REAL ORIENTATIONS OF MORAVA E-THEORIES

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Abstract. We show that Morava E-theories at the prime 2 are Real oriented and Real Landweber exact. The proof is an application of the Goerss–Hopkins–Miller theorem to algebras with involution. For each height n, we compute the entire homotopy fixed point spectral sequence for E_n with its C_2-action by the formal inverse. We study, as the height varies, the Hurewicz images of the stable homotopy groups of spheres in the homotopy of these C_2-fixed points.

Contents

1. Introduction

Let $\widehat{E(n)}$ denote a 2-periodic version of completed Johnson–Wilson theory, with

$$\pi_*(\widehat{E(n)}) = \mathbb{Z}_2[[v_1, v_2, ..., v_{n-1}]][u^{\pm}],\ |u| = 2.$$

This spectrum is a version of Morava E-theory. In particular, it is a complex-oriented and $\mathbb{E}_\infty$-ring spectrum. Work of Goerss, Hopkins, and Miller [GH04, Rez98] identifies the space of $\mathbb{E}_\infty$-ring automorphisms of $\widehat{E(n)}$, and in particular ensures the existence of a central Galois C_2-action by $\mathbb{E}_\infty$-ring maps. At the level of homotopy groups, C_2 acts as the formal inverse of the canonical formal group law.

There is also a natural C_2-action on MU, by complex-conjugation, and one can ask if these C_2-actions are compatible. To this end, the main theorem of our work is:

Theorem 1.1. *The spectrum $\widehat{E(n)}$, with its central Galois C_2-action, is* ***Real oriented****. That is to say, it receives a C_2-equivariant homotopy ring map*

$$MU_{\mathbb{R}} \longrightarrow \widehat{E(n)}$$

from the Real cobordism spectrum $MU_{\mathbb{R}}$.

Leveraging the Hill–Hopkins–Ravenel norm functor [HHR16], we use Theorem 1.1 to conclude:

Theorem 1.2. *Let k be a perfect field of characteristic 2, $\mathbb{G}$ a height n formal group over k, and $E_{(k,\mathbb{G})}$ the corresponding Morava E-theory. Furthermore, let G be a finite subgroup of the Morava stabilizer group of $\mathbb{E}_\infty$-automorphisms of $E_{(k,\mathbb{G})}$ that contains the central subgroup C_2. Then there is a G-equivariant homotopy ring map*

$$N_{C_2}^G MU_{\mathbb{R}} \longrightarrow E_{(k,\mathbb{G})}.$$

In the second half of our paper, we look toward computational applications of the above results. For simplicity, we write these sections using a specific Morava E-theory E_n that is defined via a lift of the height n Honda formal group law over $\mathbb{F}_{2^n}$. Its homotopy groups are

$$\pi_* E_n = W(\mathbb{F}_{2^n})[[u_1, u_2, \dots, u_{n-1}]][u^\pm].$$

Leveraging Hu and Kriz's computation of the homotopy fixed point spectral sequence for $MU_{\mathbb{R}}$ [HK01], we are able to compute the full $RO(C_2)$-graded homotopy fixed point spectral sequence for this Morava E theory:

Theorem 1.3. *The E_2-page of the $RO(C_2)$-graded homotopy fixed point spectral sequence of E_n is*

$$E_2^{s,t}(E_n^{hC_2}) = W(\mathbb{F}_{2^n})[[\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{n-1}]][\bar{u}^\pm] \otimes \mathbb{Z}[u_{2\sigma}^\pm, a_\sigma]/(2a_\sigma).$$

The classes $\bar{u}_1$, …, $\bar{u}_{n-1}$, $\bar{u}^\pm$, and a_σ are permanent cycles. All the differentials in the spectral sequence are determined by the differentials

$$\begin{aligned} d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) &= \bar{u}_k \bar{u}^{2^k-1} a_\sigma^{2^{k+1}-1}, \ 1 \le k \le n-1, \\ d_{2^{n+1}-1}(u_{2\sigma}^{2^{n-1}}) &= \bar{u}^{2^n-1} a_\sigma^{2^{n+1}-1}, \ k = n, \end{aligned}$$

and multiplicative structures.

As a corollary, we learn that as a C_2-spectrum, E_n is *strongly even* and *Real Landweber exact* in the sense of Hill–Meier [HM16].

Theorem 1.4. *E_n is strongly even and Real Landweber exact. More precisely, $\underline{\pi}_{k\rho-1} E\mathbb{R} = 0$ and $\underline{\pi}_{k\rho} E\mathbb{R}$ is a constant Mackey functor for all $k \in \mathbb{Z}$. The Real orientation $MU_{\mathbb{R}} \to E_n$ induces a map*

$$MU_{\mathbb{R}\bigstar}(X) \otimes_{MU_{2*}} (E_n)_{2*} \to E_{n\bigstar}(X)$$

which is an isomorphism for every C_2-spectrum X.

The second author's detection theorem for $MU_{\mathbb{R}}^{hC_2}$, joint with Li, Wang, and Xu [LSWX17] allows us to conclude a detection theorem for $E_n^{hC_2}$. Roughly speaking, as the height grows, an increasing amount of the Kervaire and $\bar{\kappa}$ families in the stable homotopy groups of spheres are detected by $\pi_* E_n^{hC_2}$. More precisely, we prove in Section 7 the following:

Theorem 1.5 (Detection theorem for $E_n^{hC_2}$)**.**

(1) *For $1 \le i, j \le n$, if the element $h_i \in Ext_{\mathcal{A}_*}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ or $h_j^2 \in Ext_{\mathcal{A}_*}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hC_2}$ is nonzero.*

(2) *For $1 \le k \le n-1$, if the element $g_k \in Ext_{\mathcal{A}_*}^{4,2^{k+2}+2^{k+3}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hC_2}$ is nonzero.*

This stands in contrast to the detecting spectrum $\Omega_{\mathbb{O}}$ of Hill–Hopkins–Ravenel, which detects the entire Kervaire family at once and is morally related to the C_8-fixed points of height 4 Morava E-theory [HHR16]. In Section 8, we make this connection more precise:

Theorem 1.6. *There is a commuting diagram*

$$\begin{tikzcd} BP^{((C_8))} \arrow[r] \arrow[d] & E_4 \\ D^{-1}BP^{((C_8))} \arrow[ur] & \end{tikzcd}$$

where $D^{-1}BP^{((C_8))}$ is the Hill–Hopkins–Ravenel detecting spectrum $\Omega_{\mathbb{O}}$.

Remark 1.7. We freely use the language of ∞-categories throughout this work, and will refer to an ∞-category simply as a category. If $\mathcal{C}$ is a symmetric monoidal category, we use $\mathbb{A}_\infty(\mathcal{C})$ to denote the category of associative algebra objects in $\mathcal{C}$, and similarly use $\mathbb{E}_\infty(\mathcal{C})$ to denote commutative algebra objects. We will use **Spaces** to denote the symmetric monoidal category of *pointed* spaces under cartesian product.

Strategy for the proof of Theorem 1.1: Before describing our proof of Theorem 1.1 it will be helpful to sketch a construction of $\widehat{E(n)}$ as a ring spectrum, not yet worrying about any C_2-actions. We describe this non-equivariant construction in detail in Section 2.

Recall that there is a periodic version of complex cobordism, denoted MUP, that is an $\mathbb{E}_\infty$-ring spectrum. We denote the symmetric monoidal category of MUP-module spectra by MUP-**Mod**. The subgroupoid spanned by the unit and its automorphisms is the space $BGL_1(MUP)$, which is naturally an infinite loop space. Associated to any map of spaces $f : X \to BGL_1(MUP)$ is a Thom MUP-module Thom(f) [ABG+14]. The category of spaces over $BGL_1(MUP)$ is symmetric monoidal, and an associative algebra object in this category gives rise to an $\mathbb{A}_\infty$-algebra structure on its Thom spectrum [ACB14].

Consider now the following diagram of categories:

$$(\star)\qquad \begin{tikzcd} \mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right) \arrow[r, "Thom"] & \mathbb{A}_\infty(\mathrm{MUP}\text{-}\mathbf{Mod}) \arrow[r, "Forget"] \arrow[d, "L_{K(n)}"] & \mathbb{A}_\infty(\mathbf{Spectra}) \arrow[d, "L_{K(n)}"] \\ \mathbf{Spaces}_{/B^2GL_1(MUP)} \arrow[u, "\Omega"] & \mathbb{A}_\infty(\mathrm{MUP}\text{-}\mathbf{Mod}) \arrow[r, "Forget"] & \mathbb{A}_\infty(\mathbf{Spectra}) \\ & & \mathbb{E}_\infty(\mathbf{Spectra}). \arrow[u, "Forget"'] \end{tikzcd}$$

In Section 2, we will construct a certain map of spaces $X \to B^2GL_1(MUP)$. Applying Ω and then the Thom spectrum construction, we obtain an $\mathbb{A}_\infty$-MUP-algebra $E(n)$ that is a 2-periodic version of Johnson–Wilson theory. The $K(n)$-localization of $E(n)$ is $\widehat{E(n)}$, equipped with the structure of an $\mathbb{A}_\infty$-MUP-algebra.

It is a consequence of work of Goerss, Hopkins, and Miller [GH04, Rez98] that we may lift the $\mathbb{A}_\infty$-ring spectrum underlying $\widehat{E(n)}$ to an $\mathbb{E}_\infty$-ring spectrum. Indeed, letting $\mathcal{C}^{\simeq}$ denote the maximal subgroupoid of a category $\mathcal{C}$, they prove that the path-component of $\mathbb{A}_\infty(\mathbf{Spectra})^{\simeq}$ containing $\widehat{E(n)}$ is *equivalent* to a path component in $\mathbb{E}_\infty(\mathbf{Spectra})^{\simeq}$, with the equivalence given by the forgetful functor.

Our strategy for the proof of Theorem 1.1 is to produce a Real orientation $MU_{\mathbb{R}} \to \overline{E(n)}$ into *some* ring spectrum $\overline{E(n)}$ with C_2-action. The $\overline{E(n)}$ we produce is obviously equivalent to $\widehat{E(n)}$ as a spectrum, and the C_2-action is obviously the Galois one up to homotopy. However, it is **not** at all obvious that the full, coherent C_2-action on $\overline{E(n)}$ is the Galois action. To prove it, we must make full use of the Goerss–Hopkins–Miller theorem.

We produce $\overline{E(n)}$ via a C_2-equivariant lift of the above construction of $\widehat{E(n)}$:

Construction 1.8. In section 3, each of the categories in the diagram $(\star)$ will be equipped with a C_2-action, yielding an *equivariant* diagram:

$$(\star\star)\qquad \begin{tikzcd}
\mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right) \arrow[loop above] \arrow[r, "Thom"] & \mathbb{A}_\infty(\mathrm{MUP}\text{-}\mathbf{Mod}) \arrow[loop above] \arrow[r, "Forget"] \arrow[d, "L_{K(n)}"] & \mathbb{A}_\infty(\mathbf{Spectra}) \arrow[loop above, "op"] \arrow[d, "L_{K(n)}"] \\
\mathbf{Spaces}_{/B^\rho GL_1(MUP)} \arrow[loop below] \arrow[u, "\Omega^\sigma"] & \mathbb{A}_\infty(\mathrm{MUP}\text{-}\mathbf{Mod}) \arrow[loop below] \arrow[r, "Forget"] & \mathbb{A}_\infty(\mathbf{Spectra}) \arrow[loop right, "\mathrm{op}"] \\
 & & \mathbb{E}_\infty(\mathbf{Spectra}). \arrow[loop below, "trivial"] \arrow[u, "Forget"]
\end{tikzcd}$$

The action on $\mathbb{E}_\infty(\mathbf{Spectra})$ will be the trivial C_2-action. The action on $\mathbb{A}_\infty(\mathbf{Spectra})$ will be the non-trivial op action that takes an algebra to its opposite.

Remark 1.9. By a homotopy fixed point in a category $\mathcal{C}$ with C_2-action we mean an object in the category $\mathcal{C}^{hC_2}$. For example, a homotopy fixed point in $\mathbb{E}_\infty(\mathbf{Spectra})$ with its trivial action is just an $\mathbb{E}_\infty$-ring spectrum with C_2-action by $\mathbb{E}_\infty$-ring maps. A homotopy fixed point for the op action on $\mathbb{A}_\infty(\mathbf{Spectra})$ is an $\mathbb{A}_\infty$-algebra A equipped with an *involution*, meaning a coherent algebra map $\sigma : A \to A^{\mathrm{op}}$. We believe the use of algebras with involution to be the most interesting feature of our construction.

In Section 4, we will refine our map $X \to B^2GL_1(MUP)$ to an equivariant map $X \to B^\rho GL_1(MUP)$. Applying Ω^σ produces a homotopy fixed point of $\mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right)$, which in turn equips $E(n)$ with an $\mathbb{A}_\infty$-involution. After $K(n)$-localizing, we obtain a C_2-action on $\widehat{E(n)}$ by $\mathbb{A}_\infty$-involutions. The Goerss–Hopkins–Miller Theorem [GH04, Rez98] proves that any such action on $\widehat{E(n)}$ may be lifted to one by $\mathbb{E}_\infty$-ring maps. Since Goerss, Hopkins, and Miller furthermore calculate the entire space of $\mathbb{E}_\infty$-ring automorphisms of $\widehat{E(n)}$, we may determine any $\mathbb{E}_\infty$-C_2-action on $\widehat{E(n)}$ by its effect on homotopy groups.

Acknowledgements: The authors would like to thank Jun-Hou Fung, Nitu Kitchloo, Guchuan Li, Vitaly Lorman, Lennart Meier, Denis Nardin, Doug Ravenel, Jay Shah, Guozhen Wang, Zhouli Xu, and Allen Yuan for helpful conversations. We also thank Hood Chatham for his comprehensive and easy to use spectral sequence package, which produced all of our diagrams. Lastly, and most importantly, both authors owe tremendous debts to Mike Hopkins, their PhD advisor, and to Mike Hill, both of whom offered crucial guidance at various stages of the project. The first author's work was supported by an NSF GRFP fellowship under Grant DGE-1144152.

2. Thom Spectra and Johnson–Wilson Theory

In this section we will describe a non-equivariant construction of $\widehat{E(n)}$, a Landweber exact Morava E-theory with

$$\pi_*(\widehat{E(n)}) \cong \mathbb{Z}_2[[v_1, v_2, \cdots, v_{n-1}]][u^\pm].$$

Our construction is a riff on Theorem 1.4 of [BSS16].

We begin with a brief review of the classical theory of Thom spectra. Useful references, in the language of ∞-categories we espouse here, include [ABG$^+$14] and [ACB14].

If R is an $\mathbb{E}_\infty$-ring spectrum, then the category of R-modules acquires a symmetric monoidal structure. The full subcategory consisting of the unit and its automorphisms is denoted $BGL_1(R)$. The symmetric monoidal structure equips $BGL_1(R)$ with an infinite loop space structure, and we write $BGL_1(R) \simeq \Omega^\infty \Sigma gl_1(R)$. The space $GL_1(R) \simeq \Omega^\infty gl_1(R)$ sits in a pullback square

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R), \end{array}$$

where $\pi_0(R)^\times$ is the subset of units of $\pi_0(R)$ under multiplication. From this latter description of $GL_1(R)$, it is clear that

$$\pi_*(BGL_1(R)) \cong \pi_{*-1}(GL_1(R)) \cong \pi_{*-1}(R), \text{ for } * > 1.$$

Given a map of spaces $X \to BGL_1(R)$, we can form the Thom R-module by taking the colimit of the composite functor $X \to BGL_1(R) \subset R\text{-}\mathbf{Mod}$. If X is a loop space and $X \to BGL_1(R)$ is a loop map, then the main theorem of [ACB14] shows that the associated Thom spectrum is an $\mathbb{A}_\infty$-R-algebra. Similarly, if X is an infinite loop space and $X \to BGL_1(R)$ an infinite loop map, then [ACB14] shows that the associated Thom spectrum is an $\mathbb{E}_\infty$-R-algebra.

Given two maps $f_1 : X_1 \to BGL_1(R)$ and $f_2 : X_2 \to BGL_1(R)$, we may use the infinite loop space structure on $BGL_1(R)$ to produce a product map

$$(f_1, f_2) : X_1 \times X_2 \to BGL_1(R) \times BGL_1(R) \to BGL_1(R).$$

The Thom R-module $\mathrm{Thom}(f_1, f_2)$ is the R-module smash product $\mathrm{Thom}(f_1) \wedge_R \mathrm{Thom}(f_2)$.

We may speak not only of $BGL_1(R)$, but also of the infinite loop space $\mathrm{Pic}(R)$. As a symmetric monoidal category, $\mathrm{Pic}(R)$ is the full subcategory of $R\text{-}\mathbf{Mod}^{\simeq}$ spanned by the invertible R-modules. It is a union of path components each of which is equivalent to $BGL_1(R)$. Again, [ACB14] explains that the colimit of an infinite loop map $X \to \mathrm{Pic(R)} \subset R\text{-}\mathbf{Mod}$ is an $\mathbb{E}_\infty$-R-algebra. Our only use of this more general construction is to recall the following classical example:

Example 2.1. The complex J-homomorphism is an infinite loop map $BU \times \mathbb{Z} \to \mathrm{Pic}(\mathbb{S})$, obtained via the algebraic K-theory construction on $\coprod BU(n) \to \mathrm{Pic}(\mathbb{S})$. The resulting Thom $\mathbb{E}_\infty$-ring spectrum is the periodic complex cobordism spectrum, denoted MUP. The 2-connective cover of spectra $bu \to ku$ is an infinite loop map $BU \to BU \times \mathbb{Z}$, which induces a map of Thom $\mathbb{E}_\infty$-ring spectra $MU \to MUP$.

The map $J : BU \times \mathbb{Z} \to \mathrm{Pic}(\mathbb{S})$ decomposes as a product of the infinite loop map $BU \to \mathrm{BGL}_1(\mathbb{S})$ and the loop map $\mathbb{Z} \to \mathrm{Pic}(\mathbb{S})$. This yields an equivalence of Thom $\mathbb{A}_\infty$-ring spectra

$$MUP \simeq MU \wedge \left(\bigvee_{n \in \mathbb{Z}} S^{2n} \right) \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU,$$

which allows us to calculate $\pi_*(MUP) \cong \pi_*(MU)[u^\pm] \cong \mathbb{Z}[x_1, x_2, \cdots][u^\pm]$, where $|u| = 2$ and $|x_i| = 2i$. The complex-conjugation action on $BU \times \mathbb{Z}$ by infinite loop maps yields a C_2-action on MUP by $\mathbb{E}_\infty$-ring homomorphisms; we will make no use of this action in the current section, but much use of it in Sections 3 and 4.

We now specialize the discussion and embark on our construction of $E(n)$. Suppose that we choose a non-zero $\alpha \in \pi_2(MUP) \cong \pi_3(BGL_1(MUP))$. Then, e.g. by [BSS16, Theorem 5.6] or [ACB14, Theorem 4.10], there is an equivalence of MUP-module spectra

$$\mathrm{Thom}(\alpha) \simeq \mathrm{Cofiber}(\Sigma^2 MUP \xrightarrow{\alpha} MUP) \simeq MUP/\alpha.$$

If we choose a sequence of elements $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \pi_2(MUP)$, we may produce a map

$$S^3 \times S^3 \times \cdots S^3 \to BGL_1(MUP)$$

and an associated Thom MUP-module

$$\begin{aligned} \mathrm{Thom}(\alpha_1, \alpha_2, \cdots, \alpha_n) &\simeq (MUP/\alpha_1) \wedge_{MUP} (MUP/\alpha_2) \wedge_{MUP} \cdots \wedge_{MUP} (MUP/\alpha_n) \\ &\simeq MUP/(\alpha_1, \alpha_2, \cdots, \alpha_n). \end{aligned}$$

If the sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is *regular* in $\pi_*(MUP)$, then the usual cofiber sequences imply that

$$\pi_*(MUP/(\alpha_1, \alpha_2, \cdots, \alpha_n)) \cong \pi_*(MUP)/(\alpha_1, \alpha_2, \cdots, \alpha_n).$$

Finally, we may even mod out an infinite regular sequence $(\alpha_1, \alpha_2, \cdots)$ by using the natural maps

$$S^3 \to S^3 \times S^3 \to S^3 \times S^3 \times S^3 \to \cdots$$

to produce a filtered colimit of MUP-modules

$$MUP/\alpha_1 \to MUP/(\alpha_1, \alpha_2) \to MUP/(\alpha_1, \alpha_2, \alpha_3) \to \cdots \to MUP/(\alpha_1, \alpha_2, \cdots).$$

Proposition 2.2. *Each map $\alpha_i : S^3 \to BGL_1(MUP)$ can be given the structure of a loop map. In other words, the above construction of the MUP-module $MUP/(\alpha_1, \alpha_2, \cdots)$ can be refined to a construction of an $\mathbb{A}_\infty$-MUP-algebra.*

Proof. It will suffice to construct a map $\widetilde{\alpha_i} : BS^3 \to B^2GL_1(MUP)$ such that $\Omega\widetilde{\alpha_i} \simeq \alpha_i$. This is equivalent to asking that the precomposition of the map $\widetilde{\alpha_i} : BS^3 \to B^2GL_1(MUP)$ with the inclusion $S^4 \to BS^3$ be adjoint to the map $\alpha_i : S^3 \to BGL_1(MUP)$. In fact, any map $S^4 \to B^2GL_1(MUP)$ automatically admits at least one factorization through BS^3. The reason is that BS^3 admits an even cell decomposition: there is a filtered colimit

$$S^4 = Y_1 \to Y_2 \to Y_3 \to \cdots \to BS^3$$

and pushouts

$$\begin{tikzcd} S^{4n-1} \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ D^{4n} \arrow[r] & Y_n. \end{tikzcd}$$

This cell decomposition is easily seen from the model $BS^3 \simeq \mathbb{HP}^\infty$, the infinite dimensional quaternionic projective space, where it is the canonical cell-decomposition corresponding to the inclusions of the $\mathbb{HP}^\ell$. The obstructions to factoring a map $Y_{n-1} \to B^2GL_1(MUP)$ through Y_n therefore live in $\pi_{4n-1}(MUP)$. This group is 0, as explained in Example 2.1. $\square$

To summarize, if we choose any regular sequence $(\alpha_1, \alpha_2, \cdots) \in \pi_*(MUP) \cong \mathbb{Z}[x_1, x_2, \cdots][u^\pm]$, each element of which lies in degree 2, then we may construct the quotient MUP-module $MUP/(\alpha_1, \alpha_2, \cdots)$ as an $\mathbb{A}_\infty$-MUP-algebra. The following standard lemma allows us to use Proposition 2.2 to build Morava E-theories as $\mathbb{A}_\infty$ algebras:

Lemma 2.3. *Let $\mathbb{G}$ denote a formal group of height n over the field $\mathbb{F}_2$, and E the associated Morava E-theory. Then there is a map $MUP \to E$, classifying a universal deformation of $\mathbb{G}$, which may be described as first taking the quotient of MUP by a regular sequence $(\alpha_1, \alpha_2, \cdots)$ of degree 2 classes and then performing $K(n)$-localization.*

Remark 2.4. If the reader prefers, they will lose no intuition by thinking of the regular sequence

$$(\alpha_1, \alpha_2, \cdots) = (x_{2^n-1}u^{2-2^n} - u, x_2u^{-1}, x_4u^{-3}, x_5u^{-4}, x_6u^{-5}, x_8u^{-7}, \cdots),$$

where the classes x_iu^{-i+1} that are included are those such that either

- i is not one less than a power of 2, or

- i is larger that $2^n - 1$.

However, since there are non-isomorphic formal groups over $\mathbb{F}_2$, not every Morava E-theory is obtained by quotienting out this particular sequence.

Definition 2.5. We denote by $E(n)$ the quotient of MUP by the regular sequence $(\alpha_1, \alpha_2, \cdots)$ of Lemma 2.3, and say that $E(n)$ is a 2-periodic form of Johnson–Wilson theory. Proposition 2.2 provides a (not necessarily unique) construction of $E(n)$ as an $\mathbb{A}_\infty$ MUP-algebra. We are deliberately vague about which formal group $\mathbb{G}$ defines $E(n)$, so that we may handle all cases at once.

Proof of Lemma 2.3. The formal group $\mathbb{G}$ is classified by some map of (ungraded) rings $\pi_*(BP) \to \mathbb{F}_2$. View this map as the solid arrow in the diagram of ring homomorphisms

$$\begin{tikzcd}
\pi_*(BP) \arrow[r, dashed, "L_2"] \arrow[dr, dashed, "L_1"] \arrow[ddr] & W(k)[[u_1, u_2, \cdots, u_{n-1}]] \arrow[d] \\
& \mathbb{F}_2[u_1, u_2, \cdots, u_{n-1}]/\mathfrak{m}^2 \arrow[d] \\
& \mathbb{F}_2.
\end{tikzcd}$$

According to [Rez98, §5.10], as long as the lift L_1 is chosen correctly, any further lift L_2 will classify a universal deformation. Furthermore, we may assume that v_i maps to u_i for $i \leq n-1$ under L_1, while v_n maps to 1. Each v_j for $j > n$ then maps to some $L_1(v_j)$ that is an $\mathbb{F}_2$-linear combination of $L_1(v_1), L_1(v_2), \cdots L_1(v_n)$. Write $\phi(v_j)$ to denote the element of $\pi_*(BP)$ that is given by the same linear combination of $v_1, v_2, \cdots v_n$. Then the map L_2 can be chosen to be the quotient by the regular sequence $(v_n - 1, v_{n+1} - \phi(v_{n+1}), v_{n+2} - \phi(v_{n+2}), \cdots)$.

Using the invertible element u to move elements by even degrees, we may identify $\pi_2(MUP)$ with $\pi_*(MU)$. Inside of $\pi_*(MU)$ we identify $\pi_*(BP)$ by viewing v_i as x_{2^i-1}. This allows us to talk about $\phi(v_j)$ as a class in $\pi_2(MUP)$.

To obtain the lemma, one mods out by the regular sequence $(\alpha_1, \alpha_2, \cdots) \in \pi_2(MUP)$, where one mods out, in order of $i \in \mathbb{N}$:

- All $x_i u^{-i+1}$ with i not one less than a power of 2.
- The class $x_i u^{-i+1} - u$ where $i = 2^n - 1$.
- The classes $x_i u^{-i+1} - \phi(v_j)$ where $i = 2^j - 1$ for $j > n$.

$\square$

3. Categories with Involution and Construction 1.8

In this section, we will construct a diagram of categories with C_2-action and equivariant functors between them:

$$
(\star\star)\qquad
\begin{tikzcd}
\mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right) \arrow[loop above] \arrow[r, "Thom"] & \mathbb{A}_\infty(\mathrm{MUP}\text{-}\mathbf{Mod}) \arrow[loop above] \arrow[r, "Forget"] \arrow[d, "L_{K(n)}"] & \mathbb{A}_\infty(\mathbf{Spectra}) \arrow[loop above, "op"] \arrow[d, "L_{K(n)}"] \\
\mathbf{Spaces}_{/B^\rho GL_1(MUP)} \arrow[loop below] \arrow[u, "\Omega^\sigma"] & \mathbb{A}_\infty(\mathrm{MUP}\text{-}\mathbf{Mod}) \arrow[loop below] \arrow[r, "Forget"] & \mathbb{A}_\infty(\mathbf{Spectra}) \arrow[loop right, "\mathrm{op}"] \\
 & & \mathbb{E}_\infty(\mathbf{Spectra}). \arrow[loop below, "trivial"] \arrow[u, "Forget"]
\end{tikzcd}
$$

Remark 3.1. An equivariant functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between categories with C_2-action is an arrow in the functor category $\mathrm{Hom}(BC_2, \mathbf{Cat}_\infty)$. Such an arrow contains a substantial amount of data, and is in particular *not* determined by its underlying functor $F_{\text{underlying}}$ of non-equivariant categories. For example, using

$$\sigma_i : \mathcal{C}_i \to \mathcal{C}_i$$

to denote the C_2-action on $\mathcal{C}_i$, part of the data of F is a choice of natural isomorphism

$$\sigma_2 \circ F_{\text{underlying}} \simeq F_{\text{underlying}} \circ \sigma_1.$$

Nonetheless, for notational convenience we will be somewhat fast and loose regarding the distinction between F and $F_{\text{underlying}}$.

Let $\mathbf{MonCat}_{Lax}$ denote the category of monoidal categories and lax monoidal functors. In the language of [Lur16, §4.1], this is the category of coCartesian fibrations of ∞-operads $\mathcal{C}^\otimes \to \mathcal{A}\mathrm{ss}^\otimes$, with morphisms maps of ∞-operads $\mathcal{C}_1 \to \mathcal{C}_2$ over $\mathcal{A}\mathrm{ss}^\otimes$ that are not required to preserve coCartesian arrows. Remark 4.1.1.8 in [Lur16] constructs a C_2-action *rev* on $\mathbf{MonCat}_{Lax}$. If $(\mathcal{C}, \otimes)$ is a monoidal ∞-category, then $(\mathcal{C}_{rev}, \otimes_{rev})$ has the same underlying category as $\mathcal{C}$ but the *opposite* $\otimes$-structure, with $X \otimes_{rev} Y$ in $\mathcal{C}_{rev}$ calculated as $Y \otimes X$ in $\mathcal{C}$. We call a homotopy fixed point for this rev action a monoidal category $(\mathcal{C}, \otimes)$ *with involution.* Such a category is equipped with a coherent equivalence $\mathcal{C} \xrightarrow{\simeq} \mathcal{C}_{rev}$.

Remark 4.1.1.8 also constructs an equivalence between $\mathbb{A}_\infty$-algebra objects A in $\mathcal{C}$ and $\mathbb{A}_\infty$-algebra objects A^{rev} in $\mathcal{C}_{rev}$. If $\mathcal{C}$ is equipped with an involution, then there is an induced C_2-action on $\mathbb{A}_\infty(\mathcal{C})$. In other words, there is an equivariant functor

$$
\begin{tikzcd}
\mathbf{MonCat}_{Lax} \arrow[loop above, "rev"] \arrow[r, "\mathbb{A}_\infty(-)"] & \mathbf{Cat}_\infty, \arrow[loop above, "trivial"]
\end{tikzcd}
$$

and so a homotopy fixed point in $\mathbf{MonCat}_{Lax}$ is sent to one in $\mathbf{Cat}_\infty$.

Finally, we also consider the category $\mathbf{SymMonCat}_{Lax}$ of symmetric monoidal categories and lax functors. The last paragraph of Remark 4.1.1.8 of [Lur16] ensures that the sequence of forgetful functors

$$\mathbf{SymMonCat}_{Lax} \xrightarrow{\text{Forget}} \mathbf{MonCat}_{Lax} \xrightarrow{\text{Forget}} \mathbf{Cat}_\infty$$

is equivariant, with the trivial C_2-action on $\mathbf{SymMonCat}_{Lax}$, the rev action on $\mathbf{MonCat}_{Lax}$, and the trivial action on $\mathbf{Cat}_\infty$.

Example 3.2. Consider the category **Set** of sets, equipped with the cartesian symmetric monoidal structure. The trivial C_2-action on **Set** by symmetric monoidal identity functors allows us to view **Set** as a homotopy fixed point for the trivial C_2-action on $\mathbf{SymMonCat}_{Lax}$. This equips the underlying monoidal category of **Set** with a canonical involution, which in turn equips the category of monoids with a C_2-action. This is the classical op action that takes a monoid M to its opposite monoid M^{op}, which has the same underlying set but the opposite multiplication.

Example 3.3. More generally, if $\mathcal{C}$ is any symmetric monoidal category, then the trivial action on $\mathcal{C}$ by symmetric monoidal identity functors induces an involution, and therefore an op action on $\mathbb{A}_\infty(\mathcal{C})$. There is an equivariant sequence of categories

$$\mathbb{E}_\infty(\mathcal{C}) \xrightarrow{Forget} \mathbb{A}_\infty(\mathcal{C}) \xrightarrow{Forget} \mathcal{C},$$

where $\mathbb{E}_\infty(\mathcal{C})$ and $\mathcal{C}$ are given the trivial C_2-actions and $\mathbb{A}_\infty(\mathcal{C})$ is given the op action.

Taking $\mathcal{C}$ in Example 3.3 to be the category **Spaces** of pointed spaces, we obtain the op action on $\mathbb{A}_\infty(\mathbf{Spaces})$. We call a homotopy fixed point for this action an $\mathbb{A}_\infty$-space *with involution*; such spaces, considered as groupoids, are special cases of categories with involution. Any spectrum E with C_2-action has an underlying $\mathbb{A}_\infty$-space with involution $\Omega^\infty E$.

Example 3.4. Suppose that X is an $\mathbb{A}_\infty$-space with involution. Then the monoidal category $\mathbf{Spaces}_{/X}$ is equipped with an involution. Concretely, this involution takes an algebra map $A \to X$ to the natural algebra map $A^{\mathrm{op}} \to X^{\mathrm{op}} \xrightarrow{\mathrm{inv}} X$.

Remark 3.5. If a monoid M happens to be a group, then there is a canonical equivalence $M \simeq M^{\mathrm{op}}$ defined by the inverse homomorphism $m \mapsto m^{-1}$. Our next few observations exploit an analogue of this equivalence for *grouplike* $\mathbb{A}_\infty$-spaces. We denote by **LoopSpaces** the full subcategory of grouplike objects in $\mathbb{A}_\infty(\mathbf{Spaces})$. Notice that the property of being grouplike is preserved under the op action on $\mathbb{A}_\infty(\mathbf{Spaces})$, so there is an op action on **LoopSpaces**.

Construction 3.6. There is a diagram of *equivalences* of equivariant categories

$$\begin{tikzcd}
& \mathbf{ConnectedSpaces} \arrow[loop above, "trivial"] \arrow[dl, "\Omega^\sigma"'] \arrow[dr, "\Omega"] & \\
\mathbf{LoopSpaces} \arrow[loop below, "\mathrm{op}"] & & \mathbf{LoopSpaces} \arrow[loop below, "\mathrm{trivial}"]
\end{tikzcd}$$

The equivariant functors Ω and Ω^σ share the same underlying, non-equivariant functor.

Proof. It is classical that Ω and the bar construction provide inverse equivalences of the non-equivariant categories **ConnectedSpaces** and **LoopSpaces**. This category has a universal property: it is the initial pointed category with all connected colimits. As such, any C_2-action on it admits an essentially unique equivalence with the trivial C_2-action. □

Corollary 3.7. *Suppose X is a grouplike $\mathbb{A}_\infty$-space with involution. Then there exists some connected space with C_2-action $B^\sigma X$ such that $\Omega^\sigma B^\sigma X \simeq X$. There is a natural C_2-equivariant functor*

$$\boldsymbol{Spaces}_{/B^\sigma X} \xrightarrow{\Omega^\sigma} \mathbb{A}_\infty\left(\boldsymbol{Spaces}_{/X}\right),$$

where the latter object is the category with C_2-action underlying the monoidal category with involution from Example 3.4.

Consider the sequence of right adjoints

$$\mathbf{Spaces} \xrightarrow{(-)_0} \mathbf{ConnectedSpaces} \xrightarrow{\Omega^\sigma} \mathbb{A}_\infty(\mathbf{Spaces}) \xrightarrow{Forget} \mathbf{Spaces}$$

By Example 3.3, this is an equivariant functor from **Spaces** with trivial action to **Spaces** with trivial action. As such it sends any space with C_2-action X to some other space with C_2-action, which by abuse of notation we denote $\Omega^\sigma X$.

Proposition 3.8. *Suppose X is a space with C_2-action. Then the space with C_2-action $\Omega^\sigma X$ is the equivariant function space $Hom(S^\sigma, X)$. In other words, the action on a loop $S^1 \to X$ is given by both precomposing with the antipode on S^1 and postcomposing with the action on X.*

Proof. By the Yoneda lemma, $\Omega^\sigma X$ must be the equivariant function spectrum $\mathrm{Hom}(S^1, X)$ for *some* C_2-action on S^1. To determine that this action is by the antipode, and not the trivial action, we look at the sequence of equivariant functors

$$\mathbf{Groups} \xrightarrow{Bar} \mathbf{ConnectedSpaces} \xrightarrow{\Omega^\sigma} \mathbf{LoopSpaces} \xrightarrow{\pi_0} \mathbf{Groups},$$

which connects groups with trivial action to groups with op action. The only *natural* isomorphism between a group and its opposite is given by $g \mapsto g^{-1}$, which is non-trivial on underlying sets. □

Specializing the discussion, recall that MUP is the Thom spectrum of the J homomorphism $BU \times \mathbb{Z} \xrightarrow{J} BGL_1(\mathbb{S})$. Since the J homomorphism is an infinite loop map, MUP acquires the structure of an $\mathbb{E}_\infty$-ring spectrum. The complex-conjugation action by infinite loop maps on $BU \times \mathbb{Z}$ gives MUP a C_2-action by $\mathbb{E}_\infty$-ring maps. This in turn induces C_2-actions by symmetric monoidal functors on the categories MUP-**Mod** and $\mathbf{Spaces}_{/BGL_1(MUP)}$. There is a diagram of lax symmetric monoidal functors

$$\begin{tikzcd}
\mathbf{Spaces}_{/BGL_1(MUP)} \arrow[r, "Thom"] & \text{MUP-}\mathbf{Mod} \arrow[r, "Forget"] \arrow[d, "L_{K(n)}"] & \mathbf{Spectra} \arrow[d, "L_{K(n)}"] \\
& \text{MUP-}\mathbf{Mod} \arrow[r, "Forget"] & \mathbf{Spectra}.
\end{tikzcd}$$

Since the C_2-action on MUP is unital, the entire diagram becomes C_2-equivariant once we equip **Spectra** with the trivial C_2-action. We may thus view the diagram as one of morphisms in the category of homotopy fixed points of $\mathbf{SymMonCat}_{Lax}$ with trivial action. This in turn induces a diagram in the homotopy fixed point category of $\mathbf{MonCat}_{Lax}$ with rev action, which yields a diagram of C_2-equivariant categories

$$\begin{tikzcd}
\mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right) \arrow[r, "Thom"] & \mathbb{A}_\infty(\text{MUP-}\mathbf{Mod}) \arrow[r, "Forget"] \arrow[d, "L_{K(n)}"] & \mathbb{A}_\infty(\mathbf{Spectra}) \arrow[d, "L_{K(n)}"] \\
& \mathbb{A}_\infty(\text{MUP-}\mathbf{Mod}) \arrow[r, "Forget"] & \mathbb{A}_\infty(\mathbf{Spectra}).
\end{tikzcd}$$

This is nearly all of our diagram ($\star\star$). To complete the diagram, we use Corollary 3.7 for $X \simeq BGL_1(MUP)$ and Example 3.3 for $\mathcal{C}$ the symmetric monoidal category of **Spectra**.

Remark 3.9. In the sequel, we will denote the space with C_2-action $B^\sigma BGL_1(MUP)$ by $B^\rho GL_1(MUP)$.

4. An Equivariant Map to $B^\rho GL_1(MUP)$

The previous Section 3 constructs a C_2-equivariant functor

$$\mathbf{Spaces}_{/B^\rho GL_1(MUP)} \to \mathbb{A}_\infty(\mathbf{Spectra}),$$

which remains C_2-equivariant after composing with $K(n)$-localization. Here, $\mathbf{Spaces}_{/B^\rho GL_1(MUP)}$ is granted its C_2-action via the one on the space $B^\rho GL_1(MUP) = B^\rho GL_1(MU_{\mathbb{R}}P)$. The category $\mathbb{A}_\infty(\mathbf{Spectra})$ is equipped with the op action of Example 3.3.

A homotopy fixed point for $\mathbf{Spaces}_{/B^\rho GL_1(MUP)}$ is just a map of spaces with C_2-action $X \to B^\rho GL_1(MUP)$, and such a map therefore gives rise to a homotopy fixed point of $\mathbb{A}_\infty(\mathbf{Spectra})$. In other words, an equivariant map of spaces with C_2-action $f : X \to B^\rho GL_1(MUP)$ gives rise to a Thom $\mathbb{A}_\infty$-algebra with involution $(\Omega^\sigma X)^{\Omega^\sigma f}$.

One can apply the construction to $* \to B^\rho GL_1(MUP)$ to obtain MUP itself as an $\mathbb{A}_\infty$-algebra with involution. Using the equivariant map $* \to X$, we obtain a map of $\mathbb{A}_\infty$-rings with involution $MUP \to (\Omega^\sigma X)^{\Omega^\sigma f}$. The canonical Real orientation

$$\Sigma^{-2}\mathbb{CP}^\infty \to MU_{\mathbb{R}} \to MUP$$

then equips $(\Omega^\sigma X)^{\Omega^\sigma f}$ with a Real orientation. If $(\Omega^\sigma X)^{\Omega^\sigma f}$ happens to also be a C_2-equivariant homotopy commutative ring, then [HK01, Theorem 2.25] implies that it receives an equivariant homotopy commutative ring map from $MU_{\mathbb{R}}$.

In this section we will be concerned with the construction of a particular map of spaces with C_2-action into $B^\rho GL_1(MUP)$; the underlying map of spaces will be the morphism

$$BS^3 \times BS^3 \times \cdots \to B^2 GL_1(MUP)$$

constructed in Section 2. Our aim is to construct both 2-periodic Johnson–Wilson theory and Morava E-theory as $\mathbb{A}_\infty$-rings with involution.

Remark 4.1. Recall that, among spaces with C_2-action, we may identify certain representation spheres $S^{a+b\sigma}$ as the one-point compactifications of real C_2-representations. We use σ to denote the sign representation, 1 to denote the trivial representation, and the shorthand ρ to denote the regular representation $1+\sigma$. If X is a space or spectrum with C_2-action, then we use $\pi_{a+b\sigma}(X)$ to denote π_0 of the space of equivariant maps $S^{a+b\sigma} \to X$. Of interest to us, Proposition 3.8 implies that $\pi_{a+b\sigma}(B^\sigma X) \cong \pi_{a+(b-1)\sigma}(X)$.

In [HK01], the equivariant homotopy groups of $MUP = MU_{\mathbb{R}}P$ are computed. For each n, $\pi_{n\rho-1}(MUP) \cong 0$. Additionally, there is a ring isomorphism

$$\pi_{*\rho}(MUP) \cong \mathbb{Z}[\bar{r}_1, \bar{r}_2, \cdots][\bar{u}^{\pm}],$$

where $\bar{r}_i$ is in degree $i\rho$ and $\bar{u}$ is in degree ρ. The forgetful map from the equivariant to ordinary homotopy groups $\pi_{*\rho}(MUP) \to \pi_{2*}(MUP)$ takes $\bar{r}_i$ to x_i and $\bar{u}$ to u.

Since $GL_1(MUP)$ is defined via a pullback square of spaces with C_2-action

$$\begin{array}{ccc} GL_1(MUP) & \longrightarrow & \Omega^\infty MUP \\ \downarrow & & \downarrow \\ \pi_0(MUP)^\times & \longrightarrow & \pi_0(MUP), \end{array}$$

we learn that $\pi_{a+b\sigma}(B^\rho GL_1(MUP)) \cong \pi_{(a-1)+(b-1)\sigma}(MUP)$ whenever $a, b > 1$.

Our next task is to understand the C_2-equivariant space $B^\sigma S^{\rho+1}$. The analogue of the even cell structure that played a prominent role in Section 2 is the following:

Proposition 4.2. *The equivariant space $B^\sigma S^{\rho+1}$ arises as a filtered colimit*
$$Y_1 = S^{2\rho} \to Y_2 \to Y_3 \to \cdots,$$
where there are homotopy pushout square of spaces with C_2-action
$$\begin{tikzcd} S^{2n\rho-1} \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\ * \arrow[r] & Y_n. \end{tikzcd}$$

Proof. This cell decomposition is due to Mike Hopkins. Recall that, non-equivariantly, the cellular filtration on BS^3 agrees with the standard filtration
$$\mathbb{HP}^1 \to \mathbb{HP}^2 \to \cdots \to \mathbb{HP}^\infty \simeq BS^3,$$
where $\mathbb{HP}^\infty$ is the infinite-dimensional quaternionic projective space. For us, the relevant C_2-action on this space is by conjugation by i. In other words, we act on a point
$$[z_0 : z_1 : z_2 : \cdots]$$
by sending it to
$$[iz_0i^{-1} : iz_1i^{-1} : \cdots].$$
From the expression $i(a + bi + cj + dk)i^{-1} = a + bi - cj - dk$ we learn both that the action is well-defined and that the C_2-cells attached are multiples of 2ρ. The non-equivariant map $S^4 \to BS^3 \simeq \mathbb{HP}^\infty$ adjoint to the identity is lifted to an equivariant map $S^{2\rho} \to \mathbb{HP}^\infty$, given by the inclusion $\mathbb{HP}^1 \to \mathbb{HP}^\infty$ under the described C_2-action. This shows in particular that $\mathbb{HP}^\infty \simeq B^\sigma S^{\rho+1}$. □

As a corollary, exactly as in Section 2, we learn that any map $S^{2\rho} \to B^\rho GL_1(MUP)$ factors through $B^\sigma S^{\rho+1}$. Using the symmetric monoidal structure on $\mathbf{Spaces}_{/B^\rho GL_1(MUP)}$, which commutes with the C_2-action, we may construct from any sequence $(\alpha_1, \alpha_2, \cdots) \in \pi_\rho MUP$ a map
$$S^{\rho+1} \times S^{\rho+1} \times \cdots \to BGL_1(MUP).$$
This then factors through at least one equivariant map
$$B^\sigma S^{\rho+1} \times B^\sigma S^{\rho+1} \times \cdots \to B^\rho GL_1(MUP).$$

We choose for $(\alpha_0, \alpha_1, \cdots)$ the same sequence as in Lemma 2.3, with the x_i replaced by $\bar{r}_i$. The reader may prefer to consider the special case in which the sequence is
$$(\bar{r}_{2^n-1}\bar{u}^{2-2^n} - \bar{u}, \bar{r}_2\bar{u}^{-1}, \bar{r}_4\bar{u}^{-3}, \bar{r}_5\bar{u}^{-4}, \bar{r}_6\bar{u}^{-5}, \bar{r}_8\bar{u}^{-7}, \cdots),$$
where the classes $\bar{r}_i\bar{u}^{-i+1}$ that are included in the sequence are all those such that either

- i is not one less than a power of 2.
- i is greater than $2^n - 1$.

In any case, applying Ω^σ and then the Thom construction we obtain a homotopy fixed point of the category $\mathbb{A}_\infty(MUP - \mathbf{Mod})$. The underlying $\mathbb{A}_\infty$-ring is $E(n)$, the 2-periodic version of Johnson–Wilson theory constructed in Section 2. Our constructions produce a coherent $\mathbb{A}_\infty$-ring map $E(n) \xrightarrow{\simeq} E(n)^{\mathrm{op}}$ lifting the complex-conjugation C_2-action $E(n) \longrightarrow E(n)$. We denote this ring with involution by $\mathbb{E}(n)$.

Remark 4.3. From this work, it seems that the natural action on $E(n)$ is by $\mathbb{A}_\infty$-involutions rather than $\mathbb{A}_\infty$-algebra maps. However, we can sketch an approach to producing an action by $\mathbb{A}_\infty$-algebra maps in the spirit of this paper.

Using Theorem 1.4 of [BSS16], $E(n)$ can be built as a Thom spectrum of a map $SU \to BGL_1(MUP)$. Obstruction theory easily lifts this to a map $BSU \to B^2GL_1(MUP)$, which produces the same involution we see above. We may go further though, and note that B^3SU also has an even cell structure. This means that it is easy to produce maps $B^3SU \to B^4GL_1(MUP)$, but as noted in [CM15, §6] it is not so easy to know which maps $BSU \to B^2GL_1(MUP)$ these lie over. If one could produce $E(n)$ as a Thom $\mathbb{E}_3$-MUP-algebra in this way, non-equivariantly, it seems likely that one could produce an $\mathbb{E}_{2\sigma+1}$-structure on the equivariant $E(n)$. In particular, this would mean the C_2-action on $E(n)$ is by $\mathbb{A}_\infty$-ring homomorphisms.

This may be of interest in light of [KLW17], in which Kitchloo, Lorman, and Wilson provide a homotopy commutative and associative ring structure *up to phantom maps* on Real Johnson-Wilson theory. We thank Kitchloo for pointing out to us that the difficulty with phantom maps disappears after $K(n)$-localization.

5. Proofs of Theorems 1.1 and 1.2

In the previous section, we constructed an $\mathbb{A}_\infty$-ring spectrum $E(n)$ with a C_2-action by $\mathbb{A}_\infty$-involutions. After $K(n)$-localizing, we obtain a C_2-action by involutions on Morava E-theory $\widehat{E(n)}$.

Now, consider the equivariant sequence of forgetful functors

$$\mathbb{E}_\infty(\mathbf{Spectra}) \to \mathbb{A}_\infty(\mathbf{Spectra}) \to \mathbf{Spectra},$$

where both $\mathbb{E}_\infty(\mathbf{Spectra})$ and $\mathbf{Spectra}$ are given the trivial C_2-action, but $\mathbb{A}_\infty(\mathbf{Spectra})$ is given the op action. We may restrict this sequence to an equivariant sequence of subcategories

$$\mathcal{C}_3 \to \mathcal{C}_2 \to \mathcal{C}_1,$$

where

- $\mathcal{C}_1$ is the category of all spectra equivalent to $\widehat{E(n)}$ and equivalences between them.
- $\mathcal{C}_2$ is the category of $\mathbb{A}_\infty$-ring spectra with underlying spectrum $\widehat{E(n)}$, and equivalences between them.
- $\mathcal{C}_3$ is the category of $\mathbb{E}_\infty$-ring spectra with underlying spectrum $\widehat{E(n)}$, and equivalences between them.

Note that a map of categories with C_2-action is equivalence if and only if the underlying non-equivariant functor is an equivalence of non-equivariant categories. The Goerss–Hopkins–Miller theorem [GH04, Rez98] says that the map $\mathcal{C}_3 \to \mathcal{C}_2$ is an equivalence of categories. It follows that any homotopy fixed point of $\mathcal{C}_2$ may uniquely be lifted to one of $\mathcal{C}_3$. Thus, the C_2-action on $\widehat{E(n)}$ by $\mathbb{A}_\infty$-involutions has a unique lift to a C_2-action by $\mathbb{E}_\infty$-ring automorphisms. According to Goerss–Hopkins–Miller [Rez98], the categories $\mathcal{C}_3$ and $\mathcal{C}_2$ are equivalent to $B\mathbb{G}$, where $\mathbb{G}$ is the Morava stabilizer group. A C_2-action on $\widehat{E(n)}$ by $\mathbb{E}_\infty$-ring maps is therefore the data of a map $BC_2 \to B\mathbb{G}$, which is the data of a group homomorphism $C_2 \to \mathbb{G}$. It follows by direct calculation that any C_2-action by $\mathbb{E}_\infty$-ring maps is determined by its effect on homotopy groups. The Real orientation $MU_\mathbb{R} \to \mathbb{E}(n) \to \widehat{E(n)}$ determines that the C_2-action we have constructed is the central one that acts by the formal inverse, proving Theorem 1.1.

Remark 5.1. Our discussion of algebras with involution, and our use of the Goerss–Hopkins–Miller Theorem, may both be entirely avoided if one only wants to know that the C_2-action on $\widehat{E(n)}$ is the Galois one *in the homotopy category of spectra*. It is, however, not a priori clear that there is a unique lift of this homotopy C_2-action on $\widehat{E(n)}$ to a fully coherent C_2-action.

To prove Theorem 1.2, recall that the assignment $(k, \mathbb{G}) \mapsto E_{(k,\mathbb{G})}$ is a functor to $\mathbb{E}_\infty(\mathbf{Spectra})$. In particular, for any field extension $\mathbb{F}_2 \subset k$ there are induced C_2-equivariant homotopy ring maps $MU_\mathbb{R} \to \widehat{E(n)} \to E_{(k,\mathbb{G})}$ involving some version of $\widehat{E(n)}$. If G is a finite subgroup of the $\mathbb{E}_\infty$-ring automorphisms of $E_{(k,\mathbb{G})}$ containing the central C_2, there then arises a sequence of homotopy ring maps

$$N_{C_2}^G MU_{\mathbb{R}} \longrightarrow N_{C_2}^G E_{(k,\mathbb{G})} \longrightarrow E_{(k,\mathbb{G})}.$$

The existence of the last homomorphism follows from the fact that the norm is an adjunction between $\mathbb{E}_\infty$-rings with C_2-action and $\mathbb{E}_\infty$-rings with G-action (see [HM16, §2.2]).

6. Real Landweber Exactness

In the remainder of the paper, for simplicity, we use a specific Morava E-theory E_n that is defined via a lift of the height n Honda formal group law over $\mathbb{F}_{2^n}$. Its homotopy groups are

$$\pi_* E_n = W(\mathbb{F}_{2^n})[[u_1, u_2, \ldots, u_{n-1}]][u^{\pm}].$$

and the 2-typical formal group law over $\pi_* E_n$ is determined by the map $\pi_* BP \to \pi_* E_n$ sending

$$v_i \mapsto \begin{cases} u_i u^{2^i - 1} & 1 \leq i \leq n-1 \\ u^{2^n - 1} & i = n \\ 0 & i > n. \end{cases}$$

Our results are all easily generalized to other variants of Morava E-theory.

In this section, we will show that E_n, as a C_2-spectrum, is Real Landweber exact in the sense of [HM16]. We do so by completely computing the $RO(C_2)$-graded homotopy fixed point spectral sequence of E_n.

6.1. $RO(C_2)$-graded homotopy fixed point spectral sequence of E_n.

So far, we have constructed a C_2-equivariant map from

$$MU_{\mathbb{R}} \to E_n.$$

Here, the C_2-action on $MU_{\mathbb{R}}$ is by complex conjugation, and the C_2-action on E_n is by the Goerss–Hopkins–Miller E_∞-action. The existence of this equivariant map will help us in computing the C_2-homotopy fixed point spectral sequence of E_n. In particular, the map $MU_{\mathbb{R}} \to E_n$ induces the map of spectral sequences

$$C_2\text{- HFPSS}(MU_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

of C_2-equivariant homotopy fixed point spectral sequences. Since both the complex conjugation action on $MU_{\mathbb{R}}$ and the Galois C_2-action on E_n are by E_∞-ring maps, both spectral sequences are multiplicative (the map between them is not necessarily a multiplicative map, but this is perfectly fine). At this point, we will replace $MU_{\mathbb{R}}$ by $BP_{\mathbb{R}}$ because everything is 2-local, and argument below is exactly the same regardless of whether we are using $MU_{\mathbb{R}}$ or $BP_{\mathbb{R}}$. Moreover, since $MU_{\mathbb{R}}$ splits as a wedge of suspensions of $BP_{\mathbb{R}}$'s, the homotopy fixed point spectral sequence of $BP_{\mathbb{R}}$ has the advantage of having less classes than $MU_{\mathbb{R}}$ but yet still retaining the important 2-local information that we need.

By [HM16, Corollary 4.7], the E_2-pages of the $RO(C_2)$-graded homotopy fixed point spectral sequences of $BP_{\mathbb{R}}$ and E_n are

$$\begin{aligned} E_2^{s,t}(BP_{\mathbb{R}}^{hC_2}) &= \mathbb{Z}[\bar{v}_1, \bar{v}_2, \ldots] \otimes \mathbb{Z}[u_{2\sigma}^{\pm}, a_\sigma]/(2a_\sigma) \\ E_2^{s,t}(E_n^{hC_2}) &= W(\mathbb{F}_{2^n})[[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-1}]][\bar{u}^{\pm}] \otimes \mathbb{Z}[u_{2\sigma}^{\pm}, a_\sigma]/(2a_\sigma). \end{aligned}$$

On the E_2-page, the class $\bar{v}_i$ is in stem $|\bar{v}_i| = i\rho$ for $i \geq 1$; the class $\bar{u}_i$ is in stem $|\bar{u}_i| = 0$ for $1 \leq i \leq n-1$; and the class $\bar{u}$ is in stem $|\bar{u}| = \rho$. The classes $u_{2\sigma}$ and a_σ are in stems $2 - 2\sigma$ and $-\sigma$, respectively. They can be defined more generally as follows:

Definition 6.1 (a_V and u_V)**.** Let V be a representation of G of dimension d.

(1) $a_V \in \pi_{-V}^G S^0$ is the map corresponding to the inclusion $S^0 \hookrightarrow S^V$ induced by $\{0\} \subset V$.

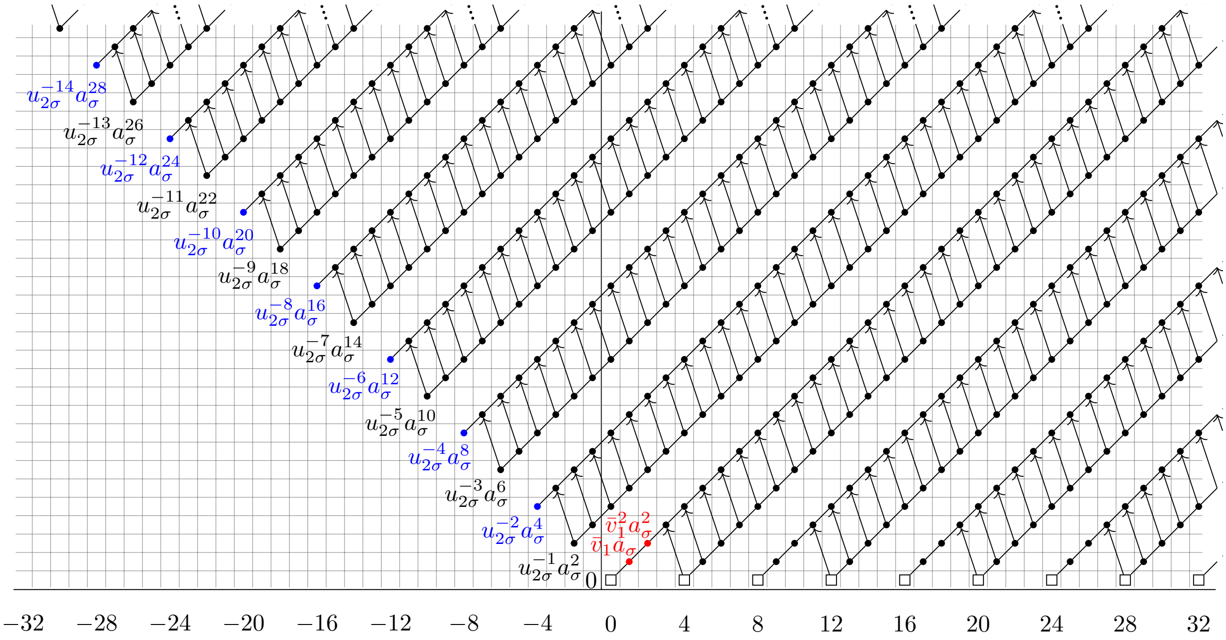

FIGURE 1. Important d_3-differentials and surviving torsion classes on the E_3-page.

(2) If V is oriented, $u_V \in \pi^G_{d-V}H\underline{\mathbb{Z}}$ is the class of the generator of $H^G_d(S^V; H\underline{\mathbb{Z}})$.

Theorem 6.2. *In the $RO(C_2)$-graded homotopy fixed point spectral sequence of E_n, the classes $\bar{u}_1, \ldots, \bar{u}_{n-1}$, $\bar{u}^{\pm}$, and a_σ are permanent cycles. All the differentials in the spectral sequence are determined by the differentials*

$$\begin{aligned} d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) &= \bar{u}_k\bar{u}^{2^k-1}a_\sigma^{2^{k+1}-1}, \ \ 1 \le k \le n-1, \\ d_{2^{n+1}-1}(u_{2\sigma}^{2^{n-1}}) &= \bar{u}^{2^n-1}a_\sigma^{2^{n+1}-1}, \ \ k = n \end{aligned}$$

and multiplicative structures.

Proof. In [HK01], Hu and Kriz completely computed the C_2-homotopy fixed point spectral sequence of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$. In particular, the classes $\bar{v}_i$ for all $i \ge 1$ and the class a_σ are permanent cycles. All the differentials are determined by the differentials

$$d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) = \bar{v}_k a_\sigma^{2^{k+1}-1}, \ k \ge 1$$

and multiplicative structures. There are no nontrivial extension problems on the E_∞-page.

On the E_2-page, the map

$$C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

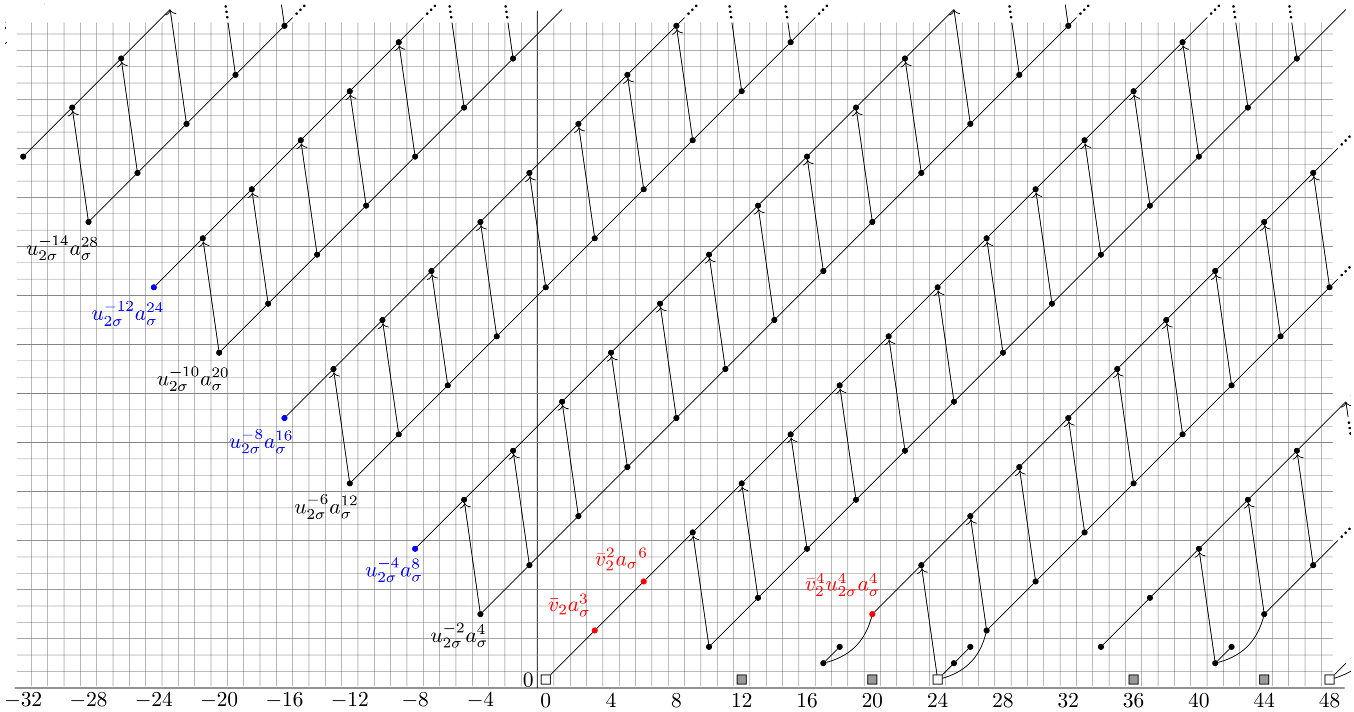

FIGURE 2. Important d_7-differentials and surviving torsion classes on the E_7-page.

of spectral sequences sends the classes $u_{2\sigma} \mapsto u_{2\sigma}$, $a_\sigma \mapsto a_\sigma$, and

$$\bar{v}_i \mapsto \begin{cases} \bar{u}_i\bar{u}^{2^i-1} & 1 \le i \le n-1 \\ \bar{u}^{2^n-1} & i = n \\ 0 & i > n. \end{cases}$$

We will first prove that the classes $\bar{u}_1$, …, $\bar{u}_{n-1}$, $\bar{u}^{\pm}$, and a_σ are permanent cycles in C_2- HFPSS(E_n). Since the classes $\bar{v}_i$, $i \ge 1$, and a_σ are permanent cycles in C_2- HFPSS$(BP_{\mathbb{R}})$, their images are also permanent cycles in C_2- HFPSS(E_n). This shows that the classes $\bar{u}_i\bar{u}^{2^i-1}$, $1 \le i \le n-1$, $\bar{u}^{2^n-1}$, and a_σ are permanent cycles in C_2- HFPSS(E_n).

Now, consider the non-equivariant map

$$u : S^2 \to i_e^* E_n.$$

Applying the Hill–Hopkins–Ravenel norm functor $N_e^{C_2}(-)$ ([HHR16]) produces the equivariant map

$$N_e^{C_2}(u) = \bar{u}^2 : S^{2\rho} \to N_e^{C_2} i_e^* E_n \to E_n,$$

where the last map is the co-unit map of the norm–restriction adjunction

$$N_e^{C_2} : \text{Commutative } C_2\text{-spectra} \rightleftarrows \text{Commutative spectra} : i_e^*.$$

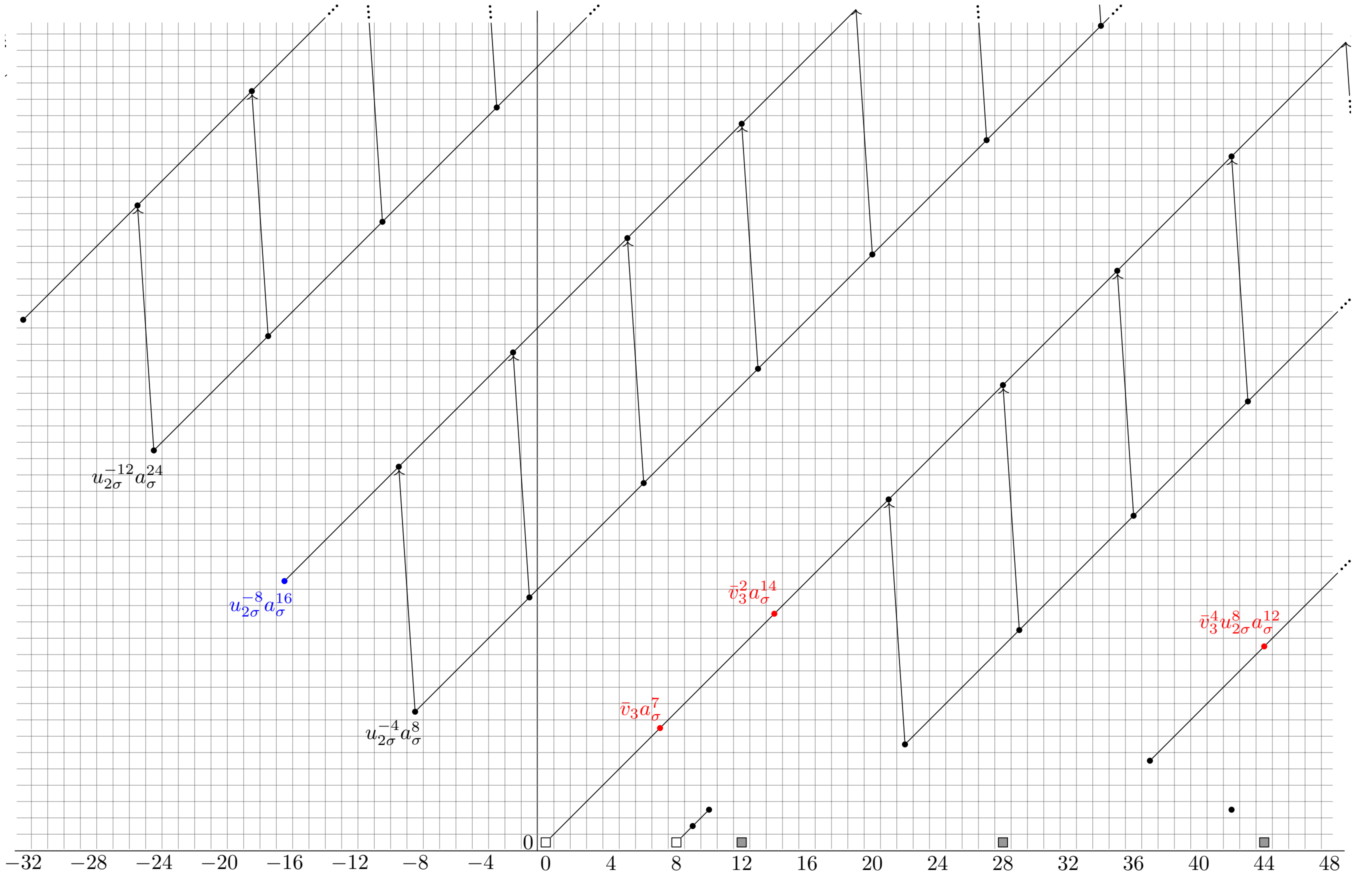

FIGURE 3. Important d_{15}-differentials and surviving torsion classes on the E_{15}-page.

Since the element $N_e^{C_2}(u) = \bar{u}^2$ is an actual element in $\pi_\bigstar^{C_2} E_n$, it is a permanent cycle. This, combined with the fact that $\bar{u}^{2^n-1}$ is a permanent cycle, shows that $\bar{u} = \bar{u}^{2^n-1} \cdot (\bar{u}^{-2})^{2^{n-1}}$ is a permanent cycle. It follows from the previous paragraph that the classes $\bar{u}_1, \ldots, \bar{u}_{n-1}$, and $\bar{u}^\pm$ are all permanent cycles in C_2- $\mathrm{HFPSS}(E_n)$.

It remains to produce the differentials in C_2- $\mathrm{HFPSS}(E_n)$. We will show by induction on k, $1 \le k \le n$, that all the differentials in C_2- $\mathrm{HFPSS}(E_n)$ are determined by the differentials

$$\begin{aligned} d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) &= \bar{u}_k \bar{u}^{2^k-1} a_\sigma^{2^{k+1}-1}, \ \ 1 \le k \le n-1, \\ d_{2^{n+1}-1}(u_{2\sigma}^{2^{n-1}}) &= \bar{u}^{2^n-1} a_\sigma^{2^{n+1}-1}, \ \ k = n \end{aligned}$$

and multiplicative structures.

For the base case, when $k = 1$, there is a d_3-differential

$$d_3(u_{2\sigma}) = \bar{v}_1 a_\sigma^3$$

in C_2- $\mathrm{HFPSS}(BP_{\mathbb{R}})$. Under the map

$$C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

of spectral sequences, the the source is mapped to $u_{2\sigma}$ and the target is mapped to $\bar{u}_1\bar{u}a_\sigma^3$. It follows that there is a d_3-differential

$$d_3(u_{2\sigma}) = \bar{u}_1\bar{u}a_\sigma^3$$

in C_2- $\mathrm{HFPSS}(E_n)$. Multiplying this differential by the permanent cycles produced before determines the rest of the d_3-differentials. These are all the d_3-differentials because there are no more room for other d_3-differentials after these differentials.

Suppose now that the induction hypothesis holds for all $1 \leq k \leq r-1 < n$. For degree reasons, after the d_{2^r-1}-differentials, the next possible differential is of length $d_{2^{r+1}-1}$. In C_2- $\mathrm{HFPSS}(BP_{\mathbb{R}})$, there is a $d_{2^{r+1}-1}$-differential

$$d_{2^{r+1}-1}(u_{2\sigma}^{2^{r-1}}) = \bar{v}_r a_\sigma^{2^{r+1}-1}.$$

The map

$$C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

of spectral sequences sends the source to $u_{2\sigma}^{2^{r-1}}$ and the target to

$$\bar{v}_r a_\sigma^{2^{r+1}-1} \mapsto \begin{cases} \bar{u}_r\bar{u}^{2^r-1}a_\sigma^{2^{r+1}-1} & r<n \\ \bar{u}^{2^n-1}a_\sigma^{2^{n+1}-1} & r=n. \end{cases}$$

In particular, both images are not zero. Moreover, the image of the target must be killed by a differential of length at most $2^{r+1}-1$. By degree reasons, the image of the target cannot be killed by a shorter differential. It follows that there is a $d_{2^{r+1}-1}$-differential

$$d_{2^{r+1}-1}(u_{2\sigma}^{2^{r-1}}) = \begin{cases} \bar{u}_r\bar{u}^{2^r-1}a_\sigma^{2^{r+1}-1} & r<n \\ \bar{u}^{2^n-1}a_\sigma^{2^{n+1}-1} & r=n. \end{cases}$$

The rest of the $d_{2^{r+1}-1}$-differentials are produced by multiplying this differential with permanent cycles. After these differentials, there are no room for other $d_{2^{r+1}-1}$-differentials by degree reasons. This concludes the proof of the theorem. $\square$

Remark 6.3. As an example, Figures 4–7 show the differentials in the integer-graded part of C_2- $\mathrm{HFPSS}(E_3)$.

6.2. **Real Landweber Exactness.** We will now use the C_2-homotopy fixed point spectral sequence of E_n to show that E_n is Real Landweber exact. First, we will recall some definitions and theorems from [HM16].

Definition 6.4 ([Ara79])**.** Let E be a C_2-equivariant homotopy commutative ring spectrum. A **Real orientation** of E is a class $\bar{x} \in \widetilde{E}^{\rho}_{C_2}(\mathbb{CP}^\infty)$ whose restriction to

$$\widetilde{E}^{\rho}_{C_2}(\mathbb{CP}^1) = \widetilde{E}^{\rho}_{C_2}(S^\rho) \cong E^0_{C_2}(pt)$$

is the unit. Here, we are viewing $\mathbb{CP}^n$ as a C_2-space via complex conjugation.

By [HK01, Theorem 2.25], Real orientations of E are in one-to-one correspondence with homotopy commutative maps $MU_{\mathbb{R}} \to E$ of C_2-ring spectra.

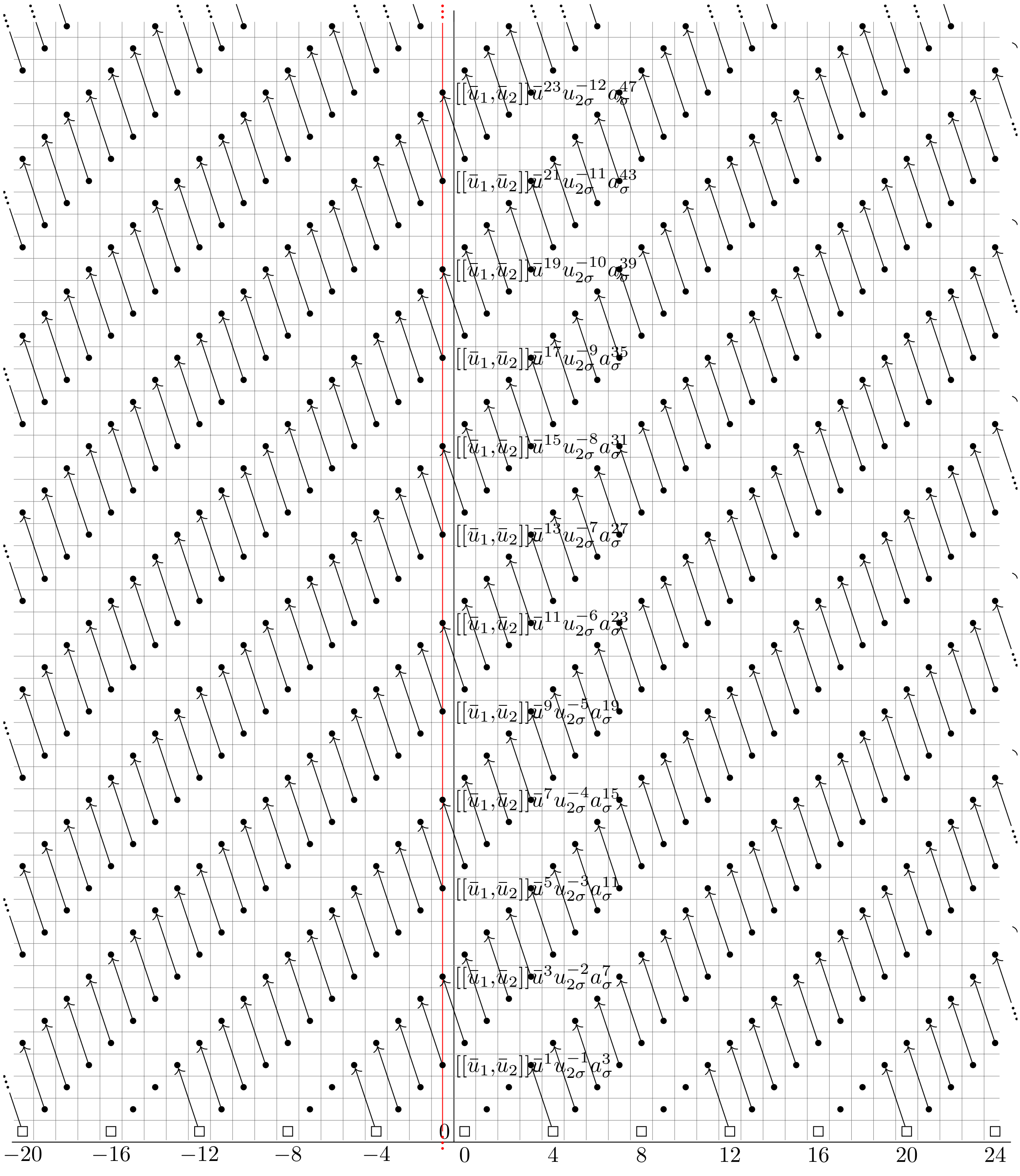

FIGURE 4. d_3-differentials in the integer graded part of C_2-HFPSS(E_3).

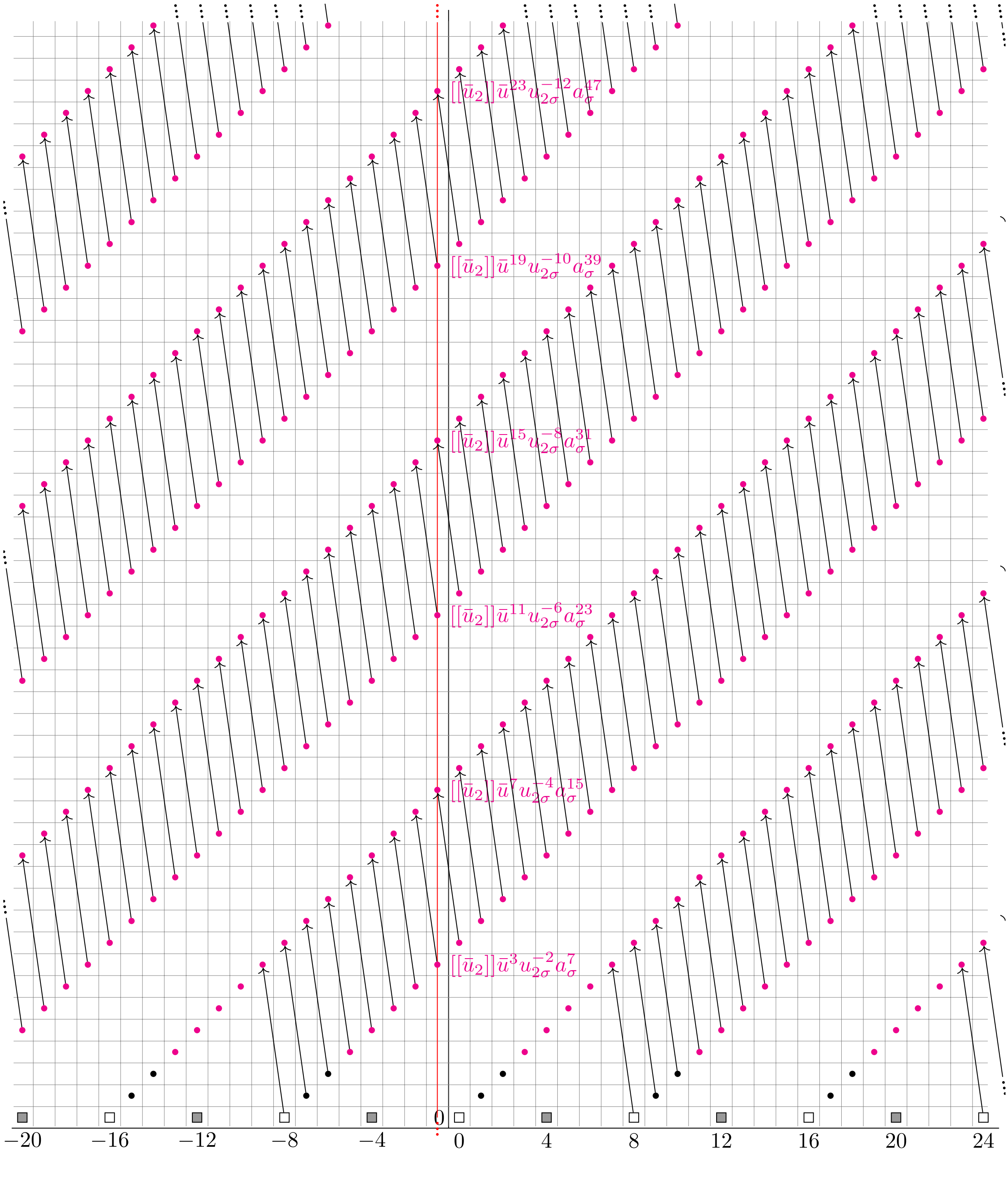

FIGURE 5. d_7-differentials in the integer graded part of C_2-HFPSS(E_3).

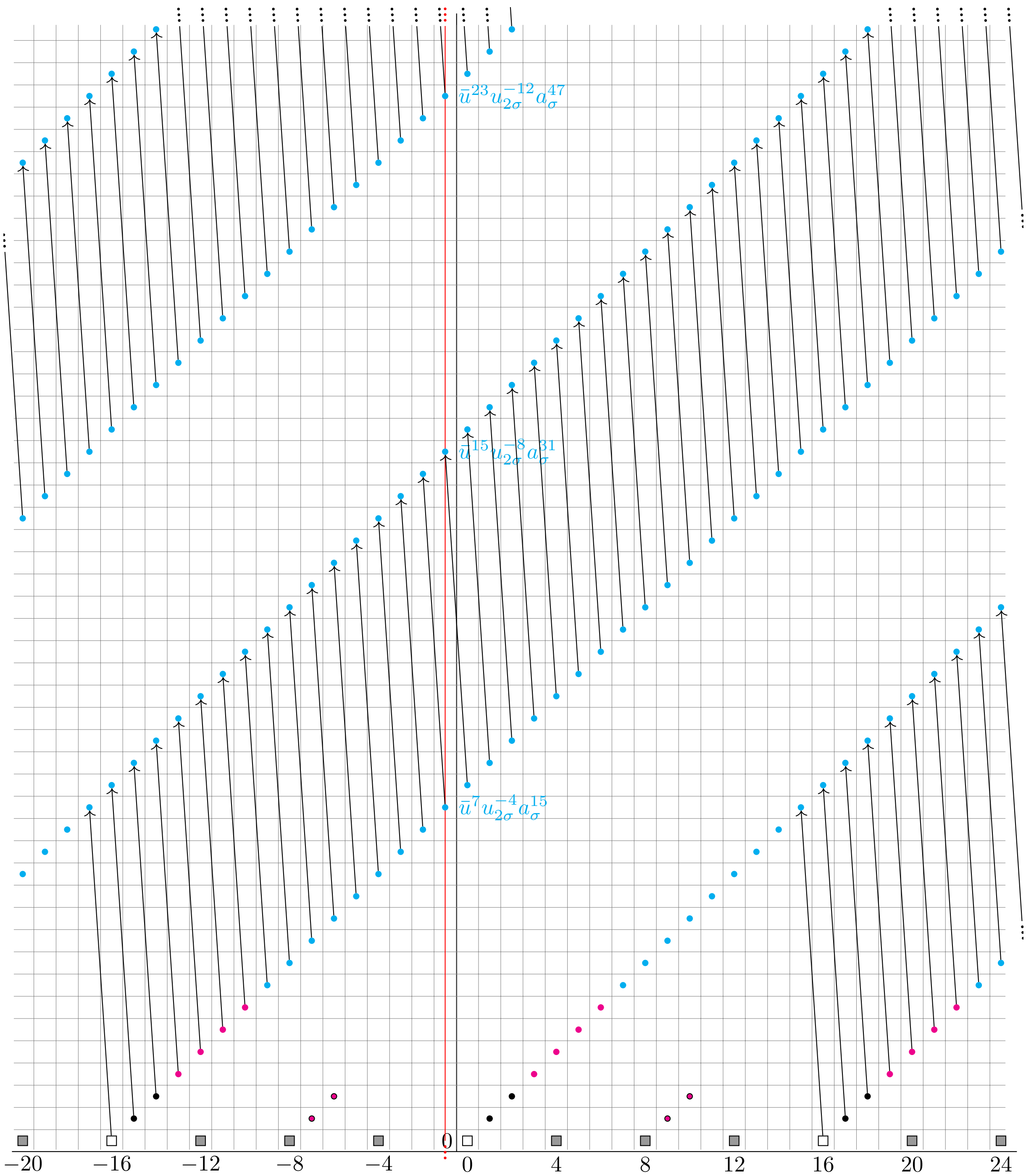

FIGURE 6. d_{15}-differentials in the integer graded part of C_2-HFPSS(E_3).

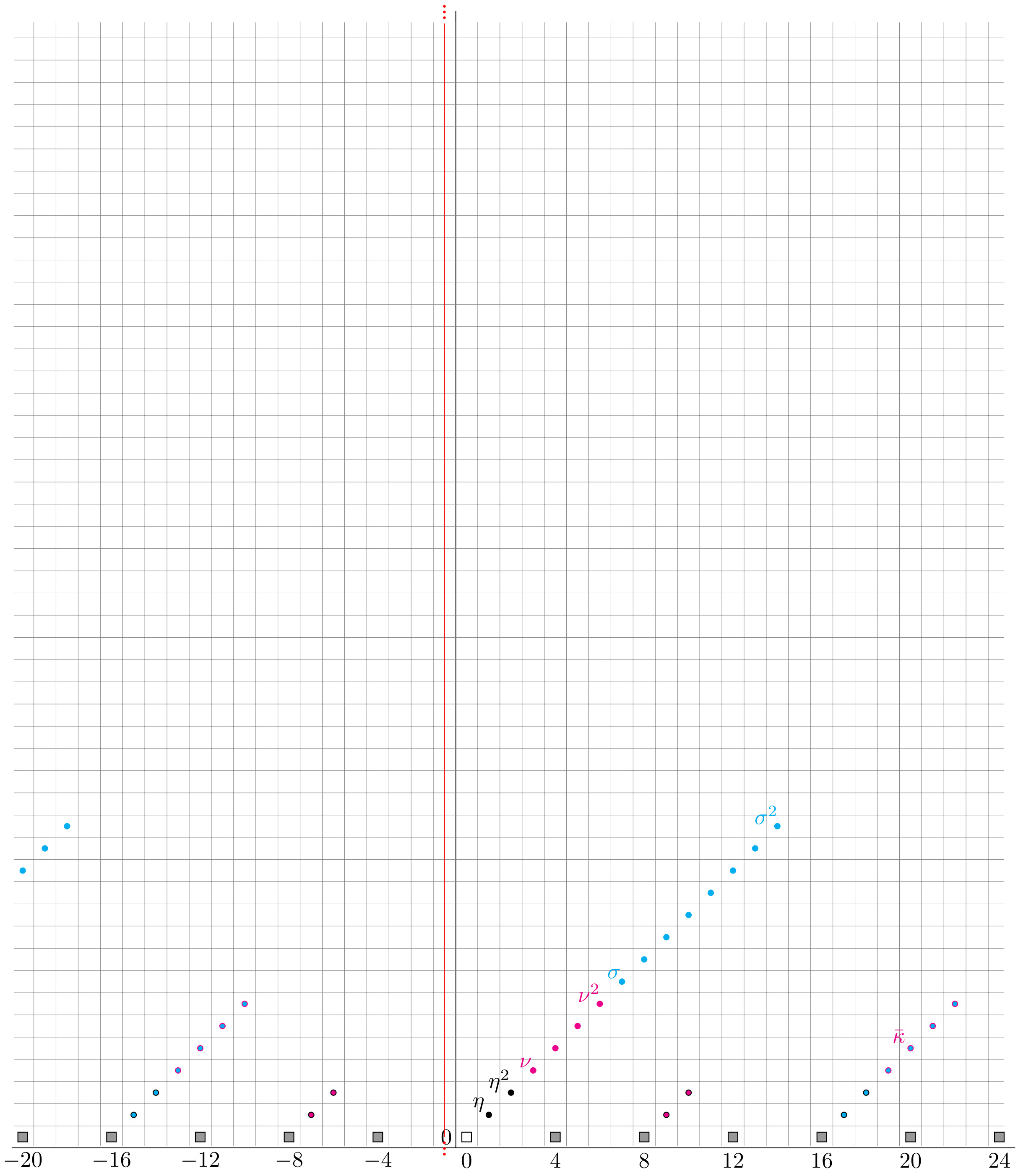

FIGURE 7. E_∞-page of the integer graded part of C_2- HFPSS(E_3).

Definition 6.5. ([HM16, Definition 3.1]). A C_2-spectrum $E\mathbb{R}$ is *even* if $\underline{\pi}_{k\rho-1}E\mathbb{R} = 0$ for all $k \in \mathbb{Z}$. It is called *strongly even* if additionally $\underline{\pi}_{k\rho}E\mathbb{R}$ is a constant Mackey functor for all $k \in \mathbb{Z}$, i.e., if the restriction

$$\pi_{k\rho}^{C_2}E\mathbb{R} \to \pi_{k\rho}^{e}E\mathbb{R} \cong \pi_{2k}^{e}E\mathbb{R}$$

is an isomorphism.

Even spectra satisfy very nice properties. In particular, Hill–Meier further proved ([HM16, Lemma 3.3]) that if a C_2-spectrum $E\mathbb{R}$ is even, then $E\mathbb{R}$ is Real orientable. They proved this by showing that all the obstructions to having a Real orientation lie in the groups $\pi_{2k-1}E\mathbb{R}$ and $\pi_{k\rho-1}^{C_2}E\mathbb{R}$, which are all 0 by definition.

Definition 6.6. ([HM16, Definition 3.5]). Let $E\mathbb{R}$ be a strongly even C_2-spectrum with underlying spectrum E. Then $E\mathbb{R}$ is called *Real Landweber exact* if for every Real orientation $MU_{\mathbb{R}} \to E\mathbb{R}$ the induced map

$$MU_{\mathbb{R}\bigstar}(X) \otimes_{MU_{2*}} E_{2*} \to E\mathbb{R}_{\bigstar}(X)$$

is an isomorphism for every C_2-spectrum X.

Here, we are treating $MU_{\mathbb{R}\bigstar}$ as a graded MU_{2*}-module because the restriction map $(MU_{\mathbb{R}})_{k\rho} \to MU_{2k}$ is an isomorphism, and it defines a graded ring morphism $MU_{2*} \to MU_{\mathbb{R}\bigstar}$ by sending elements of degree $2k$ to elements of degree $k\rho$.

Theorem 6.7 ([HM16], Real Landweber exact functor theorem)**.** *Let $E\mathbb{R}$ be a strongly even C_2-spectrum whose underlying spectrum E is Landweber exact. Then $E\mathbb{R}$ is Real Landweber exact.*

For E_n, its underlying spectrum is clearly Landweber exact. In light of Theorem 6.7, we prove the following:

Theorem 6.8. *E_n is a Real Landweber exact spectrum.*

Proof. By Theorem 6.7, it suffices to show that E_n is strongly even. By Thereom 6.2, the classes $\bar{u}_1, \ldots, \bar{u}_{n-1}$, and $\bar{u}^{\pm}$ are permanent cycles in C_2-HFPSS(E_n). The restriction of these classes to $\pi_{2*}^{e}E_n$ are $u_1, \ldots, u_{n-1}$, and $u^{\pm}$, respectively. Furthermore, there are no other classes in $\pi_{*\rho}^{C_2}E_n$. This shows that the restriction map

$$\pi_{*\rho}^{C_2}E_n \to \pi_{2*}^{e}E_n$$

is an isomorphism, hence $\underline{\pi}_{k\rho}E_n$ is a constant Mackey functor for all $k \in \mathbb{Z}$.

Classically, we already know that $\pi_{2k-1}^{e}E_n = 0$. The following lemma shows that $\underline{\pi}_{k\rho-1}E_n = 0$ for all $k \in \mathbb{Z}$. $\square$

Lemma 6.9. *The groups $\pi_{k\rho-1}^{C_2}E_n = 0$ for all $k \in \mathbb{Z}$.*

Proof. In C_2-HFPSS(E_n), the classes $\bar{u}^{\pm}$ are permanent cycles. Since $|\bar{u}| = \rho$, multiplying by $\bar{u}^k$ produces an isomorphism

$$\pi_{\bigstar}^{C_2}E_n \xrightarrow{\cong} \pi_{\bigstar+k\rho}^{C_2}E_n.$$

It follows that in order to show $\pi_{k\rho-1}^{C_2}E_n = 0$ for $k \in \mathbb{Z}$, it suffices to prove $\pi_{-1}^{C_2}E_n = 0$.

Recall that the E_2-page of C_2-HFPSS(E_n) is

$$E_2^{s,t}(E_n^{hC_2}) = W(\mathbb{F}_{2^n})[[\bar{u}_1, \ldots, \bar{u}_{n-1}]][\bar{u}^{\pm}] \otimes \mathbb{Z}[u_{2\sigma}^{\pm}, a_\sigma]/(2a_\sigma).$$

As in Figure 7, every class on the 0-line is of the form

$$W(\mathbb{F}_{2^n})[[\bar{u}_1, \ldots, \bar{u}_{n-1}]]\bar{u}^a u_{2\sigma}^b,$$

where $a, b \in \mathbb{Z}$, and every class of filtration greater than 0 is of the form

$$\mathbb{F}_{2^n}[[\bar{u}_1, \ldots, \bar{u}_{n-1}]]\bar{u}^a u_{2\sigma}^b a_\sigma^c,$$

where $a, b \in \mathbb{Z}$, and $c > 0$. For degree reasons, the classes on the (-1)-stem are all of the form

$$\mathbb{F}_{2^n}[[\bar{u}_1, \ldots, \bar{u}_{n-1}]]\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1},$$

where $\ell \geq 1$. The relevant differentials that have source or target in the (-1)-stem are all generated by

$$d_{2^{r+1}-1}(u_{2\sigma}^{-2^{r-1}}) = d_{2^{r+1}-1}(u_{2\sigma}^{-2^r} \cdot u_{2\sigma}^{2^{r-1}}) = u_{2\sigma}^{-2^r} \cdot d_{2^{r+1}-1}(u_{2\sigma}^{2^{r-1}}) = \begin{cases} \bar{u}_r \bar{u}^{2^r-1} u_{2\sigma}^{-2^r} a_\sigma^{2^{r+1}-1} & 0 < r < n \\ \bar{u}^{2^n-1} u_{2\sigma}^{-2^n} a_\sigma^{2^{n+1}-1} & r = n. \end{cases}$$

We will analyze these differentials one-by-one:

(1) The relevant d_3-differentials are all generated by the differential

$$d_3(u_{2\sigma}^{-1}) = \bar{u}_1 \bar{u} u_{2\sigma}^{-2} a_\sigma^3.$$

The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 1 \pmod 2$, are the sources of these differentials, and hence they die after the E_3-page. The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 0 \pmod 2$, are the targets. These differentials quotient out the principal ideal $(\bar{u}_1)$ at these targets. The remaining classes at these targets are of the form

$$\mathbb{F}_{2^n}[[\bar{u}_2, \ldots, \bar{u}_{n-1}]]\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1},$$

with $\ell \equiv 0 \pmod 2$.

(2) The relevant d_7-differentials are all generated by the differential

$$d_7(u_{2\sigma}^{-2}) = \bar{u}_2 \bar{u}^3 u_{2\sigma}^{-4} a_\sigma^7.$$

The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 2 \pmod 4$, are the sources of these differentials, and hence they die after the E_7-page. The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 0 \pmod 4$, are the targets. These differentials quotient out the principal ideal $(\bar{u}_2)$ at these targets. The remaining classes at these targets are of the form

$$\mathbb{F}_{2^n}[[\bar{u}_3, \ldots, \bar{u}_{n-1}]]\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1},$$

with $\ell \equiv 0 \pmod 4$.

(3) In general, for $0 < r < n$, the relevant $d_{2^{r+1}-1}$-differentials are all generated by the differential

$$d_{2^{r+1}-1}(u_{2\sigma}^{-2^{r-1}}) = \bar{u}_r \bar{u}^{2^r-1} u_{2\sigma}^{-2^r} a_\sigma^{2^{r+1}-1}.$$

The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 2^{r-1} \pmod{2^r}$, are the sources of these differentials, and hence they die after the $E_{2^{r+1}-1}$-page. The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 0 \pmod{2^r}$, are the targets. These differentials quotient out the principal ideal $(\bar{u}_r)$ at these targets. The remaining classes at these targets are of the form

$$\mathbb{F}_{2^n}[[\bar{u}_{r+1}, \ldots, \bar{u}_{n-1}]]\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1},$$

with $\ell \equiv 0 \pmod{2^r}$.

(4) The relevant $d_{2^{n+1}-1}$-differentials are all generated by the differential

$$d_{2^{n+1}-1}(u_{2\sigma}^{-2^{n-1}}) = \bar{u}^{2^n-1} u_{2\sigma}^{-2^n} a_\sigma^{2^{n+1}-1}.$$

The classes at $\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1}$, with $\ell \equiv 2^{n-1} \pmod{2^n}$, are the sources of these differentials, and hence they die after the $E_{2^{n+1}-1}$-page. The classes at $\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1}$, with $\ell \equiv 0 \pmod{2^n}$, are the targets. They also die after these differentials because the only classes at these targets now are $\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1}$.

It follows that every class at the (-1)-stem vanish after the $E_{2^{n+1}-1}$-page. This implies $\pi_{-1}^{C_2}E_n = 0$, as desired.

□

7. Hurewicz Images

In this section, we will prove that $\pi_* E_n^{hC_2}$ detects the Hopf elements, the Kervaire classes, and the $\bar{\kappa}$-family. The case when $n = 1$ and $n = 2$ are previously known. When $n = 1$, $E_1 = KU_2^\wedge$ and $E_1^{hC_2} = KO_2^\wedge$. It is well-known that $\pi_* KO_2^\wedge$ detects $\eta \in \pi_1\mathbb{S}$ and $\eta^2 \in \pi_2\mathbb{S}$ ([Ati66]). When $n = 2$, the Mahowald–Rezk transfer argument ([MR09]) shows that $\pi_* E_2^{hC_2}$ detects η, η^2, $\nu \in \pi_3\mathbb{S}$, $\nu^2 \in \pi_6\mathbb{S}$, and $\bar{\kappa} \in \pi_{20}\mathbb{S}$.

The Hopf elements are represented by the elements

$$h_i \in \mathrm{Ext}_{\mathcal{A}_*}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$$

on the E_2-page of the classical Adams spectral sequence at the prime 2. By Adam's solution of the Hopf invariant one problem [Ada60], only h_0, h_1, h_2, and h_3 survive to the E_∞-page. By Browder's work [Bro69], the Kervaire classes $\theta_j \in \pi_{2^{j+1}-2}\mathbb{S}$, if they exist, are represented by the elements

$$h_j^2 \in \mathrm{Ext}_{\mathcal{A}_*}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$$

on the E_2-page. For $j \leq 5$, h_j^2 survive. The case $\theta_4 \in \pi_{30}\mathbb{S}$ is due to Barratt–Mahowald–Tangora [MT67, BMT70], and the case $\theta_5 \in \pi_{62}\mathbb{S}$ is due to Barratt–Jones–Mahowald [BJM84]. The fate of h_6^2 is unknown. Hill–Hopkins–Ravenel's result [HHR16] shows that the h_j^2, for $j \geq 7$, do not survive to the E_∞-page.

To introduce the $\bar{\kappa}$-family, we appeal to Lin's complete classification of $\mathrm{Ext}_{\mathcal{A}_*}^{\leq 4,t}(\mathbb{F}_2, \mathbb{F}_2)$ in [Lin08]. In his classification, Lin showed that there is a family $\{g_k \mid k \geq 1\}$ of indecomposable elements with

$$g_k \in \mathrm{Ext}_{\mathcal{A}_*}^{4,2^{k+2}+2^{k+3}}(\mathbb{F}_2, \mathbb{F}_2).$$

The first element of this family, g_1, is in bidegree $(4, 24)$. It survives the Adams spectral sequence to become $\bar{\kappa} \in \pi_{20}\mathbb{S}$. It is for this reason that we name this family the $\bar{\kappa}$-family. The element g_2 also survives to become the element $\bar{\kappa}_2 \in \pi_{44}\mathbb{S}$. For $k \geq 3$, the fate of g_k is unknown.

In [LSWX17], the second author, together with Li, Wang, and Xu, proved detection theorems for the Hurewicz images of $MU_\mathbb{R}^{C_2} \approx MU_\mathbb{R}^{hC_2}$ and $BP_\mathbb{R}^{C_2} \approx BP_\mathbb{R}^{hC_2}$ (the equivalences between the C_2-fixed points and the C_2-homotopy fixed points for $MU_\mathbb{R}$ and $BP_\mathbb{R}$ are due to Hu and Kriz [HK01, Theorem 4.1]).

Theorem 7.1. *(Li–Shi–Wang–Xu, Detection Theorems for $MU_\mathbb{R}$ and $BP_\mathbb{R}$). The Hopf elements, the Kervaire classes, and the $\bar{\kappa}$-family are detected by the Hurewicz maps $\pi_*\mathbb{S} \to \pi_* MU_\mathbb{R}^{C_2} \cong \pi_* MU_\mathbb{R}^{hC_2}$ and $\pi_*\mathbb{S} \to \pi_* BP_\mathbb{R}^{C_2} \cong \pi_* BP_\mathbb{R}^{hC_2}$.*

Given the discussion above, Theorem 7.1 shows that the elements η, ν, σ, and θ_j, for $1 \leq j \leq 5$, are detected by $\pi_*^{C_2} MU_\mathbb{R}^{hC_2}$ and $\pi_*^{C_2} BP_\mathbb{R}^{hC_2}$. The last unknown Kervaire class θ_6 and the classes g_k for $k \geq 3$ will also be detected, should they survive the Adams spectral sequence.

The proof of Theorem 7.1 requires the C_2-equivariant Adams spectral sequence developed by Greenlees [Gre85, Gre88, Gre90] and Hu–Kriz [HK01]. Since $MU_{\mathbb{R}}$ splits as a wedge of suspensions of $BP_{\mathbb{R}}$ 2-locally, we only need to focus on $BP_{\mathbb{R}}$. There is a map of Adams spectral sequences

$$\begin{tikzcd}
\text{classical Adams spectral sequence of } \mathbb{S} \arrow[r, Rightarrow] \arrow[d] & (\pi_*\mathbb{S})^\wedge_2 \arrow[d] \\
C_2\text{-equivariant Adams spectral sequence of } \mathbb{S} \arrow[r, Rightarrow] \arrow[d] & (\pi^{C_2}_\bigstar F(EC_{2+}, \mathbb{S}))^\wedge_2 \arrow[d] \\
C_2\text{-equivariant Adams spectral sequence of } BP_{\mathbb{R}} \arrow[r, Rightarrow] & (\pi^{C_2}_\bigstar F(EC_{2+}, BP_{\mathbb{R}}))^\wedge_2.
\end{tikzcd}$$

It turns out that for degree reasons, the C_2-equivariant Adams spectral sequence for $BP_{\mathbb{R}}$ degenerates at the E_2-page. From this, Theorem 7.1 follows easily from the following algebraic statement:

Theorem 7.2 (Li–Shi–Wang–Xu, Algebraic Detection Theorem)**.** *The images of the elements* $\{h_i \,|\, i \geq 1\}$, $\{h_j^2 \,|\, j \geq 1\}$, *and* $\{g_k \,|\, k \geq 1\}$ *on the* E_2*-page of the classical Adams spectral sequence of* $\mathbb{S}$ *are nonzero on the* E_2*-page of the* C_2*-equivariant Adams spectral sequence of* $BP_{\mathbb{R}}$*.*

The proof of Theorem 7.2 requires an analysis of the algebraic maps

$$\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_{\mathcal{A}^{cc}_\bigstar}(H^c_\bigstar, H^c_\bigstar) \to \mathrm{Ext}_{\Lambda^{cc}_\bigstar}(H^c_\bigstar, H^c_\bigstar).$$

These are the maps on the E_2-pages of the Adams spectral sequences above. Here, $\mathcal{A}_* := (H\mathbb{F}_2 \wedge H\mathbb{F}_2)_*$ is the classical dual Steenrod algebra; $H^c_\bigstar := F(EC_{2+}, H\mathbb{F}_2)_\bigstar$ is the Borel C_2-equivariant Eilenberg–MacLane spectrum; $\mathcal{A}^{cc}_\bigstar := F(EC_{2+}, H\mathbb{F}_2 \wedge H\mathbb{F}_2)_\bigstar$ is the Borel C_2-equivariant dual Steenrod algebra; and $\Lambda^{cc}_\bigstar$ is a quotient of $\mathcal{A}^{cc}_\bigstar$. Hu and Kriz [HK01] studied $\mathcal{A}^{cc}_\bigstar$ and completely computed the Hopf algebroid structure of $(H^c_\bigstar, \mathcal{A}^{cc}_\bigstar)$. Using their formulas, it is possible to compute the map

$$(H\mathbb{F}_2, \mathcal{A}_*) \to (H^c_\bigstar, \mathcal{A}^{cc}_\bigstar) \to (H^c_\bigstar, \Lambda^{cc}_\bigstar)$$

of Hopf-algebroids. Filtering these Hopf algebroids compatibly produces maps of May spectral sequences:

$$\begin{tikzcd}
\text{May spectral sequence of } \mathbb{S} \arrow[r, Rightarrow] \arrow[d] & \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \arrow[d] \\
C_2\text{-equivariant May spectral sequence of } \mathbb{S} \arrow[r, Rightarrow] \arrow[d] & \mathrm{Ext}_{\mathcal{A}^{cc}_\bigstar}(H^c_\bigstar, H^c_\bigstar) \arrow[d] \\
C_2\text{-equivariant May spectral sequence of } BP_{\mathbb{R}} \arrow[r, Rightarrow] & \mathrm{Ext}_{\Lambda^{cc}_\bigstar}(H^c_\bigstar, H^c_\bigstar).
\end{tikzcd}$$

There is a connection between the C_2-equivariant May spectral sequence of $BP_{\mathbb{R}}$ and the homotopy fixed point spectral sequence of $BP_{\mathbb{R}}$:

Theorem 7.3 (Li–Shi–Wang–Xu)**.** *The* C_2*-equivariant May spectral sequence of* $BP_{\mathbb{R}}$ *is isomorphic to the associated-graded homotopy fixed point spectral sequence of* $BP_{\mathbb{R}}$ *as* $RO(C_2)$*-graded spectral sequences.*

By the "associated-graded homotopy fixed point spectral sequence", we mean that whenever we see a $\mathbb{Z}$-class on the E_2-page, we replace it by a tower of $\mathbb{Z}/2$-classes. Since the equivariant Adams spectral sequence of $BP_{\mathbb{R}}$ degenerates, the E_∞-page of the C_2-equivariant May spectral sequence of $BP_{\mathbb{R}}$ is an associated-graded of $\pi_{\bigstar}^{C_2} F(EC_{2+}, BP_{\mathbb{R}})$. The isomorphism in Theorem 7.3 allows us to identify the classes in C_2- HFPSS(E_n) that detects the Hopf elements, the Kervaire classes, and the $\bar{\kappa}$-family. This is crucial for tackling detection theorems of $E_n^{hC_2}$.

Using Hu–Kriz's formulas, one can compute the maps on the E_2-pages of the May spectral sequences above, as well as all the differentials in the C_2-equivariant May spectral sequence of $BP_{\mathbb{R}}$.

Theorem 7.4 (Li–Shi–Wang–Xu)**.** *On the E_2-page of the map*

$$\mathit{MaySS}(\mathbb{S}) \to C_2\text{-}\mathit{MaySS}(\mathbb{S}) \to C_2\text{-}\mathit{MaySS}(BP_{\mathbb{R}}) \cong C_2\text{-}\mathit{HFPSS}(BP_{\mathbb{R}}),$$

The classes

$$\begin{array}{rcl} h_i & \mapsto & \bar{v}_i a_\sigma^{2^i-1}, \\ h_j^2 & \mapsto & \bar{v}_j^2 a_\sigma^{2(2^j-1)}, \\ h_{2k}^4 & \mapsto & \bar{v}_{k+1}^4 u_{2\sigma}^{2^{k+1}} a_\sigma^{4(2^k-1)}. \end{array}$$

These classes all survive to the E_∞-page in C_2- HFPSS$(BP_{\mathbb{R}})$.

Since E_n is Real oriented and everything is 2-local, a Real orientation gives us a C_2-equivariant homotopy commutative map

$$BP_{\mathbb{R}} \to E_n,$$

which induces a multiplicative map

$$C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

of spectral sequences. On the E_2-page, this map sends the classes $u_{2\sigma} \mapsto u_{2\sigma}$, $a_\sigma \mapsto a_\sigma$, and

$$\bar{v}_i \mapsto \begin{cases} \bar{u}_i \bar{u}^{2^i-1} & 1 \leq i \leq n-1 \\ \bar{u}^{2^n-1} & i = n \\ 0 & i > n. \end{cases} \tag{1}$$

Theorem 7.5 (Detection Theorem for $E_n^{hC_2}$)**.**

(1) *For $1 \leq i, j \leq n$, if the element $h_i \in \mathrm{Ext}_{\mathcal{A}_*}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ or $h_j^2 \in \mathrm{Ext}_{\mathcal{A}_*}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hC_2}$ is nonzero.*

(2) *For $1 \leq k \leq n-1$, if the element $g_k \in \mathrm{Ext}_{\mathcal{A}_*}^{4,2^{k+2}+2^{k+3}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hC_2}$ is nonzero.*

Proof. By Theorem 7.4 and (1), the composite map

$$\mathrm{MaySS}(\mathbb{S}) \to C_2\text{-MaySS}(BP_{\mathbb{R}}) \cong C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

on the E_2-pages sends the classes

$$h_i \mapsto \bar{v}_i a_\sigma^{2^i-1} \quad \mapsto \quad \begin{cases} \bar{u}_i \bar{u}^{2^i-1} a_\sigma^{2^i-1} & 1 \le i \le n-1 \\ \bar{u}^{2^n-1} a_\sigma^{2^n-1} & i = n \\ 0 & i > n, \end{cases}$$

$$h_j^2 \mapsto \bar{v}_j^2 a_\sigma^{2(2^j-1)} \quad \mapsto \quad \begin{cases} \bar{u}_j^2 \bar{u}^{2(2^j-1)} a_\sigma^{2(2^j-1)} & 1 \le j \le n-1 \\ \bar{u}^{2(2^n-1)} a_\sigma^{2(2^n-1)} & j = n \\ 0 & j > n, \end{cases}$$

$$h_{2k}^4 \mapsto \bar{v}_{k+1}^4 u_{2\sigma}^{2^{k+1}} a_\sigma^{4(2^k-1)} \quad \mapsto \quad \begin{cases} \bar{u}_{k+1}^4 \bar{u}^{4(2^{k+1}-1)} u_{2\sigma}^{2^{k+1}} a_\sigma^{4(2^k-1)} & 1 \le k \le n-2 \\ \bar{u}^{4(2^n-1)} u_{2\sigma}^{2^n} a_\sigma^{4(2^{n-1}-1)} & k = n-1 \\ 0 & k > n-1. \end{cases}$$

We know all the differentials in C_2-$\mathrm{HFPSS}(E_n)$ from Section 6. From these differentials, it is clear that all the nonzero images on the E_2-page survive to the E_∞-page to represent elements in $\pi_* E_n^{hC_2}$. The statement of the theorem follows. □

Corollary 7.6 (Detection Theorem for E_n^{hG})**.** *Let G be a finite subgroup of the Morava stabilizer group $\mathbb{S}_n$ containing the centralizer subgroup C_2.*

(1) *For $1 \le i, j \le n$, if the element $h_i \in Ext_{\mathcal{A}_*}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ or $h_j^2 \in Ext_{\mathcal{A}_*}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hG}$ is nonzero.*

(2) *For $1 \le k \le n-1$, if the element $g_k \in Ext_{\mathcal{A}_*}^{4,2^{k+2}+2^{k+3}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hG}$ is nonzero.*

Proof. Consider the following factorization of the unit map $\mathbb{S} \to E_n^{hC_2}$:

$$\begin{tikzcd} E_n^{hG} = F(EG_+, E_n)^G \arrow[r] & F(EG_+, E_n)^{C_2} = E_n^{hC_2} \\ \mathbb{S} \arrow[u] \arrow[ur] & \end{tikzcd}$$

The claims now follow easily from Theorem 7.5. □

8. The Hill–Hopkins–Ravenel Detecting Spectrum

For the rest of this section, let $n = 2^{m-1}\ell$, where $\ell \equiv 1 \pmod 2$. It is well-known that there is a subgroup $C_{2^m} \subset \mathbb{S}_n$ containing the centralizer subgroup C_2. Under the action of this subgroup, we can view E_n as a commutative C_{2^m}-spectrum. The Real orientation of E_n produces a C_2-equivariant map

$$MU_\mathbb{R} \to i_{C_2}^* E_n.$$

Applying the Hill–Hopkins–Ravenel norm functor $N_{C_2}^{C_{2^m}}(-)$, we obtain a C_{2^m}-equivariant map

$$MU^{((C_{2^m}))} \to N_{C_2}^{C_{2^m}} i_{C_2}^*(E_n) \to E_n,$$

where the last map is the counit map of the norm-restriction adjunction

$$N_{C_2}^{C_{2^m}} : \mathrm{Comm}_{C_2} \rightleftarrows \mathrm{Comm}_{C_{2^m}} : i_{C_2}^*$$

For simplicity, if we 2-localize, we can consider the C_{2^m}-equivariant map

$$BP^{((C_{2^m}))} \to E_n$$

instead, where $BP^{((C_{2^m}))} := N_{C_2}^{C_{2^m}} BP_\mathbb{R}$.

Theorem 8.1 ([HHR16]). *Let $G = C_{2^m}$ and γ be the generator of G.*

(1) *There exist generators $\bar{r}_i^G \in \pi_{i\rho}^{C_2} MU^{((G))}$ such that*
$$\pi_*^u MU^{((G))} \cong \pi_{*\rho}^{C_2} MU^{((G))} = \mathbb{Z}[G \cdot \bar{r}_1^G, G \cdot \bar{r}_2^G, \ldots],$$
where $G \cdot \bar{r}_i^G := \{\bar{r}_i^G, \gamma \bar{r}_i^G, \ldots, \gamma^{2^{m-1}-1} \bar{r}_i^G\}$.

(2) *Let $\bar{v}_i^G := \bar{r}_{2^i-1}^G$, then*
$$\pi_*^u BP^{((G))} \cong \pi_{*\rho}^{C_2} BP^{((G))} = \mathbb{Z}[G \cdot \bar{v}_1^G, G \cdot \bar{v}_2^G, \ldots].$$

Hill, Hopkins, and Ravenel have shown using discrete valuation that the generators in $C_{2^m} \cdot \bar{r}_k^{C_{2^m}}$ encode information about the height $(2^{m-1}k)$ formal group law.

Theorem 8.2. *There is a factorization*
$$\begin{tikzcd} BP^{((C_{2^m}))} \arrow[r] \arrow[d] & E_n \\ D^{-1} BP^{((C_{2^m}))} \arrow[ur] & \end{tikzcd}$$
where
$$D = \prod_{1 \le i \le m} N_{C_2}^{C_{2^m}} (\bar{v}_{n/2^{i-1}}^{C_{2^i}}) = N_{C_2}^{C_{2^m}} (\bar{v}_n^{C_2}) \cdot N_{C_2}^{C_{2^m}} (\bar{v}_{n/2}^{C_4}) \cdots N_{C_2}^{C_{2^m}} (\bar{v}_{n/2^{m-1}}^{C_{2^m}}).$$

Proof. Using discrete valuation and the definition of the $\bar{v}_i^G$-generators, it is not hard to show that after taking $\pi_{*\rho}^{C_2}(-)$, the images of the generators $\bar{v}_n^{C_2}$, $\bar{v}_{n/2}^{C_4}$, …, $\bar{v}_{n/2^{m-1}}^{C_{2^m}}$ are all inverted under the map
$$\pi_{*\rho}^{C_2} BP^{((C_{2^m}))} \to \pi_{*\rho}^{C_2} E_n.$$
The factorization follows. □

Remark 8.3. In [HHR16], the detecting spectrum $\Omega_{\mathbb{O}}$, whose C_8-equivariant homotopy groups detects the Kervaire invariant elements, is defined as
$$\Omega_{\mathbb{O}} := D^{-1} BP^{((C_8))},$$
where $D = N_{C_2}^{C_8}(\bar{v}_4^{C_2}) \cdot N_{C_2}^{C_8}(\bar{v}_2^{C_4}) \cdot N_{C_2}^{C_8}(\bar{v}_1^{C_8})$. Letting $m = 3$ and $n = 2$ in Theorem 8.2 shows that there is a factorization
$$\begin{tikzcd} BP^{((C_8))} \arrow[r] \arrow[d] & E_4 \\ \Omega_{\mathbb{O}} \arrow[ur] & \end{tikzcd}$$
of E_4 through the detecting spectrum (cf. [HHR11, Section 8.2]).

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