CHAPTER 8: THE ELEMENTS OF ARF INVARIANT ONE

1. Introduction

One of the most important open problems in homotopy theory is whether or not there exist elements \( \theta_N \in \pi_{2N+1}^{S} \) of Arf invariant one. These elements arose in the work of Kervaire [26] and Kervaire and Milnor [27] as obstructions in surgery theory. Browder [14] showed that the nonvanishing of these obstructions is equivalent to the elements \( h_2^N \in E_2^{N+1-2,2} = \text{Ext}^2_2(Z_2, Z_2) \) being infinite cycles in the classical Adams spectral sequence for \( \pi_{*}^{S} \). Thus, an element \( \theta_N \in \pi_{2N+1}^{S} \) has Arf invariant one if and only if the secondary cohomology operation \( \Phi_N \) defined by the following Adem relation is nonzero in the mapping cone of \( \theta_N \):

\[
0 = \sum_{i=0}^{N} Sq^{N+1-2i} Sq^{1}.
\]

The first three elements of Arf invariant one are merely \( \eta^2, \nu^2 \) and \( \sigma^2 \). The next two elements of Arf invariant one, \( \theta_4 \in \pi_{30}^{S} \) and \( \theta_5 \in \pi_{62}^{S} \), have been shown to exist using the classical Adams spectral sequence [37], [11]. It is not known whether \( \theta_N \) exists for \( N \geq 6 \). The reader can find a more detailed exposition of this problem in [12] and [13].

In Section 2 we show that the element \( A[30] \in \pi_{30}^{S} \) has Arf invariant one by calculating that the secondary operation \( \Phi_4 \) is nonzero in the mapping cone of \( A[30] \). In Section 3 we identify \( \theta_5 \) as \( A[62,1] \) by showing that \( \theta_4^2 = 0 \) using an argument of Mahowald based upon a generalization of [34A, Theorem 16]. The construction of Barratt, Jones and Mahowald [11] shows that \( \theta_5 \) exists but does not determine the order of \( \theta_5 \). The argument of Section 3 shows that there are choices of \( \theta_5 \) of order two.
In [35] Mahowald showed that the elements \( h_1 \eta_1 \in E_2^{2,2} \) of the Adams spectral sequence are infinite cycles which are represented by the elements \( \eta_1 \in E_2^{2,2} \).

In Section 2 we identify \( \eta_5 \) as \( A[32,1] \). In Section 3, we identify \( \eta_6 \) as \( B[64,1] \).

2. The Existence of \( \theta_4 \)

Recall from Theorem 5.3.10 that \( \pi_3^S = Z A[30] \) and \( A[30] = d^{12}(2\sigma M_1^{12}) \). We will show that the secondary operation \( \psi_4 \) is nonzero in the mapping cone of \( A[30] \).

It follows that \( A[30] = \theta_4 \) has Arf invariant one. We will assume the definitions and basic properties of secondary cohomology operations and functional cohomology operations [47].

Let \( f: S^{23} \rightarrow BP^{(16)}(14) \) be the attaching map of the cell represented in homology by \( <M_1^4>^3 \). This map has image in the 16-skeleton because \( <M_1^4>^3 \) survives to \( E_8 \). Let \( i: BP^{(14)} \rightarrow BP^{(16)} \) be the natural inclusion. Since \( d^8(2\sigma<M_1^4>^3) = 0 \), there is a lifting \( F \) of \( 2\sigma f \) to \( BP^{(14)} \):

\[
\begin{array}{ccc}
S^{23} & \xrightarrow{1} & BP^{(14)} \\
\downarrow{F} & & \downarrow{BP^{(16)}} \\
2\sigma f & \xrightarrow{i} & BP^{(16)}
\end{array}
\]

**Figure 8.2.1: Definition of F**

The next step is to define a map \( G: \Sigma^7 C_f \rightarrow C_f \). We begin by defining

\[
G_1 = G|\Sigma^7 BP^{(8)}: \Sigma^7 BP^{(8)} \rightarrow BP^{(8)}
\]

as the composite of the projection map

\[
P: \Sigma^7 BP^{(8)} \rightarrow \Sigma^7 BP^{(8)} / \Sigma^7 BP^{(6)} = S^{15} V S^{15}
\]

followed by \( g \), where the first sphere is represented by \( <M_1^4> \) in homology, the second sphere is represented by \( M_1^2 \) in homology and \( g \) is the attaching map of the cell represented by \( <M_1^4>^2 \) in homology. Since \( <M_1^4>^2 \) survives to \( E_8 \) and \( d^8(<M_1^4>^2) = 2\sigma<M_1^4> \), the following diagram commutes:
\[ \Sigma^7 \text{BP}(6) \quad \xrightarrow{G_1} \quad \text{BP}(8) \]

\[ \xrightarrow{P} \]

\[ \Sigma^7 \text{BP}(8) / \Sigma^7 \text{BP}(6) = S^{15} \vee S^{15} \xrightarrow{2\sigma \vee \ast} \text{BP}(8) / \text{BP}(6) = S^8 \vee S^8 \]

**FIGURE 8.2.2: Definition of G_1**

P' is the natural projection map above. Now define \( G_2 : \Sigma^7 \text{BP}(16) \to \text{BP}(16) \) as the composite of the projection map

\[ P' : \Sigma^7 \text{BP}(16) \to \Sigma^7 \text{BP}(16) / \Sigma^7 \text{BP}(14) = S^{23} \vee 3S^{23} \]

followed by \((g \land \alpha)' : S^{15} \wedge S^8 \to \text{BP}(16)\). Here the first copy of \( S^{23} \) is represented by \( <M^4_1>^3 \) in homology and \( \alpha : (D^8, S^7) \to (\text{BP}(8), S) \) represents \( <M^4_1> \). Also,

\( (g \land \alpha)' : S^{15} \wedge S^8 \to \text{BP}(16) \) is an extension of

\[ S^{15} \wedge D^8 \xrightarrow{g\land\alpha} \text{BP}(8) \wedge \text{BP}(8) \xrightarrow{\nu} \text{BP}(16), \]

thinking of \( D^8 \) as the upper hemisphere of \( S^8 \). This extension to \( S^{15} \) smash the bottom hemisphere exists as a map into \( \text{BP}(8) \) because \( 2\sigma^2 = 0 \) in \( \pi^S_8 \). The top square in Figure 8.2.3 commutes because \( G_2 \) restricts to \( 2\sigma \) on \( \Sigma^7 \) of the cell \( C \) represented by \( <M^4_1>^2 \) in homology. Thus, the map \( G_3 \) must exist making the bottom square commute.

Now \( G_3 \) maps all cells into \( \text{BP}(14) \) except for the cell \( \Sigma^7 C \), and \( G_2 \) on this cell is \( 2\sigma \land 1 \). In \( \Sigma^7 C \), \( CE^7 S^{23} \) is attached to this cell by \( \Sigma^7 \nu \). Therefore \( G_3 \) on this cell is \( 2\sigma \land \nu \) which, as in Figure 8.2.1, lifts to \( \text{BP}(14) \). Thus \( G_3 \) lifts to a map \( G \).

\[ \Sigma^7 S^{23} \xrightarrow{1} \Sigma^7 \text{BP}(16) \]

\[ \xrightarrow{G_2} \text{BP}(16) \]

\[ \xrightarrow{G_3} \text{BP}(16) \]

\[ \xrightarrow{G} \]

\[ \Sigma^7 \text{BP}(16) \xrightarrow{G_2} \text{BP}(16) \]

\[ \xrightarrow{G_3} \text{BP}(16) \]

\[ \xrightarrow{G} \]

**FIGURE 8.2.3: Definition of G**
In $H^*(G; \mathbb{Z}_2)$, let $u(X)$ denote the element dual to $X \in H_*(G; \mathbb{Z}_2)$. Let $Y \in H^i(G; \mathbb{Z}_2)$ denote the element determined by the first sphere in Figure 8.2.2. By the definition of $G$, $Y$ represents a cell with the same attaching map as $<M^4_1>^2$. Therefore, $Sq^1 u(1) = u(Y)$. Thus, the functional secondary cohomology operation $Sq^6$ is defined on $u(1) \in H^0(G; \mathbb{Z}_2)$ and equals $S^7 u(M^4_1) \in H^8(G; \mathbb{Z}_2)$. By the Peterson-Stein formula: $G \circ \phi^*(u(1))$

$$= Sq^{31} Sq^6(u(1)) + Sq^{30} Sq^6(u(1)) + Sq^{28} Sq^6(u(1)) + Sq^{24} Sq^6(u(1)) + Sq^{16} Sq^6(u(1)).$$

Since $H^k(G; \mathbb{Z}_2) = 0$ for $k = 0, 1, 3$, we must have $Sq^1 u(1) = 0$, $Sq^2 u(1) = 0$ and $Sq^4 u(1) = 0$. Since $G_1|\Sigma^7 BP(6) = \ast$, $Sq^8 u(1)$ must be 0 not $S^7 u(1)$.

Thus, $G \circ \phi^*(u(1)) = Sq^6 Sq^6(u(1)) = Sq^6 (S^7 u(M^4_1)) = S^7 u(<M^4_1>^3) \neq 0$. Thus, $\phi^*(u(1)) \neq 0$ in $H^31(G; \mathbb{Z}_2)$. Note that there is a unique top dimensional cell of degree 31 in $C_F$ which determines a nononzero element $\tau \in H^31(G; \mathbb{Z}_2)$. Hence $\tau = \phi^*(u(1)) \neq 0$. Since $d^{12}(2\sigma M^1_1) = A[30]$ and $F$ represents the boundary of $2\sigma M^1_1$, the triangle in the following diagram must commute up to homotopy.

Therefore, there is an induced map $J$ making the square commute.

\[ (** ) \]

$F$ $\downarrow$ $BP^{(14)}$ $\rightarrow$ $C_F$

$\downarrow J$

$A[30] \downarrow S$

$\rightarrow$ $C_A[30]$

Now $\phi^*(u(1)) = \phi^*J^*(u(1)) = J^* \phi^*(u(1)) = J^*(\tau) \neq 0$. Thus, $A[30]$ must have Arf invariant one. We have thus proved the following theorem.

**Theorem 8.2.1** $A[30]$ has Arf invariant one.

We derive several Toda brackets involving elements related to $\theta_4$. The first Toda bracket below was proved by Hoffman [24]. We give a proof using our spectral sequence.
THEOREM 8.2.2  (a) \( \theta_4 = A[30] \in <\sigma, 2\sigma, 2\sigma, \sigma> \)

(b) \( \nu \theta_4 = \nu A[30] \in <C[18], \sigma, 2\sigma> \)

(c) \( \theta_4 = A[30] \in <\sigma, 2\sigma, \sigma^2, 2> = <\sigma^2, 2, \sigma^2, 2> \)

(d) \( \eta \theta_4 = \eta A[30] \in <A[16], 2, \sigma^2> \)

PROOF.  (a) Represent \(<M_1^{-1}^{-1}^\theta>^2 \) by \( \mu_8 \) such that \( \partial(\mu_8) = (\sigma \wedge 2\mu_4) \cup (B \sigma_2 \sigma) \).

Since \( 2^4 \sigma_2 = 0 \), \( \sigma A[14] = 0 \) and \( \sigma \gamma_1 = 0 \), it follows that \( <\sigma, 2\sigma, 2\sigma> = 2<\sigma, 2\sigma, \sigma> = 0 \). Thus, \( 2\sigma_1^{12} \in E_{24}^{24} \) is represented by

\[
M = (\mu_4 \wedge \sigma \wedge 2\mu_6) \cup (B \wedge \sigma_6) \cup (\mu_4 \wedge B \sigma_2 \sigma \wedge 2\mu_4) \cup (B \sigma_2 \sigma_2 \wedge \mu_4) \]

\[
\cup (\mu_4 \wedge B \sigma_2 \sigma_2 \sigma_2) \]

because \( \partial M = (B \wedge B \sigma_2 \sigma) \cup (B \sigma_2 \sigma_2 \wedge \sigma) \cup (\sigma \wedge B \sigma_2 \sigma_2 \sigma) \). Since \( d^4(2\sigma_1^{12}) = A[30] \), \( \partial M \) represents \( A[30] \) and clearly \( \partial M \in <\sigma, \sigma^2, \sigma^2, \sigma> \).

(b), (c) The four-fold Toda bracket \( <\sigma, 2\sigma, \sigma^2, 2> \) is defined by

Theorem 2.2.7(a) because \( <\sigma, 2\sigma, \sigma^2> \in \pi_2^S = 0 \) and \( <2\sigma, \sigma^2, 2> = \sigma 2\sigma + 2^4 \sigma_2^S \)

\[
= \sigma(\sigma^2) = 0. \text{ Now } \nu A[30] \in \nu <\sigma, 2\sigma, 2\sigma, \sigma> \subset <\nu, \sigma, 2\sigma, 2\sigma >, \sigma > = <C[18], 2\sigma, \sigma> \).

Since \( \text{Cok} J_{20} = \nu^{23} \nu[20] \), \( \nu A[30] \in <C[18], \sigma, 2\sigma> \subset <C[18], \sigma^2, 2> \)

\[
= <\nu, \sigma, 2\sigma, \sigma^2, 2> = \nu <\sigma, 2\sigma, \sigma^2, 2, 2> \text{. Thus, } <\sigma, 2\sigma, \sigma^2, 2> \text{ contains } A[30]. \text{ Note}\]

that \( <\sigma^2, 2, \sigma^2, 2> \) is defined by Theorem 2.2.7(a) because \( <\sigma^2, 2, \sigma^2> \in \pi_2^S = 0 \) and \( <2, \sigma^2, 2> = \sigma \sigma^2 = 0. \text{ Now } <\sigma^2, 2, \sigma^2, 2> \subset <\sigma, 2\sigma, \sigma^2, 2> = \{A[30]\}. \)

(d) \( \eta A[30] \in \eta <\sigma, 2\sigma, 2\sigma, \sigma> \subset <\eta, 2, \sigma^2, 2\sigma, \sigma> = <A[16], 2\sigma, \sigma> + <\eta \gamma_1, 2\sigma, \sigma>. \text{ Now}\)

\( <\eta \gamma_1, 2\sigma, \sigma> \subseteq \eta <\gamma_1, 2\sigma, \sigma>. \text{ Since } \nu <\gamma_1, 2\sigma, \sigma> = <\nu, \gamma_1, 2\sigma, \sigma> = 0, \text{ }<\gamma_1, 2\sigma, \sigma> \text{ can not}

\text{equal } A[30] \text{ and must therefore equal zero. It follows that } <\eta \gamma_1, 2\sigma, \sigma> = \eta \gamma_1 \sigma \sigma^2 + \sigma \sigma^2 \sigma = \eta \xi \text{ where } \nu \xi = 0. \text{ Thus, } <\eta \gamma_1, 2\sigma, \sigma> = 0 \text{ and}

\( \eta A[30] \in <A[16], 2\sigma, \sigma> \). \]

We conclude this section by identifying the Mahowald element \( \eta_5 \in \pi^S_{32} \) as \( A[32, 1] \).
THEOREM 8.2.3 Let $\eta_5$ be any element of $\pi^S_{32}$ which projects to $h_5 h_5$ in $E_2^{32,2}$ of the Adams spectral sequence. Then $\eta_5$ projects to $A[32,1]$ in $E_4^{24,0,32}$ of the Atiyah-Hirzebruch spectral sequence.

PROOF. From the computation of $E_2$ of the Adams spectral sequence by Tangora [59], it follows from the fact that $h_5 h_5$ is an infinite cycle that $h_5^3 h_5$ is a nonbounding infinite cycle. Thus, if $\eta_5$ is any element that projects to $h_5 h_5$ then $\eta_5^2 \neq 0$. Since $\eta_5^2 \pi^S_{32} = Z_2 \eta^2 A[32,1]$ for any choice of $A[32,1]$ modulo $Z_2 A[32,2] \oplus Z_2 A[32,3] \oplus Z_2 \eta_3$, it follows that $\eta_5 \in A[32,1] + (Z_2 A[32,2] \oplus Z_2 A[32,3] \oplus Z_2 \eta_3)$. Now the theorem follows from the observation that $Z_2 A[32,2] \oplus Z_2 A[32,3] \oplus Z_2 \eta_3$ projects to zero in $E_4^{24,0,32}$ of the Atiyah-Hirzebruch spectral sequence.

3. The Existence of $\theta_5$

In this section we show that $A[62,1]$ has Arf invariant one and is thus entitled to be denoted as $\theta_5$. We also identify the Mahowald element $\eta_6$ as $B[64,1]$. In addition, we derive a few miscellaneous results which are relevant to the Arf invariant problem. We begin with the following well known lemma which can be proved from a computation of $\text{Ext}_y(Z_2, Z_2)$ as the homology of the $\Lambda$-algebra.

LEMMA 8.3.1 The following elements are nonzero in $\text{Ext}_y(Z_2, Z_2)$:

(a) $h_N^2$ for $N \geq 4$;
(b) $h_0 h_N^2$ and $h_1 h_N^2$ for $N \geq 3$;
(c) $h_1 h_N^2$ for $N \geq 4$;
(d) $h_1^2 h_N^2$ for $N \geq 5$.

Adams's proof [2] of the nonexistence of elements of Hopf invariant one in degrees $2^{N-1}$, $N \geq 4$, is equivalent to the following differentials in the Adams spectral sequence. The elements listed in Lemma 8.3.1 and the differentials
of Theorem 8.3.2 for \( N \geq 4 \) are depicted in Figure 8.3.1. Note that there are other elements in the bidegrees of that figure which are not depicted.

**THEOREM 8.3.2** \( d^2(h_N) = h_N h_{0, N-1}^2 \) for \( N \geq 4 \).

![Diagram](image)

FIGURE 8.3.1: Part of \( E^2 \) of the Adams Spectral Sequence (\( N \geq 6 \))

The following lemmas will be used to identify \( \Theta_S \) as \( A[62,1] \). The entire argument is based upon ideas of Mahowald [34A] and is a rewording of a detailed proof which he sent to me.

**LEMMA 8.3.3** \( <\sigma^2, 2, A[30]> \in \mathbb{Z}_2(\eta C[44]) \oplus \mathbb{Z}_2(8D[45]) \)

**PROOF.** Note that \( \eta^2 <\sigma^2, 2, A[30]> = \sigma^2 <2, A[30], \eta^2 > \in \sigma^2 \pi^S_{33} = 0 \). Also, \( \nu^2 <\sigma^2, 2, A[30]> = \sigma^2 <2, A[30], \nu^2 > \in \sigma^2 \pi^S_{37} = \sigma(4C[44]) = 0 \). In addition, \( 2 <\sigma^2, 2, A[30]> = \sigma^2 <2, A[30], 2> = 0 \). The only elements of \( \pi^S_{45} \) which satisfy these three conditions are \( \mathbb{Z}_2(\eta C[44]) \oplus \mathbb{Z}_2(8D[45]) \).

**LEMMA 8.3.4** If \( \xi \in \pi^S_{45} \) and \( \xi A[36] = 0 \) then

\[ \xi C[44] \in <\eta \xi, \eta A[30], \nu, \sigma>. \]
PROOF. By Theorem 2.4.6(a), if \( \langle \eta, \eta A[30], \nu, \sigma \rangle \) were defined then it would contain \( C[44] \). Now \( \langle \eta A[30], \nu, \sigma \rangle \supset A[30] \langle \eta, \nu, \sigma \rangle = 0, \eta A[30] \cdot \pi_{11}^S = 0 \) and \( \sigma \cdot \pi_{35}^S = 0 \). Thus, \( \langle \eta A[30], \nu, \sigma \rangle = 0 \). However, \( A[36] \in \langle \eta, \eta A[30], \nu \rangle \). Since \( \xi A[36] = 0, \eta \xi A[30], \nu, \sigma \rangle \) is defined by Theorem 2.2.7(b). Thus, \( \xi C[44] \in \langle \eta, \eta A[30], \nu, \sigma \rangle \).

**Lemma 8.3.5**
(c) \( A[16]C[44] = 0 \).

**Proof.** (a) \( A[16]A[36] \in A[36] \langle \eta, 2, \sigma^2 \rangle = \langle A[36], \eta, 2, \sigma^2 \rangle \in \sigma^2 \cdot \pi_{38}^S = 0 \).
(b) \( \eta A[16]A[30] \in \eta A[30] \langle \eta, 2, \sigma^2 \rangle \subset 0,2, \sigma^2 \rangle = 0, \pi_{33}^S = 0 \).
(c) Since \( A[16]A[36] = 0, A[16]C[44] \in \langle \eta A[16], \eta A[30], \nu, \sigma \rangle \supset 0, \eta, \nu, \sigma \rangle \).

(Note that \( \eta A[16]A[30], \eta, \nu, \sigma \rangle \) is defined by Theorem 2.2.7(b) because \( 0 \in \langle \eta A[16]A[30], \eta, \nu \rangle \) and \( 0 = \langle \eta, \nu, \sigma \rangle \).) Since \( \sigma \cdot \pi_{53}^S = 0, A[16]C[44] \in \pi_{48}^S, \nu, \sigma \rangle + \eta A[16], \pi_{53}^S + \langle \eta A[16], \pi_{35}^S, \sigma \rangle = \langle \alpha_6, \nu, \sigma \rangle + \eta A[16], \nu A[14]C[20], \sigma \rangle = \eta A[16], \nu A[32, 3], \sigma \rangle + \eta A[16], \beta_4, \sigma \rangle = \langle \alpha_6, \nu, \sigma \rangle + \eta A[16], \nu A[20], 0 \rangle + A[16]A[30, 3] \langle \eta, \nu, \sigma \rangle + A[16] \langle \eta, \beta_4, \sigma \rangle = \langle \alpha_6, \nu, \sigma \rangle \). By Theorem 4.2.3 and Figure 4.2.2, it follows that \( \langle \alpha_6, \nu, \sigma \rangle \) projects to an element of filtration degree at least 26 in the Adams spectral sequence. The only such element is \( h_0^2 P_g = d_0^2 (h_0 P_4 k) \). Thus, \( 0 = \langle \alpha_6, \nu, \sigma \rangle = A[16]C[44] \).

**Lemma 8.3.6** \( A[30]^2 = 0 \)

**Proof.** \( A[30]^2 \in A[30] \langle 2, \sigma^2, 2, \sigma^2 \rangle \subset \langle \eta C[44], 2, \sigma^2 \rangle \supset C[44], \eta A[30], 2, \sigma^2, 2, \sigma^2 \rangle \subset 0, \eta A[30], 2, \sigma^2, 2, \sigma^2 \rangle \supset \sigma \cdot \pi_{53}^S = 0 \). Since \( \eta A[36] = 0 \), \( A[30]^2 \in \eta C[44] \cdot \pi_{15}^S + 8 D[45], \pi_{35}^S + \sigma^2 \cdot \pi_{48}^S = \eta \gamma C[44] \in \gamma \langle \eta, \eta A[30], \nu, \sigma \rangle \supset \langle \eta^2, \eta A[30], \nu, \sigma, \gamma \rangle = \langle \eta^2, A[30], \eta, \nu, \sigma, \gamma \rangle \).
\[
= \eta^2, A[30], 0 = \eta^2, \pi_{58} = 0. \text{ Thus, } A[30]^2 \in \langle \nu, \sigma, \gamma \rangle \cdot \pi_{34}^S \subset \{\nu^2 C[20], \eta \alpha \}, \pi_{34}^S
\]
\[
= \eta \alpha^3 A[14] C[20] = \eta A[14] C[20] = (8 \sigma, 2, \alpha^2) = \eta A[14] \alpha_2 C[20], 8 \sigma, 2 \in (\eta \cdot \pi_{31}^S) \cdot \pi_{28}^S
\]
\[
= (\eta \gamma_3)(A[8] C[20]) = 0. \]

**Theorem 8.3.7**


are all the elements of \( \pi_{62}^S \) of Arf invariant one. In particular, there are choices of \( \theta_5 \) of order two.

**Proof.**

Since \( \theta_4 = A[30] \) exists, \( 2 \theta_4 = 0 \) and \( \theta_4^2 = 0 \), it follows from [12, Theorem 2.1] that \( \theta_5 \) exists and has order two. From Figure 8.3.1, we see that any element \( \theta_5 \) of Arf invariant one satisfies \( \eta^2 \theta_5 \neq 0 \). Since

\[ \eta^2 \cdot \pi_{62}^S = Z_2^2 A[62, 1] \oplus Z_2^2 A[62, 4], \text{ Span } \{A[62, 2], A[62, 3], B[62], \eta^2 B[60] \} \]

has Adams filtration at least three. Since


and \( C[20] = d^{12}(\nu^2 M_{12}^2) \), \( \nu A[62, 4] = C[20] A[45, 1] \). From Figure 8.3.1, \( \nu \theta_5 \) is nonzero and is represented in the Adams spectral sequence by \( h_2 h_5^2 \) in filtration degree three while \( C[20] A[45, 1] \) has Adams filtration at least nine. Thus, \( A[62, 4] \) has Adams filtration at least three. Now all the elements of \( \{A[62, 2], A[62, 3], A[62, 4], B[62], \eta^2 B[60] \} \) have Adams filtration at least three. Therefore, all the elements of


have Arf invariant one.

Next we identify the Mahowald element \( \eta_6 \) in terms of the Atiyah-Hirzebruch spectral sequence. Recall that \( \eta_6 \) denotes any element of \( \pi_{64}^S \) which projects to \( h_2 h_6^2 \) in \( E_{64, 2}^1 \) of the Adams spectral sequence.
THEOREM 8.3.8  (a) Any choice of $\eta_6$ projects to $B[64,1]$ in $E_{0,64}^{54}$ of the Atiyah-Hirzebruch spectral sequence.

(b) All the choices of $\eta_6$ are


(c) All the values of $2\eta_6$ are $\eta_5^2 + Z_2 \eta^2 A[62,4]$, and $4\eta_6 = 0$.

(d) There are choices of $\eta_5$ and $\eta_6$ such that $2\eta_6 = \eta_6^2$.

PROOF. Since $A[64,1], A[64,2], A[64,3], B[64,2], \eta^2 A[62,1]$ and $\eta_5 \gamma$ project to zero in $E_{2,64}^{54}$ of the Adams spectral sequence, all the choices for $\eta_6$ are


All of these elements project to $B[64,1]$ in $E_{0,64}^{54}$. Moreover, $2\eta_6 = 2B[64,1] + 2sB[64,2] = \eta^2 A[62,1] + s\eta^2 A[62,4] = \eta_5^2 + s\eta^2 A[62,4]$ and $4\eta_6 = 0$. Note that $\eta_5^2$ projects to $h_1^2$ in the Adams spectral sequence. Thus, $\eta_5^2$ is not zero, and by Mahowald [36] there are choices of $\eta_5$ and $\eta_6$ such that $2\eta_6 = \eta_5^2$.\]