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**NEW COHOMOLOGICAL RELATIONSHIPS AMONG  
LOOPSPACES, SYMMETRIC PRODUCTS, AND EILENBERG  
MACLANE SPACES**

NICHOLAS J. KUHN

June 14, 1996

ABSTRACT. Let  $T(j)$  be the dual of the  $j^{\text{th}}$  Brown-Gitler spectrum (at the prime 2) with top class in dimension  $j$ . Then it is known that  $T(j)$  is a retract of a suspension spectrum, is dual to a stable summand of  $\Omega^2 S^3$ , and that the homotopy colimit of a certain sequence  $T(j) \rightarrow T(2j) \rightarrow \dots$  is a wedge of stable summands of  $K(V, 1)$ 's, where  $V$  denotes an elementary abelian 2 group. In particular, when one starts with  $T(1)$ , one gets  $K(\mathbb{Z}/2, 1) = RP^\infty$  as one of the summands.

Refining a question posed by Doug Ravenel, I discuss a generalization of this picture. I consider certain finite spectra  $T(n, j)$  for  $n, j \geq 0$  (with  $T(1, j) = T(j)$ ), dual to summands of  $\Omega^{n+1} S^N$ , conjecture generalizations of all of the above, and prove that all these conjectures are correct in cohomology. So, for example,  $T(n, j)$  has unstable cohomology, and the cohomology of the colimit of a certain sequence  $T(n, j) \rightarrow T(n, 2j) \rightarrow \dots$  agrees with the cohomology of the wedge of stable summands of  $K(V, n)$ 's corresponding to the wedge occurring in the  $n = 1$  case above.

One can also map the  $T(n, j)$  to each other as  $n$  varies, and the cohomological calculations suggest conjectures related to symmetric products of spheres.

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## 1. INTRODUCTION

With all spaces and spectra localized at 2, let  $T(j)$  be the  $(2j)^{th}$  dual of the  $j^{th}$  stable summand of  $\Omega^2 S^3$ . These finite complexes have remarkable properties that were explored in the 1970's and 1980's in work by M.Mahowald, E.Brown, S.Gitler, F.Peterson, R.Cohen, G.Carlsson, H.Miller, J.Lannes, and P.Goerss, among others. (Entries into the extensive literature include [Mah, BC, Ca, Mi2, L2, GLM].) These properties played an essential role in a number of the major achievements in homotopy theory during this time: Mahowald's construction [Mah] of an infinite family of 2-primary elements in  $\pi_*^S(S^0)$  having Adams filtration 2; Goerss, Lannes, and F.Morel's work [GLM] on representing mod 2 homology by maps from (desuspensions of) the  $T(j)$ 's; and Miller's proof of the Sullivan conjecture [Mi2], which led to Lannes' work on  $\text{Map}(BV, X)$  [L2].

[Mah] and [GLM] are reflections (and extensions) of the fact that  $T(j)$  is a dual Brown-Gitler spectrum, and, as such, has unexpected "unstable" properties. For example (following Miller in [Mi2]),  $T(j)$  is the projective cover of  $S^j$  in the category in which the objects are wedge summands of suspension spectra, and "epis" are maps inducing epimorphisms in mod 2 homology.

[Mi2] then reflects connections between the  $T(j)$  and the classifying spaces  $BV$  of elementary abelian 2 groups  $V$ . For example, the homotopy colimit of certain sequences

$$T(j) \rightarrow T(2j) \rightarrow T(4j) \rightarrow \dots$$

is always an infinite wedge of stable wedge summands of  $BV$ 's. In particular, if one starts with  $T(1)$ , one gets  $B(\mathbf{Z}/2)$  as a summand.

Refining a question posed by D.Ravenel, the goal of this paper is to show that, at least on the level on cohomology, certain finite complexes  $T(n, j)$  arising from  $\Omega^{n+1} S^N$  appear to be unstable, and are related to the Eilenberg-MacLane spaces  $K(V, n)$  in the same way that the  $T(j)$  are related to the spaces  $BV$ . Furthermore, one can let " $n$  go to  $\infty$ ", and obtain connections between these finite complexes and symmetric powers of spheres.

What I *can* prove seems substantial, and involves, on one hand, some new observations about loop-space machinery and the Nishida relations, and, on the other, much of what the author knows about the relationship between the category of unstable modules over the Steenrod algebra and the "generic representation" category of [K5, K6, K7]. What I *can't* yet prove, but only conjecture, seems even more intriguing, and seems to suggest that there is maybe a remarkable "naturally occurring" infinite loop-space waiting to be discovered.

To explain our main results, we need to introduce our cast of characters. Recall that [May], if  $X$  is path connected, there is a stable decomposition

$$\Sigma^\infty \Omega^n \Sigma^n X \simeq \bigvee_{j \geq 1} \Sigma^\infty D_{n,j} X,$$

where  $D_{n,j}X = \mathcal{C}(n,j)_+ \wedge_{\Sigma_j} X^{[j]}$ . Here  $\mathcal{C}(n,j)$  is the configuration space of  $j$  tuples of distinct ‘little cubes’ in  $I^n$ , a space acted on freely by the  $j^{\text{th}}$  symmetric group  $\Sigma_j$ , and  $X^{[j]}$  denotes the  $j$ -fold smash product of  $X$  with itself.

For a given  $n$  and  $j$ , there is a natural number  $d$  and a natural equivalence

$$D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j}X,$$

thus allowing  $D_{n,j}X$  to be defined for a finite *spectrum*<sup>1</sup>.

**Definition 1.1.** For  $n \geq 0, j \geq 0$ , let  $T(n,j)$  be the S-dual of  $D_{n+1,j}(S^{-n})$ .

$T(n,j)$  is a finite spectrum with top cell in dimension  $nj$ , and with bottom mod 2 homology in dimension  $n\alpha(j)$ , where  $\alpha(j)$  denotes the number of 1’s in the 2-adic expansion of  $j$ . As examples, we note that, for all  $j$  and  $n$ ,  $T(0,j) = S^0 = T(n,0)$ ,  $T(n,1) = S^n$ ,  $T(1,j) = T(j)$  as above, and  $T(n,2) = \text{cofiber } \{\Sigma^n RP_+^{n-1} \rightarrow S^n\}$ .

This bigraded family of finite spectra has some extra structure we will need. The H-space structure on loopspaces induces copairings

$$\Psi : T(n,k) \rightarrow \bigvee_{i+j=k} T(n,i) \wedge T(n,j).$$

Evaluation on loopspaces induces maps

$$\delta : T(n,j) \rightarrow \Sigma^{-1}T(n+1,j).$$

Finally, looping Hopf invariants, together with the above periodicity, induces ‘Frobenious’ maps

$$\Phi : T(n,j) \rightarrow T(n,2j).$$

These three families of maps will be shown to be compatible in the expected ways. In particular,  $\delta$  and  $\Phi$  commute up to homotopy.

Our first result is a description of  $H^*(T(n,j); \mathbf{Z}/2)$  as a module over the mod 2 Steenrod algebra  $\mathcal{A}$ . Following the lead of others in the  $n = 1$  case [Ca, Mi2, LZ], we describe the bigraded object  $H^*(T(n,*); \mathbf{Z}/2)$ , with the extra structure afforded by  $\Psi^*$  and  $\Phi^*$ . We need first to define variants on the category  $\mathcal{U}$  of unstable  $\mathcal{A}$  modules, and the category  $\mathcal{K}$  of unstable  $\mathcal{A}$  algebras.

Let  $\mathcal{U}_\rho$  be the category whose objects are pairs  $(M, \rho)$ :  $M = M_{*,*}$  is an  $\mathbf{N} \times \mathbf{N}[\frac{1}{2}]$  graded  $\mathbf{Z}/2$  vector space<sup>2</sup> whose columns  $M_{*,j}$  are unstable  $\mathcal{A}$  modules, and  $\rho : M \rightarrow M$  is a collection of  $\mathcal{A}$  linear maps  $\rho : M_{*,2j} \rightarrow M_{*,j}$ . Morphisms in  $\mathcal{U}_\rho$  are just maps  $f : M \rightarrow N$  preserving all structure.

Let  $\mathcal{K}_\rho$  be the category of ‘restricted algebras in  $\mathcal{U}_\rho$ ’, i.e. commutative, unital algebras  $K$  in  $\mathcal{U}_\rho$  (a category with a tensor product) satisfying the ‘restriction axiom’:  $Sq^{|x|}x = (\rho(x))^2$  for all  $x \in K$ .

Let  $U_\rho : \mathcal{U}_\rho \rightarrow \mathcal{K}_\rho$  be the free functor, left adjoint to the forgetful functor. Explicitly,  $U_\rho(M, \rho) = S^*(M)/(Sq^{|x|}x - (\rho(x))^2)$ .

If  $I = (i_1, \dots, i_l)$ , we set  $Sq^I = Sq^{i_1} \dots Sq^{i_l}$ ,  $l(I) = l$ , and  $e(I) = (i_1 - 2i_2) + \dots + (i_{l-1} - 2i_l) + i_l$ .  $I$  is called *admissible* if  $i_s \geq 2i_{s+1}$  for all  $s$ . Define  $E(n), L(k) \subset \mathcal{A}$  by

$$E(n) = \langle Sq^I \mid I \text{ is admissible and } e(I) > n \rangle$$

<sup>1</sup>There are more sophisticated ways to do this. See §2.

<sup>2</sup>Often  $\mathbf{N} \times \mathbf{N}$  graded vector spaces will be considered  $\mathbf{N} \times \mathbf{N}[\frac{1}{2}]$  graded by setting  $M_{*,j} = \{0\}$  for  $j \notin \mathbf{N}$ .

$$L(k) = \langle Sq^I \mid I \text{ is admissible and } l(I) > k \rangle.$$

Both of these are known to be left  $\mathcal{A}$  modules [S, Prop.1.6.2], [Mi1]. Now let  $F(n, k)$  be the unstable  $\mathcal{A}$  module  $\Sigma^n(A/(E(n) + L(k)))$ , and then let  $F_\rho(n) \in \mathcal{U}_\rho$  be the pair  $(\bigoplus_{k \geq 0} F(n, k), \rho)$ , where  $F(n, k)$  has second grading  $2^k$ , and  $\rho : F(n, k+1) \rightarrow F(n, k)$  is the projection.

**Theorem 1.2.** *Let  $n \geq 1$ . With multiplication and restriction given by  $\Psi^*$  and  $\Phi^*$ ,*

$$H^*(T(n, *); \mathbf{Z}/2) \simeq U_\rho(F_\rho(n))$$

*as objects in  $\mathcal{K}_\rho$ . In particular,  $H^*(T(n, j); \mathbf{Z}/2)$  is an unstable  $\mathcal{A}$  module.*

This theorem suggests

**Conjecture 1.3.**  *$T(n, j)$  is a stable wedge summand of a suspension spectrum.*

This is known to be true when  $n = 1$  [L1, Goe].

To discuss stabilizing  $T(n, j)$  with respect to  $\delta$ , we make the following definition.

**Definition 1.4.**  $T(\infty, j) = \text{hocolim} \{ T(0, j) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Sigma^{-n}T(n, j) \xrightarrow{\delta} \cdots \}$ .

**Theorem 1.5.**

- (1)  $T(\infty, j) \simeq *$  unless  $j$  is a power of 2.
- (2)  $H^*(T(\infty, 2^k); \mathbf{Z}/2) \simeq \mathcal{A}/L(k)$  as  $\mathcal{A}$  modules.

The  $\mathcal{A}$  module  $\mathcal{A}/L(k)$  is already known to arise as the cohomology of a spectrum: it is the cohomology of  $SP_\Delta^{2^k}(S^0)$ , the cofiber of the diagonal map  $\Delta : SP^{2^{k-1}}(S^0) \rightarrow SP^{2^k}(S^0)$  between symmetric products of the sphere spectrum  $S^0$  [MP]. Thus we have

**Conjecture 1.6.**  $T(\infty, 2^k) \simeq SP_\Delta^{2^k}(S^0)$ .

As will be discussed in §9, work by the author in [K2] and S. Mitchell and S. Priddy in [MP] implies that this conjecture would follow formally from the existence of certain pairings between the  $T(\infty, 2^k)$ 's. At any rate, Theorem 1.5 immediately implies

**Corollary 1.7.**  $\text{hocolim}_{n, k \rightarrow \infty} \Sigma^{-n}T(n, 2^k) = \text{hocolim}_{k \rightarrow \infty} T(\infty, 2^k) \simeq H\mathbf{Z}/2$ .

We now turn our discussion to how  $T(n, j)$  stabilizes with respect to  $\Phi$ .

**Definition 1.8.**  $\Phi^{-1}T(n, j) = \text{hocolim} \{ T(n, j) \xrightarrow{\Phi} T(n, 2j) \xrightarrow{\Phi} T(n, 4j) \xrightarrow{\Phi} \cdots \}$

Our third theorem identifies  $H^*(\Phi^{-1}T(n, j); \mathbf{Z}/2)$  as the cohomology of an infinite wedge of certain stable summands of the Eilenberg MacLane spaces  $K(V, n)$ , in a manner that is independent of  $n$ . In particular, just as  $H^*(K(\mathbf{Z}/2, 1); \mathbf{Z}/2)$  was shown in [Ca] to be an  $\mathcal{A}$  module direct summand of  $H^*(\Phi^{-1}T(1, 1); \mathbf{Z}/2)$ , so is  $H^*(K(\mathbf{Z}/2, n); \mathbf{Z}/2)$  an  $\mathcal{A}$  module summand of  $H^*(\Phi^{-1}T(n, 1); \mathbf{Z}/2)$ .

To be more precise, we need yet more notation. As in [K5, K6, K7], let  $\mathcal{F}$  be the category with objects the functors

$$F : \text{finite dimensional } \mathbf{Z}/2 \text{ vector spaces} \rightarrow \mathbf{Z}/2 \text{ vector spaces,}$$

and with morphisms the natural transformations. For example,  $S^j$  and  $S_j$ , defined by  $S^j(V) = V^{\otimes j}/\Sigma_j$  and  $S_j(V) = (V^{\otimes j})^{\Sigma_j}$ , are objects in  $\mathcal{F}$ .

Let  $\Lambda$  be an indexing set for the simple objects in this abelian category: algebraic group considerations suggest a number of  $\Lambda$ 's, e.g. the set of 2-regular partitions [K6, Sections 5 and 6]. Given  $\lambda \in \Lambda$ , let  $F_\lambda \in \mathcal{F}$  be the corresponding simple object,  $V_\lambda$  a vector space large enough so that  $F_\lambda(V_\lambda) \neq 0$ ,  $e_\lambda \in \mathbf{Z}_2[\text{End}(V_\lambda)]$  an idempotent chosen so that  $\mathbf{Z}/2[\text{End}(V_\lambda)]e_\lambda$  is the projective cover of the  $\mathbf{Z}/2[\text{End}(V_\lambda)]$  module  $F_\lambda(V_\lambda)$ , and  $K(\lambda, n) = e_\lambda \Sigma^\infty K(V_\lambda, n)$  the corresponding stable summand of  $K(V_\lambda, n)$ . Finally, given  $\lambda \in \Lambda$  and  $j = 0, 1, \dots$ , define  $a(\lambda, j) \in \mathbf{N}$  by

$$a(\lambda, j) = \dim_{\mathbf{Z}/2} \text{Hom}_{\mathcal{F}}(F_\lambda, S^{2^k j}), \text{ for } k \gg 0.$$

**Theorem 1.9.**  $H^*(\Phi^{-1}T(n, j); \mathbf{Z}/2) \simeq H^*(\bigvee_{\lambda \in \Lambda} a(\lambda, j)K(\lambda, n); \mathbf{Z}/2)$  as  $\mathcal{A}$  modules.

(Here  $\bigvee_i b_i Y_i$  means that each  $Y_i$  occurs in the wedge sum with multiplicity  $b_i$ .)  
We remark that these large  $\mathcal{A}$  modules are nevertheless of finite type.

**Conjecture 1.10.**  $\Phi^{-1}T(n, j) \simeq \bigvee_{\lambda \in \Lambda} a(\lambda, j)K(\lambda, n)$ .

Some form of the following has been known to the experts<sup>3</sup> since the late 1980's.

**Proposition 1.11.** *This conjecture is true when  $n = 1$ . In particular,  $\Phi^{-1}T(1, 1)$  has  $B(\mathbf{Z}/2)$  as a stable summand.*

The organization of the rest of the paper is as follows.

§2, §3, and §4 are devoted to the geometric constructions used to define the three families of maps  $\Psi, \Phi, \delta$  on the  $T(n, j)$ . In hopes that these will be useful in other settings, we develop this material with perhaps more care than is traditional (at one point, proving a lemma using ideas from “Goodwillie calculus”). Theorem 2.4 summarizes our main geometric results. In §5, properties of these constructions are combined with standard formula [CLM] for the homology of iterated loopspaces to give descriptions of  $H^*(T(n, j); \mathbf{Z}/2)$ ,  $\Psi^*$ ,  $\Phi^*$ , and  $\delta^*$  in terms of Dyer-Lashof-like operations. The standard Nishida relations then yield recursive formulae for how  $\chi(Sq^i)$  acts on  $H^*(T(n, j); \mathbf{Z}/2)$ ; we deduce more useful formulae for how  $Sq^i$  acts in §6. These should be of some independent interest. Theorem 1.2 and Theorem 1.5 are then deduced in §7.

The proof of Theorem 1.9 is rather different. Recall [K5] that there are adjoint functors

$$\mathcal{U} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{r} \end{array} \mathcal{F},$$

<sup>3</sup>By “experts” here I mean at least the authors of [LS], as well as myself.

where  $r(F) = \text{Hom}_{\mathcal{F}}(S_*, F)$ , with the Steenrod operations acting on the right of the  $S_j$  in the obvious way. Let  $I_\lambda \in \mathcal{F}$  be the injective envelope of the simple functor  $F_\lambda$ , and let  $\Phi^{-1}S^j \in \mathcal{F}$  be defined by

$$\Phi^{-1}S^j = \text{colim} \left\{ S^j \xrightarrow{\Phi} S^{2j} \xrightarrow{\Phi} S^{4j} \xrightarrow{\Phi} S^{8j} \dots \right\},$$

where  $\Phi : S^j \rightarrow S^{2j}$  is the squaring map.

The ‘‘Vanishing Theorem’’ of [K6] says that  $\Phi^{-1}S^j$  is an injective object in the category  $\mathcal{F}_\omega \subset \mathcal{F}$  of locally finite functors. It follows formally that there is a decomposition in  $\mathcal{F}$

$$\Phi^{-1}S^j \simeq \bigoplus_{\lambda \in \Lambda} a(\lambda, j) I_\lambda.$$

Precomposing this with the functor  $S_n$ , and then applying the functor  $r$ , yields a decomposition in  $\mathcal{U}$

$$\Phi^{-1}r(S^j \circ S_n) \simeq \bigoplus_{\lambda \in \Lambda} a(\lambda, j) r(I_\lambda \circ S_n).$$

The classical description of  $H^*(K(V, n); \mathbf{Z}/2)$  reveals that  $r(I_\lambda \circ S_n) = H^*(K(\lambda, n); \mathbf{Z}/2)$ , so the righthand side of this last decomposition agrees with the righthand side of the isomorphism in Theorem 1.9. Meanwhile, the lefthand side of the isomorphism of Theorem 1.9 is known by Theorem 1.2; this is then shown to agree with  $\Phi^{-1}r(S^j \circ S_n)$  by using a new result of ours [K8] that calculates  $r(S^j \circ F)$  as a functor of  $r(F)$ .

§8 contains the details of this outline of the proof of Theorem 1.9. Finally in §9, we comment on Proposition 1.11, as well as possible ways of attacking the conjectures.

## 2. GEOMETRIC CONSTRUCTIONS

We begin by being a bit more specific about some notation introduced in the introduction. A point  $\mathbf{c} \in \mathcal{C}(n, j)$  is a  $j$  tuple  $\mathbf{c} = (c_1, \dots, c_j)$  in which each  $c_i : I^n \rightarrow I^n$  is a product of  $n$  linear embeddings from the unit interval  $I$  to itself, and the interiors of the images of the  $c_i$  are disjoint. Then the book of Gaunce Lewis, et. al. [LMMS] shows that the functor

$$D_{n,j}X = \mathcal{C}(n, j)_+ \wedge_{\Sigma_j} X^{[j]}$$

is well defined in the category of spectra.

Standard properties of equivariant homotopy then allow us to write

$$\begin{aligned} T(n, j) &= F(D_{n,j}S^{-n}, S^0) \\ &= F(\mathcal{C}(n, j)_+ \wedge_{\Sigma_j} S^{-nj}, S^0) \\ &= F(\mathcal{C}(n, j)_+, S^{nj})^{\Sigma_j}. \end{aligned}$$

This gives an interesting alternative (and technically simpler) definition of the spectra  $T(n, j)$ , reminiscent of some of the constructions recently occurring in the ‘‘Goodwillie Calculus’’ literature [AM]<sup>4</sup>

<sup>4</sup>In work in progress, we are exploring this idea more thoroughly than is needed here.

**Definition 2.1.** Let  $\tilde{D}_{n,j}X = F(\mathcal{C}(n,j)_+, X^{[j]})^{\Sigma_j}$ .

With this definition, we have  $T(n,j) = \tilde{D}_{n+1,j}S^n$ , and, more generally, if  $X$  is a finite spectrum, then  $\tilde{D}_{n,j}X = \text{S-dual}(D_{n,j}(\text{S-dual}(X)))$ .

In the usual way, the little cubes operad structure on the spaces  $\mathcal{C}(n,j)$  induces natural maps

$$\mu : D_{n,i}X \wedge D_{n,j}X \rightarrow D_{n,i+j}X,$$

$$\Theta : D_{n,i}D_{n,j}X \rightarrow D_{n,ij}X,$$

and dually, natural maps

$$\Psi : \tilde{D}_{n,i+j}X \rightarrow \tilde{D}_{n,i}X \wedge \tilde{D}_{n,j}X,$$

and

$$\Gamma : \tilde{D}_{n,ij}X \rightarrow \tilde{D}_{n,i}\tilde{D}_{n,j}X.$$

In particular, we obtain maps

$$\Psi : T(n, i+j) \rightarrow T(n, i) \wedge T(n, j),$$

and

$$\Gamma : T(n, 2j) \rightarrow \tilde{D}_{n+1,2}T(n, j).$$

These two families of maps provide sufficient structure for the purposes of computing the mod 2 cohomology of the  $T(n, j)$ .

We turn our attention to constructing the maps

$$\delta : T(n, j) \rightarrow \Sigma^{-1}T(n+1, j).$$

In [K1] we noted that the evaluation map

$$\epsilon : \Sigma\Omega^{n+1}\Sigma^{n+1}X \rightarrow \Omega^n\Sigma^{n+1}X$$

induces maps

$$\epsilon : \Sigma D_{n+1,j}X \rightarrow D_{n,j}\Sigma X.$$

We note that the same geometric construction also yields natural maps

$$\delta : \tilde{D}_{n,j}X \rightarrow \Sigma^{-1}\tilde{D}_{n+1,j}\Sigma X.$$

Both of these families are induced by explicit  $\Sigma_j$  equivariant maps

$$\beta : \mathcal{C}(n+1, j)_+ \wedge S^1 \rightarrow \mathcal{C}(n, j)_+ \wedge S^j,$$

defined as follows.

Given a linear embedding  $c : I \rightarrow I$ , let  $c^* : I \rightarrow I$  be the associated ‘‘Thom-Pontryagin collapse’’ map. Explicitly,

$$c^*(t) = \begin{cases} 0 & \text{if } t \leq \text{Im}(c) \\ s & \text{if } c(s) = t \\ 1 & \text{if } t \geq \text{Im}(c). \end{cases}$$

Note that  $(c \circ d)^* = d^* \circ c^*$ .

Given a little  $n+1$  cube  $c : I^{n+1} \rightarrow I^{n+1}$ , we write  $c = c' \times c''$ , where  $c' : I^n \rightarrow I^n$ , and  $c'' : I \rightarrow I$ . Regarding  $S^1$  as  $I/\partial I$ , and  $S^j$  as  $(I/\partial I)^{[j]}$ , we have the following definition.

**Definition 2.2.** (Compare with [May, page 47].)

$$\beta(c_1, \dots, c_j, t) = (c'_1, \dots, c'_j, c_1''(t), \dots, c_j''(t)).$$

A straightforward check of definitions yields the next proposition, which shows how  $\delta$  is related to the maps  $\Psi$  and  $\Gamma$ .

**Proposition 2.3.**

(1) *The composite  $\Sigma \tilde{D}_{n,i+j} X \xrightarrow{\delta} \tilde{D}_{n+1,i+j} \Sigma X \xrightarrow{\Psi} \tilde{D}_{n+1,i} \Sigma X \wedge \tilde{D}_{n+1,j} \Sigma X$  is null if  $i > 0$  and  $j > 0$ .*

(2) *There are commutative diagrams:*

$$\begin{array}{ccc} \Sigma \tilde{D}_{n,ij} X & \xrightarrow{\delta} & \tilde{D}_{n+1,ij} X \\ \downarrow \Sigma \Gamma & & \downarrow \Gamma \\ \Sigma \tilde{D}_{n,i} \tilde{D}_{n,j} X & \xrightarrow{\delta} & \tilde{D}_{n+1,i} \Sigma \tilde{D}_{n,j} X \xrightarrow{\tilde{D}_{n+1,i} \delta} \tilde{D}_{n+1,i} \tilde{D}_{n+1,j} \Sigma X. \end{array}$$

Our last and most delicate construction is of the family

$$\Phi : T(n, j) \rightarrow T(n, 2j).$$

The next theorem summarizes the properties we need to know.

**Theorem 2.4.** *There exist maps  $\Phi_{n,j} : T(n, j) \rightarrow T(n, 2j)$  such that the following five properties hold.*

(1)  $\Phi_{0,j} : T(0, j) = S^0 \rightarrow T(0, 2j) = S^0$  is multiplication by  $(2j)!/j!2^j$ .

(2) *There are commutative diagrams:*

$$\begin{array}{ccc} \Sigma T(n, j) & \xrightarrow{\Sigma \Phi_{n,j}} & \Sigma T(n, 2j) \\ \downarrow \delta & & \downarrow \delta \\ T(n+1, j) & \xrightarrow{\Phi_{n+1,j}} & T(n+1, 2j). \end{array}$$

(3) *For  $n \geq 1$ , there are commutative diagrams:*

$$\begin{array}{ccc} T(n, i+j) & \xrightarrow{\Phi_{n,i+j}} & T(n, 2(i+j)) \\ \downarrow \Psi & & \downarrow \Psi \\ T(n, i) \wedge T(n, j) & \xrightarrow{\Phi_{n,i} \wedge \Phi_{n,j}} & T(n, 2i) \wedge T(n, 2j). \end{array}$$

(4) *If  $n \geq 1$ ,  $i$  and  $j$  are odd, and  $i+j = 2k$ , the composite*

$$T(n, k) \xrightarrow{\Phi_{n,k}} T(n, 2k) \xrightarrow{\Psi} T(n, i) \wedge T(n, j)$$

is null.

(5) For  $n \geq 1$ , there are commutative diagrams:

$$\begin{array}{ccc} T(n, 2j) & \xrightarrow{\Phi_{n,2j}} & T(n, 4j) \\ \downarrow \Gamma & & \downarrow \Gamma \\ \tilde{D}_{n,2}T(n, j) & \xrightarrow{\tilde{D}_{n,2}\Phi_{n,j}} & \tilde{D}_{n,2}T(n, 2j). \end{array}$$

*Proof.* Fix  $N \geq 0, J \geq 0$ . Let  $\mathcal{S}(N, J)$  be the collection of sets of maps  $S = \{ \Phi_{n,j} \mid n \leq N, j \leq J \}$  such that properties (1) – (5) are true whenever the maps  $\Phi_{n,j}$  appearing in those statements are chosen from  $S$ . (In other words,  $S \in \mathcal{S}(N, J)$  makes true a finite number of the infinite lists of statements in (1) – (5).)

There are restriction maps  $\mathcal{S}(N, J) \rightarrow \mathcal{S}(N-1, J)$  and  $\mathcal{S}(N, J) \rightarrow \mathcal{S}(N, J-1)$ . The theorem amounts to saying that the inverse limit,  $\lim \mathcal{S}(N, J)$ , taken over all  $N$  and  $J$ , is nonempty.

Since (1) and (2) determine  $\Phi_{0,j}$  and  $\Phi_{n,0}$ ,  $\mathcal{S}(N, J)$  can be regarded as a subset of  $\prod_{n=1}^N \prod_{j=1}^J \{T(n, j), T(n, 2j)\}$ , which is finite, as each  $T(n, j)$  is a finite complex, and each  $T(n, j)$  with  $n \geq 1, j \geq 2$  is torsion. Since the inverse limit of nonempty finite sets is nonempty<sup>5</sup>, the next theorem completes the proof of the theorem.  $\square$

**Theorem 2.5.**  $\mathcal{S}(N, J)$  is nonempty.

There are two ingredients in our construction of a set  $\{\Phi_{n,j}\} \in \mathcal{S}(N, J)$ . The first is the use of vector bundle trivializations to construct natural equivalences

$$\omega_{n,j} : D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j} X,$$

for  $n$  and  $j$  in any finite range, compatible with the structure maps  $(\epsilon, \mu, \Theta)$ . The second is the use of Hopf invariants to construct maps, for  $d > n$ ,

$$h_{n,j}^d : D_{n+1,2j} S^{d-n} \rightarrow D_{n+1,j} S^{2d-n}$$

with appropriate properties.

The next two theorems, whose proofs occupy the next two sections, more precisely describe what we need.

**Theorem 2.6.** Fix  $N$  and  $J$ . Then there exists  $d > 0$ , and natural equivalences

$$\omega_{n,j} : D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j} X,$$

defined for  $1 \leq n \leq N, 1 \leq j \leq J$ , such that the following diagrams commute:

(1) for all  $1 \leq n \leq N-1, 1 \leq j \leq J$ ,

$$\begin{array}{ccc} \Sigma D_{n+1,j}(\Sigma^d X) & \xrightarrow{\omega_{n+1,j}} & \Sigma^{1+dj} D_{n+1,j}(X) \\ \downarrow \epsilon & & \downarrow (-1)^{d(j-1)} \epsilon \\ D_{n,j}(\Sigma^{d+1} X) & \xrightarrow{\omega_{n,j}} & \Sigma^{dj} D_{n,j}(\Sigma X), \end{array}$$

<sup>5</sup>A standard application of the Tychonoff Theorem.

(2) for all  $1 \leq n \leq N, i + j \leq J$ ,

$$\begin{array}{ccc} D_{n,i}(\Sigma^d X) \wedge D_{n,j}(\Sigma^d X) & \xrightarrow{\omega_{n,i} \wedge \omega_{n,j}} & \Sigma^{di} D_{n,i}(X) \wedge \Sigma^{dj} D_{n,j}(X) \\ \downarrow \mu & & \downarrow \mu \\ D_{n,i+j}(\Sigma^d X) & \xrightarrow{\omega_{n,i+j}} & \Sigma^{d(i+j)} D_{n,i+j}(X), \end{array}$$

(3) for all  $1 \leq n \leq N, ij \leq J$ ,

$$\begin{array}{ccc} D_{n,i} D_{n,j}(\Sigma^d X) & \xrightarrow{D_{n,i} \omega_{n,j}} & D_{n,i} \Sigma^{dj} D_{n,j}(X) & \xrightarrow{\omega_{n,i}^j} & \Sigma^{dij} D_{n,i} D_{n,j}(X) \\ \downarrow \Theta & & & & \downarrow \Theta \\ D_{n,ij}(\Sigma^d X) & \xrightarrow{\omega_{n,ij}} & \Sigma^{dij} D_{n,ij}(X). \end{array}$$

**Theorem 2.7.** For all  $0 \leq n < d$  and for all  $j$ , there exist maps

$$h_{n,j}^d : D_{n+1,2j} S^{d-n} \rightarrow D_{n+1,j} S^{2d-n}$$

with the following properties.

(1) If  $d$  is even,  $h_{0,j}^d : D_{1,2j} S^d = S^{2jd} \rightarrow D_{1,j} S^{2d} = S^{2jd}$  is multiplication by  $(2j)!/j!2^j$ .

(2) There are commutative diagrams:

$$\begin{array}{ccc} \Sigma D_{n+1,2j} S^{d-n} & \xrightarrow{\Sigma h_{n,j}^d} & \Sigma D_{n+1,j} S^{2d-n} \\ \downarrow \epsilon & & \downarrow \epsilon \\ D_{n,2j} S^{d-n+1} & \xrightarrow{h_{n-1,j}^d} & D_{n,j} S^{2d-n+1}. \end{array}$$

(3) There are commutative diagrams:

$$\begin{array}{ccc} D_{n+1,2i} S^{d-n} \wedge D_{n+1,2j} S^{d-n} & \xrightarrow{h_{n,i}^d \wedge h_{n,j}^d} & D_{n+1,i} S^{2d-n} \wedge D_{n+1,j} S^{2d-n} \\ \downarrow \mu & & \downarrow \mu \\ D_{n+1,2(i+j)} S^{d-n} & \xrightarrow{h_{n,i+j}^d} & D_{n+1,i+j} S^{2d-n}. \end{array}$$

(4) If  $i$  and  $j$  are odd, and  $i + j = 2k$ , the composite

$$D_{n+1,i} S^{d-n} \wedge D_{n+1,j} S^{d-n} \xrightarrow{\mu} D_{n+1,2k} S^{d-n} \xrightarrow{h_{n,k}^d} D_{n+1,k} S^{2d-n}$$

is null.

(5) There are commutative diagrams:

$$\begin{array}{ccc} D_{n,2} D_{n+1,2j} S^{d-n} & \xrightarrow{D_{n,2} h_{n,j}^d} & D_{n,2} D_{n+1,j} S^{2d-n} \\ \downarrow \Theta & & \downarrow \Theta \\ D_{n+1,4j} S^{d-n} & \xrightarrow{h_{n,2j}^d} & D_{n+1,2j} S^{2d-n}. \end{array}$$

Assuming these two theorems, we note that Theorem 2.5 follows easily. First choose  $d$  as in Theorem 2.6 (but with  $J$  replaced by  $2J$ ). We can also assume  $d$  is even. Then, with  $h_{n,j}^d$  as in Theorem 2.7, we define  $\Phi_{n,j} : T(n, j) \rightarrow T(n, 2j)$  to be the S-dual of the composite

$$D_{n+1,2j}S^{-n} \xrightarrow{\omega_{n,2j}^{-1}} \Sigma^{-2dj}D_{n+1,2j}S^{d-n} \xrightarrow{h_{n,j}^d} \Sigma^{-2dj}D_{n+1,j}S^{2d-n} \xrightarrow{\omega_{n,j}^2} D_{n+1,j}S^{-n}.$$

Courtesy of Theorem 2.6, each statement in Theorem 2.7 translates immediately into the corresponding statement in Theorem 2.4, proving Theorem 2.5.

### 3. STRUCTURED PERIODICITY

In this section we prove Theorem 2.6, which asserts that given  $N$  and  $J$ , there exists  $d > 0$  and natural equivalences

$$\omega_{n,j} : D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj}D_{n,j}X,$$

defined for  $1 \leq n \leq N, 1 \leq j \leq J$  which are appropriately compatible with the three families of structure maps

$$\begin{aligned} \epsilon &: \Sigma D_{n+1,j}X \rightarrow D_{n,j}\Sigma X, \\ \mu &: D_{n,i}X \wedge D_{n,j}X \rightarrow D_{n,i+j}X, \text{ and} \\ \Theta &: D_{n,i}D_{n,j}X \rightarrow D_{n,ij}X. \end{aligned}$$

To put this theorem in context, recall that as an aid to constructing power operations and studying Thom isomorphisms, the authors of [BMMS] defined the notion of an  $H_\infty^d$ -ring spectrum. For the sphere spectrum  $S^0$  to admit an  $H_\infty^d$  structure would be roughly equivalent to natural equivalences  $\omega_{n,j}$  as in the theorem for all  $n < \infty, j < \infty$ . Though it is easy to see that this cannot be done, our theorem says that it partially *can* be. If one defines the notion of an  $H_n^d$  structure in the obvious way, we know of no reason why the following conjecture might not be true.

**Conjecture 3.1.** Localized at a prime  $p$ , for each  $n$ ,  $S^0$  admits the structure of an  $H_n^d$ -ring spectrum for some  $d > 0$ .

The origin of the natural equivalences is as follows.

Suppose  $\xi$  and  $\zeta$  are two  $r$  dimensional vector bundle over a space  $B$ , respectively classified by maps  $f_\xi, f_\zeta : B \rightarrow BO$ . Then a homotopy  $H : B \times I \rightarrow BO$  between  $f_\xi$  and  $f_\zeta$  induces a bundle isomorphism  $\omega_H : \xi \rightarrow \zeta$  and thus a homeomorphism  $\omega_H : M(\xi) \rightarrow M(\zeta)$  of Thom spaces. In particular, given a map  $i : B \rightarrow C$  to a contractible space  $C$ , and an extension  $F : C \rightarrow BO$  of  $f_\xi$ , there is an induced homeomorphism of spaces

$$\omega_F : M(\xi) \rightarrow \Sigma^r(B_+).$$

Furthermore, given a second extension  $F' : C' \rightarrow BO$ ,  $\omega_F$  and  $\omega_{F'}$  will be homotopic if the map

$$F \cup_{f_\xi} F' : C \cup_B C' \rightarrow BO$$

is null. This last map can be regarded an obstruction  $o(F, F') : \Sigma B \rightarrow BO$ .

We apply these general remarks to the case of interest. Let  $\xi_{n,j}$  be the vector bundle

$$\mathcal{C}(n, j) \times_{\Sigma_j} \mathbf{R}^j \rightarrow B(n, j) = \mathcal{C}(n, j)/\Sigma_j,$$

with classifying map  $f_{n,j} : B(n,j) \rightarrow BO$ . This is easily seen to be a bundle of finite order, and an extension  $F : CB(n,j) \rightarrow BO$  of  $df_{n,j}$  to the cone on  $B(n,j)$  induces a homeomorphism

$$\omega_F : \mathcal{C}(n,j)_+ \wedge_{\Sigma_j} S^{dj} \rightarrow \Sigma^{ds}(B(n,j)_+),$$

and thus a  $\Sigma_j$ -equivariant homeomorphism

$$\omega_F : \mathcal{C}(n,j)_+ \wedge S^{dj} \rightarrow \Sigma^{ds}(\mathcal{C}(n,j)_+),$$

and finally a natural equivalence

$$\omega_F : D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j} X.$$

A straightforward check of definitions shows

**Lemma 3.2.** *In this situation, if  $F : CB(n,j) \rightarrow BO$  is the restriction of a map  $F' : CB(n+1,j) \rightarrow BO$  extending  $df_{n+1,j}$  then the following diagram commutes:*

$$\begin{array}{ccc} \Sigma D_{n+1,j}(\Sigma^d X) & \xrightarrow{\omega_{F'}} & \Sigma^{1+dj} D_{n+1,j}(X) \\ \downarrow \epsilon & & \downarrow (-1)^{d(j-1)\epsilon} \\ D_{n,j}(\Sigma^{d+1} X) & \xrightarrow{\omega_F} & \Sigma^{dj} D_{n,j}(\Sigma X). \end{array}$$

Now fix  $N$  and  $J$  as in Theorem 2.6. Let  $d > 0$  and let  $\mathcal{F} = \{F_j : CB(N,j) \rightarrow BO \mid j = 1, \dots, J\}$  be a collection of extensions of the maps  $df_{N,j}$ . We define the obstruction set  $o(\mathcal{F})$  to be the following set of maps:

$$o_{i,j}^\mu(\mathcal{F}) : \Sigma(B(N,i) \times B(N,j)) \rightarrow BO,$$

for  $i+j = J$ , and

$$o_{i,j}^\Theta(\mathcal{F}) : \Sigma(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i) \rightarrow BO,$$

for  $ij = J$ , where these maps are defined as follows.

For  $o_{i,j}^\mu(\mathcal{F})$ , we regard  $\Sigma(B(N,i) \times B(N,j))$  as

$$C(B(N,i) \times B(N,j)) \cup_{B(N,i) \times B(N,j)} CB(N,i) \times CB(N,j),$$

and we let

$$o_{i,j}^\mu(\mathcal{F}) = \begin{cases} F_{i+j} \circ \mu & \text{on } C(B(N,i) \times B(N,j)), \\ \mu_{BO} \circ (F_i \times F_j) & \text{on } CB(N,i) \times CB(N,j). \end{cases}$$

Here  $\mu_{BO} : BO \times BO \rightarrow BO$  is the H-space structure map.

For  $o_{i,j}^\Theta(\mathcal{F})$ , we regard  $\Sigma(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i)$  as

$$C(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i) \cup_{\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i} \mathcal{C}(N,i) \times_{\Sigma_i} CB(N,j)^i,$$

and we let

$$o_{i,j}^\Theta(\mathcal{F}) = \begin{cases} F_{ij} \circ \Theta & \text{on } C(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i), \\ \Theta_{BO} \circ (Id \times_{\Sigma_i} (F_j)^i) & \text{on } \mathcal{C}(N,i) \times_{\Sigma_i} CB(N,j)^i. \end{cases}$$

Here  $\Theta_{BO} : \mathcal{C}(n,i) \times_{\Sigma_i} BO^i \rightarrow BO$  is the infinite loop space structure map.

Theorem 2.6 will follow if we can show that there is a choice of  $d$  and  $\mathcal{F}$  for which  $o(\mathcal{F})$  is a set of null maps. Firstly, we note that there *do* exist collections

$\mathcal{F}$  as above: we just need to choose  $d$  equal to a common multiple of the orders of the bundles  $\xi_{N,1}, \dots, \xi_{N,J}$ . By making  $d$  possibly bigger, we can even ensure that  $\mathcal{F}$  is the restriction of a similar family  $\tilde{\mathcal{F}}$  defined for the pair  $(N+1, J)$ , and the obstruction set  $o(\mathcal{F})$  is the restriction of  $o(\tilde{\mathcal{F}})$ .

Given a family  $\mathcal{F}$ , let  $r\mathcal{F}$  be the family with  $j^{\text{th}}$  function equal to  $rF_j$ . Note that if  $F_j$  extends  $df_{N,j}$ , then  $rF_j$  extends  $(rd)f_{N,j}$ . It is easy to check

**Lemma 3.3.**

- (1)  $o_{i,j}^\mu(r\mathcal{F}) = ro_{i,j}^\mu(\mathcal{F}) \in K^1(B(N, i) \times B(N, j))$ .
- (2)  $o_{i,j}^\ominus(r\mathcal{F}) = ro_{i,j}^\ominus(\mathcal{F}) \in K^1(\mathcal{C}(N, i) \times_{\Sigma_i} B(N, j)^i)$ .

**Proposition 3.4.** *Let  $X(N)$  be one of the spaces  $B(N, j)$ ,  $B(N, i) \times B(N, j)$ , or  $\mathcal{C}(n, i) \times_{\Sigma_i} B(N, j)^i$ . If  $x \in K^*(X(N))$  is in the image of the restriction from  $K^*(X(N+1))$ , then  $x$  is torsion.*

Postponing the proof of this proposition for the moment, we show that there is a choice of  $d$  and  $\mathcal{F}$  for which  $o(\mathcal{F})$  is a set of null maps. Start with any family  $\mathcal{F}$  (and associated  $d$ ) as above. Let  $r$  be a common multiple of the orders of the obstructions  $o_{i,j}^\mu(\mathcal{F})$  and  $o_{i,j}^\ominus(\mathcal{F})$ . (Proposition 3.4 tells us that these elements *do* have finite order.) Then the family  $r\mathcal{F}$  has an obstruction set consisting only of null maps, as needed.

It remains to prove Proposition 3.4. This will follow from three lemmas.

**Lemma 3.5.** *Let  $f : X \rightarrow Y$  be a map between finite complexes. If  $H_*(f; \mathbf{Q}) = 0$ , then  $\text{Im}\{E^*(f) : E^*(Y) \rightarrow E^*(X)\}$  is torsion for all generalized cohomology theories  $E^*$ .*

*Proof.* For finite complexes  $Z$ ,  $E^*(Z_{\mathbf{Q}}) \simeq E^*(Z) \otimes \mathbf{Q}$ .  $H_*(f; \mathbf{Q}) = 0$  implies that  $f_{\mathbf{Q}} \simeq *$ , and thus that  $E^*(f) \otimes \mathbf{Q} = 0$ .  $\square$

**Lemma 3.6.** *If  $X(N)$  is as in Proposition 3.4,  $X(N)$  has the homotopy type of a finite complex.*

*Proof.* There are many ways to see this. The author's favorite is to note that the explicit cell decomposition for  $B(2, j)$  given by Fox and Neuwirth in [FN] generalizes to  $B(n, j)$ :  $B(n, j)$  has the homotopy type of an  $(n-1)(j-1)$  dimensional cell complex with exactly  $n^{j-1}$  cells.  $\square$

**Lemma 3.7.** *With  $X(N)$  as in Proposition 3.4,*

$$H_*(X(N); \mathbf{Q}) \rightarrow H_*(X(N+1); \mathbf{Q})$$

*is 0.*

*Proof.* This follows from standard homology calculations [CLM].  $\square$

## 4. HOPF INVARIANTS

In this section we use Hopf invariants to define maps

$$h_{n,j}^d : D_{n+1,2j}S^{d-n} \rightarrow D_{n+1,j}S^{2d-n},$$

for  $0 \leq n < d$ , and then show that they have the properties listed in Theorem 2.7.

The maps are not hard to define. Let

$$H_Y : \Omega\Sigma Y \rightarrow \Omega\Sigma(Y \wedge Y)$$

be the classic Hopf invariant. Replacing  $Y$  by  $\Sigma^n X$ , and looping  $n$  times, defines an unstable natural map

$$\Omega^n H_{\Sigma^n X} : \Omega^{n+1}\Sigma^{n+1}X \rightarrow \Omega^{n+1}\Sigma^{n+1}(\Sigma^n X \wedge X).$$

Now let  $D_n X$  denote  $\bigvee_{j=1}^{\infty} D_{n,j}X$ , and, for connected  $X$ , let

$$s_n : D_n X \simeq \Omega^n \Sigma^n X$$

be the natural *stable* Snaith equivalence as studied in [LMMS, Chapter VII]. Finally,

$$H_n(X) : D_{n+1}X \rightarrow D_{n+1}(\Sigma^n X \wedge X)$$

will be the stable map given by the composite  $s_{n+1}^{-1} \circ (\Omega^n H_{\Sigma^n X}) \circ s_{n+1}$ .

**Definition 4.1.** For all  $0 \leq n < d$ , and for all  $j$ ,

$$h_{n,j}^d : D_{n+1,2j}S^{d-n} \rightarrow D_{n+1,j}S^{2d-n}$$

is defined to be the  $(2j, j)^{th}$  component of  $H_n(S^{d-n})$ .

The first of the properties in Theorem 2.7 is easily checked. If  $d$  is even,  $h_{0,j}^d : S^{2jd} \rightarrow S^{2jd}$  is multiplication by  $(2j)!/j!2^j$ , as cup product considerations easily show that  $H : \Omega S^{d+1} \rightarrow S^{2d+1}$  induces multiplication by this number in cohomology in dimension  $2dj$  [H, p.294].

Property (2) of Theorem 2.7, the compatibility of  $h_{n,j}^d$  with the maps  $\epsilon$ , follows from the main result of [K1]: under the Snaith equivalence, the evaluation

$$\epsilon : \Sigma\Omega^{n+1}\Sigma^{n+1}X \rightarrow \Omega^n \Sigma^{n+1}X$$

is carried to

$$\bigvee_{j=1}^{\infty} \epsilon : \bigvee_{j=1}^{\infty} \Sigma D_{n+1,j}X \rightarrow \bigvee_{j=1}^{\infty} D_{n,j}\Sigma X.$$

The remaining three properties follow from the next two propositions.

**Proposition 4.2.** *There is a commutative diagram:*

$$\begin{array}{ccc} D_n D_{n+1} X & \xrightarrow{D_n H_n(X)} & D_n D_{n+1}(\Sigma^n X \wedge X) \\ \downarrow \ominus & & \downarrow \ominus \\ D_{n+1} X & \xrightarrow{H_n(X)} & D_{n+1}(\Sigma^n X \wedge X). \end{array}$$

Here  $\Theta : D_n D_{n+1} X \rightarrow D_{n+1} X$  is the restriction of the structure map  $\Theta : D_{n+1} D_{n+1} X \rightarrow D_{n+1} X$ .

**Proposition 4.3.** *The  $(i, j)^{th}$  component of  $H_n(X)$  is null unless  $i \leq 2j$ .*

This tells us that  $H_n(X)$  can be regarded as an "upper triangular matrix" of maps. With this information fed into Proposition 4.2, the three last properties of Theorem 2.7 can be read off immediately.

*Proof of Proposition 4.3.* The  $(i, j)^{th}$  component of  $H_n(X)$  is a natural transformation

$$D_{n+1, i} X \rightarrow D_{n+1, j} (\Sigma^n X \wedge X).$$

In the terminology of [Goo], the domain is a homogeneous functor of degree  $i$ , while the range is a functor of degree  $2j$ . Thus there are *no* nontrivial natural transformations from the former to the latter if  $i > 2j$ .  $\square$

*Remark 4.4.* This proposition presumably has a direct proof, along the lines of the proofs of similar results in [K4].

*Proof of Proposition 4.2.* This is a consequence of the fact that  $H_n(X)$  corresponds to an  $n$  fold loop map. Let  $C_n X$  denote the usual approximation to  $\Omega^n \Sigma^n X$ , with monad structure map  $\Theta : C_n C_n \rightarrow C_n$ , and let  $Y$  denote  $\Sigma^n X \wedge X$ .

With this notation, we assert that there is a commutative digram:

$$\begin{array}{ccccc}
 D_n D_{n+1} X & \xrightarrow{D_n H_n(X)} & & & D_n D_{n+1} Y \\
 \downarrow \Theta & \searrow^{D_n s_{n+1}} & D_n C_{n+1} X & \xrightarrow{D_n(\Omega^n H)} & D_n C_{n+1} Y & \nearrow^{D_n s_{n+1}^{-1}} & \downarrow \Theta \\
 & & \downarrow s_n & & \downarrow s_n & & \\
 & & C_n C_{n+1} X & \xrightarrow{C_n(\Omega^n H)} & C_n C_{n+1} Y & & \\
 & & \downarrow \Theta & & \downarrow \Theta & & \\
 & & C_{n+1} X & \xrightarrow{\Omega^n H} & C_{n+1} Y & & \\
 \downarrow \Theta & \nearrow^{s_{n+1}} & & & & & \downarrow \Theta \\
 D_{n+1} X & \xrightarrow{H_n(X)} & & & & & D_{n+1} Y \\
 & & & & & & \nearrow^{s_{n+1}^{-1}}
 \end{array}$$

The lower central square commutes since  $\Omega^n H$  is a  $C_n$ -map. The upper square commutes by naturality. Finally the argument in [K3, §4] shows that the two side trapezoids commute.  $\square$

## 5. COHOMOLOGY CALCULATIONS

We use the following notational conventions in the next three sections.  $H_*(X)$  and  $H^*(X)$  will denote homology and cohomology with  $\mathbf{Z}/2$  coefficients. The binomial coefficient  $\binom{b}{a}$  is defined, for *all* integers  $a$  and  $b$ , as the  $a^{\text{th}}$  Taylor coefficient of  $(x+1)^b$  if  $a \geq 0$ , and 0 otherwise. We will use, without further comment, that  $\binom{b}{a} = \binom{a-b-1}{a}$ .

In this section we describe  $H^*(T(n, *))$ , and the maps  $\Psi^*$ ,  $\Phi^*$ , and  $\delta^*$ , in terms of “dual” Dyer-Lashof operations. We begin by remarking that since  $T(n, j)$  is the S-dual of  $D_{n+1, j}S^{-n}$ , and  $H_*(D_{n+1, j}S^{-n})$  embeds in  $H_*(D_{\infty, j}S^{-n})$ , we will not need to confront the Browder operations, and the “top” Dyer-Lashof operation will be additive (as are the others).

As part of the general theory [CLM], the product maps  $\mu$  induce a bigraded product on  $H_*(D_{n+1, *}S^{-n})$ , and associated to the structure maps  $\Theta$ , there are Dyer-Lashof operations

$$Q^s : H_q(D_{n+1, j}S^{-n}) \rightarrow H_{q+s}(D_{n+1, 2j}S^{-n}).$$

These are defined for  $s \leq q + n$ , and are 0 for  $s < q$ . Furthermore, these satisfy the Cartan formula, Adem relations, and restriction axiom:  $Q^{|x|}x = x^2$ .  $H_*(D_{n+1, *}S^{-n})$  is the free object with all this structure, generated by a class in degree  $-n$ .

There is a canonical isomorphism  $H^q(T(n, j)) = H_q(D_{n+1, j}S^{-n})$ . Under this isomorphism,  $\Psi^*$  will correspond to  $\mu_*$ , and will induce a bigraded product (occasionally denoted “ $*$ ”) on  $H^*(T(n, *))$ . We define operations

$$\tilde{Q}^s : H^q(T(n, j)) \rightarrow H^{q+s}(T(n, 2j))$$

to correspond to

$$Q^{-s} : H_{-q}(D_{n+1, j}S^{-n}) \rightarrow H_{-q-s}(D_{n+1, 2j}S^{-n}).$$

These are defined for  $s \geq q - n$ , and are 0 for  $s > q$ . These satisfy the Cartan formula,

$$\tilde{Q}^t(x * y) = \sum_{r+s=t} \tilde{Q}^r x * \tilde{Q}^s y,$$

Adem relations,

$$\tilde{Q}^r \tilde{Q}^s x = \sum_i \binom{s-i-1}{r-2i} \tilde{Q}^{r+s-i} \tilde{Q}^i x,$$

and restriction axiom,

$$\tilde{Q}^{|x|}x = x^2.$$

(We note that in the Adem relations, whenever the iterated operation on the left is defined, so are those appearing with nonzero coefficient on the right, though not conversely<sup>6</sup>.)

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<sup>6</sup>The relation  $\tilde{Q}^1 \tilde{Q}^2 = \tilde{Q}^3 \tilde{Q}^0$  illustrates this.

**Theorem 5.1.**  $H^*(T(n, *))$  is the free object with all this structure, generated by a class  $x_n$  in degree  $n$ . Explicitly, if

$$\tilde{R}_n = \langle \tilde{Q}^I x_n \mid I \text{ is admissible} \rangle / \langle \tilde{Q}^I x_n \mid I \text{ is admissible and } e(I) > n \rangle,$$

$H^*(T(n, *)) = S^*(\tilde{R}_n) / (\tilde{Q}^{|x|} x - x^2)$ . Thus, as a bigraded algebra,  $H^*(T(n, *))$  is a polynomial algebra on the set  $\{\tilde{Q}^I x_n \mid I \text{ is admissible and } e(I) < n\}$ , with  $\tilde{Q}^I x_n \in H^*(T(n, 2^{l(I)}))$ .

Here, if  $I = (i_1, \dots, i_l)$ ,  $\tilde{Q}^I = \tilde{Q}^{i_1} \dots \tilde{Q}^{i_l}$ , and  $e(I)$ ,  $l(I)$ , and *admissible* mean what they did in §1. There is a little wrinkle here however: as  $\tilde{Q}^0$  is not the identity, an admissible sequence can end with 0's.

The geometric results of §2 allow us to quickly deduce the behavior of  $\delta^*$  and  $\Phi^*$ .

**Proposition 5.2.**  $\delta^* : H^{*+1}(T(n+1, j)) \rightarrow H^*(T(n, *))$  is determined by

- (1)  $\delta^*(\tilde{Q}^I x_{n+1}) = \tilde{Q}^I x_n$ , and
- (2)  $\delta^*$  is 0 on decomposables.

*Proof.* This follows from Proposition 2.3, and the fact that Dyer–Lashof operations commute with the evaluation [CLM, p.6, p.218].  $\square$

**Proposition 5.3.**  $\Phi^* : H^*(T(n, *)) \rightarrow H^*(T(n, *))$  is determined by

- (1) When  $n = 0$ ,  $\Phi^*(x_0^{2^j}) = x_0^j$ .
- (2)  $\Phi^*(\tilde{Q}^s x) = \tilde{Q}^s(\Phi^* x)$  if  $s > |x| - n$ .
- (3) Whenever the iterated operation  $\tilde{Q}^I x_n$  is defined,  $\Phi^*(\tilde{Q}^I x_n) = \tilde{Q}^{I'} x_n$  if  $I = (I', 0)$ , and is 0 otherwise.
- (4) When  $n \geq 1$ ,  $\Phi^*$  is an algebra map (with the second grading in the domain of  $\Phi^*$  doubled).

*Proof.* This follows from Theorem 2.4 and the last proposition. As  $(2j)!/j!2^j$  is always odd, statement (1) of Theorem 2.4 implies that statement (1) here is true. Statement (2) here is implied by statement (5) of Theorem 2.4. To see that statement (3) is true, we first prove this in the special case when  $I$  consists only of 0's. Note that (1) includes the  $n = 0$  subcase of this special case, and then the statement for general  $n$  follows by combining the last proposition, with statement (2) of Theorem 2.4 (which implies that  $\Phi^*$  and  $\delta^*$  commute). Now use (2) to deduce (3) for general  $I$  from the special case already established. Finally, (4) follows from statements (3) and (4) of Theorem 2.4.  $\square$

Note that as a corollary of Proposition 5.2, we have partially proved Theorem 1.5.

**Corollary 5.4.**

- (1)  $T(\infty, j) \simeq *$  unless  $j$  is a power of 2.
- (2)  $H^*(T(\infty, 2^k)) = \tilde{R}[k]$ , where  $\tilde{R}[k] = \langle \tilde{Q}^I x_0 \mid I \text{ is admissible and } l(I) = k \rangle$ .

## 6. NEW NISHIDA RELATIONS

In the last section, we determined  $H^*(T(n, *))$  in terms of dual Dyer–Lashof operations. Here we describe the Steenrod algebra action.

The standard Nishida relations [CLM, p.6, p.214] tell us how  $(Sq^r)_*$  commutes with  $Q^s$  in  $H_*(D_{n+1, *}S^{-n})$ . Since  $\chi(Sq^r)^7$  acting on  $H^*(T(n, *))$  corresponds to  $(Sq^r)_*$  acting on  $H_{-*}(D_{n+1, *}S^{-n})$ , we immediately have the following formula.

**Lemma 6.1.**

$$\chi(Sq^r)\tilde{Q}^s x = \sum_i \binom{-r-s}{r-2i} \tilde{Q}^{r+s-i} \chi(Sq^i) x.$$

Though this does completely specify the  $\mathcal{A}$  module structure on  $H^*(T(n, *))$ , it is in a form completely unsuitable for proving theorems like those in the introduction. The point of this section is to prove

**Theorem 6.2.**

$$Sq^r \tilde{Q}^s x = \sum_i \binom{s-i-1}{r-2i} \tilde{Q}^{r+s-i} Sq^i x.$$

The reader may find it amusing to compare this formula to the Adem relation of the last section,

$$\tilde{Q}^r \tilde{Q}^s x = \sum_i \binom{s-i-1}{r-2i} \tilde{Q}^{r+s-i} \tilde{Q}^i x,$$

the Adem relations in  $\mathcal{A}$ ,

$$Sq^r Sq^s x = \sum_i \binom{s-i-1}{r-2i} Sq^{r+s-i} Sq^i x,$$

and the formula defining the ‘‘Singer construction’’ [Si]<sup>8</sup>

$$Sq^r(t^{s-1} \otimes x) = \sum_i \binom{s-i-1}{r-2i} t^{r+s-i-1} \otimes Sq^i x.$$

*Proof of Theorem 6.2.* With  $Sq$  denoting the total square  $1 + Sq^1 + Sq^2 + \dots$ , to verify the formula, it suffices to check that it is consistent with the identity  $Sq(\chi(Sq)) = 1$  and Lemma 6.1 above. Fixing  $n$  and  $s$ , we compute

$$\begin{aligned} \sum_r Sq^{n-r} \chi(Sq^r) \tilde{Q}^s x &= \sum_r Sq^{n-r} \left[ \sum_i \binom{-r-s}{r-2i} \tilde{Q}^{r+s-i} \chi(Sq^i) x \right] \\ &= \sum_{i,j} \left[ \sum_r \binom{r+s-i-j-1}{n-r-2j} \binom{-r-s}{r-2i} \right] \tilde{Q}^{n+s-i-j} Sq^j \chi(Sq^i) x \\ &= \sum_{i,j} \left[ \sum_p \binom{i+s-j-1+p}{n-2i-2j-p} \binom{-2i-s-p}{p} \right] \tilde{Q}^{n+s-i-j} Sq^j \chi(Sq^i) x \end{aligned}$$

<sup>7</sup> $\chi$  is the antiautomorphism of the connected Hopf algebra  $\mathcal{A}$ .

<sup>8</sup>Yet another provocative loose end we are investigating.

(letting  $p = r - 2i$ )

$$= \sum_{i,j} \binom{-(i+j)}{n-2(i+j)} \tilde{Q}^{n+s-(i+j)} Sq^j \chi(Sq^i)x$$

(using J. Adem's formula [A, (25.3)]:  $\sum_p \binom{b+p}{c-p} \binom{a-p}{p} \equiv \binom{a+b+1}{c} \pmod{2}$ )

$$\begin{aligned} &= \sum_k \binom{-k}{n-2k} \tilde{Q}^{n+s-k} \left[ \sum_i Sq^{k-i} \chi(Sq^i)x \right] \\ &= \binom{0}{n} \tilde{Q}^{n+s} x = \begin{cases} \tilde{Q}^s x & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

*Remark 6.3.* Our method of proof also shows that the analogues of the formula in Lemma 6.1,

$$\chi(Sq^r)Sq^s x = \sum_i \binom{-r-s}{r-2i} Sq^{r+s-i} \chi(Sq^i)x,$$

and

$$\chi(Sq^r)(t^{s-1} \otimes x) = \sum_i \binom{-r-s}{r-2i} t^{r+s-i-1} \otimes Sq^i x,$$

respectively hold in the Steenrod algebra and Singer construction. The formula in  $\mathcal{A}$  already appears in the literature as [BaMi, (4.4)], where it is given a proof in the style of Bullett and MacDonald [BuMacD].

## 7. THE PROOFS OF THEOREM 1.2 AND THEOREM 1.5

To prove Theorem 1.2, first recall the description of  $H^*(T(n, *))$  given in Theorem 5.1:

$$H^*(T(n, *)) = S^*(\tilde{R}_n) / (\tilde{Q}^{|x|}x - x^2),$$

where

$$\tilde{R}_n = \langle \tilde{Q}^I x_n \mid I \text{ is admissible} \rangle / \langle \tilde{Q}^I x_n \mid I \text{ is admissible and } e(I) > n \rangle.$$

Note that  $\tilde{R}_n$  is closed under both the action of  $\mathcal{A}$  and  $\Phi^*$ , thanks to our Nishida relations and Proposition 5.3, i.e.  $(\tilde{R}_n, \Phi^*)$  is an object in  $\mathcal{U}_\rho$ . Thus Theorem 1.2 will follow from the next two proposition.

**Proposition 7.1.**  $(\tilde{R}_n, \Phi^*) \simeq F_\rho(n)$  as objects in  $\mathcal{U}_\rho$ .

**Proposition 7.2.** Let  $n \geq 1$ . In  $S^*(\tilde{R}_n)$ , the ideal generated by elements of the form  $\tilde{Q}^{|x|}x - x^2$  equals the ideal generated by elements of the form  $Sq^{|y|}y - (\Phi^*y)^2$ .

Both propositions will follow from the next result.

**Theorem 7.3.**  $Sq^I \tilde{Q}^J x_n = (\Phi^*)^{l(I)}(\tilde{Q}^I \tilde{Q}^J x_n)$ , whenever the iterated operation  $\tilde{Q}^I \tilde{Q}^J x_n$  is defined.

Proposition 7.1 then follows from

**Corollary 7.4.** *If  $I$  is admissible,  $Sq^I(\tilde{Q}^0)^k x_n = \begin{cases} \tilde{Q}^I(\tilde{Q}^0)^{k-l(I)} & \text{if } l(I) \leq k, \\ 0 & \text{if } l(I) > k. \end{cases}$*

This same corollary, together with Corollary 5.4 proves Theorem 1.5.

*Proof of Proposition 7.2.* Let  $F(x) = \tilde{Q}^{|x|}x - x^2$  and  $G(x) = Sq^{|x|}x - (\Phi^*x)^2$ . Using the fact that  $\tilde{R}_n$  is unstable, it is easy to deduce that the two ideals in question are generated by elements of the form  $F(x)$  and  $G(x)$  respectively, where  $x \in \tilde{R}_n$ . We claim that the sets of such elements are the same; more precisely,  $F(\tilde{Q}^I x_n) = G(\tilde{Q}^I \tilde{Q}^0 x_n)$  and  $G(\tilde{Q}^I x_n) = F(\Phi^*(\tilde{Q}^I x_n))$ .

To see that these hold, we let  $d = |I| + n$  and compute:

$$\begin{aligned} F(\tilde{Q}^I x_n) &= \tilde{Q}^d \tilde{Q}^I x_n - (\tilde{Q}^I x_n)^2 \\ &= Sq^d \tilde{Q}^I \tilde{Q}^0 x_n - (\Phi^*(\tilde{Q}^I \tilde{Q}^0 x_n))^2, \end{aligned}$$

using Theorem 7.3 and Proposition 5.3,

$$= G(\tilde{Q}^I \tilde{Q}^0 x_n).$$

Similarly,

$$\begin{aligned} G(\tilde{Q}^I x_n) &= Sq^d \tilde{Q}^I x_n - (\Phi^*(\tilde{Q}^I x_n))^2 \\ &= \Phi^*(\tilde{Q}^d \tilde{Q}^I x_n) - (\Phi^*(\tilde{Q}^I \tilde{Q}^0 x_n))^2, \end{aligned}$$

using Theorem 7.3 and Proposition 5.3,

$$= \tilde{Q}^d \Phi^*(\tilde{Q}^I x_n) - (\Phi^*(\tilde{Q}^I \tilde{Q}^0 x_n))^2,$$

using part (2) of Proposition 5.3 (since  $n \geq 1$ ),

$$= F(\Phi^*(\tilde{Q}^I x_n)).$$

□

It remains to prove Theorem 7.3. This will follow from a couple of lemmas.

**Lemma 7.5.**  *$Sq^r \tilde{Q}^J \tilde{Q}^0 x_n = \tilde{Q}^r \tilde{Q}^J x_n$ , whenever the iterated operation  $\tilde{Q}^r \tilde{Q}^J x_n$  is defined.*

*Proof.* This is proved by induction on  $l(J)$ . The induction is started by using the Nishida relations to verify that  $Sq^r \tilde{Q}^0 x_n = \tilde{Q}^r x_n$ .

For the inductive step, suppose  $J = (j, J')$ . Then

$$\begin{aligned} Sq^r \tilde{Q}^J \tilde{Q}^0 x_n &= Sq^r \tilde{Q}^j \tilde{Q}^{J'} \tilde{Q}^0 x_n \\ &= \sum_i \binom{r-j-1}{r-2i} \tilde{Q}^{r+j-i} Sq^i \tilde{Q}^{J'} \tilde{Q}^0 x_n \quad (\text{using the Nishida relations}) \\ &= \sum_i \binom{r-j-1}{r-2i} \tilde{Q}^{r+j-i} \tilde{Q}^i \tilde{Q}^{J'} x_n \quad (\text{by induction}) \\ &= \tilde{Q}^r \tilde{Q}^j \tilde{Q}^{J'} x_n \quad (\text{using the Adem relations}) \\ &= \tilde{Q}^r \tilde{Q}^J x_n. \end{aligned}$$

□

**Lemma 7.6.**  $Sq^I \tilde{Q}^J (\tilde{Q}^0)^{l(I)} x_n = \tilde{Q}^I \tilde{Q}^J x_n$ , whenever the iterated operation  $\tilde{Q}^I \tilde{Q}^J x_n$  is defined.

*Proof.* This is proved by induction on  $l(I)$ , and the last lemma is the case  $l(I) = 1$ . Let  $I = (I', i)$ . Then

$$\begin{aligned} Sq^I \tilde{Q}^J (\tilde{Q}^0)^{l(I)} x_n &= Sq^{I'} Sq^i \tilde{Q}^J (\tilde{Q}^0)^{l(I)} x_n \\ &= Sq^{I'} \tilde{Q}^i \tilde{Q}^J (\tilde{Q}^0)^{l(I)-1} x_n \quad (\text{by the case } l(I) = 1) \\ &= \tilde{Q}^{I'} \tilde{Q}^i \tilde{Q}^J x_n \quad (\text{by induction}) \\ &= \tilde{Q}^I \tilde{Q}^J x_n. \end{aligned}$$

□

*Proof of Theorem 7.3.* Applying  $(\Phi^*)^{l(I)}$  to the formula in the previous lemma yields

$$(\Phi^*)^{l(I)} (Sq^I \tilde{Q}^J (\tilde{Q}^0)^{l(I)} x_n) = (\Phi^*)^{l(I)} (\tilde{Q}^I \tilde{Q}^J x_n).$$

As it has a topological origin,  $(\Phi^*)^{l(I)}$  commutes with Steenrod operations. By Proposition 5.3,  $(\Phi^*)^{l(I)} (\tilde{Q}^J (\tilde{Q}^0)^{l(I)} x_n) = \tilde{Q}^J x_n$ . The theorem follows. □

## 8. THE PROOF OF THEOREM 1.9

This sections contains the details of the proof of Theorem 1.9, which was outlined at the end of §1.

As in [K5],  $F \in \mathcal{F}$  is said to be *finite* if it has a finite length composition series with simple subquotients, and is said to be *locally finite* (written  $F \in \mathcal{F}_\omega$ ) if it is the union of its finite subfunctors. Recall that  $I_\lambda \in \mathcal{F}$  is the injective envelope of the simple functor  $F_\lambda$ . The  $I_\lambda$  are locally finite [K5]. Then the general theory of locally Noetherian abelian categories [S, p.92] [P, Theorem 5.8.11] implies that, if  $J \in \mathcal{F}_\omega$  is any injective, then there is a decomposition in  $\mathcal{F}$

$$J \simeq \bigoplus_{\lambda \in \Lambda} a(\lambda, J) I_\lambda,$$

where  $a(\lambda, J) = \dim_{\mathbf{Z}/2} \text{Hom}_{\mathcal{F}}(F_\lambda, J)$ .

Applying this to the case  $J = \Phi^{-1} S^j$ , and noting [KK] that

$$\dim_{\mathbf{Z}/2} \text{Hom}_{\mathcal{F}}(F_\lambda, \Phi^{-1} S^j) = \dim_{\mathbf{Z}/2} \text{Hom}_{\mathcal{F}}(F_\lambda, S^{2^k j}), \text{ for } k \gg 0,$$

we deduce that

$$\Phi^{-1} S^j \simeq \bigoplus_{\lambda \in \Lambda} a(\lambda, j) I_\lambda,$$

with  $a(\lambda, j)$  as in the introduction.

Recall that  $r : \mathcal{F} \rightarrow \mathcal{U}$  is defined by letting  $r(F)_j = \text{Hom}_{\mathcal{F}}(S_j, F)$ . The fact that  $S_j$  is finite implies that  $r$  will commute with filtered direct limits. In particular, we can deduce the decomposition in  $\mathcal{U}$

$$\Phi^{-1} r(S^j \circ S_n) \simeq \bigoplus_{\lambda \in \Lambda} a(\lambda, j) r(I_\lambda \circ S_n).$$

**Proposition 8.1.**  $r(I_\lambda \circ S_n) \simeq H^*(K(\lambda, n); \mathbf{Z}/2)$  as  $\mathcal{A}$  modules.

Momentarily postponing the proof of this, to prove Theorem 1.9, we need to show

$$H^*(\Phi^{-1}T(n, j); \mathbf{Z}/2) \simeq \Phi^{-1}r(S^j \circ S_n) \text{ as } \mathcal{A} \text{ modules.}$$

Note that this asserts that a certain inverse limit of finite dimensional modules is isomorphic to a certain direct limit of *nilclosed* modules (i.e. modules of the form  $r(F)$ ).

To show this, observe that  $\Phi^{-1}r(S^* \circ S_n)$  is  $\mathbf{N} \times \mathbf{N}[\frac{1}{2}]$  graded. It is even an object in  $\mathcal{K}_\rho$ , using  $\Phi^{-1} : \Phi^{-1}r(S^{2^j} \circ S_n) \rightarrow \Phi^{-1}r(S^j \circ S_n)$  as the restriction.

**Theorem 8.2.**  $H^*(\Phi^{-1}T(n, *); \mathbf{Z}/2) \simeq \Phi^{-1}r(S^* \circ S_n)$  as objects in  $\mathcal{K}_\rho$ .

Returning to the proof of Proposition 8.1, we first note that  $H^*(K(\lambda, n); \mathbf{Z}/2) = H^*(K(V_\lambda, n); \mathbf{Z}/2)e_\lambda$  and  $r(I_\lambda \circ S_n) = r(I_{V_\lambda} \circ S_n)e_\lambda$ , where  $I_W \in \mathcal{F}$  is the injective defined by  $I_W(V) = (\mathbf{Z}/2)^{\text{Hom}(V, W)}$ . Thus we need just show that

$$r(I_W \circ S_n) = H^*(K(W, n); \mathbf{Z}/2).$$

Now one has the classic calculation [S, p.184]  $H^*(K(\mathbf{Z}/2, n); \mathbf{Z}/2) = U(F(n))$ , where  $F(n) = \mathcal{A}/E(n)$  is the free unstable module on an  $n$  dimensional class, and where  $U : \mathcal{U} \rightarrow \mathcal{K}$  is the free functor, left adjoint to the forgetful functor. Explicitly,  $U(M) = S^*(M)/(Sq^{|x|}x - x^2)$ . Similarly,  $H^*(K(W, n); \mathbf{Z}/2) = U_W(F(n))$  where  $U_W : \mathcal{U} \rightarrow \mathcal{K}$  is given by  $U_W(M) = U(M \otimes W^*)$ .

A simple calculation reveals that  $F(n) = r(S_n)^9$  (see e.g. [K7, Prop.8.1]), so the proof of Proposition 8.1 is completed with

**Lemma 8.3.** [K8] *There are natural isomorphisms  $U_W(r(F)) \simeq r(I_W \circ F)$ , for all  $F \in \mathcal{F}_\omega$ .*

*Sketch proof.* It is easy to reduce to the case when  $W = \mathbf{Z}/2$ . Let  $I = I_{\mathbf{Z}/2}$ . By filtering  $U(M)$  one then verifies that if  $M$  is nilclosed, so is  $U(M)$ . Thus to identify  $U(r(F))$  with  $r(I \circ F)$ , it suffices to check that  $l(U(r(F))) = I \circ F$ , where  $l : \mathcal{U} \rightarrow \mathcal{F}$  is left adjoint to  $r$ . The functor  $l$  is exact, preserves tensor products, and can be regarded as localization away from nilpotent modules [HLS, K5]. Thus it carries

$$S^*(r(F))/(Sq^{|x|}x - x^2)$$

to the functor that sends  $V$  to

$$S^*(l(r(F))(V))/(x - x^2).$$

Since  $l(r(F)) = F$ , and  $I(V) = S^*(V)/(x - x^2)$  [K5], this functor is just  $I \circ F$ .  $\square$

To prove Theorem 8.2, we need to use the main result of [K8].

As in [K7], let  $\mathcal{U}^2$  be the category of  $\mathbf{N} \times \mathbf{N}$  graded modules over the bigraded algebra  $A \otimes A$ , unstable in each grading. For  $M \in \mathcal{U}^2$ , there are natural maps  $\Phi_1 : M_{m,*} \rightarrow M_{2m,*}$  and  $\Phi_2 : M_{*,n} \rightarrow M_{*,2n}$ <sup>10</sup>, and we let  $\mathcal{K}^2$  denote the category of commutative algebras  $M$  in  $\mathcal{U}^2$  satisfying the ‘‘restriction’’ axiom: for all  $x \in$

<sup>9</sup>This is false at odd primes:  $F(n)$  is not nilclosed in the odd prime case.

<sup>10</sup>These are the Steenrod squares in the right degree

$M$ ,  $(\Phi_1 \otimes \Phi_2)(x) = x^2$ . Let  $U_2 : \mathcal{U}^2 \rightarrow \mathcal{K}^2$  be left adjoint to the forgetful functor: explicitly,  $U_2(M) = S^*(M)/((\Phi_1 \otimes \Phi_2)(x) - x^2)$ .

Given  $M \in \mathcal{U}$ ,  $M \otimes F(1)$  is an object in  $\mathcal{U}^2$ .  $F(1)$  can be regarded as the module  $\langle x_1, \dots, x^{2^k}, \dots \rangle$ , with  $x^{2^k}$  having bidegree  $(1, 2^k)$ . Now define

$$\mathrm{Hom}_{\mathcal{F}}(S_*, F) \otimes F(1) \rightarrow \mathrm{Hom}_{\mathcal{F}}(S_*, S^* \circ F)$$

by sending  $(S_i \xrightarrow{\alpha} F) \otimes x^{2^k}$  to the composite  $S_i \xrightarrow{\alpha} F \rightarrow S^{2^k} \circ F$ . Since  $\mathrm{Hom}_{\mathcal{F}}(S_*, S^* \circ F)$  is easily checked to be in  $\mathcal{K}^2$ , this map extends to a natural map in  $\mathcal{K}^2$ :

$$\Theta_F : U_2(\mathrm{Hom}_{\mathcal{F}}(S_*, F) \otimes F(1)) \rightarrow \mathrm{Hom}_{\mathcal{F}}(S_*, S^* \circ F).$$

**Theorem 8.4.** [K8] *For all  $F \in \mathcal{F}_\omega$ ,  $\Theta_F$  is an isomorphism.*

This is proved in a manner similar to the way Lemma 8.3 is proved.

**Corollary 8.5.**  $r(S^* \circ S_n) \simeq U_2(F(n) \otimes F(1))$ , as objects in  $\mathcal{K}^2$ .

**Corollary 8.6.**  $\Phi^{-1}r(S^* \circ S_n) \simeq U_\rho(F(n) \otimes \Phi^{-1}F(1))$ , as objects in  $\mathcal{K}_\rho$ .

Here  $\Phi^{-1}F(1) = \langle x^{2^k} \mid k \in \mathbf{Z} \rangle$ , with the restriction map (part of the  $\mathcal{K}_\rho$  structure), taking  $x^{2^k}$  to  $x^{2^{k-1}}$ .

By Theorem 1.2,  $H^*(\Phi^{-1}T(n, *); \mathbf{Z}/2) \simeq U_\rho(F_\rho(n)_{\hat{\Phi}})$  as objects in  $\mathcal{K}_\rho$ , where  $F_\rho(n)_{\hat{\Phi}}$  denotes the inverse limit

$$F_\rho(n) \xleftarrow{\rho} F_\rho(n) \xleftarrow{\rho} F_\rho(n) \xleftarrow{\rho} \dots$$

The following observation completes the proof of Theorem 8.2, and thus the proof of Theorem 1.9.

**Lemma 8.7.**  $F_\rho(n)_{\hat{\Phi}} = F(n) \otimes \Phi^{-1}F(1)$ , as objects in  $\mathcal{U}_\rho$ .

## 9. REMARKS AND QUESTIONS

In this section we outline some possible approaches to the conjectures of the introduction, and discuss the possible “meaning” of the main theorems.

Before launching into heuristics, we start with a rigorous proof of Proposition 1.11.

*Proof of Proposition 1.11.* Let  $X(j) = \bigvee_{\lambda \in \Lambda} a(\lambda, j)K(\lambda, 1)$ , and recall that we wish to topologically realize an  $\mathcal{A}$  module isomorphism:

$$H^*(\Phi^{-1}T(j); \mathbf{Z}/2) \simeq H^*(X(j); \mathbf{Z}/2).$$

Here  $T(j) = T(1, j)$  is an appropriate dual of the  $j^{\mathrm{th}}$  Brown–Gitler spectrum, and as such has the remarkable property that, if  $X$  is any wedge summand of a suspension spectrum, then any  $\mathcal{A}$  module map  $H^*(X; \mathbf{Z}/2) \rightarrow H^*(T(j); \mathbf{Z}/2)$  can be realized by a map  $T(j) \rightarrow X$ .

Fixing  $j$ , let  $\mathcal{S}(k) \subset \{T(2^k j), X(j)\}$  be the set of maps realizing the natural projection

$$H^*(X(j); \mathbf{Z}/2) \simeq H^*(\Phi^{-1}T(j); \mathbf{Z}/2) \rightarrow H^*(T(2^k j); \mathbf{Z}/2).$$

This is a finite, nonempty set, and  $\Phi : T(2^k j) \rightarrow T(2^{k+1} j)$  induces a restriction map  $\mathcal{S}(k+1) \rightarrow \mathcal{S}(k)$ . Then an element in the nonempty inverse limit  $\lim_k \mathcal{S}(k)$  is a family of maps  $T(2^k j) \rightarrow X(j)$ , compatible under  $\Phi$ . Such a family then yields a map  $\Phi^{-1}T(j) \rightarrow X(j)$  realizing the cohomology isomorphism.  $\square$

Thus far, we have been unable to find any way in which this proof, or the related proofs of Conjecture 1.3 [L1, Goe] in the  $n = 1$  case, could generalize to prove the  $n > 1$  cases of the conjectures. These proofs rely on magical properties of the spectra  $T(j)$ , which, in turn, are (partly) due to the fact that  $H^*(T(j); \mathbf{Z}/2)$  is injective in  $\mathcal{U}$ .

**Question 9.1.** For  $n > 1$ , do  $H^*(T(n, j); \mathbf{Z}/2)$  and  $H^*(K(\mathbf{Z}/2, n); \mathbf{Z}/2)$  have any sort of injectivity properties in some well chosen subcategory of  $\mathcal{U}$ ?

We now turn to Conjecture 1.6, which we feel is our most accessible conjecture. Our reason for feeling this is that our work in [K2], together with Mitchell and Priddy's work in [MP], lead to various criteria for showing that a sequence of (2 complete, connective) spectra

$$(9.1) \quad Z_0 \xrightarrow{\Phi} Z_1 \xrightarrow{\Phi} Z_2 \xrightarrow{\Phi} \dots$$

is equivalent to the sequence

$$(9.2) \quad SP_{\Delta}^1(S^0) \rightarrow SP_{\Delta}^2(S^0) \rightarrow SP_{\Delta}^4(S^0) \rightarrow \dots$$

when they agree in cohomology. (One should, of course, be thinking of the case  $Z_k = T(\infty, 2^k)$ .)

**Theorem 9.2.** *If (9.1) and (9.2) agree in mod 2 cohomology, then any of the following suffice to ensure that the sequences are equivalent.*

- (1)  $\Sigma^{-k}(Z_k/Z_{k-1})$  is a wedge summand of a suspension spectrum.
- (2)  $Z(1) \simeq SP_{\Delta}^2(S^0)$ , and there exist pairings  $Z_k \wedge Z_l \rightarrow Z_{k+l}$ , compatible with  $\Phi$ , and nonzero in degree 0.
- (3) There exist maps  $\Sigma^{-k}Z_k \rightarrow B(\mathbf{Z}/2)_+^k$  that in cohomology are injective when restricted to  $M(k)$ , the Steinberg idempotent stable summand [MP] of  $B(\mathbf{Z}/2)_+^k$ .

*Sketch proof.* (1) is proved in [K2]. The method of proof in [MP] shows that (2) implies (1). Finally, if (3) holds, we claim that the composite  $\Sigma^{-k}Z_{k-1} \rightarrow \Sigma^{-k}Z_{k-1} \rightarrow B(\mathbf{Z}/2)_+^k$  is null, thus the maps induce composites  $\Sigma^{-k}(Z_k/Z_{k-1}) \rightarrow B(\mathbf{Z}/2)_+^k \rightarrow M(k)$  which are equivalences, establishing (1). To prove this claim, inductively one can assume that  $Z_{k-1} = SP_{\Delta}^{2^{k-1}}(S^0)$ . The next lemma finishes the proof.  $\square$

**Lemma 9.3.**  $\{\Sigma^{-l}SP_{\Delta}^{2^{k-1}}(S^0), BV_+\} = 0$  if  $l \geq k$ .

*Sketch proof.* One proves this by induction on  $k$ . It reduces to the statement that  $\{\Sigma^{-l}M(k), BV_+\} = 0$  if  $l > 0$ , which is a consequence of the Segal conjecture.  $\square$

*Remark 9.4.* It strikes the author that the geometry of the situation may make it possible to check condition (3) when  $Z_k = T(\infty, 2^k)$ .

Finally we discuss Conjectures 1.3 and 1.10. The key is to rearrange the untidy right side of the isomorphism

$$H^*(\Phi^{-1}T(n, j); \mathbf{Z}/2) \simeq H^*\left(\bigvee_{\lambda \in \Lambda} a(\lambda, j)K(\lambda, n); \mathbf{Z}/2\right).$$

We know that this module corresponds to the functor  $(\Phi^{-1}S^j) \circ S_n \in \mathcal{F}$ . The proof in [K7] that  $\Phi^{-1}S^j$  is injective in  $\mathcal{F}_\omega$  reveals that

$$\Phi^{-1}S^j \simeq \lim_{s \rightarrow \infty} I_{(\mathbf{F}_{2^s})^*}[j],$$

where  $(\mathbf{F}_{2^s})^*$  is the  $\mathbf{F}_2$  linear dual of the finite field  $\mathbf{F}_{2^s}$ , and  $I_{(\mathbf{F}_{2^s})^*}[j]$  is the  $j^{\text{th}}$  eigenspace of  $I_{(\mathbf{F}_{2^s})^*}$  under the action of  $\mathbf{F}_{2^s}^\times$ . Furthermore, if we extend the scalars to the algebraic closure  $\bar{\mathbf{F}}_2$ , this isomorphism is well behaved with respect to pairings (between various  $j$ 's).

It follows that

$$H^*(\Phi^{-1}T(n, *); \bar{\mathbf{F}}_2) \simeq H_{\text{cont}}^*(K(\bar{\mathbf{F}}_2, n); \bar{\mathbf{F}}_2)[*]$$

as  $\mathbf{N}[\frac{1}{2}]$  graded algebras in  $\mathcal{U}$ , where we write

$$H_{\text{cont}}^*(K(\bar{\mathbf{F}}_2, n); \bar{\mathbf{F}}_2) = \lim_{s \rightarrow \infty} H^*(K(\mathbf{F}_{2^s}, n); \bar{\mathbf{F}}_2).$$

Nowadays, one has learned to call an  $E_\infty$  ring spectrum an  $S$ -module. Similarly, one can discuss  $SW(\bar{\mathbf{F}}_2)$ -modules, where  $W(\bar{\mathbf{F}}_2)$  are the Witt vectors of  $\bar{\mathbf{F}}_2$ .

**Question 9.5.** Let  $\Lambda$  denote a divided power algebra over  $\mathbf{Z}_2$ . Does there exist an  $\mathbf{N}$ -graded commutative  $S$ -algebra structure on  $T = \bigvee_{j \geq 0} T(0, j) = \bigvee_{j \geq 0} S^0$  such that

- (1)  $\pi_0(T) = \Lambda$ ,
- (2)  $\Phi : T \rightarrow T$  is a map of  $S$ -algebras, and
- (3)  $\Phi^{-1}T \wedge_S SW(\bar{\mathbf{F}}_2) \simeq \Sigma^\infty((\bar{\mathbf{F}}_2)^*)_+ \wedge_S SW(\bar{\mathbf{F}}_2)$ , as  $\mathbf{N}[\frac{1}{2}]$  graded  $SW(\bar{\mathbf{F}}_2)$ -algebras?

An affirmative answer to this formidable question would presumably yield a proof of Conjecture 1.10 upon applying the ‘‘bar construction’’ to the equivalence of (3),  $n$  times.

We end with a question about the most straightforward way to try to get at these sorts of things.

**Question 9.6.** Does there exist a ‘‘naturally occurring’’ spectrum  $E$ , with a group action, such that the group action can be used to establish a splitting

$$\Sigma E_n \simeq \bigvee_j T_1(n, j),$$

where  $E_n$  is the  $n^{\text{th}}$  infinite loop space of the spectrum  $E$ , and  $T_1(n, j)$  is a desuspension of  $\Sigma T(n, j)$ ?

When  $n = 1$ , this would be consistent with [GLM]. We note that, since  $H^*(E_n; \mathbf{Z}/2)$  would be a locally finite  $\mathcal{A}$  module for each  $n$ , a reasonable  $E$  would satisfy the perhaps unreasonable condition  $\tilde{E}^*(B\mathbf{Z}/2) = 0$ .

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