Exotic spheres and the Kervaire invariant

Addendum to the slides

Michel Kervaire’s work in surgery and knot theory


Andrew Ranicki (Edinburgh)

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The Kervaire-Milnor braid for $m$ I.

- For any $m \geq 5$ there is a commutative braid of 4 interlocking exact sequences (slide 46)
The Kervaire-Milnor braid for $m$ II.

- $\Theta_m$ is the K-M group of oriented $m$-dimensional exotic spheres.
- $P_m = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}, 0, \mathbb{Z}_2, 0, \ldots$ is the $m$-dimensional simply-connected surgery obstruction group. These groups only depend on $m \pmod{4}$.
- $a : A_m = \pi_m(G/O) \to P_m$ sends an $m$-dimensional almost framed differentiable manifold $M$ to the surgery obstruction of the corresponding normal map $(f, b) : M^m \to S^m$.
- For even $m$ $b : P_m \to \Theta_{m-1}$ sends a nonsingular $(-)^{m/2}$-quadratic form over $\mathbb{Z}$ of rank $r$ to the boundary $\Sigma^{m-1} = \partial W$ of the Milnor plumbing $W$ of $r$ copies of $\tau_{S^{m/2}}$ realizing the form.
- The image of $b$ is the subgroup $bP_m \subseteq \Theta_{m-1}$ of the $(m-1)$-dimensional exotic spheres $\Sigma^{m-1}$ which are the boundaries $\Sigma^{m-1} = \partial W$ of $m$-dimensional framed differentiable manifolds $W$.
- $c : \Theta_m \to \pi_m(G/O)$ sends an $m$-dimensional exotic sphere $\Sigma^m$ to its fibre-homotopy trivialized stable normal bundle.
The Kervaire-Milnor braid for $m$ III.

- $J : \pi_m(O) \to \pi_m(G) = \pi^S_m$ is the $J$-homomorphism sending
  $\eta : S^m \to O$ to the $m$-dimensional framed differentiable manifold
  $(S^m, \eta)$.

- The map $\circ : \pi_m(G/O) = A_m \to \pi_{m-1}(O)$ sends an $m$-dimensional
  almost framed differentiable manifold $M$ to the framing obstruction
  $$\circ(M) \in \pi_m(BO) = \pi_{m-1}(O).$$

- The isomorphism $\pi_m(PL/O) \to \Theta_m$ sends a vector bundle
  $\alpha : S^m \to BO(k)$ ($k$ large) with a $PL$ trivialization
  $\beta : \alpha^{PL} \simeq \ast : S^m \to BPL(k)$ to the exotic sphere $\Sigma^m$ such that
  $\Sigma^m \times \mathbb{R}^k$ is the smooth structure on the $PL$-manifold $E(\alpha)$ given by
  smoothing theory, with stable normal bundle
  $$\nu_{\Sigma^m} : \Sigma^m \simeq S^m \xrightarrow{\alpha} BO(k).$$

- $\pi_m(PL) = \Theta_m^{fr}$ is the K-M group of framed $n$-dimensional exotic
  spheres.
The Kervaire-Milnor braid for $m = 4k + 2$ I.

For $m = 4k + 2 \geq 5$ the braid is given by

\[
P_{4k+3} = 0 \quad \Theta_{4k+2} \quad \pi_{4k+1}(O) \quad \pi_{4k+1}(G)
\]

\[
\pi_{4k+2}(PL) \quad \pi_{4k+2}(G/O) \quad \pi_{4k+2}(G) \quad \pi_{4k+1}(PL)
\]

\[
\pi_{4k+2}(O) = 0 \quad \pi_{4k+2}(G) \quad P_{4k+2} = \mathbb{Z}_2 \quad \Theta_{4k+1}
\]

with $K$ the Kervaire invariant map.
The Kervaire-Milnor braid for $m = 4k + 2$. II.

$K$ is the Kervaire invariant on the $(4k + 2)$-dimensional stable homotopy group of spheres

$$K : \pi_{4k+2}(G) = \pi_{4k+2}^S = \lim_{\longrightarrow j} \pi_{j+4k+2}(S^j)$$

$$= \Omega^fr_{4k+2} = \{\text{framed cobordism}\} \rightarrow P_{4k+2} = \mathbb{Z}_2$$

$K$ is the surgery obstruction: $K = 0$ if and only if every $(4k + 2)$-dimensional framed differentiable manifold is framed cobordant to a framed exotic sphere.

The exotic sphere group $\Theta_{4k+2}$ fits into the exact sequence

$$0 \rightarrow \Theta_{4k+2} \rightarrow \pi_{4k+2}(G) \xrightarrow{K} \mathbb{Z}_2 \rightarrow \ker(\pi_{4k+1}(PL) \rightarrow \pi_{4k+1}(G)) \rightarrow 0$$
The Kervaire-Milnor braid for \( m = 4k + 2 \) III.

- \( a : \pi_{4k+2}(G/O) \to \mathbb{Z}_2 \) is the surgery obstruction map, sending a normal map \((f, b) : M^{4k+2} \to S^{4k+2} \) to the Kervaire invariant of \( M \).

- \( b : P_{4k+2} = \mathbb{Z}_2 \to \Theta_{4k+1} \) sends the generator \( 1 \in \mathbb{Z}_2 \) to the boundary \( b(1) = \Sigma^{4k+1} = \partial W \) of the Milnor plumbing \( W \) of two copies of \( \tau S^{2k+1} \) using the standard rank 2 quadratic form \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) over \( \mathbb{Z} \) with Arf invariant 1.

- The image of \( b \) is the subgroup \( bP_{4k+2} \subseteq \Theta_{4k+1} \) of the \((4k+1)\)-dimensional exotic spheres \( \Sigma^{4k+1} \) which are the boundaries \( \Sigma^{4k+1} = \partial W \) of framed \((4k+2)\)-dimensional differentiable manifolds \( W \). If \( k \) is such that \( K = 0 \) (e.g. \( k = 2 \)) then \( bP_{4k+2} = \mathbb{Z}_2 \subseteq \Theta_{4k+1} \), and if \( \Sigma^{4k+1} = 1 \in bP_{4k+2} \) (as above) then \( M^{4k+2} = W \cup_{\Sigma^{4k+1}} D^{4k+2} \) is the \((4k+2)\)-dimensional Kervaire PL manifold without a differentiable structure.

- \( c : \Theta_{4k+2} \to \pi_{4k+2}(G/O) \) sends a \((4k + 2)\)-dimensional exotic sphere \( \Sigma^{4k+2} \) to its fibre-homotopy trivialized stable normal bundle.
What if $K = 0$?

- For any $k \geq 1$ the following are equivalent:
  - $K : \pi_{4k+2}(G) = \pi_{4k+2}^S \to \mathbb{Z}_2$ is 0,
  - $\Theta_{4k+2} \cong \pi_{4k+2}(G)$,
  - $\ker(\pi_{4k+1}(PL) \to \pi_{4k+1}(G)) \cong \mathbb{Z}_2$,
  - Every simply-connected $(4k + 2)$-dimensional Poincaré complex $X$ with a vector bundle reduction $\tilde{\nu}_X : X \to BO$ of the Spivak normal fibration $\nu_X : X \to BG$ is homotopy equivalent to a closed $(4k + 2)$-dimensional differentiable manifold.

When is $K \neq 0$?

- **Theorem** (Browder 1969)
  - If $K \neq 0$ then $4k + 2 = 2^j - 2$ for some $j \geq 2$.
  - It is known that $K \neq 0$ for $4k + 2 \in \{2, 6, 14, 30, 62\}$.
- **Theorem** (Hill-Hopkins-Ravenel 2009)
  - If $K \neq 0$ then $4k + 2 \in \{2, 6, 14, 30, 62, 126\}$.
  - It is not known if $K = 0$ or $K \neq 0$ for $4k + 2 = 126$. 
The exotic spheres home page

http://www.maths.ed.ac.uk/~aar/exotic.htm

The Kervaire invariant home page

http://www.math.rochester.edu/u/faculty/doug/kervaire.html