ENRICHED MODEL CATEGORIES IN EQUIVARIANT CONTEXTS

BERTRAND GUILLOU AND J.P. MAY

Abstract. We collect in one place a variety of results and examples concerning enriched model category theory in equivariant contexts.

Contents

We discuss the applicability of the general theory in [4] to equivariant contexts. We start in §1.1 with a well understood motivating example, the equivalence of model categories of G-spaces and model categories of presheaves of spaces defined on the orbit category of G. In the rest of §1, we ignore model categories and explain slowly and carefully the several contexts of double enrichment that arise naturally in equivariant homotopical algebra. For a $\mathscr{V}$-category $\mathscr{M}$, the fixed point and orbit objects of a G-object in $\mathscr{M}$ must lie in $\mathscr{M}$, and then the relevant presheaves should take values in $\mathscr{M}$ rather than in the enriching category $\mathscr{V}$, which is where we want them. The relevant categorical perspective is missing from the literature. It is elementary and perhaps tedious, but it is necessary to the applications.

We return to equivariant model categories in §2. Here the first questions to ask concern comparisons between $\mathscr{V}$-model categories of G-objects in $\mathscr{M}$ and $\mathscr{V}$-model categories of presheaves with values in $\mathscr{M}$. This is discussed in §2.1. One can then

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ask for comparisons between $\mathscr{V}$-model categories of G-objects in $\mathscr{M}$ and $\mathscr{V}$-model categories of presheaves with values in $\mathscr{V}$, which is what one really wants. This is discussed in §2.2. It turns out that these questions have immediate equivariant answers when there is a nonequivariant answer for $\mathscr{M}$, but there is an unsurprising surprise: the domain category for the presheaves in $\mathscr{V}$ needed to model $G\mathscr{M}$ is generally *not* a full subcategory of $G\mathscr{M}$.

We then describe two contexts to which the theory applies. In the first, we display equivalences between model categories of DG G-modules over DGAs and model categories of presheaves of chain complexes defined on appropriate algebraic orbit categories. In the second, we show that our theory has something new to say in the context of simplicial model categories, where we start with a simplicial group G and a simplicial model category $\mathscr{M}$ and display equivalences between model categories of G-objects in $\mathscr{M}$ and appropriate presheaf categories. The well-informed reader will immediately notice that other examples in algebraic geometry and category theory will work in much the same way as these do.

We are in large part motivated by questions about G-spectra, to which we turn in §3. The context is different from §1 and §2 in that G-spectra are not just G-objects in a category of spectra: those are called *naive* G-spectra, and for them the results of those sections apply as they stand. However, the discussion of stable model categories in [4] applies directly to *genuine* G-spectra, as we explain in §3. The work in §3 applies to all compact Lie groups, and some of our comparison results in [4] will come into play. Starting from §3, much sharper results for finite groups G are given in the sequel [5], where other results from [4] are applied.

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1. Equivariant enriched categories

1.1. **Enriched model categories of G-spaces.** We describe the motivating example [2, 15, 18] for a general theory relating equivariant categories to presheaf categories. Since the example is specified topologically, we use topological enrichments.

Let G be a topological group and let $\mathscr{F}$ be a (nonempty) family of closed subgroups, so that subconjugates of groups in $\mathscr{F}$ are in $\mathscr{F}$. With the notations of the general theory in [4], take $\mathscr{V}$ to be the cartesian monoidal category $\mathscr{U}$ of compactly generated spaces with its standard Quillen model structure. The generating cofibrations $\mathcal{I}$ are the cells $S^{n-1} \longrightarrow D^n$ and the generating acyclic cofibrations $\mathcal{J}$ are the inclusions $i_0 \colon D^n \longrightarrow D^n \times I$, where $n \geq 0$. Take $\mathscr{M}$ to be the $\mathscr{U}$-category $G\mathscr{U}$ of G-spaces and G-maps.[1] For G-spaces X and Y, we write $G\mathscr{U}(X,Y)$ for the space of G-maps $X \longrightarrow Y$. These spaces are the hom objects that give $G\mathscr{U}$ its enrichment in $\mathscr{U}$.

We may view $G\mathscr{U}$ as a closed cartesian monoidal category, with G acting diagonally on cartesian products. Its objects are the G-spaces. For G-spaces X and Y, let $\mathscr{U}_G(X,Y)$ be the G-space of all maps $X \longrightarrow Y$, with G acting by conjugation. These G-spaces specify the internal hom that gives $G\mathscr{U}$ its closed structure. We have a natural identification

$$G\mathscr{U}(X,Y) = \mathscr{U}_G(X,Y)^G.$$

[1] We prefer not to use the underline notation of [4] for enriched hom objects here.

As a closed symmetric monoidal category, $G\mathscr{U}$ is enriched over itself. Since Y^G can be identified with $G\mathscr{U}(*,Y)$, we have the expected identification

$$G\mathscr{U}(X,Y) = G\mathscr{U}(*,\mathscr{U}_G(X,Y)).$$

We emphasize that although $\mathscr{U}_G(X,Y)$ is defined using all maps $X \longrightarrow Y$, it is of course only functorial with respect to G-maps $X \longrightarrow X'$ and $Y \longrightarrow Y'$.

We regard $G\mathscr{U}(X,Y)$ as the space rather than just the set of G-maps $X \longrightarrow Y$, and we regard $\mathscr{U}_G(X,Y)$ as the G-space rather than just the G-set of maps $X \longrightarrow Y$. When we enrich a category in spaces or G-spaces, as here, it seems reasonable to use the same notation for sets and spaces of maps, since the latter are just given by a topology on the given sets.[2] Formally, we have a double enrichment of the underlying category $G\mathscr{U}$ with sets of morphisms: it is enriched in spaces, but as a closed symmetric monoidal category it is also enriched over itself, that is, it is enriched in G-spaces. We shall be more categorically precise when we generalize in §1.3.

Take $\mathscr{D}$ to be $\mathscr{O}_{\mathscr{F}}$, the full $\mathscr{U}$-subcategory of $G\mathscr{U}$ whose objects are the orbit G-spaces G/H with $H \in \mathscr{F}$. The most important example is $\mathscr{F} = \mathscr{A}\ell\ell$, the set of all subgroups of G, and we write $\mathscr{O}_G$ for the orbit category. We have the adjunction

$$\mathbf{Pre}(\mathscr{O}_{\mathscr{F}},\mathscr{V}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{U}. \tag{1.1}$$

For a G-space Y, $G\mathscr{U}(G/H,Y)$ can be identified with the fixed point space Y^H, so that $\mathbb{U}(Y)$ is the presheaf of fixed point spaces Y^H for $H \in \mathscr{F}$. As is easily checked, the left adjoint $\mathbb{T}$ takes a presheaf X to the G-space $X_{G/e}$. Therefore it takes the represented presheaf $\mathbb{Y}(G/H)$ to the G-space G/H. Clearly $\varepsilon\colon \mathbb{T}\mathbb{U} \longrightarrow Id$ is the identity functor, hence so is $\eta\mathbb{U}\colon \mathbb{U} \longrightarrow \mathbb{U}\mathbb{T}\mathbb{U}$. Moreover, $\mathbb{U}$ is full and faithful.

Define the $\mathscr{F}$-equivalences in $G\mathscr{U}$ to be the G-maps f such that the fixed point map f^H is a weak equivalence for $H \in \mathscr{F}$. These are the weak G-equivalences when $\mathscr{F} = \mathscr{A}\ell\ell$, and the G-Whitehead theorem says that a weak G-equivalence between G-CW complexes is a G-homotopy equivalence. When $\mathscr{F} = \{e\}$, a weak $\mathscr{F}$-equivalence is just a G-map which is a nonequivariant weak equivalence, giving a naive version of equivariant homotopy theory. Similarly, define the $\mathscr{F}$-fibrations to be the G-maps f such that f^H is a (Serre) fibration for $H \in \mathscr{F}$. We consider the following three theorems, proven in [13, 15, 18], from the point of view of [4, Questions 0.1 and 0.2]. Compactly generated model categories are discussed in [4, 17].

Theorem 1.2. *$G\mathscr{U}$ is a compactly generated proper $\mathscr{U}$-model category with respect to the $\mathscr{F}$-equivalences, $\mathscr{F}$-fibrations, and the resulting cofibrations. The sets of maps $\mathcal{I}_{\mathscr{F}} = \{G/H \times i\}$ and $\mathcal{J}_{\mathscr{F}} = \{G/H \times j\}$, where $H \in \mathscr{F}$, $i \in \mathcal{I}$, and $j \in \mathcal{J}$, are generating sets of cofibrations and acyclic cofibrations.*

For $H \in \mathscr{F}$, we have the functor $F_{G/H}\colon G\mathscr{U} \longrightarrow \mathbf{Pre}(\mathscr{O}_{\mathscr{F}},\mathscr{U})$ from [4, 1.5]; it is given by $(F_{G/H}Y)(G/K) = Y \times \mathscr{O}_{\mathscr{F}}(G/K,G/H)$ for a G-space Y. When G acts trivially on Y, this is

$$(Y \times G/H)^K \cong \mathbb{U}(G/H \times Y)^K \cong \mathbb{U}(G/H \times Y)(G/K).$$

Therefore

$$F_{G/H}Y \cong \mathbb{U}(G/H \times Y).$$

[2]This is the principle reason for not using underlines in this context.

We apply this identification with Y replaced by the maps in $\mathcal{I}_{\mathscr{F}}$ and $\mathcal{J}_{\mathscr{F}}$.

Theorem 1.3. $\mathbf{Pre}(\mathscr{O}_{\mathscr{F}}, \mathscr{U})$ *is a compactly generated proper* $\mathscr{V}$*-model category with respect to the level* $\mathscr{F}$*-equivalences, level* $\mathscr{F}$*-fibrations, and the resulting cofibrations. The sets of maps* $F_{G/H}i$ *and* $F_{G/H}j$*, where* $H \in \mathscr{F}$*,* $i \in \mathcal{I}$*, and* $j \in \mathcal{J}$*, generate the cofibrations and acyclic cofibrations, and these sets are isomorphic to* $\mathbb{U}\mathcal{I}_{\mathscr{F}}$ *and* $\mathbb{U}\mathcal{J}_{\mathscr{F}}$*.*

Theorem 1.4. $(\mathbb{T}, \mathbb{U})$ *is a Quillen* $\mathscr{U}$*-equivalence between* $G\mathscr{U}$ *and* $\mathbf{Pre}(\mathscr{O}_{\mathscr{F}}, \mathscr{U})$*.*

Theorem 1.3 holds by [4, 4.31]. The only point requiring verification is that, in the language of [4, 4.13], $\mathcal{J}_{\mathscr{F}}$ satisfies the acyclicity condition for the level $\mathscr{F}$-equivalences. This means that any relative cell complex $A \longrightarrow X$ constructed from $\mathbb{U}\mathcal{J}_{\mathscr{F}}$ is a level $\mathscr{F}$-equivalence.

We can now view Theorems 1.2 and 1.3 as answers to [4, Question 0.1], starting from the inclusion $\delta\colon \mathscr{O}_{\mathscr{F}} \longrightarrow G\mathscr{U}$. By definition, $\mathbb{U}$ creates the $\mathscr{F}$-equivalences and $\mathscr{F}$-fibrations in $G\mathscr{U}$, as in [4, 1.5], and [4, 1.16] applies. In this example, its acyclicity condition means that any relative $\mathcal{J}_{\mathscr{F}}$-cell complex is an $\mathscr{F}$-equivalence.

As observed in [13, p. 40], $\mathbb{U}$ carries $\mathcal{I}_{\mathscr{F}}$-cell complexes and $\mathcal{J}_{\mathscr{F}}$-cell complexes in $G\mathscr{U}$ bijectively to $\mathbb{U}\mathcal{I}_{\mathscr{F}}$-cell complexes and $\mathbb{U}\mathcal{J}_{\mathscr{F}}$ cell complexes in $\mathbf{Pre}(\mathscr{O}_{\mathscr{F}}, \mathscr{U})$ since $\mathbb{U}$ preserves the relevant colimits and is full and faithful. Therefore the acyclicity conditions needed to prove Theorems 1.2 and 1.3 are identical. Since the maps in $\mathcal{J}$ are inclusions of deformation retracts, the maps in $\mathcal{J}_{\mathscr{F}}$ are inclusions of G-deformation retracts. By passage to coproducts, pushouts, and sequential colimits, all relative $\mathcal{J}_{\mathscr{F}}$-cell complexes $A \longrightarrow X$ are inclusions of G-deformation retracts.

To prove Theorem 1.4, we observe that, for a G-space Y and a space V, the maps η of [4, 1.11] are the evident homeomorphisms $Y^H \times V \cong (Y \times V)^H$. This implies that $\eta\colon X \longrightarrow \mathbb{U}\mathbb{T}X$ is an isomorphism when $X \in \mathcal{I}_{\mathscr{F}}$. Again using that $\mathbb{U}$ preserves the relevant colimits, it follows (as in [4, 1.19]) that $\eta\colon X \longrightarrow \mathbb{U}\mathbb{T}X$ is an isomorphism in $\mathbf{Pre}(\mathscr{O}_{\mathscr{F}}, \mathscr{U})$ for all cofibrant X.

Alternatively, of course, we can first prove Theorem 1.2 directly, and then Theorems 1.3 and 1.4 answer [4, Question 0.2] about the $\mathscr{U}$-model category $G\mathscr{U}$.

Using smash products instead of cartesian products and giving orbit G-spaces disjoint basepoints, everything above works just as well using the categories $\mathscr{T}$ and $G\mathscr{T}$ of based spaces and based G-spaces instead of $\mathscr{U}$ and $G\mathscr{U}$. We can combine the results of this section with [4, 3.6] to show that good symmetric monoidal categories of G-spectra are enriched over $\mathscr{T}$, but we are more interested in the enrichment over a well-chosen category $\mathscr{S}$ of spectra. We shall return to G-spectra in §3. We consider equivariant categories in general in the rest of this section.

1.2. **Enriched categories of G-objects for a discrete group G.** There are several generalized versions of the results of §1.1 in different contexts. To be precise about this and to try to anticipate sources of confusion, we start at a very elementary categorical level. We ignore model categorical structures and presheaves in this and the following two sections, where we establish two variant equivariant contexts for applications of our general theory.

In this section, we take G to be an ordinary (discrete) group, given independently of any enriched context. Just as we have the notion of a category, we have the notion of a G-category, by which we understand a category enriched in the cartesian monoidal category of G-sets. Thus a G-category $\mathscr{G}$ has for each pair of objects a G-set $\mathscr{G}(X, Y)$ of morphisms and an equivariant composition, where G acts diagonally on products of G-sets; the identity morphisms in $\mathscr{G}$ are given by G-maps $* \longrightarrow \mathscr{G}(X,X)$ and so must be G-fixed. For a G-set Z the set of maps of G-sets $* \longrightarrow Z$ is the fixed point set Z^G. Therefore the underlying category of $\mathscr{G}$ has morphism sets

$$G\text{-Set}(*,\mathscr{G}(X,Y)) = \mathscr{G}(X,Y)^G.$$

Note that we have not yet mentioned possible G-actions on the objects of $\mathscr{G}$.

Now let $\mathscr{C}$ be a category. A G-object (X,α) in $\mathscr{C}$ is an object X of $\mathscr{C}$ together with a group homomorphism α from G to to the group of automorphisms of X in $\mathscr{C}$. Again, this notion is independent of any enrichment in sight. We let $G\mathscr{C}$ denote the category of G-objects and G-maps of G-objects in $\mathscr{C}$. It is bicomplete if $\mathscr{C}$ is. We also have a G-category $\mathscr{C}_G$ whose objects are the G-objects. Its G-set of morphisms $\mathscr{C}_G((X,\alpha),(Y,\beta))$ is just the set $\mathscr{C}(X,Y)$ together with the conjugation action γ of G; that is, $\gamma(g)(f) = \beta(g)\circ f \circ \alpha(g)^{-1}$. The category $\mathscr{C}_G$ is not bicomplete when $\mathscr{C}$ is, and it plays a minor auxiliary role. Since it is apparent that

$$G\mathscr{C}(X,Y) = \mathscr{C}_G(X,Y)^G,$$

we may view $G\mathscr{C}$ as the G-fixed category of $\mathscr{C}_G$ or, equivalently, as the underlying category of $\mathscr{C}_G$ in the sense of enriched category theory, as described above.

We have functors

$$\varepsilon^*\colon \mathscr{C} \to G\mathscr{C} \quad \text{and} \quad \varepsilon_*\colon G\mathscr{C} \longrightarrow \mathscr{C}.$$

The first sends an object in $\mathscr{C}$ to the same object with trivial G-action; the second forgets the G-action. Assuming that $\mathscr{C}$ is bicomplete, ε^* is left adjoint to the G-fixed point functor $(-)^G\colon G\mathscr{C} \to \mathscr{C}$, which is constructed using equalizers, and is right adjoint to the orbit functor $(-)/G\colon G\mathscr{C} \to \mathscr{C}$, which is constructed using coequalizers. To clarify a potential point of confusion later, we give a pedantically precise way of thinking about the adjunction $(\varepsilon^*, (-)^G)$.

Remark 1.5. Let X be an object of $\mathscr{C}$. The functor $\mathscr{C}(X,-)$ from $\mathscr{C}$ to sets restricts on G-objects and G-maps to a functor $\mathscr{C}(X,-)$ from $G\mathscr{C}$ to G-sets. It commutes with fixed points; for a G-object Y, $\mathscr{C}(X,Y^G)$ is isomorphic to $\mathscr{C}(X,Y)^G$, naturally on $G\mathscr{C}$. Tautologically, $\mathscr{C}(X,Y)$ is exactly the same G-set as $\mathscr{C}_G(\varepsilon^*X,Y)$: since G acts trivially on X, both are the set $\mathscr{C}(X,Y)$ with G-action induced by the action of G on Y. Therefore, passage to G-fixed points gives

$$G\mathscr{C}(\varepsilon^*X,Y) = \mathscr{C}_G(\varepsilon^*X,Y)^G = \mathscr{C}(X,Y)^G \cong \mathscr{C}(X,Y^G).$$

Remark 1.6. Any cocomplete category $\mathscr{C}$ is tensored over the category of sets. The tensor $S \odot X$ of a set S and an object X of $\mathscr{C}$ is the coproduct of copies of X indexed by the elements of S. An action of G on X can be defined equivalently in terms of actions $G \odot X \longrightarrow X$ with the evident properties.

Now take $\mathscr{C}$ to be a symmetric monoidal category and rename it $\mathscr{V} = (\mathscr{V}, \otimes, \mathbf{I})$. As above, we have the G-category $\mathscr{V}_G$ and its G-fixed category $G\mathscr{V}$ of G-objects in $\mathscr{V}$. For G-objects (X,α) and (Y,β) in $\mathscr{V}$, $X\otimes Y$ is a G-object with action given by $g \mapsto \alpha(g)\otimes\beta(g)$. Since $\otimes$ is only functorial on G-maps, this does not give $\mathscr{V}_G$ a monoidal structure, but $(G\mathscr{V}, \otimes, \varepsilon^*\mathbf{I})$ is a symmetric monoidal category under this product. The functors ε^* and ε_* are strong symmetric monoidal.

Now add the assumption that $\mathscr{V}$ is closed with internal hom objects $\underline{\mathscr{V}}(X,Y)$. Of course, the defining adjunction

$$\mathscr{V}(X\otimes Y, Z) \cong \mathscr{V}(X, \underline{\mathscr{V}}(Y,Z))$$

specializes to give

$$\mathscr{V}(X,Y) \cong \mathscr{V}(\mathbf{I}, \underline{\mathscr{V}}(X,Y)),$$

showing that $\mathscr{V}$ is enriched over itself.

For G-objects (X,α) and (Y,β) in $\mathscr{V}$, $\underline{\mathscr{V}}(X,Y)$ is a G-object in $\mathscr{V}$ with conjugation action γ given by $\gamma(g) = \underline{\mathscr{V}}(\alpha(g), \beta(g)^{-1})$. We denote this G-object by $\underline{\mathscr{V}}_G(X,Y)$, and we always remember that this is just notation for the pair $(\underline{\mathscr{V}}(X,Y), \gamma)$. When X, Y, and Z have specified actions by G, so do all objects in the previous two displays. Inspection of definitions shows that the displayed isomorphisms are then isomorphisms of G-objects. Therefore

$$\mathscr{V}_G(X \otimes Y, Z) \cong \mathscr{V}_G(X, \underline{\mathscr{V}}_G(Y,Z))$$

and

$$\mathscr{V}_G(X,Y) \cong \mathscr{V}_G(\varepsilon^*\mathbf{I}, \underline{\mathscr{V}}_G(X,Y)).$$

These isomorphisms are only natural with respect to G-maps. We reiterate that the subscript G just says "remember the given action of G".

Passing to G-fixed points from these isomorphisms of G-sets, we find

$$G\mathscr{V}(X \otimes Y, Z) \cong G\mathscr{V}(X, \underline{\mathscr{V}}_G(Y,Z))$$

and

$$G\mathscr{V}(X,Y) \cong G\mathscr{V}(\varepsilon^*\mathbf{I}, \underline{\mathscr{V}}_G(X,Y)) \cong \mathscr{V}(\mathbf{I}, \underline{\mathscr{V}}_G(X,Y)^G).$$

These isomorphisms show that $G\mathscr{V}$ is a closed symmetric monoidal category with internal hom $\underline{\mathscr{V}}_G(X,Y)$ and is therefore enriched over itself. The last isomorphism shows that $G\mathscr{V}$ is also enriched over $\mathscr{V}$ with hom objects the $\underline{\mathscr{V}}_G(X,Y)^G$. We agree to use the alternative notations

$$\underline{G\mathscr{V}}(X,Y) = \underline{\mathscr{V}}_G(X,Y)^G$$

interchangeably for the enrichment in $\mathscr{V}$. For example, when $\mathscr{V} = \mathscr{U}$, this is just the space of G-maps $X \longrightarrow Y$. The enrichment in $G\mathscr{V}$ is just the automatic enrichment from the closed symmetric monoidal structure given by the $\underline{\mathscr{V}}_G(X,Y)$, and that is the only notation we shall use for it. Our primary interest is always the enrichment in the nonequivariant category $\mathscr{V}$ we start with.

Since $G\mathscr{V}$ is the G-fixed category of $\mathscr{V}_G$, we also have the isomorphism of sets

$$G\mathscr{V}(X,Y) \cong G\text{-Set}(*, \mathscr{V}_G(\varepsilon^*\mathbf{I}, \underline{\mathscr{V}}_G(X,Y))).$$

Thinking of $\mathscr{V}_G$ as enriched in G-sets, which is not formally "correct" since $\underline{\mathscr{V}}_G(X,Y)$ is only functorial on G-maps, we can think of this isomorphism as exhibiting a kind of composite of "underlying category" constructions.

Unlike $\mathscr{V}_G$, $G\mathscr{V}$ is bicomplete and is thus a good category in which to enrich other categories. In serious equivariant work, it is very often convenient to work in terms of the enriched hom objects $\underline{\mathscr{V}}_G(X,Y)$ in $G\mathscr{V}$ as long as possible, even though the main interest is in the enriched hom objects $\underline{\mathscr{V}}(X,Y)^G$ in $\mathscr{V}$. This is especially true in this paper since it is the enriched hom objects in $\mathscr{V}$ that are the focus of attention in the presheaf categories that we are interested in. We shall say more clearly how $\mathscr{V}_G$ fits into the picture in Remark 1.9

In line with this focus, it is essential to expand on Remark 1.6; see also [4, 4.35].

Definition 1.7. Let $\mathbf{I}[-]$ be the functor from sets to $\mathscr{V}$ that sends S to $S \odot \mathbf{I}$, which is the coproduct of copies of $\mathbf{I}$ indexed on the elements of S; it sends a function $f\colon S \longrightarrow T$ to the map that sends the s^{th} copy of $\mathbf{I}$ by the identity map to the $f(s)^{\text{th}}$ copy of $\mathbf{I}$. Regarding the category of sets as cartesian monoidal, the functor $\mathbf{I}[-]$ is strong monoidal via the evident isomorphisms $\mathbf{I}[*] \cong \mathbf{I}$ and $\mathbf{I}[S] \otimes \mathbf{I}[T] \cong \mathbf{I}[S \times T]$. Therefore $\mathbf{I}[G]$ is a "Hopf algebra in $\mathscr{V}$". That is, using the product $\otimes$ in $\mathscr{V}$, $\mathbf{I}[G]$ has a multiplication, unit, and inverse map, denoted ϕ, η, and χ, and it also has a diagonal map Δ and an augmentation ε induced from the diagonal map on G and the projection $G \longrightarrow *$. These maps fit into the usual commutative diagrams that define the notion of a Hopf algebra. We usually ignore Δ and ε in this paper.

This construction is the generalization to arbitrary symmetric monoidal categories $\mathscr{V}$ of the group ring construction, $G \mapsto R[G]$, from groups to Hopf algebras, to which it specializes when $(\mathscr{V}, \otimes, R)$ is the symmetric monoidal category of modules over a commutative ring R. It is important to notice right away that if S is a G-set, then $\mathbf{I}[S^G]$ is in general quite different from $\mathbf{I}[S]^G$, to which it maps.

Finally, let $\mathscr{M}$ be a bicomplete $\mathscr{V}$-category with hom objects $\underline{\mathscr{M}}(M, N)$ in $\mathscr{V}$. Ignoring the enrichment, we have the G-category $\mathscr{M}_G$ and its G-fixed category $G\mathscr{M}$ and we have functors ε^* and ε_* relating $\mathscr{M}$ to $G\mathscr{M}$. The category $G\mathscr{M}$ is bicomplete since $\mathscr{M}$ is bicomplete. Its limits and colimits are created in $\mathscr{M}$.

For G-objects M and N in $\mathscr{M}$, G acts by conjugation on $\underline{\mathscr{M}}(M, N)$, and we write $\underline{\mathscr{M}}_G(M, N)$ for the resulting G-object in $\mathscr{V}$. The category $G\mathscr{M}$ is enriched over $G\mathscr{V}$ with hom objects the $\underline{\mathscr{M}}_G(M, N)$, and it is also enriched over $\mathscr{V}$ with hom objects the $\underline{\mathscr{M}}_G(M, N)^G$. Indeed, generalizing the isomorphisms above for $\mathscr{V}$, we have

$$G\mathscr{M}(M, N) \cong G\mathscr{V}(\varepsilon^*\mathbf{I}, \underline{\mathscr{M}}_G(M, N)) \cong \mathscr{V}(\mathbf{I}, \underline{\mathscr{M}}_G(M, N)^G). \tag{1.8}$$

As for $\mathscr{V}$, we agree to use the notations

$$\underline{G\mathscr{M}}(M, N) = \underline{\mathscr{M}}_G(M, N)^G$$

interchangeably for the enriched hom objects in $\mathscr{V}$.

Remark 1.9. Here is how to think about the G-category $\mathscr{M}_G$ in terms of enrichment. It is not essential to do so, but the question is a standard source of confusion. For $M, N \in \mathscr{M}$, we of course have

$$\mathscr{M}(M, N) \cong \mathscr{V}(\mathbf{I}, \underline{\mathscr{M}}(M, N)).$$

If M and N have given G actions, the conjugation action by G on both sides gives the isomorphism in

$$\mathscr{M}_G(M, N) \cong \mathscr{V}_G(\varepsilon^*\mathbf{I}, \underline{\mathscr{M}}_G(M, N)) = \mathscr{V}(\mathbf{I}, \underline{\mathscr{M}}_G(M, N)). \tag{1.10}$$

The isomorphism is natural with respect to maps in $G\mathscr{M}$ but not $\mathscr{M}_G$. The equality is explained in Remark 1.5, with $\mathscr{C} = \mathscr{V}$ and $X = \mathbf{I}$, and that remark also makes clear that (1.10) becomes (1.8) on passage to G-fixed points.

We conclude that even though $\mathscr{V}_G$ is not monoidal under $\otimes$, it is entirely reasonable to pretend that $\mathscr{M}_G$ is the underlying category of a $\mathscr{V}_G$-category with hom objects $\underline{\mathscr{M}}_G(M, N)$ in $\mathscr{V}_G$. Categorically, we can define a new notion of a $\mathscr{V}_G$-category $\mathscr{M}_G$, meaning precisely the constellation of data that we see in this remark: conceptually, after all, categorical language should describe the phenomena we encounter, and if it does not it should be rebuilt to do so. Since the idea should be clear, we desist. With this in mind, we shall feel free to refer to $\mathscr{V}_G$-categories in the sequel [5].

For G-objects $V \in \mathscr{V}$ and $M \in \mathscr{M}$, the tensor $V \odot M$ and cotensor $F(V, M)$ in $\mathscr{M}$ inherit G-actions from those of V and M, the latter by conjugation. These tensors and cotensors make $G\mathscr{M}$ a bicomplete $G\mathscr{V}$-category. Restricting them to objects ε^*V, they make $G\mathscr{M}$ a bicomplete $\mathscr{V}$-category. A comparison of group actions show that we have isomorphisms

$$\underline{\mathscr{M}}_G(M \odot V, N) \cong \underline{\mathscr{V}}_G(V, \underline{\mathscr{M}}_G(M,N)) \cong \underline{\mathscr{M}}_G(M, F(V,N)) \tag{1.11}$$

in $G\mathscr{V}$. Applying the G-fixed point functor, we obtain the bitensor adjunctions

$$\underline{G\mathscr{M}}(M \odot V, N) \cong \underline{G\mathscr{V}}(V, \underline{\mathscr{M}}_G(M,N)) \cong \underline{G\mathscr{M}}(M, F(V,N)) \tag{1.12}$$

in $\mathscr{V}$. When G acts trivially on V, this specializes to give the tensors and cotensors of the bicomplete $\mathscr{V}$-category $G\mathscr{M}$. Again, categorically, we could define new notions of tensored and cotensored $\mathscr{V}_G$-categories $\mathscr{M}_G$ to capture the data we see here.

1.3. **Enriched categories of G-objects for a $\mathscr{V}$-group G.** There is an important variant of this discussion when $\mathscr{V}$ is a closed cartesian monoidal category, such as the category of spaces or simplicial sets. We can now start, not with a discrete group, but rather with a group object G in $\mathscr{V}$, which we shall call a $\mathscr{V}$-group. Thus G is an object of $\mathscr{V}$ equipped with maps (ϕ, η, χ); together with the maps (Δ, ε) implicit in the cartesian product, the structure is exactly like that in Definition 1.7. The evident diagrams commute. For uniformity of notation when we treat our two contexts together, we shall use the alternative notation $G = \mathbf{I}[G]$. This is reasonable since the unit object $\mathbf{I}$ is a zero object (initial and terminal), so that G and $\mathbf{I}[G]$ are isomorphic objects of $\mathscr{V}$; we could write $\mathbf{I} = *$ for emphasis, but we continue to write $\mathbf{I}$ for uniformity of notation.

We have an underlying (discrete) group G^δ specified by $G^\delta = \mathscr{V}(\mathbf{I}, G)$, with the group structure induced from that of G. To define the product, we use that the functor $\mathscr{V}(\mathbf{I}, -)$ from $\mathscr{V}$ to sets is symmetric monoidal. The relevance of G^δ should be clear since the hom sets of the underlying category of a $\mathscr{V}$-category $\mathscr{M}$ are defined by $\mathscr{M}(M,N) = \mathscr{V}(\mathbf{I}, \underline{M}(M,N))$.

For example, this construction codifies the difference between topological groups G and their underlying discrete groups G^δ. If $\mathscr{V}$ is the category of simplicial sets, a $\mathscr{V}$-group is a simplicial group, and then G^δ is just its group of vertices. In later sections, the group G^δ is of negligible importance and in fact it will never again be mentioned. While the previous section applies to G^δ, that is not a situation of particular interest and the category $G^\delta\mathscr{V}$ is not relevant to us. Our underlying category $G\mathscr{V}$ is the category of G-objects in $\mathscr{V}$ and G-maps between them.

The reader may want to assume that $\mathscr{V}$ is concrete, meaning that it has a faithful underlying set functor. That allows subgroups and conjugate groups in G to be interpreted concretely. Otherwise, to be precise, we should understand a subgroup to be an isomorphism class of monomorphisms $\iota \colon H \longrightarrow G$ of groups in $\mathscr{V}$. The assumption is not necessary, but the added categorical care would be digressive.

From here, the categorical background we need is similar to that of the previous section, provided that we think there in terms of the "group ring" $\mathbf{I}[G]$ in $\mathscr{V}$ rather than the originally given discrete group G.

If V and W are G-objects in $\mathscr{V}$, then $V \times W$ and $\underline{\mathscr{V}}(V,W)$ inherit group actions. We shall make the conjugation action precise in a moment. We write $\underline{\mathscr{V}}_G(V,W)$ for $\underline{\mathscr{V}}(V,W)$ with this group action. With these definitions, $G\mathscr{V}$ is a closed cartesian monoidal category. It is enriched over $\mathscr{V}$ with hom objects in $\mathscr{V}$ written in two ways:

$$\underline{G\mathscr{V}}(V,W) = \underline{\mathscr{V}}_G(V,W)^G.$$

Now let $\mathscr{M}$ be a bicomplete $\mathscr{V}$-category with hom objects $\underline{\mathscr{M}}(M,N)$ in $\mathscr{V}$. An action of G on an object $M \in \mathscr{M}$ is a homomorphism $\alpha\colon G \longrightarrow \underline{\mathscr{M}}(M,M)$ of monoids in $\mathscr{V}$. Equivalently, by adjunction, α can be viewed as an action map

$$\alpha\colon G \odot M \longrightarrow M$$

in $\mathscr{M}$ such that the usual unit and transitivity diagrams commute. We have the category $G\mathscr{M}$ of G-objects and G-maps in $\mathscr{M}$.

When M and N are **G**-objects in $\mathscr{M}$, $\underline{M}(M,N)$ has a conjugation action by G. We denote the resulting G-object in $\mathscr{V}$ by $\underline{\mathscr{M}}_G(M,N)$. Explicitly, the conjugation action is given by the composite

$$\begin{tikzcd}
G \times \underline{\mathscr{M}}(M,N) \arrow[d, "((\mathrm{id}\times\chi)\circ\Delta)\times\mathrm{id}"] \\
G \times G \times \underline{\mathscr{M}}(M,N) \arrow[d, "\mathrm{id}\times\tau"] \\
G \times \underline{\mathscr{M}}(M,N) \times G \arrow[d, "\alpha\times\mathrm{id}\times\alpha"] \\
\underline{\mathscr{M}}(M,M) \times \underline{\mathscr{M}}(M,N) \times \underline{\mathscr{M}}(M,M) \arrow[d, "\circ"] \\
\underline{\mathscr{M}}(M,N)
\end{tikzcd}$$

in our cartesian model category $G\mathscr{V}$, where τ denotes the transposition.

This gives an enrichment of $G\mathscr{M}$ in $G\mathscr{V}$, but we are more interested in its enrichment in $\mathscr{V}$, which is given by the fixed point hom objects

$$\underline{G\mathscr{M}}(M,N) = \underline{\mathscr{M}}_G(M,N)^G.$$

We again have the double enrichment isomorphisms

$$G\mathscr{M}(X,Y) \cong G\mathscr{V}(\varepsilon^*\mathbf{I}, \underline{\mathscr{M}}_G(X,Y)) \cong \mathscr{V}(\mathbf{I}, \underline{\mathscr{M}}_G(X,Y)^G).$$

Remark 1.9 applies verbatim to explain how $\mathscr{M}_G$ fits into the context of enrichment.

As in the previous section, for G-objects V in $\mathscr{V}$ and M in $\mathscr{M}$, the tensors and cotensors $M \odot V$ and $F(V,M)$ in $\mathscr{M}$ give tensors and cotensors in $G\mathscr{M}$ when endowed with the actions of G induced by the given actions on V and M; the conjugation action on $F(V,M)$ is defined analogously to the conjugation action on $\underline{\mathscr{M}}_G(M,N)$. We again have isomorphisms (1.11) and $\mathscr{V}$-adjunctions (1.12).

1.4. **Orbit tensors and fixed point cotensors.** Let $\mathscr{V}$ be a bicomplete closed symmetric monoidal category and let $\mathscr{M}$ be a bicomplete $\mathscr{V}$-category. Let G be a group or, when $\mathscr{V}$ is cartesian monoidal, a $\mathscr{V}$-group. We have the "Hopf ring" $\mathbf{I}[G]$ in $\mathscr{V}$ when G is a group, and we write $G = \mathbf{I}[G]$ when G is a $\mathscr{V}$-group.

The essential starting point for enriched equivariant homotopy theory is an understanding of the fixed point objects M^H and orbit objects M/H in $\mathscr{M}$ for objects

$M \in G\mathscr{M}$ and subgroups H of G.[3] We also need induction and coinduction functors $H\mathscr{M} \longrightarrow G\mathscr{M}$. If we view G as a $\mathscr{V}$-category with a single object, these can be specified as suitable limits and colimits (weighted when G is a group object in $\mathscr{V}$) defined on the subcategory H of G, but we want a better enriched categorical perspective. Note in particular that $G\mathscr{M}$ will generally not contain "orbit objects G/H".

We get around this by constructing "orbit tensors" $V \odot_H M$ for left H-objects $M \in \mathscr{M}$ and right H-objects $V \in \mathscr{V}$ and "fixed point cotensors" $F_H(V, N)$ for left H-objects $N \in \mathscr{M}$ and $V \in \mathscr{V}$. These are objects of $\mathscr{M}$, and they specialize to give change of group functors that are entirely analogous to those in familiar examples.

We expand on Definition 1.7. When G is discrete and S is a G-set, we have an object $\mathbf{I}[S]$ in $G\mathscr{V}$, constructed as in Definition 1.7 and given the G-action that permutes coproduct summands as G permutes elements of S. When $\mathscr{V}$ is cartesian monoidal, G is a $\mathscr{V}$-group, and S is in $G\mathscr{V}$, we agree to interpret $\mathbf{I}[S]$ to be S itself. In both contexts, a left action of H on M can be viewed as a map $\mathbf{I}[H] \odot M \longrightarrow M$ in $\mathscr{M}$.

Definition 1.13. Let V be a right H-object in V and M be a left H-object in $\mathscr{M}$. Using [4, 4.7] implicitly, define $V \odot_H M$ in $\mathscr{M}$ to be the coequalizer

$$V \odot \mathbf{I}[H] \otimes M \rightrightarrows V \odot M \longrightarrow V \odot_H M.$$

Dually, for left H-objects V in $\mathscr{V}$ and N in $\mathscr{M}$, define $F_H(V, N)$ in $\mathscr{M}$ to be the equalizer

$$F_H(V, N) \longrightarrow F(V, N) \rightrightarrows F(\mathbf{I}[H] \otimes V, N).$$

One of each of the parallel pairs of arrows is induced by the action of H on V and the other is induced by the action of H on M or on N.

Observe that the left G-object $\mathbf{I}[G/H]$ in $\mathscr{V}$ can be identified with $\mathbf{I}[G] \otimes_H \mathbf{I}$, where $\mathbf{I}$ is viewed as a trivial left H-object in $\mathscr{V}$. More generally, for a left H-object M in $\mathscr{M}$, we can use the left action of G on $\mathbf{I}[G]$ to give $\mathbf{I}[G] \odot_H M$ a left G-action. Similarly, we can give $F_H(\mathbf{I}[G], N)$ the left action induced by the right action of G on $\mathbf{I}[G]$. The inclusion $\iota\colon H \longrightarrow G$ induces a forgetful $\mathscr{V}$-functor $\iota^*\colon G\mathscr{M} \longrightarrow H\mathscr{M}$. The following result identifies $\mathbf{I}[G] \odot_H (-)$ and $F_H(\mathbf{I}[G], -)$ as the enriched left and right adjoints, called induction and coinduction, of the functor ι^*.

Lemma 1.14. *There are $\mathscr{V}$-adjunctions*

$$\underline{G\mathscr{M}}(\mathbf{I}[G] \odot_H N, M) \cong \underline{H\mathscr{M}}(N, \iota^* M)$$

and

$$\underline{H\mathscr{M}}(\iota^* M, N) \cong \underline{G\mathscr{M}}(M, F_H(\mathbf{I}[G], N)),$$

where $M \in G\mathscr{M}$ and $N \in H\mathscr{M}$.

For $N \in H\mathscr{M}$, such as $N = \iota^* M$, we define the orbit objects N/H and fixed point objects N^H in $\mathscr{M}$ to be

$$N/H = \mathbf{I} \odot_H N \quad \text{and} \quad N^H = F_H(\mathbf{I}, N), \tag{1.15}$$

[3]In [2], the authors start with a simplicially enriched category $\mathscr{N}$ and a set $\mathscr{O}$ of objects, which they call 'orbits', in $\mathscr{N}$. For $O \in \mathscr{O}$ and $N \in \mathscr{N}$, they view the simplicial sets $\mathscr{N}(O, N)$ as analogues of fixed point objects. When $\mathscr{N} = G$-sSet, their context leads to the simplicial analogue of §1.1. However, their general context is not relevant to the equivariant theory discussed here since the natural fixed point objects N^H are in $\mathscr{N}$ and not *sSet*, so play no role in their theory.

where $\mathbf{I}$ has trivial H-action. These functors actually take values in $WH\mathscr{M}$, where $WH = NH/H$, but we shall ignore that. The trivial homomorphism $\varepsilon\colon H \longrightarrow \{e\}$ induces a $\mathscr{V}$-functor $\varepsilon^*\colon \mathscr{M} \longrightarrow H\mathscr{M}$ that assigns the trivial H-action to an object $M \in \mathscr{M}$, and we have the expected enriched adjunctions.

Lemma 1.16. *There are $\mathscr{V}$-adjunctions*

$$\underline{H\mathscr{M}}(N, \varepsilon^* L) \cong \underline{\mathscr{M}}(N/H, L)$$

and

$$\underline{H\mathscr{M}}(\varepsilon^* M, N) \cong \underline{\mathscr{M}}(M, N^H),$$

where $N \in H\mathscr{M}$ and $M \in \mathscr{M}$.

As in familiar examples, for $M \in G\mathscr{M}$ we have the natural isomorphism

$$\mathbf{I}[G] \odot_H \varepsilon^* M \cong \mathbf{I}[G/H] \odot M \tag{1.17}$$

in $G\mathscr{M}$, where the diagonal G-action is used on the right. Composing adjunctions and omitting ι^* from the notations, we obtain

$$\underline{G\mathscr{M}}(\mathbf{I}[G/H] \odot M, N) \cong \underline{\mathscr{M}}(M, N^H), \tag{1.18}$$

where M,$N \in G\mathscr{M}$. When $\mathscr{M} = \mathscr{V}$ and $M = \mathbf{I}$, this specializes to give

$$\underline{G\mathscr{V}}(\mathbf{I}[G/H], V) \cong V^H.$$

A further comparison of definitions gives the following expected identifications.

$$M/H \cong \mathbf{I}[G/H] \odot_G M \quad \text{and} \quad M^H \cong F_G(\mathbf{I}[G/H], M). \tag{1.19}$$

2. Equivariant enriched model and presheaf categories

We return to model category theory. In addition to the assumptions on $\mathscr{V}$ and $\mathscr{M}$ of [4, §1.1], we assume that $\mathscr{M}$ is a cofibrantly generated $\mathscr{V}$-model category with generating cofibrations $\mathcal{I}_{\mathscr{M}}$ and acyclic cofibrations $\mathcal{J}_{\mathscr{M}}$. Again let G be a group or, when $\mathscr{V}$ is cartesian monoidal, a $\mathscr{V}$-group. We shall see that the context of discrete groups and general enriching categories behaves quite differently from the context of group objects in cartesian monoidal categories. In the latter context, things work in much the same way as in §1.1. In particular, when $\mathscr{V} = \mathscr{U}$, the results to follow generalize the results there from $\mathscr{U}$ to general topological categories $\mathscr{M}$. We focus primarily on the former context; it is especially interesting in algebraic situations, as we illustrate in §2.3. In view of the essential role played by the functor $\mathbf{I}[-]$, we assume once and for all that $\mathbf{I}$ is cofibrant; compare [4, 4.35].

2.1. **Equivariant model categories and presheaf categories in $\mathscr{M}$.** Again let $\mathscr{F}$ be a family of subgroups of G. As in §1.1, the most important example is $\mathscr{F} = \mathscr{A}\ell\ell$, which leads to genuine equivariant homotopy theory, but the example $\mathscr{F} = \{e\}$, which leads to naive equivariant homotopy theory, is also of interest.

Definition 2.1. A G-map $f\colon M \longrightarrow N$ between objects of $G\mathscr{M}$ is an $\mathscr{F}$-equivalence or $\mathscr{F}$-fibration if $f^H\colon M^H \longrightarrow N^H$ is a weak equivalence or fibration in $\mathscr{M}$ for all $H \in \mathscr{F}$; f is an $\mathscr{F}$-cofibration if it satisfies the LLP with respect to all acyclic $\mathscr{F}$-fibrations. Define $\mathscr{F}\mathcal{I}_{\mathscr{M}}$ and $\mathscr{F}\mathcal{J}_{\mathscr{M}}$ to be the sets of maps obtained by applying the functors $\mathbf{I}[G/H] \odot (-)$ to the maps in $\mathcal{I}_{\mathscr{M}}$ and $\mathcal{J}_{\mathscr{M}}$, where $H \in \mathscr{F}$.

Specializing [4, 4.16], we obtain the following result.

Theorem 2.2. *If the sets $\mathscr{F}\mathcal{I}_{\mathscr{M}}$ and $\mathscr{F}\mathcal{J}_{\mathscr{M}}$ admit the small object argument and $\mathscr{F}\mathcal{J}_{\mathscr{M}}$ satisfies the acyclicity condition for the $\mathscr{F}$-equivalences, then $G\mathscr{M}$ is a cofibrantly generated $\mathscr{V}$-model category with generating cofibrations $\mathscr{F}\mathcal{I}_{\mathscr{M}}$ and acyclic cofibrations $\mathscr{F}\mathcal{J}_{\mathscr{M}}$.*

Here (ii) of [4, 4.16] follows formally from (1.18), and (1.18) also reduces the small object argument to a question about colimits of (transfinite) sequences in $\mathscr{M}$ that are obtained by passing to H-fixed points from relative cell complexes in $G\mathscr{M}$. The acyclicity condition (i) will hold provided that passage to H-fixed points from a relative $\mathscr{F}\mathcal{J}_{\mathscr{M}}$-cell complex gives a weak equivalence in $\mathscr{M}$.

To show that $G\mathscr{M}$ is a $\mathscr{V}$-model category, we must show that for every cofibration $i\colon A \longrightarrow X$ and fibration $p\colon E \longrightarrow B$ in $G\mathscr{M}$, the map [4, 4.19] is a fibration and is an $\mathscr{F}$-equivalence if either i or p is an $\mathscr{F}$-equivalence. It is enough to consider the case of a generating cofibration $\mathbf{I}[G/H] \odot M \longrightarrow \mathbf{I}[G/H] \odot N$, where $H \in \mathscr{F}$. The map [4, 4.19] then takes the form

$$\begin{array}{c} \underline{G\mathscr{M}}(\mathbf{I}[G/H] \odot N, E) \\ \big\downarrow \\ \underline{G\mathscr{M}}(\mathbf{I}[G/H] \odot M, E) \otimes_{\underline{G\mathscr{M}}(\mathbf{I}[G/H]\odot M, B)} \underline{G\mathscr{M}}(\mathbf{I}[G/H] \odot N, B). \end{array}$$

This map is isomorphic to the map

$$\underline{\mathscr{M}}(N, E^H) \longrightarrow \underline{\mathscr{M}}(M, E^H) \times_{\underline{\mathscr{M}}(M,B^H)} \underline{\mathscr{M}}(N, B^H).$$

The conclusion holds since $E^H \longrightarrow B^H$ is a fibration and $\mathscr{M}$ is a $\mathscr{V}$-model category.

We can compare the model structures on $G\mathscr{M}$ of Theorem 2.2 to model categories of presheaves in $\mathscr{M}$, generalizing Theorem 1.4. We view a discrete group G as a category with a single object and have the "group ring" $\mathscr{V}$-category $\mathbf{I}[G]$. When R is a commutative ring and $\mathscr{V}$ is the category of R-modules or, more interestingly, the category of chain complexes over R, $\mathbf{I}[G]$ is the group ring $R[G]$ regarded as a $\mathscr{V}$-category with a single object. Similarly, we view a $\mathscr{V}$-group $G = \mathbf{I}[G]$ as a $\mathscr{V}$-category with a single object.

In both contexts, we then have the $\mathscr{V}$-category $\mathbf{Fun}(\mathbf{I}[G]^{op}, \mathscr{M})$ of $\mathscr{V}$-enriched presheaves in $\mathscr{M}$.[4] These presheaves are just G-objects in $\mathscr{M}$,[5] and the underlying category is just $G\mathscr{M}$. This puts us in the case $\mathscr{F} = \{e\}$. Evaluation at the single object of our domain category forgets the G-action, and its left adjoint, $F_{G/e}$, sends an object $M \in \mathscr{M}$ to the free G-object $\mathbf{I}[G] \odot M$ in $G\mathscr{M}$. Here the level $\mathscr{V}$-model structure of [4, 4.31] coincides with the $\{e\}$-model structure on $G\mathscr{M}$ of Theorem 2.2. We regard this as a naive model structure, rather than a truly equivariant one.

For larger families $\mathscr{F}$, such as $\mathscr{A}\ell\ell$, we need orbit categories in order to compare the $\mathscr{F}$-model structure on $G\mathscr{M}$ to a presheaf model category. When G is discrete, we have the usual category $\mathscr{O}_{\mathscr{F}}$ of orbits G/H with $H \in \mathscr{F}$ and G-maps between them, so that $\mathscr{O}_{\mathscr{F}}(G/H, G/K) = (G/K)^H$. This is a subcategory of the category of sets. Here [4, 4.35] gives a $\mathscr{V}$-category $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$ with objects $\mathbf{I}[G/H]$ and

$$\mathbf{I}[\mathscr{O}_{\mathscr{F}}](\mathbf{I}[G/H], \mathbf{I}[G/K]) = \mathbf{I}[(G/K)^H].$$

[4] As in [4, 1.4], we use **Fun** for functor categories and **Pre** for presheaves with values in $\mathscr{V}$.

[5] Strictly speaking, since presheaves are contravariant functors, these are right G-objects unless we use the opposite multiplication on G when regarding it as a category with a single object.

When G is a $\mathscr{V}$-group in a cartesian monoidal category $\mathscr{V}$, we have the orbit category $\mathscr{O}_{\mathscr{F}}$ of orbits $G/H \in \mathscr{V}$ and G-maps between them. It is the underlying category of a $\mathscr{V}$-category with morphism objects $\underline{G\mathscr{V}}(G/H, G/K)$. For uniformity of notation, we also denote this $\mathscr{V}$-category by $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$, with objects $\mathbf{I}[G/H]$; this is consistent since $\mathbf{I}$ is a zero object. In both contexts, we have the $\mathscr{V}$-category $\mathbf{Fun}(\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op}, \mathscr{M})$ of $\mathscr{V}$-enriched presheaves in $\mathscr{M}$; the morphism sets of the underlying category are the sets of maps $X \longrightarrow Y$ of presheaves $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op} \longrightarrow \mathscr{M}$.

We also have the *full* $\mathscr{V}$-subcategory, denoted $\mathscr{V}\mathscr{O}_{\mathscr{F}}$, of $G\mathscr{V}$ whose objects are again the $\mathbf{I}[G/H]$ for $H \in \mathscr{F}$. Its hom objects in $\mathscr{V}$ are

$$\mathscr{V}\mathscr{O}_{\mathscr{F}}(\mathbf{I}[G/H], \mathbf{I}[G/K]) = \underline{G\mathscr{V}}(\mathbf{I}[G/H], \mathbf{I}[G/K]).$$

When $\mathscr{V}$ is cartesian monoidal, we use that its unit is a zero object to see that the $\mathscr{V}$-categories $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$ and $\mathscr{V}\mathscr{O}_{\mathscr{F}}$ can be identified.

When $\mathscr{V}$ is a general symmetric monoidal category, the $\mathscr{V}$-categories $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$ and $\mathscr{V}\mathscr{O}_{\mathscr{F}}$ are generally quite different. For example, when $\mathscr{V}$ is the category of R-modules,

$$R[\mathscr{O}_{\mathscr{F}}](R[G/H], R[G/K])] \cong R[(G/K)^H]$$

is generally smaller than

$$\underline{G\mathscr{M}_R}(R[G/H], R[G/K]) = \underline{R[G]}(R[G/H], R[G/K]) \cong (R[G/K])^H.$$

We have a $\mathscr{V}$-functor $\delta\colon \mathbf{I}[\mathscr{O}_{\mathscr{F}}] \longrightarrow \mathscr{V}\mathscr{O}_{\mathscr{F}}$. The maps

$$\delta\colon \mathbf{I}[\mathscr{O}_{\mathscr{F}}](I[G/H], I[G/K]) \longrightarrow \underline{G\mathscr{V}}(\mathbf{I}[G/H], \mathbf{I}[G/K])$$

are the adjoints of the evaluation maps

$$\mathbf{I}[\mathscr{O}_{\mathscr{F}}](I[G/H], I[G/K]) \otimes I[G/H] \cong \mathbf{I}[\mathscr{O}_{\mathscr{F}}(G/H, G/K) \times G/H] \longrightarrow \mathbf{I}[G/K].$$

When $\mathscr{V}$ is cartesian monoidal, δ may be viewed as an identification. In general, the maps δ of hom objects in $\mathscr{V}$ need not be weak equivalences.

We described the $\mathscr{F}$-model structure on $G\mathscr{M}$ in Theorem 2.2. For comparison, using the level $\mathscr{F}$-classes of weak equivalences and fibrations as in [4, 4.30], [4, 4.31] gives level $\mathscr{F}$-model structures on both $\mathbf{Fun}(\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op}, \mathscr{M})$ and $\mathbf{Fun}(\mathscr{V}\mathscr{O}_{\mathscr{F}}^{op}, \mathscr{M})$. To describe the generating cofibrations and acyclic cofibrations, observe that in general the presheaf $F_{G/H}$ in $\mathscr{V}$ represented by the object $I[G/H]$ has different interpretations in $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$ and in $\mathscr{V}\mathscr{O}_{\mathscr{F}}$. For $I[G/H]$ viewed as an object of $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$, the value of this presheaf on $I[G/K]$ is

$$\mathbf{I}[\mathscr{O}_{\mathscr{F}}(G/K, G/H)] = \mathbf{I}[(G/H)^K]. \tag{2.3}$$

For $I[G/H]$ viewed as an object of $\mathscr{V}\mathscr{O}_{\mathscr{F}}$, its value on $I[G/K]$ is

$$\underline{G\mathscr{V}}(\mathbf{I}[(G/K], \mathbf{I}[G/H]) \cong (\mathbf{I}[G/H])^K. \tag{2.4}$$

Relying on context, we use the notation $F_{G/H}$ for this presheaf in both cases; when $\mathscr{V}$ is cartesian monoidal, there is only one case. As observed in [4, 5.1], for a presheaf X in $\mathbf{Pre}(\mathscr{D}, \mathscr{V})$ and an object $M \in \mathscr{M}$, application of $\odot$ levelwise gives a presheaf $X \odot M$ in $\mathbf{Fun}(\mathscr{D}^{op}, \mathscr{M})$, and this construction is functorial.

Definition 2.5. Let $F_{\mathscr{F}}\mathcal{I}_{\mathscr{M}}$ and $F_{\mathscr{F}}\mathcal{J}_{\mathscr{M}}$ denote the sets of presheaves $F_{G/H} \odot i$ and $F_{G/H} \odot j$ in either $\mathbf{Fun}(\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op}, \mathscr{M})$ or $\mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op}, \mathscr{M})$, where $H \in \mathscr{F}$, $i \in \mathcal{I}_{\mathscr{M}}$, and $j \in \mathcal{J}_{\mathscr{M}}$.

Theorem 2.6. *When the sets $F_{\mathscr{F}}\mathcal{I}_{\mathscr{M}}$ and $F_{\mathscr{F}}\mathcal{J}_{\mathscr{M}}$ admit the small object argument and every relative $F_{\mathscr{F}}\mathcal{J}_{\mathscr{M}}$-cell complex is a level $\mathscr{F}$-equivalence,* $\mathbf{Fun}(\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op},\mathscr{M})$ *and* $\mathbf{Fun}(\mathscr{V}\mathscr{O}_{\mathscr{F}}^{op},\mathscr{M})$ *are cofibrantly generated $\mathscr{V}$-model categories with generating cofibrations $F_{\mathscr{F}}\mathcal{I}_{\mathscr{M}}$ and acyclic cofibrations $F_{\mathscr{F}}\mathcal{J}_{\mathscr{M}}$.*

When G is discrete, our assumption that $\mathbf{I}$ is cofibrant guarantees the acyclicity condition in the case of $\mathbf{Fun}(\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op},\mathscr{M})$. In the cartesian monoidal case, the acyclicity condition is often an elaboration of the simple argument that applied to spaces in §1.1. The smallness condition is generally inherited from $\mathscr{M}$, often reducing to a compactness observation in contexts of compactly generated model categories.

The verification of the $\mathscr{V}$-model category structure is similar to that given in Theorem 2.2. We must show that the map $\underline{\mathbf{Fun}}(\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op},\mathscr{M})(i,p)$ of [4, 4.19] is a fibration and is acyclic if i or p is so, where $i\colon F_{G/H}\odot M\longrightarrow F_{G/H}\odot N$ is a generating cofibration and $p\colon E\longrightarrow B$ is a fibration. The map in question is isomorphic to

$$\underline{\mathscr{M}}(N,E(G/H))\longrightarrow \underline{\mathscr{M}}(M,E(G/H))\times_{\underline{\mathscr{M}}(M,B(G/H))}\underline{\mathscr{M}}(N,B(G/H)).$$

The conclusion holds since $p\colon E(G/H)\longrightarrow B(G/H)$ is a fibration and $\mathscr{M}$ is a $\mathscr{V}$-model category.

Assuming the hypotheses of Theorems 2.2 and 2.6, we have the $\mathscr{F}$-model categories $G\mathscr{M}$, $\mathbf{Fun}(\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op},\mathscr{M})$, and $\mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op},\mathscr{M})$. When $\mathscr{V}$ is cartesian monoidal, we may identify the two presheaf categories. In general, the $\mathscr{V}$-functor δ induces a Quillen adjunction, not usually an equivalence, between them. It is $\mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op},\mathscr{M})$ and not $\mathbf{Fun}(\mathbf{I}[\mathscr{O}_{\mathscr{F}}]^{op},\mathscr{M})$ that correctly models the $\mathscr{F}$-model structure on $G\mathscr{M}$.

Theorem 2.7. *There is a Quillen $\mathscr{V}$-adjunction*

$$\mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op},\mathscr{M})\underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}}G\mathscr{M}$$

and it is a Quillen equivalence if the functors $(-)^H$ preserve the tensors, coproducts, pushouts, and sequential colimits that appear in the construction of cell complexes.

Proof. We have displayed the adjunction on underlying categories; on the enriched level, the corresponding adjunction is a comparison of equalizer diagrams. For $N\in G\mathscr{M}$, we define $\mathbb{U}(N)_{G/H}=N^H$. For $X\in\mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op},\mathscr{M})$, we define $\mathbb{T}X=X_{G/e}$. Using the canonical G-maps $G/e\longrightarrow G/H$, we easily check the claimed adjunction. Since $\mathbb{U}$ creates the $\mathscr{F}$-equivalences and $\mathscr{F}$-fibrations in $G\mathscr{M}$, $(\mathbb{T},\mathbb{U})$ is a Quillen adjunction, and it is a Quillen equivalence if and only if $\eta\colon X\longrightarrow\mathbb{U}\mathbb{T}X$ is a level equivalence when $X\in\mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op},\mathscr{M})$ is cofibrant. First consider $X=F_{G/H}\odot M$, where $M\in\mathscr{M}$ (not $G\mathscr{M}$). Evaluated at G/e, this gives $\mathbf{I}[G/H]\odot M$, by (2.4). Now take K-fixed points. The assumption that $(-)^K$ preserves tensors means that the result is $(\mathbf{I}[G/H])^K\odot M$. This agrees with $X_{G/K}$, and η is an isomorphism. Now the assumed commutation of passage to K-fixed points and the relevant colimits ensures that $\mathbb{U}$ maps relative cell complexes to relative cell complexes bijectively and that η is an isomorphism for any cell complex X, just as for topological spaces in §1.1. □

2.2. **Equivariant model categories and presheaf categories in $\mathscr{V}$.** Now that we understand equivariant model categories as presheaf categories in $\mathscr{M}$, we can understand them as presheaf categories in $\mathscr{V}$ whenever we can understand $\mathscr{M}$ itself as a presheaf category in $\mathscr{V}$. That is, if we have an answer to one of [4, Questions 0.1 – 0.4] for $\mathscr{M}$, then we have an answer to an analogous question with $\mathscr{M}$ replaced by $G\mathscr{M}$. This is immediate from the observation that a presheaf category in a presheaf category is again a presheaf category.

Proposition 2.8. *Let $\mathscr{D}$ and $\mathscr{E}$ be small $\mathscr{V}$-categories and let $\mathscr{N}$ be any $\mathscr{V}$-category. Then there is a canonical isomorphism of $\mathscr{V}$-categories*

$$\mathbf{Fun}(\mathscr{D}, \mathbf{Fun}(\mathscr{E}, \mathscr{N})) \cong \mathbf{Fun}(\mathscr{D} \otimes \mathscr{E}, \mathscr{N}).$$

If we have level $\mathscr{V}$-model structures induced by a $\mathscr{V}$-model structure on $\mathscr{N}$ on all functor categories in sight, then this is an isomorphism of $\mathscr{V}$-model categories.

Proof. The isomorphism can be written $(X_d)_e = X_{d,e}$ on objects. That is, if X is given as a functor of either type, then the equality specifies a functor of the other type. It requires just a bit of thought to see how this works on the maps of enriched homs that specify the $\mathscr{V}$-category structure. One point is that for a presheaf $X\colon \mathscr{D} \otimes \mathscr{E} \longrightarrow \mathscr{N}$, we obtain maps

$$\mathscr{E}(e, e') \cong \mathbf{I} \otimes \mathscr{E}(e, e') \longrightarrow \mathscr{D}(d, d) \otimes \mathscr{E}(e, e') \xrightarrow{X} \mathscr{N}(X_{d,e'}, X_{d,e})$$

which specify a $\mathscr{V}$-functor $X_d\colon \mathscr{E} \longrightarrow \mathscr{N}$. Comparisons of equalizers show that the hom objects in $\mathscr{V}$ between presheaves are also isomorphic. $\square$

Now return to the equivariant context. Suppose that we have a $\mathscr{V}$-model category $\mathscr{M}$ together with a $\mathscr{V}$-functor $\delta\colon \mathscr{D} \longrightarrow \mathscr{M}$ that gives rise to a Quillen equivalence $\mathscr{M} \longrightarrow \mathbf{Pre}(\mathscr{D}, \mathscr{V})$, as in [4, Question 0.2 or 0.3]. Retaining the assumptions of the previous section, for any family of subgroups $\mathscr{F}$ we also have a Quillen equivalence $G\mathscr{M} \longrightarrow \mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op}, \mathscr{M})$. Composing, these give a composite Quillen equivalence

$$G\mathscr{M} \longrightarrow \mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op}, \mathscr{M}) \longrightarrow \mathbf{Fun}((\mathscr{V}\mathscr{O}_{\mathscr{F}})^{op}, \mathbf{Pre}(\mathscr{D}, \mathscr{V})).$$

Proposition 2.8 allows us to rewrite this, giving the following general conclusion.

Theorem 2.9. *The $\mathscr{F}$-model category $G\mathscr{M}$ is Quillen equivalent to the presheaf category $\mathbf{Pre}(\mathscr{V}\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D}, \mathscr{V})$.*

The objects of $\mathscr{V}\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D}$ are pairs $(\mathbf{I}[G/H], d)$, where $H \in \mathscr{F}$ and $d \in \mathscr{D}$. We have a canonical $\mathscr{V}$-functor $\tau\colon \mathscr{V}\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D} \longrightarrow G\mathscr{M}$ that sends $(\mathbf{I}[G/H], d)$ to $\mathbf{I}[G/H] \odot \delta d$. The maps of enriched hom objects are given by the tensor bifunctor

$$\odot\colon \underline{G\mathscr{V}}(\mathbf{I}[G/H], \mathbf{I}[G/K]) \otimes \mathscr{D}(d, e) \longrightarrow \underline{G\mathscr{M}}(\mathbf{I}[G/H] \odot \delta d, \mathbf{I}[G/K] \odot \delta e).$$

Let $\mathscr{F}\mathscr{D}$ denote the full $\mathscr{V}$-subcategory of $G\mathscr{M}$ whose objects are the $\mathbf{I}[G/H] \odot \delta d$ with $H \in \mathscr{F}$. Since τ lands in $\mathscr{F}\mathscr{D}$, it specifies a $\mathscr{V}$-functor

$$\tau\colon \mathscr{V}\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D} \longrightarrow \mathscr{F}\mathscr{D}.$$

Even when δ is the inclusion of a full subcategory, it is unclear to us whether or not τ is a weak equivalence. In any case, this is an important example where the domain of the presheaf category that arises most naturally in answering [4, Question

0.2 or 0.4] is not a full $\mathscr{V}$-subcategory. We have Quillen adjunctions of $\mathscr{F}$-model categories

$$\mathbf{Pre}(\mathscr{V}\mathscr{O}_{\mathscr{F}}\otimes\mathscr{D},\mathscr{V}) \underset{\tau^*}{\overset{\tau_*}{\rightleftarrows}} \mathbf{Pre}(\mathscr{F}\mathscr{D},\mathscr{V}) \quad \text{and} \quad \mathbf{Pre}(\mathscr{F}\mathscr{D},\mathscr{V}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{M}.$$

A check of definitions using (1.19) shows that the composite Quillen adjunction is the Quillen equivalence of Theorem 2.9, but we do not know whether or not these Quillen adjunctions themselves can also be expected to be Quillen equivalences.

2.3. **Enriched model categories of DG G-modules.** Let R be a commutative ring. We specialize the general theory to the category $\mathscr{V}=\mathscr{M}_R$ of ($\mathbb{Z}$-graded) chain complexes over R and the $\mathscr{V}$-category $\mathscr{M}=\mathscr{M}_A$ of DG A-modules, where A is a differential graded R-algebra (DGA). Differentials lower degree; replacing X_n by X^{-n} would reverse this convention.

We give $\mathscr{M}_R$ the model structure whose weak equivalences, fibrations, and cofibrations are the quasi-isomorphisms, the degreewise epimorphisms, and the degreewise split monomorphisms with cofibrant cokernel. Cofibrant objects are degreewise projective, and the converse holds for bounded below objects. This model structure is compactly generated. Canonical generating sets $\mathcal{I}_R$ and $\mathcal{J}_R$ are given by the inclusions $S_R^{n-1}\longrightarrow D_R^n$ and $0\longrightarrow D_R^n$ for $n\in\mathbb{Z}$. Here S_R^n is R-free on one generator of degree n, with zero differential, and D_R^n is R-free on generators of degrees n and $n-1$, with $d_n=\mathrm{id}$. See for example [17, §18.4] for details and alternative model structures.

We give $\mathscr{M}_A$ the model structure whose weak equivalences and fibrations are the maps which are weak equivalences and fibrations when regarded as maps in $\mathscr{M}_R$. That is, we take the model structure induced by the underlying R-module functor $\mathbb{R}\colon \mathscr{M}_A\longrightarrow\mathscr{M}_R$. Then $(\mathbb{F},\mathbb{R})$ is a Quillen adjunction, where $\mathbb{F}\colon\mathscr{M}_R\longrightarrow\mathscr{M}_A$ is the extension of scalars functor that sends X to $A\otimes_R X$. This model structure is also compactly generated. Generating sets $\mathcal{I}_A$ and $\mathcal{J}_A$ are obtained by applying $\mathbb{F}$ to the maps in $\mathcal{I}_R$ and $\mathcal{J}_R$. Other model structures defined in [1] could also be used.

Let G be a group and $\mathscr{F}$ a family of subgroups. We have the categories $G\mathscr{M}_R$, $(\mathscr{M}_R)_G$, $G\mathscr{M}_A$, and $(\mathscr{M}_A)_G$ as in §1.2. The unit of $\mathscr{M}_R$ is $R=S_R^0$ and thus $R[G]$ in the general theory is just the group ring $R[G]$. Similarly $R[G/H]$ is the free R-module generated by G/H and is a left $R[G]$-module regarded as a chain complex concentrated in degree zero. Let $\mathscr{M}_R\mathscr{O}_{\mathscr{F}}$ be the full $\mathscr{M}_R$-subcategory of $\mathscr{M}_R$ whose objects are the $R[G/H]$ for $H\in\mathscr{F}$.

Obviously $\mathscr{M}_A$ is enriched over $\mathscr{M}_R$ since the $\mathrm{Hom}_A(M,N)$ are chain complexes of R-modules. The general theory specializes to give results analogous to those in the topological context of §1.1, but we now need the more general context that we have developed to deal with the $\mathscr{M}_R$-category $\mathscr{M}_A$. The acyclicity conditions required to prove the theorems below are verified by the same simple arguments as in §1.1, using deformation retractions of chain complexes.

Define the $\mathscr{F}$-equivalences and $\mathscr{F}$-fibrations in $G\mathscr{M}_R$ and $G\mathscr{M}_A$ to be the G-maps f (of chain complexes or DG A-modules with an action of G) such that the fixed point map f^H is a quasi-isomorphism or degreewise epimorphism for $H\in\mathscr{F}$. With these definitions, we have the following theorems. In the first two, which are specializations of Theorems 2.2 and 2.6, respectively, the case $A=R$ gives the specialization to R.

Theorem 2.10. *The $\mathscr{F}$-equivalences, $\mathscr{F}$-fibrations, and the resulting cofibrations give $G\mathscr{M}_A$ a compactly generated proper $\mathscr{M}_R$-model category structure such that the sets $\mathscr{F}\mathcal{I}_A = \{R[G/H] \odot i\}$ and $\mathscr{F}\mathcal{J}_A = \{R[G/H] \odot j\}$, where $H \in \mathscr{F}$, $i \in \mathcal{I}_A$, and $j \in \mathcal{J}_A$, are generating sets of cofibrations and acyclic cofibrations.*

Now recall (2.4) and Definition 2.5, using the case $\mathscr{M}_R\mathscr{O}_{\mathscr{F}}$. Here the functor $\odot$ is given by tensoring over R. Thus, for $M \in G\mathscr{M}_A$,

$$(F_{G/H} \odot M)(R[G/K]) = R[G/H]^K \otimes_R M.$$

When G acts trivially on M, we have a natural inclusion of this A-module in

$$(R[G/H] \otimes_R M)^K \cong \mathbb{U}(R[G/H] \otimes_R M)(R[G/K]),$$

where $\mathbb{U}\colon G\mathscr{M}_A \longrightarrow \mathbf{Pre}(\mathscr{M}_R\mathscr{O}_{\mathscr{F}}, \mathscr{M}_A)$ is the represented presheaf functor. This inclusion is often an isomorphism, and we assume that this is so when M is the domain or target of a map in $\mathcal{I}_{\mathscr{F}}$ or $\mathcal{J}_{\mathscr{F}}$.

Theorem 2.11. *the level $\mathscr{F}$-equivalences, level $\mathscr{F}$-fibrations, and the resulting cofibrations give $\mathbf{Fun}(\mathscr{M}_R\mathscr{O}_{\mathscr{F}}^{op}, \mathscr{M}_A)$ a compactly generated proper $\mathscr{M}_R$-model structure such that the sets $F_{\mathscr{F}}\mathcal{I}_A$ and $F_{\mathscr{F}}\mathcal{J}_A$ generate the cofibrations and acyclic cofibrations, and these sets are isomorphic to the sets $\mathbb{U}\mathscr{F}\mathcal{I}_A$ and $\mathbb{U}\mathscr{F}\mathcal{J}_{\mathscr{F}}$.*

Just as for G-spaces, the functor $\mathbb{U}$ is full and faithful and preserves those colimits used to construct relative cell complexes, hence the acyclicity conditions needed to prove the previous two theorems are identical. The following comparison now follows from Theorem 2.7; compare Theorem 1.4.

Theorem 2.12. $(\mathbb{T}, \mathbb{U})$ *is a Quillen $\mathscr{M}_R$-equivalence*

$$\mathbf{Fun}((\mathscr{M}_R\mathscr{O}_{\mathscr{F}})^{op}, \mathscr{M}_A) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{M}_A$$

Remark 2.13. Since $\mathscr{M}_A$ can be identified with $Fun(A, \mathscr{M}_R)$, where A is regarded as an $\mathscr{M}_R$-category with a single object, we can apply Proposition 2.8 and Theorem 2.9 to identify the functor model category here with $\mathbf{Pre}(\mathscr{M}_R\mathscr{O}_{\mathscr{F}} \otimes A^{op}, \mathscr{M}_R)$.

Remark 2.14. We can generalize the theory of this section by demanding an action of G on A through automorphisms of DG algebras and using twisted modules (e.g. [6, 7]) or by generalizing from equivariant DG algebras to equivariant DG categories. This section can be viewed as a modest contribution to the nascent field of equivariant homological algebra.

2.4. **Equivariant simplicial model categories.** Since simplicial enrichment is the one most commonly used, we would be remiss not to show how our theory applies to equivariant simplicial model categories. Here we take $\mathscr{V}$ to be the closed cartesian monoidal category of simplicial sets which we denote by $s\mathscr{S}$ in this section. We give $s\mathscr{S}$ its usual model structure; other choices are possible. We take G to be a simplicial group. This places us in context B. Less generally, we could take a group G and regard it as a discrete simplicial group, which according to our general theory would be denoted $\mathbf{I}[G]$. That places us in context A.

We take $\mathscr{M}$ to be any cofibrantly generated bicomplete simplicial model category. We have the category $G\mathscr{M}$ of G-objects in $\mathscr{M}$. For G-objects M and N, $\underline{\mathscr{M}}(M, N)$ with the induced G-action is denoted $\underline{\mathscr{M}}_G(M, N)$. It gives the hom objects for an enrichment of $\mathscr{M}$ in $Gs\mathscr{S}$. The G-fixed objects

$$\underline{G\mathscr{M}}(M, N) = \underline{\mathscr{M}}_G(M, N)^G$$

give the hom objects for the enrichment of $G\mathscr{M}$ in $s\mathscr{S}$ that we are interested in.

The notion of a family of subgroups remains meaningful for simplicial groups, and we fix such a family $\mathscr{F}$; as usual, the most interesting examples are $\mathscr{A}\ell\ell$ and $\{e\}$. Theorem 2.6 applies. The applicability of the small object argument and the acyclicity condition are both inherited from $\mathscr{M}$.

Theorem 2.15. *The category* $\mathbf{Fun}(\mathscr{O}_{\mathscr{F}}^{op}, \mathscr{M})$ *is a cofibrantly generated simplicial model category.*

Here by $\mathscr{O}_{\mathscr{F}}$ we understand the category whose objects are the simplicial sets G/H for simplicial subgroups H of G in $\mathscr{F}$; the morphisms are the simplicial sets

$$\underline{Gs\mathscr{S}}(G/H, G/K) = \underline{s\mathscr{S}}_G(G/H, G/K) \cong (G/H)^K,$$

which is the simplicial set whose n-simplices are $(G_n/H_n)^{K_n}$, with the induced faces and degeneracies. When G is a group regarded as a discrete simplicial set, $\underline{G\mathscr{V}}(G/H, G/K)$ is the usual orbit category $\mathscr{O}_{\mathscr{F}}$ of G-sets G/H, $H \in \mathscr{F}$, regarded as a category of discrete simplicial sets. The generating cofibrations and acyclic cofibrations of the functor category are obtained by applying the functors $G/H \odot (-)$ to the generating cofibrations and acyclic cofibrations of $\mathscr{M}$. Thus the generalities unravel into something quite simple and explicit in this situation.

Theorem 2.7 also applies.

Theorem 2.16. *There is a simplicial Quillen adjunction*

$$\mathbf{Fun}((\mathscr{O}_{\mathscr{F}})^{op}, \mathscr{M}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{M},$$

and it is a Quillen equivalence if the functors $(-)^H$ *preserve the tensors, coproducts, pushouts, and sequential colimits that appear in the construction of cell complexes.*

If we assume that the category $\mathscr{M}$ is Quillen equivalent to a presheaf category $\mathbf{Pre}(\mathscr{D}, s\mathscr{S})$ for some small simplicial category $\mathscr{D}$, then Theorem 2.9 applies to give the following conclusion. We assume the hypothesis about the functors $(-)^H$.

Theorem 2.17. *The* $\mathscr{F}$*-model category* $G\mathscr{M}$ *is Quillen equivalent to the presheaf category* $\mathbf{Pre}(\mathscr{O}_{\mathscr{F}} \times \mathscr{D}, s\mathscr{S})$.

Since $s\mathscr{S}$ is itself the presheaf category $\mathbf{Pre}(\Delta, \mathscr{S})$, where $\mathscr{S}$ is the category of sets, the proof of Theorem 2.9 applies to express the presheaf category here as

$$\mathbf{Pre}(\mathscr{O}_{\mathscr{F}} \times \mathscr{D} \times \Delta, \mathscr{S}) \cong \mathbf{Pre}(\Delta, \mathbf{Pre}(\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D}, \mathscr{S})),$$

which is the category of simplicial objects in $\mathbf{Pre}(\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D}, \mathscr{S})$.

3. Enriched model categories of G-spectra

To illustrate our interest in some of the comparison results developed in [4] and to prepare for the sequel [5], we show how our results play out in model categories of G-spectra. We shall rely on [3, 11, 13, 14] for definitions of the relevant categories.

We can work with a general topological group G and categories of *naive* G-spectra, which are just spectra with G-actions. Here we can take $\mathscr{V} = \mathscr{M}$ to be a good category of spectra, such as symmetric spectra (of spaces), orthogonal spectra, or S-modules [3, 14] and apply the theory already developed to $(\mathscr{V}, \wedge, S)$. The topology on G introduces no serious difficulties, and nor does that fact that S is not cofibrant in the category of S-modules. Theorems 2.2, 2.6, and 2.7 apply without added hypotheses and with little extra work to give $\mathscr{V}$-model categories and Quillen equivalences

$$\mathbf{Pre}(\mathscr{V}\mathscr{O}_{\mathscr{F}}, \mathscr{V}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{V}$$

for any family $\mathscr{F}$ of (closed) subgroups of G. Here $\mathscr{V}\mathscr{O}_{\mathscr{F}}$ is the full $\mathscr{V}$-subcategory of $G\mathscr{V}$ whose objects are bifibrant approximations of the naive G-spectra $\Sigma_G^\infty(G/H_+)$ for $H \in \mathscr{F}$. Here X_+ is the disjoint union of a G-space X and a G-fixed basepoint.

However, our main interest is in compact Lie groups G. Here G-spectra are indexed on a preferred G-universe U, which is a sum of countably many copies of each of a set of representations of G. We are mainly interested in a complete G-universe, which contains all representations of G. The resulting G-spectra are then said to be *genuine*. Our model categories are all stable, and the arguments below work as stated for naive G-spectra for any topological group G, but we focus on G-spectra indexed on some G-universe U, where G is compact Lie. We can again work relative to a family $\mathscr{F}$ of (closed) subgroups, but for notational simplicity we specialize to the case $\mathscr{F} = \mathscr{A}\ell\ell$.

3.1. **Presheaf models for categories of G-spectra.** We focus on two categories of G-spectra treated in detail in [13]. We have the closed symmetric monoidal category $\mathscr{S}$ of nonequivariant orthogonal spectra [14]. Its function spectra are denoted $F(X, Y)$. We also have the closed symmetric monoidal category $G\mathscr{S}$ of orthogonal G-spectra (for a fixed G-universe U as above) [13]. Its function G-spectra are denoted $F_G(X, Y)$. Then $G\mathscr{S}$ is enriched over $\mathscr{S}$ via the G-fixed point spectra $F_G(X, Y)^G$. In terms of the general context of [4], we are taking $\mathscr{V} = \mathscr{S}$ and $\mathscr{M} = G\mathscr{S}$. We have stable model structures on $\mathscr{S}$ and $G\mathscr{S}$ [13, 14], and we have the following specialization of [4, 1.35].

Theorem 3.1. *Let $G\mathscr{D}$ be the full $\mathscr{S}$-subcategory of $G\mathscr{S}$ whose objects are fibrant approximations of the orbit suspension G-spectra $\Sigma_G^\infty(G/H_+)$, where H runs over the closed subgroups of G. Then there is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D}, \mathscr{S}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{S},$$

and it is a Quillen equivalence.

We have a second specialization of [4, 1.35]. We have the closed symmetric monoidal category $\mathscr{Z}$ of nonequivariant S-modules [3].[6] Its function spectra are again denoted $F(X, Y)$. We also have the closed symmetric monoidal category $G\mathscr{Z}$ of S_G-modules (for a fixed G-universe U as above) [13]. Its function G-spectra are denoted $F_G(X, Y)$. Then $G\mathscr{Z}$ is enriched over $\mathscr{Z}$ via the G-fixed point spectra $F_G(X, Y)^G$. We are taking $\mathscr{V} = \mathscr{Z}$ and $\mathscr{M} = G\mathscr{Z}$. We have stable model structures on $\mathscr{Z}$ and $G\mathscr{Z}$ [3, 13].

Theorem 3.2. *Let $G\mathscr{D}$ be the full $\mathscr{Z}$-subcategory of $G\mathscr{Z}$ whose objects are cofibrant approximations of the orbit suspension G-spectra (= S_G-modules) $\Sigma_G^\infty(G/H_+)$,*

[6]The notation $\mathscr{S}$ is short for $\mathscr{I}\mathscr{S}$ and the notation $\mathscr{Z}$ is short for $\mathscr{M}_S$ in the original sources; as a silly mnemonic device, $\mathscr{Z}$ stands for the Z in the middle of Elmendorf-KriZ-Mandell-May.

where H runs over the closed subgroups of G. Then there is an enriched Quillen adjunction

$$\mathbf{Pre}(G\mathscr{D},\mathscr{Z}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{Z},$$

and it is a Quillen equivalence.

Remark 3.3. We stated Theorems 3.1 and 3.2 in terms of orbits G/H. We could equally well shrink the category $G\mathscr{D}$ by choosing one H in each conjugacy class.

When G is finite, we can instead expand $G\mathscr{D}$ to the full subcategory of $G\mathscr{S}$ or $G\mathscr{Z}$ whose objects are bifibrant approximations of the suspension G-spectra $\Sigma^\infty_G(A_+)$, where A runs over the finite G-sets. By [4, 2.5], [4, 1.35] applies to *any* set of compact generators, hence Theorems 3.1 and 3.2 remain true for these expanded versions of the categories $G\mathscr{D}$.

Alternatively, still defining $\mathscr{D}$ using finite G-sets, we can restrict attention to additive presheaves, namely those that take finite wedges in $G\mathscr{D}$ to finite products (which are weakly equivalent to finite wedges). The original categories $\mathbf{Pre}(G\mathscr{D}_{\mathscr{S}},\mathscr{S})$ and $\mathbf{Pre}(G\mathscr{D}_{\mathscr{Z}},\mathscr{Z})$ are equivalent to the respective categories of additive presheaves defined using finite G-sets. One point is that the represented presheaves $F_G(-,Y)^G$ are additive, so that additivity drops out of the proofs and need not be assumed.

Either way, when G is finite Theorems 3.1 and 3.2 remain valid with $G\mathscr{D}$ reinterpreted to allow general finite G-sets rather than just orbits.

Homotopically, Theorems 3.1 and 3.2 are essentially the same result since $G\mathscr{S}$ and $G\mathscr{Z}$ are Quillen equivalent. On the point set level they are quite different, and they have different virtues and defects. Since we now have both results, we write $G\mathscr{D}_{\mathscr{S}}$ or $G\mathscr{D}_{\mathscr{Z}}$ instead of $G\mathscr{D}$ when it is unclear from context which is intended.

We say just a bit about the proofs of these theorems. By [4, 4.31], the presheaf categories used in them are well-behaved model categories. The acyclicity condition there holds in Theorem 3.1 because $\mathscr{S}$ satisfies the monoid axiom, by [13, 7.4]. It holds in Theorem 3.2 by use of the "Cofibration Hypothesis" of [3, p. 146], which also holds equivariantly. The orbit G-spectra give compact generating sets in both $\mathrm{Ho}(G\mathscr{S})$ and $\mathrm{Ho}(G\mathscr{Z})$. We require bifibrant representatives. In Theorem 3.1, the orbit G-spectra are cofibrant, and fibrant approximation makes them bifibrant. We say more about the relevant functors in §3.3.

By contrast, in Theorem 3.2, all S_G-modules are fibrant, and cofibrant approximation makes them bifibrant. Here cofibrant approximation is given by a well understood left adjoint that very nearly preserves smash products, as we shall explain in §3.4.

Technically, [4, 1.35] requires *either* that the unit object of the enriching category $\mathscr{V}$ be cofibrant *or* that every object in $\mathscr{V}$ be fibrant. The first hypothesis holds in $\mathscr{S}$ and the second holds in $\mathscr{Z}$. It is impossible to have both of these conditions in the same symmetric monoidal model category for the stable homotopy category [10, 16]. That is a key reason that both of these results are of interest.

3.2. **Comparison of presheaf models of G-spectra.** Theorems 3.1 and 3.2 are related by the following result, which is [13, IV.1.1]; the nonequivariant special case is [13, I.1.1]. In this result, $G\mathscr{S}$ is given its positive stable model structure from [13] and is denoted $G\mathscr{S}_{pos}$ to indicate the distinction; in that model structure, the sphere G-spectrum in $G\mathscr{S}$, like the sphere G-spectrum in $G\mathscr{Z}$ is not cofibrant.

The cited result is proven for genuine G-spectra for compact Lie groups G, but the same proof applies to naive G-spectra for any topological group G.

Theorem 3.4. *There is a Quillen equivalence*

$$G\mathscr{S}_{pos} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} G\mathscr{Z}.$$

The functor $\mathbb{N}$ *is strong symmetric monoidal, hence* $\mathbb{N}^{\#}$ *is lax symmetric monoidal.*

The identity functor is a left Quillen equivalence $G\mathscr{S}_{pos} \longrightarrow G\mathscr{S}$. Therefore Theorems 3.1, 3.2, and 3.4, have the following immediate consequence.

Corollary 3.5. *The categories* $\mathbf{Pre}(G\mathscr{D}_{\mathscr{S}}, \mathscr{S})$ *and* $\mathbf{Pre}(G\mathscr{D}_{\mathscr{Z}}, \mathscr{Z})$ *are Quillen equivalent. More precisely, there are left Quillen equivalences*

$$\mathbf{Pre}(G\mathscr{D}_{\mathscr{S}}, \mathscr{S}) \longrightarrow G\mathscr{S} \longleftarrow G\mathscr{S}_{pos} \longrightarrow G\mathscr{Z} \longleftarrow \mathbf{Pre}(G\mathscr{D}_{\mathscr{Z}}, \mathscr{Z}).$$

In fact, we can compare the $\mathscr{S}$-category $G\mathscr{D}_{\mathscr{S}}$ with the $\mathscr{Z}$-category $G\mathscr{D}_{\mathscr{Z}}$ via the right adjoint $\mathbb{N}^{\#}$. The adjunction

$$G\mathscr{S}_{pos} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} G\mathscr{Z}$$

is tensored over the adjunction

$$\mathscr{S}_{pos} \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} \mathscr{Z}$$

in the sense of [4, 3.20]. Indeed, since $G\mathscr{S}$ is a bicomplete $\mathscr{S}$-category, it is tensored over $\mathscr{S}$. While a more explicit definition is easy enough, we can define $Y \odot X$ to be $Y \wedge i_*\varepsilon^*X$, where $i_*\varepsilon^*\colon \mathscr{S} \longrightarrow G\mathscr{S}$ is the change of group and universe functor associated to $\varepsilon\colon G \longrightarrow e$ that assigns a genuine G-spectrum to a nonequivariant spectrum. The same is true with $\mathscr{S}$ replaced by $\mathscr{Z}$. These functors are discussed in both contexts and compared in [13]. Results there (see [13, IV.1.1]) imply that

$$\mathbb{N}Y \odot \mathbb{N}X \cong \mathbb{N}(Y \odot X),$$

which is the defining condition for a tensored adjunction. Now [4, 3.24] gives that the $\mathscr{S}$-category $\mathbb{N}^{\#}G\mathscr{D}_{\mathscr{Z}}$ is quasi-equivalent to $G\mathscr{D}_{\mathscr{S}}$. Using [4, 2.15 and 3.17], this implies a direct proof of the Quillen equivalence of Corollary 3.5. Therefore Theorems 3.1 and 3.2 are equivalent: each implies the other.

We reiterate the generality: the results above do not require G to be finite. In that generality, we do not know how to simplify the description of the domain category $G\mathscr{D}$ to transform it into a weakly equivalent $\mathscr{S}$-category or $\mathscr{Z}$-category that is intuitive and perhaps even familiar, something accessible to study independent of knowledge of the category of G-spectra that we seek to understand. When G is finite, we show how to do just that in the sequel [5].

3.3. **Suspension spectra and fibrant replacement functors in $G\mathscr{S}$.** We here give some observations relevant to understanding the category $G\mathscr{D}_{\mathscr{S}}$ of Theorem 3.1. We start with a parenthetical observation about fibrant approximations that is immediate from Theorem 3.4 but does not appear in the literature.

Proposition 3.6. *The unit* $\eta\colon E \longrightarrow \mathbb{N}^{\#}\mathbb{N}E$ *of the adjunction between* $G\mathscr{S}$ *and* $G\mathscr{Z}$ *specifies a lax monoidal fibrant replacement functor for the positive stable model structure on* $G\mathscr{S}$.

Remark 3.7. Nonequivariantly, Kro [9] has given a different lax monoidal positive fibrant replacement functor for orthogonal spectra. As he notes, his construction does not apply to symmetric spectra. However, by [14, 3.3], the unit $E \longrightarrow \mathbb{N}^\sharp\mathbb{U}\mathbb{P}\mathbb{N}E$ of the composite of the adjunction $(\mathbb{P},\mathbb{U})$ between symmetric and orthogonal spectra and the adjunction $(\mathbb{N},\mathbb{N}^\sharp)$ gives a lax monoidal positive fibrant replacement functor for symmetric spectra.

Unfortunately the restriction to the positive model structure is necessary, and the only fibrant approximation functor we know of for use in Theorem 3.1 is that given by the small object argument. The point is that the suspension G-spectra $\Sigma^\infty_G(G/H_+)$ are cofibrant but not positive cofibrant. For an inner product space V and a based G-space X, the V^{th} space of $\Sigma^\infty_G X$ is $X\wedge S^V$. The functor Σ^∞_G, also denoted F_0, is left adjoint to the zeroth space $(-)_0\colon G\mathscr{S}\longrightarrow G\mathscr{T}$. Nonequivariantly, it is part of [14, 1.8] that for based spaces X and Y, $F_0X\wedge F_0Y$ is naturally isomorphic to $F_0(X\wedge Y)$. The categorical proof of that result in [14, §21] applies equally well equivariantly to give the following complement to Proposition 3.6.

Proposition 3.8. *The functor $\Sigma^\infty_G\colon G\mathscr{T}\longrightarrow G\mathscr{S}$ is strong symmetric monoidal.*

Therefore the zeroth space functor is lax symmetric monoidal, but of course that functor is not homotopically meaningful except on objects that are fibrant in the stable model structure. There is no known fibrant replacement functor in that model structure that is well-behaved with respect to smash products.

Nonequivariantly, a homotopically meaningful version of the adjunction $(\Sigma^\infty,\Omega^\infty)$ has been worked out for symmetric spectra by Sagave and Schlichtkrull [19] and for symmetric and orthogonal spectra by Lind [12], who compares his constructions with the adjunction $(\Sigma^\infty,\Omega^\infty)$ in $\mathscr{S}p$ (see below) and with its analogue for $\mathscr{Z}$. This generalizes to the equivariant context, although details have not been written down.

3.4. **Suspension spectra and smash products in $G\mathscr{Z}$.** We here give some observations relevant to understanding the category $G\mathscr{D}_{\mathscr{Z}}$ of Theorem 3.2. In particular, we give properties of cofibrant approximations of suspension spectra that will be needed in [5]. For more information, see [15, XXIV], [13, §IV.2], and the nonequivariant precursor [3].

We have a category $G\mathscr{P}$ of (coordinate-free)-prespectra. Its objects Y are based G-spaces $Y(V)$ and based G-maps $Y(V)\wedge S^W\longrightarrow Y(W-V)$ for $V\subset W$. Here V and W are sub inner product spaces of a G-universe U. A G-spectrum is a G-prespectrum Y whose adjoint G-maps $Y(V)\longrightarrow \Omega^{W-V}Y(W)$ are homeomorphisms. The (Lewis-May) category $G\mathscr{S}p$ of G-spectra is the full subcategory of G-spectra in $G\mathscr{P}$. The suspension G-prespectrum functor Π sends a based G-space X to $\{X\wedge S^V\}$. There is a left adjoint spectrification functor $L\colon G\mathscr{P}\longrightarrow G\mathscr{S}p$, and the suspension G-spectrum functor $\Sigma^\infty_G\colon G\mathscr{T}\longrightarrow G\mathscr{S}p$ is $L\circ\Pi$. Explicitly, let

$$Q_GX = \operatorname{colim}\Omega^V\Sigma^V X,$$

where V runs over the finite dimensional subspaces of a complete G-universe U. Then the V^{th} G-space of $\Sigma^\infty_G X$ is $Q_G\Sigma^V X$.

All objects of $G\mathscr{S}p$ are fibrant, and the zeroth space functor $\Omega^\infty_G\colon G\mathscr{S}p\longrightarrow G\mathscr{T}$ is now homotopically meaningful. For a based G-CW complex X (with based attaching maps), $\Sigma^\infty_G X$ is cofibrant in $G\mathscr{S}p$. In particular, the sphere G-spectrum $S_G=\Sigma^\infty_G S^0$ is cofibrant. At least when G is a compact Lie group, the orbits

G/H are G-CW complexes, hence the $\Sigma_G^\infty(G/H_+)$ are cofibrant. However, $G\mathscr{S}p$ is not symmetric monoidal under the smash product. The implicit trade off here is intrinsic to the mathematics, as was explained by Lewis [10]; see [16] for a more recent discussion.

We summarize some constructions in [3] that work in exactly the same fashion equivariantly as nonequivariantly. We have the G-space $\mathscr{L}(j)$ of linear isometries $U^j \longrightarrow U$, with G acting by conjugation. These spaces form an E_∞ G-operad when U is complete. The G-monoid $\mathscr{L}(1)$ gives rise to a monad $\mathbb{L}$ on $G\mathscr{S}p$. Its algebras are called $\mathbb{L}$-spectra, and we have the category $G\mathscr{S}p[\mathbb{L}]$ of $\mathbb{L}$-spectra. It has a smash product $\wedge_{\mathscr{L}}$ which is associative and commutative but not unital. The action map $\xi\colon \mathbb{L}Y \longrightarrow Y$ of an $\mathbb{L}$-spectrum Y is a stable equivalence.

Suspension G-spectra are naturally $\mathbb{L}$-spectra. In particular, the sphere G-spectrum S_G is an $\mathbb{L}$-spectrum. There is a natural stable equivalence $\lambda\colon S_G \wedge_{\mathscr{L}} Y \longrightarrow Y$ for $\mathbb{L}$-spectra Y. The S_G-modules are those Y for which λ is an isomorphism, and they are the objects of $G\mathscr{Z}$. All suspension G-spectra are S_G-modules, and so are all $\mathbb{L}$-spectra of the form $S_G \wedge_{\mathscr{L}} Y$. The smash product $\wedge$ on S_G-modules is just the restriction of the smash product $\wedge_{\mathscr{L}}$, and it gives $G\mathscr{Z}$ its symmetric monoidal structure.

We have a sequence of Quillen left adjoints

$$G\mathscr{T} \xrightarrow{\Sigma_G^\infty} G\mathscr{S}p \xrightarrow{\mathbb{L}} G\mathscr{S}p[\mathbb{L}] \xrightarrow{\mathbb{J}} G\mathscr{Z},$$

where $\mathbb{L}X$ is the free $\mathbb{L}$-spectrum generated by a G-spectrum X and $\mathbb{J}Y = S_G \wedge_{\mathscr{L}} Y$ is the S_G-module generated by an $\mathbb{L}$-spectrum Y. We let $\mathbb{F} = \mathbb{J}\mathbb{L}$; then $\mathbb{L}$, $\mathbb{J}$, and $\mathbb{F}$ are Quillen equivalences. The composite $\gamma = \xi \circ \lambda\colon \mathbb{F}Y \longrightarrow Y$ is a stable equivalence for any $\mathbb{L}$-spectrum Y. We define $\mathbf{\Sigma_G^\infty}$ to be the composite functor $\mathbb{F}\Sigma_G^\infty$, and we have the natural stable equivalence of S_G-modules $\gamma\colon \mathbf{\Sigma_G^\infty} X \longrightarrow \Sigma_G^\infty X$.

The tensor $Y \odot X$ of a G-prespectrum and a based G-space X has V^{th} G-space $Y(V) \wedge X$. When Y is a G-spectrum, the G-spectrum $Y \odot X$ is $L(\ell Y \odot X)$, where ℓY is the underlying G-prespectrum of Y [11, I.3.1]. Tensors in $G\mathscr{S}p[\mathbb{L}]$ and $G\mathscr{Z}$ are inherited from those in $G\mathscr{S}p$. All of our left adjoints are enriched in $\mathscr{T}$ and preserve tensors. This leads to the following relationship between $\wedge$ and $\mathbf{\Sigma_G^\infty}$.

Proposition 3.9. *For based G-spaces X and Y, there are natural isomorphisms*

$$\mathbf{\Sigma_G^\infty} X \wedge \mathbf{\Sigma_G^\infty} Y \cong (\mathbf{S}_G^\infty \wedge \mathbf{S}_G^\infty) \odot (X \wedge Y) \cong \mathbf{S}_G^\infty \wedge \mathbf{\Sigma_G^\infty}(X \wedge Y).$$

Proof. We have $\Sigma_G^\infty X \cong S_G \odot X$ and therefore

$$\mathbf{\Sigma_G^\infty} X = \mathbb{F}\Sigma_G^\infty X \cong \mathbb{F}(S_G \odot X) \cong (\mathbb{F}S_G) \odot X = \mathbf{S}_G^\infty \odot X.$$

We also have

$$(\mathbf{S}_G^\infty \odot X) \wedge (\mathbf{S}_G^\infty \odot Y) \cong (\mathbf{S}_G^\infty \wedge \mathbf{S}_G^\infty) \odot (X \wedge Y)$$

and the conclusion follows. □

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E-mail address: `bertguillou@uky.edu`

Department of Mathematics, University of Kentucky, Lexington, KY 40506

E-mail address: `may@math.uchicago.edu`

Department of Mathematics, The University of Chicago, Chicago, IL 60637