algebra structure on $\mathbb{Z}_p[x_0, x_1, \ldots]$. The group $W(k)$ acquires a commutative ring structure represented by a map

$$\mathbb{Z}_p[x_0, x_1, \ldots] \to \mathbb{Z}_p[x_0, x_1, \ldots] \otimes \mathbb{Z}_p[x_0, x_1, \ldots]$$

sending $w_n(x)$ to $w_n(x \otimes 1) w_n(1 \otimes x)$, which exists by the Lemma 1.1. The inclusion $\mathbb{Z}_p[x_0, x_1, \ldots, x_n] \to \mathbb{Z}_p[x_0, x_1, \ldots]$ defines a quotient map to the Witt vectors of length n,

$$W(k) \to W_n(k) = \mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}_p[x_0, x_1, \ldots, x_n], k).$$

Evidently, $W(k) = \lim W_n(k)$ and $W_1(k) \cong k$. If k is a perfect field, $W(k)$ is the unique complete discrete valuation ring so that $W(k)/pW(k) \cong k$; in particular, if $k = \mathbb{F}_p$, $W(\mathbb{F}_p) \cong \mathbb{Z}_p$. See [**6**], p. 58.

4. The Hopf algebra structure of Lemma 1.4 passes to a Hopf algebra structure in the graded case, if we use the grading conventions of Remark 1.2

It is convenient to have a name for these Hopf algebras.

DEFINITION 1.6. For $0 \le k \le \infty$, let $CW(k)$ be the Hopf algebra with underlying algebra

$$\mathbb{Z}_p[x_0, \ldots, x_k]$$

and coalgebra structure defined by Corollary 1.3. If we are working in a graded situation, write $CW_m(k)$ for the graded analog of this Hopf algebra with the degree of x_i equal to $p^i m$.

The Dwork Lemma 1.1 has a uniqueness clause in it, but we are often interested in a weaker version of uniqueness; in particular we will be interested in knowing when two algebra maps are equal when reduced modulo p. This is Lemma 1.7 below.

First, some notation. Suppose A is a torsion free algebra equipped with an algebra map $\varphi : A \to A$ so that $\varphi(x) \equiv x^p \mod p$. Suppose one has two sequences of elements in A, g_n and h_n so that $\varphi g_{n-1} \equiv g_n \mod p^n$ and $\varphi h_{n-1} \equiv h_n \mod p^n$. Then there are unique elements q_n and r_n in A so that

$$\begin{aligned} w_n(q_0, \ldots, q_n) &= g_n \\ w_n(r_0, \ldots, r_n) &= h_n. \end{aligned}$$

LEMMA 1.7. *If, for all n, $g_n \equiv h_n \mod p^{n+1}$ then $q_n \equiv r_n \mod p$. In other words, the two induced maps of algebras*

$$\mathbb{Z}_p[x_0, x_1, \ldots] \to A$$

agree modulo p.

Proof. Note that if $x \equiv y \mod p^n$, then $x^{p^k} \equiv y^{p^k} \mod p^{n+k}$. Then from formula 1.1 we have $p^n q_n \equiv p^n r_n \mod p^{n+1}$ and the result follows. □

Proposition 1.8. *1. The Hopf algebra $CW(\infty)$ has a lift of the Frobenius φ which is a Hopf algebra map and so that $\varphi w_n = w_{n+1}$.*

2. Let $[p] : CW(\infty) \to CW(\infty)$ be p-times the identity map in the abelian group of Hopf algebra maps of $CW(\infty)$ to itself. Then

$$[p](x_i) \equiv x_{i-1}^p \mod p.$$

Proof. 1) Certainly Lemma 1.1 supplies an algebra map $\varphi : CW(\infty) \to CW(\infty)$ so that $\varphi w_n = w_{n+1}$. This can be seen to be a coalgebra map by applying the uniqueness clause of Lemma 1.1 to the two possible maps

$$CW(\infty) \to CW(\infty) \otimes CW(\infty).$$

To see that φ is a lift of the Frobenius, note that $\varphi x_n = q_n$ where

$$\begin{aligned} w_n(q_0, \ldots, q_n) &= w_{n+1}(x_0, \ldots, x_n) \\ &\equiv w_n(x_0^p, \ldots, x_n^p) \mod p^{n+1}. \end{aligned}$$

So Lemma 1.7 implies $q_n \equiv x_n^p \mod p$ and $\varphi(x) \equiv x^p \mod p$.

2) Virtually the same argument applies, using that

$$pw_n(x_0, \ldots, x_n) \equiv w_n(0, x_0^p, \ldots, x_{n-1}^p)$$

$\mod p^{n+1}$. □

The following property of the Hopf algebra $CW(\infty)$ will be extremely useful. Let $\mathcal{HF}$ be the category of pairs (H, φ) where H is a torsion free bicommutative Hopf algebra over $\mathbb{Z}_p$ and $\varphi : H \to H$ is a morphism of Hopf algebras which is a lift of the Frobenius. Morphisms in $\mathcal{HF}$ must commute with lifts of the Frobenius. Proposition 1.8 produces a pair $(CW(\infty), \varphi)$ in $\mathcal{HF}$. Hopf algebras withs such lifts of the Frobenius are fairly rare; for example the primitively generated Hopf algebra $\mathbb{Z}_p[x]$ cannot support a lift of the Frobenius as there is no primitive which reduces to x^p modulo p.

For any Hopf algebra H, let PH denote the primitives.

Proposition 1.9. *Let $H \in \mathcal{HF}$ be a Hopf algebra equipped with a lift of the Frobenius. Then there is a natural isomorphism*

$$\Phi : \mathrm{Hom}_{\mathcal{HF}}(CW(\infty), H) \cong PH$$

given by $f \mapsto f(x_0)$.

Proof. Let φ and φ_H denote the lifts of the Frobenius in $CW(\infty)$ and H respectively. First note if $y = f(x_0) \in PH$, then

$$fw_n = \varphi_H^n(y).$$

Thus Φ is an injection by the uniqueness clause of Lemma 1.1.

Next, if $y \in PH$ is primitive, then so is $g_n = \varphi_H^n y$, $n \geq 0$. Since $g_n = \varphi_H g_{n-1}$, Lemma 1.1 supplies $f : CW(\infty) \to H$ so that

$$fw_n = g_n.$$

In particular we have $fx_0 = y$. So we need only show f is a morphism in $\mathcal{HF}$; that is $f\varphi = \varphi_H f$ and that f is a morphism of Hopf algebras. For the first, we have

$$(f\varphi)w_n = f(w_{n+1}) = g_{n+1} = (\varphi_H f)w_n$$

so the uniqueness clause of Lemma 1.1 applies. For the second, again apply this uniqueness clause to the two possible compositions

$$CW(\infty) \to H \otimes H.$$

□

There is a graded version of this result, which yields that there is a natural isomorphism

$$\Phi : \mathrm{Hom}_{\mathcal{HF}_*}(CW_m(\infty), H) \cong (PH)_m.$$

Here $\mathcal{HF}_*$ are graded torsion free Hopf algebras with a Hopf algebra lift of the Frobenius and $(PH)_m$ is the degree m part of the primitives.

2. Hopf algebras with a lift of the Verschiebung.

While our primary interest is in abelian Hopf algebras over fields, the natural generators of that category are the reductions, modulo p, of the Hopf algebras $CW_n(k)$ of the previous section. These are not only Hopf algebras over $\mathbb{Z}_p$, but they support a curious bit of extra structure, which we now explore. We will work in the graded case, as it is considerably simpler.

Let A be a torsion-free bicommutative Hopf algebra over $\mathbb{Z}_p$. Then A will be said to have a lift of the Verschiebung if there is a Hopf algebra endomorphism $\psi : A \to A$ so that for all $x \in A$ there is a congruence

$$\psi(x) \equiv \xi(x) \qquad \mathrm{mod}\ p$$

where $\xi : \mathbb{F}_p \otimes A \to \mathbb{F}_p \otimes A$ is the Verschiebung.

This is a very restrictive condition on a Hopf algebra. For example, the divided power algebra on a primitive generator cannot be equipped with a lift of the Verschiebung. For if x is the primitive generator, we would have $\psi(x) = py$ for some y; hence,

$$\psi(\gamma_p(x)) = \psi(\frac{x^p}{p!}) = \frac{p^p}{p!}y^p \equiv 0 \qquad \mod \quad p.$$

But

$$\psi(\gamma_p(x)) \equiv \xi(\gamma_p(x)) = x \qquad \mathrm{mod} p.$$

This kind of simple example can be expanded into the following observation. A graded Hopf algebra A over a commutative ring k will be called connected if $A_0 \cong k$. If such a Hopf algebra over $\mathbb{Z}_p$ has a lift of the Verschiebung $\psi : A \to A$, then ψ necessarily divided degree by p and is the identity on degree 0.

Proposition 2.1. *Let A be a graded, connected torsion-free Hopf algebra over $\mathbb{Z}_p$ equipped with a lift of the Verschiebung. If A is finitely generated as an algebra, then A is a polynomial algebra.*

This will be proved in the next section, as it takes us a bit afield.

The next result supplies our main examples. Let $CW_n(k)$ be the Hopf algebras of Definition 1.6.

Proposition 2.2. *The graded Witt Hopf algebras $CW_n(k) = \mathbb{Z}_p[x_0, \dots , x_k]$, $0 \le k \le \infty$ have a unique lift of the Verschiebung $\psi : CW_n(k) \to CW_n(k)$ so that*

$$\psi(x_i) = \begin{cases} x_{i-1} & i \ge 1; \\ 0 & i = 0. \end{cases}$$

PROOF. The Dwork Lemma 1.1 supplies a map of algebras

$$\psi : \mathbb{Z}_p[x_0, x_1, \dots] \longrightarrow \mathbb{Z}_p[x_0, x_1, \dots]$$

so that $\psi(w_i) = pw_{i-1}$, where $w_{-1} = 0$. Since the polynomials pw_{i-1} are primitive this yields a morphism of Hopf algebras

$$\psi : CW_n(\infty) \longrightarrow CW_n(\infty)$$

and a simple induction argument shows $\psi(x_i) = x_{i-1}$. Since $\mathbb{F}_p \otimes CW(\infty)$ is a polynomial algebra the equation

$$(\xi(x_i))^p = [p](x_i) = (x_{i-1})^p$$

from Proposition 1.8 shows that ψ is indeed a lift of the Verschiebung. The case of $CW(k)$ with k finite follows by restriction. $\square$

We define the categories of coalgebras and Hopf algebras we are interested in.

DEFINITION 2.3. The category $\mathcal{HV}$ is the category of pairs (A, ψ) where

1. A is a graded connected, torsion-free Hopf algebra over $\mathbb{Z}_p$, and
2. $\psi : A \to A$ is a lift of the Verschiebung.

We will call this category the category of Hopf algebras with a lift of the Verschiebung.

There is an associated notion of a coalgebra with a lift of the Verschiebung, which we will use in Lemma 2.5 and in section 9.

DEFINITION 2.4. The category $\mathcal{CV}$ of coalgebras with a lift of the Verschiebung consists of pairs (C, ψ) where

1. C is a graded, connected, torsion-free, cocommutative coalgebra over $\mathbb{Z}_p$;
2. $\psi : C \to C$ is a lift of the Verschiebung.

We will call this category the category of coalgebras with a lift of the Verschiebung.

The category $\mathcal{HV}$ is additive and has all limits and colimits, but it is not abelian. For example, the cokernel in $\mathcal{HV}$ of the map

$$[p] : \mathbb{Z}_p[x] \longrightarrow \mathbb{Z}_p[x]$$

of primitively generated Hopf algebras with trivial lift of the Verschiebung is simply $\mathbb{Z}_p$.

We now insert a technical lemma that says that the category $\mathcal{HV}$ has a set of generators.

LEMMA 2.5. *Let $H \in \mathcal{HV}$. Then H is the union of its sub-objects $H_\alpha \subseteq H$ in $\mathcal{HV}$ which are finitely generated as algebras over $\mathbb{Z}_p$.*

PROOF. It is enough to show that if $x \in H$ is a homogeneous element, then there is an H_α containing x. Let $C \subseteq H$ be a finitely generated sub-coalgebra with a lift of the Verschiebung containing x. Such a C is easily constructed by a downwards degree argument. Let $S(C)$ be the symmetric algebra on the coaugmentation ideal of C endowed with induced structure as an object in $\mathcal{HV}$. Then let H_α be the image of the evident map

$$S(C) \longrightarrow H.$$

$\square$

If $(H, \psi) \in \mathcal{HV}$, let IH denote the augmentation ideal. Then the indecomposables $QH = IH/IH^2$ form a torsion-free $\mathbb{Z}_p$ module by Proposition 2.1 and Lemma 2.5. It is, in fact, a $\mathbb{Z}_p[V]$ module with

$$V(x + IH^2) = \psi(x) + IH^2.$$

The action of V is nilpotent on QH for degree reasons. This leads to the following definition.

DEFINITION 2.6. Let $\mathcal{M}_V$ denote the category of positively graded $\mathbb{Z}_p[V]$ modules M so that

1. M is torsion-free as a $\mathbb{Z}_p$ module and
2. the action of V divided degree by p; that is

$$V : M_{pn} \to M_n.$$

Implicit in the second statement of this definition is that $V = 0$ on M_n if p does not divide n.

The main result of this section is the following:

THEOREM 2.7. *The functor*

$$Q : \mathcal{HV} \longrightarrow \mathcal{M}_V$$

is an equivalence of categories.

This will be proved by a sequence of lemmas. To begin we have the following result.

LEMMA 2.8. *For all H and K in $\mathcal{HV}$, the natural map*

$$\mathrm{Hom}_{\mathcal{HV}}(H, K) \to \mathrm{Hom}_{\mathcal{M}_V}(QH, QK)$$

is an injection.

PROOF. Let $\mathcal{H}\mathbb{Q}_p$ be the category of graded, connected Hopf algebras over $\mathbb{Q}_p$. Consider the diagram, where each of the maps is the obvious one:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{HV}}(H, K) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}_V}(QH, QK) \arrow[d] \\
\mathrm{Hom}_{\mathcal{H}\mathbb{Q}_p}(\mathbb{Q}_p \otimes H, \mathbb{Q}_p \otimes K) \arrow[r] & \mathrm{Hom}_{\mathcal{M}_V}(Q(\mathbb{Q}_p \otimes H), Q(\mathbb{Q}_p \otimes K))).
\end{tikzcd}$$

Then the left vertical map is an injection because the Hopf algebras are torsion free, and the right vertical map is an injection because of Proposition 2.1 and Lemma 2.5. The bottom map is an isomorphism, [**17**]. Thus the top map is an injection. $\square$

Theorem 2.7 would assert, among other things, that the natural map of the previous result is an isomorphism.

We now supply an algebraic result. Let $\mathbb{Z}_p[n]$ be the the graded $\mathbb{Z}_p$ module free on one generator in degree n.

PROPOSITION 2.9. *1.) If $M \in \mathcal{M}_V$ then there is a natural ismorphism for all $n \geq 1$*

$$\mathrm{Hom}_{\mathcal{M}_V}(M, QCW_n(\infty)) \cong \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p[n]) = (M_n)^*.$$

2.) Suppose n is relatively prime to p and $k \geq 0$. Then there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{M}_V}(QCW_n(k), M) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[p^k n], M) \cong M_{p^k n}.$$

PROOF. If $y_i \in QCW_n(\infty)$ is the residue class of x_i, then Lemma 2.2 implies that $Vy_i = y_{i-1}$. If $g : M \to \mathbb{Z}_p[n]$ is a $\mathbb{Z}_p$ module homomorphism, define $f : M \to CW_n(\infty)$ by

$$f(x) = \sum_i gV^i(x)y_i.$$

For degree reasons, $gV^i(x) \neq 0$ for at most one i. Conversely, given a morphism $h : M \to QCW_n(\infty)$ in $\mathcal{M}_V$, one can write

$$h(x) = \sum_i g_i(x)y_i$$

where $g_i : M_{p^i n} \to \mathbb{Z}_p$ and define a module homomorphism $M \to \mathbb{Z}_p$ by g_0. The functions $g \to f$ and $h \to g_0$ are inverse to each other. Part 2 is a simple calculation with Proposition 2.2. □

The key to Theorem 2.7 is the following result.

PROPOSITION 2.10. *If $H \in \mathcal{HV}$, then the natural map*

$$\mathrm{Hom}_{\mathcal{HV}}(H, CW_n(\infty)) \to \mathrm{Hom}_{\mathcal{M}_V}(QH, QCW_n(\infty)) \cong (QH)^*_m$$

is an isomorphism.

PROOF. By Lemma 2.8 we know that this map is an injection. To complete the argument we must prove surjectivity.

First note that we may assume that H is finitely generated as a $\mathbb{Z}_p$ module in each degree. For, by Lemma 2.5 we may write a general H as colimit:

$$\mathrm{colim}_\alpha H_\alpha \cong H$$

where H_α runs over the finitely generated sub-objects in $\mathcal{HV}$ of H. If the result holds for H_α, then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{HV}}(H, CW_n(\infty)) &\cong \lim_\alpha \mathrm{Hom}_{\mathcal{HV}}(H_\alpha, CW_n(\infty))\\ &\cong \lim_\alpha \mathrm{Hom}_{\mathcal{HV}}(QH_\alpha, QCW_n(\infty))\\ &\cong \mathrm{Hom}_{\mathcal{HV}}(QH, QCW_n(\infty)).\end{aligned}$$

So assume H is finitely generated in each degree as a $\mathbb{Z}_p$ module. Let $g \in \mathrm{Hom}_{\mathcal{M}_V}(QH, QCW(\infty))$. As Proposition 2.9 we may write

$$g(x) = \sum_i g_i(x)y_i$$

where $g_i : (QH)_{p^i n} \to \mathbb{Z}_p$. The isomorphism

$$Hom_{\mathcal{M}_V}(QH, QCW(\infty)) \cong (QH)^*_n$$

sends g to g_0. The $\mathbb{Z}_p$ dual of H, which we write H^* is a Hopf algebra with a lift of the Frobenius given by $\varphi = \psi^*$. Since $g_0 \in (QH)^* \cong P(H)^*$, Lemma 1.9 supplies a morphism of Hopf algebras equipped with a lift of the Frobenius

$$f : CW_n(\infty) \longrightarrow H^*$$

with $f(x_0) = g_0$. Since f commutes with the lifts of the Frobenius, Lemma 1.8 and the fact $g_i = g_{i-1}\psi^*$ imply that $f(w_i) = g_i \in (QH)^* \cong P(H^*)$. Dualizing f, we obtain a morphism of Hopf algebras

$$\lambda : H \longrightarrow CW_n(\infty)^*.$$

If we define a lift of the Verschiebung on $CW_n(\infty)^*$ by φ^*, then λ is a morphism in $\mathcal{HV}$. We conclude the argument by identifying $CW_n(\infty)^*$. As a matter of notation if $a \in H$ and $x \in CW_n(\infty)$ let us write

$$\langle \lambda(a), x\rangle$$

for the value of $\lambda(a) \in CW_n(\infty)^*$ evaluated at x. The above considerations imply that

$$\langle \lambda(a), w_i\rangle = h_i(a).$$

If we take H to be $CW_n(\infty)$ and g to be the indentity map, we obtain a morphism of Hopf algebras with a lift of Verschiebung

$$\lambda_0 : (CW_n(\infty), \psi) \longrightarrow (CW_n(\infty)^*, \varphi^*)$$

with the property that

$$\langle \lambda_0(x_i), w_i\rangle = h_i(x_i) = 1.$$

According to [**22**], this implies that λ_0 is an isomorphism. To finish we must show that is an equality

$$Q(\lambda_0^{-1}\lambda) = g_i : QH \to QCW_n(\infty).$$

This is equivalent to asserting that for all $a \in QH$,

$$g_i(a)Q(\lambda_0)(y_i) = Q\lambda(a)$$

in $(PCW(\infty))^*$. Recall that y_i is the residue class of x_i in $QCW_n(\infty)$. Since w_i generates $PCW(\infty)$ in this degree we may write $a = x + IH^2$ and calculate

$$\begin{aligned}\langle Q\lambda(a), w_i\rangle &= \langle Q\lambda(x), w_i\rangle\\ &= g_i(x) = g_i(a)\\ &= g_i(a)\langle Q\lambda(x_i), w_i\rangle\\ &= \langle g_i(a)Q\lambda(y_i), w_i\rangle.\end{aligned}$$

□

The following is the main techinical result behind the proof of Theorem 2.7.

THEOREM 2.11. *The indecomposables functor $Q : \mathcal{HV} \to \mathcal{M}_V$ has a right adjoint S^*. Furthermore the natural map $M \to QS^*(M)$ in $\mathcal{M}_V$ is an isomorphism.*

Note that Theorem 2.7 follows immediately. The one natural map $M \to QS^*(M)$ is an isomorphism by this result; the other natural map $S^*(QH) \to H$ is an isomorphism because of Proposition 2.1 and Lemma 2.5, and the composite

$$QH \xrightarrow{\cong} QS^*(QH) \to QH$$

forces $QS^*(QH) \to QH$ to be an isomorphism.

PROOF. We begin by constructing $S^*(M)$ for M finitely generated in each degree. Define functors

$$\Phi_0, \Phi_1 : \mathcal{M}_V \longrightarrow \mathcal{M}_V$$

as follows.

$$\Phi_0(M)_n = M_n \oplus M_{n/p} \oplus M_{n/p^2} \oplus \cdots$$

where is is understood that $M_k = 0$ if k is a fraction. Define V on $\Phi_0(M)$ by

$$V(x_0, x_1, x_2, \cdots) = (x_1, x_2, \cdots).$$

There is a natural injection $\eta : M \to \Phi_0(M)$ given by

$$\eta(x) = (x, Vx, V^2x, \cdots).$$

Define $\Phi_1(M)$ by the formula

$$\Phi_1(M)_n = M_{n/p} \oplus M_{n/p^2} \oplus \cdots$$

and V again defined by projection. There is a natural map

$$d : \Phi_0(M) \to \Phi_1(M)$$

given by

$$d(x_0, x_1, x_2, \cdots) = (x_1 - Vx_0, x_2 - Vx_1, \cdots).$$

This map is surjective and the sequence

$$0 \to M \xrightarrow{\eta} \Phi_0(M) \xrightarrow{d} \Phi_1(M) \to 0$$

is short exact in $\mathcal{M}_V$. In fact, in each degree it is split short exact. Furthermore, choosing a set of generators (over $\mathbb{Z}_p$) for M_n for each n defines an isomorphism

$$\Phi_0(M) \cong \oplus_\alpha QCW_{k_\alpha}(\infty).$$

The number of times each positive integer appears as a k_α is finite. There is a similar isomorphism for $\Phi_1(M)$:

$$\Phi_1(M) \cong \oplus_\beta QCW_{k_\beta}(\infty).$$

Define

$$S^*(\Phi_0(M)) = \otimes_\alpha CW_{k_\alpha}(\infty).$$

Note that in any given degree this tensor product is the tensor product of finitely many groups. Similary define $S^*(\Phi_1(M))$ and note that Proposition 2.10 implies that there is a map in $\mathcal{HV}$

$$f : S^*(\Phi_0(M)) \longrightarrow S^*(\Phi_1(M))$$

with $Qf = d$. Define $S^*(M)$ by the pull-back diagram in $\mathcal{HV}$

$$\begin{tikzcd} S^*(M) \arrow[r] \arrow[d] & S^*(\Phi_0(M)) \arrow[d, "f"] \\ \mathbb{Z}_p \arrow[r] & S^*(\Phi_1(M)). \end{tikzcd}$$

Note that $S^*(M)$ is the usual Hopf algebra kernel of f.

We must now examine this construction. The first thing to do is to show $QS^*(M) \to QS^*(\Phi_0(M))$ is an injection. In fact there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & PS^*(M) \arrow[r] \arrow[d] & PS^*(\Phi_0(M)) \arrow[d] \\ & QS^*(M) \arrow[r] & QS^*(\Phi_0(M)) \end{tikzcd}$$

where the vertical maps are the canonical maps from the primitives to indecomposables. Since these are Hopf algebras in $\mathcal{HV}$, Lemma 2.1 implies that these maps are injections with torsion cokernel. Hence $QS^*(M) \to QS^*(\Phi_0(M))$ is one-to-one.

Next let H be any object in $\mathcal{HV}$. Then there is a diagram with columns exact

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}_{\mathcal{HV}}(H, S^*(M)) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}_V}(QH, QS^*(M)) \arrow[d] \\
\mathrm{Hom}_{\mathcal{HV}}(H, S^*(\Phi_0(M))) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathcal{M}_V}(QH, QS^*(\Phi_0(M))) \arrow[d] \\
\mathrm{Hom}_{\mathcal{HV}}(H, S^*(\Phi_1(M))) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{M}_V}(QH, QS^*(\Phi_1(M)))
\end{tikzcd}$$

Note that, for example

$$\mathrm{Hom}_{\mathcal{M}_V}(QH, QS^*(\Phi_0(M))) \cong \mathrm{Hom}_{\mathcal{M}_V}(QH, \Phi_0(M))$$

and the horizontal maps labelled as isomorphims are so by Lemma 2.10. This diagram implies that

$$\mathrm{Hom}_{\mathcal{HV}}(H, S^*(M)) \cong \mathrm{Hom}_{\mathcal{M}_V}(QH, QS^*(M)).$$

Furthermore, if we set $H = CW_n(k)$ for various n and k, Lemma 2.9 implies that $M \cong QS^*(M)$. These two equations together yield the desired adjunction formula

$$\mathrm{Hom}_{\mathcal{HV}}(H, S^*(M)) \cong \mathrm{Hom}_{\mathcal{M}_V}(QH, M).$$

To complete the proof, we must extend S^* to M which are not finitely generated in each degree. To do this write such a general M as a filtered colimit of sub-objects M_i which are finitely generated in each degree then define

$$S^*(M) = \operatorname{colim} S^*(M_i).$$

Note that this colimit can be calculated in graded $\mathbb{Z}_p$ modules. If $H \in \mathcal{HV}$ is finitely generated as an algebra

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{HV}}(H, S^*(M)) &\cong \operatorname{colim} \mathrm{Hom}_{\mathcal{HV}}(H, S^*(M_i)) \\
&\cong \operatorname{colim} \mathrm{Hom}_{\mathcal{M}_V}(QH, M_i) \\
&\cong \mathrm{Hom}_{\mathcal{M}_V}(QH, M),
\end{aligned}$$

as needed. In particular, setting $H = CW_n(k)$ for various n and k, Lemma 2.9 implies that $M \cong QS^*(M)$. Finally, if H is general, Lemma 2.5 implies that H is the filtered colimit of its finitely generated sub-objects which are finitely generated as algebras. Then, using an argument similar to that given at the beginning of Proposition 2.10, it follows that

$$\mathrm{Hom}_{\mathcal{HV}}(H, S^*(M)) \cong \mathrm{Hom}_{\mathcal{M}_V}(QH, M).$$

□

EXAMPLE 2.12. (The Husemoller Splitting [**14**]). Consider the Hopf algebra

$$H_*BU \cong \mathbb{Z}_p[a_1, a_2, a_3, \dots]$$

with the degree of a_i equal to $2i$ and

$$\Delta a_k = \sum_{i+j=k} a_i \otimes a_j.$$

This represents the functor on graded $\mathbb{Z}_p$ algebras

$$\Lambda(A) = (1 + tA[[t]])_0$$

where $A[[t]]$ is the graded power series on A with $\deg(t) = -2$. $\Lambda(A)$ becomes a group under power series multiplication. If $f(t) = \Sigma a_i t^i \in \Lambda(A)$, then, modulo p,

$$f(t)^p \equiv \Sigma a_i^p t^{pi}$$

from which it follows that on $H_*(BU, \mathbb{F}_p)$, $[p]a_i = (a_{i/p})^p$ or

$$\xi a_i = a_{i/p}$$

where $a_{i/p} = 0$ if i/p is a fraction. Now define $\psi : H_*BU \to H_*BU$ to be the unique Hopf algebra map inducing

$$\psi^* : \Lambda(A) \to \Lambda(A)$$

where $\psi^* f(t) = f(t^p) = \Sigma a_i t^{pi}$. Then $\psi(a_i) = a_{i/p}$ and ψ is a lift of the Frobenius. Thus, from Proposition 2.2 one has

$$QH_*BU \cong \bigoplus_{p \nmid n} QCW_{2n}(\infty)$$

in $\mathcal{M}_V$, hence one has the Husemoller splitting

$$H_*BU \cong \bigotimes_{p \nmid n} CW_{2n}(\infty).$$

To be fair, these arguments are not so different that than the original ones.

3. The Implications of a lift of the Frobenius

The main purpose of this section is to prove Lemma 2.1: that a Hopf algebra with a lift of Verschiebung which is also finitely generated as an algebra is a polynomial algebra. This requires as examination of the dual Hopf algebra, and here I am very indebted to ideas of Bousfield [**1**].

To keep the record straight note that some finiteness hypothesis is necessary; for example, if H is the primitively generated symmetric algebra over $\mathbb{Z}_p$ on $\mathbb{Q}_p$ with trivial lift of the Verschiebung, then H is not a polynomial algebra.

The following is the algebra input. It is a consequence of Nakayama's Lemma.

LEMMA 3.1. *Let $f : M \to N$ be a homomorphism of finitely generated $\mathbb{Z}_p$ modules. Then f is an isomorphism if and only if*

$$\mathbb{F}_p \otimes f : \mathbb{F}_p \otimes M \to \mathbb{F}_p \otimes N$$

is an isomorphism.

This result has the following immediate corollary.

PROPOSITION 3.2. *Let A is a graded, connected torsion-free $\mathbb{Z}_p$ algebra, and suppose A is finitely generated as an algebra. Then A is a polynomial algebra if and only of $\mathbb{F}_p \otimes A$ is a polynomial algebra.*

If H is a finitely generated torsion-free Hopf algebra over $\mathbb{Z}_p$, then we may apply Borel's structure theorem to $\mathbb{F}_p \otimes H$ and conclude that there is an isomorphism of algebras

$$\mathbb{F}_p \otimes H \cong \bigotimes_i \mathbb{F}_p[a_i]/(a^{p^{n_i}})$$

where n_i is an integer $1 \leq n_i \leq \infty$. Thus $\mathbb{F}_p \otimes H$ is a polynomial algebra if and only if the Frobenius is injective. Equivalently, one need only show that the Verschiebung on the dual Hopf algebra $(\mathbb{F}_p \otimes H)^*$ is surjective. This will be proved below; see Corollary 3.9.

Because we are considering the dual $\mathbb{F}_p$ Hopf algebra $(\mathbb{F}_p \otimes H)^*$ we begin by considering the dual $\mathbb{Z}_p$ Hopf algebra $H^* = \mathrm{Hom}_{\mathbb{Z}_p}(H, \mathbb{Z}_p)$. Since H has a lift of the Verschiebung $\psi : H \to H$, the dual H^* has a lift of the Frobenius $\phi = \psi^*$. We now give a short investigation into the category of such objects.

Let $\mathcal{AF}$ be the category of torsion-free graded connected $\mathbb{Z}_p$ algebra equipped with a lift of the Frobenius. Let $\mathcal{M}$ be the category of torsion-free, positively graded $\mathbb{Z}_p$ modules.

LEMMA 3.3. *The augmentation ideal functor $I : \mathcal{AF} \to \mathcal{M}$ has a left adjoint S_φ. Furthermore, for all finitely $M \in \mathcal{M}$, $S_\varphi(M)$ is isomorphic to a polynomial algebra.*

PROOF. First suppose M is finitely generated. Choose a homogeneous set of generators $\{y_i\}$ for M. If $A \in \mathcal{AF}$ and $f : M \to IA$ is given, the Dwork Lemma 1.1 supplies a unique map

$$g_i : \mathbb{Z}_p[x_0, x_1, x_2, \dots] \to A \tag{3.1}$$

so that $w_n \mapsto \varphi^n f y_i$. Thus one obtains a unique map in $\mathcal{AF}$

$$g : S_\varphi(M) = \bigotimes_i \mathbb{Z}_p[x_0, x_1, x_2, \dots] \to A$$

so that the evident composite

$$M \to IS_\varphi(M) \to IA$$

is f. The clause stipulating that $S_\varphi(M)$ is a polynomial algebra follows from Equation 3.1.

To finish the argument, note that the construction of $S_\varphi(M)$ is natural in M, at least where so far defined; that is, for finitely generated M. For general M, write $M = \operatorname{colim}_\alpha M_\alpha$ where $M_\alpha \subseteq M$ runs over the diagram of finitely generated sub-modules of M. Define $S_\varphi(M) = \operatorname{colim}_\alpha S_\varphi(M_\alpha)$; since the diagram is filtered the colimit in $\mathcal{AF}$ is isomorphic to the colimit as graded modules. One easily checks $S_\varphi(M)$ has the requisite universal property. □

Now let $\mathcal{HF}$ be the category of Hopf algebra with a lift of the Frobenius and $\mathcal{CA}$ the category of graded, connected, torsion free coalgebras over $\mathbb{Z}_p$. Let $J : \mathcal{CA} \to \mathcal{M}$ be the "coaugmentation coideal functor"; that is, $JC = \operatorname{coker}(\mathbb{Z}_p \to C)$.

PROPOSITION 3.4. *The forgetful functor $\mathcal{HF} \to \mathcal{CA}$ has a left adjoint F. Furthermore, for if $C \in \mathcal{CA}$ if finitely generated as a $\mathbb{Z}_p$ module in each degree, then $F(C)$ is a polynomial algebra; indeed, as algebras*

$$F(C) \cong S_\varphi(JC).$$

PROOF. Define $F(C)$ to be the algebra $S_\varphi(JC)$ with the diagonal induced by completing the following diagram using the universal property of S_φ:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d] & C \otimes C \arrow[d] \\ S_\varphi(JC) \arrow[r, dashed] & S_\varphi(JC) \otimes S_\varphi(JC). \end{tikzcd}$$

Then one easily checks $F(C) \in \mathcal{HF}$ fulfills the conclusions of the result. $\square$

If A is any torsion-free $\mathbb{Z}_p$ algebra equipped with a lift φ of the Frobenius define a function $\theta : A \to A$ by the formula

$$\varphi(x) = x^p + p\theta(x)$$

Compare [**1**] for the following result.

LEMMA 3.5. *1) For all x and y in A,*

$$\theta(xy) = \theta(x)y^p + x^p\theta(y) + p\theta(x)\theta(y)$$

2) For all x and y in A,

$$\theta(x + y) = \theta(x) + \theta(y) - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^{p-i}y^i.$$

PROOF. These are both consequences of the fact that φ is an algebra map. $\square$

Note that these formulas imply that if A is equipped with an augmentation $A \to \mathbb{Z}_p$, then the augmentation ideal is closed under θ.

The operation θ may clarify the structure of free objects in $\mathcal{AF}$. Compare [**1**]

LEMMA 3.6. *Suppose M is a free $\mathbb{Z}_p$ module in each degree. Let $\{y_i\} \subseteq M$ be a set of generators for M. Then there is an isomorphism of $\mathbb{Z}_p$ algebras*

$$S_\varphi(M) \cong \mathbb{Z}_p[\theta^n y_i \mid n \geq 0].$$

PROOF. By the construction of Lemma 3.3, it is sufficient to consider the case where M has one generator in degree m. Then

$$S_\varphi(M) \cong \mathbb{Z}_p[x_0, x_1, x_2, \ldots] = CW_m(\infty)$$

in $\mathcal{AF}$, where $\varphi w_n = w_{n+1}$. Thus one need only show

$$\theta x_n \equiv x_{n+1}$$

modulo p and indecomposables. The formula

$$w_{n+1} = \varphi w_n = w_n^p + p\theta w_n$$

and the formulas of Lemma 3.5 imply

$$\theta(p^n x_n) \equiv p^n x_{n+1}$$

modulo p^{n+1} and decomposables. Since $p^n = \varphi(p^n) = p^{pn} + p\theta(p^n)$, Lemma 3.5.1 implies

$$p^n\theta x_n \equiv p^n x_{n+1}$$

modulo p^{n+1} and decomposables, as required. $\square$

Next let $H \in \mathcal{HF}$ be a Hopf algebra with a lift of the Frobenius and let $\xi : \mathbb{F}_p \otimes H \to \mathbb{F}_p \otimes H$ be the Verschiebung. If $x \in H$, let $\{x\}$ denote the class of x in $\mathbb{F}_p \otimes H \cong H/pH$.

LEMMA 3.7. *For all $x \in H$, there is a congruence*

$$\xi\{\theta(x)\} \equiv \{x\}$$

modulo the decomposables in $\mathbb{F}_p \otimes H$.

PROOF. First suppose H is a polynomial algebra. Then $\mathbb{F}_p \otimes H$ is a polynomial algebra. We will use the formula $(\xi y)^p = [p](y)$, where $[p]$ is p times the identity map in the abelian group of Hopf algebra endomorphisms of $\mathbb{F}_p \otimes H$. Thus it is sufficient to show

$$[p]\{\theta(x)\} \equiv \{x\}^p$$

modulo decomposables. Since $[p]$ is a morphism of Hopf algebras and commutes with the lift of the Frobenius, one has $[p]\theta(x) = \theta([p]x)$. Now $[p]x = px + z$ where $z \in IH^2$. Hence

$$\theta[p]x = \theta(px) + \theta(z) - \sum_{i=1}^{p} \frac{1}{p} \left(\begin{array}{c} p \\ i \end{array} \right) (px)^{p-i} z^i.$$

The last term is zero modulo p. Lemma 3.5.1 implies that $\theta(IH)^2 \subseteq IH^2$. Hence $\theta[p]x \equiv \theta(px)$ modulo p and indecomposables. But

$$\begin{aligned} \theta(px) &= \theta(p)x^p + p^p\theta(x) + p\theta(p)\theta(x) \\ &= \frac{p - p^p}{p}x^p + p^p\theta(x) + (p - p^p)\theta(x) \\ &\equiv x^p \mod p. \end{aligned}$$

For the general case of H, fix $x \in H$ and let $C \subseteq H$ be a coalgebra so that $x \in C$ and so that C is finitely generated over $\mathbb{Z}_p$. Consider the induced map $F(C) \to H$ given by Corollary 3.4. Then $F(C)$ is a polynomial algebra, and this result follows from the naturality of θ and ξ. $\square$

This has the following immediate consequence:

PROPOSITION 3.8. *Let H be a graded, connected torsion-free Hopf algebra over $\mathbb{Z}_p$ equipped with a lift of the Frobenius. The the Verschiebung*

$$\xi : \mathbb{F}_p \otimes H \longrightarrow \mathbb{F}_p \otimes H$$

is surjective.

PROOF. The previous result shows that ξ is surjective on indecomposables. This implies ξ is surjective. $\square$

For completeness we now add:

COROLLARY 3.9. *Let $H \in \mathcal{HV}$ be a graded, connected, torsion-free Hopf algebra over $\mathbb{Z}_p$ equipped with a lift of the Frobenius. If H is finitely generated as an algebra, then H is a polynomial algebra.*

PROOF. Combining Lemma 3.2 with the remarks following that result, we need only show that the Verschiebung is surjective on $(\mathbb{F}_p \otimes H)^*$. This follows from the previous result, and the fact that H^* has a lift of the Frobenius. $\square$

4. Dieudonné theory.

Positively graded bicommutative Hopf algebras over a perfect field form an abelian category with a set of projective generators. As such this category is equivalent to a category of modules over some ring. Dieudonné theory says which modules over which ring. In this section we use the results of the previous sections to elucidate the case $k = \mathbb{F}_p$. Here the main classification result is due to Schoeller [**23**]; the main advance is that we gave a formula of Theorem 4.8 for computations.

Let $\mathcal{HA}$ be the category of graded, connected, bicommutative Hopf algebras over $\mathbb{F}_p$. We describe a good set of generators for this category. In this context, a set of objects $\{A_\alpha\}$ is a set of generators if every object is a quotient of a coproduct of the A_α.

Let $n > 0$ be a positive integer, $n = p^k m$ where $(p, m) = 1$. Consider the torsion free Hopf algebra $CW_m(k) = \mathbb{Z}_p[x_0, x_1, \ldots, x_k]$ of Definition 1.6. It has the Witt vector diagonal. Define

$$H(n) = \mathbb{F}_p \otimes CW_m(k) \cong \mathbb{F}_p[x_0, x_1, \ldots, x_k].$$

There are many proofs of the following result. A very conceptual argument, due to Fabien Morel, can be found in [**9**].

Lemma 4.1. *The Hopf algebras $H(n)$ are projective in $\mathcal{HA}$ and form a set of generators. Furthermore is $\xi : H(n) \to H(n)$ is the Verschiebung, then*

$$\xi(x_i) = x_{i-1}.$$

Here and elsewhere in this document, one takes x_{-1} to be zero.

Of all the morphisms in $\mathcal{HA}$ between the various $H(n)$ we emphasize two. The first is in the inclusion

$$v : H(n) = \mathbb{F}_p[x_0, \ldots, x_k] \to \mathbb{F}_p[x_0, \ldots, x_{k+1}] = H(pn).$$

For the second, we note that Proposition 1.8.2 implies that there is a unique map of Hopf algebras $f : H(pn) \to H(n)$ so that the following diagram commutes

$$\begin{tikzcd} H(pn) \arrow[r, "f"] \arrow[dr, "{[p]}"'] & H(n) \arrow[d, "v"] \\ & H(pn). \end{tikzcd}$$

Note that, by construction, $vf = [p]$ and that Proposition 1.8.2 also implies that $fv = [p]$.

Definition 4.2. If $H \in \mathcal{HA}$, the Dieudonné module D_*H is the graded abelian group $\{D_nH\}_{n \geq 1}$ with

$$D_nH = \mathrm{Hom}_{\mathcal{HA}}(H(n), H)$$

and homomorphisms

$$F = f^* : D_nH \to D_{pn}H$$

and

$$V = v^* : D_{pn}H \to D_nH.$$

It follows from the remarks on v and f above that $FV = VF = p$. Also Proposition 1.8.2 implies that the order of the identity in $\mathrm{Hom}_{\mathcal{HA}}(H(n), H(n))$ is p^{s+1} if $n = p^s k$ with $(p, k) = 1$; hence, we also have $p^{s+1} D_n H = 0$. This suggests the following definition.

DEFINITION 4.3. Let $\mathcal{D}$ be the category of graded modules M equipped with operators F and V so that $FV = VF = p$ and $p^{s+1} M_{p^s k} = 0$ if $(k, p) = 1$. This is the category of Dieudonné modules.

Thus we have a functor $D_* : \mathcal{HA} \to \mathcal{D}$.

Some familiar functors on Hopf algebras can be recovered from the Dieudonné module. Let Φ be the "doubling" functor on graded modules; that is, $\Phi(M)_{pn} = M_n$ and $\Phi(M)_k = 0$ of p does not divide k. Then, for example, if M is a Dieudonné module, V defines a homomorphism of graded modules $V : M \to \Phi(M)$.

LEMMA 4.4. *1) Let $H \in \mathcal{HA}$, then there is an exact sequence of graded abelian groups*

$$0 \to PH \to D_* H \xrightarrow{V} \Phi D_* H.$$

2) There is also an exact sequence

$$\Phi D_* H \xrightarrow{F} D_* H \to QH \to 0.$$

PROOF. For 1), notice there is a short exact sequence of Hopf algebras

$$\mathbb{F}_p \to H(n) \xrightarrow{v} H(pn) \to \mathbb{F}_p[x_{k+1}] \to \mathbb{F}_p$$

where v defines V.

Part 2) is proved in a similar manner, after the introduction of some auxiliary technology. See [**10**], for example. □

PROPOSITION 4.5. *Let $f : H \to K$ be a morphism in $\mathcal{HA}$. If the induced homomorphism $D_* f : D_* H \to D_* K$ is an isomorphism, then f is an isomorphism.*

PROOF. From Lemma 4.4, we have that both $Qf : QH \to QK$ and $Pf : PH \to PK$ are isomorphisms. Hence f is an isomorphism. □

A crucial calculational result is the following.

PROPOSITION 4.6. *The homomorphism*

$$D_* : \mathrm{Hom}_{\mathcal{HA}}(H(n), H(m)) \to \mathrm{Hom}_{\mathcal{D}}(D_* H(n), D_* H(m))$$

is an isomorphism.

This is a consequence of a more general calculation, which we give below in Theorem 4.8. See Corollary 4.11. An immediate consequence of Lemma 4.1 and Proposition 4.6 is the following result, which is Schoeller's Theorem [**23**]. The proof follows from standard techniques in abelian category theory. See, for example, the proof of Mitchell's Theorem in [**7**].

THEOREM 4.7. *The functor $D_* : \mathcal{HA} \to \mathcal{D}$ has a right adjoint U and the pair (D_*, U) form an equivalence of categories.*

As mentioned, Proposition 4.6 follows from a much more general calculation. To set the stage, let R be the graded ring

$$R = \mathbb{Z}_p[V, F]/(VF - p)$$

where the degree of V is -1 and the degree of F is $+1$. Then an object $M \in \mathcal{D}_*$ may be regarded as a graded R module, provided one adopts the convention that for $a \in R$ and $x \in M$,

$$\deg(ax) = p^{\deg(a)}x. \tag{4.1}$$

With this convention, $\mathcal{D}_*$ is exactly the category of positively graded R modules, provided we agree that $ax = 0$ if $\deg(ax)$ is a fraction. Similarly, we have a category of positively graded $\mathbb{Z}_p[V]$ modules M, with $\deg(V) = -1$ and the same convention on degrees. Let us call this category $\mathcal{D}_V$. The category $\mathcal{M}_V$ of Definition 2.6 is the full subcategory of $\mathcal{D}_V$ with torsion free objects. There is a forgetful functor $\mathcal{D} \to \mathcal{D}_V$ and it has a left adjoint given by

$$M \mapsto R \otimes_{\mathbb{Z}_p[V]} M.$$

Now let $H \in \mathcal{HV}$ be a Hopf algebra with a lift of the Verschiebung. We wish to give a formula to calculate $D_*(\mathbb{F}_p \otimes H)$. Since H is determined by $QH \in \mathcal{M}_V \subseteq \mathcal{D}_V$, we'd like this formula to be functorial in QH.

We can define a natural homorphism $\eta : QH \to D_*(\mathbb{F}_p \otimes H)$ as follows. Write $n = p^k m$ with $(p, m) = 1$. Then Theorem 2.7 and Proposition 2.9.2 supply an isomorphism

$$(QH)_n \to \mathrm{Hom}_{\mathcal{HV}}(CW_m(k), H)$$

and the morphism $V : (QH)_{pn} \to (QH)_n$ is induced by the inclusion $CW_m(k) \subseteq CW_m(k+1)$. In degree n define η to be the composition

$$(QH)_n \cong \mathrm{Hom}_{\mathcal{HV}}(CW_m(k), H) \to \mathrm{Hom}_{\mathcal{HA}}(H(n), \mathbb{F}_p \otimes H) = D_n(\mathbb{F}_p \otimes H).$$

Then η is a morphism in $\mathcal{D}_V$ and, hence, it extends to a natural map of Dieudonné modules

$$\varepsilon_H : R \otimes_{\mathbb{Z}_p[V]} QH \to D_*(\mathbb{F}_p \otimes H).$$

The following is the main result of this section.

THEOREM 4.8. *The map ε_H is a natural isomorphism of Dieudonné modules for all $H \in \mathcal{HV}$.*

This will be proved below after some preliminary calculations.

LEMMA 4.9. *Let $M \in \mathcal{D}_V$ be torsion-free and $R = \mathbb{Z}_p[V, F]/(VF - p)$ the Dieudonné ring. Then for all $s > 0$*

$$\mathrm{Tor}^s_{\mathbb{Z}_p[V]}(R, M) = 0.$$

PROOF. Because Tor commutes with filtered colimits, we may assume M is finitely generated. Then, if $M(n) \subseteq M$ is the sub-module of elements in degree n or less, one has exact sequences

$$0 \to M(n-1) \to M(n) \to M(n)/M(n-1) \to 0$$

in $\mathcal{M}_V$ and $M(n)/M(n-1)$ is isomorphic to a direct sum of modules, which $\mathbb{Z}_p[n]$, meaning a single copy of $\mathbb{Z}_p$ in degree n. If $\mathrm{Tor}^s_{\mathbb{Z}_p[V]}(R, \mathbb{Z}_p[n]) = 0$ for all n, then a simple induction argument finishes the proof.

Define a complex of right $\mathbb{Z}_p[V]$ modules

$$0 \to \bigoplus_{s \geq 0} y_s \otimes \mathbb{Z}_p[V] \xrightarrow{\partial} \bigoplus_{s \geq 0} x_s \otimes \mathbb{Z}_p[V] \xrightarrow{\varepsilon} R \to 0$$

with $\varepsilon(x_s \otimes 1) = F^s$ and $\partial(y_s \otimes 1) = x_s \otimes p - x_{s+1} \otimes V$. This is a projective resolution of R as a right $\mathbb{Z}_p[V]$ module. Tensoring with $\mathbb{Z}_p[n]$ yields a complex

$$0 \to \bigoplus_{s\geq 0} y_s \otimes \mathbb{Z}_p[n] \xrightarrow{\partial} \bigoplus_{s\geq 0} x_s \otimes \mathbb{Z}_p[n] \xrightarrow{\varepsilon} R \otimes_{\mathbb{Z}_p[V]} \mathbb{Z}_p[n] \to 0.$$

One calculates that $\partial(y_s \otimes 1) = x_s \otimes p$. Hence ∂ is an injection and the result follows. $\square$

We also need a calculation.

LEMMA 4.10. *Let $\mathbb{F}_p[x]$ be the primitively generated Hopf algebra on an element of degree n and $f : H(n) \to \mathbb{F}_p[x]$ the unique map of Hopf algebras so that $f(x_k) = x$ and $f(x_i) = 0$ for $i < s$. Then $D_*\mathbb{F}_p[x]$ is the free module over $\mathbb{F}_p[F]$ on f.*

PROOF. Since the Verschiebung is zero on $\mathbb{F}_p[x]$, Lemma 4.1 and the definition of V on $D_*(\mathbb{F}_p[x])$ show that $V = 0$ on $D_*(\mathbb{F}_p[x])$. The result now follows from Lemma 4.4.1. $\square$

We now give the proof of Theorem 4.8.

PROOF. Since both the source and target of ε_H commute with filtered colimits we may assume that H is finitely generated and hence a polynomial algebra. (See Proposition 2.1). Since QH has a finite filtration

$$0 = M_s \subseteq \cdots \subseteq M_1 \subseteq M_0 = QH$$

so that M_k/M_{k+1} is a direct sum modules of the form $\mathbb{Z}_p[n]$, Theorem 2.7 implies that H has a finite filtration

$$\mathbb{Z}_p = H_s \subseteq \cdots \subseteq H_1 \subseteq H_0 = H$$

so that $QH_k = M_k$ and $\mathbb{Z}_p \otimes_{H_{k+1}} H_k$ is a primitively generated polynomial algebra with indecomposables isomorphic to M_k/M_{k+1}. Both the source and target of ε_H are exact on this filtration (here we use Lemma 4.9) and we are reduced to the case where H is a primitively generated polynomial algebra. Since both the source and target of ε_H commute with coproducts we may assume $H = \mathbb{Z}_p[x]$ with $\deg(x) = n$. Then it is a matter of direct calculation. The module $QH \cong Z_p[n] \cong \mathrm{Hom}_{\mathcal{HV}}(CW_m(k), H)$ and we can choose as generator of the latter group the map

$$CW_m(k) = \mathbb{Z}_p[x_0, x_1, \ldots, x_k] \to \mathbb{Z}_p[x] = H$$

sending x_k to x and x_i to 0 if $i < k$. Let $f \in D_n(\mathbb{F}_p \otimes H)$ be the reduction. Now $R \otimes_{\mathbb{Z}_p[V]} QH$ is the free module over $\mathbb{F}_p[F] \cong R \otimes_{\mathbb{Z}_p[V]} \mathbb{Z}_p$ on f. Hence the result follows from Lemma 4.10 $\square$

Now write $n = p^k m$ with $(m, p) = 1$ and let $\iota_n \in D_n H(n)$ be the identity.

COROLLARY 4.11. *Let $H(n) = \mathbb{F}_p[x_0, x_1, \ldots, x_k]$ be one of the projective generators of $\mathcal{HA}$. Then for all Dieudonné modules M, then the natural map*

$$\mathrm{Hom}_{\mathcal{D}_*}(D_*H(n), M) \to M_n$$

given by sending f to $f(\iota_n)$ is an isomorphism. Furthermore

$$D_*H(n) \cong R \otimes_{\mathbb{Z}_p[V]} K(n)$$

where $K(n)$ is the $\mathbb{Z}_p[V]$ module with $K(n)_m = 0$ unless $m = p^i k$, $0 \leq i \leq s$, $K(n)_{p^i s} \cong \mathbb{Z}/p^{i+1}\mathbb{Z}$ and V is onto.

PROOF. This follows from Theorem 4.8 and Proposition 2.9. □

EXAMPLE 4.12. Let's calculate the Dieudonné module of $H_*BU = H_*(BU, \mathbb{F}_p)$. We could use Example 2.12 and Theorem 4.8, but here's another way more suited to a later application. Consider the coalgebra $H_*(\mathbb{C}P^\infty, \mathbb{Z}_p)$. This is dual to the algebra $H^*(\mathbb{C}P^\infty, \mathbb{Z}_p) \cong \mathbb{Z}_p[x]$, which has a lift of the Frobenius given by $x \mapsto x^p$. Thus $H_*(\mathbb{C}P^\infty, \mathbb{Z}_p)$ has a lift of the Verschiebung ψ; indeed, if we define $\beta_i \in H_{2i}(\mathbb{C}P^\infty, \mathbb{Z}_p)$ to be dual to x^i, then $\psi(\beta_i) = \beta_{p/i}$ where $\beta_{p/i} = 0$ if p/i is a fraction. Now

$$H_*(BU, \mathbb{Z}_p) = S_*(H_*(\mathbb{C}P^\infty, \mathbb{Z}_p)) = \mathbb{Z}_p[b_1, b_2, \cdots]$$

where we write b_i for the image of β_i. Theorem 4.8 now implies that

$$D_*H_*BU \cong R \otimes_{\mathbb{Z}_p[V]} \tilde{H}_*(\mathbb{C}P^\infty, \mathbb{Z}_p)$$

with $V(\beta_i) = \beta_{p/i}$. In particular, if we also write b_i for the image of β_i in $D_{2i}H_*BU$, then

$$D_{2i}H_*BU \cong \mathbb{Z}/p^{\nu(i)}\mathbb{Z}$$

generated by b_i where $\nu(i) = k$ if $i = p^k j$ with $(j, p) = 1$. Furthermore, $Vb_i = b_{i/p}$, and this forces $F(b_i) = pb_{pi}$.

Part II: Bilinear Maps, Pairings, and Ring Objects

5. Bilinear maps and the tensor product of Hopf algebras

First we establish a categorical framework for tensor products, then specialize to our main interest—the category of Hopf algebras over over a field. Much of the ideas about tensor products in the category of coalgebras can be found in [**13**].

Let $\mathcal{C}$ be a category and $\mathcal{A} \subseteq \mathcal{C}$ a sub-category of abelian objects in $\mathcal{C}$. Thus, for all $A \in \mathcal{A}$,

$$F_A = \mathrm{Hom}_{\mathcal{C}}(\cdot, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$$

is a functor to abelian groups. Morphisms $f : A \to B$ in $\mathcal{A}$ induce natural transformations $F_A \to F_B$ of group-valued functors.

We will assume, for simplicity, that

5.1.1)both $\mathcal{C}$ and $\mathcal{A}$ have all limits and colimits;

5.1.2)the forgetful functor $\mathcal{A} \to \mathcal{C}$ has a left adjoint $S(\cdot)$.

DEFINITION 5.2. Let A, B, and C be objects in $\mathcal{A}$. A morphism $\varphi : A \times B \to C$ in $\mathcal{C}$ is a bilinear map if for all $X \in \mathcal{C}$, the induced map

$$F_A(X) \times F_B(X) \to F_C(X)$$

is a natural bilinear map of abelian groups.

This is equivalent to demanding that the following diagrams commute:

$$\begin{tikzcd}
A \times B \times B \arrow[rr, "1\times m_B"] \arrow[d, "(1\times T\times 1)(\Delta_A\times 1)"'] & & A \times B \arrow[d, "\varphi"] \\
A \times B \times A \times B \arrow[dr, "\varphi\times\varphi"'] & & C \\
& C \times C \arrow[ur, "m_C"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \times A \times B \arrow[rr, "m_A\times 1"] \arrow[d, "(1\times T\times 1)(1\times\Delta_B)"'] & & A \times B \arrow[d, "\varphi"] \\
A \times B \times A \times B \arrow[dr, "\varphi\times\varphi"'] & & C \\
& C \times C \arrow[ur, "m_C"'] &
\end{tikzcd}$$

Here $m_A : A \times A \to A$ is the multiplication, Δ_A is the diagonal, and T is the twist map interchanging factors.

We now define tenor products. The symbol "$\otimes$" will be reserved for the tensor product of modules over rings and, hence, for the product of coalgebras.

Definition 5.3. A tensor product of $A,\ B \in \mathcal{A}$ is an initial bilinear map $\varepsilon : A \times B \to A \boxtimes B$ in $\mathcal{A}$. Specifically, if $\varphi : A \times B \to C$ is any bilinear map there is a unique morphism $\psi : A \boxtimes B \to C$ in $\mathcal{A}$ making the diagram

$$\begin{tikzcd}
A \times B \arrow[rr, "\varepsilon"] \arrow[dr, "\varphi"'] & & A \boxtimes B \arrow[dl, "\psi"] \\
& C &
\end{tikzcd}$$

commute in $\mathcal{C}$.

Remark 5.4. If tensor products exist they are unique up to isomorphism in $\mathcal{A}$. Also, there is then a natural transformation of group-valued functors

$$(F_A \otimes F_B)(X) = F_A(X) \otimes F_B(X) \to F_{A\boxtimes B}(X).$$

This need not be an isomorphism: consider the case where $\mathcal{C}$ is the category of sets, $\mathcal{A}$ the category of abelian groups and $A = B = \mathbb{Z}$.

The assumptions made above make it easy to show tensor products exist. See [**13**].

Proposition 5.5. *Under the assumptions of (5.1) any two objects $A,\ B \in \mathcal{A}$ have a tensor product $A \boxtimes B$ in $\mathcal{A}$.*

Proof. This is an adaptation of the case where $\mathcal{A}$ is the category of R-modules for some ring R. Define $A \boxtimes B$ to be the colimit of the diagram in $\mathcal{A}$

$$S(A \times A \times B) \underset{g_A}{\overset{f_A}{\rightrightarrows}} S(A \times B) \underset{g_B}{\overset{f_B}{\leftleftarrows}} S(A \times B \times B)$$

where f_A is adjoint to

$$A \times A \times B \stackrel{m_A \times 1}{\to} A \times B \to S(A \times B)$$

and g_A is adjoint to

$$A \times A \times B \to A \times B \times A \times B \to S(A \times B) \times S(A \times B) \stackrel{m}{\to} S(A \times B)$$

where the first map is $(1 \times T \times 1)(1 \times \Delta_B)$. The morphism f_B and g_B are defined similarly. Let $\varepsilon : A \times B \to A \boxtimes B$ be defined by the composite

$$A \times B \to S(A \times B) \to A \boxtimes B;$$

it is bilinear by construction. Also if $\varphi : A \times B \to C$ is any bilinear map, the induced morphism

$$S(A \times B) \to C$$

factors uniquely through $A \boxtimes B$. □

We now begin to specialize to the case where $\mathcal{A}$ is a category of bicommutative Hopf algebras over a commutative ring k and $\mathcal{C}$ is a category of coalgebras. In this case a bilinear map is a morphism of coalgebras

$$\varphi : H_1 \otimes H_2 \to K$$

where H_1, H_2, and K are Hopf algebras. It is convenient to write $x \circ y$ for $\varphi(x \otimes y)$.

Lemma 5.6. *For Hopf algebras the following formulas hold.*

1. *For all $x\ y \in H_1$ and $z \in H_2$*

$$xy \circ z = \sum_i (x \circ z_i)(y \circ z_i')$$

where $\Delta z = \Sigma z_i \otimes z_i'$. A similar formula holds for $x \in H_1$, y, $z \in H_2$.

2. *If $1 \in H_1$ is the unit of H_1 regarded as an algebra, and $x \in H_2$, then $1 \circ x = \varepsilon(x)1$, where $\varepsilon : H_2 \to k$ is the augmentation.*

Proof. Part 1.) merely rewrites in formulas what it means to be bilinear. For 2.), it is sufficient to show $k \boxtimes H_2 \cong k$. More generally, given $\mathcal{A}$ and $\mathcal{C}$ as in 5.1, let $* \in \mathcal{A}$ be the terminal object. Then for all $A \in \mathcal{A}$, $* \boxtimes A \cong *$. To see this notice that if $* \times A \to C$ is a bilinear map, the induced bilinear map

$$F_*(X) \times F_A(X) \to F_C(X)$$

is the trivial map, since it factors through $F_*(X) \otimes F_A(X)$ and $F_*(X) = 0$. Now consider the pair

$$(\varepsilon, 1_A) \in F_*(A) \times F_A(A)$$

where $\varepsilon : A \to *$ is the unique map. This maps to 0 in $F_C(A)$ so there is a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\cong"'] & * \arrow[d, "\eta"] \\ * \times A \arrow[r] & C \end{tikzcd}$$

where η is the unit map in $\mathcal{C}$ from $*$ to C. This implies that the unique map $* \times A \to *$ is the initial bilinear map, as required. □

For the next few results we will stipulate only that $\mathcal{C}$ is a category of cocommutative coalgebras over a ring k and that $\mathcal{A}$ is a category of bicommutative Hopf algebras over k. We will abbreviate this by saying $\mathcal{A}$ is a category of Hopf algebras and $\mathcal{C}$ is a category of coalgebras.

LEMMA 5.7. *Let $\mathcal{A}$ be a category of Hopf algebras and $\mathcal{C}$ a category of coalgebras. Then for any diagram in $\mathcal{A}$, the natural map*

$$\operatorname*{colim}_{\alpha}(A_\alpha \boxtimes B) \to (\operatorname*{colim}_{\alpha} A_\alpha) \boxtimes B$$

is an isomorphism.

PROOF. This result is true in much more general contexts; however, it does require that the abelian category $\mathcal{A}$ satisfy certain axioms which can be uncovered by examining the argument.

First consider pushouts. Let $A_1 \xleftarrow{f_1} A_{12} \xrightarrow{f_2} A_2$ be a diagram of Hopf algebras. Then the push-out in $\mathcal{A}$ is $A_1 \otimes_{A_{12}} A_2$; that is, the algebra $A_1 \otimes A_2$ modulo the ideal generated by elements of the form

$$f_1(a)x \otimes y - x \otimes f_2(a)y. \tag{5.1}$$

Suppose one has a diagram

$$\begin{tikzcd} A_{12} \otimes B \arrow[r, "f_2 \otimes 1"] \arrow[d, "f_1 \otimes 1"'] & A_2 \otimes B \arrow[d, "\varphi_2"] \\ A_1 \otimes B \arrow[r, "\varphi_1"'] & C \end{tikzcd}$$

where φ_1 and φ_2 are bilinear. Define a bilinear map

$$\varphi : (A_1 \otimes_{A_{12}} A_2) \otimes B \to C$$

by

$$\varphi(x \otimes y \otimes b) = \sum_i \varphi_1(x \otimes b_i)\varphi_2(y \otimes c_i) \tag{5.2}$$

where $\Delta_B b = \Sigma b_i \otimes c_i$. The associativity of the diagonal, the formula 5.2, and Lemma 5.6.1 imply φ is well-defined. Now take

$$C = A_1 \boxtimes B \otimes_{A_{12} \boxtimes B} A_2 \boxtimes B$$

and φ_1 and φ_2 the evident bilinear maps. Then there is a resulting bilinear map

$$\varphi : (A_1 \otimes_{A_{12}} A_2) \otimes B \to A_1 \boxtimes B \otimes_{A_{12} \boxtimes B} A_2 \boxtimes B.$$

We leave it as an exercise to show φ has the requisite universal property to prove the result in this case.

Since the result is true for push-outs, it is true for finite coproducts and coequalizers. I claim it is true for all coproducts. In fact, I claim it is true for filtered colimits, so the claim about coproducts follows from the fact that any coproduct is the filtered colimit of its finite sub-coproducts. To see that it is true for filtered colimits, one uses the construction given in the proof of Proposition 5.5 and the fact that, in this case, the left adjoint S commutes with filtered colimits in the sense that for any diagram of coalgebras, the natural map in $\mathcal{C}$

$$\operatorname*{colim}_{\alpha}{}^{\mathcal{C}} S(C_\alpha) \to S(\operatorname*{colim}_{\alpha}{}^{\mathcal{C}} C_\alpha) \cong \operatorname*{colim}_{\alpha}{}^{\mathcal{A}} S(C_\alpha)$$

is an isomorphism; that is, the forgetful functor from $\mathcal{A}$ to $\mathcal{C}$ makes filtered colimits.

Finally the result is true for all colimits because there is, for any diagram A_α in $\mathcal{A}$, a coequalizer diagram

$$\coprod_{\alpha\to\beta} A_\alpha \rightrightarrows \coprod_\alpha A_\alpha \to \operatorname*{colim}_\alpha A_\alpha.$$

Hence the general result follows from the result on coproducts and coequalizers. $\square$

We next calculate the tensor product of free objects.

Corollary 5.8. *Let $\mathcal{A}$ be a category of Hopf algebras and $K \in \mathcal{A}$. Then the functor $\mathcal{A} \mapsto \mathcal{A}$ given by $H \mapsto H \boxtimes K$ has a right adjoint.*

Proof. This is a consequence of the special adjoint functor theorem [**15**] and the fact that $\cdot \boxtimes K$ commutes with all colimits. $\square$

It is appropriate to call this functor $\hom(K, \cdot)$ so that one has a formula

$$\mathrm{Hom}_{\mathcal{A}}(H_1 \boxtimes K, H_2) \cong \mathrm{Hom}_{\mathcal{A}}(H_1, \hom(K, H_2)). \tag{5.3}$$

It would interesting to have a concrete description of this homomorphism object and its Dieudonné module.

6. Bilinear pairings for Hopf Algebras with a lift of the Verschiebung.

We now examine the case where $\mathcal{A} = \mathcal{HV}$, the category of connected, torsion-free Hopf algebras over $\mathbb{Z}_p$ equipped with a lift of the Verschiebung, and $\mathcal{C} = \mathcal{CV}$, the category of connected, torsion free coalgebras over $\mathbb{Z}_p$, also equipped with a lift of the Verschiebung. These objects were discussed in some detail in section 2. The forgetful functor $\mathcal{HV} \to \mathcal{CV}$ has left adjoint S_*: if $C \in \mathcal{CV}$, S_*C is the symmetric algebra on

$$JC = \ker\{\varepsilon : C \to \mathbb{Z}_p\}$$

with diagonal induced from C and lift of the Verschiebung given by extending the left on C. Thus $\mathcal{HV}$ has the pairing $\boxtimes$.

The indecomposables functor Q defines a functor from $\mathcal{HV}$ to the category $\mathcal{M}_V$ of graded $\mathbb{Z}_p[V]$ modules M which are torsion-free as $\mathbb{Z}_p$ modules. If $x \in M_k$, then $Vx \in M_{k/p}$, wwhere we mean $Vx = 0$ if p does not divide k. In Theorem 2.11 it was noted that this indecomosables functor has a right adjoint $S^*(\cdot)$ and in Theorem 2.7 we showed that these two functors give an equivalence of categories.

Now let H_1, H_2, and K be in $\mathcal{HV}$ and

$$\varphi : H_1 \otimes H_2 \to K$$

be a bilinear map in $\mathcal{CV}$. Then, because φ is a morphism in $\mathcal{CV}$, φ commutes with the lifts of the Verschiebung by definition. Lemma 5.6.1 now implies that φ induces a pairing

$$Q\varphi : QH_1 \otimes QH_2 \to QK \tag{6.1}$$

where, writing $x \circ y$ for $Q\varphi(x \otimes y)$, one must have

$$V(x \circ y) = Vx \circ Vy.$$

Thus we should give $QH_1 \otimes QH_2$ the structure of an object in $\mathcal{M}_V$ by defining $V(x \otimes y) = Vx \otimes Vy$.

Now let $H_1 = S^*(M)$ and $H_2 = S^*(N)$ be two objects in $\mathcal{HV}$.

PROPOSITION 6.1. *There is a natural isomorphism in $\mathcal{HV}$*

$$S^*(M) \boxtimes S^*(N) \cong S^*(M \otimes N).$$

PROOF. We define a bilinear pairing $S^*(M) \otimes S^*(N) \to S^*(M \otimes N)$, and then we will show it has the correct universal property.

First note that if C is a coalgebra with a lift of the Verschiebung, one can regard the coaugmentation ideal JC as an object in $\mathcal{M}_V$ and the functor $J : \mathcal{CV} \to \mathcal{M}_V$ has as right adjoint $S^*(-)$, where we forget the algebra structure. To see this, let $S_* : \mathcal{CV} \to \mathcal{HV}$ be left adjoint to the forgetful functor. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{CV}}(C, S^*(M)) &\cong \mathrm{Hom}_{\mathcal{HV}}(S_*(C), S^*(M)) \\ &\cong \mathrm{Hom}_{\mathcal{M}V}(QS_*(C), M) \\ &\cong \mathrm{Hom}_{\mathcal{M}_V}(JC, M). \end{aligned}$$

With this in hand, let

$$\phi : S^*(M) \otimes S^*(N) \to K = S^*(QK)$$

be any bilinear map in $\mathcal{CV}$. The adjoint

$$J(S^*(M) \otimes S^*(N)) \to QK$$

factors, by Lemma 5.6, as

$$J(S^*(M) \otimes S^*(N)) \to J(S^*(M)) \otimes J(S^*(N)) \to M \otimes N \xrightarrow{Q\phi} QK$$

where $Q\phi$ is an in equation 6.1. This supplies a factoring of ϕ

$$S^*(M) \otimes S^*(N) \xrightarrow{\eta} S^*(M \otimes N) \xrightarrow{S^*Q\phi} S^*(QK) \cong K.$$

If we can show the first map is bilinear, we'll be done. To do this, we give another construction of η.

Note there is an obvious bilinear pairing

$$\mathrm{Hom}_{\mathcal{M}_V}(JC, M) \times \mathrm{Hom}_{\mathcal{M}_V}(JC, N) \to \mathrm{Hom}_{\mathcal{M}_V}(JC, M \otimes N)$$

sending a pair (f, g) to the composition

$$JC \xrightarrow{J\Delta} J(C \otimes C) \to JC \otimes JC \xrightarrow{f \otimes g} M \otimes N \tag{6.2}$$

Thus we get a bilinear pairing

$$\mathrm{Hom}_{\mathcal{CV}}(C, S^*(M)) \times \mathrm{Hom}_{\mathcal{CV}}(C, S^*(N)) \to \mathrm{Hom}_{\mathcal{CV}}(C, S^*(M \otimes N)) \tag{6.3}$$

and, hence, a bilinear pairing

$$\eta' : S^*(M) \otimes S^*(N) \to S^*(M \otimes N).$$

Note that η' is obtained by applying 6.3 to the two projections $S^*(M) \otimes S^*(N) \to S^*(M)$ and $S^*(M) \otimes S^*(N) \to S^*(N)$. Hence the morphism of coalgebras η' is adjoint to the morphism in $\mathcal{M}_V$

$$J(S^*(M) \otimes S^*(N)) \to JS^*(M) \otimes JS^*(N) \to M \otimes N$$

where the last map is projection onto the indecomposables. Since η and η' are adjoint to the same map, and since η' is bilinear, η must be bilinear. $\square$

The universal bilinear map

$$\eta : S^*(M) \otimes S^*(N) \to S^*(M \otimes N)$$

induces a map, via equation 6.1

$$Q\eta : QS^*(M) \otimes QS^*(N) \to QS^*(M \otimes N)$$

which fits into the following diagram:

$$\begin{tikzcd} QS^*(M) \otimes QS^*(N) \arrow[r] \arrow[d] & QS^*(M \otimes N) \arrow[d] \\ M \otimes N \arrow[r, "="'] & M \otimes N. \end{tikzcd}$$

The vertical maps are induced by the isomorphism of functors $QS^* \to 1$.

Thus we have proved

COROLLARY 6.2. *If H_1 and H_2 are two Hopf algebras in $\mathcal{HV}$, the universal bilinear map*

$$H_1 \otimes H_2 \to H_1 \boxtimes H_2$$

induces an isomorphism in $\mathcal{M}_V$

$$QH_1 \otimes QH_2 \to Q(H_1 \boxtimes H_2).$$

In the next result we are concerned with free Hopf algebras. If $C \in \mathcal{CV}$ is a torsion-free coalgebra with a lift of the Verschiebung, let $S_*(C)$ be the free Hopf algebra in $\mathcal{HV}$ on C. It is the symmetric algebra on the coaugmentation ideal of C, equipped with the coproduct and the lift of the Verschiebung induced from C.

Given C_1 and C_2 in $\mathcal{CV}$ let $C_1 \wedge C_2$ be the algebraic smash product of C_1 and C_2. This is defined as follows. First $C_1 \vee C_2$ is defined by the push-out diagram

$$\begin{tikzcd} \mathbb{Z}_p \times \mathbb{Z}_p \arrow[r] \arrow[d] & C_1 \times C_2 \arrow[d] \\ \mathbb{Z}_p \arrow[r] & C_1 \vee C_2 \end{tikzcd}$$

where $\mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$ is addition. Then $C_1 \wedge C_2$ is defined by the push-out diagram

$$\begin{tikzcd} C_1 \vee C_2 \arrow[r] \arrow[d, "\varepsilon"] & C_1 \otimes C_2 \arrow[d] \\ \mathbb{Z}_p \arrow[r] & C_1 \wedge C_2 \end{tikzcd}$$

If $S_*(C_1) \otimes S_*(C_2) \to K$ is a bilinear map in $\mathcal{HV}$, then the composite

$$C_1 \otimes C_2 \to S_*(C_1) \otimes S_*(C_2) \to K$$

factors, in $\mathcal{CV}$, through $C_1 \wedge C_2$, by Lemma 5.6. Thus one gets a map in $\mathcal{HV}$

$$S_*(C_1 \wedge C_2) \to K.$$

COROLLARY 6.3. *The induced map in $\mathcal{HV}$*

$$S_*(C_1 \wedge C_2) \to S_*(C_1) \boxtimes S_*(C_2)$$

is an isomporphism.

PROOF. There is a natural isomorphism $QS_*(C) \cong JC$, where JC is the coaugmentation ideal of C. The result now follows from Corollary 6.2 and Theorem 2.7. □

7. Universal bilinear maps for Hopf algebras over $\mathbb{F}_p$.

Let $\mathcal{HA}$ be the category of graded, connected Hopf algebras over $\mathbb{F}_p$. The purpose of this section is to calculate the Dieudonné module of the target of the universal bilinear map $H \otimes K \to H \boxtimes K$ as a functor of D_*H and D_*K in $\mathcal{HA}$. In summary, there will be an induced pairing $D_*H \otimes D_*K \to D_*(H \boxtimes K)$ which, while not an isomorphism, can be modified in a simple way to produce an isomorphism. The exact result is below in Theorem 7.7.

The first observation is this. Let H and K be objects in $\mathcal{HV}$, the category of graded connected torsion-free Hopf algebras over the $\mathbb{Z}_p$ equipped with a lift of the Verschiebung. Let $H \otimes K \to H \boxtimes K$ be the universal bilinear map in that category.

LEMMA 7.1. *Suppose H and K in $\mathcal{HV}$ are finitely generated as $\mathbb{Z}_p$ modules in each degree. Then the induced map*

$$(\mathbb{F}_p \otimes H) \boxtimes (\mathbb{F}_p \otimes K) \to \mathbb{F}_p \otimes (H \boxtimes K)$$

is an isomorphism.

PROOF. By construction $(\mathbb{F}_p \otimes H) \boxtimes (\mathbb{F}_p \otimes K)$ is a quotient of of the symmetric algebra $S_*((\mathbb{F}_p \otimes H) \otimes (\mathbb{F}_p \otimes K))$. See Proposition 5.5. This fact and Lemma 5.6.1) imply that if $\{x_i\}$ is a homogeneous basis of $Q(\mathbb{F}_p \otimes H) \cong \mathbb{F}_p \otimes QH$ and $\{y_j\}$ is a homogeneous basis of $Q(\mathbb{F}_p \otimes K)$, then $\{x_i \circ y_j\}$ spans $Q((\mathbb{F}_p \otimes H) \boxtimes (\mathbb{F}_p \otimes K))$. But Corollary 6.1 implies that the elements $x_i \circ y_j$ are linearly independent in $Q(\mathbb{F}_p \otimes (H \boxtimes K))$. Hence

$$Q((\mathbb{F}_p \otimes H) \boxtimes (\mathbb{F}_p \otimes K)) \to Q(\mathbb{F}_p \otimes (H \boxtimes K))$$

is an isomorphism. Since $H \boxtimes K$ is a polynomial algebra by Proposition 2.1, the result follows. □

REMARK 7.2. The finite type hypothesis can be removed by a filtered colimit argument, or by an application of Theorem 7.7 below.

In the following result, we are primarily concerned with generators $H(n)$ of $\mathcal{HA}$. Recall from Equation 4.1 that

$$H(n) \cong \mathbb{F}_p \otimes CW_m(k) \cong \mathbb{F}_p[x_0, \dots, x_k]$$

where $n = p^k m$ with $(p, m) = 1$. For simplicity write

$$G(n) = QCW_m(k).$$

COROLLARY 7.3. *Suppose H and K in $\mathcal{HV}$ are finitely generated as $\mathbb{Z}_p$ modules in each degree. Then there is an isomorphism*

$$R \otimes_{\mathbb{Z}_p[V]} (QH \otimes QK) \to D_*((\mathbb{F}_p \otimes H) \boxtimes (\mathbb{F}_p \otimes K)).$$

In particular if $H(n) \in \mathcal{HA}$ are the projective generators,

$$R \otimes_{\mathbb{Z}_p[V]} (G(n) \otimes G(m)) \to D_*(H(n) \boxtimes H(m))$$

is an isomorphism.

PROOF. The first isomorphism follows from Theorem 4.8 and Corollary 6.2. The second isomorphism follows from the first. □

We now can define the pairing $D_*H \otimes D_*K \to D_*(H \boxtimes K)$, using the method of universal examples. By Proposition 2.9.2, if $M \in \mathcal{M}_V$, then $\mathrm{Hom}_{\mathcal{M}_V}(G(n), M) \cong M_n$. Let $\iota_n \in G(n)_n$ correspond to the identity. By abuse of notation, also write ι_n for the reduction of this class in

$$D_*H(n) \cong R \otimes_{\mathbb{Z}_p[V]} G(n).$$

Then Corollary 4.11 says the function

$$\mathrm{Hom}_{\mathcal{D}}(D_*H(n), M) \to M_n$$

given $f \mapsto f(\iota_n)$ is an isomorphism. In particular, if $H \in \mathcal{HA}$, the natural isomorphism

$$\mathrm{Hom}_{\mathcal{HA}}(H(n), H) \cong D_nH$$

is defined by $f \mapsto D_*f(\iota_n)$.

Now let $\iota_n \otimes \iota_m \in D_*(H(n) \boxtimes H(m)) \cong R \otimes_{\mathbb{F}_p[V]} (G(n) \otimes G(m))$ be the evident class. If $H_1 \otimes H_2 \to K$ is a bilinear pairing of objects in $\mathcal{HA}$, we get a pairing

$$\circ : D_nH_1 \times D_mH_2 \to D_{n+m}K \tag{7.1}$$

as follows. If $x \in D_nH_1$ and $y \in D_mH_2$ we get a diagram

$$\begin{tikzcd} H(n) \otimes H(m) \arrow[r] \arrow[d, "f_x \otimes f_y"'] & H(n) \boxtimes H(m) \arrow[d, "g"] \\ H_1 \otimes H_2 \arrow[r] & K \end{tikzcd} \tag{7.2}$$

where $D_*f_x(\iota_n) = x$ and $D_*f_y(\iota_m) = y$, and g is the unique Hopf algebra map filling the diagram. Then

$$x \circ y = D_*g(\iota_n \otimes \iota_m). \tag{7.3}$$

Notice that element $\iota_n \circ \iota_m \in D_{n+m}[H(n) \boxtimes H(m)]$ is represented by the map g in the above commutative diagram.

LEMMA 7.4. *This pairing is bilinear and induces a pairing of graded modules*

$$\circ : D_*H_1 \otimes D_*H_2 \to D_*K.$$

PROOF. It is sufficient to examine the universal example

$$(H(n) \otimes H(n)) \otimes H(m) \to (H(n) \otimes H(n)) \boxtimes H(m). \tag{7.4}$$

If $\iota_n^1 + i_n^2 \in D_n(H(n) \otimes H(m)) \cong D_nH(n) \oplus D_nH(n)$, we need

$$(\iota_n^1 + \iota_n^2) \circ \iota_m = \iota_n^1 \circ \iota_m + \iota_n^2 \circ \iota_m. \tag{7.5}$$

But the bilinear map of 7.4 is the reduction modulo p of a bilinear map of objects in $\mathcal{HV}$. Indeed, write $n = p^js$ and $m = p^kt$ where $(p,s) = 1 = (p,t)$. Then the pairing of 7.4 is the reduction of pairing

$$(CW_s(j) \otimes CW_s(j)) \otimes CW_t(k) \to (CW_s(j) \otimes CW_s(j)) \boxtimes CW_t(k)$$

which induces a pairing on indecomposables (which is an isomorphism)

$$\circ : (G(n) \oplus G(n)) \otimes G(m) \to Q[(CW_s(j) \otimes CW_s(j)) \boxtimes CW_t(k)].$$

Here the formula 7.6 is obvious. □

Because the isomorphism

$$QCW_s(j) \otimes QCW_t(k) \to Q(CW_s(j) \boxtimes CW_t(k))$$

is an isomorphism in $\mathcal{M}_V$, similar methods imply

LEMMA 7.5. *If $H_1 \otimes H_2 \to K$ is a bilinear pairing of Hopf algebras over $\mathbb{F}_p$, then the pairing*

$$\circ : D_*H_1 \otimes D_*H_2 \to D_*K$$

has the property that

$$V(x \circ y) = Vx \circ Vy.$$

The operator F behaves differently. In fact, if F is supposed to reflect the Frobenius $(\cdot)^p$ and V the Verschiebung ξ, then one calculates

$$x^p \circ y = (x \circ \xi y)^p$$

using Lemma 5.6.1. Then one might expect

LEMMA 7.6. *The pairing $\circ : D_*H_1 \otimes D_*H_2 \to D_*K$ has the property that*

$$Fx \circ y = F(x \circ Vy) \quad \text{and} \quad x \circ Fy = F(Vx \circ y).$$

PROOF. Again, one need only calculate

$$F\iota_n \circ \iota_m = F(\iota_n \circ V\iota_m)$$

in $D_*(H(n) \boxtimes H(m))$. To do this, write $m = p^k t$ and $n = p^j s$, where $(s,p) = 1 = (t,p)$. Let $CW_s(\infty) = \mathbb{Z}_p[x_0, x_1, \dots]$ and $CW_t(\infty) = \mathbb{Z}_p[y_0, y_1, \dots]$ be the Hopf algebras of Definition 1.6. Then

$$CW_s(j) = \mathbb{Z}_p[x_0, x_1, \dots, x_j] \subseteq CW_s(\infty)$$

and, similarly, $CW_t(k) \subseteq CW_t(\infty)$. Now

$$QCW_s(\infty)_i \cong \begin{cases} \mathbb{Z}_p & i = p^r s \\ 0 & \text{otherwise} \end{cases}$$

and x_r (we confuse x_r with its image in the indecomposables) generates the module of indecomposables $QCW_s(\infty)_{p^r s}$. Also $Vx_r = x_{r-1}$, by Lemma 4.9. Finally $QCW_s(j) \subseteq CW_s(\infty)$ is a split injection of $\mathbb{Z}_p$ modules and $\iota_n = x_j$.

Notice that $H(n) \boxtimes H(m) \to (\mathbb{F}_p \otimes CW_s(\infty)) \boxtimes (\mathbb{F}_p \otimes CW_t(\infty))$ is an injection since

$$QCW_s(j) \otimes QCW_t(k) \to QCW_s(\infty) \otimes QCW_t(\infty)$$

is a split injection and both source and target are polynomial algebras. Thus

$$D_*H(n) \boxtimes H(m) \to D_*[(\mathbb{F}_p \otimes CW_s(\infty)) \boxtimes (\mathbb{F}_p \otimes CW_t(\infty))]$$

is an injection. Now one calculates

$$\begin{aligned} F\iota_n \circ \iota_m = Fx_k \circ y_j &= FVx_{k+1} \circ y_j \\ &= px_{k+1} \circ y_j \\ &= FV(x_{k+1} \circ y_j) \\ &= F(x_k \circ Vy_j) \\ &= F(\iota_n \circ V\iota_m). \end{aligned}$$

□

These results prompt the following definitions. A function $f : M_1 \times M_2 \to N$ of graded abelian groups will be called a *graded pairing* if

$$\deg(f(x,y)) = \deg(x) + \deg(y).$$

A graded pairing $f : M_1 \times M_2 \to N$ of Dieudonné modules will be called *bilinear in $\mathcal{D}$* if it is bilinear as a pairing of graded abelian groups and

$$\begin{aligned} Vf(x,y) &= f(Vx,Vy) \\ Ff(x,Vy) &= f(Fx,y) \qquad f(x,Fy) = Ff(Vx,y). \end{aligned}$$

Then the content of Lemmas 7.4–7.6 is that a bilinear map of Hopf algebras over $\mathbb{F}_p$ induces a bilinear pairing in $\mathcal{D}$

$$D_*H_1 \times D_*H_2 \to D_*K.$$

A universal bilinear pairing in $\mathcal{D}$, written $\eta : M_1 \times M_2 \to M_1 \boxtimes_{\mathcal{D}} M_2$ is an initial bilinear pairing in $\mathcal{D}$ out of $M_1 \times M_2$, in the obvious sense. If it exists, it is unique. We will show it exists and give a concrete description.

Indeed, for the purposes of the next few paragraphs, define a Dieudonné module $M \boxtimes_{\mathcal{D}} N$ as follows. For the graded tensor product $M \otimes N$, then this is a $\mathbb{Z}_p[V]$ module with

$$V(x \otimes y) = Vx \otimes Vy.$$

Our grading conventions are the exponential conventions given in Equation 4.1. Let $R = \mathbb{Z}_p[F,V]$ and

$$M \boxtimes_{\mathcal{D}} N = R \otimes_{\mathbb{Z}_p[V]} (M \otimes N)/K \tag{7.6}$$

where K is the sub-Dieudonné module generated by the relations

$$\begin{aligned} F \otimes x \otimes Vy &= 1 \otimes Fx \otimes y \\ F \otimes Vx \otimes y &= 1 \otimes x \otimes Fy. \end{aligned}$$

If $M_1 \times M_2 \to N$ is any bilinear pairing in $\mathcal{D}$, there is a morphism of Dieudonné modules $M_1 \boxtimes_{\mathcal{D}} M_2 \to N$. Also the function

$$\eta : M_1 \times M_2 \to M_1 \boxtimes_{\mathcal{D}} M_2$$

given by $(x,y) \mapsto x \otimes y$ is a bilinear pairing in $\mathcal{D}$. One now easily checks that η has the requisite universal property, so $\mathcal{D}$ has universal bilinear maps.

The following is the main result of this section.

THEOREM 7.7. *Let H and K be Hopf algebras over $\mathbb{F}_p$. Then the induced map of Dieudonné modules*

$$D_*H \boxtimes_{\mathcal{D}} D_*K \to D_*(H \boxtimes K)$$

is an isomorphism.

PROOF. We reduce to a special case. Let $S_* : \mathcal{CA} \to \mathcal{HA}$ be left adjoint to the forgetful functor. Then the Hopf algebras $S_*(\mathbb{F}_p \otimes C)$ with $C \in \mathcal{CV}$ finitely generated in each degree generate $\mathcal{HA}$. This follows from the fact that the Hopf algebras $H(n) = \mathbb{F}_p \otimes CW_m(k)$ generate $\mathcal{HA}$, so the Hopf algebras $\mathbb{F}_p \otimes H$, $H \in \mathcal{HV}$ with H finitely generated in each degree generate $\mathcal{HA}$ and $\mathcal{HV}$ is, in turn, generated by Hopf algebras of the form $S_*(C)$ with $C \in \mathcal{CV}$, finitely generated in each degree. See Lemma 2.5.

Thus we may write equations

$$\operatorname*{colim}_{\alpha} S_*(\mathbb{F}_p \otimes C_\alpha) \cong H \qquad \text{and} \qquad \operatorname*{colim}_{\beta} S_*(\mathbb{F}_p \otimes C_\beta) \cong K$$

for suitable diagrams, where C_α and C_β are objects in $\mathcal{CV}$. Since both the source and target of the natural map

$$D_*H \boxtimes_{\mathcal{D}} D_*K \to D_*(H \boxtimes K)$$

commute with colimits (see Lemma 5.7), we may assume that $H = S_*(\mathbb{F}_p \otimes C_1)$ and $K = S_*(\mathbb{F}_p \otimes C_2)$, with C_1 and C_2 in $\mathcal{CV}$, and finitely generated in each degree.

To prove the result in this case, Lemma 7.1 implies there is a natural isomorphism

$$\mathbb{F}_p \otimes [S_*(C_1) \boxtimes S_*(C_2)] \to S_*(\mathbb{F}_p \otimes C_1) \boxtimes S_*(\mathbb{F}_p \otimes C_2).$$

Then Corollary 6.3 and Theorem 4.8 complete the calculation:

$$\begin{aligned} D_*(S_*(\mathbb{F}_p \otimes C_1) \boxtimes S_*(\mathbb{F}_p \otimes C_2)) &\cong D_*(\mathbb{F}_p \otimes [S_*(C_1) \boxtimes S_*(C_2)]) \\ &\cong D_*(\mathbb{F}_p \otimes S_*(C_1 \wedge C_2)) \\ &\cong R \otimes_{\mathbb{Z}_p[V]} (JC_1 \otimes JC_2) \\ &\cong [R \otimes_{\mathbb{Z}_p[V]} (JC_1)] \boxtimes_{\mathcal{D}} [R \otimes_{\mathbb{Z}_p[V]} (JC_2)]. \end{aligned}$$

□

The pairings $\boxtimes$ on $\mathcal{HA}$ and $\boxtimes_{\mathcal{D}}$ on $\mathcal{D}$ are symmetric and the isomorphism of Theorem 7.7 reflects the symmetry. Note that if $t : H_1 \otimes H_2 \to H_2 \otimes H_1$ is the switch map, the composite

$$H_1 \otimes H_2 \xrightarrow{t} H_2 \otimes H_1 \to H_2 \boxtimes H_1$$

is bilinear and induces an isomorphism

$$t : H_2 \boxtimes H_1 \xrightarrow{\cong} H_2 \boxtimes H_1.$$

The following can be proved by reducing to the case of universal examples, as in Lemma 7.4.

LEMMA 7.8. *Let H_1 and H_2 be Hopf algebras in $\mathcal{HA}$. Then the following diagram commutes:*

$$\begin{tikzcd} D_*H_1 \boxtimes_{\mathcal{D}} D_*H_2 \arrow[r, "\varepsilon"] \arrow[d, "t"'] & D_*(H_1 \boxtimes H_2) \arrow[d, "D_*(t)"] \\ D_*H_2 \boxtimes_{\mathcal{D}} D_*H_1 \arrow[r, "\varepsilon"] & D_*(H_2 \boxtimes H_1). \end{tikzcd}$$

A similar statement holds for $\mathcal{HV}$.

EXAMPLE 7.9. Suppose M, N are Dieudonné modules and V is surjective on M and N. Then one can define the structure of a Dieudonné module on $M \otimes N$ as follows. First $V(x \otimes y) = Vx \otimes Vy$. Second, if $x \in M$, write $x = Vz$. Then one defines

$$F(x \otimes y) = z \otimes Fy.$$

If $Vz = Vz'$, then $z' = z + w$ where $pw = 0$. Write $y = Vy'$. Then

$$z' \otimes Fy = z \otimes Fy + w \otimes FVy' = z \otimes Fy$$

so F is well-defined. One easily checks $VF = FV = p$. Now consider the inclusion

$$j : M \otimes N \to M \boxtimes_{\mathcal{D}} N.$$

To start, this is only a morphism of $\mathbb{Z}_p[V]$ modules. However,

$$\begin{aligned} j(F(x \otimes y)) &= 1 \otimes z \otimes Fy = F \otimes x \otimes y \\ &= Fj(x \otimes y) \end{aligned} \tag{7.7}$$

so this is a morphism of Dieudonné modules. I claim it is an isomorphism. To see this we use the universal property of $M \boxtimes_{\mathcal{D}} N$. Suppose $f : M \times N \to K$ is a bilinear pairing in $\mathcal{D}$. Then we need only show there is a unique map of Dieudonné modules

$$g : M \otimes N \to K$$

making the obvious diagram commute. It is required, then, that $g(x \otimes y) = f(x, y)$. It follows that g commutes with V. That g commutes with F follows exactly as in Equation 7.7

8. Symmetric monoidal structures.

The categories of bicommutative Hopf algebras considered in the previous two sections are nearly symmetric monoidal categories; what is missing is a unit object e so that $e \boxtimes H \cong H \boxtimes e \cong H$. In order to supply this object we must extend the category somewhat.

Let H be a non-negatively graded Hopf algebra over a commutative ring k. Then $H_0 \subseteq H$—the elements of degree 0—form a Hopf algebra over k. With an eye to topological applications we insist that H_0 be a group ring. Put another way, let

$$X(H) = \mathrm{Hom}_{\mathcal{CA}_k}(k, H)$$

where $\mathcal{CA}_k$ is the category of non-negatively graded coalgebras over k. Then $X(H)$ is an abelian group and evaluation at $1 \in k$, defines an injection $X(H) \to H$ and hence an injection of Hopf algebras

$$k[X(H)] \to H$$

where $k[X(H)]$ is the group algebra. We will insist that $k[X(H)] \to H_0$ be an isomorphism, and say H is *group-like* in degree 0.

More generally, if $C \in \mathcal{CA}_k$, we can define

$$X(C) = \mathrm{Hom}_{\mathcal{CA}_k}(k, C)$$

and get an injection $k[X(C)] \to C_0$, where $k[X(C)]$, the free k module on $X(C)$, is now only a coalgebra. If this is an isomorphism, we will say C is *set-like* in degree 0.

Finally, if H is a non-negatively graded Hopf algebra over k, let $H_c = k \otimes_{H_0} H$ be the connected component of the identity. Then there is a natural isomorphism

$$H \to H_c \otimes_k H_0. \tag{8.1}$$

Such a splitting fails for coalgebras. The map $H \to H_c$ is an isomorphism if and only if H is connected.

We begin with a category of torsion-free Hopf algebras over $\mathbb{Z}_p$.

DEFINITION 8.1. Let $\mathcal{HV}^+$ be the category of torsion-free bicommutative Hopf algebras

i) group-like in degree 0;
ii) are equipped with a Hopf algebra map ψ lifting the Verschiebung; and
iii) $\psi = 1$ in degree 0.

The last hypothesis is innocuous as for all $H \in \mathcal{HV}^+$, the Verschiebung $\xi : \mathbb{F}_p \otimes H \to F_p \otimes H$ is the identity in degree zero. The underlying coalgebra category will be called $\mathcal{CV}^+$; it consists of torsion-free coalgebras C over $\mathbb{Z}_p$ equipped with a coalgebra map $\psi : C \to C$ lifting the Verschiebung. Again we ask that $\psi = 1$ in degree 0.

LEMMA 8.2. *The forgetful functor $\mathcal{HV}^+ \to \mathcal{CV}^+$ has a left adjoint F.*

PROOF. Let $C \in \mathcal{CV}^+$ and S_*C be the symmetric algebra on $\mathcal{C}$ endowed with the obvious diagonal. This is not yet a Hopf algebra. In fact, let $X = X(C)$. Then there is an isomorphism

$$S_*(C)_0 \cong \mathbb{Z}_p[\mathbb{N}X]$$

where $\mathbb{N}X$ is the free abelian monoid on X. Set

$$FC = \mathbb{Z}_p[\mathbb{Z}X] \otimes_{\mathbb{Z}_p\mathbb{N}X} S_*(C).$$

Note that FC is an appropriate group completion of S_*C. □

To classify $\mathcal{HV}^+$ as a category of modules, we extend the category $\mathcal{M}_V$ of section 2 as follows: define a new category $\mathcal{M}_V^+$ to be the category of graded abelian groups M equipped with a shift map $V : M_{pn} \to M_n$ so that M_n is a torsion free $\mathbb{Z}_p$ module if $n \geq 1$ and $V = 1 : M_0 \to M_0$. If $M \in \mathcal{M}_V^+$ then the elements $M_c \subseteq M$ of positive degree form a sub-object and the natural isomorphism $M \cong M_c \times M_0$ defines an equivalence of categories

$$\mathcal{M}_V^+ \cong \mathcal{M}_V \times \mathcal{A}b.$$

Here $\mathcal{A}b$ is the category of abelian groups. This will reflect the isomorphism of Equation 8.1

We next define a functor $Q^+ : \mathcal{HV}^+ \to \mathcal{M}_V^+$ by the equation

$$Q^+(H) = QH_c \times X(H).$$

In degree n, this functor is representable. If $n > 0$, let $n = p^k s$ with $(n, s) = 1$ and $CW_s(k)$ the Witt vector Hopf algebra of Definition 1.6. Then we have, by Theorem 2.7 and Proposition 2.9, that

$$[Q^+(H)]_n \cong QH_c \cong \mathrm{Hom}_{\mathcal{HV}}(CW_s(k), H_c) \cong Hom_{\mathcal{HV}^+}(CW_s(k), H).$$

If $n = 0$, then

$$[Q^+(H)]_0 = X(H) \cong \mathrm{Hom}_{\mathcal{HV}^+}(\mathbb{Z}_p[\mathbb{Z}], H).$$

The splitting of Equation 8.1 and Theorem 2.7 immediately imply

PROPOSITION 8.3. *The functor Q^+ defines an equivalence of categories $Q^+ : \mathcal{HV}^+ \to \mathcal{M}_V^+$.*

Now we turn to bilinear maps. Propositions 5.5 and Lemma 8.2 imply that the category $\mathcal{HV}^+$ has universal bilinear maps $H \otimes K \to H \boxtimes K$. Here is the first result on these.

LEMMA 8.4. *1.) Let A and B be abelian groups. Then in $\mathcal{HV}^+$*

$$\mathbb{Z}_p[A] \boxtimes \mathbb{Z}_p[B] \cong \mathbb{Z}_p[A \otimes_{\mathbb{Z}} B].$$

2.) Let $H \in \mathcal{HV}^+$ be connected and A an abelian group. Then $\mathbb{Z}_p[A] \boxtimes H$ is connected and

$$Q(\mathbb{Z}_p[A] \boxtimes H) \cong [A \otimes_{\mathbb{Z}} QH]/T$$

where $T \subseteq A \otimes_{\mathbb{Z}} QH$ is the torsion subgroup.
3.) For all $H \in \mathcal{HV}^+$ there is a natural isomorphism

$$\mathbb{Z}_p[\mathbb{Z}] \boxtimes H \cong H.$$

PROOF. Part 1.) is immediate from the universal property and the requirement that objects on $\mathcal{HV}^+$ are group-like in degree 0.

We next prove 3.) Let $\tau \in \mathbb{Z} \subseteq \mathbb{Z}_p[\mathbb{Z}]$ be the generator. Write $\mathbb{Z}$ multiplicatively. If $f : \mathbb{Z}_p[\mathbb{Z}] \otimes H \to K$ is any bilinear map, let $h : H \to K$ be the map $h(x) = f(\tau \otimes x) = \tau \circ x$. This is a Hopf algebra map since τ is group-like. See Lemma 5.6 Also f is determined by h and bilinearity. The claim is that there is a bilinear map g making the following diagram commute

$$\begin{tikzcd}
\mathbb{Z}_p[\mathbb{Z}] \otimes H \arrow[r, "g"] \arrow[d, "f"'] & H \arrow[dl, "h"] \\
K. &
\end{tikzcd}$$

If so, the result will follow. If $x \in H$ write the n-fold diagonal of x

$$\Delta_n x = \Sigma x_{i1} \otimes \cdots \otimes x_{in}.$$

Then g is defined by bilinearity and

$$g(\tau^n \circ x) = \begin{cases} \Sigma x_{i1} x_{i2} \cdots x_{in} & n > 0 \\ \eta\varepsilon(x) & n = 0 \\ \Sigma \chi(x_{i1}) \cdots \chi(x_{in}) & n < 0 \end{cases}$$

Here $\chi : H \to H$ is the canonical anti-automorphism arising from the fact, that as a group object, H must support inverses. One easily checks g is bilinear.

For part 2.), we first prove $\mathbb{Z}_p[A] \boxtimes H$ is connected. Given any bilinear map $\mathbb{Z}_p[A] \otimes H \to K$ we get a diagram of bilinear maps

$$\begin{tikzcd}
\mathbb{Z}_p[A] \otimes H \arrow[r, "\varphi"] \arrow[d] & K \arrow[d] \\
\mathbb{Z}_p[A] \otimes \mathbb{Z}_p \arrow[r] & K_0;
\end{tikzcd}$$

hence, a diagram of Hopf algebras

$$\begin{tikzcd}
\mathbb{Z}_p[A] \boxtimes H \arrow[r, "\overline{\varphi}"] \arrow[d, "\varepsilon"'] & K \arrow[d] \\
\mathbb{Z}_p \arrow[r] & K_0
\end{tikzcd}$$

since $\mathbb{Z}_p[A] \boxtimes \mathbb{Z}_p \to \mathbb{Z}_p$ is an isomorphism. Hence $\overline{\varphi}$ factors through K_c. Now let φ be universal bilinear map $\mathbb{Z}_p[A] \otimes H \to \mathbb{Z}_p[A] \boxtimes H$; then $\overline{\varphi} = 1$, and $\mathbb{Z}_p[A] \boxtimes H$ is connected.

To prove the assertion on indecomposables, note that part 3.) implies the result for $A = \mathbb{Z}$. For general A, take a free resolution

$$F_1 \to F_0 \to A \to 0.$$

Then since $Q(\mathbb{Z}_p[\cdot] \boxtimes H)$ commutes with all colimits, one gets a short exact sequence in $\mathcal{M}_V$

$$F_1 \otimes_{\mathbb{Z}} QH \to F_0 \otimes_{\mathbb{Z}} QH \to Q(\mathbb{Z}_p[A] \boxtimes H) \to 0.$$

Since all objects in $\mathcal{M}_V$ are torsion free, the result follows. □

This last result suggests how to define and analyze universal bilinear maps in $\mathcal{M}_V^+$. A bilinear pairing in $\mathcal{M}_V$

$$f : M_1 \times M_2 \to N$$

is a bilinear map of graded abelian groups (that is, deg $f(x, y) = \deg(x) + \deg(y)$) so that

$$Vf(x, y) = f(Vx, Vy).$$

There is a universal bilinear map $M \times N \to M \boxtimes_{\mathcal{M}} N$. Write $M \cong M_0 \oplus M_c$ where M_0 and M_c are the elements of degree 0 and positive degree respectively. Then $M \boxtimes_{\mathcal{M}} N$ is nearly $M \otimes N$ modulo torsion in positive degrees:

$$M \boxtimes_{\mathcal{M}} N \cong M_0 \otimes N_0 \oplus M_0 \otimes N_c/T_1 \oplus M_c \otimes N_0/T_2 \oplus M_c \otimes_{\mathbb{Z}_p} N_c,$$

where T_1 and T_2 are the torsion sub-modules. Lemma 8.4 and Corollary 6.2 now imply

PROPOSITION 8.5. *Let $H \otimes K \to H \boxtimes K$ be the universal bilinear pairing in $\mathcal{HV}^+$ and $Q^+ : \mathcal{HV}^+ \to \mathcal{M}_V^+$ the equivalence of categories. Then there is a bilinear pairing in $\mathcal{M}_V^+$*

$$Q^+H \times Q^+K \to Q^+(H \boxtimes K)$$

and the induced map in $\mathcal{M}_V$

$$Q^+H \boxtimes \mathcal{M} Q^+K \to Q^+(H \boxtimes K)$$

is an isomorphism.

For the following result, we will say that a functor $F : (\mathcal{C}, \otimes, e) \to (\mathcal{C}', \otimes', e')$ is an equivalance of categories with symmetric monoidal structure if F is an equivalence of categories $F(e) \cong e'$ and there are natural isomorphisms

$$F(X) \otimes' F(Y) \cong F(X \otimes Y).$$

COROLLARY 8.6. *1.) The category of Hopf algebras $\mathcal{HV}^+$ with pairing $\boxtimes$ is a symmetric monoidal category with unit $e = \mathbb{Z}_p[\mathbb{Z}]$.*
2.) The category $\mathcal{M}_V^+$ with pairing $\boxtimes_{\mathcal{M}}$ is a symmetric monoidal category with unit $e = \mathbb{Z}$, concentrated in degree 0.
3.) $Q^+ : \mathcal{HV}^+ \to \mathcal{M}_V^+$ is an equivalence of categories with symmetric monoidal structure.

Similar results hold over $\mathbb{F}_p$. Let $\mathcal{HA}^+$ be the category of bicommutative Hopf algebras over $\mathbb{F}_p$ group-like in degree 0, and let $\mathcal{CA}_p^+$ be the category of cocommutative coalgebras over $\mathbb{F}_p$ which are set-like in degree 0. Then:

LEMMA 8.7. *The forgetful functor $\mathcal{HA}^+ \to \mathcal{CA}^+$ has a left adjoint.*

The corresponding category of Dieudonné modules is easily defined.

DEFINITION 8.8. An object $M \in \mathcal{D}^+$ is a non-negatively graded abelian group equipped with operators $V : M_{pn} \to M_n$, $F : M_n \to M_{pn}$ so that

1. $VF = FV = p$;
2. $M_c \in \mathcal{D}$; and
3. $V = 1 : M_0 \to M_0$.

Notice, as in Equation 4.1, $M \in \mathcal{D}^+$ may be regarded as a $\mathbb{Z}_p[V,F]/(VF-p)$ module.

Define $H(n) \in \mathcal{HA}^+$ by $H(0) = \mathbb{F}_p[\mathbb{Z}]$ and $H(n) = \mathbb{F}_p[x_0, x_1, \ldots, x_s] = CW_m(k)$ for $n = p^k m$ with $(m,p) = 1$. Then one has a functor $D_* : \mathcal{HA}^+ \to \mathcal{D}^+$ given by

$$D_n H = \mathrm{Hom}_{\mathcal{HA}^+}(H(n), H)$$

with V and F given as in Definition 4.2 (for $n > 0$) or Defintion 8.8 for $n = 0$.

PROPOSITION 8.9. *The functor $D_* : \mathcal{HA}^+ \to \mathcal{D}^+$ is an equivalence of categories.*

Lemma 8.7 and Proposition 5.5 imply $\mathcal{HA}^+$ has universal bilinear pairings $H \otimes K \to H \boxtimes K$. Just as in Lemma 8.4 one has:

LEMMA 8.10. *1.) Let A and B be abelian group. Then in $\mathcal{HA}^+$*

$$\mathbb{F}_p[A] \boxtimes \mathbb{F}_p[B] \cong \mathbb{F}_p[A \otimes_{\mathbb{Z}} B].$$

2.) Let $H \in \mathcal{HA}^+$ be connected and A an abelian group. Then $\mathbb{F}_p[A] \boxtimes H$ is connected and

$$D_*(\mathbb{Z}_p[A] \boxtimes H) \cong A \otimes D_* H.$$

3.) For all $H \in \mathcal{HA}^+$, there is a natural isomorphism $\mathbb{F}_p[\mathbb{Z}] \boxtimes H \cong H$.

To characterize bilinear pairings via Dieudonné modules, we introduce the notion of a $\mathcal{D}^+$ bilinear map. It is a mild generalization of the notion introduced in section 7. A pairing $f : M \times N \to K$ of objects in $\mathcal{D}^+$ is a *bilinear pairing* in $\mathcal{D}^+$ if it is a bilinear pairing of graded abelian groups and

$$f(Vx, Vy) = Vf(x,y)$$
$$f(Fx, y) = Ff(x, Vy) \quad \text{and} \quad f(x, Fy) = Ff(Vx, y).$$

There is a universal bilinear pairing

$$M \times N \to M \boxtimes_{\mathcal{D}} N.$$

In fact, we can write

$$M \boxtimes_{\mathcal{D}^+} N \cong R_0 \otimes_{\mathbb{Z}[V]} (M \otimes N)/K$$

where $R_0 = \mathbb{Z}[V,F]/(VF - p)$ and K is the submodule generated by

$$F \otimes (x \otimes Vy) - 1 \otimes (Fx \otimes y) \quad \text{and} \quad F \otimes (Vx \otimes y) - 1 \otimes (x \otimes Fy).$$

This mimics the construction $M \boxtimes_{\mathcal{D}} N$ of the previous section. In fact if we write $M \cong M_0 \oplus M_c$ and $N \cong N_0 \oplus N_c$ where the 0 and c indicate elements of degree 0 and positive degree respectively, we have

LEMMA 8.11. *There is a natural isomorphism in $\mathcal{D}^+$*

$$M \boxtimes_{\mathcal{D}^+} N \cong M_0 \otimes_{\mathbb{Z}} N_0 \oplus M_0 \otimes_{\mathbb{Z}} N_c \oplus M_c \otimes_{\mathbb{Z}} N_0 \oplus M_c \boxtimes_{\mathcal{D}} N_c.$$

PROOF. One has

$$M \boxtimes_{\mathcal{D}^+} N \cong M_0 \boxtimes_{\mathcal{D}^+} N_0 \oplus M_0 \boxtimes_{\mathcal{D}^+} N_c \oplus M_c \boxtimes_{\mathcal{D}^+} N_0 \oplus M_c \boxtimes_{\mathcal{D}^+} N_c.$$

Now $M_c \boxtimes_{\mathcal{D}^+} N_c \cong M_c \boxtimes_{\mathcal{D}} N_c$, so one must identify the other three summands. I claim $M_0 \otimes_{\mathbb{Z}} N_c \to M_0 \boxtimes_{\mathcal{D}^+} N_c$ is an isomorphism. To see this note that $M_0 \otimes_{\mathbb{Z}} N_c$ has a structure of a Dieudonné module with $V(x \otimes y) = Vx \otimes Vy = x \otimes Vy$ and $F(x \otimes y) = x \otimes Fy$. The bilinear pairing $M_0 \times N_c \to M_0 \otimes_{\mathbb{Z}} N_c$ is a bilinear pairing in $\mathcal{D}^+$ and any bilinear pairing in $\mathcal{D}^+$ factors through this one. Hence $M_0 \otimes_{\mathbb{Z}} N_c$ has the required universal property. The other summands are handled the same way. □

Now let $H_1 \otimes H_2 \to K$ be a bilinear pairing in $\mathcal{HA}^+$. The one gets an induced pairing

$$\mu : D_*H_1 \times D_*H_2 \to D_*K.$$

Indeed, there are universal maps

$$H(n+m) \to H(n) \boxtimes H(m)$$

given by Equation 7.2 for $n > 0$ and $m > 0$, and by Lemma 8.10.3 for $n = 0$ or $m = 0$. If $f : H(n) \to H_1$ and $g : H(m) \to H_2$ represent $x \in D_n H_1$ and $y \in D_m H_2$ respectively, then $\mu(x, y)$ is represented by

$$H(n+m) \to H(n) \boxtimes H(m) \xrightarrow{f \boxtimes g} H_1 \boxtimes H_2 \to K.$$

LEMMA 8.12. *This pairing is a bilinear pairing in $\mathcal{D}^+$.*

PROOF. If $n > 0$ and $m > 0$, this follows from Lemmas 7.4–7.6. If $n = 0$ (or $m = 0$), one only has to check the universal examples. Thus, for example, the pairing is bilinear because one has a diagram

$$\begin{array}{ccc}
\mathbb{F}_p[\mathbb{Z}] \boxtimes H(n) & \xrightarrow{\Delta \boxtimes 1} & (\mathbb{F}_p[\mathbb{Z}] \otimes \mathbb{F}_p[\mathbb{Z}]) \boxtimes H(n) \\
\downarrow \cong & & \downarrow \cong \\
H(n) & \xrightarrow{\Delta} & H(n) \otimes H(n)
\end{array}$$

using Lemma 8.10.3 (or, rather, the proof of that statement—see Lemma 8.4.3) and the fact that $\boxtimes$ commutes with coproducts.

That $\mu(Vx, Vy) = V\mu(x, y)$ follows from the diagram

$$\begin{array}{ccc}
\mathbb{F}_p[\mathbb{Z}] \boxtimes H(n) & \xrightarrow{1 \boxtimes v} & \mathbb{F}_p[\mathbb{Z}] \boxtimes H(pn) \\
\downarrow \cong & & \downarrow \cong \\
H(n) & \xrightarrow{v} & H(pn)
\end{array} \tag{8.2}$$

where v induces V (see Definition 4.2). This implies $\mu(Fx, y) = F\mu(x, Vy)$ because $Fx = px$ and

$$\mu(px, y) = p\mu(x, y) = FV\mu(x, y) = F\mu(x, Vy).$$

Finally $\mu(x, Fy) = F\mu(Vx, y) = F\mu(x, y)$ by using a diagram similar to of Equation 8.2, using $f : H(pn) \to H(n)$ (which defines F) in place of the morphism v. □

That completed, note that Lemmas 8.10, 8.11, and Theorem 7.7 will now imply the following result. The proof is the same as for Proposition 8.5.

PROPOSITION 8.13. *Let $H \otimes K \to H \boxtimes L$ be the universal bilinear pairing in $\mathcal{HA}^+$. Then the induced bilinear pairing in $\mathcal{D}^+$*

$$D_*H \times D_*K \to D_*(H \boxtimes K)$$

induces an isomorphism

$$D_*H \boxtimes_{\mathcal{D}^+} D_*K \to D_*(H \boxtimes K).$$

We now can write down the analog of Corollary 8.6.

COROLLARY 8.14. *The categories $\mathcal{HA}^+$ and $\mathcal{D}^+$ are symmetric monoidal categories and the equivalence of categories*

$$D_* : \mathcal{HA}^+ \to \mathcal{D}^+$$

is an equivalence of categories with symmetric monoidal sturcture.

Part III: Hopf rings associated to homology theories

9. Skew commutatative Hopf algebras

If X is a double loop space, the homology Hopf algebra $H_*X = H_*(X, \mathbb{F}_p)$ is not commutative, but skew commutative, meaning that if $x \in H_mX$ and $y \in H_nX$, then

$$xy = (-1)^{mn}yx. \tag{9.1}$$

Similary, the homology coalgebra of a space Y is skew cocommuative. This turns out to be only a mild variation on the commutative case considered up to now, and this section adapts Dieudonné theory and the theory of bilinear pairings to this new situation.

To make the definitions precise, we work with non-negatively graded vector spaces over the field $\mathbb{F}_p$, $p > 2$. The case $p = 2$ is the commutative case. Then the signed twist map of the tensor product of two such objects

$$t : V \otimes W \longrightarrow W \otimes V$$

is given on homogeneous elements by $t(x \otimes y) = (-1)^{mn}y \otimes x$, with $x \in V_m$ and $y \in W_n$. A *skew commutative coalgebra* over $\mathbb{F}_p$ is a coassociative coalgebra C, set-like in degree zero, so that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & C \otimes C \arrow[d, "t"] \\ & C \otimes C. \end{tikzcd}$$

The category of such will be written $\mathcal{CA}_\pm$. Similary, a *skew commutative Hopf algebra H* is a Hopf algebra in the category of skew commutative coalgebras so that the multiplication satisfies the formula of Equation 9.1 The category of such Hopf algebras will be written $\mathcal{HA}_\pm$.

The reason this adaptation is simple is the following result, known as the splitting principle. See [**18**].

Proposition 9.1. *Let H be a skew commutative Hopf algebra. Then there is a natural isomorphism in $\mathcal{HA}_\pm$*

$$H \cong H_{ev} \otimes H_{odd}$$

where H_{ev} is concentrated in even degrees and H_{odd} is an exterior algebra on primitive generators in odd degrees.

Notice that a skew commutative Hopf algebra concentrated in even degrees is, in fact, commutative. In effect, Proposition 9.1 implies that there is a natural equivence of categories

$$\mathcal{HA}_\pm \simeq \mathcal{HA}_+ \times \mathcal{V}_* \tag{9.2}$$

where $\mathcal{V}_*$ is the category of non-negatively graded $\mathbb{F}_p$ vector spaces and $\mathcal{HA}_+$ is the category of commutative Hopf algebras which are group-like in degree zero.

The splitting principle immediately translates into a Dieudonné theory for this situation.

Definition 9.2. The category $\mathcal{D}_\pm$ of skew commutative Dieudonné modules has, as objects, non-negatively graded abelian groups M equipped with homomorphisms

$$F : M_{2n} \to M_{2pn} \qquad \text{and} \qquad V : M_{2pn} \to M_{2n}$$

so that

1. $FV = VF = p$;
2. $V = 1 : M_0 \to M_0$;
3. if $n = p^k m > 0$ and $(m,p) = 1$, then $p^{k+1}M_{2n} = 0$; and
4. $pM_{2n+1} = 0$ for all $n \geq 0$.

Note that the category of skew commutative Dieudonné modules can be regarded as a category of modules over the ring $R_0 = \mathbb{Z}[V,F]/(VF - p)$, subject to the exponential grading conventions of Equation 4.1 and the requirement that $Vx = Fx = 0$ if x is of odd degree.

Proposition 9.1, the assumption that our Hopf algebras are group-like in degree zero, and Schoeller's Theorem 4.7 now immediately imply:

Proposition 9.3. *There is an equivalence of categories*

$$D_* : \mathcal{HA}_\pm \longrightarrow \mathcal{D}_\pm.$$

The functor $H \mapsto D_nH$ is representable, just as in the commutative case. Indeed, if $n = 2p^k m$, with $(p,m) = 1$, then

$$D_nH \cong \mathrm{Hom}_{\mathcal{HA}_\pm}(H(n), H) \cong \mathrm{Hom}_{\mathcal{HA}}(H(n), H_{ev})$$

where $H(n) = \mathbb{F}_p \otimes CW_{2m}(k)$ is the Witt vector Hopf algebra of Definition 4.2. A similar remark holds in degree zero with $H(0) = \mathbb{F}_p[\mathbb{Z}]$. If $n = 2m+1$, then let $\Lambda(\mathbb{F}_p[n])$ be the exterior algebra on a single primitive generator of degree n. Then

$$D_nH \cong \mathrm{Hom}_{\mathcal{HA}_\pm}(\Lambda(\mathbb{F}_p[n]), H) \cong \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p[n]), PH) \cong (PH)_n.$$

Here PH are the primitives. If we write $H_\pm(2n) = H(2n)$ and $H_\pm(2m+1) = \Lambda(\mathbb{F}_p[2m+1])$, then these formulas combine to read

$$D_nH \cong \mathrm{Hom}_{\mathcal{HA}_\pm}(H_\pm(n), H). \tag{9.3}$$

We now come to bilinear pairings. The forgetful functor from $\mathcal{HA}_\pm \to \mathcal{CA}_\pm$ has a left adjoint; it is the free skew-commutative algebra functor suitably group-completed in degree zero. Compare Lemma 8.2. As a result, Proposition 5.5 implies that $\mathcal{HA}_\pm$ has bilinear pairings. We'd like to compute these via Dieudonné modules. The following is the crucial result. Let $S_\pm(\cdot)$ denote the free skew commutative algebra functor. If W is a vector spaces, the $S_\pm(W)$ can be made into a primitively generated skew commutative Hopf algebra.

PROPOSITION 9.4. *Let $\Lambda(V)$ be the primitively generated exterior algebra on a vector space V concentrated in odd degrees. If $H \in \mathcal{HA}_\pm$ is connected, then there is a natural ismorphism of Hopf algebras*

$$\Lambda(V) \boxtimes H \cong S_\pm(V \otimes QH).$$

PROOF. Because $(\cdot) \boxtimes H$ commutes with colimits, by Lemma 5.7, we may assume that $V = \mathbb{F}_p[n]$ for some odd integer n. Let $x \in \mathbb{F}_p[n]$ a generator. If

$$\phi : \Lambda(\mathbb{F}_p[n]) \otimes H \to K$$

is any bilinear map, then the formulas of Lemma 5.6 imply that for all y and z in the augmentation ideal of H,

$$\phi(x \otimes yz) = x \circ yz = 0.$$

Furthermore, for any y, the element $x \circ y \in K$ is primitive. In fact, there is a factoring of ϕ

$$\begin{tikzcd} \Lambda(\mathbb{F}_p[n]) \otimes H \arrow[r, "q"] \arrow[dr, "\phi"'] & \mathbb{F}_p[n] \otimes QH \arrow[d] \\ & PK \end{tikzcd}$$

where q is the composite

$$\Lambda(\mathbb{F}_p[n]) \otimes H \to I\Lambda(\mathbb{F}_p[n]) \otimes IH \to Q\Lambda(\mathbb{F}_p[n]) \otimes QH \cong \mathbb{F}_p[n] \otimes QH.$$

The vertical map extends to a Hopf algebra map $S_\pm(\mathbb{F}_p[n] \otimes QH) \to K$ and one checks that

$$\Lambda(\mathbb{F}_p[n]) \otimes H \xrightarrow{q} \mathbb{F}_p[n] \otimes QH \to S_\pm(\mathbb{F}_p[n] \otimes QH)$$

is bilinear. The result follows from the uniqueness of the universal bilinear maps. □

Exterior algebras and group rings behave in the expected way. The proofs are the same as those of Lemma 8.4.

LEMMA 9.5. *Let $\Lambda(V)$ a primitively generated exterior algebra on a vector space concentrated in odd degrees.*

1. *There is a natural isomorphism $\mathbb{F}_p[\mathbb{Z}] \boxtimes \Lambda(V) \cong \Lambda(V)$;*
2. *For all abelian groups A, $\mathbb{F}_p[A] \boxtimes \Lambda(V) \cong \Lambda(A \otimes V)$.*

Proposition 9.4 and Lemma 9.5 also allow one to come to term with $H \boxtimes \Lambda(V)$. In fact, if $t : H_1 \otimes H_2 \to H_2 \otimes H_1$ is the signed switch map, then t is an isomorphism of Hopf algebras and the composition

$$H_1 \otimes H_2 \xrightarrow{t} H_2 \otimes H_1 \to H_2 \boxtimes H_1$$

is bilinear and induces an ismorphism

$$t : H_1 \boxtimes H_2 \xrightarrow{\cong} H_2 \boxtimes H_1.$$

We now notice that the bilinear pairings defined on the various categories of Dieudonné modules of the previous sections extends to $\mathcal{D}_\pm$ as well. Regarding an object in $\mathcal{D}_\pm$ as a $R_0 = \mathbb{Z}[V,F]/(VF-p)$ module, we define, for $M, N \in \mathcal{D}_\pm$

$$M \boxtimes_{\mathcal{D}_\pm} M = R_0 \otimes_{\mathbb{Z}[V]} (M \otimes N)/K$$

where K is the submodule generated by

$$F \otimes (x \otimes Vy) - 1 \otimes (Fx \otimes y) \qquad \text{and} \qquad F \otimes (Vx \otimes y) - 1 \otimes (x \otimes Fy).$$

This mimics the construction $M \boxtimes_{\mathcal{D}} N$ of the two previous sections. The techniques of section 7 now imply that for all $H_1, H_2 \in \mathcal{HA}_\pm$ there is a natural map of Dieudonné modules

$$\varepsilon : D_*H_1 \boxtimes_{\mathcal{D}_\pm} D_*H_2 \to D_*(H_1 \boxtimes H_2).$$

The expected result is the following:

THEOREM 9.6. *This natural map*

$$\varepsilon : D_*H_1 \boxtimes_{\mathcal{D}_\pm} D_*H_2 \to D_*(H_1 \boxtimes H_2).$$

is an isomorphism in $\mathcal{D}_\pm$.

PROOF. As in the proof of Proposition 8.13, one splits up the bilinear pairing in $\mathcal{D}_\pm$ and reduces the result to previous calculations.

If $M \in \mathcal{D}_\pm$, then there is a natural splitting in $\mathcal{D}_\pm$

$$M \cong M_{ev} \oplus M_{odd}$$

where M_{ev} and M_{odd} are the elements of even and off degrees respectively. This reflects the splitting principle of Proposition 9.1. From this it follows that

$$M \boxtimes_{\mathcal{D}_\pm} N \cong M_{ev} \boxtimes_{\mathcal{D}_+} N_{ev} \oplus M_{odd} \otimes N_{ev}/FN_{ev} \oplus M_{ev}/FM_{ev} \otimes N_{odd} \oplus M_{odd} \otimes N_{odd}.$$

The result now follows from the splitting principle Proposition 9.1, the analagous result for $\mathcal{HA}_+$ Proposition 8.13, Proposition 9.4, Lemma 9.5, and Lemma 4.4 which says that for $H \in \mathcal{HA}$, there is a natural isomorphism

$$QH \cong D_*H/FD_*H.$$

□

It is worth pointing out that the natural isomorphism of the previous result reflects the skew symmetric nature of the category $\mathcal{HA}_\pm$. In the following let t stand for any of the signed switch maps; that is if the degree of x is m and the degree of y in n, then

$$t(x \otimes y) = (-1)^{nm}(y \otimes x).$$

Also let t also be the induced isomorphism of Hopf algebras

$$t : H_1 \boxtimes H_2 \to H_2 \boxtimes H_1.$$

For the following, compare Lemma 7.8.

LEMMA 9.7. *Let H_1 and H_2 be Hopf algebras in $\mathcal{HA}_\pm$. Then the following diagram commutes:*

$$\begin{tikzcd} D_*H_1 \boxtimes_{\mathcal{D}_\pm} D_*H_2 \arrow[r, "\varepsilon"] \arrow[d, "t"'] & D_*(H_1 \boxtimes H_2) \arrow[d, "D_*(t)"] \\ D_*H_2 \boxtimes_{\mathcal{D}_\pm} D_*H_1 \arrow[r, "\varepsilon"] & D_*(H_2 \boxtimes H_1). \end{tikzcd}$$

PROOF. It is only necessary to check this for the universal examples $H_1 = H(n)$ and $H_2 = H(m)$ of Equation 9.3 (Compare the proof of Lemma 7.4.) If n and m are even this follows from Lemma 7.8. If either n or m is odd, the result follows from Proposition 9.4 and Lemma 9.5. □

10. The Hopf ring of complex oriented cohomology theories.

Let E^* be a multiplicative cohomology theory represented by a ring spectrum E. Define spaces $E(n)$ by the formula

$$E(n) = \Omega^\infty \Sigma^n E.$$

The spaces $E(n)$ are generalized Eilenberg-Mac Lane spaces in the sense that if X is a CW complex, then there is a natural isomorphism

$$E^n X \cong [X, E(n)].$$

The cup product pairings

$$E(m) \wedge E(n) \to E(m+n)$$

induce bilinear maps

$$H_*E(m) \otimes H_*E(n) \to H_*E(m+n)$$

and, hence, $D_*H_*\mathbf{E} = \{D_*H_*E(n)\}_{n\in\mathbb{Z}}$ is a graded ring object in the category $\mathcal{D}_\pm$ of skew-commutative Dieudonné modules. This means that the ring multiplication satisfies the formulas of Lemmas 7.4–7.6. Such an object will be called a Dieudonné ring. Since

$$D_0H_*E(n) \cong \pi_0 E(n) \cong \pi_0 \Sigma^n E \cong \pi_{-n} E \cong E^n,$$

$D_*H_*\mathbf{E}$ is an E^* algebra and the operators V and F act on E^* as the identity and multiplication by p. We will call such an object an E^* Dieudonné algebra.

We will be particularly interested in homotopy commutative ring spectra E. This implies that that for all integers n and m there is a homotopy commutative diagram

$$\begin{tikzcd} E(m) \wedge E(n) \arrow[r] \arrow[d, "T"] & E(m+n) \arrow[d, "\chi_{m,n}"] \\ E(n) \wedge E(m) \arrow[r] & E(m+n) \end{tikzcd}$$

where the horizontal maps are the cup product maps, T is topological switch map, and $\chi_{m,n}$ is

$$\chi_{m,n} = \Omega^\infty[(-1)^{nm}\mathrm{id}] \in [E(n+m), E(n+m)].$$

This observation and Lemma 9.7 immediately imply the following result.

LEMMA 10.1. *Let E be a homotopy commutative ring spectrum. Suppose $x \in D_iH_*E(m)$ and $y \in D_jH_*E(n)$. Then*

$$x \circ y = (-1)^{ij+mn} y \circ x \in D_{i+j}H_*E(m+n).$$

Now suppose E is complex oriented; thus, there is a chosen element

$$x \in E^2\mathbb{C}P^\infty = [\mathbb{C}P^\infty, E(2)]$$

so that the composite

$$S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \xrightarrow{x} E(2)$$

represents the multiplicative identity in $\pi_0 E = \pi_2 E(2)$. The map x induces a morphism of coalgebras

$$H_*\mathbb{C}P^\infty \to H_*E(2);$$

hence, by adjointness, a morphism of Hopf algebras

$$H_*BU \cong S(H_*\mathbb{C}P^\infty) \to H_*E(2).$$

Since H_*BU is the reduction modulo p of a torsion-free Hopf algebra over $\mathbb{Z}_p$ with a lift of the Verschiebung we have, as in Example 4.12,

$$D_*H_*BU \cong R \otimes_{\mathbb{Z}_p[V]} QH_*(BU;\mathbb{Z}_p).$$

In fact, if $\beta_i \in H_{2i}(\mathbb{C}P^\infty, \mathbb{Z}_p)$ is the standard generator, we obtain an induced element $b_i \in D_{2i}H_*BU$ under the composition

$$\widetilde{H}_*(\mathbb{C}P^\infty, \mathbb{Z}_p) \to \widetilde{H}_*(BU, \mathbb{Z}_p) \to R \otimes_{\mathbb{Z}_p[V]} QH_*(BU, \mathbb{Z}_p) \cong D_*H_*BU;$$

Furthermore, $Vb_{pi} = b_i$. Note that in $H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; \mathbb{Z}_p)$

$$\Delta_*\beta_i = \sum_{j+k=i} \beta_j \times \beta_k \tag{10.1}$$

We will also use later that if $m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the multiplication, then

$$m_*(\beta_j \times \beta_k) = \binom{j+k}{j} \beta_{j+k}. \tag{10.2}$$

The complex orientation induces a map

$$D_*x_* : D_*H_*BU \to D_*H_*E(2).$$

Define $b_i^E \in D_{2i}H_*E(2)$ by

$$b_i^E = (D_*x_*)(b_i).$$

If there is no ambiguity, we may abuse notation and write b_i for b_i^E.

Now define an E^* Dieudonné algebra with underlying E^* algebra

$$R_0(E) = E^*[b_1, b_2, \dots]$$

with E^* in Dieudonné-degree 0 and b_i in bidegree $(2i, 2)$ where $2i$ is the Dieudonné degree. We require $Vb_{pi} = b_i$ and $V = 1$ on E^*, and that V be multiplicative. This and the formulas of Lemma 5.6 determine the action of F. The existence of the elements b_i^E determine a morphism of E^* Dieudonné algebras

$$R_0(E) \to D_*H_*\mathbf{E}.$$

This brings us to the Ravenel-Wilson relation. Let

$$x +_F y \in E^*[[x, y]]$$

be the formal group law for E^* and let

$$b(t) = \sum_{i=1}^{\infty} b_i^E t^i$$

be the evident power series over the ring $D_*H_*\mathbf{E}$. Since $D_*H_*\mathbf{E}$ is an E_* algebra we can form the power series in two variables

$$b(s) +_F b(t) \in (D_*H_*E)[[s,t]]$$

and, in this context, the Ravenel-Wilson relation becomes:

PROPOSITION 10.2. *Over $D_*H_*\mathbf{E}$ there is a formula*

$$b(s) +_F b(t) = b(s+t).$$

PROOF. The argument here is not so different than the one in [**21**]; the idea of using the total unstable operation as an organizing principle is due to Neil Strickland.

For any CW complex X, there is a total unstable operation

$$\mu = \mu_X : E^*X \to \mathrm{Hom}_{\mathcal{CA}}(H_*X, H_*\mathbf{E})$$

sending $f \in E^nX = [X, E(n)]$ to

$$f_* : H_*X \to H_*E(n)$$

this is continuous E^* algebra homomorphism and natural in X. If

$$m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

is the H-space multiplication we then get a commutative diagram

$$\begin{tikzcd}
E^*[[x]] \cong E^*(\mathbb{C}P^\infty) \arrow[r, "\mu"] \arrow[d, "E^*m"] & \mathrm{Hom}_{\mathcal{CA}}(H_*\mathbb{C}P^\infty, H_*\mathbf{E}) \arrow[d, "\mathrm{Hom}(m_*,1)"] \\
E^*[[x,y]] \cong E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \arrow[r, "\mu"] & \mathrm{Hom}_{\mathcal{CA}}(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty), H_*\mathbf{E})
\end{tikzcd}$$

so

$$(\mathrm{Hom}(m_*,1) \circ \mu)(x) = (\mu \circ E^*m)(x). \tag{10.3}$$

Since μ is an E^* algebra homomorphism

$$(\mu \circ E^*m)(x) = \mu(x +_F y) = \mu(x) +_F \mu(y) \tag{10.4}$$

where $\mu(x)$ is the coalgebra map

$$H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \xrightarrow{\pi_1} H_*\mathbb{C}P^\infty \xrightarrow{x_*} H_*E(2)$$

and $\mu(y)$ uses π_2 instead of π_1. On the other hand,

$$(\mathrm{Hom}(m_*,1) \circ \mu)(x)$$

is the composite

$$H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \xrightarrow{m_*} H_*\mathbb{C}P^\infty \xrightarrow{x_*} H_*E(2). \tag{10.5}$$

Given any map of coalgebras

$$f : H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \to H_*E(n)$$

it extends to a map of Hopf algebras

$$f^\natural : S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)) \to H_*E(n)$$

and the isomorphisms

$$
\begin{aligned}
(10.6)\quad \mathrm{Hom}_{\mathcal{CA}}(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty), &H_*\mathbf{E}) \\
&\cong \mathrm{Hom}_{\mathcal{HA}}(S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)), H_*\mathbf{E}) \\
&\cong \mathrm{Hom}_{\mathcal{D}^+}(D_*SH_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty), D_*H_*\mathbf{E})
\end{aligned}
$$

are isomorphisms of E^* algebras.

Since $S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty))$ is the mod p reduction of $S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty); \mathbb{Z}_p)$, which has a lift of the Frobenius, we can calculate, by Theorem 4.8

$$
\begin{aligned}
D_*S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)) &\cong R \otimes_{\mathbb{Z}_p[V]} QS(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty, \mathbb{Z}_p)) \\
&\cong R \otimes_{\mathbb{Z}_p[V]} \widetilde{H}_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; \mathbb{Z}_p).
\end{aligned}
$$

Let $b_{i,j} \in D_{2(i+j)}S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty))$ be the image of $\beta_i \times \beta_j$.

Then naturality and 10.2 implies that

$$D_*S(m_*) : D_*S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)) \to D_*S(H_*\mathbb{C}P^\infty)$$

sends $b_{i,j}$ to $\binom{i+j}{j} b_{i+j}$. Similarly if π_1 is projection on the first factor

$$D_*S(\pi_1)_* : D_*S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)) \to D_*S(H_*\mathbb{C}P^\infty)$$

is given by sending $b_{i,j}$ to b_i if $j = 0$ and 0 otherwise. There is an analogous formula involving π_2. Now let $f : H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \to H_*E(2)$ be either of the maps of (10.3) and

$$D_*f^\natural : D_*S(H_*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)) \to D_*H_*E(2)$$

the induced map. Then $D_*f^\natural$ extends to a map of E_* Dieudonné algebras

$$E^*[b_{i,j}] \to D_*H_*\mathbf{E}$$

which we will also call $D_*f^\natural$. Let $b^E_{i,j} = D_*f^\natural(b_{i,j})$ and consider the power series over $D_*H_*\mathbf{E}$:

$$b(s,t) = \Sigma b^E_{i,j} s^i t^j.$$

Note $i \geq 1$ or $j \geq 1$. Using the expression for f given in (10.5) we have

$$
\begin{aligned}
b(s,t) = \sum b^E_{i,j} s^i t^j &= \sum D_*f^\natural(b_{i,j}) s^i t^j \\
&= \sum D_*x_* \circ D_*S(m_*)(b_{i,j}) s^i t^j \\
&= \sum D_*x_* \binom{i+j}{j} b_{i+j} s^i t^j \\
&= \sum \binom{i+j}{j} b^E_{i+j} s^i t^j = b(s+t).
\end{aligned}
$$

This rewrites $b(s,t)$ in one way. Next, using the expression for f given in (10.4) and that the isomorphisms of (10.6) are E_* algebra maps we have

$$b(s,t) = D_*\mu(x)^\natural b(s,t) +_F D_*\mu(y)^\natural b(s,t).$$

However, since $\mu(x)$ involves projection on the first factor

$$
\begin{aligned}
D_*\mu(x)^\natural b(s,t) &= \sum D_*\mu(x)^\natural(b_{i,j}) s^i t^j \\
&= \sum D_*x_* \circ D_*S(\pi_1)_*(b_{i,j}) s^i t^j \\
&= \sum D_*x_*(b_i) s^i = b(s).
\end{aligned}
$$

Similarly $D_*\mu(y)^\natural b(s,t) = b(t)$, so $b(s,t) = b(s) +_F b(t)$. Combining the two expressions for $b(s,t)$ yields the result. $\square$

Now let $I \subseteq E_*[b_1, b_2, \ldots] = R_0(E)$ be the ideal of relations forced by requiring that $b(s+t) = b(s) +_F b(t)$. Then we get an induced map

$$R_E = R_0(E)/I \to D_* H_* \mathbf{E} \tag{10.7}$$

of E_* Dieudonné algebras. Under favorable circumstances this is almost an isomorphism, but not quite. To see what's missing, note that $R(E)$ is concentrated in bidegrees (s,t) with both s and t even. To account for odd degree groups we proceed as follows.

If $S^1 = \Sigma S^0$ is the stable 1-sphere, let $e \in D_1 H_* \Omega^\infty S^1 \cong [PH_*\Omega^\infty S^1]_1$ be the image of the generator of $\pi_1 S^1$ under the Hurewicz map. For any spectrum X, there is a bilinear pairing

$$D_*H_*\Omega^\infty S^1 \otimes D_*H_*\Omega^\infty\Sigma^{-1}X \to H_*\Omega^\infty(S^1 \wedge \Sigma^{-1}X) = H_*\Omega^\infty X$$

and we can define a degree raising map

$$D_*H_*\Omega^\infty\Sigma^{-1}X \to D_{n+1}H_*\Omega^\infty X \tag{10.8}$$

by

$$x \mapsto e \circ x. \tag{10.9}$$

Note that since $Ve_1 = 0$ we have, by Lemma 9.5 and Theorem 9.6

$$V(e \circ x) = 0 = e \circ Fx. \tag{10.10}$$

Hence there is a factorization (by Lemma 4.4)

$$\begin{tikzcd} D_*\Omega^\infty\Sigma^{-1}X \arrow[r, "e"] \arrow[d] & D_*H_*\Omega^\infty X \\ QH_*\Omega^\infty\Sigma^{-1}X \arrow[r, "\sigma"] & PH_*\Omega^\infty X \arrow[u] \end{tikzcd} \tag{10.11}$$

The map labeled σ is the homology suspension induced by the evaluation

$$\epsilon : \Sigma\Omega^\infty\Sigma^{-1}X \to \Omega^\infty X.$$

To see this note that there is a commutative diagram

$$\begin{tikzcd} S^1 \wedge \Omega^\infty\Sigma^{-1}X \arrow[r, "\epsilon"] \arrow[d] & \Omega^\infty X \arrow[d, "="] \\ \Omega^\infty S^1 \wedge \Omega^\infty\Sigma^{-1}X \arrow[r] & \Omega^\infty X. \end{tikzcd}$$

So, at $p > 2$, there is a diagram of bilinear pairings of Hopf algebras

$$\begin{tikzcd} \Lambda(e) \otimes H_*\Omega^\infty\Sigma^{-1}X \arrow[r] \arrow[d] & H_*\Omega^\infty X \arrow[d] \\ H_*\Omega^\infty S^1 \otimes H_*\Omega^\infty\Sigma^{-1}X \arrow[r] & H_*\Omega^\infty X \end{tikzcd}$$

which induces a diagram (using Proposition 9.4)

$$\begin{tikzcd} S(\Sigma QH_*\Omega^\infty X) \cong \Lambda(e) \boxtimes H_*\Omega^\infty\Sigma^{-1}X \arrow[r] \arrow[d] & H_*\Omega^\infty X \arrow[d] \\ H_*\Omega^\infty S^1 \boxtimes H_*\Omega^\infty\Sigma^{-1}X \arrow[r] & H_*\Omega^\infty X \end{tikzcd}$$

and the top map in this diagram is induced by the homology suspension. The argument at $p = 2$ is similar.

If E is a ring spectrum and $X = \Sigma^{n+1}E$, then $\Omega^\infty\Sigma^{n+1}E = E(n+1)$ and the natural map $\Omega^\infty S^1 \wedge \Omega^\infty E(n) \to \Omega^\infty E(n+1)$ fits into a diagram

$$\begin{tikzcd} \Omega^\infty S^1 \wedge E(n) \arrow[r] \arrow[d, "\Omega^\infty\eta\wedge 1"] & E(n+1) \arrow[d, "="] \\ E(1) \wedge E(n) \arrow[r, "\mu"] & E(n+1) \end{tikzcd}$$

where $\eta : S^1 \to \Sigma E$ is the suspension of the unit and the bottom map is the cup product pairing. Let $e_E \in D_1H_*E(1)$ be the image of e under $D_*H_*\Omega^\infty\eta$. If E is complex oriented, then there is a diagram

$$\begin{tikzcd} S^1 \wedge S^1 \arrow[r] \arrow[d] & \Omega^\infty S^1 \wedge \Omega^\infty S^1 \arrow[r] & E(1) \wedge E(1) \arrow[d] \\ S^2 \arrow[r] & \mathbb{C}P^\infty \arrow[r, "x"] & E(2) \end{tikzcd}$$

where x is the complex orientation. By definition of x the bottom composite must be the unit in $\pi_2E(2) \cong \pi_0E$. It follows that

$$e_E^2 = b_1^E \in D_2H_*E(2)$$

and, hence, we can extend 10.7 to a map of E_* Dieudonné algebras

$$\phi : R_E[e]/(e^2 - b_1) \to D_*H_*\mathbf{E}. \tag{10.12}$$

The equation $e^2 = b_1$ also appears in [**2**] and [**21**].

The main theorems on Hopf rings [**11, 13, 21**] can now be rephrased as follows:

THEOREM 10.3. *Suppose E_* is a Landweber exact homology theory with coefficient ring E_* concentrated in even degrees. Then*

$$\phi : R_E[e]/(e^2 - b_1) \to D_*H_*\mathbf{E}$$

is an isomorphism.

There is another proof of this fact in the next section.

It might be worth pointing out that the ring R_E has an interpretation in the language of formal groups. If Γ is an E^* Dieudonné algebra, the formal group law over E^* passes to a formal group law over Γ via the ring homomorphism $E^* \to \Gamma$. This might be called the E^* formal group law over Γ. Then the set of E^* Dieudonné algebra homomorphisms

$$R_E \longrightarrow \Gamma$$

is in one-to-one correspondence with homomorphisms of the additive formal group law over Γ to the E^* formal group law. Compare Paul Turner's interpretation of QBP_*BP [**25**].

EXAMPLE 10.4. Suppose $E = K$ is complex oriented K-theory; hence $K^* = \mathbb{Z}[\mu, \mu^{-1}]$ where $\mu \in K^{-2}$. The formal group law for K is

$$x +_F y = x + y + \mu xy,$$

so the equation $b(s+t) = b(s) +_F b(t)$ implies

$$\mu b_i b_j = \binom{i+j}{j} b_{i+j}$$

in $D_* H_* \mathbf{K}$.

Thus multiplication by μ^{-1} induces isomorphisms

$$\begin{aligned} \mu^{-n} : D_* H_* K(0) &\rightarrow D_* H_* K(2n) \\ \mu^{-n} : D_* H_* K(1) &\rightarrow D_* H_* K(2n+1) \end{aligned}$$

and, with ν the p-adic valuation,

$$\begin{aligned} D_i H_* K(0) &= D_i H_* (\mathbb{Z} \times BU) \\ &\cong \begin{cases} \mathbb{Z}/(p^{\nu(j)}+1), & i = 2j > 0 \\ 0, & i = 2j+1 \end{cases} \end{aligned}$$

The generator of $D_{2j} H_* K(0)$ is μb_i.

$$D_i H_* K(1) = D_i H_* U = \begin{cases} 0 & i = 2j \\ \mathbb{Z}/p & i = 2j+1. \end{cases}$$

The generator of $D_{2j+1} H_* K(1)$ is e if $j = 0$ and $\mu e b_j$ if $j > 0$.

EXAMPLE 10.5. Because of the connection between the element $e \in D_1 H_* E(1)$ and the homology suspension given in Equation 10.11, one can use Theorem 10.3 to give a description of $H_* E$ for certain E. Indeed, there is an isomorphism

$$H_* E \cong D_* H_* \mathbf{E}[\frac{1}{e}]$$

where $D_* H_* \mathbf{E}[1/e]$ in degree k is the colimit of

$$\cdots \rightarrow D_{n+k} H_* E(n) \xrightarrow{e} D_{n+1+k} H_* E(n+1) \xrightarrow{e} \cdots$$

Note that $E_n = D_0 H_* E(-n)$ maps to $H_n E$; this is the image of the Hurewicz homomorphism $E_* \rightarrow H_* E$. If we let $a_i \in H_{2i} E$ be the image of b_{i+1} (note the shift in indices), then we get a surjective map

$$E_*[a_0, a_1, \ldots]/(p, a_0 - 1) \rightarrow H_* E$$

and the kernel is determined by the relation

$$a(s+t) = a(s) +_F a(t)$$

where $a = \sum_{i=0}^{\infty} a_i t^{i+1}$. In short, a is a strict isomorphism between F modulo p and the additive formal group law; that is, a is an exponential for F after reducing mod p. Compare Corollary 4.1.9 of [**20**]

11. The role of $E_*\Omega^2S^3$.

If X is a spectrum, then $H_*\Omega^\infty X$ is a graded bicommutative Hopf algebra and one can study the functor

$$X \mapsto D_nH_*\Omega^\infty X, \qquad n \geq 0.$$

The following was proved in [**9**].

PROPOSITION 11.1. *There is a spectrum $B(n)$ and a natural surjection*

$$B(n)_nX \to D_nH_*\Omega^\infty X$$

which is an isomorphism if $n \not\equiv \pm 1 \mod 2p$.

As the notation suggests, the spectra $B(n)$ are the Brown-Gitler spectra. This is to say, if $p = 2$,

$$H^*B(n) \cong A/A\{\chi(Sq^i) : 2i > n\} \tag{11.1}$$

or if $p > 2$,

$$H^*B(n) \cong A/A\{\chi(\beta^\epsilon P^i) : 2pi + 2\epsilon > n\}, \tag{11.2}$$

and, furthermore, if $B(n) \to H\mathbb{Z}/p\mathbb{Z}$ classifies the generator of $H^0B(n)$, then the induced map

$$B(n)_nZ \to H_nZ \tag{11.3}$$

is surjective for all CW complexes Z.

The group homomorphisms

$$V : D_{2pn}H_*\Omega^\infty X \to D_{2n}H_*\Omega^\infty X \qquad \text{and} \qquad F : D_{2n}H_*\Omega^\infty X \to D_{2pn}H_*\Omega^\infty X$$

are induced, respectively, by maps

$$\phi : B(2pn) \to \Sigma^{2n(p-1)}B(n) \tag{11.4}$$

and

$$\psi : \Sigma^{2n(p-1)}B(n) \to B(2pn). \tag{11.5}$$

The map ϕ is very familiar, as it is the one that fits into the "Mahowald exact sequence"—the cofibration sequence

$$B(2pn-1) \longrightarrow B(2pn) \xrightarrow{\phi} \Sigma^{2n(p-1)}B(n),$$

explored, at $p = 2$, in [**16**] and [**3**], and at odd primes in [**5**]. The map ψ is less familiar—it is, for example, zero in cohomology.

At $p = 2$, the maps

$$V : D_{4n+2}H_*\Omega^\infty X \to D_{2n+1}H_*\Omega^\infty X$$

and

$$F : D_{2n+1}H_*\Omega^\infty X \to D_{4n+2}H_*\Omega^\infty X$$

are induced by

$$\phi : B(4n+2) \to \Sigma^{2n+1}B(2n+1)$$

and

$$\psi : \Sigma^{2n+1}B(2n+1) \to B(4n+2)$$

respectively, but the maps are not uniquely determined.

If X and Y are spectra, there is a natural bilinear pairing in $\mathcal{HA}$

$$H_*\Omega^\infty X \otimes H_*\Omega^\infty Y \to H_*\Omega^\infty(X \wedge Y)$$

induced by the map

$$\Omega^\infty X \wedge \Omega^\infty Y \to \Omega^\infty(X \wedge Y)$$

adjoint to the evaluation $\Sigma^\infty(\Omega^\infty X \wedge \Omega^\infty Y) \to X \wedge Y$. Thus, we get a pairing

$$D_n H_*\Omega^\infty X \otimes D_m H_*\Omega^\infty Y \to D_{n+m} H_*\Omega^\infty(X \wedge Y).$$

This yields pairings

$$\mu : B(n) \wedge B(m) \to B(n+m) \tag{11.6}$$

which are uniquely determined if $n + m \not\equiv \pm 1 \mod 2p$ and, in particular if n and m are both even. These pairings are also familiar, at least in cohomology, as

$$\mu^* : H^*B(n+m) \to H^*B(n) \otimes H^*B(m)$$

sends the generator to the tensor product of the two generators.

The ambiguity in the definition of μ when $n+m \equiv \pm 1 \mod 2p$ can be removed by noting that there there are canonical maps $B(n) \to B(n+1)$ which is a weak equivalence if n is even. Thus if n is odd we could require $\mu : B(n) \wedge B(m) \to B(n+m)$ to be the composite

$$B(n) \wedge B(m) \xrightarrow{\simeq} B(n-1) \wedge B(m) \to B(n+m-1) \xrightarrow{\simeq} B(n+m).$$

This said, the object $\mathcal{B} = \{B(n)\}_{n \geq 0}$ becomes a graded, commutative ring spectrum.

Now let E be a ring spectrum representing a cohomology theory E^* with products. Then

$$E_k B(n) \cong B(n)_k E \cong B(n)_n \Sigma^{n-k} E$$

so there is a surjective homomorphism

$$E_k B(n) \to D_n H_* \Omega^\infty E(n-k).$$

This induces a surjection

$$h : E_*\mathcal{B} \to D_* H_* \mathbf{E} \tag{11.7}$$

which skews degrees. Note that when $k = 0$, we get an isomorphism

$$E_k \cong E_k B(0) \cong D_0 H_* E(-k) = E^{-k}.$$

This is the standard isomorphism $E_* \cong E^{-*}$ and makes the homomorphism h is a morphism of $E_* \cong E^{-*}$ modules. It is for this reason that we will speak of E_* Dieudonné algebras in the sequel, rather than E^* Dieudonné algebras.

The properties of the map h of Equation 11.7 are recorded in the following sequence of results.

Proposition 11.2. *At primes $p > 2$, the bigraded E_* module $E_*\mathcal{B}$ is an E_* Dieudonné algebra. At the prime 2, $E_*\mathcal{B}$ has homomorphisms*

$$V : E_*B(4n) \to E_*B(2n) \qquad \textit{and} \qquad F : E_*B(2n) \to E_*B(4n)$$

satisfying the formulas of Lemmas 7.4 – 7.6

PROPOSITION 11.3. *At odd primes, the map*

$$h : E_*\mathcal{B} \to D_*H_*\mathbf{E}$$

is a surjective homomorphism of E_* Dieudonné algebras. At the prime 2, the map h respects the homomorphisms V and F.

REMARK 11.4. 1) At $p = 2$, $D_*H_*\Omega^\infty E(\cdot)$ has operators

$$V : D_{4n+2}H_*\Omega^\infty E(\cdot) \to D_{2n+1}H_*\Omega^\infty E(\cdot)$$

and

$$F : D_{2n+1}H_*\Omega^\infty E(\cdot) \to D_{4n+2}H_*\Omega^\infty E(\cdot)$$

which are not unambiguously defined in $E_*\mathcal{B}$.

2) The failure of the $B(2n+1)_{2n+1}X \to D_{2n+1}H_*\Omega^\infty X$ to be an isomorphism can be measured by the operator e introduced in Equation 10.8. Since $B(2n+1) = B(2n)$, we have a diagram

$$\begin{tikzcd} B(2n)_{2n}\Sigma^{-1}X \arrow[r, "\cong"] \arrow[d, "\cong"] & D_{2n}H_*\Omega^\infty\Sigma^{-1}X \arrow[d, "e"] \\ B(2n+1)_{2n+1}X \arrow[r] & D_{2n+1}H_*\Omega^\infty X \end{tikzcd}$$

and we see that these kernel of $B(2n+1)_{2n+1}X \to D_{2n+1}H_*\Omega^\infty X$ is isomorphic to the kernel of

$$e : D_{2n}H_*\Omega^\infty\Sigma^{-1}X \to D_{2n+1}H_*X.$$

We now begin our analysis of specific ring spectra. The following result will allow a careful examination of the kernel of $E_*\mathcal{B} \to D_*H_*\mathbf{E}$ in some cases, for example any Landweber exact theory with coefficients in even degrees.

For the following compare [**21**] and [**2**].

LEMMA 11.5. *Suppose E is a ring spectrum with E_* torsion free and concentrated in even degrees. Suppose further that $E_*B(n)$ is concentrated in even degrees. Then for all k,*

1) *$H_*E(2k)$ is concentrated in even degrees.*
2) *On $H_*E(2k)$, the Frobenius is injective and the Verschiebung surjective.*
3) *There is an isomorphism of primitively generated Hopf algebras*

$$H_*E(2k+1) \cong \Lambda(\Sigma QH_*E(2k)).$$

PROOF. These will be proved together. The surjection

$$0 = E_{2n-2k-1}B(2n) \to D_{2n}H_*E(2k+1)$$

shows $D_*H_*E(2k+1)$ is concentrated in odd degrees, and, hence, $H_*E(2k+1)$ is an exterior algebra. Let $E_0(2n) \subseteq E(2n)$ be the component of the basepoint. Then the Rothenberg-Steenrod spectral sequence,

$$\Gamma(QH_*E(2k-1)) \cong \mathrm{TOR}_*^{H_*E(2k-1)}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_*E_0(2n)$$

is a spectral sequence of Hopf algebras and, hence, must collapse. Since the Verschiebung is surjective at E_2 on this spectral sequence, it is on $H_*E_0(2n)$ and hence on $H_*E(2n)$.

Similarly, consider

$$\mathrm{Tor}_s^{H_*E(2k)}(\mathbb{F}_p,\mathbb{F}_p)_t \Rightarrow H_{s+t}E(2k+1). \tag{11.8}$$

We work at $p > 2$. The argument for $p = 2$ is similar. Let $QH_*E(2k)$ denote the indecomposables and Q_+ and Q_- the even and odd degree sub-vector spaces of $QH_*E(2k)$. Let $L_1QH_*E(2k)$ be the first derived functor of Q applied to $H_*E(2k)$. Then $L_1QH_*E(2k) = L_1Q$ is concentrated in degrees congruent to zero modulo $2p$. Then

$$\mathrm{Tor}_*^{H_*E(2k)}(\mathbb{F}_p,\mathbb{F}_p) \cong \Lambda(Q_+)\otimes\Gamma(Q_-)\otimes\Gamma(L_1Q)$$

where

$$QH_*E(2k) \cong \mathrm{Tor}_1^{H_*E(2k)}(\mathbb{F}_p,\mathbb{F}_p)$$

and

$$L_1Q \subseteq \mathrm{Tor}_2^{H_*E(2k)}(\mathbb{F}_p,\mathbb{F}_p)$$

are the primitives. If $Q_- \neq 0$ or $(L_1Q) \neq 0$, the lowest degree non-zero class in $Q_-\oplus L_1Q$ would produce a non-zero even degree class in $H_*E(2k+1)$, a contradiction, so $Q_- = 0 = L_1Q$ and

$$QH_*E(2k) = Q_+.$$

The spectral sequence of Equation 11.8 now collapses, proving part 3. Since the module of indecomposables $QH_*E(2k)$ is in even degrees part 1 follows. Half of part 2 has already been proved, and since $L_1QH_*E(2k) = 0$, the Frobenius is injective on $H_*E(2k)$. See [**10**]. In fact, the kernel of F in Lemma 4.4 is exactly L_1Q. □

Now let $D_E \subseteq D_*H_*\mathbf{E}$ be the Dieudonné ring

$$D_E = \{D_{2m}H_*E(2n)\}.$$

If E_* is concentrated in even degrees. Let $e \in D_1H_*E(1)$ be the suspension element of Equation 10.8

PROPOSITION 11.6. *Suppose E is a ring spectrum with E_* torsion free and concentrated in even degrees. Suppose $E_*B(n)$ is concentrated is even degrees for all n. Then the natural map of E_* Dieudonné algebras*

$$D_E[e]/(e^2-b_1) \to D_*H_*\mathbf{E}$$

is an isomorphism.

PROOF. Since $D_{2m+1}H_*E(2n) = 0$ (by Lemma 11.5.1), the induced map

$$(D_E[e]/(e^2-b_1))_{*,2n} \to D_*H_*E(2n)$$

is an isomorphism. The result will follow once we show

$$eD_*H_*E(2n) = \{\ e\circ x\ :\ x \in D_*H_*E(2n)\ \} \to D_*H_*E(2n+1)$$

is an isomorphism. Since $e(Fx) = F(Ve\circ x) = 0$, and $D_*H/FD_*H \cong QH$

$$eD_*H_*E(2n) = \Sigma QH_*E(2n)$$

and the result follows from Lemma 11.5. □

If $E_*B(n)$ is concentrated in even degrees for all n, then we may define

$$E_*\mathcal{B}(ev) = \{E_*B(2n)\} = \{E_{2m}B(2n)\}$$

and the homomorphism of E_* Dieudonné algebras

$$E_*\mathcal{B} \to D_*H_*\mathbf{E}$$

restricts to an isomorphism of E_* Dieudonné algebras

$$E_*\mathcal{B}(ev) \to D_E.$$

Calculating the source of this map is where $E_*\Omega^2S^3_+$ comes in.

Write $\Omega^2S^3_+$ for the suspension spectrum of the space Ω^2S^3 with a disjoint basepoint. The May-Milgram filtration $\{F_k\Omega^2S^3\}$ of Ω^2S^3 suspends to a filtration $\{F_k = F_k\Omega^2S^3_+\}$ of $\Omega^2S^3_+$. The Snaith splitting implies this stable filtration is trivial: $\Omega^2S^3_+ \simeq \vee_k F_k/F_{k-1}$. At a prime p, the filtration quotients are Brown-Gitler spectra. Specifically, at $p=2$,

$$F_k/F_{k-1} \simeq \Sigma^k B(k)$$

and if $p>2$,

$$F_k/F_{k-1} \simeq \begin{cases} \Sigma^{2n(p-1)}B(2n); & \text{if } k = pn; \\ \Sigma^{2n(p-1)+1}B(2n+1), & \text{if } k = pn+1; \\ * & \text{if } k \not\equiv 0,1 \mod p. \end{cases} \tag{11.9}$$

This previous paragraph summarizes work of [**3, 5, 12, 16, 24**]. In short, the associated graded spectrum of $\Omega^2S^3_+$ is a regraded version of the graded spectrum $\mathcal{B} = \{B(n)\}$.

It is convenient, for our purposes, to eliminate the odd Brown-Gitler spectra. If we write $\Omega^2S^3\langle 3\rangle$ for the universal cover of Ω^2S^3 and $F_k\Omega^2S^3\langle 3\rangle$ for the universal cover of $F_k\Omega^2S^3$, then $\{F_k\Omega^2S^3\langle 3\rangle\}$ is a filtration of $\Omega^2S^3\langle 3\rangle$ and we get a filtration $\{F'_k = F_k\Omega^2S^3\langle 3\rangle_+\}$ of the suspension spectrum. Let $\mathcal{B}'$ be the associated graded spectrum. Then Equation 11.9 and a homology calculation shows that at any prime

$$\mathcal{B}' \simeq \{\Sigma^{2k(p-1)}B(2k)\}.$$

In short, $\mathcal{B}'$ is a regraded version of $\mathcal{B}(ev)$. In particular, there is a degree shearing isomorphism of E_* modules

$$E_*\mathcal{B}(ev) \cong E_*\mathcal{B}'.$$

We next observe that the loop space multiplication on $\Omega^2S^3\langle 3\rangle$ gives $\Omega^2S^3\langle 3\rangle_+$ the structure of a ring spectrum and, since the May-Milgram filtration is multiplicative, gives $\mathcal{B}'$ the structure of a graded ring spectrum. I don't know whether $\mathcal{B}(ev) \simeq \mathcal{B}'$ as graded ring spectra; this would settle Ravenel's Conjecture 3.4 ([**19**]), for example. However, the standard calculations ([**4**]) show that

$$H_*\mathcal{B}(ev) \cong H_*\mathcal{B}' \tag{11.10}$$

as $\mathbb{F}_p$ algebras. We can use this fact and Ravenel's Adams spectral sequence calculations ([**19**]) to calculate

$$D_{BP} \subseteq D_*H_*\mathbf{BP}.$$

Let R_E be the ring of Equation 10.7

THEOREM 11.7. *The graded BP_* modules $BP_*B(n)$ are concentrated in even degrees and the natural maps*

$$BP_*\mathcal{B}(ev) \to D_{BP} \leftarrow R_{BP}$$

are isomorphisms of BP_* Dieudonné algebras.

PROOF. Let $I = (p, v_1, v_2, \dots) \subseteq BP_*$ be maximal ideal. If M is an BP_* module we can filter M by powers of I and form the associated graded object E^0M. Then

$$E^0BP_* \cong \mathrm{Ext}_A^{**}(\mathbb{F}_p, H_*BP) = \mathrm{Ext}_E^*(\mathbb{F}_p, \mathbb{F}_p),$$

where A is the dual Steenrod algebra and $E = \mathbb{F}_p \otimes_{*BP} A$. If X is a spectrum, then the Adams spectral sequence

$$\mathrm{Ext}_E^{*;*}(\mathbb{F}_p, H_*X) \Rightarrow BP_*X$$

is a spectral sequence of E_0BP_* modules. The calculations of Theorem 3.3 of [**19**] combined with Theorem 3.14 of [**21**] (which is algebraic and precedes their computation of $H_*\mathbf{BP}$) now imply

$$E^0BP_*\mathcal{B}' = E_\infty^{*;*} = \mathrm{Ext}_E^{*;*}(\mathbb{F}_p, H_*\mathcal{B}') \cong E^0R_{BP},$$

all as E^0BP_* algebras. As a result we have a commutative square of E^0BP_* algebras

$$\begin{tikzcd}
E^0BP_*\mathcal{B}(ev) \arrow[r, "\cong"] & E^0D_*H_*\mathbf{BP} \\
E^0BP_*\mathcal{B}' \arrow[u, "\cong"] \arrow[r, "\cong"] & E^0R_{BP}. \arrow[u]
\end{tikzcd}$$

This finishes the proof. ∎

Landweber exactness now implies the following result, which finishes the proof of Theorem 10.3

COROLLARY 11.8. *Let E_* be any Landweber exact theory with coefficients concentrated in even degrees. Then $E_*B(n)$ is in even degrees and the natural maps*

$$E_*\mathcal{B}(ev) \to D_E \leftarrow R_E$$

are isomorphisms of E_* Dieudonné algebras.

PROOF. Note that Landweber exactness implies that E_* is torsion free. Since $\mathcal{B}$ is p-local,

$$MU_*\mathcal{B}(ev) \cong \bigoplus_{(p,n)\neq 1} (\Sigma^{2n}BP)_*\mathcal{B}(ev)$$

and the result follows from Theorem 11.7. For general E we have a commutative diagram

$$\begin{tikzcd}
E_* \otimes_{MU_*} MU_*\mathcal{B}(ev) \arrow[r, "\cong"] \arrow[d, "\cong"] & E_* \otimes_{MU_*} R_{MU} \cong R_E \arrow[d] \\
E_*\mathcal{B}(ev) \arrow[r, "\cong"] & D_E.
\end{tikzcd}$$

∎

Glossary

Categories of Hopf Algebras

$\mathcal{HA}$: graded, connected bicommutative Hopf algebras over $\mathbb{F}_p$: section 4

$\mathcal{HA}^+$: graded, bicommutative Hopf algebras over $\mathbb{F}_p$ that are group-like in degree 0: section 8.

$\mathcal{HA}_\pm$: graded, skew-commutative Hopf algebra over $\mathbb{F}_p$ that are group-like in degree 0; for example, the homology of a double loop space: section 9.

$\mathcal{HV}$: graded, connected, torsion-free Hopf algebra over $\mathbb{Z}_p$ equipped with a lift of the Verschiebung: section 2.

$\mathcal{HV}^+$: graded, torsion-free Hopf algebra over $\mathbb{Z}_p$ equipped with a lift of the Verschiebungand group-like in degree 0: section 8.

$\mathcal{HF}$: graded, connected Hopf algebras equipped with a lift of the Frobenius: sections 1 and 3.

Categories of Modules

$\mathcal{D}$: Dieudonné modules for $\mathcal{HA}$: section 4.

$\mathcal{D}^+$: Dieudonné modules for $\mathcal{HA}^+$: section 8.

$\mathcal{D}_pm$: Dieudonné modules for $\mathcal{HA}_\pm$: section 9.

$\mathcal{M}_V$: Torsion-free graded $\mathbb{Z}_p$ modules with an endomorphism V; for example, QH, $H \in \mathcal{HV}$: section 2.

$\mathcal{M}_V^+$: the analog of M_V for $\mathcal{HV}^+$.

$\mathcal{D}_V$: similar to $\mathcal{M}_V$, dropping the torsion-free hypothesis: section 4.

Certain Hopf Algebras

$CW(k)$ and $CW_n(k)$: the torsion free Hopf algebras with Witt-vector diagonal: Definition 1.6.

$H(n)$: the projective generators of $\mathcal{HA}$: Lemma 4.1.

H_*BU: Examples 2.12 and 4.12

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