PERGAMON

Topology 40 (2001) 317–399

TOPOLOGY

www.elsevier.com/locate/top

Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence

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Received 1 November 1996

Abstract

Using the Landweber–Araki theory of Real cobordism and Real-oriented spectra, we define a Real analogue of the Adams–Novikov spectral sequence. This is a new spectral sequence with a potentially calculable E_2-term. It has versions converging to either the $\mathbb{Z}/2$-equivariant or the non-equivariant stable 2-stems. We also construct a Real analogue of the Miller–Novikov 'algebraic' spectral sequence.

MSC: 55P91; 55P42

Keywords: Equivariant stable homotopy theory; Cobordism; Adams–Novikov spectral sequence; Real K-theory

1. Introduction

The Adams–Novikov spectral sequence calculating stable homotopy groups of spheres

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi_*(S^0)_{(p)} \tag{1.1}$$

behaves differently at the prime 2 than at other primes. For $p = 2$, (1.1) contains extra families of elements which do not occur for $p > 2$. One can see this phenomenon in two ways: (1) The strict automorphism group of the Lubin–Tate formal group law Φ_n reduced modulo 2 (the Morava stabilizer group) contains a subgroup generated by the formal inverse (which is the identity for $p > 2$). This produces additional families of elements in its cohomology, which, by the chromatic

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0040-9383/01/$ - see front matter
PII: S0040-9383(99)00065-8

spectral sequence of Miller, Ravenel and Wilson, translate to additional families of elements in (1.1). (2) While at $p > 2$ the Adams–Novikov E_2-term is smaller than the classical Adams E_2-term, this does not happen at $p = 2$. This is explained by the algebraic Novikov spectral sequence, whose E_1 term for $p > 2$ (but not for $p = 2$) can be identified with the Adams E_2-term. This is related to the fact that the dual of the Steenrod algebra for $p > 2$ (but not $p = 2$) splits as a semidirect product (in the Hopf algebra sense) of an exterior and a polynomial algebra.

Geometrically, the prime $p = 2$ is special also. The complex cobordism spectrum has an automorphism of order 2, given by complex conjugation. So, one could conjecture that there might be an analogue of the Adams–Novikov spectral sequence taking into account the complex conjugation structure, which would avoid the anomalies (1) and (2).

In this paper, we do, in fact, construct a candidate for such a spectral sequence. Recall Atiyah's Real KR-theory [7] of complex bundles with anti-linear involution. Then KR is a $\mathbb{Z}/2$-equivariant spectrum. Similarly, there exists a Real cobordism spectrum (defined by Landweber [27,28]). We construct an Adams-type spectral sequence based on Real cobordism. We prove that it converges to $\mathbb{Z}/2$-equivariant stable homotopy, which is essentially $\pi_*(B\mathbb{Z}/2_+ \vee S^0)$. We also construct another version of this spectral sequence which converges to the non-equivariant stable homotopy groups of spheres $\pi_*(S^0)$. Concrete calculations with these spectral sequences were done by the first author [21].

The behaviour of the new spectral sequences with respect to problems (1) and (2) above is as follows: The non-equivariant version of our spectral sequence eliminates (1) (see Section 5 below). In other words, we show that the real Morava stabilizer group at $p = 2$ does not contain the formal inverse. For (2), it is simpler to consider the equivariant version of the real spectral sequence. In this case, we construct an algebraic spectral sequence. We conjecture that its E_1-term can be identified with Greenlees' equivariant Adams E_2-term [18]. We give some evidence in that direction. In particular, we show that the splitting of the dual Steenrod algebra into a semidirect product for $p > 2$ has an analogue in the $\mathbb{Z}/2$-equivariant Steenrod algebra at $p = 2$.

To explain the results of this paper in more detail we have to start from the beginning. First consider a larger picture. Let E be a complex-oriented spectrum with formal group law F. Let G be a finite group of automorphisms of the ring E_* together with underlying strict isomorphisms of F. Then sometimes it can be shown that G also acts on E, i.e. that E has the structure of a G-equivariant spectrum (see [20]). Now there is one such group $G \cong \mathbb{Z}/2$ which occurs for $E = MU$: the generator $MU_* \to MU_*$ is $(-1)^k$ in dimension $2k$. The underlying strict isomorphism is $-[-1]_F(x)$.

This group has a very natural geometric interpretation: it corresponds to the action of $\mathbb{Z}/2$ on $\mathbb{CP}^\infty$ by complex conjugation. In fact, complex conjugation also acts on the classifying spaces $BU(n)$ and the corresponding universal Thom spaces. Consequently, this action very naturally rigidifies to a $\mathbb{Z}/2$-equivariant structure on MU. This was first noticed by Landweber [27,28], who called the resulting spectrum *Real cobordism* $M\mathbb{R}$. This was prompted by Atiyah's discovery of Real KR-theory [7]. Later, Araki [4–6] observed that complex conjugation preserves essentially all of the notions of the general theory of complex-oriented spectra. He defined the notion of a Real-oriented spectrum, and associated formal group laws with Real-oriented spectra. He also used Quillen's idempotent to construct a Real analogue $BP\mathbb{R}$ of the Brown–Peterson spectrum. However, it was not known if Morava $K(n)$-theories and other complex-oriented spectra have Real analogues (although for Landweber-exact complete spectra, this is a consequence of

Hopkins–Miller theory [20]). A general construction of Real analogues of complex-oriented spectra is a by-product of the present paper (see below).

We want to develop and investigate the analogue of the Adams–Novikov spectral sequence based on the Landweber–Araki Real cobordism. Realizing this goal turns out to be a more formidable task than it may seem at first. First of all, Real cobordism $M\mathbb{R}$, like Atiyah's Real K-theory KR ([7]), is an $RO(\mathbb{Z}/2)$-graded (co)homology theory, or, in other words, a $\mathbb{Z}/2$-equivariant spectrum indexed over the complete universe in the sense of Lewis et al. [31]. We review the main aspects of $M\mathbb{R}$, $BP\mathbb{R}$ and the theory of Real-oriented spectra from the point of view of modern foundations of equivariant stable homotopy theory in Section 2 below.

Section 3 contains some constructions and calculations suggested by this theory. Using the fact that $M\mathbb{R}$ is an E_∞-ring spectrum, we construct, by the methods of [14], Real analogues of all the usual derived spectra of MU, including $E\mathbb{R}(n)$, $P\mathbb{R}(n)$, $B\mathbb{R}(n)$, Morava K-theories $K\mathbb{R}(n)$, etc. We also give a calculation of the coefficients of Real Morava K-theories, and show that the Real version $K\mathbb{R}$ of K-theory constructed by our method is Atiyah's Real KR-theory.

In Section 4, we give a calculation of the coefficients $BP\mathbb{R}_\star$. This has been announced, but as far as we know not published, by Araki [4].

Next, we show that $BP\mathbb{R}, M\mathbb{R}$ satisfy a *strong completion theorem* in the sense that the canonical maps to the corresponding Borel cohomology theories are equivalences. It turns out that the Hopf algebroid $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$ is flat and almost completely determined by its Borel cohomology spectral sequence. In Section 4, we will prove this up to a certain error term, which we later eliminate in Section 7 by different methods. What we unfortunately do not know is a "moduli interpretation" of $BP\mathbb{R}_\star$, which would say that ring maps from $BP\mathbb{R}_\star$ represent "Real formal group laws" in some natural sense.

One way or another, we obtain the spectral sequence

$$\mathrm{Ext}_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star) \Rightarrow \pi_\star(S^0_{\mathbb{Z}/2}). \tag{1.2}$$

We also prove convergence using the completion theorem for $BP\mathbb{R}$.

Section 5 is devoted to ways of modifying the spectral sequence (1.2) so that it would converge to the non-equivariant stable 2-stems. To this end, we apply a "Galois descent" method. The key notion is a *graded spectrum*. This is simply a $\mathbb{Z}$-graded sequence of spectra. (The idea that it may be advantageous to consider objects consisting of several spectra was inspired by the work of Bousfield [9].) In effect, we show in Section 5, that the Hopf algebroids corresponding to $E\mathbb{R}(n)$, considered in our category of graded spectra, are flat. Unfortunately, this is not the case for $BP\mathbb{R}$ (the difficulty is caused by the fact that the geometric fixed points of $M\mathbb{R}$ are non-trivial and equal to MO). Nevertheless, motivated by the case of $E\mathbb{R}(n)$, we define an Adams–Novikov-type spectral sequence based on the graded version of $BP\mathbb{R}$, and show that it converges to the 2-completion of $\pi_* S^0_{\{e\}}$. We also explain algebro-geometric interpretations of our discussions, and their connections with the work of Hopkins and Miller [20].

In Section 6, we develop the theory of $\mathbb{Z}/2$-equivariant Steenrod algebra. This partially overlaps with previous work of Greenlees [16–18] in considering the Borel (co)homology Steenrod algebra. We introduce a slightly different setup and notation, which are more suitable for investigating the connections with our Real Adams–Novikov spectral sequence.

We also consider the $\mathbb{Z}/2$-equivariant Steenrod algebra $A^m_\star$ based on the 'honest' $\mathbb{Z}/2$-equivariant Eilenberg–MacLane spectrum $H\mathbb{Z}/2_m$ associated with the constant Mackey functor $\mathbb{Z}/2$. We show that this Steenrod algebra has some remarkable properties. For example, $H\mathbb{Z}/2_m$ is a *flat spectrum*, and there is an $A^m_\star$-based Adams spectral sequence converging to the homotopy of X for any finite $\mathbb{Z}/2$-equivariant spectrum X. Also, $A^m_\star$ splits into a semidirect product of Hopf algebras of extended powers and Milnor primitives in the same way as the non-equivariant Steenrod algebra A_* for $p > 2$. The Steenrod algebra $A^m_\star$ is closely tied to $BP\mathbb{R}$: the description of $A^m_\star$ intrinsicly involves $H_\star BP\mathbb{R}$, unlike the Borel cohomology Steenrod algebra, which can be described directly using the non-equivariant Steenrod algebra and coefficients of Borel cohomology.

For other work on the Mackey functor Steenrod algebra $A^m_\star$, see [11,12]. There is also an important connection between our work and the work of Voevodsky [53] on the motivic Steenrod algebra at the prime 2. In several comments throughout this paper, we will point out analogies between $\mathbb{Z}/2$-equivariant Mackey-functor cohomology and Voevodsky's motivic cohomology of schemes. This analogy culminates in a surprising similarity of Steenrod algebras. More details on relations between $\mathbb{Z}/2$-equivariant and motivic cohomology can be found in [25].

In Section 7, we first write down the $\mathbb{Z}/2$-equivariant Adams spectral sequence for $BP\mathbb{R}$, and use it to eliminate the possible error terms in our structure formulas of the Hopf algebroid $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$ given in Section 4. Then, we construct the Real algebraic spectral sequence. This is our analogue of the Miller–Novikov spectral sequence [38,39,44]. It is the spectral sequence converging to the Real Adams–Novikov spectral E_2-term, based on the Adams filtration on $BP\mathbb{R}_\star$. Similar to the classical case of the Miller–Novikov spectral sequence for $p > 2$, the E_1-term of our algebraic spectral sequence coincides with the E_2-term of a certain Cartan–Eilenberg spectral sequence associated with the $\mathbb{Z}/2$-equivariant Steenrod algebra. We conjecture that the Cartan–Eilenberg spectral sequence collapses, thus showing that our Real Adams–Novikov E_2-term is sparser than the $\mathbb{Z}/2$-equivariant Adams E_2-term.

2. Preliminaries on real-oriented spectra

In this section, we shall review the theory of real cobordism and real-oriented spectra discovered by Landweber [27,28], and Araki [4–6]. The reader might find this review of known results useful, as the language of equivariant stable homotopy theory has changed quite substantially since Landweber's and Araki's work.

In this paper, $\mathbb{Z}/2$-equivariant spectra will always be indexed over the complete universe (abbr. $\mathbb{Z}/2$-spectra).

Definition 2.1. Let $\mathbb{S}^1$ denote the group $S^1 \subset \mathbb{C}^*$ with complex conjugation as an involution. We shall consider $\mathbb{S}^1$ a based space with the base point 1. A Real space (group) in the sense of Atiyah and Segal is a space (group) with involution. Let $B\mathbb{S}^1$ be the classifying space of $\mathbb{S}^1$, which is the space of all complex lines in $\mathbb{C}^\infty$ with complex conjugation as involution.

Note that we do have

$$\Omega B\mathbb{S}^1 \cong \mathbb{S}^1$$

in the category of based $\mathbb{Z}/2$-spaces. Thus, by adjunction, we obtain a canonical equivariant-based map

$$\eta : S^{1+\alpha} \to B\mathbb{S}^1,$$

where α is the non-trivial one-dimensional real representation of $\mathbb{Z}/2$.

Definition 2.2. Let E be a $\mathbb{Z}/2$-equivariant commutative associative ring spectrum. A Real orientation of E is a map

$$u : B\mathbb{S}^1 \to \Sigma^{1+\alpha}E$$

which makes the following diagram commute:

$$\begin{array}{ccc} S^{1+\alpha} & \xrightarrow{\eta} & B\mathbb{S}^1 \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle u} \\ & & \Sigma^{1+\alpha}E \end{array}$$

Remark. According to the terminology of Adams [1], this would be called a *strict orientation*.

Examples. First consider the $\mathbb{Z}/2$-equivariant Eilenberg–MacLane spectrum $H\mathbb{Z}_m$ associated with the constant Mackey functor $\mathbb{Z}$.

Lemma 2.3. *We have* $B\mathbb{S}^1 \simeq (H\mathbb{Z}_m)_{1+\alpha}$.

Proof. Consider the cofibration of spectra

$$\begin{array}{ccccc} \Sigma\mathbb{Z}/2_+ \wedge H\mathbb{Z}_m & \longrightarrow & \Sigma H\mathbb{Z}_m & \longrightarrow & \Sigma^{1+\alpha} H\mathbb{Z}_m \\ & & & & \downarrow \\ & & \Sigma^2 H\mathbb{Z}_m & \longleftarrow & \Sigma^2\mathbb{Z}/2_+ \wedge H\mathbb{Z}_m \end{array} \tag{2.4}$$

Passing to infinite loop spaces and fixed points, the last two maps give a fibration

$$(H\mathbb{Z}_m)^{\mathbb{Z}/2}_{1+\alpha} \to \mathbb{C}P^\infty \to \mathbb{C}P^\infty, \tag{2.5}$$

where the right-hand map is the squaring of a line bundle, which is 2 on homotopy groups (this is by the definition of the constant Mackey functor $\mathbb{Z}$). By (2.5), $(H\mathbb{Z}_m)^{\mathbb{Z}/2}_{1+\alpha} \simeq B\mathbb{Z}/2$. Moreover, from (2.4) we see that the first map

$$B\mathbb{Z}/2 \overset{\subset}{\to} \mathbb{C}P^\infty$$

of (2.5) coincides with the inclusion

$$(H\mathbb{Z}_m)^{\mathbb{Z}/2}_{1+\alpha} \to (H\mathbb{Z}_m)_{1+\alpha} \tag{2.6}$$

given by forgetting the $\mathbb{Z}/2$-action. We see (by $\mathbb{Z}/2$-equivariant CW approximation) that this characterizes the space $B\mathbb{S}^1$ up to $\mathbb{Z}/2$-equivariant weak homotopy type. □

Comment. This result has an analogue in Voevodsky's motivic cohomology [43,52]. Since we plan to follow this analogy throughout this paper, we introduce some of its basic notions. Morel and Voevodsky [43] introduce a category of spaces over a field k (or k-spaces), which are simplicial sheaves in the category of smooth schemes over *Spec*(k) with the Nisnevich topology. One can put

$$S^1 = \mathbb{A}^1/\{0,1\}, \qquad S^\alpha = \mathbb{G}_m = \mathbb{A}^1 - \{0\}.$$

(Voevodsky's original notation is $S^1_s = S^1$, $S^1_t = S^\alpha$.) A fundamental fact is that in Voevodsky's category of k-spaces, one has

$$S^1 \wedge S^\alpha \simeq \mathbb{P}^1.$$

Now Voevodsky defines a notion of spectra in his category, i.e. k-spectra. If we do not insist on understanding point set rigidity questions (which were worked out in [22,24,26]), one can characterize (a generalized cohomology theory associated with) a k-spectrum by a sequence of k-spaces $E_{k(1+\alpha)}$ with

$$E_{k(1+\alpha)} \simeq Map(\mathbb{P}^1, E_{(k+1)(1+\alpha)})$$

(where *Map* (X, Y), for the moment, denotes the k-space of based maps from X to Y).

Now Morel and Voevodsky [43,52] show that one can define the motivic cohomology k-spectrum $H\mathbb{Z}_{\mathrm{Mot}}$ by

$$(H\mathbb{Z}_{\mathrm{Mot}})_{k(1+\alpha)} = SP^\infty(\mathbb{P}^k/\mathbb{P}^{k-1})$$

(where SP^∞ is an algebraic version of the symmetric product). In particular,

$$(H\mathbb{Z}_{\mathrm{Mot}})_{1+\alpha} \simeq \mathbb{P}^\infty.$$

We remark that the motivic cohomology k-spectrum has some remarkable properties. For example, its coefficients are Bloch's higher Chow groups [54].

Now for $k = \mathbb{R}$, there is a forgetful functor t from k-spaces to $\mathbb{Z}/2$-spaces, which takes a smooth scheme X to the space of its complex points $X(\mathbb{C})$ with the analytic topology, and the natural action of

$$\mathbb{Z}/2 \cong Gal(\mathbb{C}/\mathbb{R}).$$

This forgetful functor t extends to spectra.

Now back in the category of $\mathbb{Z}/2$-spaces, following Voevodsky's method, one can actually show that

$$(H\mathbb{Z}_m)_{k(1+\alpha)} = SP^\infty(S^{k(1+\alpha)}).$$

It follows that

$$t(H\mathbb{Z}_{\mathrm{Mot}}) = H\mathbb{Z}_m.$$

The reader is referred to [25] for other interesting properties of the functor t.

By Lemma 2.3, $H\mathbb{Z}_m$ is Real-oriented (by the identity $B\mathbb{S}^1 \to B\mathbb{S}^1$).

In the sequel, we will also refer to the mod 2 motivic cohomology k-spectrum

$$H_{\mathrm{Mot}} = H\mathbb{Z}/2_{\mathrm{Mot}} = H\mathbb{Z}_{\mathrm{Mot}} \wedge M\mathbb{Z}/2,$$

where $M\mathbb{Z}/2$ is the cofiber of the self-map 2 of the sphere k-spectrum, and the corresponding $\mathbb{Z}/2$-equivariant spectrum of cohomology with coefficients in the constant Mackey functor $\mathbb{Z}/2$:

$$H\mathbb{Z}/2_m = H\mathbb{Z}_m \wedge M\mathbb{Z}/2.$$

Next, recall some more of Atiyah's terminology: A Real bundle is a complex bundle with a $\mathbb{Z}/2$-action which acts by an antilinear map. Following Atiyah and Segal, we denote by $\mathbb{U}$ the unitary group with the complex conjugation as an involution. We denote by $B\mathbb{U}$ the classifying space of this Real group, which is the Real space of all infinite-dimensional subspaces of $\mathbb{C}^\infty$ with complex conjugation as an involution. Using Real Bott periodicity, $B\mathbb{U}$ is the 0th space of a canonical $\mathbb{Z}/2$-spectrum, called Real K-theory KR. Moreover, following analogous notation, we can consider Thom spaces $B\mathbb{U}(n)^{\gamma_n}_{\ddagger}$ where the involution is, again, complex conjugation. Thus, we obtain maps

$$\Sigma^{1+\alpha} B\mathbb{U}(n)^{\gamma_n}_{\ddagger} \to B\mathbb{U}(n+1)^{\gamma_{n+1}}_{\ddagger}.$$

There arises a $\mathbb{Z}/2$-spectrum $M\mathbb{R}$, which was originally defined by Landweber [28] (who also introduced the notation $M\mathbb{R}$, perhaps to avoid the potentially more ambiguous $M\mathbb{U}$).

Proposition 2.7 (Araki [6]). *The spectrum $M\mathbb{R}$ is Real-oriented.*

Proof. Same as in the complex case. $B\mathbb{S}^1$ is the Thom space of the canonical Real line bundle on $B\mathbb{S}^1$, and therefore the first term of the prespectrum defining $M\mathbb{R}$. □

Theorem 2.8. *The spectrum KR (Real K-theory) is Real-oriented.*

Proof. First, one can use the tensor product of Real bundles to show that KR is a commutative associative ring spectrum. Next, Atiyah [7] proved that the Bott map refines to produce both a homotopy equivalence

$$b : \Sigma^{1+\alpha} KR \to KR,$$

and a KR-orientation on every Real line bundle. □

Convention 2.9. For an equivariant spectrum E, by E_*, E^* we shall mean coefficients, homology and cohomology in integral dimensions. By $E_\star, E^\star$ we shall mean coefficients, homology and cohomology in all the dimensions $k + \ell\alpha$, $k, \ell \in \mathbb{Z}$.

Comment. It is more common to denote (at least the $\mathbb{Z}$-graded) G-equivariant coefficient groups by $\pi^G_*(E)$ or E^G_*. However, we find the simplified Notation 2.9 much more convenient in calculations. To avoid confusion in cases when we have to consider the same spectrum E as a G-spectrum for different groups G, we shall make this explicit by replacing E with E_G.

Theorem 2.10 (Araki [6]). *If E is a Real-oriented spectrum, then*

$$E^{\star}B\mathbb{S}^1 = E^{\star}[[u]], \qquad \dim(u) = -(1+\alpha),$$

$$E^{\star}(B\mathbb{S}^1 \times B\mathbb{S}^1) = E^{\star}[[u\otimes 1, 1\otimes u]].$$

Moreover, $E \wedge B\mathbb{S}^1$ is a wedge of suspensions of E by $k(1+\alpha)$, $k = 0,1,2,\ldots$. Thus, using the map on classifying spaces induced by the product $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$, a Real-oriented spectrum specifies a formal group law over the ring $E^{\star}$.

Comment 2.11. The grading in $E_{\star}$ is such that the FGL $F(x,y)$ is homogeneous of degree 0 if both x, y are of dimension $-(1+\alpha)$. Also, isomorphisms of formal group laws are formal power series homogeneous in dimension 0, where x has dimension $-(1+\alpha)$.

However, the grading raises an even more serious issue. Note that even non-equivariantly, coefficient rings of commutative ring spectra are not necessarily commutative: they are *graded-commutative*. $RO(G)$-graded coefficient rings of equivariant commutative ring spectra need not be even graded-commutative, at least in the ordinary non-equivariant sense (see [2]).

For the purposes of this paper, we specialize to the case $G = \mathbb{Z}/2$. Recall that

$$\pi_0^{\mathbb{Z}/2}(S^0) = A(\mathbb{Z}/2),$$

where $A(G)$ denotes the Burnside ring, i.e. the Grothendieck ring associated with the semiring of finite G-sets with respect to disjoint sum and Cartesian product.

Lemma 2.12. *If E is a commutative $\mathbb{Z}/2$-equivariant ring spectrum (indexed over the complete universe), and $x \in E_{k+\ell\alpha}$, $y \in E_{m+n\alpha}$, then*

$$xy = (-1)^{km}\varepsilon^{\ell n} yx \in E_{(k+m)+(\ell+n)\alpha},$$

where in the Burnside ring $A(\mathbb{Z}/2)$, ε is represented by $[1]-[\mathbb{Z}/2]$. (Note that, in particular, $\varepsilon^2 = 1$.)

Comment. We explain why there is no sign attached to switching trivial representations with α. This is because, precisely speaking,

$$\begin{aligned} xy &\in [S^k \wedge S^{\ell\alpha} \wedge S^m \wedge S^{n\alpha}, E],\\ yx &\in [S^m \wedge S^{n\alpha} \wedge S^k \wedge S^{\ell\alpha}, E]. \end{aligned} \tag{2.13}$$

The groups on the right-hand side are not identical. To identify them, we allow composition with the permutation maps

$$T_{1\alpha}: S^{\alpha} \wedge S^1 \to S^1 \wedge S^{\alpha}. \tag{2.14}$$

Using such compositions, the two groups on the right-hand side are *uniquely* identified with

$$E_{k+m+(\ell+n)\alpha} = [S^{k+m} \wedge S^{(\ell+n)\alpha}, E].$$

However, the choice of the maps $T_{1\alpha}$ as identifications in (2.14) is a matter of convention.

Proof. This can be found in [2], and also in Morel's work on motivic ring spectra [42]. We reproduce the argument here to keep our reasoning self-contained. It suffices to show that the map

$$S^{2\alpha} \to S^{2\alpha} \tag{2.15}$$

obtained by switching the two copies of α is stably homotopic to ε. First, (2.15) is stably homotopic to the map

$$v : S^{\alpha} \to S^{\alpha} \tag{2.16}$$

which reverses the sign of the coordinate. Thus, we need to show that v is stably homotopic to ε. But recalling the Pontrjagin–Thom construction, we see that the map $[\mathbb{Z}/2] \in A(\mathbb{Z}/2)$ is realized by the map

$$\lambda : S^{\alpha} \to S^{\alpha}$$

which is squaring when we identify S^{α} with $\mathbb{S}^1$, and $-[\mathbb{Z}/2] \in A(\mathbb{Z}/2)$ is represented by

$$\mu : S^{\alpha} \to S^{\alpha}$$

which is $z \mapsto z^{-2}$ when we identify S^{α} with $\mathbb{S}^1$. But now if

$$\iota : S^{\alpha} \to S^{\alpha}$$

is the identity, we easily see that stably

$$v = \mu + \iota,$$

and hence

$$\varepsilon = -[\mathbb{Z}/2] + 1 \in A(\mathbb{Z}/2),$$

as claimed. □

To properly interpret the above proposition, we assert the following.

Lemma 2.17. *If E is a Real-oriented spectrum, then*

$$\varepsilon = -1 \in E_{\star}. \tag{2.18}$$

Thus, $E_{*(1+\alpha)} \subset E_{\star}$ is a commutative ring.

Remark. (1) If we choose the maps $-T_{1\alpha}$ instead of $T_{1\alpha}$ in (2.14) to identify the right-hand sides of (2.13), then $E_{\star}$ becomes a graded-commutative ring, where the grading is by $k + \ell$.

(2) By Theorem 2.25 below, it suffices to consider $E = M\mathbb{R}$. We will later see that $M\mathbb{R}_0 = MU_0 = \mathbb{Z}$. Since (2.18) holds non-equivariantly (i.e. in the target of the forgetful map $E_{\star} \to (E_{\{e\}})_*$), it therefore holds equivariantly because the forgetful map is iso in that dimension. However, we will give a more geometric proof.

Proof. First of all, both of the elements $[\mathbb{Z}/2]$ and 2 of the Burnside ring come from the unstable homotopy group $\pi_{1+\alpha}S^{1+\alpha}$. We have a canonical diagram

$$\begin{array}{ccc} \pi_{1+\alpha}S^{1+\alpha} & \longrightarrow & \pi_0^{stable}S^0_{Z/2} \\ \downarrow & & \downarrow \\ \pi_{1+\alpha}(B\mathbb{S}^1)^{\gamma^1} & \longrightarrow & \pi_0^{stable}M\mathbb{R}, \end{array} \tag{2.19}$$

because $(B\mathbb{S}^1)^{\gamma^1}$ is the $(1+\alpha)$'s space of the prespectrum defining $M\mathbb{R}$.

Thus, it suffices to show that $[\mathbb{Z}/2]$ and 2 represent the same element in

$$\pi_{1+\alpha}(B\mathbb{S}^1)^{\gamma_1}. \tag{2.20}$$

But $(B\mathbb{S}^1)^{\gamma^1} = B\mathbb{S}^1$, so (2.20) is

$$\pi_{1+\alpha}(B\mathbb{S}^1). \tag{2.21}$$

But (2.21) classifies Real line bundles on $S^{1+\alpha}$, and is equal to $\mathbb{Z}$. □

Before proving Theorem 2.10, we introduce one additional notion: by a Real CW-complex we shall mean a $\mathbb{Z}/2$-space $K = \bigcup_n K_n, K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$, where K_0 is a discrete fixed set, and K_n is a pushout of a diagram

$$\begin{array}{ccc} K_{n-1} & & \\ \uparrow & & \\ \coprod \mathbb{S}^{2n-1} & \longrightarrow & \coprod \mathbb{D}^{2n}, \end{array} \tag{2.22}$$

where

$$\mathbb{S}^{2n-1} = \{z \in \mathbb{C}^n \mid |z| = 1\},$$
$$\mathbb{D}^{2n} = \{z \in \mathbb{C}^n \mid |z| \leqslant 1\}$$

have $\mathbb{Z}/2$-action by conjugation, and the maps in (2.22) are equivariant.

We call the filtration

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K$$

the Real CW-filtration of K and call K_n the Real n-skeleton of K. A real CW-map is a map-preserving Real CW-filtrations.

Lemma 2.33. *Let K, L be Real CW-complexes. Any equivariant continuous map $f: K \to L$ is homotopic to a Real CW-map $\phi: K \to L$.*

Proof. The map ϕ is obtained inductively by iterating the following process: suppose $K' \subseteq K$ is a Real subcomplex including K_{n-1} and suppose $f|K'$ is Real CW. Consider an n-cell e in K and an

m-cell e' in L such that $n < m$ and

$$f(\mathrm{Int}(e)) \cap \mathrm{Int}(e') \neq 0.$$

We introduce a filtration on $\mathbb{D}^{2n}$ by

$$\mathbb{D}_0^{2n} \subset \cdots \subset \mathbb{D}_n^{2n} = \mathbb{D}^{2n},$$

where $\mathbb{D}_i^{2n}$ is the unit disk of a real subspace spanned by n real and i imaginary coordinates. By induction, we will construct maps f_i homotopic to f such that

$$f_i(K' \cup \mathbb{D}_i^{2n}) \subseteq L_n.$$

To construct f_0, we have $f(\mathbb{D}_0^{2n}) \subseteq \mathbb{D}_0^{2m}$, and

$$\dim \mathbb{D}_0^{2n} = n < m = \dim \mathbb{D}_0^{2m}.$$

Thus, f is homotopic rel K' to a map g such that

$$\exists z \in \mathbb{D}_0^{2m} - g(\mathbb{D}_0^{2n}).$$

Contracting via a radial projection from z and then extending by the cofibration property gives the map f_0.

Suppose f_i is constructed. Notice that $\mathbb{D}_{i+1}^{2n}$ is obtained from $\mathbb{D}_i^{2m}$ by attaching an equivariant cell $\mathbb{Z}/2_+ \wedge D^{n+i+1}$. This equivariant $(n+i+1)$-cell is contracted into L_n in the same way as the non-equivariant cell D^{n+i+1}. Thus, f_{i+1} is constructed. □

Now let E be a $\mathbb{Z}/2$-spectrum and let X be a Real CW-complex. Then, similarly as in the non-equivariant case, the Real n-skeleta X_n of X form a sequence of $\mathbb{Z}/2$-cofibrations

$$X_{n-1} \to X_n \to \bigvee_{\text{Real } n\text{-cells}} S^{n(1+\alpha)}.$$

Applying E-cohomology, we obtain an exact couple which gives a conditionally convergent spectral sequence

$$E_1^{p,q} = \bigoplus_{\substack{e \text{ real} p\text{-cell} \\ \text{of } X}} E^q \Rightarrow E^{p(1+\alpha)+q}(X). \tag{2.24}$$

Here $q \in RO(\mathbb{Z}/2)$, so $q = q' + q''\alpha$, $q', q'' \in \mathbb{Z}$. The differentials are

$$d_r : E_r^{p,q} \to E_r^{p+r,\, q+1-r(1+\alpha)}.$$

Further, if E is a ring spectrum, (2.24) has a ring structure in the obvious way. Also, by Lemma 2.23, the spectral sequence (2.24) is functorial with respect to equivariant maps in X. It should be pointed out that (2.24) is not the usual equivariant Atiyah–Hirzebruch spectral sequence.

Proof of Theorem 2.10. Using Schubert cells, we see that $B\mathbb{S}^1$ is a Real CW-complex. $E^\star[[u]]$ is the E^1-term of the spectral sequence (2.24) for $X = B\mathbb{S}^1$. Because E is Real-oriented, u is a permanent cycle. Similarly for $X = B\mathbb{S}^1 \times B\mathbb{S}^1$. □

Theorem 2.25. *For a Real oriented spectrum E, every Real bundle is E-oriented. Moreover, we can identify*

$$E^{\star}M\mathbb{R} = E^{\star}[[b_1, b_2, \ldots]], \tag{2.26}$$

where the b_i's are the elementary symmetric series in infinitely many indeterminates $x_1, x_2, \ldots$. Moreover, in this notation, those maps from (2.26) *which are maps of ring spectra $M\mathbb{R} \to E$ correspond under* (2.26) *to series in the b_i's which, when expressed in terms of the x_i's, are of the form*

$$\prod_i \frac{f(x_i)}{x_i},$$

where f is a strict isomorphism of formal group laws. Therefore, Real orientations of E are in bijective correspondence with maps of ring spectra $M\mathbb{R} \to E$, which, again, are in bijective correspondence with strict isomorphisms whose source is the formal group law of E.

Proof. To prove the assertion about Real-oriented bundles, it suffices to consider the case of the universal Real n-bundle, which is the canonical n-bundle on $B\mathbb{U}(n)$. Note that the Thom space $B\mathbb{U}^{\gamma_n}$ is a Real CW-complex (using Schubert cells). Consider the spectral sequences (2.24) for $X = B\mathbb{U}^{\gamma_n}$ and $X' = ((B\mathbb{S}^1)^n)^{\gamma_n}$. By functoriality, the canonical inclusion $X' \to X$ induces a map of spectral sequences. Further, this map of spectral sequences is injective on E_1-terms. (The E_1-term for X' injects as the algebra of symmetric polynomials.) Now since the spectral sequence for X' collapses (because E is Real-oriented and by the Künneth theorem), the spectral sequence for X also collapses.

Passing to the limit, we obtain the calculation of $E^{\star}M\mathbb{R}$. To verify the statement about products, we notice that by a similar method, we can calculate $E^{\star}(M\mathbb{R} \wedge M\mathbb{R})$. The reader is referred to the non-equivariant analogue of this result (e.g. [1]) for the rest of the proof. □

To proceed further, we need some initial information about the coefficient ring $M\mathbb{R}_{\star}$. A complete calculation will be given later in Section 4. Recall [31] that for any G-spectrum E there is a canonical map

$$E^G \to E_{\{e\}}.$$

Proposition 2.27. *The canonical map*

$$M\mathbb{R}_{\star} \to MU_*$$

splits by a map of rings

$$MU_* \to M\mathbb{R}_{\star}$$

which sends the generator $x_i \in MU_*$ into an element of dimension $i(1+\alpha)$. (By abuse of notation, this element of $M\mathbb{R}_{\star}$ will be also denoted by x_i.)

Proof. Use Theorem 2.10 and the fact that MU_* is the universal formal group law. □

Remark. (1) By a (weakly) Real manifold of dimension $k + \ell\alpha$ we shall mean a smooth manifold M with smooth $\mathbb{Z}/2$-action together with a Real bundle ν of Real (= complex) dimension m, together with an isomorphism of $\mathbb{Z}/2$-bundles

$$\tau_M \oplus v \cong k + \ell\alpha + m(1+\alpha).$$

While we do not have a transversality theorem which would say that Real cobordism actually is the cobordism theory of Real manifolds, the Pontrjagin–Thom construction still gives, for a Real manifold of dimension $k + \ell\alpha$, a class in $\pi_{k+\ell\alpha} M\mathbb{R}$.

Now we claim that the $\mathbb{C}$-points M of an n-dimensional smooth projective variety *defined over* $\mathbb{R}$ are a Real manifold of dimension $n(1+\alpha)$. This is easy: the tangent bundle comes with a Real structure by complex conjugation. Embed τ_M as real bundles into $M \times \mathbb{C}^n, n >> 0$ (this can be done by equivariant partition of unity). Then the unitary complement of τ_M in $M \times \mathbb{C}^n$ is a Real bundle of the required dimension.

Thus, we have explicit representatives for the elements $x_i \in M\mathbb{R}_\star$ if the elements $x_i \in MU_*$ are represented by $\mathbb{C}$-points of (not necessarily connected) smooth algebraic varieties defined over $\mathbb{R}$. Such varieties are given in the literature (see, e.g., [48, p. 130]).

(2) The simplest example showing that $M\mathbb{R}_\star$ does *not*, in fact calculate the cobordism ring of Real manifolds is provided by Lemma 2.17. That lemma asserts that the Real manifolds $\mathbb{Z}/2$, $* \amalg *$ represent the same element in $M\mathbb{R}_\star$. On the other hand, it is easy to see that a fixed Real manifold can only be Real-cobordant to another fixed manifold. For more details, see [23].

(3) There is yet another approach to the generators

$$x_i \in M\mathbb{R}_\star.$$

In Section 4 below, we will construct a spectral sequence converging to $M\mathbb{R}_\star$, where differentials are determined by primary $\mathbb{Z}/2$-equivariant Steenrod operations (see Theorem 4.1). These operations can be computed directly, in fact, yielding a stronger result (see Section 4 below, after Lemma 4.5):

Theorem 2.28. *For* $k \in \mathbb{Z}$, *we have*

$$M\mathbb{R}_{k(1+\alpha)} \cong MU_{2k}.$$

Corollary 2.29. *The formal group law F associated with Real cobordism* (*via Theorem* 2.10) *is Lazard's universal* (*one-dimensional commutative*) *formal group law* (*see Comment* 2.11 *for a discussion of the grading*).

Next, recall some of the standard methods of equivariant stable homotopy theory. Recall the cofibration of $\mathbb{Z}/2$-spaces

$$E\mathbb{Z}/2_+ \to S^0 \to \widetilde{E\mathbb{Z}/2}.$$

Now smash with an equivariant $\mathbb{Z}/2$-spectrum E indexed over the complete universe and take $\mathbb{Z}/2$-fixed points. There are two facts to recall: first, the Adams isomorphism [31] gives

$$(E\mathbb{Z}/2_+ \wedge E)^{\mathbb{Z}/2} \simeq E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} E.$$

(More precisely, the right-hand side is the pullback of $E\mathbb{Z}/2_+ \wedge E$ to the trivial universe, factored through the action of $\mathbb{Z}/2$ in the sense of [31].) If E were a split spectrum in the sense of Lewis et al. [31], this would be further equal to $E_{\{e\}} \wedge B\mathbb{Z}/2_+$, but we shall soon see that $E = M\mathbb{R}$ is not a split spectrum (if $M\mathbb{R}$ were split, Real-oriented spectra would essentially give nothing new).

Next, we have

$$\widetilde{E\mathbb{Z}/2} \simeq S^V, \tag{2.30}$$

where $V = \infty\alpha$. Thus, $(\widetilde{E\mathbb{Z}/2} \wedge E)^{\mathbb{Z}/2}$ is the spectrum of geometrical fixed points of E, which is usually denoted by $\Phi^{\mathbb{Z}/2}E$. (More generally, for any compact Lie group G, one has

$$\Phi^G E = (\widetilde{E\mathscr{P}} \wedge E)^G$$

where $\mathscr{P}$ is the family of all proper subgroups of G — see [31].) Therefore, we obtain a cofibration sequence of non-equivariant spectra

$$E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} E \to E^{\mathbb{Z}/2} \to \Phi^{\mathbb{Z}/2}E. \tag{2.31}$$

We now turn to the case $E = M\mathbb{R}$.

Recall that geometrical fixed points can be calculated on the prespectrum level by taking fixed points both on the spaces and the structure maps. Using this method, Araki [4] found that

$$\Phi^{\mathbb{Z}/2}M\mathbb{R} = MO.$$

On the other hand, $E\mathbb{Z}/2_+$ has the usual filtration (for example skeletal in the usual simplicial model). We obtain a spectral sequence

$$E^2_{p,q} = H_p(\mathbb{Z}/2, \pi_q MU) \Rightarrow \pi_{p+q}(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} M\mathbb{R}). \tag{2.32}$$

A closer inspection shows that $\mathbb{Z}/2$ acts trivially on $\pi_q MU$ for $q = 0 \bmod 4$ and by minus one for $q = 2 \bmod 4$. (Thus, $M\mathbb{R}$ cannot be split, for then $\mathbb{Z}/2$ would act trivially.)

Theorem 2.33 (Araki [6]). *Localize at $p = 2$. Let*

$$MU = \bigvee_{m_i} \Sigma^{2m_i} BP$$

for integers m_i. Then there exists a spectrum $BP\mathbb{R}$ such that

$$M\mathbb{R} = \bigvee_{m_i} \Sigma^{m_i(1+\alpha)} BP\mathbb{R}.$$

Further, $\Phi^{\mathbb{Z}/2}BP\mathbb{R} = H\mathbb{Z}/2$ and there exists a spectral sequence

$$E^2_{p,q} = H_p(\mathbb{Z}/2, BP_q) \Rightarrow (E\mathbb{Z}/2_+ \wedge BP\mathbb{R})_*.$$

and the action of $\mathbb{Z}/2$ on BP_* is obtained by restriction from the case of MU.

Remark. The numbers m_i range over dimensions of additive free generators of a symmetric algebra — not in the graded sense — on generators in dimensions i, $i \neq 2^n - 1$, see [48].

By considering the cofibration sequence (2.31) for $\Sigma^{k(1+\alpha)}$, $k \geqslant 0$, one sees that on the level of coefficients (in integral dimensions), there is no room for extensions or connecting homomorphisms, except in the case $k = 0$ when there is an easily detectible extension ($1 \in H\mathbb{Z}/2_*$ is the actual unit). In particular, we obtain the following result:

Corollary 2.34. *The connecting homomorphism of* (2.31) *is zero on coefficients* (*in integral dimensions*) *for* $E = M\mathbb{R}$.

Proof of Theorem 2.33. The Real-oriented spectrum $BP\mathbb{R}$ is obtained by applying the Quillen idempotent [45] (which works for any formal group law), using Theorem 2.25. On geometric fixed points, we obtain, by (2.31), the usual splitting of MO by the Quillen idempotent, which produces $H\mathbb{Z}/2$. Therefore, we have $\Phi^{\mathbb{Z}/2}BP\mathbb{R} = H\mathbb{Z}/2$, the element $1 \in \pi_* H\mathbb{Z}/2$ is the actual unit and on π_0, the cofibration sequence (2.31) becomes the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Now the Quillen idempotent provides a splitting

$$M\mathbb{R} \xrightarrow{r} BP\mathbb{R} \xrightarrow{\gamma} M\mathbb{R}, \quad r\gamma \simeq Id.$$

Next, we use Proposition 2.27. Consider the wedge of all compositions

$$BP\mathbb{R} \xrightarrow{\gamma} M\mathbb{R} \xrightarrow{\mu} M\mathbb{R},$$

where μ is multiplication by products of x_n, $n \neq 2^k - 1$. Note that $\vee \mu\gamma$ is an equivalence non-equivariantly, but also on geometric fixed points (since it simply defines the standard splitting of MO into copies of $H\mathbb{Z}/2$). Thus, $\vee \mu\gamma$ is a $\mathbb{Z}/2$-equivalence. □

3. Some constructions and calculations

In this section, we prove some of the more advanced results on spectra constructed from $M\mathbb{R}$. First, using the E_∞-module theory of Elmendorf et al. [14], we define real analogues of all the standard complex oriented spectra. We show that real K-theory defined in this way coincides with Atiyah's real K-theory [7]. In the second part of the Section, we calculate the coefficients of real Morava K-theories.

Proposition 3.1. *$M\mathbb{R}$ is an E_∞ ring spectrum and, after taking fixed points, the right-hand arrow $M\mathbb{R}^{\mathbb{Z}/2} \to \Phi^{\mathbb{Z}/2}M\mathbb{R}$ of* (2.31) *is a map of E_∞ ring spectra.*

Proof. The proof that $M\mathbb{R}$ is E_∞ is exactly the same as for MU (that proof was given in May et al. [35] and in Lewis's thesis, see [31]). To prove the second statement of the proposition, recall that if $\mathscr{U}$ is a complete universe, then the spaces of isometries $\mathscr{I}(\mathscr{U}^n, \mathscr{U})$ form the linear isometries operad. Note that isometries $\mathscr{I}(\mathscr{U}^n, \mathscr{U})^G$ is also a contractible operad. McClure [37] observed that we have a canonical action

$$\mathscr{I}(\mathscr{U}^n, \mathscr{U})^G \ltimes (\underbrace{S^V \wedge \cdots \wedge S^V}_{n \text{ times}}) \to S^V$$

(see (2.30)). There is, of course, also an action

$$\mathscr{I}(\mathscr{U}^n,\mathscr{U})^G \ltimes (\underbrace{S^0 \wedge \cdots \wedge S^0}_{n\text{ times}}) \to S^0,$$

and the inclusion $S^0 \to S^V$ preserves the action. Smashing with $M\mathbb{R}$ and taking fixed points, we obtain the result. □

Now we have, directly by construction,

$$M\mathbb{R}_{\{e\}} = MU.$$

Let $MU_* = \mathbb{Z}[x_1, \ldots, x_n, \ldots]$. By a derived spectrum of MU, we mean any E_∞ MU-module obtained by killing off a regular sequence

$$(z_1, z_2, \ldots) \quad \text{in } MU_*$$

and localizing at elements of MU_* (see [13,14]).

Now by Proposition 2.27, we have a map $\mathbb{Z}[x_1, \ldots, x_n, \ldots] \to M\mathbb{R}_\star$. Thus, for a derived spectrum E of MU, we can construct an E_∞ $M\mathbb{R}$-module $E\mathbb{R}$ by killing off and localizing at the lifts of the relevant elements to $M\mathbb{R}_\star$. In more detail, for each element $z \in M\mathbb{R}_\star$, every $M\mathbb{R}$-module M has a self-map of $M\mathbb{R}$-modules

$$z: M \to M.$$

(Here the notion of module is used in the same sense as in [14], i.e. we mean E_∞-modules, not module spectra.) Now denoting the canonical lifts of the elements z_i to $M\mathbb{R}_\star$ by the same symbols, let $M_0 = M\mathbb{R}$ and let

$$M_{i-1} \xrightarrow{z_i} M_{i-1} \to M_i$$

be a cofibration of $M\mathbb{R}$-modules. Localization is handled similarly. Observe that the lifts $z_i \in M\mathbb{R}_\star$ may not form a regular sequence in $M\mathbb{R}_\star$. Because of that, we do not get an immediate calculation of $E\mathbb{R}_\star$. Non-equivariantly, however, we have

$$E\mathbb{R}_{\{e\}} = E$$

(see also [36]). This will enable us, at least in some cases (when there is an appropriate completion theorem) to calculate $E\mathbb{R}_\star$ by means of a Borel cohomology spectral sequence. We agree to write $K\mathbb{R}(n), E\mathbb{R}(n)$, etc., instead of $K(n)\mathbb{R}, E(n)\mathbb{R}$, etc. These spectra are unfortunately not always Real-oriented in the sense of our definition, because they may not be ring spectra (example: $K\mathbb{R}(n)$ for $n \geqslant 1$). Note that the same difficulty arises also non-equivariantly, although because of the simple structure of MU_*, we can get ring structures on derived spectra of MU much more generically (see [14]).

Our first result concerns a well-known spectrum, the "fixed" version of $\mathbb{Z}/2$-equivariant cohomology, constructed by change of universe.

Proposition 3.2. *If $i : \mathbb{R}^\infty \to \mathscr{U}$ denotes the embedding from the trivial to the complete universe, then the spectrum $i_* H\mathbb{Z}/2$ (which we also denote by $H\mathbb{Z}/2$) is Real-oriented.*

Proof. First construct $BP\mathbb{R}$ by killing all $x_i \in \pi_{i(1+\alpha)} M\mathbb{R}$, $i \neq 2^k - 1$. (These do actually form a regular sequence, by Theorem 2.33.) Next, construct inductively a sequence of $M\mathbb{R}$-modules $\mathscr{P}\mathbb{R}(n)$, where $\mathscr{P}\mathbb{R}(0) = BP\mathbb{R}$, and for $n \geqslant 0$, $\mathscr{P}\mathbb{R}(n+1)$ is the cofiber of the composition

$$\begin{array}{ccc} \Sigma^{(2^n-1)(1+\alpha)} E\mathbb{Z}/2_+ \wedge \mathscr{P}\mathbb{R}(n) & \longrightarrow & \Sigma^{(2^n-1)(1+\alpha)} \mathscr{P}\mathbb{R}(n) \\ & & \big\downarrow{\scriptstyle v_n} \\ & & \mathscr{P}\mathbb{R}(n). \end{array} \tag{3.3}$$

Now let

$$H = ho\lim_{\rightarrow} \mathscr{P}\mathbb{R}(n). \tag{3.4}$$

We claim that $H = H\mathbb{Z}/2$. To this end, first, by definition, we have

$$H_{\{e\}} = H\mathbb{Z}/2_{\{e\}}.$$

Thus, by adjunction, we obtain a ($\mathbb{Z}/2$-equivariant) map

$$\mathbb{Z}/2_+ \wedge H\mathbb{Z}/2 \to H. \tag{3.5}$$

The obstructions to extending this to

$$E\mathbb{Z}/2_+ \wedge H\mathbb{Z}/2 \to H \tag{3.6}$$

lie in the $(1-n)$th non-equivariant $\mathbb{Z}/2$-valued cohomology group of $H\mathbb{Z}/2_{\{e\}}$, $n = 1,2,3,\ldots$. Thus, the only possible non-trivial obstruction is for $n = 1$. But that group is isomorphic to the 0th non-equivariant cohomology of S^0, so it is detected by homotopy and consequently vanishes, since the canonical map $\mathbb{Z}/2_+ \to H$ extends to $S^0 \to H$ (the latter map is induced from the unit $S^0 \to BP\mathbb{R}$).

By looking at non-equivariant coefficients, we see immediately that (3.6) induces a $\mathbb{Z}/2$-equivalence

$$E\mathbb{Z}/2_+ \wedge H\mathbb{Z}/2 \xrightarrow{\simeq} E\mathbb{Z}/2_+ \wedge H. \tag{3.7}$$

Now we proceed to study the 'Tate diagram' of the spectra $H_{\mathbb{Z}/2}$, $H\mathbb{Z}/2_{\mathbb{Z}/2}$. More precisely, we will make use of the fact that H is the homotopy pullback of the diagram

$$\begin{array}{ccc} & & S^{\infty\alpha} \wedge H \\ & & \big\downarrow \\ F(E\mathbb{Z}/2_+, H) & \longrightarrow & \hat{H} \end{array} \tag{3.8}$$

and $H\mathbb{Z}/2$ is the homotopy pullback of the diagram

$$\begin{array}{ccc} & & S^{\infty\alpha}\wedge H\mathbb{Z}/2 \\ & & \downarrow \\ F(E\mathbb{Z}/2_+, H\mathbb{Z}/2) & \longrightarrow & \widehat{H\mathbb{Z}/2} \end{array} \tag{3.9}$$

Here the hat denotes the Tate spectrum, i.e., for any spectrum E,

$$\hat{E} = \widetilde{E\mathbb{Z}/2} \wedge F(E\mathbb{Z}/2_+, E). \tag{3.10}$$

Thus, to prove the proposition, it suffices to give an isomorphism of the diagrams (3.8), (3.9) over the $\mathbb{Z}/2$-equivariant stable homotopy category. We will construct, say, an equivalence from (3.8) to (3.9). Note that, on the bottom row of the diagrams, such equivalence is supplied by (3.7).

On the other hand,

$$S^{\infty\alpha} \wedge H \simeq S^{\infty\alpha} \wedge BP\mathbb{R} \simeq S^{\infty\alpha} \wedge H\mathbb{Z}/2$$

by the construction (3.3) and Theorem 4.1 below.

Thus, we are done if we can choose the cited equivalence on the top and bottom parts of (3.8), (3.9) in such a way that the diagram

$$\begin{array}{ccc} S^{\infty\alpha}\wedge H\mathbb{Z}/2 & \longrightarrow & S^{\infty\alpha}\wedge H \\ \downarrow & & \downarrow \\ \widehat{H\mathbb{Z}/2} & \longrightarrow & \hat{H} \end{array} \tag{3.11}$$

commutes (in the $\mathbb{Z}/2$-equivariant stable category). To this end,

$$[S^{\infty\alpha} \wedge H\mathbb{Z}/2, \widehat{H\mathbb{Z}/2}] \cong [H\mathbb{Z}/2, \widehat{H\mathbb{Z}/2}] \cong [H\mathbb{Z}/2_{\{e\}}, (\widehat{H\mathbb{Z}/2})] \cong \hat{A}^*. \tag{3.12}$$

Here the last term denotes the non-equivariant Steenrod algebra considered as an ungraded $\mathbb{Z}/2$-module, completed at the augmentation ideal I: this is because

$$(\widehat{H\mathbb{Z}/2})^{\mathbb{Z}/2} \simeq \prod_{n\in\mathbb{Z}} \Sigma^n H\mathbb{Z}/2_{\{e\}}.$$

Note that the $\mathbb{Z}/2$-submodule generated by 1 in $\hat{A}^*$ is characterized as the set of all maps

$$S^{\infty\alpha} \wedge H \to \hat{H}, \tag{3.13}$$

where the composition

$$H\mathbb{Z}/2_{\{e\}} \xrightarrow{\simeq} (S^{\infty\alpha} \wedge H)^{\mathbb{Z}/2} \to \hat{H}^{\mathbb{Z}/2}$$

lifts to $F(E\mathbb{Z}/2_+, H)^{\mathbb{Z}/2}$. Since this is clearly true for the element given by passage through the lower-left corner in (3.11), and since 0 can be distinguished from 1 by π_0, it suffices to prove the following claim. □

Claim 3.14. *The map $H\mathbb{Z}/2_{\{e\}} \to \hat{H}^{\mathbb{Z}/2}$ obtained by applying '$\hat{\ }$' to the canonical map $BP\mathbb{R} \to H$ lifts to a map $H\mathbb{Z}/2_{\{e\}} \to F(E\mathbb{Z}/2_+, H)^{\mathbb{Z}/2}$.*

Proof. Clearly, it suffices to prove the statement for $M\mathbb{R}$. In other words, we must show that after applying $\mathbb{Z}/2$-fixed points, the composition

$$S^{\infty\alpha} \wedge M\mathbb{R} \xrightarrow{\simeq} \widehat{M\mathbb{R}} \to \widehat{BP\mathbb{R}} \to \hat{H} \tag{3.15}$$

lifts to $F(E\mathbb{Z}/2_+, H)$. But $M\mathbb{R} \to BP\mathbb{R} \to H$ were chosen as maps of (E_∞)-$M\mathbb{R}$-modules. Therefore, (3.15) comes by free $M\mathbb{R}$-module extension from a map

$$S^{\infty\alpha} \to \hat{H},$$

for which the statement is automatic (the analogue of the group (3.12) is $\mathbb{Z}/2$). □

Remark. Note that $i_*H\mathbb{Z}$ cannot be Real-oriented, because a Real orientation would, on geometric fixed points, induce a map

$$MO = \Phi^{\mathbb{Z}/2}M\mathbb{R} \to \Phi^{\mathbb{Z}/2}i_*H\mathbb{Z} = H\mathbb{Z}_{\{e\}}$$

which would send 1 to 1 (it would be a ring map). Clearly, there is no such map: the source is $\mathbb{Z}/2$, and the target is $\mathbb{Z}$.

On the other hand, the inclusion of universes induces an equivalence of naive spectra

$$i^*H\mathbb{Z}_m \simeq H\mathbb{Z},$$

and hence we get a diagram

$$i_*H\mathbb{Z} \to H\mathbb{Z}_m \to F(E\mathbb{Z}/2_+, i_*H\mathbb{Z}). \tag{3.16}$$

The second map comes from the fact that the first map induces a non-equivariant equivalence, and hence an equivalence on Borel cohomology theories. Similarly, reducing mod 2, we have a diagram of equivariant maps which are homotopy equivalences non-equivariantly

$$i_*H\mathbb{Z}/2 \to H\mathbb{Z}/2_m \to F(E\mathbb{Z}/2_+, i_*H\mathbb{Z}/2). \tag{3.17}$$

Note that we now have two reasons why $H\mathbb{Z}/2_m$ is Real-oriented: Proposition 3.2 and Lemma 2.3.

Formula (3.4) is interesting in and of itself. A similar formula can be obtained also for $H\mathbb{Z}/2_m$. We postpone this until we have more precise calculations (see Proposition 4.9 below).

Our next result is a comparison of Real K-theory with the derived $M\mathbb{R}$-spectrum $K\mathbb{R}$.

Theorem 3.18. *We have $K\mathbb{R} \simeq KR$.*

Proof. We will make use of the following fact:

Lemma 3.19 (Fajstrup [15]). *The spectra*

$$KR, K\mathbb{R}, F(E\mathbb{Z}/2_+, KR), F(E\mathbb{Z}/2_+, K\mathbb{R})$$

are free.

Proof. A $\mathbb{Z}/2$-spectrum E is free if and only if

$$0 = a^{-1}E = \lim_{\rightarrow} (E \stackrel{a}{\rightarrow} E \stackrel{a}{\rightarrow} E \rightarrow \cdots),$$

where

$$a : S^0 \rightarrow S^\alpha$$

is the inclusion. But multiplying a by the Bott periodicity element in dimension $1 + \alpha$, we obtain $\eta \in \pi_1(S^0)$, which is nilpotent. □

Now by Theorems 2.8 and 2.25, we have a Real orientation

$$M\mathbb{R} \rightarrow KR. \tag{3.20}$$

On the other hand, by construction, we get a map

$$M\mathbb{R} \rightarrow K\mathbb{R}. \tag{3.21}$$

Also, KR and $K\mathbb{R}$ both enjoy periodicity self-maps:

$$u : \Sigma^{(1+\alpha)} KR \stackrel{\cong}{\rightarrow} KR,$$
$$u : \Sigma^{(1+\alpha)} K\mathbb{R} \stackrel{\cong}{\rightarrow} K\mathbb{R}.$$

Next, consider the free spectra $E\mathbb{Z}/2_+ \wedge K\mathbb{R}$, $E\mathbb{Z}/2_+ \wedge KR$ and their connective covers $E\mathbb{Z}/2_+ \wedge k\mathbb{R}$, $E\mathbb{Z}/2_+ \wedge kR$. (Note: we just proved in Lemma 3.19 that $K\mathbb{R}$ and KR are, in fact, free spectra.) Now for free spectra, there is an equivariant analogue of the Hurewicz map. We will call this map the *free Hurewicz map*, to avoid conflict with [29], which considers a different concept of equivariant Hurewicz map. Indeed, recall that the derived categories of free spectra over the complete and fixed universes are equivalent via the functors $i_*?$, $EG_+ \wedge (i^*?)$. Thus, it suffices to consider the question over a fixed universe. But there, if

$$\pi_i^{\{e\}} E = 0 \quad \text{for } i < n,$$

obstruction theory gives a map

$$E \rightarrow H_{\text{free}}(\pi_n E, n) \tag{3.22}$$

(the right-hand side denotes the Eilenberg–MacLane spectrum over the fixed universe corresponding to the coefficient system whose value is $\pi_n^{\{e\}} E$ with the prescribed G-action on $\{e\}$ and 0 elsewhere). Moreover, (3.22) induces an isomorphism on π_n.

Therefore, one can develop a theory of Postnikov towers of connective free spectra, analogous to the non-equivariant case. In more detail, we first get a tower of the form

$$\begin{array}{ccc} E^{k+1} & & \\ \downarrow & & \\ E^{k} & \xrightarrow{\phi_k} & H_{\text{free}}(\pi_k E,k) \\ \downarrow & & \\ \vdots & & \\ \downarrow & & \\ E^{0} & \xrightarrow{\phi_0} & H_{\text{free}}(\pi_0 E,0) \end{array} \tag{3.23}$$

where $E^0 = E$, ϕ_k are free Hurewicz maps, H_{free} are as above, E^{k+1} is the fibre of ϕ_k. We have

$$\underset{\leftarrow}{\text{holim}}\, E^k \simeq *.$$

As usual, because of stability (i.e. because fibration and cofibration sequences coincide), there is no difficulty in rearranging (3.23) into a Postnikov-type tower. Concretely, let E_k be the cofibre of the map $E^{k+1} \to E^0$. Then we have a (co)fibration diagram

$$\begin{array}{ccccc} E^{n+1} & \longrightarrow & E^{0} & \longrightarrow & E_n \\ \downarrow & & \downarrow = & & \downarrow \\ E^{n} & \longrightarrow & E^{0} & \longrightarrow & E^{n-1} \\ \downarrow & & \downarrow & & \downarrow k_n \\ H_{\text{free}}(\pi_n E,n) & \longrightarrow & * & \longrightarrow & H_{\text{free}}(\pi_n E,n+1) \end{array}$$

Then $E_0 = H_{\text{free}}(\pi_0 E, 0)$, and k_n are the k-invariants (also known as Postnikov invariants),

$$\underset{\leftarrow}{\text{holim}}\, E_n = E.$$

Now the maps (3.20), (3.21) induce maps

$$E\mathbb{Z}/2_+ \wedge M\mathbb{R} \to E\mathbb{Z}/2_+ \wedge k\mathbb{R},$$
$$E\mathbb{Z}/2_+ \wedge M\mathbb{R} \to E\mathbb{Z}/2_+ \wedge kR,$$

which are equivalences on the first two stages of Postnikov towers. More concretely, the first two free Eilenberg–MacLane spectra involved in these towers are

$$E\mathbb{Z}/2_+ \wedge H\mathbb{Z}$$

and

$$\Sigma^2 E\mathbb{Z}/2_+ \wedge H\tilde{\mathbb{Z}},$$

where $\tilde{\mathbb{Z}}$ is $\mathbb{Z}$, with $\mathbb{Z}/2$-action by -1. Now returning to complete universes, we have

$$\pi_\star(E\mathbb{Z}/2_+ \wedge H\mathbb{Z}) = H_*(\mathbb{Z}/2, \mathbb{Z})\{\sigma^{2i}, i \in \mathbb{Z}\} \oplus H_*(\mathbb{Z}/2, \tilde{\mathbb{Z}})\{\sigma^{2i+1}, i \in \mathbb{Z}\},$$

where σ is a formal element of dimension $\alpha - 1$. Also, we have

$$E\mathbb{Z}/2_+ \wedge H\tilde{\mathbb{Z}} = \Sigma^{\alpha-1} E\mathbb{Z}/2_+ \wedge H\mathbb{Z}.$$

It follows that the first k-invariants

$$\mathbb{Q}_1 : E\mathbb{Z}/2_+ \wedge H\mathbb{Z} \to \Sigma^{2+\alpha} E\mathbb{Z}/2_+ \wedge H\mathbb{Z} \tag{3.24}$$

coincide.

Next, the maps (3.20), (3.21) induce the expected maps on non-equivariant coefficients. Thus, $\mathbb{Z}/2$-actions on non-equivariant coefficients of KR and $K\mathbb{R}$ also agree. Thus, we can calculate

$$\pi_*(E\mathbb{Z}/2_+ \wedge KR), \quad \pi_*(E\mathbb{Z}/2_+ \wedge K\mathbb{R}). \tag{3.25}$$

To this end, we shall use the Borel homology spectral sequences

$$E^2 = H_*(\mathbb{Z}/2, \pi_* K) \Rightarrow \pi_*(E\mathbb{Z}/2_+ \wedge KR), \quad \pi_*(E\mathbb{Z}/2_+ \wedge K\mathbb{R}). \tag{3.26}$$

First, the E_2-terms are

$$\bigoplus_{i \in \mathbb{Z}} \{v_1^{2i}\} \otimes \left(\mathbb{Z} \oplus \bigoplus_{j \geq 0} \mathbb{Z}/2\{e_{2j+1}\} \right) \oplus \bigoplus_{i \in \mathbb{Z}} \{v_1^{2i+1}\} \otimes \left(\mathbb{Z} \oplus \bigoplus_{j \geq 0} \mathbb{Z}/2\{e_{2j}\} \right),$$

where $\dim(v_1) = (0, 2)$, $\dim(e_j) = (j, 0)$. Note that, by sparsity, the first possible higher differential is d^3. Now, in the next section, we will see that the first k-invariant (3.24) implies that the differential

$$d^3 : E^3_{i,2j} \to E^3_{i-3,2j+2} \tag{3.27}$$

is

$$0 \text{ if } i + j = 0, 1 \bmod 4,$$
$$1 \text{ if } i + j = 2, 3 \bmod 4.$$

(The same differentials also occur in the Borel cohomology and Tate spectral sequences; the sequences are compared by the standard maps [19].) We see that the differential d_3 wipes out the Tate spectral sequence, so all of the spectral sequences collapse to E^4 and the coefficients (3.25) are the same. (For a slightly different argument with the same conclusion, see [47, Theorem 3.1].)

Now even though KR, $K\mathbb{R}$ have the same homotopy groups, periodicity and first k-invariant, this still does not give us a map between them. To get a map, we introduce our final trick: the spectrum $F(KR, K\mathbb{R})$.

First recall that

$$KR_{\{e\}} \simeq K\mathbb{R}_{\{e\}}. \tag{3.28}$$

This is shown as follows. Denote $K' = K\mathbb{R}_{\{e\}}$, $K = KR_{\{e\}}$. By definition, K is Atiyah's K-theory. K' is, by definition, the MU-module constructed in the following way: we kill a regular sequence of elements in MU_* generating the kernel of the complex orientation map $MU_* \to K_*$, thus constructing successively a sequence of MU-modules $N_1, N_2, \ldots$. Let

$$k' = ho\lim_{\rightarrow} N_i.$$

Then K' is obtained from k' by inverting the Bott class u. But then both K, K' are Landweber flat, and hence

$$K'_* K \cong K_* \otimes_{MU_*} MU_* MU \otimes_{MU_*} K_* \cong K_* K.$$

But by Adams–Clarke [3], this is a free K_*-module, and hence

$$K^* K' \cong Hom_{K_*}(K_* K', K_*) \cong K^* K$$

by the universal coefficient theorem in K-theory. To obtain an equivalence $K \simeq K'$, choose the element in $K^* K'$ corresponding to $1 \in K^* K$.

Back to the equivariant situation. By Lemma 3.19, $\pi_* F(K\mathbb{R}, KR)$ can be calculated by the Borel cohomology spectral sequence

$$H^*(\mathbb{Z}/2, F(K, K)^*) \Rightarrow F(K\mathbb{R}, KR)^*. \tag{3.29}$$

Now $F(K, K)^*$ was calculated by Adams and Clarke [3]. Concretely, they show that

$$F(K, K)_* = Hom_{K_*}(K_* K, K_*)$$

and that $K_* K$ is a free K_*-module. We claim that (3.29) is a product of copies of the Borel cohomology spectral sequence

$$H^*(\mathbb{Z}/2, K^*) \Rightarrow KR^* \tag{3.30}$$

(which collapses to E_4, as calculated above). In effect, the d_3 differential of (3.29) can be calculated by comparing with the spectral sequence

$$H^*(\mathbb{Z}/2, F(MU, K)^*) \Rightarrow F(M\mathbb{R}, KR)^*. \tag{3.31}$$

(Concretely, $F(K\mathbb{R}, KR)$ is mapped into $F(M\mathbb{R}, KR)$ by mapping the source via

$$M\mathbb{R} \to K\mathbb{R} \stackrel{(v_1)^i}{\to} K\mathbb{R},$$

$i \in \mathbb{Z}$, where the first map comes from the construction of $K\mathbb{R}$.)

Now d_3 in (3.31) can be deduced from the fact that the target has been calculated in Theorem 2.25. Consequently, (3.29) is a product of copies of (3.30). (In fact, it can be shown more precisely that $KR_\star KR$ is a free $KR_\star$-module and

$$F(K\mathbb{R}, KR)_\star = Hom_{KR_\star}(KR_\star KR, KR_\star).)$$

Thus, we conclude that the map $1: K\mathbb{R}_{\{e\}} \to KR_{\{e\}}$ is a permanent cycle in the spectral sequence (3.29), representing an equivalence $K\mathbb{R} \to KR$. □

We next turn to Real Morava K-theories.

Theorem 3.32. *The coefficients* $K\mathbb{R}(n)_*$ *are periodic with period* $2^{n+2}(2^n-1)$. *Moreover, there is a decreasing filtration on* $K\mathbb{R}(n)_\star$ *such that*

$$E^0 K\mathbb{R}(n)_* = \bigoplus_{k\in\mathbb{Z}} \Sigma^{2k(2^n-1)} M(k)$$

where

$$M(k) = \bigoplus_{\substack{i,\, j: 1-2^{n+1} < i \leqslant 0 \\ 0 \leqslant j < 2^n \\ k(2^n-1)+i = j \mod 2^{n+1}}} \Sigma^i \mathbb{Z}/2.$$

Comment. Note that there are some non-trivial extensions: For example, it is well known that $\pi_2 KO/2 = \pi_2 M\mathbb{Z}/2 = \mathbb{Z}/4$. However, we do not know these extensions completely.

We investigate the Borel cohomology spectral sequence

$$H^{-p}(\mathbb{Z}/2, \pi_q k(n)) \Rightarrow \pi_{p+q} F(E\mathbb{Z}/2_+, k\mathbb{R}(n)). \tag{3.33}$$

(As indexed, the spectral sequence is homological.) Note that, by v_n-periodicity, Theorem 3.32 follows from the following lemma.

Lemma 3.34. *The spectral sequence* (3.33) *collapses to* $E^{2^{n+1}}$. *The only non-trivial differentials are* $d^{2^{n+1}-1}$ *originating in* $E^{2^{n+1}}_{i,\, j(2^{n+1}-2)}$ *where*

$$i \in [\, j(1-2^n) + 2^n + k2^{n+1}, j(1-2^n) + 2^{n+1} - 1 + k2^{n+1}]$$

for some $k \in \mathbb{Z}$.

Proof. An induction on n. First note that multiplication by v_n induces a periodicity operator

$$E^*_{i,\, j} \to E^*_{i-(2^n-1),\, j+(2^{n+1}-2)},$$

compatible with the differentials. Thus, it suffices to consider differentials originating at $E^*_{*,0}$. Now, by sparsity, $d^{2^{n+1}-1}$ is the first possible (= primary) differential. It is well known that primary differentials in spectral sequences tend to be given by cohomology operations.

To make this more precise in the present case, we return to the theory of free Postnikov towers introduced in the proof of Theorem 3.18. We shall consider the Postnikov tower of $k\mathbb{R}(n) \wedge E\mathbb{Z}/2_+$. Let

$$\mathbb{Q}_n : H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+ \to \Sigma^{(2^n-1)(1+\alpha)+1} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+ \tag{3.35}$$

be the first k-invariant in that tower.

Remark. The spectrum $H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+$ is $(1-\alpha)$-periodic — see Section 6 below. So, one can write $\Sigma^{2(2^n-1)+1}H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+$ on the right-hand side of (3.35). This, in fact, is more correct from the point of view of the concept of free Postnikov towers introduced in the proof of Theorem 3.18 (where suspensions by non-trivial representations were not explicitly allowed in k-invariants). However, we prefer to keep the notation (3.35), because it makes the $\mathbb{Q}_n$'s behave better as cohomological operations — see below. For a brief explanation, we point out that the dimensional reindexing, while almost transparent, is also a cohomological operation, which must be taken into account (see also Section 6 below).

Lemma 3.36. *We have*

$$d^{2^{n+1}-1}(e_i) = \mathbb{Q}_n(e_i), \tag{3.37}$$

where

$$0 \neq e_i \in \pi_{-i}F(E\mathbb{Z}/2_+, H\mathbb{Z}/2).$$

(*Note that the force of the above formula does not change if we replace* $H\mathbb{Z}/2$ *by* $H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+$.)

Proof. Let k_i be the $2(2^n-1)(i+1)$th stage of the free Postnikov tower of the free spectrum $k\mathbb{R}(n) \wedge E\mathbb{Z}/2_+$ (i.e. a free spectrum with $(i+1)$ non-trivial non-equivariant homotopy groups). Then we have a cofibration

$$\Sigma^{2(2^n-1)}H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+ \to k_1 \to H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+. \tag{3.38}$$

We have a map

$$k\mathbb{R}(n) \wedge E\mathbb{Z}/2_+ \to k_1,$$

which induces a map of Borel cohomology spectral sequences. Thus, it suffices to prove the statement with $k\mathbb{R}(n) \wedge E\mathbb{Z}/2_+$ replaced by k_1. Now we turn to (3.38). On equivariant homotopy groups, this induces a long exact sequence, whose connecting map, by definition, is $\mathbb{Q}_n$. On the other hand, (3.38) also induces maps on Borel cohomology spectral sequences. These sequences for the first and last terms, of course, collapse to E_2 (because in each case, there is only one complementary degree). Moreover, we see that (3.38) in fact induces a short exact sequence on Borel cohomology E_2-terms. Because $d_{2^{n+1}-1}$ is the only differential in this case, if an element $x \in \pi_*^{\mathbb{Z}/2}F(E\mathbb{Z}/2_+, H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+)$ does not lift to $x \in \pi_*^{\mathbb{Z}/2}F(E\mathbb{Z}/2_+, k_1)$ (which means that $\mathbb{Q}_n x \neq 0$), it must support a $d_{2^{n+1}-1}$ in the Borel cohomology spectral sequence of k_1. Note that there is only one non-trivial element in the dimensions of the possible targets of $\mathbb{Q}_n$, $d_{2^{n+1}-1}$. □

Lemma 3.39. $\mathbb{Q}_n(e_i) = e_{i+2^{n+1}}$.

This clearly is a calculation in (one version of) the equivariant Steenrod algebra

$$A^{\star c} = F(E\mathbb{Z}/2_+ \wedge H\mathbb{Z}/2, H\mathbb{Z}/2)_\star.$$

To avoid a forward reference, we postpone until Section 6 (after the proof of Theorem 6.10).

Lemma 3.40. *There is an operation*

$$\tilde{\mathbb{Q}}_k : E^*_{i,j} \to E^*_{i-2^k,j}$$

which commutes with $d^{2^{n+1}-1}$ *for* $k<n$. *Moreover*,

$$\tilde{\mathbb{Q}}_k e_i \neq 0 \quad \textit{iff}\ \mathbb{Q}_k e_i \neq 0.$$

Proof. First consider the cofibration sequence

$$\mathbb{Z}/2_+ \to S^0 \to S^\alpha.$$

This represents an element

$$a \in \pi^{\mathbb{Z}/2}_{-\alpha}(S^0).$$

We put

$$\tilde{\mathbb{Q}}_k = \mathbb{Q}_k \cdot a^{2^k-1},$$

where $\mathbb{Q}_k$ is written as in Lemma 3.36. To see that $\tilde{\mathbb{Q}}_k e_i \neq 0$ iff $\mathbb{Q}_k e_i \neq 0$, note that $E^*_{i,j}$ is embedded into the Tate spectral sequence (see [19])

$$\hat{H}^*(\mathbb{Z}/2, \pi_*(\Sigma^k k(n))) \Rightarrow \pi^{\mathbb{Z}/2}_* \Sigma^{k\alpha} \widehat{k\mathbb{R}(n)}. \tag{3.41}$$

Here recall that the hat denotes the Tate spectrum (see (3.29)). The point is that in (3.41), multiplication by a is an isomorphism, while

$$\mathbb{Q}_k \text{ commutes with } a. \tag{3.42}$$

Finally, to see that $\tilde{\mathbb{Q}}_k$ (or $\mathbb{Q}_k$ of Lemma 3.36) is compatible with differentials, it suffices to show that $\mathbb{Q}_k$ lifts to an operation

$$\mathbb{Q}_k : k_i \to \Sigma^{(2^k-1)(1+\alpha)+1} k_i, \tag{3.43}$$

and hence

$$\mathbb{Q}_k \mathbb{Q}_i = \mathbb{Q}_i \mathbb{Q}_k. \tag{3.44}$$

To this end, we return to the method in [14]. Let $BP\mathbb{R}_{*(1+\alpha)} \cong BP_{2*} = \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$. Now $k_i = E\mathbb{Z}/2_+ \wedge \kappa_i$ where

$$(\kappa_i)_{*(1+\alpha)} = \mathbb{Z}/2[v_n]/(v_n^{i+1}).$$

Now this spectrum can be constructed from $M\mathbb{R}$ by killing off an appropriate regular sequence in $M\mathbb{R}_{*(1+\alpha)}$. But in the same way, one can also construct a spectrum E with

$$E_{*(1+\alpha)} = \mathbb{Z}/2[v_n, v_k]/(v_n^{i+1}, v_k^2),$$

and a map

$$E \to \kappa_i. \tag{3.45}$$

Then the cofiber of (3.45), smashed with $E\mathbb{Z}/2_+$, is, by definition, (3.43). □

We now resume the proof of Lemma 3.34. To begin the induction, we have $\tilde{\mathbb{Q}}_0 = \mathbb{Q}_0 = \beta$. (Precisely speaking, the definition of $\tilde{\mathbb{Q}}_0 = \mathbb{Q}_0$ above has to be handled separately: we put $k\mathbb{R}(0) = H\mathbb{Z}$, while $K\mathbb{R}(0) = H\mathbb{Q}$; so $\tilde{\mathbb{Q}}_0 = \mathbb{Q}_0$ is really the first k-invariant of the tower

$$H\mathbb{Z}_2^\wedge = \operatorname{holim}_{\leftarrow} H\mathbb{Z}/2^n,$$

which is β.) Thus, by non-triviality of the operations $\mathbb{Q}_i$ for $i \leqslant n$ (the induction hypothesis), it suffices to show that

$$0 \neq d^{2^{n+1}-1} : E^{2^{n+1}-1}_{-2^n,0} \to E^{2^{n+1}-1}_{-2^{n+1}+1,2^{n+1}-2}. \tag{3.46}$$

But by Lemma 3.36, $\mathbb{Q}_k(e_i)$, $k < n$, can be calculated from the differentials in the Borel cohomology spectral sequence for $K\mathbb{R}(k)$. Now if $\mathbb{Q}_k(e_j) = e_{j'}$ for some $k < n$, j, j', then, by commutation of $\mathbb{Q}_k$ with $d^{2^{n+1}-1}$,

$$d^{2^{n+1}-1}(e_j) \neq 0 \text{ if and only if } d^{2^{n+1}-1}(e_{j'}) \neq 0. \tag{3.47}$$

In more detail, we have $j' = j + 2^k$ so

$$\mathbb{Q}_n(e_{j'}) \neq 0 \text{ iff } \mathbb{Q}_n(e_{j+2^k}) \neq 0 \text{ iff } \mathbb{Q}_n\mathbb{Q}_k(e_j) \neq 0 \text{ iff } \mathbb{Q}_k\mathbb{Q}_n(e_j) \neq 0.$$

This certainly implies $\mathbb{Q}_n(e_j) \neq 0$. On the other hand, $\mathbb{Q}_n(e_j) \neq 0$ implies $\mathbb{Q}_n(e_j) = e_{j+2^n}$, and since we assumed $\mathbb{Q}_k(e_j) \neq 0$, $\mathbb{Q}_k\mathbb{Q}_n(e_j) = \mathbb{Q}_k(e_{j+2^n}) \neq 0$, since $\mathbb{Q}_k(e_j)$ is 2^{k+1}- (and hence 2^n-) periodic in j.

By (3.47), if we know (3.46), to establish non-triviality of $d^{2^{n+1}-1}(e_j)$ for all $j \in [1, 2^k]$, it suffices to show that the graph G_n on vertices e_j, $1 \leqslant j \leqslant 2^n$ obtained by drawing an edge whenever $\mathbb{Q}_k$ (for some $k < n$) is non-trivial, is connected.

But the subgraphs of G_n on $e_j : j \in [1, 2^{n-1}]$ and $e_j : j \in [2^{n-1}+1, 2^n]$ are isomorphic to G_{n-1}, which can be assumed connected by induction. In addition,

$$\mathbb{Q}_{n-1}(e_{2^{n-1}}) = e_{2^n},$$

by the induction hypothesis for $\mathbb{Q}_{n-1}$. Thus, G_n is connected.

To prove (3.46), we note that the non-triviality of this differential is best visible outside of the Borel cohomology spectral sequence range. In fact, (3.46) is equivalent to the corresponding differential in the Tate cohomology spectral sequence for $K\mathbb{R}(n)$ (see [19]). There, however, the same periodicities occur as in the Borel cohomology spectral sequence, and therefore it suffices to show that

$$0 \neq d^{2^{n+1}-1} : E^{2^{n+1}-1}_{2^{n+1}-1,-2^{n+1}+2} \to E^{2^{n+1}-1}_{0,0}. \tag{3.48}$$

In fact, consider the integral version $\tilde{K}\mathbb{R}(n)$ of $K\mathbb{R}(n)$. Let ${}_{\mathrm{f}}E$ be the Borel homology spectral sequence for $\tilde{K}\mathbb{R}(n)$ and let ${}_{\mathrm{t}}E$ be the Tate spectral sequence for $\tilde{K}\mathbb{R}(n)$.

Lemma 3.49. $0 \neq \xi \in {}_{\mathrm{f}}E_{2^{n+1}-2,-2^{n+1}+2}$ *supports an extension: In the abutment,* 2ξ *is equal to* $\iota \in {}_{\mathrm{f}}E_{0,0}$. *Here* ι *is the bottom class in* $H_*(\mathbb{Z}/2, \mathbb{Z})$.

Proof. By periodicity, it suffices to prove that the element $v_n \iota$ in filtration degree 0 of the Borel homology spectral sequence

$$H_*(\mathbb{Z}/2, \Sigma^{2^n-1}\tilde{K}(n)_*) \Rightarrow \pi_*^{\mathbb{Z}/2}(\Sigma^{(2^n-1)\alpha}\tilde{K}\mathbb{R}(n) \wedge E\mathbb{Z}/2_+) \tag{3.50}$$

is the target of an extension in the sense that it is a 2-multiple of an element in filtration degree $2^{n+1}-2$. Now by the same arguments as above, one can replace $\tilde{K}\mathbb{R}(n) \wedge E\mathbb{Z}/2_+$ in (3.50) by its connective cover, in fact by the second stage of its free Postnikov tower $\tilde{k}_1$. Now consider the cofibration

$$\Sigma^{(2^n-1)(1+\alpha)}H\mathbb{Z} \wedge E\mathbb{Z}/2_+ \xrightarrow{u} \tilde{k}_1 \xrightarrow{v} H\mathbb{Z} \wedge E\mathbb{Z}/2_+. \tag{3.51}$$

The connecting map of (3.51) is the integral lift Q of $\mathbb{Q}_n$. Now we claim that

$$2Q = 0. \tag{3.52}$$

(In fact, note that Q is in the $RO(\mathbb{Z}/2)$-graded coefficients of $F(E\mathbb{Z}/2_+ \wedge H\mathbb{Z}, H\mathbb{Z})$). But $F(H\mathbb{Z}, H\mathbb{Z})_{\mathbb{Z}/2}$ is a split spectrum. Hence, it follows from the calculation of $F(H\mathbb{Z}, H\mathbb{Z})_e$ that multiplication by 2 is 0 on $F(E\mathbb{Z}/2_+ \wedge H\mathbb{Z}, H\mathbb{Z})_\star$ except in the dimensions $2k(1-\alpha)$, $k \in \mathbb{Z}$.)

By (3.52),

$$Q = \beta P$$

(where β is the Bockstein) for some

$$P \in F(E\mathbb{Z}/2_+ \wedge H\mathbb{Z},\ H\mathbb{Z}/2)_\star.$$

Now the Borel homology spectral sequence for $\Sigma^{(2^n-1)\alpha}\tilde{k}_1$ is, in fact, a reindexing of the long exact sequence in homotopy associated with the cofibration (3.51) in dimensions $k + (2^n-1)\alpha$. ιv_n is represented by

$$\iota \in \pi^{\mathbb{Z}/2}_{(2^n-1)(1+\alpha)}\Sigma^{(2^n-1)(1+\alpha)}H\mathbb{Z} \wedge E\mathbb{Z}/2_+.$$

On the other hand, if we consider the Hurewicz image $h(\iota v_n)$ of ιv_n in integral homology (meaning $\pi_*^{\mathbb{Z}/2}(\Sigma^{(2^n-1)(1+\alpha)}H\mathbb{Z} \wedge E\mathbb{Z}/2_+ \wedge H\mathbb{Z})$, then the Bockstein lemma ([10]) implies that

$$u(h(\iota v_n)) = 2\varepsilon,$$

where the mod 2 reduction of ε is

$$P\iota \in \pi_*^{\mathbb{Z}/2}(H\mathbb{Z} \wedge E\mathbb{Z}/2_+ \wedge H\mathbb{Z}/2).$$

Therefore, we see that the extension we are looking for takes place after taking images under the Hurewicz homomorphism to integral homology. But, on the other hand, $P\iota$ cannot support a differential in the Borel homology spectral sequence for dimensional reasons (its filtration degree is $2^{n+1}-2$)). Thus, both classes participating in the extension survive to homotopy. □

Now consider the map

$${}_{t}E_{p,q} \to {}_{f}E_{p-1,q}.$$

In ${}_{t}E$, the element ι does not exist in the E^2-term. Thus, ξ must support a differential in ${}_{t}E$! By comparing with the Tate spectral sequence for the second stage of the free Postnikov tower for the $(-2^{n+1}+1)$-connected cover of $\tilde{K}\mathbb{R}(n) \wedge E\mathbb{Z}/2_+$, where similar arguments apply (see the proof of the above lemma), we see that the differential must indeed be $d^{2^{n+1}-1}$. Reducing modulo 2, we obtain the differential (3.48). □

4. The Real Adams–Novikov spectral sequence

We are now ready to calculate the coefficients of the Real Brown–Peterson spectrum $BP\mathbb{R}$. The latter calculation was announced by Araki [5], but, as far as we know, not published. We also prove a strong completion theorem for $BP\mathbb{R}$. We then construct an equivariant Adams–Novikov-type spectral sequence based on $BP\mathbb{R}$, and show that this spectral sequence converges to $\pi_\star S^0_{\mathbb{Z}/2}$. We also calculate the Hopf algebroid

$$(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$$

modulo a certain possible error term, which will be eliminated later in Section 7. Recall that for a G-spectrum E, we denoted the corresponding Tate spectrum $\widetilde{EG} \wedge F(EG_+, E)$ by $\hat{E}$.

Theorem 4.1.

(1) *The canonical map*

$$\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R} \to \widehat{BP\mathbb{R}}$$

is an equivalence (*recall also that* $(\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R})^{\mathbb{Z}/2} = H\mathbb{Z}/2$). *Thus*, $BP\mathbb{R}$ *satisfies a strong completion theorem in the sense that*

$$BP\mathbb{R} \stackrel{\cong}{\to} F(E\mathbb{Z}/2_+, BP\mathbb{R}).$$

Similar results hold with $BP\mathbb{R}$ *replaced by* $M\mathbb{R}$.

(2) $\pi_* BP\mathbb{R} \cong \pi_* F(E\mathbb{Z}/2_+, BP\mathbb{R})$ *can be described as follows*: *For a sequence of non-negative integers* $\underline{m} = (m_0, m_1, m_2, \dots)$ *with at most finitely many non-zero entries*, *put*

$$\dim(\underline{m}) = \sum_i 2m_i(2^i - 1).$$

Let k *be the smallest number such that* $m_k \neq 0$. *Let* q *be the remainder of* $\dim(\underline{m})/2$ *modulo* 2^{k+1}. *Put*

$$N(\underline{m}) = \begin{cases} \mathbb{Z}/2\{0\} & \text{if } \underline{m} = 0, \\ \mathbb{Z}/2\{\dim(\underline{m}) - q\} & \text{if } q \neq 2^{k+1} - 1, \\ 0 & \text{if } q = 2^{k+1} - 1. \end{cases}$$

Then there is a filtration on $\pi_* BP\mathbb{R}$ *such that the associated graded object is*

$$\bigoplus_{\underline{m}} N(\underline{m}).$$

(*Note*: *The extension corresponding to* $v_0 = 2$ *arises for* $k = 0$.)

(3) *Analogous formulas are valid for the spectra* $E\mathbb{R}(n)$ *if we index over all sequences* $\underline{m} = (m_0, m_1, m_2, \dots)$ *such that* $v_0^{m_0} v_1^{m_1} \dots$ *is a monomial in the non-equivariant homotopy of* $E(n)$.

Proof. By the same argument as in Lemma 3.34, the same differentials will arise whenever possible. For example, consider the Tate spectral sequence for $\widehat{BP\mathbb{R}}$. We claim that $E^{2^{n+1}-1}$ can be described as follows:

Let $\underline{m} = (m_n, m_{n+1}, m_{n+2}, \dots)$ be a sequence of non-negative integers, only finitely many of which are allowed to be non-zero. Define

$$\dim \underline{m} = \sum_{k \geqslant n} 2m_k(2^k - 1)$$

(which is the dimension of

$$\prod_{k \geqslant n} v_k^{m_k} \in MU_*,)$$

similarly as above. Let b be the remainder of $\dim(\underline{m})/2$ modulo 2^n. Put

$$Q(m) = \Sigma^{(-\dim(\underline{m})/2, \dim(\underline{m}))} \mathbb{Z}/2[a_n, a_n^{-1}], \ \dim(a_n) = (2^n, 0).$$

(In the suspension notation, the two coordinates denote filtration and complementary degrees, respectively; the grading is as in the Borel cohomology spectral sequence graded homologically. This means that the filtration degrees are non-positive.) Then

$$E_{*,*}^{2^{n+1}-1} = \bigoplus_{\underline{m}} Q(\underline{m}).$$

The differential is

$$d^{2^{n+1}-1}(a_n^{2i+1})_{Q(\underline{m})} = (a_n^{2i})_{Q(\underline{m}+\Delta_n)}, \qquad d^{2^{n+1}-1}(a_n^{2i})_{Q(\underline{m})} = 0. \tag{4.2}$$

(As usual, Δ_n is the word with 1 in position n and 0 elsewhere.) Note that we need to show that the sources of the differentials will survive to the terms where the differentials occur. First, the sources of complementary degree 0 certainly survive for reasons of sparsity (the next lowest complementary degree is $2^{n+1} - 2$). However, since $BP\mathbb{R}$ is a ring spectrum (although we do not know if it is E_∞), we have maps

$$v_n^{m_n} v_{n+1}^{m_{n+1}} \dots : \Sigma^{\frac{\dim \underline{m}}{2}(1+\alpha)} BP\mathbb{R} \to BP\mathbb{R},$$

which induce maps of Tate spectral sequences. Consequently, sources in all other complementary degrees will survive to the terms predicted by (4.2), by naturality.

Now observe that the only element surviving to E^∞ is $\mathbb{Z}/2\{0,0\}$. Thus, the map

$$H\mathbb{Z}/2 = (S^V \wedge BP\mathbb{R})^{\mathbb{Z}/2} \to \widehat{BP\mathbb{R}}^{\mathbb{Z}/2} \tag{4.3}$$

is an equivalence. But now a map of $\mathbb{Z}/2$-equivariant spectra indexed over the complete universe which is an equivalence both non-equivariantly and after taking fixed points is an equivalence.

Note that the geometric and Tate spectra are zero non-equivariantly. Thus, the first statement is proved.

To prove the second statement, we consider the Borel cohomology spectral sequnce ${}_{\mathrm{c}}E$ (associated with $F(E\mathbb{Z}/2_+, E)$), the Borel homology spectral sequence ${}_{f}E$ (associated with $E\mathbb{Z}/2_+ \wedge E$) and the Tate spectral sequence ${}_{\mathrm{t}}E$ for $BP\mathbb{R}$. We have maps of spectral sequences

$$ {}_{\mathrm{c}}E \to {}_{\mathrm{t}}E \to {}_{\mathrm{f}}E. \tag{4.4}$$

Now we know all the differentials in ${}_{\mathrm{t}}E$ (in fact, in ${}_{\mathrm{t}}E^{\infty}$, all elements with the exception of bidegree (0,0) are wiped out). By (4.4), all differentials of ${}_{\mathrm{t}}E$ where the filtration degrees of the source and target are non-positive (graded homologically) determine differentials in ${}_{\mathrm{c}}E$, and all differentials of ${}_{\mathrm{t}}E$ where the degree of the source and target are positive determine differentials in ${}_{\mathrm{f}}E$. We claim that no other differentials in ${}_{\mathrm{c}}E$ and ${}_{\mathrm{f}}E$ occur. (Note that this immediately implies part (2) of the Theorem.)

To prove the claim, we state it in the following more precise form: Denote by

$$N : (E\mathbb{Z}/2_+ \wedge BP\mathbb{R})_*^{\mathbb{Z}/2} \to F(E\mathbb{Z}/2_+, BP\mathbb{R})_*^{\mathbb{Z}/2}$$

the norm map (coming from the equivalence $E\mathbb{Z}/2_+ \wedge BP\mathbb{R} \simeq E\mathbb{Z}/2_+ \wedge F(E\mathbb{Z}/2_+, BP\mathbb{R})$).

Lemma 4.5 (Bückstedt and Madsen [8, (2.15)]). *Suppose an element $x \in {}_{\mathrm{t}}E^2_{p,q}$ satisfies $d^r x = y$ where $p > 0, (p-r) \leqslant 0$. Then in ${}_{\mathrm{f}}E^2_{p,q}$, x is a permanent cycle which represents an element $\bar{x} \in (E\mathbb{Z}/2_+ \wedge BP\mathbb{R})_*^{\mathbb{Z}/2}$ such that, in ${}_{\mathrm{c}}E^2_{*,*}$, $N\bar{x}$ is represented by y.*

Proof. By the definition of the exact couple determining the differentials in ${}_{\mathrm{t}}E^*_{*,*}$, x survives to $x' \in \pi_*(\Sigma^{(-p+r+1)\alpha} E\mathbb{Z}/2_+ \wedge BP\mathbb{R})_*^{\mathbb{Z}/2}$ (hence, in particular, to $\pi_*(E\mathbb{Z}/2_+ \wedge BP\mathbb{R})_*^{\mathbb{Z}/2}$). Now the differential d^r is realized by taking the map on homotopy induced from the connecting map

$$\Sigma^{(-p+r+1)\alpha} E\mathbb{Z}/2_+ \wedge BP\mathbb{R} \overset{\partial}{\to} \Sigma^{-p+r+1}\mathbb{Z}/2_+ \wedge BP\mathbb{R} \tag{4.6}$$

associated with the cofibration

$$\Sigma^{(-p+r)}\mathbb{Z}/2_+ \to \Sigma^{(-p+r)\alpha} E\mathbb{Z}/2_+ \to \Sigma^{(-p+r+1)\alpha} E\mathbb{Z}/2_+.$$

But now we have a commutative diagram

$$\begin{array}{ccccc}
\Sigma^{(-p+r)\alpha} E\mathbb{Z}/2_+ \wedge BP\mathbb{R} & \longrightarrow & \Sigma^{(-p+r+1)\alpha} E\mathbb{Z}/2_+ \wedge BP\mathbb{R} & \longrightarrow & \Sigma^{-p+r+1}\mathbb{Z}/2_+ \wedge BP\mathbb{R} \\
\downarrow = & & \downarrow & & \downarrow \\
\Sigma^{(-p+r)\alpha} E\mathbb{Z}/2_+ \wedge BP\mathbb{R} & \longrightarrow & E\mathbb{Z}/2_+ \wedge BP\mathbb{R} & \xrightarrow{N'} & \Sigma^{(-p+r+1)\alpha+1} S((r-p)\alpha)_+ \wedge BP\mathbb{R}.
\end{array} \tag{4.7}$$

We call the bottom right map N', because it factors through the norm map $N : E\mathbb{Z}/2_+ \wedge BP\mathbb{R} \to F(E\mathbb{Z}/2_+, BP\mathbb{R})$. If we apply $\pi_*^{\mathbb{Z}/2}$ to the right-hand square (4.7), we obtain on elements a diagram

$$\begin{array}{ccc} x' & \longrightarrow & y \\ \downarrow & & \downarrow \\ \bar{x} & \longrightarrow & N'\bar{x}. \end{array}$$

Now $N'\bar{x}$ is the element reesenting $N\bar{x}$ in the Borel cohomology spectral sequence, as claimed. $\square$

To prove the second statement, consider the Borel cohomology spectral sequence. The differentials are exactly the same, except that the differentials which go from positive to non-positive filtration degrees are missing. It is easy to see that adding the resulting elements to E^{∞} gives the formula of (2). Note that the new elements are permanent cycles, because the "missing differentials" from the Tate spectral sequence identify these elements as images of the source elements under the canonical map

$$\pi_k(E\mathbb{Z}/2_+ \wedge BP\mathbb{R}) \to \pi_k F(E\mathbb{Z}/2_+, BP\mathbb{R}).$$

The treatment of $E\mathbb{R}(n)$ is analogous. $\square$

Proof of Theorem 2.28. By the previous theorem, $MU_{k(1+\alpha)}$ can be computed by Borel cohomology spectral sequences. Select an element $y \in MU_*$. Now consider the horizontal line ℓ corresponding to y in the Tate spectral sequence E'' for $M\mathbb{R}$. (We mean only the elements represented by $\hat{H}^*(\mathbb{Z}/2, \mathbb{Z}\{y\}) \subset \hat{H}^*(\mathbb{Z}/2, MU_*)$: there may be other elements of the same complementary degree.) The Borel cohomology spectral sequence E for $\Sigma^{-k\alpha}M\mathbb{R}$ is obtained by considering the elements in E'' of filtration degree $\leqslant -k$. Now for each value of k, the proof of Theorem 4.1 shows that at most one element on the line ℓ will survive. For $k = \dim y/2$, we see that the surviving element is on the edge of the Borel cohomology spectral sequence, and has, in fact, dimension $k(1+\alpha)$. If $k > \dim y/2$, we see that the surviving element on ℓ (if any) will be of filtration degree $< -k$, thus having a dimension $m + n\alpha$ with $n > m$. On the other hand, recalling the differentials in E'', we see that also if $k < \dim y/2$, the surviving element on ℓ (if any) will have filtration degree $\geqslant -k$, and hence dimension $m + n\alpha$ with $n < m$. $\square$

Comments.

(1) Theorem 4.1 is a *strong* completion theorem. In most completion theorems, one only claims that the map

$$E \to F(EG_+, E)$$

is an algebraic completion, not an equivalence. However, a strong completion theorem is not unprecedented; it holds, for example, for KR.

(2) The analogue of (3) of Theorem 4.1 is false for $BP\mathbb{R}\langle n\rangle$ (the Real version of $BP\langle n\rangle$), because additional elements remain on the line of complementary degree 0. However, the right formula can be worked out by the same method.

(3) The reader is encouraged to check that the formula of Theorem 4.1 gives the right answer for $E\mathbb{R}(1) = K\mathbb{R}$.

(4) Note that our calculations give completion theorems for

$$M\mathbb{R}, BP\mathbb{R}, K\mathbb{R}(n), E\mathbb{R}(n).$$

On the other hand, $E \to F(E\mathbb{Z}/2_+, E)$ does not even induce an isomorphism of 2-completed homotopy groups if E is one of the spectra

$$H\mathbb{Z}/2, H\mathbb{Z}, k\mathbb{R}(n), BP\mathbb{R}\langle n\rangle.$$

The point is that the fixed point spectra of the Tate spectra in these cases are non-connective (in fact their 2-completed homotopy groups are not bounded below), while the corresponding geometric fixed point spectra are connective.

(5) By the same method, one can also calculate $E\mathbb{R}(n)^*(E\mathbb{R}(n) \wedge E\mathbb{R}(n))$ and show that $E\mathbb{R}(n)$ is a commutative associative ring spectrum.

(6) Hopkins and Miller [20] found a purely obstruction-theoretic way to obtain G-actions on certain spectra E_n related to $E(n)$, where G is a finite subgroup of the group S_n of automorphisms of a formal group law Φ_n of height n over the finite field F_{p^n}. Here S_n acts on $(E_n)_*$ by Lubin–Tate theory [34]. The spectra $E\mathbb{R}_n$ are equivalent to the Hopkins–Miller spectra for a special subgroup $\mathbb{Z}/2 \subset S_n$.

More concretely, denote by F the Lazard universal formal group law on MU_*, and denote by F' the formal group law obtained by the map $MU_* \to MU_*$ which is the identity in dimensions $4k$ and reverses signs in dimensions $4k+2$. Then $-x$ defines an isomorphism $F \cong F'$. It is, however, not a strict isomorphism. The strict isomorphism $F \to F'$ is

$$-i(x), \tag{4.8}$$

where $i(x)$ is the inverse function for F. Note that this is also easy to see geometrically, since the complex conjugation $B\mathbb{S}^1 \to B\mathbb{S}^1$ is $B(-1)$, and the additional minus sign enters when we pass to Thom spaces of the corresponding complex line bundles.

To obtain the automorphism corresponding to (4.8) by Lubin–Tate theory, we simply reduce modulo 2: The resulting automorphism is $i_{\Phi_n}(x)$.

For more comments on this, see the remarks in Section 5 below.

We are now ready for the promised $H\mathbb{Z}/2_m$-counterpart of (3.2):

Proposition 4.9. *Define inductively a sequence of $M\mathbb{R}$-modules $P\mathbb{R}(n)$ by $P\mathbb{R}(0) = BP\mathbb{R}$, and for $n \geqslant 0$, $P\mathbb{R}(n+1)$ is the homotopy cofiber of*

$$v_n : \Sigma^{(2^n-1)(1+\alpha)} P\mathbb{R}(n) \to P\mathbb{R}(n).$$

Then

$$H\mathbb{Z}/2_m \simeq ho\lim_{\rightarrow} P\mathbb{R}(n). \tag{4.10}$$

Proof. Let H_m be the homotopy colimit (4.10). Since homotopy direct colimits commute with fixed points, we have

$$(H_m^{\mathbb{Z}/2})_* \cong \lim_{\rightarrow} P\mathbb{R}(n)_*^{\mathbb{Z}/2}.$$

By induction, $P\mathbb{R}(n)$ satisfies a completion theorem analogous to Theorem 4.1. Moreover, by the same methods, the Borel cohomology spectral sequence for $P\mathbb{R}(n)_*^{\mathbb{Z}/2}$ gives a filtration on $P\mathbb{R}(n)_*$ whose associated graded object is

$$\bigoplus_{\underline{m}} N_n(\underline{m}),$$

where $\underline{m}$ is as in Theorem 4.1(2), with $k \geqslant n$ and $N_n(\underline{m})$ is $\mathbb{Z}/2\{0\}$ if $\underline{m} = 0$ and $\oplus\{\mathbb{Z}/2\{\dim(\underline{m}) - s\} \mid 0 \leqslant s < 2^{k+1} - 1,\ 0 \leqslant j < 2^n,\ \dim(\underline{m})/2 \equiv s + j \bmod 2^{k+1}\}$ else. In particular, we see that for $\underline{m} \neq 0$,

$$\dim(\underline{m})/2 - s > \dim(\underline{m})/2 - (2^{k+1} - 1) \geqslant (2^k - 1) - (2^{k+1} - 1) = -2^k.$$

Therefore, if, for $0 \leqslant j < 2^n$ we have $\dim(\underline{m})/2 \equiv s + j \bmod 2^{k+1}$, then

$$\dim(\underline{m}) - s > \dim(\underline{m})/2 - s \geqslant -2^k + 2^{k+1} = 2^k.$$

One way or another, the minimum dimension of any non-zero element of $N_n(\underline{m})$ tends to infinity with n. Thus,

$$(H_{\underline{m}}^{\mathbb{Z}/2})_* = \mathbb{Z}/2.$$

From the edge map of the Borel cohomology spectral sequence, we then see that

$$(H_{\underline{m}}^{\mathbb{Z}/2})_* \to ((H_{\underline{m}})_{\{e\}})_* \to (H\mathbb{Z}/2_{\{e\}})_* = \mathbb{Z}/2$$

is an isomorphism. But any spectrum $H_{\underline{m}}$ satisfying these conditions is $H\mathbb{Z}/2_{\underline{m}}$. □

Theorem 4.11. *In the Hopf algebroid* $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$, *we have*

$$\mathrm{BP}\mathbb{R}_\star = \mathbb{Z}_{(2)}[v_{n,\ell}, a \mid n \geqslant 0, \ell \in \mathbb{Z}] \Bigg/ \left(\begin{array}{l} v_{0,0} = 2, \\ a^{2^{n+1}-1} v_{n,\ell} = 0, \\ \text{for } n \leqslant m\text{: } v_{m,k} \cdot v_{n,\ell 2^{m-n}} = v_{m,k+\ell} \cdot v_{n,0} \end{array} \right),$$

$$|a| = -\alpha,\ |v_{n,\ell}| = (2^n - 1)(1 + \alpha) + \ell 2^{n+1}(\alpha - 1),$$

$$BP\mathbb{R}_\star BP\mathbb{R} = BP\mathbb{R}_\star[t_1, t_2, \ldots],\ |t_n| = (2^n - 1)(1 + \alpha). \tag{4.12}$$

Further, write $v_{n,\ell} = \sigma^{\ell 2^{n+1}} v_n$. *Then the coproduct and right unit formulas in* $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$ *are the same as in* $(BP_*, BP_* BP)$. *Further, a and σ commute with the operations*

$$\eta_R(ax) = a\eta_R(x), \tag{4.13}$$

$$\eta_R(\sigma^{k2^{n+1}} v_k) = \sigma^{k2^{n+1}} \eta_R(v_k). \tag{4.14}$$

Remark. Instead of the third multiplicative relation in (4.12), we will now only prove that

$$\text{for } n \leqslant m\text{: } v_{m,k} \cdot v_{n,\ell 2^{m-n}} = v_{m,k+\ell} \cdot v_{n,0} + \Delta,$$

where the error term Δ is a sum of a^i-multiples of monomials of the form

$$v_{n_1,\ell_1}.\cdots.v_{n_k,\ell_k} \tag{4.15}$$

with $n_i > m$. The proof that $\Delta = 0$ requires different methods and will be deferred to Section 7 below (see Theorem 7.4).

Comment. The elements $\sigma^{\ell 2^{n+1}} v_n$ can be geometrically constructed as follows. Consider the cofibration

$$S((2^{n+1}-2)\alpha)_+ \to S^0 \xrightarrow{a^{2^{n+1}-1}} S^{(2^{n+1}-1)\alpha}.$$

Mapping to $M\mathbb{R}$, we obtain a long exact sequence

$$\ldots M\mathbb{R}^{\star+1} \overset{\kappa}{\leftarrow} M\mathbb{R}^{\star+1-(2^{n+1}-1)\alpha} \overset{\delta}{\leftarrow} M\mathbb{R}^{\star}(S((2^{n+1}-2)\alpha)) \leftarrow M\mathbb{R}^{\star} \ldots .$$

Now the element $v_n \in M\mathbb{R}^{\star}$ exists, because it is realized by the Milnor manifold (real algebraic variety dual to $\gamma^1 \otimes \gamma^1$ in $\mathbb{C}P^{2^n} \times \mathbb{C}P^{2^n}$). In $M\mathbb{R}^{\star+1-(2^{n+1}-1)\alpha}$, v_n has dimension

$$(2^n-1)(1+\alpha)+1-(2^{n+1}-1)\alpha = 2^n - 2^n\alpha.$$

Further,

$$\kappa(v_n) = 0. \tag{4.16}$$

Indeed, recall the cofibration

$$M\mathbb{R} \wedge E\mathbb{Z}/2_+ \to M\mathbb{R} \to M\mathbb{R} \wedge \widetilde{E\mathbb{Z}/2}.$$

Now $a^{2^{n+1}-1}v_n$ is dimension $k+\ell\alpha$ where $k+\ell<0$. Since $M\mathbb{R}\wedge E\mathbb{Z}/2_+$ is connective, $a^{2^{n+1}-1}v_n \neq 0$ would imply that $a^{2^{n+1}-1}v_n$ has a non-trivial image in $\Phi^{\mathbb{Z}/2}M\mathbb{R} = MO$. However, we know that v_n has a trivial image in MO, because the formal group law of MO is additive. Thus, we have proven (4.16).

Thus,

$$v_n = \delta(s)$$

for some

$$s \in M\mathbb{R}^{2^n - 2^n\alpha}(S((2^{n+1}-2)\alpha)). \tag{4.17}$$

In fact, we see that s is a unit (invertible element) in the ring $M\mathbb{R}^{\star}(S((2^{n+1}-2)\alpha))$. To this end, note that, by comparison with the Borel cohomology spectral sequence of $M\mathbb{R}$, the Borel cohomology spectral sequence of $F(S((2^{n+1}-2)\alpha)_+, M\mathbb{R})$ is $|s|$-periodic, and moreover s realizes the periodicity operator, since, as was shown in the previous section, in the Borel cohomology spectral sequence of $M\mathbb{R}$ we have the differential

$$d_{2^{n+1}-1}\sigma^{2^n} = v_n a^{2^{n+1}-1}.$$

But now also by comparing with Tate cohomology,

$$F(S((2^{n+1}-2)\alpha)_+, M\mathbb{R}) \simeq \Sigma^{2^n(1-\alpha)} M\mathbb{R} \wedge S((2^{n+1}-2)\alpha)_+,$$

and thus $M\mathbb{R}_\star S((2^{n+1}-2)\alpha)$ is also σ^{2^n}-periodic, where the periodicity can be realized by cap product with the cohomology class s.

Finally, by (4.16), $v_n \in M\mathbb{R}_\star$ lifts to $M\mathbb{R}_\star S((2^{n+1}-2)\alpha)$ via the map $S((2^{n+1}-2)\alpha)_+ \to S^0$. Thus, we can consider the elements

$$\sigma^{\ell 2^n} v_n \in M\mathbb{R}_\star S(((2^{n+1}-2)\alpha)), \tag{4.18}$$

and their images in $M\mathbb{R}_\star$. The reader will notice a discrepancy in that the elements v_n are $\sigma^{\ell 2^n}$-periodic in $M\mathbb{R}_\star S(((2^{n+1}-2)\alpha))$, but $\sigma^{\ell 2^{n+1}}$-periodic in $M\mathbb{R}_\star$. This is due to the fact that by the differentials in the Borel cohomology spectral sequence of $M\mathbb{R}_\star$, the images of $\sigma^{\ell 2^n} v_n$ in $M\mathbb{R}_\star$ for ℓ odd are 0.

Now for a monomial

$$x = v_R \sigma^{i 2^{\ell+1}} a^j,$$

put

$$k(x) = \min R.$$

(For $R = (r_0, r_1, \dots)$, by $\min R$ we denote the minimum i such that $r_i \neq 0$.) For a sum of monomials z, let $k(z)$ denote the minimum of $k(x)$ over the monomial summands x of z. For a homogeneous element $x \in BP\mathbb{R}_\star$ of dimension $r + s\alpha$, put

$$\delta(x) = s - r, \qquad r(x) = r, \qquad t(x) = r + s.$$

Lemma 4.19. *For a monomial*

$$\delta(x) \equiv i \bmod 2^{k(x)+2}$$

for some

$$i \in \{2 - 2^{k(x)+1}, \dots, 0\}.$$

Proof. If

$$x = v_R \sigma^{i 2^{k(x)+1}} a^j,\ 0 \leqslant j \leqslant 2^{k(x)+1} - 2,$$

then

$$\delta(x) = 2 \cdot i \cdot 2^{k(x)+1} - j$$

which is congruent to one of the numbers $\{2 - 2^{k(x)+1}, \dots, 0\}$ modulo $2^{k(x)+2}$. □

Proof of Theorem 4.11 except $\Delta = 0$ (*see the Remark after its statement*)**.** First, (4.13) is obvious, since a is realized by the right-hand map of the cofibration sequence

$$\mathbb{Z}/2_+ \to S^0 \to S^\alpha.$$

Next, by the Comment preceding Lemma 4.19,

$$a^{2^{n+1}-1}v_{n,\ell} = 0. \tag{4.20}$$

Next, an associated graded ring of $BP\mathbb{R}_\star$ is the E_∞-term of the Borel cohomology spectral sequence

$$H^*(\mathbb{Z}/2, BP^*)[\sigma, \sigma^{-1}] \Rightarrow BPR^\star.$$

Because $BP\mathbb{R}_{*(1+\alpha)} = BP_*$, we have an embedding of Hopf algebroids

$$(BP_*, BP_*BP) \subset (BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$$

in dimensions $k(1+\alpha)$. In particular, we have a multiplicative extension

$$v_0 = 2.$$

Next, (4.11) holds with the error terms of the described form because other terms are mapped injectively by the map

$$BP\mathbb{R}_\star \to F(S(2^{m+1}-1)\alpha)_+, BP\mathbb{R})_\star,$$

while the right-hand side contains an element $\sigma^{2^{m+1}}$.

To conclude the proof of Theorem 4.11, it remains to prove the right unit formula. To this end, once again, by the Borel cohomology spectral sequence we know that

$$\eta_R(v_k\sigma^{i2^{k+1}}) = \sigma^{i2^{k+1}}\eta_R(v_k) + ax.$$

Lemma 4.19 implies that

$$k(x) > k$$

(if $k(x) \leqslant k$, then, by the lemma, $\delta(ax) \neq 0 \bmod 2^{k+2}$, while $\delta(v_k\sigma^{i2^{k+1}}) \equiv 0 \bmod 2^{k+2}$). Now suppose that $x \neq 0$. Consider a monomial summand y of x. Note that $k(y) \geqslant k(x) > k$. We distinguish two possibilities:

Case 1: $y = v_{k(y)}\sigma^{i'2^{k(y)+1}}a^j\prod t_m$. But then

$$(2^{k(y)}-1) - i'2^{k(y)+1} = r(v_{k(y)}\sigma^{i'2^{k(y)+1}}) \leqslant r(v_k\sigma^{i2^{k+1}} = 2^k - 1 - i2^{k+1}.$$

Therefore,

$$2^{k(y)} - 2^k \leqslant i'2^{k(y)+1} - i2^{k+1}. \tag{4.21}$$

On the other hand, the right-hand side of (4.21) is divisible by 2^{k+1}. Therefore,

$$2^{k(y)} \leqslant i'2^{k(y)+1} - i2^{k+1}.$$

But now since $\delta(ay) = \delta(v_k\sigma^{i2^{k+1}})$, we must have

$$j+1 = \delta(v_{k(y)}\sigma^{i'2^{k(y)+1}}) - \delta(v_k\sigma^{i2^{k+1}}) = 2(i'2^{k(y)+1} - i2^{k+1}) \geqslant 2^{k(y)+1},$$

and hence

$$a^{j+1} v_{k(y)} \sigma^{i' 2^{k(y)+1}} = 0,$$

so $ay = 0$.

Case 2: $y = v_{k(y)} \prod v_\ell \sigma^{i' 2^{k(y)+1}} a^j \prod t_m$, $\min(\ell) \geqslant k(y)$. But then, since

$$t(v_{k(y)} \sigma^{i' 2^{k(y)+1}} a^{j+1}) < 0$$

implies $v_{k(y)} \sigma^{i' 2^{k(y)+1}} a^{j+1} = 0$,

$$t(ay) \geqslant t(v_{\min(\ell)}) \geqslant 2^{k(y)+1} - 2 > 2^{k+1} - 2 = t(v_k \sigma^{i 2^{k+1}}).$$

a contradiction. □

Next, note that $BP\mathbb{R}_\star BP\mathbb{R}$ is a flat module over $BP\mathbb{R}_\star$.

Theorem 4.22. *There is a convergent spectral sequence*

$$\mathrm{Ext}_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star) \Rightarrow \pi_\star^{\mathbb{Z}/2}(S^0) \otimes \mathbb{Z}_2. \tag{4.23}$$

Proof. As in the non-equivariant case, the Quillen idempotent defines a Morita equivalence between $((M\mathbb{R}_\star)_2^\wedge, (M\mathbb{R}_\star M\mathbb{R})_2^\wedge)$ and $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$. Thus, it suffices to prove the corresponding statement for $M\mathbb{R}$, i.e.

$$\mathrm{Ext}_{M\mathbb{R}_\star M\mathbb{R}}(M\mathbb{R}_\star, M\mathbb{R}_\star)_2^\wedge \Rightarrow \pi_\star^{\mathbb{Z}/2}(S^0) \otimes \mathbb{Z}_2. \tag{4.24}$$

Indeed, for $r \geqslant 2$, the E_r-term of the spectral sequence (4.23) is obtained from the E_r-term of the spectral sequence (4.24) by applying Quillen's idempotent.

Now consider the adjunction

$$E_\infty\text{-}M\mathbb{R}\text{-modules} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{Z}/2\text{-spectra (complete universe)}. \tag{4.25}$$

The spectral sequence (4.24) is the Borel cohomology spectral sequence of the cosimplicial spectrum

$$C(U, FU, FS^0).$$

The identification of the E_2-term is a direct consequence of flatness. Thus, if we denote the cosimplicial realization functor by $\underleftarrow{real}$, our statement is equivalent to

The canonical map

$$\underleftarrow{real}\, C(U, FU, FS^0) \xrightarrow{p} S^0 \tag{4.26}$$

is an equivalence after 2-*completion.*

Now consider a simplicial model E for $E\mathbb{Z}/2_+$. Consider the map of cosimplicial spectra

$$\underset{\overset{\leftarrow}{C}}{real}\, F(E, C(U, FU, FS^0)) \leftarrow F(E, S^0). \tag{4.27}$$

Here

$$\underset{\overset{\leftarrow}{C}}{real}$$

denotes the realization of the bi-cosimplicial spectrum on the left with respect to the cobar coordinates. Note that, by the non-equivariant Adams–Novikov spectral sequence, the map (4.27) is an equivalence of cosimplicial spectra. Finally, consider the diagram of spectra

$$\begin{array}{ccc}
\underset{\leftarrow}{real}\, C(U, FU, FS^0) & \xleftarrow{\;p\;} & S^0 \\
\Big\downarrow r & & \Big\downarrow s \\
\underset{\leftarrow}{real}\, F(E, C(U, FU, FS^0)) & \xleftarrow[\;t\;]{} & \underset{\leftarrow}{real}\, F(E, S^0).
\end{array}$$

The map t is an equivalence because (4.27) is an equivalence. The map r is an equivalence because

$$\underset{\overset{\leftarrow}{E}}{real}\, F(E, C(U, FU, FS^0)) \leftarrow C(U, FU, FS^0))$$

is a cosimplicial equivalence by the Completion Theorem 4.1. The map s is an equivalence after completing at 2 by Lin's theorem [32,33]. Therefore, p is an equivalence after completing at 2, as claimed. □

5. Galois descent. The *BPO*-based Adams-type spectral sequence

In this section, we will refine the methods of Theorem 4.22 to give a spectral sequence converging directly to $\pi_*^{\{e\}}(S^0)$. The first idea is that $BPO = BP\mathbb{R}^{\mathbb{Z}/2}$ is a non-equivariant spectrum, and one could therefore consider the Adams spectral sequence based on BPO. But, as we shall see, BPO is not a flat spectrum, and therefore the E_2-term of the BPO-based Adams spectral sequence is not easily identifiable. In fact, even $EO(n) = E\mathbb{R}(n)^{\mathbb{Z}/2}$ is not a flat spectrum (for $n > 1$). The point is that suspensions by all representations are relevant, not just the trivial ones (this is invisible for $n = 1$, since $E\mathbb{R}(1)$ is $(1+\alpha)$-periodic. We will construct a machinery which will allow us to consider all the fixed points of suspensions by non-trivial representations $(\Sigma^{k\alpha}EO(n))^{\mathbb{Z}/2}$ at the same time, and show that then, in some sense, $EO(n)$ becomes flat. We will also show that the "Morava stabilizer groups" for $BP\mathbb{R}$ do not have the extra subgroup $\mathbb{Z}/2$ generated by the formal inverse, which occurs for BP at $p = 2$. From that point of view, they are more similar to the Morava stabilizer groups at $p > 2$.

The situation of $BP\mathbb{R}$, which we need to consider to obtain our spectral sequence converging to $\pi_*(S^0)$, is more complicated than the situation for $E\mathbb{R}(n)$. Very roughly speaking, when considered globally, the "subgroup generated by the formal inverse" in the Morava stabilizer group is no longer injective. To see this, note for example that for the additive formal group law on $\mathbb{F}_2$, the inverse is the identity. Thus, instead of factoring through a subgroup, one must consider an appropriate bar construction. As a result, the E_2-term of our spectral sequence is the cohomology of a certain simplicial algebraic groupoid (instead of just algebraic groupoid).

The expository approach of this section is as follows: We first develop all of the necessary theory in the "pedestrian" language of algebra. We then explain what the constructions mean in the more appealing dual language of geometry, and how they further connect with some of the ideas of Hopkins and Miller [20]. In the process of that explanation, we shall see that the geometric language, while giving better understanding, presents technical difficulties which are better understood in algebra. A couple of technical proofs in this section will be postponed until its end.

Our starting point is the notion of a *graded* homology theory. Let $\mathscr{D}$ be the derived category of spectra (= the 'stable category'). Define the category $\mathscr{D}[a, a^{-1}]$ of *graded spectra* to be the category of formal infinite sums of the form

$$\sum_{i \in \mathbb{Z}} E_i a^i \tag{5.1}$$

where E_i are objects, (resp. morphisms) in $\mathscr{D}$. Thus, the category of graded spectra is isomorphic to $\mathscr{D}^{\mathbb{Z}}$. The extra structure we are interested in is the smash product. For graded spectra E, F, put

$$(E \wedge F)_k = \sum_{i \in \mathbb{Z}} E_i \wedge F_{k-i}.$$

Note that $S^0 = S^0.1$ is a unit of the smash product of graded spectra, which is also commutative and associative. The notion of (commutative, associative, etc.) graded ring spectra is introduced in the standard way.

An example of a commutative associative graded ring spectrum is

$$BPO = \sum_{k \in \mathbb{Z}} BPO[k\alpha]a^k,$$

where

$$BPO[k\alpha] = (\Sigma^{k\alpha} BP\mathbb{R})^{\mathbb{Z}/2}.$$

The *coefficients* of a graded spectrum E are

$$E_\star = \sum_{i \in \mathbb{Z}} (E_i)_* a^i.$$

Put $E_\star F = (E \wedge F)_\star$, $E^\star F = \sum_{n \in \mathbb{Z}} [Ea^n, F]a^n$ where '[ ]' denotes maps in $\mathscr{D}^{\mathbb{Z}}$ and Ea^n denotes the graded spectrum obtained from E by multiplying by a^n in (5.1). Note that the coefficients of a graded spectrum are bigraded in the same way as the coefficients of a $\mathbb{Z}/2$-equivariant spectrum. This resemblance in notation is deliberate, and helpful in comparing the Hopf algebroids constructed out of graded spectra to those constructed from $\mathbb{Z}/2$-equivariant spectra. For an element of the coefficients of bidegree $k + \ell\alpha$, the number $\ell \in \mathbb{Z}$ will be sometimes referred to as *twist*. Following this notation,

$$BPO_\star = BP\mathbb{R}_\star,$$

but

$$BPO_\star BPO \neq BP\mathbb{R}_\star BP\mathbb{R}.$$

Note that graded spectra satisfy Whitehead's theorem and Brown's representability theorem (the proofs are analogous to those for spectra), and therefore we can define graded *function spectra* by the adjunction formula

$$Z^\star(X \wedge Y) = F(Y, Z)^\star X.$$

We have now set up the machinery for constructing the $BPO_\star$-based non-equivariant Adams spectral sequence. We first explain the analogous construction for the Real-oriented spectrum $E\mathbb{R}(n)$. This case is easier, because $E\mathbb{R}(n)$ is a free spectrum (at least when completed at 2). Analogously as in the case of BPO, we let

$$EO(n) = \sum_i (\Sigma^{i\alpha} E\mathbb{R}(n))^{\mathbb{Z}/2}.a^k.$$

We have

$$EO(n) \wedge EO(n) = \sum_{k,\ell \in \mathbb{Z}} (\Sigma^{k\alpha} E\mathbb{R}(n))^{\mathbb{Z}/2} \wedge (\Sigma^{\ell\beta} E\mathbb{R}(n))^{\mathbb{Z}/2}.a^{k+\ell}. \tag{5.2}$$

We will find ourselves in need to distinguish between the two copies of $\mathbb{Z}/2$ involved in (5.2). To this end, put $A = B = \mathbb{Z}/2$, and let $\mathbb{Z}/2 \cong C \subset A \times B$ be the diagonal. Also, let α (resp. β) be the irreducible real representation of $A \times B$ obtained by composing the sign representation with the projection $\pi : A \times B \to A$ (resp. $\varrho : A \times B \to B$). Denote the complete G-universe for any finite group G by U_G. Now rather than (5.2), it will turn out easier to investigate the graded spectrum

$$EO(n) \wedge EO(n)^\wedge = \sum_{k,\ell \in \mathbb{Z}} F(E(A \times B)_+, \Sigma^{k\alpha} E\mathbb{R}(n)_A \wedge \Sigma^{\ell\beta} E\mathbb{R}(n)_B)^{A \times B}.a^{k+\ell}. \tag{5.3}$$

Here $EO(n)_A, EO(n)_B$ are copies of $EO(n)$, considered as $A \times B$ spectra by the projections π, ϱ, respectively. Note that the smash product on the right hand side of (5.3) can be considered an $A \times B$-spectrum indexed over $U_A \oplus U_B$ (which is not $U_{A \times B}$, but this is irrelevant, since we are taking homotopy fixed points). We will eventually establish the following

Lemma 5.4. *The $A \times B$-spectrum*

$$\Sigma^{k\alpha} E\mathbb{R}(n)_A \wedge \Sigma^{\ell\beta} E\mathbb{R}(n)_B$$

is free. Furthermore, we have

$$\Sigma^{k\alpha} E\mathbb{R}(n)_A \wedge \Sigma^{\ell\beta} E\mathbb{R}(n)_B \simeq F(E(A \times B)_+, \Sigma^{k\alpha} E\mathbb{R}(n)_A \wedge \Sigma^{\ell\beta} E\mathbb{R}(n)_B)$$

and hence

$$EO(n) \wedge EO(n) \simeq EO(n) \wedge EO(n)^\wedge.$$

Now we can write

$$EO(n)\wedge EO(n)^{\wedge} = \sum F(EA_+, F(EC_+, \Sigma^{k\alpha}E\mathbb{R}_A \wedge \Sigma^{\ell\beta}E\mathbb{R}_B)^C)^A.a^{k+\ell}. \tag{5.5}$$

But, by the theory of Real-oriented spectra,

$$F(EC_+, E\mathbb{R}_A \wedge E\mathbb{R}_B)^C_\star = E\mathbb{R}(n)_\star E\mathbb{R}(n) = (E(n)\wedge E(n))\mathbb{R}_\star = E\mathbb{R}(n)_\star[t_i]/(R) \tag{5.6}$$

where the relations (R) are the same as in $E(n)_* E(n)$. The difference is that the dimension of t_i is $(1+\alpha)(2^i-1)$. Note, further, that (5.6) does not change when we suspend by $S^{\alpha-\beta}$. In other words, there is a periodicity operator μ of total dimension 0 on the graded spectrum

$$E\mathbb{R}(n)_\star E\mathbb{R}(n)[\mu,\mu^{-1}] = \sum F(EC_+, \Sigma^{k\alpha}E\mathbb{R}(n)_A \wedge \Sigma^{\ell\beta}E\mathbb{R}(n)_B)^C.a^{k+\ell}.$$

By considering, in an analogous way, external smash products of several copies of $E\mathbb{R}(n)$ with actions of distinct copies of $\mathbb{Z}/2$, and taking fixed points with respect to the diagonals, we obtain a Hopf algebroid structure

$$(E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star E\mathbb{R}(n)[\mu,\mu^{-1}]). \tag{5.7}$$

Lemma 5.8.

$$\mathrm{Ext}_{E\mathbb{R}(n)_\star E\mathbb{R}(n)[\mu,\mu^{-1}]}(E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star)$$

is the coefficient at a^0 *of*

$$\mathrm{Ext}_{E\mathbb{R}(n)_\star E\mathbb{R}(n)}(E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star).$$

Proof. The cobar resolution of $E\mathbb{R}(n)_\star$ over (5.7) is

$$\sum(\Sigma^{k_0\alpha_0}E\mathbb{R}(n)_{A_0}\wedge\cdots\wedge\Sigma^{k_n\alpha_n}E\mathbb{R}(n)_{A_n}\wedge\Sigma^{k_{n+1}\alpha_{n+1}}E\mathbb{R}(n)_{A_{n+1}})^{C_{n+1}}_\star.a^{\sum k_i}$$

where C_{n+1} is the diagonal $\mathbb{Z}/2$-subgroup of $A_0\times\cdots\times A_{n+1}$. But now another relatively injective resolution of $E\mathbb{R}(n)_\star$ over (5.7) is

$$\sum(E\mathbb{R}(n)_{A_0}\wedge\cdots\wedge E\mathbb{R}(n)_{A_n}\wedge\Sigma^{k_{n+1}\alpha_{n+1}}E\mathbb{R}(n)_{A_{n+1}})^{C_{n+1}}_\star.a^{k_{n+1}}.$$

The (5.7)-primitives of the second resolution coincide with the $(\mathbb{Z}+0\alpha)$-dimensional summand of

$$C(E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star E\mathbb{R}(n), E\mathbb{R}(n)_\star). \qquad \square$$

Now our main result on $EO(n)_\star EO(n)$ is the following

Theorem 5.9. $E\mathbb{R}(n)_\star E\mathbb{R}(n)$ *is a free* $E\mathbb{R}(n)_\star[A]$*-module by the A-action arising from* (5.5). *Furthermore, the A-action is multiplied by the sign representation when we multiply by* μ. *Consequently, the*

Hopf algebroid $(EO(n)_\star, EO(n)_\star EO(n))$ *is flat and we have*

$$EO(n)_\star EO(n) = (E\mathbb{R}(n)_\star E\mathbb{R}(n)[\mu, \mu^{-1}])^A. \tag{5.10}$$

Proof. Let $(E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star[A])$ denote the group algebra of $A(=\mathbb{Z}/2)$ over the ring $E\mathbb{R}(n)_\star$. Denote by $(E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star[A]^\vee)$ the dual $E\mathbb{R}(n)_\star$-Hopf algebra. Then, by appropriately dualizing, the A-action on $E\mathbb{R}(n)_\star E\mathbb{R}(n)$ gives a map of Hopf algebroids

$$\phi: (E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star E\mathbb{R}(n)[\mu, \mu^{-1}]) \to (E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star[A]^\vee).$$

Concretely, if the action of the generator q of A on $E\mathbb{R}(n)_\star E\mathbb{R}(n)$ is by a map χ, then we have the formulas

$$\phi(x)[1] = \varepsilon(x), \qquad \phi(x)[q] = \varepsilon\chi(x).$$

An explicit formula for χ follows from the corresponding formula for χ on

$$BP\mathbb{R}_\star BP\mathbb{R} = BP\mathbb{R}_\star[t_i]. \tag{5.11}$$

for $x \in E\mathbb{R}(n)_\star E\mathbb{R}(n)$. First, χ in (5.11) is identity on $BP\mathbb{R}_\star$. The formulas for the action on the t_i's are the same as in BP_*BP, because the forgetful homomorphism

$$BP\mathbb{R}_\star BP\mathbb{R} = (\Sigma^{k\alpha}BP\mathbb{R} \wedge \Sigma^{\ell\beta}BP\mathbb{R})^C_\star \to ((\Sigma^{k\alpha}BP\mathbb{R} \wedge \Sigma^{\ell\beta}BP\mathbb{R})_e)_\star = BP_*BP[\sigma,\sigma^{-1}][\mu,\mu^{-1}]$$

is an isomorphism in dimensions $k(1+\alpha)$. For the same reason, the A-action on μ can be detected by the A-action on

$$BP_*BP[\sigma,\sigma^{-1}][\mu, \mu^{-1}].$$

But non-equivariantly, the action of q on α is -1, while on β it is $+1$. Thus,

$$q(\mu) = -\mu.$$

The formula for the action on the t_i's in BP_*BP, by ordinary formal group law theory, represents sending a map of formal group laws $f: G \to F$ (where F corresponds to η_R) to $f(i_G(-x))$. Thus, we have

$$\Sigma^F \chi(t_i)x^{2^i} = \Sigma^F t_i i_G(-x)^{2^i}. \tag{5.12}$$

Applying the augmentation ε, we get

$$\sum^F \varepsilon\chi(t_i)x^{2^i} = i_G(-x) = -i_F(x). \tag{5.13}$$

Now

$$(x - i_F(x)) \,|\, [2]_F x, \tag{5.14}$$

because setting $x = i_F(x)$ sets $[2]_F(x) = 0$. Recursively, (5.14) implies that

$$\mathbb{Z}[\varepsilon\chi(t_i)] = \mathbb{Z}[v_i].$$

Defining $x \in E\mathbb{R}(n)_\star[A]^\vee$ by $x[1] = 1$, $x[\alpha] = 0$, we conclude that

$$E\mathbb{R}(n)_\star \oplus \{x\}.(v_1, v_2, \dots) \subseteq \operatorname{Im} \phi$$

(where $(v_1, v_2, \dots)$ denotes the ideal in $E\mathbb{R}(n)_\star$ generated by the v_i's). However, because in $E\mathbb{R}(n)_\star$ there exists an element v_n^{-1}, we conclude that

$$(v_1, v_2, \dots) = E\mathbb{R}(n)_\star.$$

In other words, the map ϕ is onto. Consequently, by the Milnor–Moore theorem (see [46, Section 6.1 and Theorem A1.1.17 of the appendix]), $E\mathbb{R}(n)_\star E\mathbb{R}(n)$ is, as an $E\mathbb{R}(n)_\star[A]^\vee$-comodule, a sum of copies of $E\mathbb{R}(n)_\star[A]^\vee$, as claimed. The other claims follow. □

Proposition 5.15. $\operatorname{Ext}_{EO(n)_\star EO(n)}(EO(n)_\star, EO(n)_\star)$ *is entirely concentrated in dimensions* $\mathbb{Z} + 0\alpha$ (*twist* 0).

Proof. Consider the usual cobar resolution of the graded spectrum $S^0.a^0$:

$$\begin{aligned} &L_0 = S^0, \\ &K_n = EO(n) \wedge L_n, \\ &L_n \to K_n \to L_{n+1} \text{ is a cofibration.} \end{aligned} \tag{5.16}$$

Applying $EO(n)_\star$ to (5.16), we obtain the cobar $EO(n)_\star EO(n)$-resolution of $EO(n)_\star$ (by flatness). On the other hand, we can also take the graded spectra $\tilde{K}_n$, $\tilde{L}_n$ obtained from K_n, L_n by restricting to the coefficient at a^0 (and setting the other coefficients equal to 0). Then, in the category of graded spectra,

$$EO(n)_\star \tilde{K}_n \tag{5.17}$$

is another relatively injective $EO(n)_\star EO(n)$ resolution of $EO(n)_\star$.

The $EO(n)_\star EO(n)$-primitives on (5.17), however, are concentrated in twist 0, thus proving our statement. □

We state one more result on the periodicities in $EO(n)_\star EO(n)$.

Lemma 5.18. *Let* $\gamma = \alpha \otimes \beta$. *Then*

$$\Sigma^{1-\gamma} EO(n) \wedge EO(n) \simeq \Sigma^{\alpha-\beta} EO(n) \wedge EO(n).$$

In particular, by symmetry, $EO(n) \wedge EO(n)$ *is* $(2\alpha - 2\beta)$*-periodic and* $(2\gamma - 2)$*-periodic.*

Proof. By Lemma 5.4, it suffices to prove

$$E(A \times B_+) \wedge_{A \times B} \Sigma^{1+\beta} E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B \simeq E(A \times B_+) \wedge_{A \times B} \Sigma^{\alpha+\gamma} E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B.$$

We will prove that, more generally, for any U_A-spectrum Z,

$$E\mathbb{R}(n)_B \wedge EA_+ \wedge_A \Sigma^{1+\beta} Z_A \simeq E\mathbb{R}(n)_B \wedge EA_+ \wedge_A \Sigma^{\alpha+\gamma} Z_A. \tag{5.19}$$

To this end, for $Z_A = \Sigma^\infty X_+$, where X is an A-space, we have

$$EA_+ \wedge_A \Sigma^{\alpha+\gamma} Z_A \simeq T((1+\alpha)\zeta),$$
$$EA_+ \wedge_A \Sigma^{1+\beta} Z_A \simeq T((1+\alpha)\xi),$$

where T denotes the Thom space (more precisely Thom spectrum), ζ is the real vector bundle on $EA \times_A X$ associated with the principal line bundle $EA \times X$, while ξ is the real vector bundle on $EA \times_A X$ associated with the trivial line bundle. Thus, in this case, (5.19) follows from the existence of Thom isomorphisms for Real-oriented spectra. The general case follows by a colimit argument. □

Before giving a proof of Lemma 5.4, we state a more general result in this situation, which, however, we use only marginally in the present paper.

Lemma 5.20. *Let A, B be finite groups, let X (resp. Y) be a A-U_A-spectrum (resp. B-U_B-spectrum). Then*

$$X^A \wedge Y^B \simeq (X \wedge Y)^{A \times B}.$$

Here, on the right-hand side, $X \wedge Y$ is considered as an $A \times B$-spectrum indexed over $U_A \oplus U_B$.

The proof of this lemma is postponed until the end of this section.

Proof of Lemma 5.4. By Lemma 5.20 and the fact that $EO(n)$ is free,

$$EO(n) \wedge EO(n) \simeq \sum_{k,\ell \in \mathbb{Z}} ((EA \times EB)_+ \wedge \Sigma^{k\alpha} E\mathbb{R}(n)_A \wedge \Sigma^{\ell\beta} E\mathbb{R}(n)_B)^{A \times B} \cdot a^{k+l}.$$

Thus, the statement of Lemma 5.4 follows from saying that the norm-map

$$N : (EA \times EB)_+ \wedge_{A \times B} E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B \to F((EA \times EB)_+, E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B)^{A \times B}$$

is an equivalence. (Note that the norm map is defined, because the universe $U_A \oplus U_B$, although not complete, does contain a copy of the $A \times B$-set $A \times B$.) First of all, we know (by freeness and Thom isomorphisms for Real-oriented spectra) that the norm map

$$N : (EA \times EB)_+ \wedge_C E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B \to F((EA \times EB)_+, E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B)^C$$

is an equivalence. Thus, it suffices to show that the norm map

$$N : EA_+ \wedge_A (F((EA \times EB)_+, E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B)^C)$$
$$\to F(EA_+, (F((EA \times EB)_+, E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B)^C)^A$$

is an equivalence. But this follows from the above established fact that the action of A on

$$(F(EA \times EB_+, E\mathbb{R}(n)_A \wedge E\mathbb{R}(n)_B)^C)_\star$$

is free. □

We now consider the case of $BP\mathbb{R}_\star$, which we are really interested in. Unfortunately, one easily sees that $BPO_\star BPO$ is not $BP\mathbb{R}_\star$-flat, as, for example,

$$BPO \wedge \lim_{\substack{\to \\ a}} \Sigma^{-k\alpha} BPO \simeq BPO \wedge H\mathbb{Z}/2[a, a^{-1}],$$

and the coefficients of the right-hand side graded spectrum certainly are not $BP\mathbb{R}_\star$-flat.

Because of non-flatness, the situation becomes more complicated. In particular, the term 'Adams spectral sequence' becomes ambiguous, and we have to select one of several possibilities.

Our approach is motivated by the proof of (5.15) above. Consider the action of $\mathbb{Z}/2 = (A \times B)/C$ on

$$BP\mathbb{R}_\star BP\mathbb{R}[\mu, \mu^{-1}] = \sum (\Sigma^{k\alpha + \ell\beta} BP\mathbb{R}_A \wedge BP\mathbb{R}_B)^C . a^{k+\ell}.$$

This gives a map of Hopf algebroids

$$(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R}[\mu, \mu^{-1}]) \to BP\mathbb{R}_\star \otimes (\mathbb{Z}, \mathbb{Z}[\mathbb{Z}/2]^\vee),$$

which, however, unlike its $EO(n)$ analogues, unfortunately is not onto (concretely, 1 is not in the image). Nevertheless, we obtain a cosimplicial $BP\mathbb{R}_\star$-coalgebroid

$$\Gamma_. = C(BP\mathbb{R}_\star BP\mathbb{R}[\mu, \mu^{-1}], \mathbb{Z}[\mathbb{Z}/2]^\vee, \mathbb{Z}),$$

which is totalized by the shuffle map into a DG-$BP\mathbb{R}_\star$-coalgebroid Γ.

We can now state our main result:

Theorem 5.21. *There exists a convergent spectral sequence*

$$\mathrm{Ext}_{(BP\mathbb{R}_\star, \Gamma)}(BP\mathbb{R}_\star, BP\mathbb{R}_\star) \Rightarrow \pi_*^{\{e\}}(S^0).1. \tag{5.22}$$

Furthermore, the left hand side of (5.22) *is entirely concentrated in twist* 0.

Proof. The fact that the left-hand side of (5.22) is concentrated in twist 0, is completely analogous to the case of $EO(n)_\star$.

Now by a *semicosimplicial spectrum* we shall mean a sequence of spectra X_i, $i \geqslant 0$ together with coface and codegeneracy structure maps, where the coface identities are satisfied on the nose, and all the other cosimplicial identities (in particular codegeneracies) are satisfied up to homotopy. Note that this is sufficient to produce a cosimplicial realization functor (by which we will mean its 'derived' version, i.e. a functor that preserves weak equivalences). Specifically, the realization is by totalization, whose definition does not involve the (co)degeneracies. The (co)degeneracies up to homotopy are used to control the quasiisomorphism type of the resulting cochain complex.

Now let $A_0 = \cdots = A_m = \mathbb{Z}/2$, let $C_m \subseteq A_0 \times \cdots \times A_m$ be the diagonal, and let $D_m = A_0 \times \cdots \times A_m / C_m$. Then there is a double semicosimplicial graded spectrum (indexed by m, p)

$$\Omega = \sum F(B_p(D_m, D_m, *), \Sigma^{k_0\alpha_0 + \cdots + k_m\alpha_m} BP\mathbb{R}_{A_0} \wedge \cdots \wedge BP\mathbb{R}_{A_m})^{A_0 \times \cdots \times A_m} . a^{k_0 + \cdots + k_m}. \tag{5.23}$$

Convergence of (5.22) is equivalent to saying that the double cosimplicial realization of Ω is homotopy equivalent to S^0. Now consider also the double semicosimplicial graded spectrum

$$\Omega' = \sum F\left(B_p\left(\prod_{i=0}^{m} A_i, \prod_{i=0}^{m} A_i, *\right), \Sigma^{k_0\alpha_0 + \cdots + k_m\alpha_m} BP\mathbb{R}_{A_0} \wedge \cdots \wedge BP\mathbb{R}_{A_m}\right)^{A_0 \times \cdots \times A_m} .a^{k_0 + \cdots + k_m}. \quad (5.24)$$

By completeness of $BP\mathbb{R} \wedge \cdots \wedge BP\mathbb{R}$ with respect to C_m, one can see that the totalizations of Ω, Ω' coincide. But now realizing the m-coordinate first in Ω' gives a constant semicosimplicial spectrum with term $S^0.a^0$, by the non-equivariant Adams–Novikov spectral sequence.

In more detail, the pth term of the cosimplicial spectrum obtained from Ω' by realizing the m-coordinate first is

$$\left| \sum F\left(\Big(\overbrace{\prod_{i=0}^{m} A_i \times \cdots \times \prod_{i=0}^{m} A_i}^{p+1 \text{ times}} \times *\Big)_+, \right.\right.$$
$$\left.\left. \Sigma^{k_0\alpha_0 + \cdots + k_m\alpha_m} BP\mathbb{R}_{A_0} \wedge \cdots \wedge BP\mathbb{R}_{A_m}\right)^{A_0 \times \cdots \times A_m} .a^{k_0 + \cdots + k_m} \right|_m$$

$$= \left| \sum F\left(\Big(\overbrace{\prod_{i=0}^{m} A_i \times \cdots \times \prod_{i=0}^{m} A_i}^{p+1 \text{ times}} \times *\Big)_+, \right.\right.$$

$$\left.\left. \Sigma^{k_0 + \cdots + k_m} \overbrace{BP \wedge \cdots \wedge BP}^{m+1 \text{ times}}\right) . a^{k_0 + \cdots + k_m} \right|_m$$

$$= \left| \sum F\left(\Big(\overbrace{\prod_{i=0}^{m} A_i \times \cdots \times \prod_{i=0}^{m} A_i}^{m+1 \text{ times}} \times *\Big)_+, \right.\right.$$

$$\left.\left. \overbrace{BP \wedge \cdots \wedge BP}^{m+1 \text{ times}}\right) \right|_m ,$$

where $| \; |_m$ denotes realization with respect to the m-coordinate. (The last equality is Morita equivalence. Similarly as above, but non-equivariantly, we can consider two variables σ, ϱ of dimension $\alpha - 1$, and the graded Hopf algebroid $(BP_*[\sigma, \sigma^{-1}]), BP_*BP[\sigma, \sigma^{-1}, \varrho, \varrho^{-1}])$. The canonical map from the cobar complex of the first Hopf algebroid to the cobar complex of the second is an equivalence.)

Now the right-hand side is the diagonal of a $(p+1)$-fold cosimplicial spectrum. The diagonal realization is homotopically equivalent to the totalization:

$$F\left(\left(*\times\prod_{i=0}^{m_1} A_i\times\cdots\times\prod_{i=0}^{m_p} A_i\times *\right)_+,\ \overbrace{BP\wedge\cdots\wedge BP}^{m_{p+1}\text{ times}}\right). \tag{5.25}$$

The realization of the m_{p+1}-coordinate is S^0, by the convergence of the non-equivariant Adams–Novikov spectral sequence. Each of the simplicial sets

$$\prod_{i=0}^{m_j} A_i,\quad j=1,\dots,p$$

is $B(A_i, A_i, *)$ written in homogeneous coordinates, and is therefore contractible. $\square$

We shall now explain the geometrical interpretation of our constructions. Let K be a degree n unramified extension of $\mathbb{Q}_p$, and let $\mathcal{O}_K$ be the ring of integers of K. Then p is a prime in $\mathcal{O}_K$ and the residue field is

$$\mathcal{O}_K/p\mathcal{O}_K\cong\mathbb{F}_{p^n}.$$

Consider the p-typical formal group law Φ_n on $\mathbb{F}_{p^n}$ given by the map

$$BP_*\to\mathbb{F}_{p^n}$$

which sends v_n to 1 and v_i to 0 for $i\neq n$. The group of strict automorphisms of Φ_n will be denoted by S_n^0. Now one can study lifts of Φ_n to complete rings with residue field $\mathbb{F}_{p^n}$. A ★-isomorphism of lifts F, G of Φ_n is a strict isomorphism

$$f: F\to G$$

such that the reduction of f modulo the maximal ideal is the identity. Lubin and Tate [34] showed that the set of ★-isomorphism classes of lifts of Φ_n to complete $\mathcal{O}_K$-algebras A with maximal ideal m containing (p) is in bijective correspondence with maps of rings

$$\phi:\mathcal{O}_K[[u_1,\dots,u_{n-1}]]\to A \tag{5.26}$$

for certain elements $u_1,\dots,u_{n-1}$. There is a *universal* lift Γ_n of Φ_n to $\mathcal{O}_K[[u_1,\dots,u_{n-1}]]$ and of each lift F of Φ_n to A, there is a unique map of rings (5.26) such that $\phi(\Gamma_n)$ is ★-isomorphic to F. Note that considering $A=\mathcal{O}_K[[u_1,\dots,u_{n-1}]]$ and the fact that a formal group law isomorphic to Φ_n always lifts to A, we obtain an action of S_n^0 on $\mathcal{O}_K[[u_1,\dots,u_{n-1}]]$.

Now in algebraic topology, one deals with graded rings and maps of rings realized by maps of spectra must preserve dimensions. Lubin–Tate theory carries over essentially without change to the category of $\mathbb{Z}$-graded commutative rings (in the ungraded sense), where formal group laws are by definition homogeneous series of degree 1. The appropriate analogue of $\mathbb{F}_{p^n}$ is

$$\mathbb{F}_{p^n}[u,u^{-1}].$$

The appropriate analogue of Φ_n is the graded p-typical formal group law (which we will also denote by Φ_n) given by the map $BP_* \to \mathbb{F}_{p^n}[u,u^{-1}]$ where $v_n \mapsto u^{p^n-1}$, $v_i \mapsto 0$ for $i \neq n$. The reason this is the right graded translation is that the group of graded strict isomorphisms of Φ_n is still S_n^0. Now the appropriate graded analogue of $\mathcal{O}_K$ is $\mathcal{O}_K[u,u^{-1}]$, and the appropriate analogue of $\mathcal{O}_K[[u_1,\dots,u_{n-1}]]$ is

$$E_{n*} = \mathcal{O}_K[[u_1,\dots,u_{n-1}]][u,u^{-1}], \tag{5.27}$$

where the u_i's are (necessarily) in degree 0. This was invented by Mike Hopkins. In particular, E_{n*} represents $\star$-isomorphism classes of graded FGL's on graded complete Noetherian E_{n*}-algebras A with (graded) maximal ideal containing (p). In particular, S_n^0 acts on E_{n*}. But, in fact, by the same argument, the action can be extended to

$$G_n = Gal(K/\mathbb{Q}_p) \rtimes (\mu_{p^n-1} \rtimes S_n^0),$$

where $\mu_{p^n-1} \rtimes S_n^0 = S_n$ is the group of non-strict (graded) isomorphisms of Φ_n, and $Gal(K/\mathbb{Q}_p)$ acts on everything by the Galois action.

Now Hopkins and Miller [20] observed that using Landweber exactness, one can construct a complex-oriented spectrum E_n with coefficients E_{n*}, such that Γ_n is the formal group law associated with the complex orientation of E_n. Further, by considering G_n as a pro-algebraic group, we see that G_n acts algebraically on $Spec(E_{n*})$. It is a result of Hopkins and Miller that $E_{n*}E_n$ can be identified with $\mathcal{O}_{G_n} \otimes E_{n*}$ after appropriately completing both sides. ($E_{n*}E_n$ must be completed for the same reason as $\mathbb{Z}_p \otimes \mathbb{Z}_p \neq \mathbb{Z}_p$; on the other hand, G_n being an inverse limit, $\mathcal{O}_{G_n}$ is a direct limit, and is not complete, either.)

Now from the point of view of Real orientations, we can construct a Real-oriented spectrum $E\mathbb{R}_n$ where

$$(E\mathbb{R}_n)_{k(1+\alpha)} = (E_n)_{2k}.$$

Now similarly as above, G_n acts on $E\mathbb{R}_{n\star}$ and after appropriately completing, ($Spec(E\mathbb{R}_{n\star})$, $Spec(E\mathbb{R}_{n\star}E\mathbb{R}_n)$) can be identified with the algebraic groupoid obtained from the group G_n acting on $Spec(E\mathbb{R}_{n\star})$. Now $E\mathbb{R}_{n\star}$, $E\mathbb{R}_{n\star}E\mathbb{R}_n$ have an additional grading by the twist. From a geometric point of view, a $\mathbb{Z}$-grading specifies an action of $\mathbb{G}_m = Spec(\mathbb{Z}[\mu,\mu^{-1}])$ by shifting of the degree (if we take advantage of the (2^n-1)-periodicity in the grading, i.e. consider the $\mathbb{Z}/(2^n-1)$-graded factor object obtained by identifying degrees modulo 2^n-1, geometrically that means we have found a subgroupoid with an action of μ_{2^n-1}, from which the present object is obtained by applying $? \times_{\mu_{2^n-1}} \mathbb{G}_m$). Our point is that, after completing,

$$(Spec\, E\mathbb{R}_{n\star}, Spec\, E\mathbb{R}_{n\star}E\mathbb{R}_n[\mu,\mu^{-1}])$$

can be identified with the groupoid obtained from the group $G_n \times \mathbb{G}_m$ acting on $Spec(E\mathbb{R}_{n\star})$. (Here the product is direct because G_n is concentrated in twist 0 or, geometrically speaking, has trivial $\mathbb{G}_m$-action.) Now to interpret

$$(Spec(EO_{n\star}), Spec(EO_{n\star}EO_n)) \tag{5.28}$$

(where EO_n is the graded spectrum associated with $E\mathbb{R}_n$ in the same way as $EO(n)$ is associated with $E\mathbb{R}(n)$), consider the map

$$\mu_2 \to G_n$$

which takes the generator ζ to the formal inverse $i_{\Phi_n}(x)$. But this, of course, does not give the trivial action on $EO_{n\star}$. In fact, to calculate the map on $EO_{n\star}$, we must lift $i_{\Phi_n}(x)$. Now such a lift, to a strict isomorphism, is $-i_\Gamma(x)$. If we denote by Γ' the image of Γ under $-i_\Gamma(x)$, it is the same as the image of Γ under the non-strict isomorphism $-x$. This is induced by the map

$$\zeta : EO_{n\star} \to EO_{n\star}$$

which is $(-1)^\ell$ in twist ℓ. But this action is the same as the action of the image of ζ under the canonical inclusion

$$\mu_2 \subset \mathbb{G}_m.$$

Thus, we have an inclusion

$$\mu_2 \subset G_n \times \mathbb{G}_m,$$

where the image acts trivially on $EO_{n\star}$, and, up to completion (5.28) is isomorphic to the groupoid obtained by the action of $(G_n \times \mathbb{G}_m)/\mu_2$ on $Spec(EO_{n\star})$.

Now the cohomology of $\mathbb{G}_m$ with coefficients in a graded module is 0 in positive dimensions (similarly as for representations). Consequently, the cohomology of $(G_n \times \mathbb{G}_m)/\mu_2$ (in a suitable category of complete modules) with coefficients in $EO_{n\star}$ is the same as the cohomology of G_n/μ_2 with coefficients in EO_{n*} (the twist 0 part of $EO_{n\star}$). This is what we mean when we say that the Real Morava stabilizer groups are equal to the complex Morava stabilizer groups modulo the subgroup μ_2 generated by the formal inverse (see (1) in the Introduction).

However, technically, we prefer to consider the uncompleted Hopf algebroid

$$(E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star E\mathbb{R}(n)), \tag{5.29}$$

to avoid problems with the completion. This, of course, results in a certain loss of symmetry: Consider (5.29) as an affine algebraic groupoid Θ_n. Then $\mathbb{G}_m$ acts on Θ_n by the grading. Then, one can form a semidirect product $\Theta_n \rtimes \mathbb{G}_m$. (The objects of $\Theta_n \rtimes \mathbb{G}_m$ are the same as the objects of Θ_n, the morphisms are the morphisms of Θ_n times $\mathbb{G}_m$, where $\mathbb{G}_m$ acts on objects and interchanges with morphisms via the specified actions.) The structure is

$$\mathcal{O}_{\Theta_n \rtimes \mathbb{G}_m} = (E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star E\mathbb{R}(n)[\mu,\mu^{-1}]).$$

Now consider the algebraic groupoid

$$M_n = (Spec(E\mathbb{R}(n)_\star),(\mu_2)_{Spec(E\mathbb{R}(n)_\star)}),$$

with structure ring

$$(E\mathbb{R}(n)_\star, E\mathbb{R}(n)_\star[\mu_2]^\vee).$$

We can describe the normal subgroupoid inclusion

$$\iota : M_n \to \Theta_n \rtimes \mathbb{G}_m \tag{5.30}$$

as follows: The composition

$$M_n \to \Theta_n \rtimes \mathbb{G}_m \to (\mathbb{G}_m)_{Spec(E\mathbb{R}(n)_\star)}$$

is induced by the standard map

$$\gamma : \mu_2 \to \mathbb{G}_m \tag{5.31}$$

(sending the generator ζ to -1). Now a map (5.30) is the same thing as an element of

$$H^1(M_n, \Theta_n),$$

by which we mean a twisted homomorphism

$$\phi : M_n \to \Theta_n, \tag{5.32}$$

where the twisting is given by (5.31). In our case, the desired map (5.32), on the generator ζ of μ_2 represents the strict isomorphism

$$F \overset{-i_F(x)}{\to} G. \tag{5.33}$$

In this case, the map (5.30) is an inclusion (Theorem 5.9), and we have an identification of the structure ring

$$\mathcal{O}_{(\Theta_n \rtimes \mathbb{G}_m)/M_n} = (EO(n)_\star, EO(n)_\star EO(n)). \tag{5.34}$$

Note that Hopkins–Miller theory makes the factor $\mathbb{G}_m$ (or, more precisely, μ_{2^n-1} when considering the periodicity) transparent, but the factor is hidden in introducing the element u (see (5.27)) which, in fact, makes the theory $E\mathbb{R}_n$ $(1+\alpha)$-periodic.

Now in the case of $BP\mathbb{R}$, the situation becomes further complicated by the fact that if we replace Θ_n by the algebraic groupoid Θ with structure ring

$$(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R}),$$

the analogue

$$\iota : M \to \Theta \rtimes \mathbb{G}_m \tag{5.35}$$

of the map (5.30) is no longer an inclusion. Here we let

$$M = (Spec(BP\mathbb{R}_\star), (\mu_2)_{Spec(BP\mathbb{R}_\star)}),$$

which has structure ring

$$(BP\mathbb{R}_\star, BP\mathbb{R}_\star[\mu_2]^\vee).$$

Note that the map (5.31) is not an inclusion, although it is an inclusion on the complement of the point 2. On the other hand, over $\mathbb{F}_2$, the map (5.31) is actually 0. Similar reasoning can be applied to (5.35): the objects of Θ represent all p-typical formal group laws, in particular the additive formal group law, where the isomorphism (5.33) is the identity.

For this reason, instead of $(\Theta \rtimes \mathbb{G}_m)/M$, we need to consider the appropriate *derived* object, which is the simplicial algebraic groupoid

$$\Phi = B(\Theta \rtimes \mathbb{G}_m, M, *) \tag{5.36}$$

(where $*$ stands for $Spec(BP\mathbb{R}_\star)$). Note that the structure ring of the simplicial algebraic groupoid (5.36) is exactly

$$(BP\mathbb{R}_\star, \Gamma)$$

(see Theorem 5.21). Therefore, the E_2-term of the spectral sequence constructed in Theorem 5.21 is the cohomology of the simplicial algebraic groupoid (5.36).

Finally, note that while the geometric language presented in the above remarks helps clarify the meaning of the objects we constructed, if one wants to make sense of what the geometrical objects really means, one has to go back to the algebra.

Proof of Lemma 5.20. Let $i:(U_A)^A \to U_A$, $j:(U_B)^B \to U_B$ be inclusions. First, we have a map

$$i^*X \wedge j^*Y \to (i \oplus j)^*(X \wedge Y), \tag{5.37}$$

adjoint to the counit map

$$(i \oplus j)_*(i^*X \wedge j^*Y) = i_*i^*X \wedge j_*j^*Y \to X \wedge Y.$$

Next, for spectra Z, T indexed over $(U_A)^A$, $(U_B)^B$, respectively, we have

$$Z^A \wedge T^B = (Z \wedge T)^{A \times B},$$

because for Σ-closed inclusion prespectra indexed over trivial universes the fixed point functor commutes with the L-functor.

Thus, we obtain a map

$$(i^*X)^A \wedge (j^*Y)^B \to ((i \oplus j)^*(X \wedge Y))^{A \times B}. \tag{5.38}$$

We need to show that (5.38) is an equivalence. But then, by induction, it suffices to show this for orbit spectra

$$X = \Sigma^\infty A/H_+, \quad Y = \Sigma^\infty B/J_+.$$

By the Wirthmüller isomorphism, however, we have a commutative diagram

$$\begin{array}{ccc}
(i_*A/H_+)^A \wedge (j_*B/J_+)^B & \longrightarrow & ((i \oplus j)_*(A \times B)/(H \times J)_+)^{A \times B} \\
\simeq \uparrow & & \uparrow \simeq \\
(i_*S^0)^H \wedge (j_*S^0)^J & \longrightarrow & ((i \oplus j)_*S^0)^{H \times J}
\end{array} \tag{5.39}$$

and hence it suffices to consider the case $X = S^0$, $Y = S^0$.

Now we claim that the wedge of transfer maps

$$\bigvee_{\substack{(H) \subset A \\ (J) \subseteq B}} B(W_A H)_+ \wedge B(W_B J)_+ \to ((i \oplus j)_* S^0)^{A \times B} \tag{5.40}$$

is an equivalence, where (H),(J) run through conjugacy classes of subgroups of A, B, and W denotes the Weyl group in the ambient group specified by the subscript. Note that this implies our statement by naturality of transfer, since the splitting principle of [31] (Theorem V.9.1 and Corollary V.11.2) says that

$$\bigvee_{(H) \subseteq A} B(W_A H)_+ \xrightarrow{\simeq} (i_* S^0)^A.$$

Obviously, (5.40) is a variant of the splitting principle for a non-complete universe. For a recent account of some non-complete universe splitting theorems, see [30].

To prove (5.40) here, we use Segal's infinite loop space approximation theorem [31, Theorem VII.5.6], which is stated for general universes. By the Approximation theorem, the infinite loop space QS^0 with respect to the universe $U_A \times U_B$ is equivalent to the group completion of the unordered configuration space $\mathrm{Conf}(U_A \times U_B)$ (by which we mean the space of unordered tuples of distinct points in the universe, with action induced by the group action on the universe). By taking fixed points, we obtain the space of configurations of $A \times B$-orbits. Now it is easy to see that

1. the orbits occurring in $U_A \times U_B$ are exactly $A/H \times B/J$,
2. $(\mathrm{Conf}(U_A \times U_B))^{A \times B}$ is homotopically equivalent to the product of the configuration spaces of orbits of the different fixed types $A/H \times B/J$,
3. the configuration space of $A/H \times B/J$-orbits is

$$\coprod_{n \geqslant 0} B(\Sigma_n \wr W_A H \times W_B J).$$

This implies (5.40). □

6. The $\mathbb{Z}/2$-equivariant Steenrod algebra

In this section, we will study the $\mathbb{Z}/2$-equivariant Steenrod algebra. Large parts of this subject were previously thoroughly investigated by Greenlees [16]. In particular, Greenlees has constructed a $\mathbb{Z}/2$-equivariant Adams spectral sequence. In this paper, however, we are mostly interested in connections between the $\mathbb{Z}/2$-equivariant Adams spectral sequence and the Real Adams–Novikov spectral sequence. For this reason, we use a somewhat different approach.

First of all, the reader should be warned of the ambiguity of the term $\mathbb{Z}/2$-equivariant Steenrod algebra. First note that there are several sets of "coefficients" around. Consider the fixed Eilenberg–MacLane spectrum $H\mathbb{Z}/2_{\mathrm{triv}}$ indexed over the trivial universe. Let $i: \mathbb{R}^\infty \to \mathscr{U}$ be the inclusion from the trivial to the complete universe. We put

$$H = H\mathbb{Z}/2 = i_*(H\mathbb{Z}/2_{\mathrm{triv}}).$$

For any $\mathbb{Z}/2$-equivariant spectrum indexed over the complete universe, we have the Tate diagram

$$\begin{array}{ccccc} E\mathbb{Z}/2_+ \wedge E & \longrightarrow & E & \longrightarrow & \widetilde{E\mathbb{Z}/2} \wedge E \\ \downarrow & & \downarrow & & \downarrow \\ E\mathbb{Z}/2_+ \wedge F(E\mathbb{Z}/2_+, E) & \longrightarrow & F(E\mathbb{Z}/2_+, E) & \longrightarrow & \widetilde{E\mathbb{Z}/2} \wedge F(E\mathbb{Z}/2_+, E) = \hat{E} \end{array} \tag{6.1}$$

(see [19], and also Section 2 above). In (6.1), coefficient groups associated with terms of the lower row are usually decorated by the letters f,c,t, in this order — see [19]. In accordance with that notation, let

$$H^{\mathrm{c}}_\star = F(E\mathbb{Z}/2_+, H)_\star = \mathbb{Z}/2[\sigma, \sigma^{-1}, a],$$

$$H^{\mathrm{t}}_\star = (F(E\mathbb{Z}/2_+, H) \wedge \widetilde{E\mathbb{Z}/2})_\star = \mathbb{Z}/2[\sigma, \sigma^{-1}, a, a^{-1}],$$

$$H^{\mathrm{f}}_\star = (E\mathbb{Z}/2_+ \wedge H)_\star = H^{\mathrm{t}}_\star / H^{\mathrm{c}}_\star = \mathbb{Z}/2[\sigma, \sigma^{-1}, a^{-1}],$$

$$H_\star = \mathbb{Z}/2[a] \oplus H^{\mathrm{f}}_\star / \sigma^{-1}.\mathbb{Z}/2[a^{-1}] = \mathbb{Z}/2[a] \oplus \mathbb{Z}/2[\sigma, \sigma^{-1}, a^{-1}]/\sigma^{-1}.\mathbb{Z}/2[a^1],$$

$$H^{\mathrm{g}}_\star = (H \wedge \widetilde{E\mathbb{Z}/2})_\star = \mathbb{Z}/2[a, a^{-1}].$$

The reader is encouraged to draw pictures to visualize these coefficient groups. Here also the superscript g stands for *geometric*. In the above calculations, $\dim(a) = -\alpha$, $\dim(\sigma) = \alpha - 1$ (which agrees with the way we used these symbols before). Additionally, the map a is defined by the standard cofibre sequence

$$\mathbb{Z}/2_+ \to S^0 \xrightarrow{a} S^\alpha.$$

The above calculations are done using the Tate diagram, the Adams isomorphism and the splitting of $H\mathbb{Z}/2$ (see [31]). Concretely, the splitting means that the canonical map

$$H\mathbb{Z}/2^{\mathbb{Z}/2} \to H\mathbb{Z}/2_{\{e\}}$$

has a left inverse in the derived (= stable) category of spectra. The Adams isomorphism in this case simply asserts that

$$F(\Sigma^{k\alpha} E\mathbb{Z}/2_+, H)^{\mathbb{Z}/2} \simeq F(B\mathbb{Z}/2^{k\gamma_1}, H_{\{e\}}),$$

$$(\Sigma^{k\alpha} E\mathbb{Z}/2_+ \wedge H)^{\mathbb{Z}/2} \simeq B\mathbb{Z}/2^{k\gamma_1} \wedge H_{\{e\}}$$

where γ_1 is the canonical one-dimensional real line bundle on $B\mathbb{Z}/2$, and the superscript denotes Thom space.

All of the above coefficient systems are rings (with structure induced by commutative ring structure on the corresponding spectra), except $H^{\mathrm{f}}_\star$. The ring structure of $H_\star$ is not immediately obvious, but can be calculated from dimensional considerations: $\mathbb{Z}/2[a]$ is a subring of $H_\star$ over which $H_\star$ is a module in the standard way (multiplication by a). The product of two elements of $H^{\mathrm{f}}_\star$ is 0, since these elements are both a-torsion and a-divisible (see also Proposition 6.2 below).

Now even ignoring the 'geometric and Tate' cases, there are at least three natural versions of the Steenrod algebra:

$$A_\star = (H \wedge H)_\star,$$

$$A^{\mathrm{c}}_\star = (F(E\mathbb{Z}/2_+, H) \wedge F(E\mathbb{Z}/2_+, H))_\star,$$

$$A^{\mathrm{cc}}_\star = (F(E\mathbb{Z}/2_+, H \wedge H))_\star.$$

Note that there are still other possible versions of the equivariant Eilenberg–MacLane spectrum and the $\mathbb{Z}/2$-equivariant Steenrod algebra. Most interestingly, one can consider the 'honest' $\mathbb{Z}/2$-equivariant Eilenberg–MacLane spectrum $H\mathbb{Z}/2_m$ associated with the constant Mackey functor $\mathbb{Z}/2$. We shall also consider the $H\mathbb{Z}/2_m$-based Steenrod algebra,

$$A^m_\star = (H\mathbb{Z}/2_m \wedge H\mathbb{Z}/2_m)_\star.$$

This is the 'honest' $\mathbb{Z}/2$-equivariant Steenrod algebra. We will show that this algebra has some remarkable properties, which make it, in some sense, analogous to the Steenrod algebra A_* for $p > 2$ [41]. We will begin by calculating the $RO(G)$-graded coeficients of $H\mathbb{Z}/2_m$.

Proposition 6.2.

$$(H\mathbb{Z}/2_m)_{k+\ell\alpha} = \begin{cases} H^{\mathrm{c}}_{k+\ell\alpha} & \text{if } k \geqslant 0, \\ H^{\mathrm{f}}_{k+\ell\alpha} & \text{if } k < -1, \\ 0 & \text{if } k = -1. \end{cases} \tag{6.3}$$

An analogous statement holds with $\mathbb{Z}/2$ replaced by $\mathbb{Z}$ throughout. The ring structure on (6.3) *is given as follows*:

$$H^{\mathrm{c}}_{k+\ell\alpha} \otimes H^{\mathrm{c}}_{k'+\ell'\alpha} \to H^{\mathrm{c}}_{k+k'+(\ell+\ell')\alpha}.$$

$(k, k' \geqslant 0)$ *coincides with the multiplication on* $H^{\mathrm{c}}_\star$,

$$H^{\mathrm{f}}_{k+\ell\alpha} \otimes H^{\mathrm{c}}_{k'+\ell'\alpha} \to H^{\mathrm{f}}_{k+k'+(\ell+\ell')\alpha}$$

$(k < 0 \leqslant k')$ *coincides with the* $H^{\mathrm{c}}_\star$*-module structure on* $H^{\mathrm{f}}_\star$, *and the multiplicative structure on*

$$H^{\mathrm{f}}_{k+\ell\alpha} \otimes H^{\mathrm{f}}_{k'+\ell'\alpha}$$

is 0.

Remark. In particular, it follows that most (de)suspensions of $H\mathbb{Z}/2_m$, $H\mathbb{Z}_m$ by non-trivial representations are not equivariant Eilenberg–MacLane spectra associated with Mackey functors.

Proof. We already know that $H\mathbb{Z}/2_m$ is split and that

$$H\mathbb{Z}/2_m \to F(E\mathbb{Z}/2_+, H\mathbb{Z}/2_m) \simeq F(E\mathbb{Z}/2_+, H\mathbb{Z}/2) \tag{6.4}$$

induces an isomorphism on non-equivariant coefficients. But (6.4), of course, does not induce an isomorphism on $\mathbb{Z}/2$-equivariant (untwisted) coefficients. In fact,

$$(H\mathbb{Z}/2_m)_*^{\mathbb{Z}/2} = \mathbb{Z}/2,$$
$$(F(E\mathbb{Z}/2_+, H\mathbb{Z}/2))_*^{\mathbb{Z}/2} = H^*B\mathbb{Z}/2_+ = \mathbb{Z}/2[x], \tag{6.5}$$

where $\dim(x) = -1$ ($x = \sigma a$). Next, consider the cofibre sequence

$$H\mathbb{Z}/2_m \to F(E\mathbb{Z}/2_+, H\mathbb{Z}/2) \stackrel{q}{\to} E.$$

Then $E_{\{e\}} \simeq *$, and hence $a\colon E \to \Sigma^\alpha E$ is an equivalence. Thus, by (6.5),

$$E_\star = x\mathbb{Z}/2[x, a, a^{-1}].$$

Now the fact that we know q on homotopy groups in twist 0 together with the $\mathbb{Z}/2[a]$-module structure and dimensional considerations gives our statement. The case of $H\mathbb{Z}$ is handled analogously.

To compute the ring structure, the assertions about $H_\star^c \otimes H_\star^c$ and $H_\star^f \otimes H_\star^c$ follow from comparisons with the respective coefficient modules. To get the multiplication on $H_\star^f \otimes H_\star^f$, note that an element $x \in H^f_{k+\ell\alpha}$, $k,\ell \geqslant 0$ is both a-divisible and a-torsion. Thus, any product of such elements satisfies

$$xy = \left(\frac{x}{a^N}\right)(ya^N),$$

which is 0 for $N \gg 0$. □

Proposition 6.6. *Additively*, (*as* $H_\star$*-modules*), *we have*

$$A_\star \cong H_\star[\xi_i \mid i \geqslant 1] \oplus J \otimes H_\star^f[\xi_i \mid i \geqslant 1],$$

where J is the augmentation ideal of the exterior algebra on generators ζ_i, $i = 0,1,2,\ldots$, where ζ_i has dimension $2^{i+1} - 1$, and ξ_i are generators of dimension $(2^i - 1)(1 + \alpha)$.

Proof. We will use the standard approach of smashing with the cofibration

$$E\mathbb{Z}/2_+ \to S^0 \to \widetilde{E\mathbb{Z}/2}$$

and taking fixed points. We obtain a long exact sequence

$$\cdots \stackrel{\delta}{\to} A_* \otimes H_\star^f \to A_\star \to A_*[a, a^{-1}] \stackrel{\delta}{\to} \cdots .$$

So we are done if we can identify the connecting map δ. To this end, recall the map

$$H_\star BP\mathbb{R} \to H_\star H = A_\star.$$

Since H is Real-oriented, we have

$$H_\star BP\mathbb{R} = H_\star \otimes \mathbb{Z}/2[\xi_i],\ \dim(\xi_i) = (2^i - 1)(1 + \alpha).$$

In particular, $A_\star$ contains the subalgebra

$$\mathbb{Z}/2[\xi_i \,|\, i \geqslant 1], \tag{6.7}$$

which, moreover, survives injectively if we forget the $\mathbb{Z}/2$-equivariant structure. Consequently, by the long exact sequence associated with smashing with the cofibration

$$\mathbb{Z}/2_+ \to S^0 \to S^\alpha,$$

the elements (6.7) cannot be multiples of a. Note that $\mathbb{Z}/2[a, a^{-1}][\xi_i \,|\, i \geqslant 1]$ maps injectively into $A_*[a, a^{-1}]$, since $\mathbb{Z}/2[\xi_i \,|\, i \geqslant 1]$ maps injectively into A_*. Thus, we may write

$$\mathbb{Z}/2[a][\xi_i] \subseteq \mathrm{Ker}(\delta). \tag{6.8}$$

But now by a count of dimensions of graded $\mathbb{Z}/2[a, a^{-1}]$-modules,

$$a^{-1}\mathbb{Z}/2[a][\xi_i \,|\, i \geqslant 1] = A_*[a, a^{-1}].$$

However, given the fact that the elements (6.7) are not divisible by a, this is only possible if equality arises in (6.8). □

Computing $A_\star^m$ is more difficult and will be postponed until later. We will first compute $A_\star^{cc}$. When computing $A_\star^{cc}$, it will turn out appropriate to work in the category $\mathscr{M}$ of bigraded $\mathbb{Z}/2[a]$-modules *complete* with respect to the topology associated with the principal ideal (a), and continuous homomorphisms. Here, the degrees will be written in the form $k + \ell\alpha$, the degree of a is $-\alpha$. The condition that M is complete means that

$$M = \lim_{\leftarrow} M/(a^k),$$

where the limit is taken in the category of graded $\mathbb{Z}/2[a]$-modules. Note that $\mathscr{M}$ is not an abelian category. The point is that a continuous inclusion in $\mathscr{M}$ may not be closed, and if it is not, it cannot be a kernel.

For example, let M be the (a)-completion of the free graded $\mathbb{Z}/2[a]$-module with basis $\{x_1, x_2, \ldots\}$ where $\dim x_k = k\alpha$, and let N be the free $\mathbb{Z}/2[a]$-module with basis $\{y_1, y_2, \ldots\}$ where $\dim y_k = 0$. Then N is complete (in the graded sense), and we have a continuous homomorphism

$$N \to M \tag{6.9}$$

given by

$$y_k \mapsto x_k a^k.$$

But the map (6.9) is not closed.

On the other hand, $\mathscr{M}$ is a reflexive subcategory of the category $\mathscr{M}_0$ of bigraded $\mathbb{Z}/2[a]$-modules, which means that the inclusion

$$\mathscr{M} \to \mathscr{M}_0$$

has a right adjoint. This is, of course, the completion functor

$$M \mapsto \lim_{\leftarrow} M/(a^k).$$

Therefore, $\mathscr{M}$ has all limits and colimits where limits are the same as in $\mathscr{M}_0$, and colimits are obtained by applying colimits in $\mathscr{M}_0$ followed by completion. Similarly, there is a tensor product in $\mathscr{M}$ which is the completion of the product $\otimes$ in $\mathscr{M}_0$.

We shall abuse notation slightly by denoting the colimit-type constructions in $\mathscr{M}$ (including the tensor product) by the same symbols as in $\mathscr{M}_0$, omitting the reflection (= completion) from the notation. (There is certainly precedent for that in reflexive subcategories: for example, the wedge sum of two spectra X and Y is denoted by $X \vee Y$, not $L(X \vee Y)$, where "$\vee$" denotes the wedge sum of prespectra.)

We can define rings and modules in $\mathscr{M}$, and the tensor product over a ring in $\mathscr{M}$. We can talk about Hopf algebroids (A, Γ) in $\mathscr{M}$, and their cohomology. This, by definition, is the (co)homology of the standard cobar complex, where the tensor products are in the category of A-modules in the category $\mathscr{M}$. Note that various results of homological algebra which use explicit cosimplicial maps and homotopies will remain in effect. In the sequel, we will assume we are in $\mathscr{M}$, unless the opposite is specified explicitly.

Now while neither of the spectra H, $F(E\mathbb{Z}/2_+, H)$ is flat, it turns out that $A^{cc}_\star$ is flat over $H^c_\star$ in the category $\mathscr{M}$. This endows $(H^c_\star, A^{cc}_\star)$ with a structure of a Hopf algebroid in the category of (a)-complete $\mathbb{Z}/2[a]$-modules.

Theorem 6.10. *The $\mathscr{M}$-Hopf algebroid*

$$(H^c_\star, A^{cc}_\star)$$

(*which we call the complete $\mathbb{Z}/2$-Borel cohomology Steenrod algebra*) *is given by the following formulas*:

$$H^c_\star = \mathbb{Z}/2[a][\sigma, \sigma^{-1}], \ \dim a = -\alpha, \ \dim \sigma = \alpha - 1,$$

$$A^{cc}_\star = H^c_\star[\zeta_i \,|\, i \geqslant 1]^\wedge_a, \ \dim \zeta_i = 2^i - 1,$$

$$\varepsilon(\zeta_i) = 0, \ i \geqslant 1,$$

$$\psi(\zeta_i) = \sum_{0 \leqslant j \leqslant i} \zeta_{i-j}^{2^j} \otimes \zeta_j \quad (\zeta_0 = 1),$$

$$\eta_R(\sigma) = \sum_{i \geqslant 0} \sigma^{2^i} \zeta_i a^{2^i - 1},$$

$$\eta_R(a) = a.$$

Proof. The formulas follow from the Borel cohomology spectral sequence, from the embedding to the corresponding Tate cohomologies

$$(\widetilde{E\mathbb{Z}/2}(E\mathbb{Z}/2_+, H\mathbb{Z}/2))_\star = \mathbb{Z}/2[a, a^{-1}][\sigma, \sigma^{-1}],$$

$$(\widetilde{E\mathbb{Z}/2}(E\mathbb{Z}/2_+, H\mathbb{Z}/2 \wedge H\mathbb{Z}/2))_\star = A_*[a, a^{-1}][\sigma, \sigma^{-1}]^\wedge,$$

and from identifying the corresponding geometric theories

$$(\widetilde{E\mathbb{Z}/2} \wedge H\mathbb{Z}/2)_\star = \mathbb{Z}/2[a, a^{-1}],$$

$$(\widetilde{E\mathbb{Z}/2} \wedge H\mathbb{Z}/2 \wedge H\mathbb{Z}/2)_\star = A_*[a, a^{-1}].$$

To identify η_R, note that this is equivalent to studying the Steenrod coaction on

$$H^c_\star = H^c_\star S^0,$$

where H^c is the Borel cohomology spectrum. Considering, for the moment, honest fixed points as opposed to Borel cohomology, the Adams isomorphism together with the splitting of S^0 shows that the equivariant Steenrod coaction on $H_\star S^0$ is compatible with the non-equivariant Steenrod coaction on $H_\star B\mathbb{Z}/2$. Thus, the element

$$r = \eta_R(\sigma)$$

satisfies

$$\psi(r) = \sum r^{2^i} \otimes \zeta_i.$$

This implies our formula for $\eta_R(\sigma)$. □

Proof of Lemma 3.39. Consider again

$$A^{\star c} = F(E\mathbb{Z}/2_+ \wedge H\mathbb{Z}/2, H\mathbb{Z}/2)_\star$$

and

$$A^{cc}_\star = F(E\mathbb{Z}/2_+, H\mathbb{Z}/2 \wedge H\mathbb{Z}/2)_\star.$$

It can be shown by the above methods that the obvious maps of spectra induce a perfect pairing

$$A^{\star c} \otimes A^{cc}_\star \to F(E\mathbb{Z}/2_+, H\mathbb{Z}/2)_\star.$$

Given that, the statement of the lemma follows from Theorem 6.10. More specifically,

$$e_i = (\sigma a)^i.$$

Now in Theorem 6.10, we have the formula

$$\eta_R(\sigma) = \sum_{i \geqslant 0} \sigma^{2^i} \zeta_i a^{2^i - 1},$$

which implies

$$\eta_R(\sigma a) = \sum_{i \geq 0} (\sigma a)^{2^i} \zeta_i,$$

which implies

$$\eta_R(e_i) = \left(\sum_{j \geq 0} e_{2^j} \zeta_j \right)^i.$$

Thus,

$$\eta_R(e_{i+2^{n+1}}) = \eta_R(e_i) \left(\sum_{j \geq 0} e_{2^j} \zeta_j \right)^{2^{n+1}} = \eta_R(e_i) \sum_{j \geq 0} e_{2^{j2^{n+1}}} (\zeta_j)^{2^{n+1}}.$$

We conclude that the statement of the lemma holds for every $\mathbb{Z}/2$-equivariant cohomology operation (in the above sense) in (non-equivariant) dimension $< 2^{n+1}$. □

The following result is not directly applied in the present paper:

Theorem 6.11 (Restatement of Greenlees [16–18]). *There is a convergent spectral sequence*

$$\mathrm{Ext}_{A_\star^{cc}}(H_\star^c, H_\star^c) \Rightarrow \pi_\star^{\mathbb{Z}/2}(S^0)_2^\wedge. \tag{6.12}$$

Comment. The map of Hopf algebroids

$$(\mathbb{Z}/2, A_*) \to (H_\star^c, A_\star^{cc})$$

gives a ring change isomorphism

$$\mathrm{Ext}_{A_\star^{cc}}(H_\star^c, H_\star^c) = \widehat{\mathrm{Ext}}_{A_*}(\mathbb{Z}/2, H_\star^c). \tag{6.13}$$

Here by $\widehat{\mathrm{Ext}}_{A_*}(\mathbb{Z}/2, M)$ we mean the cohomology of

$$\lim_{\leftarrow} C(\mathbb{Z}/2, A_*, M/(a)^n).$$

(This is the dual formulation of a theorem of Greenlees [17, Lemma 5.2]). Now in twist 0, the right hand side of (6.13) is

$$\widehat{\mathrm{Ext}}_{A_*}(\mathbb{Z}/2, H^* B\mathbb{Z}/2) \tag{6.14}$$

(we mean unreduced cohomology). In (6.14), we mean the cohomology of

$$\lim_{\leftarrow} C(\mathbb{Z}/2, A_*, H^* \mathbb{R}P^n).$$

Note that if we replace $H^*B\mathbb{Z}/2$ by $H_*B\mathbb{Z}/2$ on the right-hand side of (6.14), we obtain the E_2-term of the non-equivariant Adams spectral sequence for $\Sigma^\infty B\mathbb{Z}/2_+$. However, (6.14) is only slightly different. We refer to the results of Lin et al. [33].

In their notation, $H^*B\mathbb{Z}/2 = \mathbb{Z}/2[x]$, $\hat{H}^*(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[x, x^{-1}] = :P$. Now we have a short exact sequence

$$0 \to H^*B\mathbb{Z}/2 \stackrel{u}{\to} P \stackrel{v}{\to} \Sigma H_*B\mathbb{Z}/2 \to 0 \tag{6.15}$$

(here we index dimensions homologically, to stay consistent with the rest of this paper). From (6.15), we get a long exact sequence

$$\begin{aligned} &\to \mathrm{Ext}^i_{A_*}(\mathbb{Z}/2, H^*B\mathbb{Z}/2) \stackrel{u_*}{\to} \mathrm{Ext}^i_{A_*}(\mathbb{Z}/2, P) \stackrel{v_*}{\to} \\ &\to \mathrm{Ext}^i_{A_*}(\mathbb{Z}/2, \Sigma H_*B\mathbb{Z}/2) \stackrel{\delta}{\to} \mathrm{Ext}^{i+1}_{A_*}(\mathbb{Z}/2, H^*B\mathbb{Z}/2) \to \cdots. \end{aligned} \tag{6.16}$$

In (6.16), similar comments as in (6.14),(6.15) apply. Additionally, by $\mathrm{Ext}^i_{A_*}(\mathbb{Z}/2, P)$, we mean the cohomology of

$$\lim_{\stackrel{\to}{m}} \lim_{\stackrel{\leftarrow}{n}} C(\mathbb{Z}/2, A_*, H^*\mathbb{R}P^n_{-m}),$$

where $\mathbb{R}P^n_{-m} = \Sigma^{-m\alpha}S((m+n+1)\alpha)_+ \wedge_{\mathbb{Z}/2} S^0$ is the stunted projective space. This may look formidable, but (6.16) is actually isomorphic to the ordinary long exact sequence of Ext groups of A^* where the second argument of the Ext groups is $\mathbb{Z}/2$, and the first argument is the dual of (6.14).

Now identifying $\mathbb{Z}/2 = H^*(S^0)$, we obtain a diagram

$$\begin{array}{ccc} H^*B\mathbb{Z}/2 & \stackrel{u}{\longrightarrow} & P \\ & \nwarrow & \uparrow \phi \\ & & \mathbb{Z}/2. \end{array}$$

One of the main results of Lin et al. [33] asserts that

$$\phi_*: \mathrm{Ext}_{A_*}(\mathbb{Z}/2, \mathbb{Z}/2) \stackrel{\cong}{\to} \mathrm{Ext}_{A_*}(\mathbb{Z}/2, P).$$

Consequently, in (6.16), u_* is a split injection, and (6.16) gives an isomorphism

$$\mathrm{Ext}^i_{A_*}(\mathbb{Z}/2, H^*B\mathbb{Z}/2) \cong \mathrm{Ext}^i_{A_*}(\mathbb{Z}/2, \mathbb{Z}/2) \oplus \mathrm{Ext}^{i-1}_{A_*}(\mathbb{Z}/2, H_*B\mathbb{Z}/2).$$

Calculations further developing this theory can be found in [21].

On the other hand, the E_2-term (6.12) is related to the non-equivariant Adams E_2-term in various interesting ways. For example, we have maps of Hopf algebroids

$$(\mathbb{Z}/2, A_*) \to (H^{c}_\star, A^{cc}_\star) \stackrel{\psi}{\to} (\mathbb{Z}/2[\sigma, \sigma^{-1}], A_*[\sigma, \sigma^{-1}]), \tag{6.17}$$

where on the right-hand side, σ is primitive. Here ψ is reduction modulo (a). Now by (6.17), passing to Ext's, we see that the non-equivariant Adams E_2-term is a direct summand of the equivariant Adams E_2-term.

In fact, $\mathrm{Ext}(\psi)$ is the edge homomorphism of the spectral sequence converging to $\mathrm{Ext}_{A_\star^{cc}}(H_\star^c, H_\star^c)$, associated with the decreasing filtration on $(H_\star^c, A_\star^{cc})$ by powers of (a). This spectral sequence has the form

$$\mathrm{Ext}_{A_*}(\mathbb{Z}/2, \mathbb{Z}/2)[\sigma, \sigma^{-1}, a] \Rightarrow \mathrm{Ext}_{A_\star^{cc}}(H_\star^c, H_\star^c).$$

Thus, we see that every non-trivial element of the $\mathbb{Z}/2$-equivariant Adams E_2-term gives rise to a non-trivial element in the non-equivariant Adams E_2-term.

For any $\mathbb{Z}/2$-equivariant spectrum X, put $H_\star^{cc}X = F(E\mathbb{Z}/2_+, H\mathbb{Z}/2 \wedge X)_\star$, $H_\star^m X = (H\mathbb{Z}/2_m \wedge X)_\star$, $H_m^\star X = F(X, H\mathbb{Z}/2_m)_{-\star}$.

Theorem 6.18. *We have*

$$H_\star^{cc}BP\mathbb{R} = H_\star^c[\xi_i | i \geqslant 1], \quad H_\star^m BP\mathbb{R} = H_\star[\xi_i | i \geqslant 1], \quad \dim \xi_i = (2^i - 1)(1 + \alpha).$$

The map

$$H_\star^{cc}BP\mathbb{R} \to A_\star^{cc}$$

induced by the characteristic class

$$BP\mathbb{R} \to H\mathbb{Z}/2$$

sends ξ_i to elements given by the recursion

$$\xi_0 = 1,$$

$$\xi_i = \frac{1}{a^{2^i}}\left(\frac{\zeta_{i-1}^2}{\eta_R(\sigma)} + \zeta_i a + \frac{\xi_{i-1}}{\sigma^{2^{i-1}}}\right),\ i \geqslant 1. \tag{6.19}$$

Proof. All of the statements follow from the existence of Real orientations, except for the identification of ξ_i. To calculate these elements, recall the way one calculates the generators of $P_* = H_* BP_{\{e\}}$: one looks at the complex orientation

$$b \in H^2 \mathbb{C}P^\infty.$$

Then we have the formula

$$\psi(b) = \sum_{i \geqslant 0} b^{2^i} \otimes \xi_i. \tag{6.20}$$

The same formula holds in the Real situation, if we let b denote the Real orientation

$$b \in (H\mathbb{Z}/2_m)^{1+\alpha} B\mathbb{S}^1.$$

The problem is to identify the element b. To this end, consider the map

$$B\mathbb{Z}/2_+ \to B\mathbb{S}^1_+.$$

In Borel cohomology, this induces the map

$$F(E\mathbb{Z}/2_+ \wedge B\mathbb{S}^1_+, H\mathbb{Z}/2) \to F(E\mathbb{Z}/2_+ \wedge B\mathbb{Z}/2_+, H\mathbb{Z}/2)$$

or, on coefficients,

$$H^c_\star[b] \to H^c_\star[r],$$

where $r \in H^1_c(B\mathbb{Z}/2_+)$. Now

$$(H^c_\star[r])_{-1-\alpha} = \mathbb{Z}/2\{\sigma a^2, ra, r^2\sigma^{-1}\}.$$

We claim that

$$b \mapsto ra + r^2\sigma^{-1}. \tag{6.21}$$

First of all, the σa^2 summand comes from the cohomology of S^0, which is a wedge summand of $B\mathbb{Z}/2_+$ as well as $B\mathbb{S}^1_+$. Thus, this summand can be eliminated by different choice of b, r. Now to detect the $r^2\sigma^{-1}$-summand, map to

$$F(\mathbb{Z}/2_+ \wedge B\mathbb{Z}/2_+, H\mathbb{Z}/2).$$

Thus, we have shown that

$$b \mapsto \varepsilon ra + r^2\sigma^{-1}, \quad \varepsilon \in \{0,1\}.$$

We have already seen that

$$\psi(r) = \sum r^{2^i} \otimes \zeta_i,$$

and thus (6.20), (6.21) give

$$\sum(\varepsilon ra + r^2\sigma^{-1})^{2^i} \otimes \xi_i = \varepsilon a \sum(r^{2^i} \otimes \zeta_i) + \eta_R(\sigma)^{-1} \sum r^{2^{i+1}} \otimes \zeta_i^2.$$

Comparing coefficients at equal powers of r, we see that $\varepsilon = 0$ leads to contradiction, while $\varepsilon = 1$ gives the recursion (6.19). $\square$

Now put $P_\star = \mathbb{Z}/2[\xi_i | i \geqslant 1]$. Then $P_\star$ is a $\mathbb{Z}/2$-Hopf algebra with

$$\psi(\xi_i) = \sum_{0 \leqslant j \leqslant i} \xi_{i-j}^{2^j} \otimes \xi_j.$$

(Note that this identity follows from (6.20) in the preceding proof.) Similarly, $(\mathbb{Z}/2[a], P_\star[a])$ is a $\mathbb{Z}/2[a]$-Hopf algebra. From now on, we shall write $\varrho = \eta_R(\sigma)$. Next, we introduce elements τ_i analogous to elements arising for $p > 2$.

Theorem 6.22. *There exist elements* $\tau_i \in A^m_\star$ *of dimension* $2^i + (2^i - 1)\alpha$ *satisfying the following relations*:

$$\psi(\tau_i) = \tau_i \otimes 1 + \sum_j \xi_{i-j}^{2^j} \otimes \tau_j. \tag{6.23}$$

Further, $\tau_i \in A^{cc}_\star$ *satisfy the following relations*:

$$\tau_i = \frac{1}{a^{2^i}}(\tau_{i-1}\sigma^{-2^{i-1}} + \zeta_i \varrho^{-1}) \quad \textit{for } i > 0,$$

$$\tau_0 = \frac{1}{a}(\varrho^{-1} + \sigma^{-1}). \tag{6.24}$$

Proof. By Lemma 2.3, $(H\mathbb{Z}_m)_{1+\alpha} = B\mathbb{S}^1$. Now $2: H\mathbb{Z}_m \to H\mathbb{Z}_m$ is, on $(H\mathbb{Z}_m)_{1+\alpha}$, represented by the map

$$B\mathbb{S}^1 \xrightarrow{z^2} BS^1 \tag{6.25}$$

representing the squaring of real line bundles. Denote by $B'\mathbb{Z}/2$ the homotopy fibre

$$B'\mathbb{Z}/2 \to B\mathbb{S}^1 \xrightarrow{z^2} B\mathbb{S}^1. \tag{6.26}$$

(*Caution*: Note that $B'\mathbb{Z}/2$ is not homotopically equivalent to $B\mathbb{Z}/2$.) Now consider again the Real orientation

$$b \in (H\mathbb{Z}_m)^{1+\alpha} B\mathbb{S}^1$$

and its image

$$b' \in (H\mathbb{Z}_m)^{1+\alpha} B'\mathbb{Z}/2.$$

Then b is represented by

$$Id: B\mathbb{S}^1 \to B\mathbb{S}^1.$$

The element b' is represented by the inclusion

$$B'\mathbb{Z}/2 \subset B\mathbb{S}^1.$$

Because the composition (6.26) is trivial, we have

$$2b' = 0,$$

and hence b' must be in the target of the Bockstein, i.e. the connecting map associated with the cofibration

$$H\mathbb{Z}_m \xrightarrow{2} H\mathbb{Z}_m \to H\mathbb{Z}/2_m.$$

Let

$$b' = \beta c, \quad c \in (H\mathbb{Z}/2_m)^{\alpha} B'\mathbb{Z}/2.$$

Lemma 6.27. *$H_m^{\star} B'\mathbb{Z}/2$ is a free $H_m^{\star}$-module with basis $c^{\varepsilon}(b')^i$, $\varepsilon \in \{0,1\}$, $i \geqslant 0$.*

Proof. The fibration (6.26) gives rise to a cofibration

$$B'\mathbb{Z}/2_+ \to B\mathbb{S}^1_+ \to (B\mathbb{S}^1)^{\gamma_1^2}, \tag{6.28}$$

where the right-hand term is the Thom space of the tensor square of the canonical line bundle. The long exact sequence associated with the cofibration (6.28) by applying $H_m^{\star}$ is the Gysin sequence. Now $H_m^{\star}(B\mathbb{S}^1)^{\gamma_1^2}$ is a free module over $H_m^{\star}(BS^1)$ because γ_1^2 is a real bundle. The generator of this free module maps to c by the connecting map. The statement follows. □

Now consider c as an element in Borel cohomology. Then, analogously as in the non-equivariant $p > 2$ case,

$$\psi(c) = c \otimes 1 + \sum_{i \geqslant 0} b'^{2^i} \otimes \tau_i,$$

and the identity (6.23) is deduced from co-associativity in the standard way.

To deduce (6.24), we must compare the element c with the element r in Borel cohomology. Analogously as in the preceding proof, we see that

$$c = a + r\varrho^{-1}.$$

Plugging into the comodule structure formulas for r and c, we obtain

$$\begin{aligned}\psi(c) &= \sum (ar + r^2\sigma^{-1})^{2^i} \otimes \tau_i + \sum (a + r\sigma^{-1}) \otimes 1 \\ &= a \otimes 1 + \sum (r^{2^i} \otimes \zeta_i)\varrho^{-1} \\ &= \psi(a) + \psi(r\sigma^{-1}).\end{aligned}$$

(Actually, it is possible to add a to c: the same conclusion is obtained: the important formula is $c^2 = b'\sigma^{-1} + ca$.) Comparing coefficients at r^{2^i}, we obtain for $i = 0$

$$a\tau_0 + \sigma^{-1} = \varrho^{-1},$$

and for $i > 0$

$$a^{2^i}\tau_i + \sigma^{-2^{i-1}}\tau_i = \zeta_i \varrho^{-1},$$

as claimed. □

Proposition 6.29. *There is an extension of $\mathcal{M}$-Hopf algebroids of the form*

$$(\mathbb{Z}/2[a], P_{\star}[a]) \to (H_{\star}^{c}, A_{\star}^{cc}) \to (H_{\star}^{c}, \Lambda), \tag{6.30}$$

where

$$\Lambda = H^{\mathrm{c}}_{\star}[\tau_i]/(\tau_i^2 = \tau_{i+1}a), \tag{6.31}$$

$$\psi(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i, \tag{6.32}$$

$$\eta_R(\sigma)^{-1} + \sigma^{-1} = \tau_0 a, \tag{6.33}$$

$$\varepsilon(\tau_i) = 0. \tag{6.34}$$

Moreover, (6.33) *holds in* $A^{\mathrm{cc}}_{\star}$.

Proof. First of all, (6.33) is (6.24) of the previous theorem, and (6.32) is (6.23) of the previous theorem. (6.34) is obvious.

To prove (6.31), we first note that by (6.19), we have in Λ

$$\zeta_{i-1}^2 = \varrho \zeta_i a \quad \text{for } i > 1, \qquad \zeta_1 = \frac{1}{a}(\sigma^{-1} + \varrho^{-1}) = \tau_0. \tag{6.35}$$

Note that (6.19) for $i = 1$ says

$$\xi_1 a^2 = \zeta_1 a + \varrho^{-1} + \sigma^{-1}.$$

Now we prove

$$\tau_i^2 = \tau_{i+1} a \tag{6.36}$$

by induction on i. Suppose the formula (6.36) is valid for $i-1$. Then, by (6.24) and (6.35), for $i > 2$ we have

$$\tau_i^2 = \frac{1}{a^{2^{i+1}}}(\tau_{i-1}^2 \sigma^{-2^i} + \zeta_i^2 \varrho^{-2}) = \frac{1}{a^{2^{i+1}}}(\tau_i a \sigma^{-2^i} + \zeta_{i+1}\varrho^{-1} a) = \tau_{i+1} a.$$

For $i = 1$, by (6.24) and (6.35),

$$\tau_1 a = \frac{1}{a}(\tau_0 \sigma^{-1} + \zeta_1 \varrho^{-1}) = \frac{1}{a}\zeta_1 (\sigma^{-1} + \varrho^{-1}) = \frac{1}{a^2}(\sigma^{-1} + \varrho^{-1})^2 = \tau_0^2. \qquad \square$$

Next, we have

Proposition 6.37. *In* $A^{\mathrm{cc}}_{\star}$,

$$\tau_i^2 = \tau_{i+1} a + \xi_{i+1} \varrho^{-1}.$$

Proof. Induction on i. First consider the case $i = 0$. By (6.19), we have

$$\xi_1 = \frac{1}{a^2}(\varrho^{-1} + \zeta_1 a + \sigma^{-1}),$$

so

$$\zeta_1 a = \varrho^{-1} + \sigma^{-1} + \xi_1 a^2. \tag{6.38}$$

Now by (6.24),

$$\tau_1 a = \frac{1}{a}(\tau_0\sigma^{-1} + \zeta_1\varrho^{-1})$$

$$= \frac{1}{a^2}(a\tau_0\sigma^{-1} + a\zeta_1\varrho^{-1})$$

$$= \frac{1}{a^2}(a\tau_0\sigma^{-1} + (\varrho^{-1} + \sigma^{-1} + \xi_1 a^2)\varrho^{-1})$$

$$= \frac{1}{a^2}(a\tau_0\sigma^{-1} + (a\tau_0 + \xi_1 a^2)\varrho^{-1})$$

$$= \frac{1}{a^2}(a^2\tau_0^2 + \xi_1 a^2\varrho^{-1}) = \tau_0^2 + \xi_1\varrho^{-1},$$

as claimed. Now assume the statement is true with i replaced by $i-1$. Then, by (6.19) (applied with i replaced by $i+1$),

$$\zeta_i^2\varrho^{-1} = \zeta_{i+1}a + \xi_i\sigma^{-2^i} + a^{2^{i+1}}\xi_{i+1}. \tag{6.39}$$

Now compute, using (6.24) and (6.39),

$$\tau_i^2 = \frac{1}{a^{2^{i+1}}}(\tau_{i-1}^2\sigma^{-2^i} + \zeta_i^2\varrho^{-2})$$

$$= \frac{1}{a^{2^{i+1}}}(\tau_{i-1}^2\sigma^{-2^i} + \zeta_{i+1}a\varrho^{-1} + \xi_i\sigma^{-2^i}\varrho^{-1} + a^{2^{i+1}}\xi_{i+1}\varrho^{-1})$$

$$= (\text{by the induction hypothesis})\ \frac{1}{a^{2^{i+1}}}(\tau_i a\sigma^{-2^i} + \xi_i\varrho^{-1}\sigma^{-2^i} + \zeta_{i+1}a\varrho^{-1}$$

$$+ \xi_i\sigma^{-2^i}\varrho^{-1} + a^{2^{i+1}}\xi_{i+1}\varrho^{-1})$$

$$= (\text{by (6.24)})$$

$$= \tau_{i+1}a + \xi_{i+1}\varrho^{-1},$$

as claimed. □

Corollary 6.40. *The $\mathbb{Z}/2$-equivariant Steenrod algebra $\mathscr{M}$-Hopf algebroid can be completely described as follows*:

$$A^{cc}_{\star} = \mathbb{Z}/2[\sigma,\sigma^{-1},\varrho,\varrho^{-1},a,\tau_i,\xi_i]/\left({\tau_0 a=\varrho^{-1}+\sigma^{-1} \atop \tau_i^2=\tau_{i+1}a+\xi_{i+1}\varrho^{-1}}\right),$$

$$\eta_R(\sigma) = \varrho,$$

$$\psi(\xi_i) = \sum \xi_{i-j}^{2^j}\otimes\xi_j,$$

$$\psi(\tau_i) = \sum \xi_{i-j}^{2^j}\otimes\tau_j + \tau_i\otimes 1. \qquad \square$$

We are now ready to describe the Mackey Steenrod algebra.

Theorem 6.41. (a) *We have*

$$A^m_{\star} = H^m_{\star}[\xi_i,\tau_i,\varrho^{-1}]/\tau_0 a = \varrho^{-1} + \sigma^{-1}\tau_i^2 = \tau_{i+1}a + \xi_{i+1}\varrho^{-1}. \tag{6.42}$$

The comultiplication is given by the same formula as in $A^{cc}_{\star}$, and

$$\eta_R(\sigma^{-1}) = \varrho^{-1}.$$

Denote by ∂ the composition

$$\partial: H^t_{\star} \to \Sigma H^b_{\star} \to \Sigma H^m_{\star}$$

the connecting map, *so that*

$$\partial\sigma^k a^{\ell} = 0 \ \text{ iff } \ \ell \geqslant 0 \text{ or } k \leqslant 0, \tag{6.43}$$

then

$$\eta_R(\partial\sigma^k a^{\ell}) = \partial((\sigma^{-1}+\tau_0 a)^{-k}a^{\ell}) \tag{6.44}$$

for $k>0$, $\ell<0$. Here the sum on the right-hand side of (6.44) *is finite by* (6.43).

(b) *We have an extension of Hopf algebroids*

$$(\mathbb{Z}/2[a], P_{\star}[a]) \to (H^m_{\star}, A^m_{\star}) \to (H^m_{\star}, \Lambda^m),$$

where

$$\Lambda^m = H^m_{\star}[\tau_i,\varrho^{-1}]/(\tau_0 a = \sigma^{-1}+\varrho^{-1}, \tau_i^2 = \tau_{i+1}a)$$

with τ_i primitive.

Corollary 6.45. *H_m is a flat spectrum*, *i.e. $H_{m\star}H_m$ is a flat* (*in fact*, *free*) *$H_{m\star}$-module.*

Remark. This corresponds to a result of Voevodsky [53] asserting that the motivic cohomology k-spectrum is a flat k-spectrum. See also remarks after the proof of Lemma 2.3 and after the proof of Corollary 6.47 below.

Proof. (a) The elements ξ_i,τ_i have been constructed in (6.18) and (6.22) above. However, the relations between them were only verified in $A^{cc}_\star$. But the relations are determined by the relation between $\sigma^{-1}b'$, ca, $c^2 \in H^{1+\alpha}_m(B'\mathbb{Z}/2)$. In fact, the relation is

$$\sigma^{-1}b' + ca = c^2. \tag{6.46}$$

(For the indexing sign, recall that $H^{1+\alpha}_m(X) = [X,\Sigma^{1+\alpha}H\mathbb{Z}/2_m]$.) We conclude that

$$H^{2\alpha}_m(B'\mathbb{Z}/2) \cong H^{2\alpha}(B'\mathbb{Z}/2) \cong \mathbb{Z}/2\{\sigma^{-1}b', ca, c^2\}.$$

Thus, (6.46) is valid in $H^{1+\alpha}_m(B'\mathbb{Z}/2)$, and this implies the relations stated in (a). This gives a map ϕ from the right-hand side of (6.42) to the left-hand side. We must now show that this map is an isomorphism. To this end, denote the right-hand side of (6.42) by R. Put also

$$A^{gm}_\star = (\widetilde{E\mathbb{Z}/2} \wedge H\mathbb{Z}/2_m \wedge H\mathbb{Z}/2_m)_\star,$$

$$R^f = R \otimes_{H^m_\star} H^f_\star$$

and

$$R^g = \mathbb{Z}/2[\varrho^{-1}, \sigma^{-1}, a, a^{-1}, \xi_i].$$

Consider the following diagram:

$$\begin{array}{ccccccccc}
\longrightarrow & R^f & \xrightarrow{i} & R & \xrightarrow{j} & R^g & \xrightarrow{\delta} & \Sigma R^{f*} & \longrightarrow \\
 & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & \\
\longrightarrow & A^f_* & \longrightarrow & A^m_* & \longrightarrow & A^{gm}_* & \longrightarrow & \Sigma A^f_* & \longrightarrow
\end{array}$$

Here the map i is induced by the natural map $H^f_\star \to H^m_\star$, j is the localization by inverting a. The bottom sequence is the top line of the Tate diagram for $H\mathbb{Z}/2_m \wedge H\mathbb{Z}/2_m$. The vertical map u is the Künneth map using the fact that $H\mathbb{Z}/2_f = E\mathbb{Z}/2_+ \wedge H\mathbb{Z}/2_m$ is an $H\mathbb{Z}/2_m$-module. The map v is obtained from u by inverting a. But then v is an isomorphism by the formula

$$\Phi^{\mathbb{Z}/2}(E \wedge F) = \Phi^{\mathbb{Z}/2}E \wedge \Phi^{\mathbb{Z}/2}F.$$

Now u is an isomorphism because both the source and target are additively direct sums of copies of $H^f_\star$ over basis elements $\xi_R\tau_E$ where $R = (r_1, r_2, \dots)$ (resp. $E = (\varepsilon_1, \varepsilon_2, \dots)$) is a sequence of non-negative integers (resp. 0's and 1's) with only finitely many non-zero entries. Thus, because u and v are isomorphisms, the connecting map ∂ is determined. By definition, the diagram commutes. Thus, ϕ is iso by the 5-lemma.

To prove (b), the first map is an inclusion of Hopf algebroids similarly as in Proposition (6.29). The algebra Λ^m is calculated by the standard Hopf algebroid formulas. □

We now have the following amusing result: in $\mathbb{Z}/2$-equivariant spectra, the honest Eilenberg–MacLane spectrum is flat, and defines an honest Adams spectral sequence, which converges analogously as its non-equivariant analogue!

Corollary 6.47. *For a finite* $\mathbb{Z}/2$*-spectrum* X, *there is a convergent Adams-type spectral sequence*

$$\mathrm{Ext}_{A^m_\star}(H^m_\star, H^m_\star X) \Rightarrow (\pi^{\mathbb{Z}/2}_\star X)^\wedge_2.$$

Proof. Form the $H\mathbb{Z}/2_m$-based Adams resolution

$$\cdots \to X_{(2)} \to X_{(1)} \to X_{(0)} = X.$$

The identification of the E_2-term follows because $H\mathbb{Z}/2_m$ is flat. To prove convergence, we must show that

$$\operatorname*{holim}_{\leftarrow} X_{(i)} = *.$$

Because homotopy limit commutes with cofibrations, it suffices to consider two cases:

$$X = \mathbb{Z}/2_+. \tag{6.48}$$

In this case, because we are using Mackey (co)homology, the statement reduces to the convergence of the non-equivariant Adams spectral sequence.

$$X = S^0. \tag{6.49}$$

In that case, we use the fact that fixed points and forgetting of $\mathbb{Z}/2$-equivariant structure commute with homotopy inverse limits. Thus, it suffices to show that

$$\operatorname*{holim}_{\leftarrow} X^{\mathbb{Z}/2}_{(i)} = *$$

and

$$\operatorname*{holim}_{\leftarrow} (X_{(i)})_{\{e\}} = *.$$

The second fact follows again from the convergence of the non-equivariant Adams spectral sequence. The first statement uses the fact that $\mathbb{Z}$-graded homotopy groups of a bounded below $\mathbb{Z}/2$-equivariant spectrum are bounded below. Thus, the proof can be done step-by-step analogously as the proof of the convergence of the non-equivariant Adams spectral sequence. □

Remark. As we pointed out in the Introduction, there is a close connection of the above results with the work of Voevodsky [53] on the (dual of the) motivic Steenrod algebra $A^{\mathrm{Mot}}_\star = (H_{\mathrm{Mot}} \wedge H_{\mathrm{Mot}})_\star$ (see the Remark after Lemma 2.3) at the prime $\ell = 2$. (In algebraic geometry, it is customary to call this prime ℓ, as p is usually reserved for the characteristic; we shall assume that the characteristic is 0.)

Voevodsky's main theorem on $A^{\mathrm{Mot}}_\star$ is identical to formula (6.42), if we replace $H^m_\star$ by $H^{\mathrm{Mot}}_\star$. (Because our work was independent of Voevodsky's, our notations are different. Our '$\sigma^{-1}, a, \tau_i, \xi_i$' correspond to Voevodsky's '$\tau, \rho, \tau_i, \xi_i$'.) Also, $A^{\mathrm{Mot}}_\star$ is a Hopf algebroid and the formula

$\eta_R(\sigma^{-1}) = \sigma^{-1} + \tau_0 a$ remains in effect. However, in the motivic situation, an analogue of formula (6.44) does not arise, since the motivic cohomology algebra in that range is 0. We have [25]

$$H^{\mathrm{Mot}}_{p+q\alpha} = H^{-p-q,-q}(Spec(k);\mathbb{Z}/2) = H^{-p-q}(Spec(k);\mathbb{Z}/2(-q)) = CH^{-q}(Spec(k), p-q;\mathbb{Z}/2).$$

Thus, by Voevodsky's confirmation of the Bloch–Kato conjecture [50–52], the observation of Suslin–Voevodsky that the Bloch–Kato conjecture implies the Beilinson–Lichtenbaum conjecture [49], the fact that motivic cohomology groups are Chow groups [54], and periodicity of étale cohomology, we have

$$H^{\mathrm{Mot}}_{k+\ell\alpha} = \begin{cases} K^M_{-k-\ell}(k)/2 & \text{if } k \geqslant 0,\ k+\ell \leqslant 0, \\ 0 & \text{else,} \end{cases}$$

where $K^M_*(k)$ denotes Milnor K-theory, i.e. the tensor algebra on $k^\times$ modulo the relations $x \otimes (1-x)$ for $x \in k - \{0,1\}$.

We will show what translations have to be made in our arguments to obtain the motivic analogues of the elements τ_i, ξ_i, and the relations between them. For the proof that the resulting algebra is isomorphic to the motivic Steenrod algebra, we refer the reader to [53].

First of all, the motivic analogue of $\mathbb{S}^1$ is $\mathbb{G}_m = Spec(k[x,x^{-1}])$, the multiplicative group over $Spec(k)$. Some attention must be given to the classifying spaces: Corresponding to the $\mathbb{Z}/2$-equivariant $B\mathbb{S}^1$, there seems to be only one reasonable classifying space of $\mathbb{G}_m$: this is the bar construction $B\mathbb{G}_m$. This, in Voevodsky's category of k-spaces, is equivalent to

$$\lim_{n\to\infty} \mathbb{G}_m * \cdots * \mathbb{G}_m/\mathbb{G}_m = \lim_{n\to\infty} \mathbb{A}^n - \{0\}/\mathbb{G}_m \simeq \mathbb{P}^\infty,$$

where $*$ denotes the join. For $\mathbb{Z}/2 = \mu_2 \subset \mathbb{G}_m$, however, Voevodsky points out that we must distinguish between the bar construction $B\mathbb{Z}/2$ and

$$B_{\mathrm{et}}\mathbb{Z}/2 = \lim_{n\to\infty} \mathbb{A}^n - \{0\}/\mathbb{Z}/2.$$

Since $B_{\mathrm{et}}\mathbb{Z}/2$, $B_{\mathrm{et}}\mathbb{G}_m = B\mathbb{G}_m$ represent étale cohomology with coefficients in $\mathbb{Z}/2$, $\mathbb{G}_m$, we see that the canonical diagram

$$B_{\mathrm{et}}\mathbb{Z}/2 \to B\mathbb{G}_m \overset{B2}{\to} B\mathbb{G}_m$$

is a fibration sequence. Recalling that for $k = \mathbb{R}$ the functor t is the $\mathbb{Z}/2$-equivariant realization of k-spaces, we conclude that

$$t(B_{\mathrm{et}}\mathbb{Z}/2) = B'\mathbb{Z}/2$$

(compare (6.26)). Now our construction of the classes c, b' has a direct motivic analogue, and there follows a construction of the motivic ξ_i, τ_i. The coproduct formulas for these elements are deduced in a formal way.

Also, the motivic analogue of Borel cohomology is étale cohomology, and there is, correspondingly, a 'completion' map

$$H^{\star}_{\mathrm{Mot}}X \to H^{\star}_{\mathrm{et}}X.$$

Analogues of the classes ζ_i in the appropriate étale cohomology Steenrod algebra can also be constructed. This does not mean, however, that all our $\mathbb{Z}/2$-equivariant calculations could be automatically carried over to the motivic case. In the $\mathbb{Z}/2$-equivariant calculations, we made extensive use of the Tate diagram (6.1). While one can construct various analogues of this diagram in the motivic case, none of them seems to immediately give such a decisive calculational tool.

Back to the motivic Steenrod algebra, to establish the multiplicative relations between the ξ_i's and τ_i's, we must first define a. We claim that this is

$$[-1] \in K^M_1(k)/2.$$

Recalling the proofs of Theorems 6.18, 6.22 and Propositions 6.29, 6.37, the crucial formula we must prove in the motivic case is

$$c^2 = b'\sigma^{-1} + ca. \tag{6.50}$$

(This also gives the right unit formula; see the last lines of the proof of Theorem 6.22.)

To prove (6.50), similarly to the $\mathbb{Z}/2$-equivariant case, it suffices to work in étale cohomology (since it coincides with motivic cohomology in the relevant dimensions [52,49]). For various background facts on étale cohomology, we refer the reader to [40]. In particular, since étale cohomology is σ^{-1}-periodic, we can choose $r \in H^{\mathrm{et}}_{\star}$ $(:= H^{-\star}_{\mathrm{et}}(Spec(k)))$ such that

$$c = \sigma^{-1} r.$$

To prove (6.50), it then suffices to prove

$$r^2\sigma^{-1} = b' + ra. \tag{6.51}$$

Thus, comparing with étale cohomology of the algebraic closure (singular cohomology if $k \subseteq \mathbb{C}$), (6.51) can be detected by restricting via the composition

$$S^1 \to B\mathbb{Z}/2 \to B\mathbb{G}_m \tag{6.52}$$

where the left-hand side of (6.51) goes away (since the product of reduced cohomology classes is 0 in a suspension of a space).

Thus, it suffices to show that in $H^{\star}_{\mathrm{et}}(S^1)$, b' restricts to ra. Here in étale cohomology of S^1, r is simply the fundamental class.

But because we are working in étale cohomology, we can use the bar construction classifying space $B\mathbb{Z}/2$ (not $B_{\mathrm{et}}\mathbb{Z}/2$) in (6.52). By analyzing the bar construction, we see that the composition (6.52) factors as

$$S^1 \to \mathbb{P}^1 \to \mathbb{P}^{\infty},$$

where the second map is the canonical inclusion, and the first map is suspension of the inclusion which takes the non-base point to -1. This is, by definition, a.

7. The Real algebraic Novikov spectral sequence

We begin by clarifying the multiplicative structure of $BP\mathbb{R}_\star$, i.e. concluding the proof of Theorem 4.11 by proving $\Delta = 0$ (see the Remark after its statement). First of all, by considering the usual cobar resolution by cofibrations

$$D_0 = BP\mathbb{R}, \qquad E_n = D_n \wedge H\mathbb{Z}/2, \qquad D_n \to E_n \to D_{n+1},$$

we obtain a $\mathbb{Z}/2$-equivariant Adams spectral sequence

$$\mathrm{Ext}_{(H^{c}_\star,\, A^{cc}_\star)}(H^{c}_\star, H_\star BP\mathbb{R}) \Rightarrow \pi_\star BP\mathbb{R}. \tag{7.1}$$

Now Proposition 6.29 and Theorem 6.18 imply

$$A^{cc}_\star \square_\Lambda H^{cc}_\star = H_\star BP\mathbb{R}, \qquad \mathrm{Cotor}^{>0}_\Lambda(A^{cc}_\star, H^{c}_\star) = 0.$$

Thus, the left-hand side of (7.1) is equal to

$$\mathrm{Ext}_{(H^{c}_\star, \Lambda)}(H^{c}_\star, H^{c}_\star). \tag{7.2}$$

Note that (7.2), as the Ext of a Hopf algebroid, is a commutative algebra. Furthermore, by filtering Λ through powers of (a), we obtain a spectral sequence converging to (7.2), which has the same differentials as the Borel cohomology spectral sequence of $BP\mathbb{R}_\star$: this is equivalent to the fact that all the differentials of the Borel cohomology spectral sequence are given by primary cohomological operations. By this discussion, the spectral sequence (7.1) collapses, and we have

$$\mathrm{Ext}_{(H^{c}_\star, \Lambda)}(H^{c}_\star, H^{c}_\star) \cong E_0 BP\mathbb{R}_\star, \tag{7.3}$$

for some decreasing filtration on $BP\mathbb{R}_\star$. Consequently, we can denote the generators of (7.2) by $v_{n,\ell} \in \mathrm{Ext}^1$.

Theorem 7.4. *The error term Δ in the formula* (4.12) *is* 0.

The proof of this theorem is contained in Propositions 7.5 and 7.7 below.

Proposition 7.5. *The error term Δ in the relation*

$$v_{n,\ell 2^{m-n}} v_{m,k} = v_n v_{m(k+\ell)} + \Delta, \tag{7.6}$$

in (4.12) *is identical to the error term in the same relation in the associated graded object* (7.2), *if we give the generators the same names.* (*In other words, as a ring, $BP\mathbb{R}_\star$ is isomorphic to* (7.2).)

Comment. Note that (7.2) is not isomorphic to $BP\mathbb{R}_\star$, since we have the extension $2 = v_0$. On the other hand, the error terms Δ of (7.6) are multiples of a, which annihilates 2.

Proof. Let, in (7.2)

$$v_{n,\ell 2^{m-n}} v_{m,k} = v_n v_{m(k+\ell)} + \Delta'.$$

Thus, we have shown that

$$\Delta - \Delta'$$

is a sum of monomials

$$a^i v_{n_1,\ell_1} \dots v_{n_k,\ell_k}$$

with $n_i > m$ (as both Δ,Δ' are). On the other hand, by the definition of the Adams spectral sequence, $\Delta - \Delta'$ must be of Adams degree (= Ext degree) $\geqslant 3$. But for a non-zero monomial

$$x = a^i v_{n_1,\ell_1} v_{n_2,\ell_2} v_{n_3,\ell_3}, \quad \min n_i > m,$$

we have

$$t(x) > t(v_{n,\ell 2^{m-n}} v_{m,k}).$$

Hence,

$$\Delta - \Delta' = 0,$$

as claimed. □

Proposition 7.7. *The error term Δ in Proposition* 7.5 *is* 0.

Proof. By Theorem 7.5, it suffices to prove the result in

$$\mathrm{Ext}_\Lambda(H^c_\star, H^c_\star) = E_0 BP\mathbb{R}_\star.$$

It is advantageous to consider only the elements $\sigma^{-l2^{n+1}} v_n$, $l \geqslant 0$, i.e. to look at the Hopf algebroid $(H^-_\star, \Lambda^-)$, where

$$H^-_\star = \mathbb{Z}/2[\sigma^{-1}, a], \qquad \Lambda^- = H^-_\star[\tau_i \mid i \geqslant 0]/\tau_i^2 = \tau_{i+1} a.$$

The structure formulas are the same as in $(H^c_\star, \Lambda)$. We claim that if the statement of Theorem 7.7 is true for $\mathrm{Ext}_{\Lambda^-}(H^-_\star, H^-_\star)$, then it is true for $\mathrm{Ext}_\Lambda(H^c_\star, H^c_\star)$, and hence $BP\mathbb{R}_\star$. The reason is that for $m > 0$, if we reduce modulo products of $\sigma^{l2^{n+1}} v_n$ with $n \geqslant m$ only (i.e. products not containing $\sigma^{l2^{n+1}} v_n$ for $n < m$), then the formula in $BP\mathbb{R}_\star$ can be verified in

$$F(S((2^m - 1)\alpha), BP\mathbb{R}_\star) \tag{7.8}$$

(see Comment after Theorem 4.11). But there is some $N \geqslant 0$, such that σ^N exists as a self-map

$$S((2^m - 1)\alpha) \to S((2^m - 1)\alpha).$$

Therefore, a structure formula in (7.8) can be multiplied by powers of σ^{-N}, to ensure that it only involves elements $\sigma^{l2^{n+1}} v_n$ with $l \leqslant 0$.

Now $\sigma^{-l2^{n+1}} v_n$ is represented by the element

$$\frac{d\sigma^{-(2l+1)2^n}}{a^{2^{n+1}-1}} \in C^1(H_\star^-, \Lambda^-, H_\star^-).$$

Now

$$C(H_\star^-, \Lambda^-, H_\star^-) \tag{7.9}$$

is an algebra. In fact, it is identified with the tensor algebra on Λ^- over $H_\star^-$. Let I be the right differential graded ideal in (7.9) generated by σ^{-1}, i.e.

$$I = (\sigma^{-1}) \cdot C(H_\star^-, \Lambda^-, H_\star^-) + d(\sigma^{-1}) \cdot C(H_\star^-, \Lambda^-, H_\star^-).$$

Let

$$\phi : C(H_\star^-, \Lambda^-, H_\star^-) \to C(H_\star^-, \Lambda^-, H_\star^-)/I$$

be the reduction map. Note that the right-hand side is not an algebra.

Lemma 7.10. *ϕ induces an isomorphism on cohomology.*

Proof. Each element of I is a sum of elements of the form $\sigma^{-i} \otimes x$ and $d(\sigma^{-j} \otimes y)$, where $i, j \geqslant 1$, and $x, y \in T_{\mathbb{Z}/2[a]}(a, \tau_i)/\tau_i^2 = \tau_{i+1} a$, where T denotes the tensor algebra. This is to say that x, y are elements of $C(H_\star^-, \Lambda^-, H_\star^-)$, containing no powers of σ^{-1}. Note that

$$d(\sigma^{-j}) = \sigma^{-j} + \rho^{-j} = \sigma^{-j} + (\sigma^{-1} + \tau_0 a)^j.$$

So $d(\sigma^{-j} \otimes y) = d\sigma^{-j} \otimes y + \sigma^{-j} \otimes d^y$ has a term that is $(\tau_0 a)^j \otimes y$. Therefore, the elements $\sigma^{-i} \otimes x, d(\sigma^{-j} \otimes y)$ are linearly independent over $\mathbb{Z}/2[a]$. In particular,

$$d(\sigma^{-j} \otimes y) \neq 0$$

for $y \neq 0$.

Each element z of I can be written uniquely as

$$z = \sum_{r=1}^{k} (\sigma^{-i_r} \otimes x_r) + \sum_{s=1}^{t} d(\sigma^{-j_s} \otimes y_s)$$

where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_t$. Hence,

$$\begin{aligned} \mathrm{d}z &= \sum_{r=1}^{k} \mathrm{d}(\sigma^{-1_r} \otimes x_r) \\ &= \sum_{r=1}^{k} (\mathrm{d}\sigma^{-i_r} \otimes x_r + \sigma^{-i_r} \otimes \mathrm{d}x_r). \end{aligned}$$

$\mathrm{d}\sigma^{-i_r} \otimes x_r$ has the term $(\tau_0 a)^{i_r}$, so if $\mathrm{d}z = 0$, then we must have that $x_r = 0$ for all r. So for all z such that $\mathrm{d}z = 0$,

$$\begin{aligned} z &= \sum_{s=1}^{t} \mathrm{d}(\sigma^{-j_s} \otimes y_s) \\ &= \mathrm{d}\left(\sum_{s=1}^{t} \sigma^{-j_s} \otimes y_s \right), \end{aligned}$$

i.e. I is acyclic. Hence, ϕ induces an isomorphism in cohomology. □

To finish the proof of Theorem 7.7, we can consider $\sigma^{-l2^{n+1}} v_n \in H^1 M$. One can show by induction on n that $(\tau_0 a)^{2^n} = \tau_n a^{2^{n+1}-1}$, and so

$$\begin{aligned} \mathrm{d}\sigma^{-(2l+1)2^n} &= \sigma^{-(2l+1)2^n} + \rho^{-(2l+1)2^n} \\ &= \sigma^{-(2l+1)2^n} + (\sigma^{-1} + \tau_0 a)^{(2l+1)2^n} \\ &= \sigma^{-(2l+1)2^n} + (\sigma^{-2^n} + \tau_n a^{2^{n+1}-1})^{2l+1} \\ &= \sum_{k=1}^{2l+1} \binom{2l+1}{k} \sigma^{-(2l+1-k)2^n} (\tau_n a^{2^{n+1}-1})^k. \end{aligned}$$

So $\sigma^{-l2^{n+1}} v_n$ is represented by

$$\sum_{k=1}^{2l+1} \binom{2l+1}{k} \sigma^{-(2l+1-k)2^n} \tau_i^k (a^{2^{n+1}-1})^{k-1}.$$

The only term in this not containing a power of σ^{-1} is

$$\tau_i^{2l+1} (a^{2^{n+1}-1})^{2l} = \frac{(\tau_0 a)^{(2l+1)2^n}}{a^{2^{n+1}-1}}.$$

Thus, for $n \leqslant m$, $(\sigma^{-l2^{n+1}} v_n)(\sigma^{-u2^{m+1}} v_m)$ is represented by

$$\frac{(\tau_0 a)^{(2l+1)2^n}}{a^{2^{n+1}-1}} \otimes \sum_{k=1}^{2u+1} \binom{2u+1}{k} \sigma^{-(2u+1-k)2^m} \tau_m^k (a^{2^{m+1}-1})^{k-1}.$$

But $(\tau_0 a)^j \otimes x = 0$ in M, and $2^{m+1}-1 \geqslant 2^{n+1}-1$, so for $k \geqslant 2$,

$$\frac{(\tau_0 a)^{(2l+1)2^n}}{a^{2^{n+1}-1}} \otimes \binom{2u+1}{k} \sigma^{-(2u+1-k)2^m} \tau_m^k (a^{2^{m+1}-1})^{k-1} = 0$$

since we can shift $a^{2^{n+1}-1}$ from the right side to the left of the tensor. Hence, $(\sigma^{-l2^{n+1}} v_n)(\sigma^{-u2^{m+1}} v_m)$ is

$$\frac{(\tau_0 a)^{(2l+1)2^n}}{a^{2^{n+1}-1}} \otimes \sigma^{-u2^{m+1}} \tau_m = \frac{(\tau_0 a)^{(2l+1)2^n}}{a^{2^{n+1}-1}} (\sigma^{-1} + \tau_0 a)^{u2^{m+1}} \otimes \tau_m \tag{7.11}$$

$$= \frac{(\tau_0 a)^{(2l+1)2^n + u2^{m+1}}}{a^{2^{n+1}-1}} \otimes \tau_m. \tag{7.12}$$

On the other hand, similarly, $(\sigma^{-(l+u2^{m-n})2^{n+1}} v_n) v_m$ is represented by

$$\frac{\mathrm{d}\sigma^{-(2(l+u2^{mn})+1)2^n}}{a^{2^{n+1}-1}} \otimes \tau_m = \frac{(\tau_0 a)^{(2l+1)2^n + u2^{m+1}}}{a^{2^{n+1}-1}} \otimes \tau_m$$

same as in (7.12). Hence

$$(\sigma^{-l2^{n+1}} v_n)(\sigma^{-u2^{m+1}} v_m) = (\sigma^{-(l+u2^{m-n})2^{n+1}} v_n) v_m$$

in $H^*(M) = Ext_{A^-}(H^-_\star, H^-_\star)$, and $\Delta = 0$. This concludes our proof of Proposition 7.7. □

We shall now construct the algebraic Novikov spectral sequence. First, similar to the non-equivariant case, we consider the Adams filtration F on

$$(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$$

(i.e. the filtration arising from the spectral sequence (7.1)).

Proposition 7.13. *There is an algebraic Novikov spectral sequence*

$$E_1 = \mathrm{Ext}_{P_\star[a]}(\mathbb{Z}/2[a], E_0 BP\mathbb{R}_\star) \Rightarrow \mathrm{Ext}_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star).$$

Proof. Analogously as in the non-equivariant case,

$$\mathrm{Ext}_{E_0 BP\mathbb{R}_\star BP\mathbb{R}}(E_0 BP\mathbb{R}_\star, E_0 BP\mathbb{R}_\star) = \mathrm{Ext}_{P_\star[a]}(\mathbb{Z}/2[a], E_0 BP\mathbb{R}_\star). \qquad \square$$

Proposition 7.14. *We have the following Cartan–Eilenberg spectral sequences*:

$$E_2 = \mathrm{Ext}_{P_\star[a]}(\mathbb{Z}/2[a], E_0 BP\mathbb{R}_\star) \Rightarrow \mathrm{Ext}_{A^{cc}_\star}(H^c_\star, H^c_\star), \tag{7.15}$$

$$E_2 = \mathrm{Ext}_{P_\star[a]}(\mathbb{Z}/2[a], E_0 BP\mathbb{R}_\star) \Rightarrow \mathrm{Ext}_{A^m_\star}(H^m_\star, H^m_\star). \tag{7.16}$$

Proof. (7.15) is simply the Cartan–Eilenberg spectral sequence associated with the extensions (6.30) and Theorem 6.41(b). □

Comment. The spectral sequences (7.15), (7.16) are actually different. Even though the E_2-terms are isomorphic, there is a filtration shift. The isomorphism

$$\mathrm{Ext}_{\Lambda}(H^{c}_{\star}, H^{c}_{\star}) \cong E_0 BP\mathbb{R}_{\star}$$

has been described above. In particular, the element $\sigma^{\ell 2^{n+1}} v_R$, $n = \min(R)$, has filtration degree $|R| = \sum r_i$. But, in the isomorphism

$$\mathrm{Ext}_{\Lambda^m}(H^{m}_{\star}, H^{m}_{\star}) \cong E_0 BP\mathbb{R}_{\star},$$

$\sigma^{\ell 2^{n+1}} v_R$ has filtration degree $|R|$ if $\ell \leqslant 0$ and $|R| - 1$ if $\ell.0$. It can be checked that this does not affect the $P_{\star}[a]$-comodule structure, even though it does change the multiplicative structure.

Conjecture 7.17. *The spectral sequences* (7.15), (7.16) *collapse to their E_2-terms. Consequently, the spectral sequence* (4.23) *converging to* $(\pi_{\star} S^0_{\mathbb{Z}/2})^{\wedge}_2$ *has a smaller E_2-term than the $\mathbb{Z}/2$-equivariant Adams spectral sequence.*

While we do not know how to prove this conjecture, we would like to present some partial evidence in its favour. To this end, we briefly analyze the non-equivariant collapse theorem at $p > 2$. Consider the extension of $\mathbb{Z}/p$-Hopf algebras

$$P_* \xrightarrow{i} A_* \xrightarrow{\pi} E_*, \tag{7.18}$$

where $P_* = \mathbb{Z}/p[\xi_i]$, $A_* = \mathbb{Z}/p[\xi_i] \otimes \Lambda[\tau_i]$, $E_* = \Lambda[\tau_i]$. The extension (7.18) splits into a semidirect product. This means that the inclusion i has a left inverse

$$j : A_* \to P_*$$

which is a map of Hopf algebras (of course $j(\xi_i) = \xi_i$, $j(\tau_i) = 0$). We claim that whenever a splitting of this form occurs, we have an equivalence of differential graded P_*-comodules

$$C(\mathbb{Z}/p, E_*, \mathbb{Z}/p) \simeq C(P_*, A_*, \mathbb{Z}/p) \tag{7.19}$$

(on the left-hand side, P_* acts by the dual of conjugation, see Lemma 7.21 below). Now the other relevant fact is that E_* is Koszul. This means that there is a splitting s of the canonical inclusion from the module of primitives to the augmentation ideal of the coalgebra E_* such that the map

$$C(\mathbb{Z}/p, E_*, \mathbb{Z}/p) \to C(\mathbb{Z}/p, E_*, \mathbb{Z}/p)/I, \tag{7.20}$$

where $I \subset C(\mathbb{Z}/p, E_*, \mathbb{Z}/p)$ is the differential graded ideal generated by

$$Ker(s) \subset E_* \subset C(\mathbb{Z}/p, E_*, \mathbb{Z}/p)_1$$

is a quasiisomorphism. In our case, s is, of course, identity on the τ_i's and 0 on their non-linear monomials. This is a map of A_*-comodules (acting by dual conjugation), and hence (7.20) is a map of A_*-comodules. Since the differential on the right-hand side of (7.20) is trivial, we have

$$C(\mathbb{Z}/p, A_*, \mathbb{Z}/p) \simeq C(\mathbb{Z}/p, P_*, C(P_*, A_*, \mathbb{Z}/p))$$
$$\simeq C(\mathbb{Z}/p, P_*, C(\mathbb{Z}/p, E_*, \mathbb{Z}/p)) \simeq C(\mathbb{Z}/p, P_*, \mathrm{Ext}_{E_*}(\mathbb{Z}/p, \mathbb{Z}/p)).$$

Hence,

$$\mathrm{Ext}_{A_*}(\mathbb{Z}/p,\ \mathbb{Z}/p) = \mathrm{Ext}_{P_*}(\mathbb{Z}/p,\ \mathrm{Ext}_{E_*}(\mathbb{Z}/p,\ \mathbb{Z}/p)),$$

which is the Miller–Novikov collapse theorem (see [38,44]). (This may be the world's most complicated proof of that fact, but it is one which might generalize.)

To justify (7.19), we state its dual in the category of sets (it is easier to state — the proof dualizes verbatim).

Lemma 7.21. *Let $H \lhd G \leftrightarrows J$ be a semidirect product of groups. Then there is an equivalence of J-spaces*

$$B(*, H, *) \simeq B(J, G, *), \tag{7.22}$$

where on the left, J acts by conjugation, and on the right J acts by left multiplication on J.

Proof. It is more convenient to deal with the homogeneous bar construction. Let $\check{B}(G)$ denote the homogeneous unreduced bar construction, i.e. the simplicial set whose nth stage is $\underbrace{G \times \cdots \times G}_{n+1 \text{ times}}$, faces are projections and degeneracies are inclusions of units. Note that this does not require G to be a group. Then we have an equivalence

$$H \backslash \check{B}(G) \simeq H \backslash \check{B}(G/J), \tag{7.23}$$

where H acts diagonally. Writing (7.23) in non-homogeneous coordinates gives (7.22). □

Now the extension (6.30) apparently does not split, but it very nearly does! Put

$$H_\star^- = \mathbb{Z}/2[\sigma^{-1}, a],$$

$$A_\star^- = \mathbb{Z}/2[\sigma^{-1}, \varrho^{-1}, a, \tau_i, \xi_i] \Big/ \begin{pmatrix} \tau_0 a = \varrho^{-1} + \sigma^{-1} \\ \tau_i^2 = \tau_{i+1} a + \xi_{i+1} \varrho^{-1} \end{pmatrix},$$

$$\Lambda^- = \mathbb{Z}/2[\sigma^{-1}, \varrho^{-1}, \tau_i, a] \Big/ \begin{pmatrix} \tau_0 a = \varrho^{-1} + \sigma^{-1} \\ \tau_i^2 = \tau_{i+1} a \end{pmatrix}.$$

(compare with Corollary 6.40). Then by the above calculations,

$$(H^-_\star, A^-_\star) \subset (H_\star, A^{cc}_\star),\ (H^-_\star, \Lambda^-) \subset (H_\star, \Lambda)$$

are sub-Hopf algebroids. Moreover, by Corollary 6.40,

$$C_{H^-_\star}(H^-_\star, A^-_\star, H^c_\star) \cong C_{H^c_\star}(H^c_\star, A^{cc}_\star, H^c_\star),$$

and hence

$$\mathrm{Ext}_{A^-_\star}(H^-_\star, H^c_\star) \cong \mathrm{Ext}_{A^{cc}_\star}(H^c_\star, H^c_\star).$$

We also have an extension of Hopf algebroids

$$(\mathbb{Z}/2[a], P_\star[a]) \to (H^-_\star, A^-_\star) \to (H^-_\star, \Lambda^-). \tag{7.24}$$

But now (7.24) does split! The splitting

$$(H^-_\star, A^-_\star) \to (\mathbb{Z}/2[a], P_\star[a])$$

is given by

$$a \mapsto a, \xi_i \mapsto \xi_i, \quad \sigma^{-1}, \varrho^{-1}, \tau_i \mapsto 0.$$

In fact, the same holds for the Mackey Steenrod algebra (and Voevodsky's motivic Steenrod algebra [53]).

Proposition 7.25. *The extension* (*Theorem* 6.41(*b*)) *splits into a semidirect product. The splitting map*

$$\psi : (H^m_\star, A^m_\star) \to (\mathbb{Z}/2[a], P_\star[a])$$

is given by

$$\psi(\sigma^{-1}) = 0, \quad \psi(a) = a, \quad \psi(\xi_i) = \xi_i, \quad \psi(\tau_i) = 0, \quad \psi\partial = 0,$$

where ∂ *is the map introduced in Theorem* 6.41(*a*).

Thus, if we could generalize Lemma 7.21 to Hopf algebroids, and if we could prove an analogue (7.20) (Koszul property), we would have a proof of the conjecture. Unfortunately, this meets with a difficulty. But first note that for groupoids, an analogue of Lemma 7.21 is no problem:

Lemma 7.26. *Let*

$$(A, H) \to (A, G) \rightleftarrows (B, J)$$

be a semidirect product of groupoids. Then we have an equivalence of (B, J)*-simplicial sets*

$$B_B(B, B \times_A H \times_A B, B) \simeq B_A(J \times_B A, G, A) \tag{7.27}$$

(*by* $\times_B$ *we mean pullback*, *by* B_B *the bar construction over* B). *Here* J *acts by left multiplication on the right-hand side*, *and by conjugation on the left-hand side.*

Proof. Let $\check{B}_B(Z)$ denote the homogenous unreduced bar construction over B of a B-set Z (this is homotopically equivalent to B). Then we have

$$H\backslash \check{B}_A(G \times {}_A B/J) \simeq H\backslash \check{B}_A(G). \tag{7.28}$$

Writing (7.28) in non-homogeneous coordinates we get (7.27). □

Now the lemma also works for Hopf algebroids (by dualizing the argument), but it requires a flatness hypothesis. In our case, however, flatness does not hold: the analogue of $(B, B \times {}_A H \times {}_A B)$ is

$$(\mathbb{Z}/2[a], L), \tag{7.29}$$

where

$$L = \mathbb{Z}/2[a] \otimes_{H_\star^-} \Lambda^- \otimes_{H_\star^-} \mathbb{Z}/2[a] = \mathbb{Z}/2[a, \tau_i]/(\tau_i a^{i+1}).$$

Obviously, (7.29) is not a flat Hopf algebroid. In fact, one can easily see that the analogue of (7.19) is false in this case.

It is an interesting problem to see if one can construct a "derived" form of L which would fit into an analogue of (7.19) and yet have a suitable Koszul property to give an analogue of (7.20). If one had such an object, one might be able to prove the conjecture.

Acknowledgements

We are very thankful to J.P. May for valuable comments and suggestions.

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