Global homotopy theory

Stefan Schwede
Mathematisches Institut
Universität Bonn, Germany

schwede@math.uni-bonn.de
www.math.uni-bonn.de/people/schwede

v0.32 / January 20, 2017
Preliminary and incomplete version

Contents

Introduction

This book introduces a context for global homotopy theory; here 'global' refers to simultaneous and compatible actions of compact Lie groups. It has been noticed since the beginnings of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class. Prominent examples of this are equivariant stable homotopy, equivariant K-theory or equivariant bordism. Various ways to formalize this idea and to obtain a category that is the home of global stable homotopy types have been explored in [**97**, Ch. II], [**64**, Sec. 5], [**22**]. We use a different approach: we work with the well-known category of orthogonal spectra, but use a much finer notion of equivalence, the *global equivalences*, then what is traditionally considered. The basic underlying observation is that every orthogonal spectrum gives rise to an orthogonal G-spectrum for every compact Lie group G, and the fact that all these individual equivariant objects come from one orthogonal spectrum implicitly encodes strong compatibility conditions as the group G varies. An orthogonal spectrum thus has G-equivariant homotopy groups for every compact Lie group, and a global equivalence is a morphism of orthogonal spectra that induces isomorphisms for all equivariant homotopy groups for all compact Lie groups (compare Definition IV.1.3). For the experts we should add here that the equivariant homotopy groups that we consider are based on 'complete G-universes'.

The structure on the equivariant homotopy groups of an orthogonal spectrum gives an idea of the information contained in a global homotopy type in our sense: the equivariant homotopy groups $\pi_k^G(X)$ are contravariantly functorial for continuous group homomorphisms ('restriction maps'), and they are covariantly functorial for inclusions of closed subgroups ('transfer maps'). The restriction and transfer maps enjoy various transitivity properties and interact via a double coset formula. This kind of algebraic structure has been studied under different names (e.g., 'global Mackey functor', 'inflation functor',...). From a purely algebraic perspective, there are various parameters here than one can vary, namely the class of groups to which a value is assigned and the classes of homomorphisms to which restriction maps respectively transfer maps are assigned, and lots of variations have been explored algebraically. However, the decision to work with orthogonal spectra and equivariant homotopy groups on complete universes dictates a canonical choice: we prove in Theorem IV.2.6 that the algebra of natural operations between the equivariant homotopy groups of orthogonal spectra is freely generated by restriction maps along continuous group homomorphisms and transfer maps along subgroup inclusion, subject to explicitly understood relations.

We define the *global stable homotopy category* $\mathcal{GH}$ by localizing the category of orthogonal spectra at the class of global equivalences. Every global equivalence is in particular a non-equivariant stable equivalence, so there is a 'forgetful' functor $U : \mathcal{GH} \longrightarrow \mathcal{SH}$ on localizations, where $\mathcal{SH}$ denotes the traditional non-equivariant stable homotopy category. By Theorem IV.5.1 this forgetful functor has a left adjoint L and a right adjoint R, both fully faithful, that participate in a recollement of triangulated categories:

$$\begin{tikzcd} \mathcal{GH}^+ \arrow[r, "i_*"] & \mathcal{GH} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "U"] & \mathcal{SH} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

Here $\mathcal{GH}^+$ denotes the the full subcategory of the global homotopy category spanned by the orthogonal spectra that are stably contractible in the traditional, non-equivariant sense.

The global sphere spectrum and suspension spectra are in the image of the left adjoint (Example IV.5.12). Global Borel cohomology theories are the image of the right adjoint (Example IV.5.21). The 'natural' global versions of Eilenberg-Mac Lane spectra (Construction VI.1.9), Thom spectra (Section VI.2), or topological K-theory (Section VI.5) are not in the image of either of the two adjoints. Periodic global K-theory, however, is right induced from finite cyclic groups i.e., in the image of the analogous right adjoint from an intermediate global homotopy category $\mathcal{GH}_{cyc}$ based on finite cyclic groups (Example VI.5.33).

Looking at orthogonal spectra through the eyes of global equivalences is a bit like using a prism: the latter breaks up white light into a spectrum of colors, and global equivalences split a traditional, non-equivariant homotopy type into many different global homotopy types. The first example of this phenomenon that we will encounter refines the classifying space of a compact Lie group G. On the one hand, there is the constant orthogonal space with value a non-equivariant model for BG; and there is the *global classifying* space $B_{\mathrm{gl}}G$ (see Definition I.2.11). The global classifying space is analogous to the 'geometric classifying space' of a linear algebraic group in motivic homotopy theory, compare [**114**, 4.2]. After reading Chapter I, most readers will probably agree that the global classifying space is the more interesting object.

Another good example is the splitting up of the non-equivariant homotopy type of the classifying space of the infinite orthogonal group O. Again there is the constant orthogonal space with value BO, the Grassmannian model $\mathbf{BO}$, the restricted Grassmannian model $\mathbf{bO}$, the bar construction model $\mathbf{BO}^\circ$, and finally a certain 'cofree' orthogonal space $R(BO)$. The restricted Grassmannian model $\mathbf{bO}$ is also a homotopy colimit, as n goes to infinity, of the global classifying spaces $B_{\mathrm{gl}}O(n)$, We discuss these different global forms of BO is some detail in Section II.4; they are related by various highly structured morphisms, compare (4.1) of Chapter II. The different global form of BO have corresponding Thom spectra, and we discuss these in Section VI.2).

In the stable global world, every non-equivariant homotopy type has two extreme global refinements, the 'left induced' (the global analog of a constant orthogonal space, see Example IV.5.11) and the 'right induced' global homotopy type (representing Borel cohomology theories, see Example IV.5.21) Many important stable homotopy types have other natural global forms. The non-equivariant Eilenberg-Mac Lane spectrum of the integers has a 'free abelian group functor' model (Construction VI.1.9), and another incarnation as the Eilenberg-Mac Lane spectrum of the constant global functor with value $\mathbb{Z}$ (see Remark IV.4.13). These two global refinements of the integral Eilenberg-Mac Lane spectrum agree on finite groups, but differ for compact Lie groups of positive dimensions; the author is uncertain which of the two is the 'better', or the more useful, global homotopy type.

As already indicated, there is a great variety of orthogonal Thom spectra, in real (or unoriented) flavors as $\mathbf{mO}$ and $\mathbf{MO}$, as complex (or unitary) versions $\mathbf{mU}$ and $\mathbf{MU}$, and there are periodic versions $\mathbf{mOP}$, $\mathbf{MOP}$, $\mathbf{mUP}$ and $\mathbf{MUP}$ of these; we discuss these spectra in Section VI.2. The theories represented by $\mathbf{mO}$ and $\mathbf{mU}$ have the closest ties to geometry; for example, the equivariant homotopy groups of $\mathbf{mO}$ receive Thom-Pontryagin maps from equivariant bordism rings, and these are isomorphisms for products of finite groups and tori (compare Theorem VI.3.40). The theories represented by $\mathbf{MO}$ are tom Dieck's homotopical bordism, isomorphic to 'stable equivariant bordism'.

Connective topological K-theory also has two fairly natural global refinements (in addition to the left and right induced ones). The 'orthogonal subspace' model $\mathbf{ku}$ (Construction VI.4.11) represents connective equivariant K-theory on the class of finite groups; on the other hand, global connective K-theory $\mathbf{ku}^c$ (Construction VI.5.34) is the global synthesis of equivariant connective K-theory in the sense of Greenlees [**63**].

The global equivalences are part of a closed model structure (see Theorem IV.3.26), so the methods of homotopical algebra can be used to study the global homotopy category. This works more generally relative to a class $\mathcal{F}$ of compact Lie groups, where we define $\mathcal{F}$-equivalences by requiring that $\pi_k^G(f)$ is an isomorphism for all integers and all groups in $\mathcal{F}$. We call a class $\mathcal{F}$ of compact Lie groups a *global family* if it is closed under isomorphism, subgroups and quotients. For global families we complement the $\mathcal{F}$-equivalences to a stable model structure, the *$\mathcal{F}$-global model structure*, see Theorem IV.3.21) These model structure is useful for showing that the forgetful functor

$$(\mathcal{F}\text{-global homotopy category}) \longrightarrow (\text{stable homotopy category})$$

has both a left and a right adjoint, and both are fully faithful. Besides all compact Lie groups, interesting global families are the classes of all finite groups, or all abelian compact Lie groups. The class of trivial groups is also admissible here, but then we just recover the 'traditional' stable category. If the family $\mathcal{F}$

is multiplicative, then the $\mathcal{F}$-global model structure is monoidal with respect to the smash product of orthogonal spectra and satisfies the 'monoid axiom' (Proposition IV.3.33). Hence this model structure lifts to modules over an orthogonal ring spectrum and to algebras over an ultra-commutative ring spectrum (Corollary IV.3.34).

Organization. In Chapter I we set up the unstable global homotopy theory using orthogonal spaces, i.e., continuous functor from the category of finite-dimensional inner product spaces and linear isometric embeddings to spaces. We introduce global equivalences (Definition I.1.2), discuss global classifying spaces of compact Lie groups (Definition I.2.11), set up the global model structures on the category of orthogonal spaces (Theorem I.3.22) and investigate the box product of orthogonal spaces from a global equivariant perspective.

Chapter II is devoted to ultra-commutative monoids (a.k.a. commutative monoids with respect to the box product, or lax symmetric monoidal functors), which we want to advertise as a rigidified notion of 'global E_∞-space'. In Section II.1 we establish a global model structure for ultra-commutative monoids (Theorem 1.13). Section II.2 introduces and studies global power monoids, the algebraic structure that an ultra-commutative multiplication gives rise to on the homotopy group Rep-functor $\underline{\pi}_0(R)$. Section II.3 contains a large collection of examples of ultra-commutative monoids and interesting morphisms between them. In Section 4 we discuss and compare different global refinements of the non-equivariant homotopy type BO, the classifying space for the infinite orthogonal group, and we formulated a global, highly structured version of Bott periodicity (Theorem 4.39). Section II.5 discusses 'units' and 'group completion' of ultra-commutative monoids.

Chapter III is a largely self-contained exposition of many basics about equivariant stable homotopy theory for a fixed compact Lie group, modeled by orthogonal G-spectra. In Section III.1 we recall orthogonal G-spectra and equivariant homotopy groups and prove their basic properties, such as the suspension isomorphism and long exact sequences of mapping cones and homotopy fibers, and the behavior of equivariant homotopy groups on sums and products. Section III.2 discusses the Wirthmüller isomorphism and the closed related transfers. In Section III.3 we introduce and study geometric fixed point homotopy groups, an alternative invariant to characterize equivariant stable equivalences. We also prove the double coset formula for the composite of a transfer followed by the restriction to a closed subgroup. Section III.5 is devoted to multiplicative aspects of equivariant stable homotopy theory.

Chapter IV sets the stage for stable global homotopy theory: We discuss free orthogonal spectra and global Ω-spectra (Definition IV.3.11), the natural concept of a 'global infinite loop object' in our setting. Two main results in Chapter IV are the calculation of algebra of natural operations on equivariant homotopy groups (Theorem IV.2.6) and the identification of certain morphisms between free orthogonal spectra as global equivalences (Theorem IV.1.31). In Section IV.3 we complement the global equivalences of orthogonal spectra by a stable model structure. Here we work more generally relative to a global family $\mathcal{F}$ and consider the $\mathcal{F}$-equivalences (i.e., equivariant stable equivalences for all compact Lie groups in the family $\mathcal{F}$). We follow the familiar outline: a certain $\mathcal{F}$-level model structure is Bousfield localized to an $\mathcal{F}$-global model structures (see Theorem IV.3.21). We use the $\mathcal{F}$-global model structure to construct and study left and right adjoints to the forgetful functors associated to a change of global family (Theorem IV.5.1). We develop some basic theory around the global stable homotopy category; since it comes from a stable model structure, this category is naturally triangulated and we show that the suspension spectra of global classifying spaces form a set of compact generators (Theorem IV.4.3). As an application of Morita theory for stable model categories [**134**], we can then deduce that rationally the global homotopy category for finite groups has an algebraic model, namely the derived category of rational global functors (Theorem IV.6.3).

Chapter V is devoted to the study of 'global power functors', the algebraic structure on the equivariant homotopy groups of ultra-commutative ring spectra. Roughly speaking, global power functors are global Green functors equipped with additional power operations, satisfying various properties reminiscent of those of the power maps $x \mapsto x^m$ in a commutative ring. We show that the 0-th equivariant homotopy groups of

an ultra-commutative ring spectrum form a global power functor (Theorem V.1.9), give a description of the category of global power functor via the comonad of 'exponential sequences' (Theorem V.2.10) and discuss localization of global power functors at a multiplicative subset of the underlying ring (Theorem V.2.15), In Section V.3 we discuss properties and interrelationship of various free functors between the categories of Rep-functors, abelian Rep-monoids, global power monoids, global functors, global Green functors and global power functors. These algebraic structures are relevant to topology because they are the homes of the 0-th equivariant homotopy sets/groups of orthogonal spaces, E_∞ orthogonal monoid spaces, ultra-commutative monoids, orthogonal spectra E_∞ orthogonal ring spectra, respectively ultra-commutative ring spectra. In Section V.4 we discuss various examples of global power functors, such as the Burnside ring global power functor, the global functor represented by an abelian compact Lie group, free global power functors, constant global power functors, and the complex representation ring global functor.

Chapter VI focuses on *ultra-commutative ring spectra*, i.e., commutative orthogonal ring spectra under multiplicative global equivalences. In Section VI.1 we establish the global model structure for ultra-commutative ring spectra (Theorem VI.1.5), show that every global power functor is realized by an ultra-commutative ring spectrum (Theorem VI.1.7), and discuss the example of Eilenberg-Mac Lane spectra (see Construction VI.1.9). Section VI.2 discusses two orthogonal Thom spectra **mO** and **MO**. The globally connective spectrum **mO** is closely related to equivariant bordism. The Thom spectrum **MO** was first considered by tom Dieck and it represents 'stable' equivariant bordism. Section 3 recalls the geometrically defined equivariant bordism theories. The Thom-Pontryagin construction maps the unoriented G-equivariant bordism ring $\mathcal{N}_*^G$ to the equivariant homotopy ring $\pi_*^G(\mathbf{mO})$, and that map is an isomorphism when G is a product of a finite group and a torus, see Theorem VI.3.40. We discuss global K-theory in Sections VI.4 and VI.5, which comes in three interesting flavors as connective global K-theory **ku**, global connective K-theory $\mathbf{ku}^c$ and periodic global K-theory **KU** (and in the real versions **ko**, $\mathbf{ko}^c$ and **KO**).

Relation to other work. The idea of global equivariant homotopy theory is not at all new and has previously been explored in different contexts. For example, in Chapter II of [**97**], Lewis and May define *coherent families* of equivariant spectra; these consists of collections of equivariant coordinate free spectra in the sense Lewis, May and Steinberger, equipped with comparison maps involving change of groups and change of universe functors.

The approach closest to ours are the *global $\mathcal{I}_*$-functors* introduced by Greenlees and May in [**64**, Sec. 5]. These objects are 'global orthogonal spectra' in that they are indexed on pairs (G, V) consisting of a compact Lie group and a G-representation V. The corresponding objects with commutative multiplication are called *global $\mathcal{I}_*$-functors with smash products* in [**64**, Sec. 5] and it is for these that Greenlees and May define and study multiplicative norm maps. Clearly, an orthogonal spectrum gives rise to a global $\mathcal{I}_*$-functors in the sense of Greenlees and May. In the second chapter of her thesis [**21**], A. M. Bohmann compares the approaches of Lewis-May and Greenlees-May; in the paper [**22**] she also relates these to orthogonal spectra.

Symmetric spectra in the sense of Hovey, Shipley and Smith [**80**] are another prominent model for the (non-equivariant) stable homotopy category. Much of what we do here with orthogonal spectra can also be done with symmetric spectra, if one is willing to restrict to finite groups (as opposed to general compact Lie groups). This restriction arises because only finite groups embed into symmetric groups, while every compact Lie group embeds into an orthogonal group. M. Hausmann [**71, 72**] has established a global model structure on the category of symmetric spectra, and he showed that the forgetful functor is a right Quillen equivalence from the category of orthogonal spectra with the $\mathcal{F}in$-global model structure to the category of symmetric with the global model structure. While some parts of the symmetric and orthogonal theories are similar, there are serious technical complications arising from the fact that for symmetric spectra the naively defined equivariant homotopy groups are not 'correct', a phenomenon that is already present non-equivariantly.

Prerequisites. This books assumes a solid background in algebraic topology and (non-equivariant) homotopy theory, including topics such as singular homology and cohomology, CW-complexes, homotopy

groups, mapping spaces, loop spaces, fibrations and fiber bundles, Eilenberg-Mac Lane spaces, smooth manifolds, Grassmannian and Stiefel manifolds. Two modern references that contain all we need (and much more) are the text books by Hatcher [**68**] and tom Dieck [**167**]. Some knowledge of non-equivariant stable homotopy theory is helpful to appreciate the equivariant and global features of the structures and examples we discuss; from a strictly logical perspective, however, the non-equivariant theory is a degenerate special case of the global theory for the global family of trivial Lie groups. In particular, by simply ignoring all group actions the examples presented in this book give models for many interesting and prominent non-equivariant stable homotopy types.

Since actions of compact Lie groups are central to the topics of this book, some familiarity with the structure and representation theory of compact Lie groups is obviously helpful, but we give precise references to the literature whenever we need any non-trivial facts. Many of our objects of study organize themselves into model categories in the sense of Quillen [**126**], so some basic background on model categories is necessary to understand the respective sections. The article [**47**] by Dwyer and Spalinski is a good introduction to model categories, and Hovey's book [**79**] is still the definitive reference. Some acquaintance with unstable equivariant homotopy theory is useful, but not logically necessary. In contrast, we do not assume any prior knowledge in equivariant *stable* homotopy theory, and Chapter III is a self-contained introduction based on equivariant orthogonal spectra. The last three sections of Chapter IV study the global stable homotopy category, and here we freely use the language of triangulated categories. The first chapter of Neeman's book [**118**] is a possible reference for the necessary background.

Acknowledgments. I would like to thank John Greenlees for being a reliable consultant on matters of equivariant stable homotopy theory; Johannes Ebert and Michael Joachim for tutorials on the C^*-algebraic approach to equivariant K-theory; and Markus Hausmann for careful reading of large parts of this book, along with numerous suggestions for improvements. I am indebted to Benjamin Böhme, Lars Hesselholt, Matthias Kreck, Jacob Lurie, Luca Pol, Steffen Sagave, Albert Schulz, Neil Strickland, Karol Szumiło, Peter Teichner, Andreas Thom and Mahmoud Zeinalian for various helpful discussions, comments and suggestions.

CHAPTER I

Unstable global homotopy theory

In this chapter we develop a framework for unstable global homotopy theory via *orthogonal spaces*, i.e., continuous functors from the linear isometries category $\mathbf{L}$ to spaces. In Section 1 we introduce global equivalences of orthogonal spaces and establish many basic properties of this class of morphisms. In Section 2 we introduce and discuss global classifying spaces of compact Lie groups, the basic building blocks of unstable global homotopy types. In Section 3 we complement the global equivalences by a global model structure on the category of orthogonal spaces. The construction follows a familiar pattern, by Bousfield localization of an auxiliary 'strong level model structure'. Section 3 also contains a discussion of cofree orthogonal spaces, i.e., global homotopy types that are 'right induced' from non-equivariant homotopy types. We also recall the box product of orthogonal spaces, a Day convolution product based on orthogonal direct sum of inner product spaces. The box product is a symmetric monoidal product, fully invariant under global equivalences, and globally equivalent to the cartesian product. Section 4 introduces an important variation of our theme, where we discuss unstable global homotopy theory for a 'global family', i.e., a class of compact Lie groups with certain closure properties. In Section 5 we introduce the G-equivariant homotopy set $\pi_0^G(Y)$ of an orthogonal space and identify the natural structure on these sets (restriction maps along continuous group homomorphisms). The study of natural operations on the sets $\pi_0^G(Y)$ is a recurring theme throughout this book, and we will revisit and extend the results in the later chapters for ultra-commutative monoids, orthogonal spectra and ultra-commutative ring spectra.

Our main reason for working with orthogonal spaces is that they are the direct unstable analog of orthogonal spectra, and in this unstable model for global homotopy theory the passage to the stable theory in Chapter IV is especially simple. However, there are other models for unstable global homotopy theory, most notably the *topological stacks* and the *orbispaces* as developed by Gepner and Henriques in their paper [**58**]. For a comparison of these two models to our orthogonal space model we refer the reader to the author's paper [**138**]. The comparison proceeds through yet another model, the global homotopy theory of 'spaces with an action of the universal compact Lie group'. Here the universal compact Lie group (which is neither compact nor a Lie group) is the topological monoid $\mathcal{L}$ of linear isometric self-embeddings of $\mathbb{R}^\infty$, and in [**138**] we establish a global model structure on the category of $\mathcal{L}$-spaces.

1. Orthogonal spaces and global equivalences

In this section we introduce *orthogonal spaces*, our basic objects of study, along with the notion of *global equivalences*. Orthogonal spaces up to global equivalences are our way to rigorously formulate the idea of 'compatible equivariant homotopy types for all compact Lie groups'. We introduce various basic techniques to manipulate global equivalences of orthogonal spaces, such as recognition criteria by homotopy or strict colimits over representations (Propositions 1.7 and 1.18), and a list of standard constructions that preserves global equivalences (see Proposition 1.9). Theorem 1.11 is a strong 'cofinality' result for orthogonal spaces, showing that fairly general changes in the linear isometries indexing category do not affect the global homotopy type.

Before we start, let us fix some notation and conventions. By a 'space' we mean a *compactly generated space* in the sense of [**112**], i.e., a k-space (also called *Kelley space*) that satisfies the weak Hausdorff

condition. We review the definition and some basic properties of these spaces in Section A.1. We denote by $\mathbf{T}$ the category of compactly generated spaces. Later we will also consider *based* spaces, and $\mathbf{T}_*$ will denote the category of based compactly generated spaces.

An *inner product space* is a finite dimensional real vector space equipped with a scalar product, i.e., a positive definite symmetric bilinear form. We denote by $\mathbf{L}$ the category with objects the inner product spaces and morphisms the linear isometric embeddings. The category $\mathbf{L}$ is a topological category in the sense that the morphism spaces come with a preferred topology: if $\varphi : V \longrightarrow W$ is one linear isometric embedding, then the action of the orthogonal group $O(W)$, by postcomposition, induces a bijection

$$O(W)/O(\varphi^\perp) \;\cong\; \mathbf{L}(V,W)\ , \quad A\cdot O(\varphi^\perp) \;\longmapsto A\circ\varphi\ ,$$

where $\varphi^\perp = W - \varphi(V)$ is the orthogonal complement of the image of φ. We topologize $\mathbf{L}(V,W)$ so that this bijection is a homeomorphism, and this topology is independent of φ. If $(v_1,\dots,v_k)$ is an orthonormal basis of V, then for every linear isometric embedding $\varphi : V \longrightarrow W$ the tuple $(\varphi(v_1),\dots,\varphi(v_k))$ is an orthonormal k-frame of W. This assignment is a homeomorphism from $\mathbf{L}(V,W)$ to the Stiefel manifold of k-frames in W.

An example of an inner product spaces is the vector space $\mathbb{R}^n$ with the standard scalar product

$$\langle x,y\rangle \;=\; \sum_{i=1}^n x_i y_i\ .$$

In fact, every inner product space V is isometrically isomorphic to the inner product space $\mathbb{R}^n$, for n the dimension of V. So the full topological subcategory with objects the $\mathbb{R}^n$ is a small skeleton of $\mathbf{L}$.

Definition 1.1. An *orthogonal space* is a continuous functor $Y : \mathbf{L} \longrightarrow \mathbf{T}$ to the category of spaces. A morphism of orthogonal spaces is a natural transformation. We denote by *spc* the category of orthogonal spaces.

The use of continuous functors from the category $\mathbf{L}$ to spaces has a long history in homotopy theory. The systematic use of inner product spaces (as opposed to numbers) to index objects in stable homotopy theory seems to go back to Boardman's thesis [**18**]. The category $\mathbf{L}$ (or its extension that also contains countably infinite dimensional inner product spaces) is denoted $\mathscr{I}$ by Boardman and Vogt [**19**], and this notation is also used in [**107**]; other sources [**98**] use the symbol $\mathcal{I}$. Accordingly, orthogonal spaces are sometimes referred to as $\mathscr{I}$-functors, $\mathscr{I}$-spaces or $\mathcal{I}$-spaces. Our justification for using yet another name is twofold: on the one hand, our use of orthogonal spaces comes with a shift in emphasis, away from a focus on non-equivariant homotopy types, and towards viewing an orthogonal space as representing compatible equivariant homotopy types for all compact Lie groups. Secondly, we want to stress the analogy between orthogonal spaces and orthogonal spectra, the former being an unstable global world with the latter a corresponding stable global world.

Now we define our main new concept, the notion of 'global equivalence' between orthogonal spaces. We let G be a compact Lie group. By a *G-representation* we mean a finite dimensional orthogonal representation, i.e., a real inner product space equipped with a continuous G-action by linear isometries. In other words, a G-representation consists of an inner product space V and a continuous homomorphism $\rho : G \longrightarrow O(V)$. In this context, and throughout the book, we will often use without explicit mentioning that continuous homomorphisms between Lie groups are automatically smooth, compare for example [**29**, Prop. I.3.12]. For every orthogonal space Y and every G-representation V, the value $Y(V)$ inherits a G-action from the G-action on V and the functoriality of Y. For a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the induced map $Y(\varphi) : Y(V) \longrightarrow Y(W)$ is G-equivariant.

We denote by

$$D^k \;=\; \{x\in\mathbb{R}^k \;:\; \langle x,x\rangle \le 1\} \qquad \text{and} \qquad \partial D^k \;=\; \{x\in\mathbb{R}^k \;:\; \langle x,x\rangle = 1\}$$

the unit disc in $\mathbb{R}^k$ respectively its boundary, a sphere of dimension $k-1$. In particular, $D^0 = \{0\}$ is a one-point space and $\partial D^0 = \emptyset$ is empty.

Definition 1.2. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *global equivalence* if the following condition holds: for every compact Lie group G, every G-representation V, every $k \geq 0$ and all continuous maps $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $\beta|_{\partial D^k} = f(V)^G \circ \alpha$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^G$ such that $\lambda|_{\partial D^k} = X(\varphi)^G \circ \alpha$ and such that $f(W)^G \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\varphi)^G \circ \beta$.

In other words, for every commutative square on the left

$$\begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[d, "f(V)^G"] \\ D^k \arrow[r, "\beta"'] & Y(V)^G \end{tikzcd} \qquad \begin{tikzcd} \partial D^k \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[r, "X(\varphi)^G"] & X(W)^G \arrow[d, "f(W)^G"] \\ D^k \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G \end{tikzcd}$$

there exists the lift λ on the right hand side that makes the upper left triangle commute on the nose, and the lower right triangle up to homotopy relative to ∂D^k. In such a situation we will often refer to the pair (α, β) as a 'lifting problem' and we will say that the pair (φ, λ) *solves the lifting problem.*

Example 1.3. If $X = \underline{A}$ and $Y = \underline{B}$ are the constant orthogonal spaces with values the spaces A respectively B, and $f = \underline{g}$ the constant morphism associated to a continuous map $g : A \longrightarrow B$, then $\underline{g}$ is a global equivalence if and only if for every commutative square

$$\begin{tikzcd} \partial D^k \arrow[r] \arrow[d, "\mathrm{incl}"'] & A \arrow[d, "g"] \\ D^k \arrow[r] \arrow[ur, dashed, "\lambda"] & B \end{tikzcd}$$

there exists a lift λ that makes the upper left triangle commute, and the lower right triangle up to homotopy relative to ∂D^k. But this is one of the equivalent ways of characterizing weak equivalences of spaces, compare [**110**, Sec. 9.6, Lemma]. So $\underline{g}$ is a global equivalence if and only if g is a weak equivalence.

Remark 1.4. The notion of global equivalence is meant to capture the idea that for every compact Lie group G, some induced morphism

$$\mathrm{hocolim}_V\, f(V) \ : \ \mathrm{hocolim}_V\, X(V) \ \longrightarrow \ \mathrm{hocolim}_V\, Y(V)$$

is a G-weak equivalence, where '$\mathrm{hocolim}_V$' is a suitable homotopy colimit over all G-representations V along all equivariant linear isometric embeddings. This is a useful way to think about global equivalences, and it could be made precise by letting V run over the poset of finite dimensional subrepresentations of a complete G-universe and using the Bousfield-Kan construction of a homotopy colimit over this poset. However, the actual definition that we work with has the advantage that we do not have to make precise what we mean by 'all' G-representations and we do not have to define or manipulate homotopy colimits. Since the 'poset of all G-representations' has a cofinal subsequence, called an *exhaustive sequence* in Definition 1.6, we can conveniently model the 'homotopy colimit over all G-representations' as the mapping telescope over an exhaustive sequence.

In many examples of interest, all the structure maps of an orthogonal space Y are closed embeddings. When this is the case, the actual colimit (over the subrepresentations of a complete universe) of the G-spaces $Y(V)$ serves the purpose of a 'homotopy colimit over all representations', and it can be used to detect global equivalences, compare Proposition 1.18 below.

We will now establish some useful criteria for detecting global equivalences. The following more technical lemma will be needed. We call a continuous map $f : A \longrightarrow B$ between compactly generated spaces an

h-cofibration if it has the homotopy extension property, i.e., given a continuous map $\varphi : B \longrightarrow X$ and a homotopy $H : [0,1] \times A \longrightarrow X$ starting with φf, there is a homotopy $\bar{H} : [0,1] \times B \longrightarrow X$ starting with φ such that $\bar{H} \circ ([0,1] \times f) = H$. All h-cofibrations in the category of compactly generated spaces are closed embeddings, compare Proposition A.1.19.

LEMMA 1.5. *Let A be a subspace of a space B such that the inclusion $A \longrightarrow B$ is an h-cofibration. Let $f : X \longrightarrow Y$ be a continuous map and*

$$H \ : \ A \times [0,1] \ \longrightarrow \ X \qquad \textit{and} \qquad K \ : \ B \times [0,1] \ \longrightarrow \ Y$$

homotopies such that $K|_{A\times[0,1]} = fH$. Then the lifting problem (H_0, K_0) has a solution if and only if the lifting problem (H_1, K_1) has a solution.

PROOF. The problem is symmetric, so we only show one direction. We suppose that the lifting problem (H_0, K_0) has a solution consisting of a continuous map $\lambda : B \longrightarrow X$ such that $\lambda|_A = H_0$ and a homotopy $G : B \times [0,1] \longrightarrow Y$ such that

$$G_0 \ = \ f \circ \lambda \ , \quad G_1 \ = \ K_0 \qquad \text{and} \qquad (G_t)|_A \ = \ f \circ H_0$$

for all $t \in [0,1]$. The homotopy extension property provides a homotopy $H' : B \times [0,1] \longrightarrow X$ such that

$$H'_0 \ = \ \lambda \qquad \text{and} \qquad H'|_{A\times[0,1]} \ = \ H \ .$$

Then the map $\lambda' = H'_1 : B \longrightarrow X$ satisfies

$$\lambda'|_A \ = \ (H'_1)|_A \ = \ H_1 \ .$$

We define a continuous map $J : B \times [0,3] \longrightarrow Y$ by

$$J_t \ = \ \begin{cases} f \circ H'_{1-t} & \text{for } 0 \le t \le 1, \\ G_{t-1} & \text{for } 1 \le t \le 2, \text{ and} \\ K_{t-2} & \text{for } 2 \le t \le 3. \end{cases}$$

In particular,

$$J_0 \ = \ f \circ \lambda' \qquad \text{and} \qquad J_3 \ = \ K_1 \ ;$$

so J almost witnesses the fact that λ' solves the lifting problem (H_1, K_1), *except* that J is not a relative homotopy.

We improve J to a relative homotopy from $f \circ \lambda'$ to K_1. We define a continuous map $L : A \times [0,3] \times [0,1] \longrightarrow Y$ by

$$L(-,t,s) \ = \ \begin{cases} f \circ H_{1-t} & \text{for } 0 \le t \le s, \\ f \circ H_{1-s} & \text{for } s \le t \le 3-s, \text{ and} \\ f \circ H_{t-2} & \text{for } 3-s \le t \le 3. \end{cases}$$

Then $L(-,-,0)$ is the constant homotopy at the map $f \circ H_1$, and

$$L(-,-,1) \ = \ J|_{A\times[0,3]} \ : \ A \times [0,3] \ \longrightarrow \ Y \ .$$

Since the inclusion of A into B is an h-cofibration, the map

$$B \times \{0\} \cup_{A\times\{0\}} A \times [0,1] \ \longrightarrow \ B \times [0,1]$$

has a retraction; hence the map

$$B \times \{0\} \times [0,1] \cup_{A\times\{0\}\times[0,1]} A \times [0,1] \times [0,1] \ \longrightarrow \ B \times [0,1] \times [0,1]$$

has a retraction as well. We abbreviate $D = [0,3] \times \{1\} \cup \{0,3\} \times [0,1]$; the pair of spaces $([0,3] \times [0,1],\ D)$ is pair homeomorphic to the pair $([0,1] \times [0,1], \{0\} \times [0,1])$. So the canonical map

$$B \times D \cup_{A\times D} A \times [0,3] \times [0,1] \ \longrightarrow \ B \times [0,3] \times [0,1]$$

has a retraction. The map L and the map

$$J \cup \mathrm{const}_{f\lambda} \cup \mathrm{const}_{K_1} \ : \ B \times D \ = \ B \times ([0,3] \times \{1\} \cup \{0,3\} \times [0,1]) \ \longrightarrow \ Y$$

agree on $A \times D$, so there is a continuous map $\bar{L} : B \times [0,3] \times [0,1] \longrightarrow Y$ such that

$$\bar{L}(-,-,1) \ = \ J \ , \qquad \bar{L}|_{A\times[0,3]\times[0,1]} \ = \ L \ ,$$

and

$$\bar{L}(-,0,s) \ = \ f \circ \lambda \qquad \text{and} \qquad \bar{L}(-,1,s) \ = \ K_1$$

for all $s \in [0,1]$. The map $\bar{J} = \bar{L}(-,-,0) : B \times [0,3] \longrightarrow Y$ then satisfies

$$\bar{J}|_{A\times[0,3]} \ = \ \bar{L}(-,-,0)|_{A\times[0,3]} \ = \ L(-,-,0) \ ,$$

which is the constant homotopy at the map $f \circ H_1$; so $\bar{J}$ is a homotopy (parametrized by $[0,3]$ instead of $[0,1]$) *relative* to i. Because

$$\bar{J}_0 \ = \ \bar{L}(-,0,0) \ = \ f \circ \lambda \qquad \text{and} \qquad \bar{J}_3 \ = \ \bar{L}(-,3,0) \ = \ K_1 \ ,$$

the homotopy $\bar{J}$ witnesses that λ' solves the lifting problem (H_1, K_1). □

Definition 1.6. Let G be a compact Lie group. An *exhaustive sequence* is a nested sequence

$$V_1 \ \subset \ V_2 \ \subset \ \dots \ \subset \ V_n \ \subset \ \dots$$

of finite dimensional G-representations such that every finite dimensional G-representation admits a linear isometric G-embedding into some V_n.

Given an exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations and an orthogonal space Y, the values of Y at the representations and their inclusions form a sequence of G-spaces and G-equivariant continuous maps.

$$Y(V_1) \ \longrightarrow \ Y(V_2) \ \longrightarrow \ \cdots \ \longrightarrow \ Y(V_i) \ \longrightarrow \ \cdots \ .$$

We denote by

$$\mathrm{tel}_i \ Y(V_i)$$

the mapping telescope of this sequence of G-spaces; this telescope inherits a natural G-action.

We recall that G-equivariant continuous map $f : A \longrightarrow B$ between G-spaces is a *G-weak equivalence* if for every closed subgroup H of G the map $f^H : A^H \longrightarrow B^H$ of H-fixed points is a weak homotopy equivalence (in the non-equivariant sense).

Proposition 1.7. *For every morphism of orthogonal spaces $f : X \longrightarrow Y$, the following four conditions are equivalent.*

(i) *The morphism f is a global equivalence.*

(ii) *For every compact Lie group G, every G-representation V, every finite G-CW-pair (B, A) and all continuous G-maps $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ such that $\beta|_A = f(V) \circ \alpha$, there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ \alpha$ and such that $f(W) \circ \lambda$ is G-homotopic, relative to A, to $Y(\varphi) \circ \beta$.*

(iii) *For every compact Lie group G and every exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations the induced map*

$$\mathrm{tel}_i \, f(V_i) \ : \ \mathrm{tel}_i \, X(V_i) \ \longrightarrow \ \mathrm{tel}_i \, Y(V_i)$$

is a G-weak equivalence.

(iv) *For every compact Lie group G there is an exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations such that the induced map*

$$\mathrm{tel}_i \, f(V_i) \ : \ \mathrm{tel}_i \, X(V_i) \ \longrightarrow \ \mathrm{tel}_i \, Y(V_i)$$

is a G-weak equivalence.

Proof. At various places in the proof we use without explicit mentioning that taking G-fixed points commutes with formation of the mapping telescope; indeed this follows from the fact that taking G-fixed points commutes with pushouts along closed embeddings and sequential colimits along closed embeddings.

(i)$\Longrightarrow$(ii) We argue by induction over the number of the relative G-cells in (B, A). If $B = A$, then $\lambda = \alpha$ solves the lifting problem, and there is nothing to show. Now we suppose that A is a proper subcomplex of B. We choose a G-CW-subcomplex B' that contains A and such that (B, B') has exactly one equivariant cell. Then (B', A) has strictly fewer cells, and the restricted equivariant lifting problem $(\alpha : A \longrightarrow X(V), \beta' = \beta|_{B'} : B' \longrightarrow Y(V))$ has a solution $(\varphi : V \longrightarrow W', \lambda' : B' \longrightarrow X(W'))$ by the inductive hypothesis.

We choose a characteristic map for the last cell, i.e., a pushout square of G-spaces

$$\begin{tikzcd} G/H \times \partial D^k \arrow[r, "\chi"] \arrow[d, "\mathrm{incl}"'] & B' \arrow[d, "\mathrm{incl}"] \\ G/H \times D^k \arrow[r, "\chi"'] & B \end{tikzcd}$$

in which H is a closed subgroup of G. We arrive at the non-equivariant lifting problem on the left:

$$\begin{tikzcd} \partial D^k \arrow[r, "(\lambda')^H \circ \bar\chi"] \arrow[d, "\mathrm{incl}"'] & X(W')^H \arrow[d, "f(V)^H"] \\ D^k \arrow[r, "\beta^H \circ \bar\chi"'] & Y(W')^H \end{tikzcd} \qquad \begin{tikzcd} \partial D^k \arrow[r, "(\lambda')^H \circ \bar\chi"] \arrow[d, "\mathrm{incl}"'] & X(W')^H \arrow[r, "X(\psi)^H"] & X(W)^H \arrow[d, "f(W)^H"] \\ D^k \arrow[r, "\beta^H \circ \bar\chi"'] \arrow[urr, dashed, "\lambda"] & Y(W')^H \arrow[r, "Y(\psi)^H"'] & Y(W)^H \end{tikzcd}$$

Here $\bar\chi = \chi(H, -) : D^k \longrightarrow B^H$. Since f is a global equivalence, there is an H-equivariant linear isometric embedding $\psi : W' \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^H$ such that $\lambda|_{\partial D^k} = X(\psi)^H \circ (\lambda')^H \circ \bar\chi$ and $f(W)^H \circ \lambda$ is homotopic, relative ∂D^k to $Y(\psi)^H \circ \beta^G \circ \bar\chi$, as illustrated by the diagram on the right above. By enlarging W, if necessary, we can assume without loss of generality that W is underlying a G-representation and ψ is even G-equivariant.

The G-equivariant extension of λ

$$G/H \times D^k \longrightarrow X(W) \ , \quad (gH, x) \ \longmapsto \ g \cdot \lambda(x)$$

and the map $X(\psi) \circ \lambda' : B' \longrightarrow X(W)$ then agree on $G/H \times \partial D^k$, so they glue to a G-map

$$\lambda \ : \ B \ \longrightarrow \ X(W) \ .$$

The pair $(\psi\varphi : V \longrightarrow W, \lambda : B \longrightarrow X(W))$ then solves the original lifting problem (α, β).

(ii)$\Longrightarrow$(iii) We suppose that f satisfies (ii), and we let G be any compact Lie group and $\{V_i\}_{i \geq 1}$ an exhaustive sequence of G-representations. We consider an equivariant lifting problem, i.e., a finite G-CW-pair (B, A) and a commutative square:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & \mathrm{tel}_i\, X(V_i) \arrow[d, "\mathrm{tel}_i\, f(V_i)"] \\ B \arrow[r, "\beta"'] & \mathrm{tel}_i\, Y(V_i) \end{tikzcd}$$

We show that every such lifting problem has an equivariant solution. Since B and A are compact, there is an $n \geq 0$ such that α has image in the truncated telescope $\mathrm{tel}_{[0,n]}\, X(V_i)$ and β has image in the truncated telescope $\mathrm{tel}_{[0,n]}\, Y(V_i)$. There is a natural equivariant homotopy from the identity of the truncated telescope

$\operatorname{tel}_{[0,n]} X(V_i)$ to the composite

$$\operatorname{tel}_{[0,n]} X(V_i) \xrightarrow{\text{proj}} X(V_n) \xrightarrow{\text{incl}} \operatorname{tel}_{[0,n]} X(V_i)$$

Naturality means that this homotopy is compatible with the same homotopy for the telescope of the G-spaces $Y(V_i)$. Lemma 1.5 (or rather its G-equivariant generalization) applies to these homotopies, so instead of the original lifting problem we may solve the homotopic lifting problem

$$\begin{tikzcd}
A \arrow[r, "\alpha'"] \arrow[d, "\text{incl}"'] & X(V_n) \arrow[r, "i_n"] \arrow[d, "f(V_n)"'] & \operatorname{tel}_i X(V_i) \arrow[d, "\operatorname{tel}_i f(V_i)"] \\
B \arrow[r, "\beta'"'] & Y(V_n) \arrow[r, "i_n"'] & \operatorname{tel}_i Y(V_i)
\end{tikzcd}$$

where α' is the composite of the projection $\operatorname{tel}_{[0,n]} X(V_i) \longrightarrow X(V_n)$ with α, viewed as a map into the truncated telescope, and similarly for β'.

Since f satisfies (ii), the lifting problem $(\alpha' : A \longrightarrow X(V_n), \beta : B \longrightarrow Y(V_n))$ has a solution after enlarging V_n along some linear isometric G-embedding. Since the sequence $\{V_i\}_{i\geq 1}$ is cofinal, we can take this embedding as the inclusion $i : V_n \longrightarrow V_m$ for some $m \geq n$, i.e., there is a continuous map $\lambda : D^k \longrightarrow X(V_m)$ such that $\lambda|_{\partial D^k} = X(i)^G \circ \alpha'$ and such that $f(V_m)^G \circ \lambda$ is homotopic, relative ∂D^k, to $Y(i)^G \circ \beta'$, compare the diagram:

$$\begin{tikzcd}
A \arrow[r, "\alpha'"] \arrow[d, "\text{incl}"'] & X(V_n) \arrow[r, "X(i)"] & X(V_m) \arrow[d, "f(V_m)"] \\
B \arrow[r, "\beta'"'] \arrow[urr, dashed, "\lambda"] & Y(V_n) \arrow[r, "Y(i)"'] & Y(V_m)
\end{tikzcd}$$

The composite

$$X(V_n) \xrightarrow{X(i)} X(V_m) \xrightarrow{i_m} \operatorname{tel}_i X(V_i)$$

does *not* agree with $i_n : X(V_n) \longrightarrow \operatorname{tel}_i X(V_i)$; so the composite $i_n \circ \lambda : B \longrightarrow \operatorname{tel}_i X(V_i)$ does not quite solve the (modified) lifting problem $(i_n \circ \alpha', i_n \circ \beta')$. But there is a G-equivariant homotopy $H : X(V_n) \times [0,1] \longrightarrow \operatorname{tel}_i X(V_i)$ between $i_m \circ X(i)$ and i_n, and a similar homotopy $K : Y(V_n) \times [0,1] \longrightarrow \operatorname{tel}_i Y(V_i)$ for Y instead of X. These homotopies satisfy

$$K \circ (f(V_n) \times [0,1]) \;=\; (\operatorname{tel}_i f(V_i)) \circ H \;,$$

so Lemma 1.5 implies that the modified lifting problem, and hence the original lifting problem, has an equivariant solution.

Condition (iii) clearly implies condition (iv).

(iv)$\Longrightarrow$(i) We let G be a compact Lie group, V a G-representation, $k \geq 0$ and $(\alpha : \partial D^k \longrightarrow X(V)^G, \beta : D^k \longrightarrow Y(V)^G)$ a lifting problem, i.e., such that $\beta|_{\partial D^k} = f(V)^G \circ \alpha$. Since the sequence $\{V_i\}$ is exhaustive we can embed V into some V_n by a linear isometric G-map and thereby assume without loss of generality that $V = V_n$.

We let $i_n : X(V_n) \longrightarrow \operatorname{tel}_i X(V_i)$ and $i_n : Y(V_n) \longrightarrow \operatorname{tel}_i Y(V_i)$ be the canonical maps. Since $\operatorname{tel}_i f(V_i) : \operatorname{tel}_i X(V_i) \longrightarrow \operatorname{tel}_i Y(V_i)$ is a G-weak equivalence, there is a continuous map $\lambda : D^k \longrightarrow (\operatorname{tel}_i X(V_i))^G$ such that $\lambda|_{\partial D^k} = i_n^G \circ \alpha$ and $(\operatorname{tel}_i f(V_i))^G \circ \lambda$ is homotopic, relative ∂D^k, to $i_n^G \circ \beta$. Since fixed points commute with mapping telescopes and since D^k is compact, there is an $m \geq n$ such that λ and the relative homotopy that witnesses the relation $(\operatorname{tel}_i f(V_i))^G \circ \lambda \simeq i_n^G \circ \beta$ both have image in $\operatorname{tel}_{[0,m]} X(V_i)^G$, the truncated

telescope of the G-fixed points. The following diagram commutes

$$\begin{tikzcd}
X(V_n)^G \arrow[r, "\mathrm{can}"'] \arrow[rrr, bend left, "X(i)^G"] \arrow[d, "f(V_n)^G"'] & \mathrm{tel}_{[0,n]}\, X(V_i)^G \arrow[r, "\mathrm{incl}"'] \arrow[d, "\mathrm{tel}\, f(V_i)^G"'] & \mathrm{tel}_{[0,m]}\, X(V_i)^G \arrow[r, "\mathrm{proj}"'] \arrow[d, "\mathrm{tel}\, f(V_i)^G"] & X(V_m)^G \arrow[d, "f(V_m)^G"] \\
Y(V_n)^G \arrow[r, "\mathrm{can}"] \arrow[rrr, bend right, "Y(i)^G"'] & \mathrm{tel}_{[0,n]}\, Y(V_i)^G \arrow[r, "\mathrm{incl}"] & \mathrm{tel}_{[0,m]}\, Y(V_i)^G \arrow[r, "\mathrm{proj}"] & Y(V_m)^G
\end{tikzcd}$$

where the right horizontal maps are the projections of the truncated telescope to the last term. So projecting from $\mathrm{tel}_{[0,m]}\, X(V_i)^G$ to $X(V_m)^G$ and from $\mathrm{tel}_{[0,m]}\, Y(V_i)^G$ to $Y(V_m)^G$ produces the desired solution to the lifting problem. □

We establish some basic facts about the class of global equivalences. We need some new vocabulary. A *homotopy* between two morphisms of orthogonal spaces $f, f' : X \longrightarrow Y$ is a morphism

$$H \ : \ [0,1] \times X \ \longrightarrow \ Y$$

such that $H(-,0) = f$ and $H(-,1) = f'$.

Definition 1.8. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *homotopy equivalence* if there is a morphism $g : Y \longrightarrow X$ such that gf and fg are homotopic to the respective identity morphisms. The morphism f is a *strong level equivalence* if for every compact Lie group G and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence.

If $f, f' : X \longrightarrow Y$ are homotopic morphisms of orthogonal spaces, then the maps $f(V)^G, f'(V)^G : X(V)^G \longrightarrow Y(V)^G$ are homotopic for every compact Lie group G and every G-representation V. So if f is a homotopy equivalence of orthogonal spaces, then the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a non-equivariant homotopy equivalence for every G-representation V. So every homotopy equivalence is in particular a strong level equivalence. By the following proposition, strong level equivalences are global equivalences, so schematically:

$$\text{homotopy equivalence} \implies \text{strong level equivalence} \implies \text{global equivalence}$$

A continuous map $\varphi : A \longrightarrow B$ is a *closed embedding* if it is injective and a closed map. Such a map is then a homeomorphism of A onto the closed subspace $\varphi(A)$ of B. If a compact Lie group G acts on two spaces A and B and $\varphi : A \longrightarrow B$ is a G-equivariant closed embedding, then the restriction $\varphi^G : A^G \longrightarrow B^G$ to G-fixed points is also a closed embedding. In particular, for every closed orthogonal space Y and every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ of G-representations, the induced map on G-fixed points $Y(\varphi)^G : Y(V)^G \longrightarrow Y(W)^G$ is also a closed embedding.

We call a morphism $f : A \longrightarrow B$ of orthogonal spaces an *h-cofibration* if it has the homotopy extension property, i.e., given a morphism of orthogonal spaces $\varphi : B \longrightarrow X$ and a homotopy $H : [0,1] \times A \longrightarrow X$ starting with φf, there is a homotopy $\bar{H} : [0,1] \times B \longrightarrow X$ starting with φ such that $\bar{H} \circ ([0,1] \times f) = H$.

Proposition 1.9. (i) *Every strong level equivalence is a global equivalence.*
(ii) *The composite of two global equivalences is a global equivalence.*
(iii) *If f, g and h are composable morphisms of orthogonal spaces such that hg and gf are global equivalences, then f, g, h and hgf are also global equivalences.*
(iv) *Every retract of a global equivalence is a global equivalence.*
(v) *A coproduct of any set of global equivalences is a global equivalence.*
(vi) *A finite product of global equivalences is a global equivalence.*

(vii) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of orthogonal spaces that are objectwise closed embeddings, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be global equivalences of orthogonal spaces that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is a global equivalence.*

(viii) *Let $f_n : Y_n \longrightarrow Y_{n+1}$ be a global equivalence of orthogonal spaces that is objectwise a closed embedding, for $n \geq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to the colimit of the sequence $\{f_n\}_{n\geq 0}$ is a global equivalence.*

(ix) *Let*

$$\begin{tikzcd} C \arrow[d, "\gamma"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ C' & A' \arrow[l, "g'"] \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a commutative diagram of orthogonal spaces such that g and g' are h-cofibrations. If the morphisms α, β and γ are global equivalences, then so is the induced morphism of pushouts

$$\gamma \cup \beta \ : \ C \cup_A B \ \longrightarrow \ C' \cup_{A'} B' \ .$$

(x) *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spaces such that f is a global equivalence. If in addition f or g is an h-cofibration, then the morphism k is a global equivalence.

(xi) *Let*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"'] & Y \end{tikzcd}$$

be a pullback square of orthogonal spaces in which f is a global equivalence. If in addition one of the morphisms f or h is a strong level fibration, then the morphism g is also a global equivalence.

Proof. (i) We let $f : X \longrightarrow Y$ be a strong level equivalence, G a compact Lie group, V a G-representation and $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ continuous maps such that $f(V)^G \circ \alpha = \beta|_{\partial D^k}$. Since f is a strong level equivalence, the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence, so there is a continuous map $\lambda : D^k \longrightarrow X(V)^G$ such that $\lambda|_{\partial D^k} = \alpha$ and $f(V)^G \circ \lambda$ is homotopic to β relative ∂D^k. So the pair (Id_V, λ) solves the lifting problem, and hence f is a global equivalence.

(ii) We let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be global equivalences, G a compact Lie group, (B, A) a finite G-CW-pair, V a G-representation and $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Z(V)$ continuous G-maps such that $(gf)(V) \circ \alpha = \beta|_A$. Since g is a global equivalence, the equivariant lifting problem $(f(V) \circ \alpha, \beta)$ has a solution $(\varphi : V \longrightarrow W,\ \lambda : B \longrightarrow Y(W))$ such that

$$\lambda|_A \ = \ Y(\varphi) \circ f(V) \circ \alpha \ = \ f(W) \circ X(\varphi) \circ \alpha \ ,$$

and $g(W) \circ \lambda$ is homotopic to $Z(\varphi) \circ \beta$ relative A. Since f is a global equivalence, the equivariant lifting problem $(X(\varphi) \circ \alpha, \lambda)$ has a solution $(\psi : W \longrightarrow U,\ \lambda' : B \longrightarrow X(U))$ such that

$$\lambda'|_A \ = \ X(\psi) \circ X(\varphi) \circ \alpha$$

and such that $f(U)\circ\lambda'$ is G-homotopic to $Y(\psi)\circ\lambda$ relative A. Then $(gf)(U)\circ\lambda'$ is G-homotopic, relative A, to

$$g(U)\circ Y(\psi)\circ\lambda \;=\; Z(\psi)\circ g(W)\circ\lambda$$

which in turn is G-homotopic to $Z(\psi\varphi)\circ\beta$, also relative A. So the pair $(\psi\varphi,\lambda')$ solves the original lifting problem for the morphism $gf : X \longrightarrow Z$.

The following proofs of parts (iii) and (iv) are somewhat lengthy, and alternative (and shorter) proofs can given by using the criterion of Proposition 1.7 and the fact that G-weak equivalences have the 2-out-of-6 property and are closed under retracts. My reason for including the longer proofs here is that they are elementary and proceed directly from the definition of global equivalences.

(iii) Step 1: We let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow Q$ be the three composable morphisms such that $gf : X \longrightarrow Z$ and $hg : Y \longrightarrow Q$ are global equivalences. We show that f is a global equivalence. We let G be a compact Lie group, V a G-representation, (B,A) a finite G-CW-pair, and $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ continuous G-maps such that $\beta|_A = f(V)\circ\alpha$. Since gf is a global equivalence and

$$(gf)(V)\circ\alpha \;=\; g(V)\circ\beta|_A \;=\; (g(V)\circ\beta)|_A\ ,$$

the equivariant lifting problem $(\alpha,\, g(V)\circ\beta)$ has a solution $(\varphi : V \longrightarrow W,\, \lambda : B \longrightarrow X(W))$ such that $\lambda|_A = X(\varphi)\circ\alpha$ and $(gf)(W)\circ\lambda$ is G-homotopic to $Z(\varphi)\circ g(V)\circ\beta$ relative A. We let

$$H \;:\; B\times[0,1] \;\longrightarrow\; Z(W)$$

be a relative G-homotopy that witnesses the latter fact. We arrive at the following commutative square:

$$\begin{tikzcd}
(A\times[0,1])\cup(B\times\{0,1\}) \arrow[r, "K\cup(f(W)\circ\lambda)\cup(Y(\varphi)\circ\beta)"] \arrow[d] & Y(W) \arrow[d, "(hg)(W)"] \\
B\times[0,1] \arrow[r, "h(W)\circ H"'] & Q(W)
\end{tikzcd}$$

Here $K : A\times[0,1] \longrightarrow Y(W)$ is the constant homotopy of the map $f(W)\circ X(\varphi)\circ\alpha = Y(\varphi)\circ f(V)\circ\alpha$. Since $(B\times[0,1], A\times[0,1]\cup B\times\{0,1\})$ admits the structure of a finite G-CW-pair, and since hg is a global equivalence, there is a G-equivariant linear isometric embedding $\psi : W \longrightarrow U$ and a continuous G-map $\lambda' : B\times[0,1] \longrightarrow Y(U)$ such that

$$\lambda'|_{A\times[0,1]\cup B\times\{0,1\}} \;=\; Y(\psi)\circ(K\cup(f(W)\circ\lambda)\cup(Y(\varphi)\circ\beta))\ .$$

So λ' is a G-homotopy, relative A, from

$$Y(\psi)\circ f(W)\circ\lambda \;=\; f(U)\circ X(\psi)\circ\lambda$$

to $Y(\psi\varphi)\circ\beta$. So the pair $(\psi\varphi, X(\psi)\circ\lambda)$ solves the original lifting problem for the morphism $f : X \longrightarrow Y$, and thus f is a global equivalence.

Step 2: We let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two composable morphisms such that gf and f are global equivalences. We show that g is a global equivalence. We let G be a compact Lie group, V a G-representation, (B,A) a finite G-CW-pair and $\alpha : A \longrightarrow Y(V)$ and $\beta : B \longrightarrow Z(V)$ continuous G-maps such that $g(V)\circ\alpha = \beta|_A$. Since f is a global equivalence and $(A,\emptyset)$ is a finite G-CW-pair, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : A \longrightarrow X(W)$ and a G-homotopy $H : A\times[0,1] \longrightarrow Y(W)$ from $f(W)\circ\lambda$ to $Y(\varphi)\circ\alpha$. Since $g(V)\circ\alpha = \beta|_A$, the equivariant homotopy extension property of the pair (B,A) provides a G-homotopy $K : B\times[0,1] \longrightarrow Z(W)$ from $g(W)\circ f(W)\circ\lambda$ to β that restrict to $g(W)\circ H$ on $A\times[0,1]$.

We define

$$\beta' \;=\; K(-,0) \;:\; B \;\longrightarrow\; Z(W)\ .$$

Then $\beta'|_A = g(W)\circ H(-,0) = (gf)(W)\circ\lambda$. Since gf is a global equivalence, there is a G-equivariant linear isometric embedding $\psi : W \longrightarrow U$ and a continuous G-map $\bar\lambda : B \longrightarrow X(U)$ such that $\bar\lambda|_A = X(\psi)\circ\lambda$

and $(gf)(U) \circ \bar{\lambda}$ is homotopic to $Z(\psi) \circ \beta'$ relative A. This means that the pair $(\psi, f(U) \circ \bar{\lambda})$ solves the equivariant lifting problem on the left:

$$\begin{tikzcd} A \arrow[r, "f(U)\circ X(\psi)\circ\lambda"] \arrow[d, "\mathrm{incl}"'] & Y(U) \arrow[d, "g(U)"] \\ B \arrow[r, "Z(\psi)\circ(gf)(W)\circ\lambda"'] \arrow[ur, dashed, "f(U)\circ\bar{\lambda}"] & Z(U) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "Y(\psi\varphi)\circ\alpha"] \arrow[d, "\mathrm{incl}"'] & Y(U) \arrow[d, "g(U)"] \\ B \arrow[r, "Z(\psi)\circ\beta"'] \arrow[ur, dashed, "\lambda'"] & Z(U) \end{tikzcd} \tag{1.10}$$

If we feed the homotopies

$$Y(\psi) \circ H \ : \ A \times [0,1] \ \longrightarrow \ Y(U) \qquad \text{and} \qquad Z(\psi) \circ K \ : \ B \times [0,1] \ \longrightarrow \ Z(U)$$

into Lemma 1.5, we conclude that also the equivariant lifting problem on the right has a solution $\lambda' : B \longrightarrow Y(U)$. The pair $(\psi\varphi, \lambda')$ then solves the original equivariant lifting problem.

Step 3: Now we prove the 2-out-of-6 property. We let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow Q$ be the three composable morphisms such that $gf : X \longrightarrow Z$ and $hg : Y \longrightarrow Q$ are global equivalences. Then f is a global equivalence by Step 1, and hence g is a global equivalence by Step 2. Applying Step 2 to g and h (instead of f and g) shows that h is a global equivalence. Finally, hgf is a global equivalence by part (ii).

(iv) Let g be a global equivalence and f a retract of g. So there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & \bar{X} \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "j"'] & \bar{Y} \arrow[r, "s"'] & Y \end{tikzcd}$$

such that $ri = \mathrm{Id}_X$ and $sj = \mathrm{Id}_Y$. We let G be a compact Lie group, V a G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ continuous maps such that $f(V) \circ \alpha = \beta|_A$. Since g is a global equivalence and

$$g(V) \circ i(V) \circ \alpha \ = \ j(V) \circ f(V) \circ \alpha \ = \ (j(V) \circ \beta)|_A \ ,$$

there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow \bar{X}(W)$ such that $\lambda|_A = \bar{X}(\varphi) \circ i(V) \circ \alpha$ and $g(W) \circ \lambda$ is G-homotopic to $\bar{Y}(\varphi) \circ j(V) \circ \beta$ relative A. Then

$$(r(W) \circ \lambda)|_A \ = \ r(W) \circ \bar{X}(\varphi) \circ i(V) \circ \alpha \ = \ X(\varphi) \circ r(V) \circ i(V) \circ \alpha \ = \ X(\varphi) \circ \alpha$$

and

$$f(W) \circ r(W) \circ \lambda \ = \ s(W) \circ g(W) \circ \lambda$$

is G-homotopic to

$$s(W) \circ \bar{Y}(\varphi) \circ j(V) \circ \beta \ = \ Y(\varphi) \circ s(V) \circ j(V) \circ \beta \ = \ Y(\varphi) \circ \beta$$

relative A. So the pair $(\varphi, r(W) \circ \lambda)$ solves the original lifting problem for the morphism $f : X \longrightarrow Y$; thus f is a global equivalence.

Part (v) holds because the disc D^k is connected, so any lifting problem for a coproduct of orthogonal spaces is located in one of the summands.

For part (vi) it suffices to consider a product of two global equivalences $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$. Because global equivalences are closed under composition (part (ii)) and $f \times f' = (f \times Y') \circ (X \times f')$, it suffices to show that for every global equivalence $f : X \longrightarrow Y$ and every orthogonal space Z the morphism $f \times Z : X \times Z \longrightarrow Y \times Z$ is a global equivalence. But this is straightforward: we let G be a compact Lie group, V a G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow (X \times Z)(V)$ and $\beta : B \longrightarrow (Y \times Z)(V)$ continuous G-maps such that $(f \times Z)(V) \circ \alpha = \beta|_A$. Because

$$(X \times Z)(V) \ = \ X(V) \times Z(V)$$

and similarly for $(Y\times Z)(V)$, we have $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ for continuous G-maps $\alpha_1 : A \longrightarrow X(V)$, $\alpha_2 : A \longrightarrow Z(V)$, $\beta_1 : B \longrightarrow Y(V)$ and $\beta_2 : B \longrightarrow Z(V)$. The relation $(f \times Z)(V) \circ (\alpha_1, \alpha_2) = (\beta_1, \beta_2)|_A$ shows that $\alpha_2 = (\beta_2)|_A$. Since f is a global equivalence, the equivariant lifting problem (α_1, β_1) for $f(V)$ has a solution $(\varphi : V \longrightarrow W, \lambda : B \longrightarrow X(W))$ such that $\lambda|_A = X(\varphi) \circ \alpha_1$ and $f(W) \circ \lambda$ is G-homotopic to $Y(\varphi) \circ \beta_1$ relative A. Then the pair $(\varphi, (\lambda, Z(\varphi) \circ \beta_2))$ solves the original lifting problem, so $f \times Z$ is a global equivalence.

(vii) We let G be a compact Lie group, V a G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow X_\infty(V)$ and $\beta : B \longrightarrow Y_\infty(V)$ continuous G-maps such that $\psi_\infty(V) \circ \alpha = \beta|_A : A \longrightarrow Y_\infty(V)$. Since A and B are compact and $X_\infty(V)$ respectively $Y_\infty(V)$ are colimits of sequences of closed embeddings, the maps α and β factor through maps

$$\bar\alpha \ : \ A \ \longrightarrow \ X_n(V) \qquad \text{respectively} \qquad \bar\beta \ : \ B \ \longrightarrow \ Y_n(V)$$

for some $n \geq 0$, see for example [**79**, Prop. 2.4.2]. Since the canonical maps $X_n(V) \longrightarrow X_\infty(V)$ and $Y_n(V) \longrightarrow Y_\infty(V)$ are injective, $\bar\alpha$ and $\bar\beta$ are again G-equivariant. Moreover, the relation $\psi_n(V)\circ\bar\alpha = \bar\beta|_A : A \longrightarrow Y_n(V)$ holds because it holds after composition with the injective map $Y_n(V) \longrightarrow Y_\infty(V)$.

Since ψ_n is a global equivalence, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X_n(W)$ such that $\lambda|_A = X_n(\varphi) \circ \bar\alpha$ and $\psi_n(W) \circ \lambda$ is G-homotopic to $Y_n(\varphi) \circ \bar\beta$ relative A. We let $\lambda' : B \longrightarrow X_\infty(W)$ be the composite of λ and the canonical map $X_n(W) \longrightarrow X_\infty(W)$. Then the pair (φ, λ') is a solution for the original lifting problem, and hence $\varphi_\infty : X_\infty \longrightarrow Y_\infty$ is a global equivalence.

(viii) This is a special case of part (vii) where we set $X_n = Y_0$, $e_n = \mathrm{Id}_{Y_0}$ and $\psi_n = f_{n-1} \circ \cdots \circ f_0 : Y_0 \longrightarrow Y_n$. The morphism ψ_n is then a global equivalence by part (ii), and Y_0 is a colimit of the constant first sequence. Since the morphism ψ_∞ induced on the colimits of the two sequences is the canonical map $Y_0 \longrightarrow Y_\infty$, part (vii) proves the claim.

(ix) Let G be a compact Lie group. We choose an exhaustive sequence $\{V_i\}_{i\geq 1}$ of finite dimensional G-subrepresentations of the complete universe $\mathcal{U}_G$. Since α, β and γ are global equivalences, the three vertical maps in the following commutative diagram of G-spaces are G-weak equivalences, by Proposition 1.7:

$$\begin{tikzcd}
\mathrm{tel}_i\, C(V_i) \arrow[d, "\mathrm{tel}\,\gamma(V_i)"'] & \mathrm{tel}_i\, A(V_i) \arrow[l, "\mathrm{tel}\, g(V_i)"'] \arrow[r, "\mathrm{tel}\, f(V_i)"] \arrow[d, "\mathrm{tel}\,\alpha(V_i)"] & \mathrm{tel}_i\, B(V_i) \arrow[d, "\mathrm{tel}\,\beta(V_i)"] \\
\mathrm{tel}_i\, C'(V_i) & \mathrm{tel}_i\, A'(V_i) \arrow[l, "\mathrm{tel}\, g'(V_i)"] \arrow[r, "\mathrm{tel}\, f'(V_i)"'] & \mathrm{tel}_i\, B'(V_i)
\end{tikzcd}$$

Since mapping telescopes also commute with product with $[0, 1]$ and retracts, the maps $\mathrm{tel}_i\, g(V_i)$ and $\mathrm{tel}_i\, g'(V_i)$ are h-cofibrations of G-spaces. The induced map of the horizontal pushouts is thus a G-weak equivalence by Proposition A.2.9. Since formation of mapping telescopes commutes with pushouts, the map

$$\mathrm{tel}_i(\gamma \cup \beta)(V_i) \ : \ \mathrm{tel}_i(C \cup_A B)(V_i) \ \longrightarrow \ \mathrm{tel}_i(C' \cup_{A'} B')(V_i)$$

is a G-weak equivalence. The claim thus follows by another application of the telescope criterion for global equivalences, Proposition 1.7.

(x) In the commutative diagram

$$\begin{tikzcd}
C \arrow[d, equal] & A \arrow[l, "g"'] \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "f"] \\
C & A \arrow[l, "g"] \arrow[r, "f"'] & B
\end{tikzcd}$$

all vertical morphisms are global equivalences. If g is an h-cofibration, we can apply part (ix) to this square to get the desired conclusion. If f is an h-cofibration, we apply part (ix) after interchanging the roles of left and right horizontal morphisms.

(xi) We let G be a compact Lie group, V a G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow P(V)$ and $\beta : B \longrightarrow Z(V)$ continuous G-maps such that $g(V) \circ \alpha = \beta|_A$. Since f is a global equivalence, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ k(V) \circ \alpha$ and such that $f(W) \circ \lambda$ is G-homotopic, relative to A, to $Y(\varphi) \circ h(V) \circ \beta$. We let $H : B \times [0,1] \longrightarrow Y(W)$ be a relative G-homotopy from $Y(\varphi) \circ h(V) \circ \beta = h(W) \circ Z(\varphi) \circ \beta$ to $f(W) \circ \lambda$. Now we distinguish two cases.

Case 1: The morphism h is a strong level fibration. We can choose a lift $\bar{H}$ in the square

$$\begin{tikzcd}
B \times 0 \cup_{A \times 0} A \times [0,1] \arrow[r, "(Z(\varphi)\circ\beta)\cup K"] \arrow[d, "\sim"'] & Z(W) \arrow[d, two heads, "h(W)"] \\
B \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\bar{H}"'] & Y(W)
\end{tikzcd}$$

where $K : A \times [0,1] \longrightarrow Z(W)$ is the constant homotopy from $g(W) \circ P(\varphi) \circ \alpha$ to itself. Since the square is a pullback and $h(W) \circ \bar{H}(-,1) = H(-,1) = f(W) \circ \lambda$, there is a unique continuous G-map $\bar{\lambda} : B \longrightarrow P(W)$ that satisfies

$$g(W) \circ \bar{\lambda} \;=\; \bar{H}(-,1) \qquad \text{and} \qquad k(W) \circ \bar{\lambda} \;=\; \lambda \;.$$

The restriction of $\bar{\lambda}$ to A satisfies

$$\begin{aligned}
g(W) \circ \bar{\lambda}|_A \;&=\; \bar{H}(-,1)|_A \;=\; g(W) \circ P(\varphi) \circ \alpha \qquad \text{and} \\
k(W) \circ \bar{\lambda}|_A \;&=\; \lambda|_A \;=\; X(\varphi) \circ k(V) \circ \alpha \;=\; k(W) \circ P(\varphi) \circ \alpha \;.
\end{aligned}$$

The pullback property thus implies that $\bar{\lambda}|_A = P(\varphi) \circ \alpha$.

Finally, the composite $g(W) \circ \bar{\lambda}$ is homotopic, relative A and via $\bar{H}$, to $\bar{H}(-,0) = Z(\varphi) \circ \beta$. This is the required lifting data, and we have verified the defining property of a global equivalence for the morphism g.

Case 2: The morphism f is a strong level fibration. The argument is similar as in the first case. Now we can choose a lift H' in the square

$$\begin{tikzcd}
B \times 1 \cup_{A \times 1} A \times [0,1] \arrow[r, "\lambda\cup K'"] \arrow[d, "\sim"'] & X(W) \arrow[d, two heads, "f(W)"] \\
B \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, "H'"'] & Y(W)
\end{tikzcd}$$

where $K' : A \times [0,1] \longrightarrow X(W)$ is the constant homotopy from $X(\varphi) \circ k(V) \circ \alpha$ to itself. Since the square is a pullback and $f(W) \circ H'(-,0) = H(-,0) = h(W) \circ Z(\varphi) \circ \beta$, there is a unique continuous map $\bar{\lambda} : B \longrightarrow P(W)$ that satisfies

$$g(W) \circ \bar{\lambda} \;=\; Z(\varphi) \circ \beta \qquad \text{and} \qquad k(W) \circ \bar{\lambda} \;=\; H'(-,0) \;.$$

The restriction of $\bar{\lambda}$ to A satisfies

$$\begin{aligned}
g(W) \circ \bar{\lambda}|_A \;&=\; Z(\varphi) \circ g(V) \circ \alpha \;=\; g(W) \circ P(\varphi) \circ \alpha \qquad \text{and} \\
k(W) \circ \bar{\lambda}|_A \;&=\; H'(-,0)|_A \;=\; X(\varphi) \circ k(V) \circ \alpha \;=\; k(W) \circ P(\varphi) \circ \alpha \;.
\end{aligned}$$

The pullback property thus implies that $\bar{\lambda}|_A = P(\varphi) \circ \alpha$. Since $g(W) \circ \bar{\lambda} = Z(\varphi) \circ \beta$, this is the required lifting data, and we have verified the defining property of a global equivalence for the morphism g. $\square$

The restriction to *finite* products is essential in part (vi) of the previous Proposition 1.9, i.e., an infinite product of global equivalences need not be a global equivalence. The following simple example illustrates this. We let Y_n denote the orthogonal space with

$$Y_n(V) \;=\; \begin{cases} \{0,1\} & \text{if } \dim(V) \le n, \text{ and} \\ \{0\} & \text{if } \dim(V) > n. \end{cases}$$

The structure maps of Y_n are either the identity of $\{0,1\}$ or the unique map to $\{0\}$. The unique morphism $Y_n \longrightarrow *$ to a terminal orthogonal space (i.e., constant with value any one-point space) is a global equivalence for every $n \ge 0$. Still, the product

$$\prod_{n\ge 0} Y_n \;\longrightarrow\; \prod_{n\ge 0} * \cong *$$

of these global equivalences is *not* a global equivalence. To see this we consider the map

$$\alpha \;:\; \partial D^1 \;=\; \{-1,1\} \;\longrightarrow\; \prod_{n\ge 0} Y_n(0) \;=\; \prod_{n\ge 0} \{0,1\}^n$$

such that $\alpha(-1) = (0,0,0,\dots)$ and $\alpha(1) = (1,1,1,\dots)$. Then no matter how large we choose the inner product space W, the composite

$$\partial D^1 \;\xrightarrow{\alpha}\; \prod_{n\ge 0} Y_n(0) \;\xrightarrow{\prod_{n\ge 0} Y_n(\varphi)}\; \prod_{n\ge 0} Y_n(W) \;=\; \prod_{n\ge 0} \{0,1\}^n$$

never admits a continuous extension to D^1.

The following proposition provides a lot of flexibility for changing an orthogonal space into a globally equivalent one by modifying the input variable. We will use it multiple times in this book.

THEOREM 1.11. *Let $F : \mathbf{L} \longrightarrow \mathbf{L}$ be a continuous endofunctor of the linear isometries category $\mathbf{L}$ and $i : \mathrm{Id} \longrightarrow F$ a natural transformation. Then for every orthogonal space Y the morphism*

$$Y \circ i \;:\; Y \;\longrightarrow\; Y \circ F$$

is a global equivalence of orthogonal spaces.

PROOF. In a first step we show an auxiliary statement. We let V be an inner product space and $z \in F(V)$ an element that is orthogonal to the subspace $i_V(V)$, the image of the linear isometric embedding $i_V : V \longrightarrow F(V)$. We claim that for every linear isometric embedding $\varphi : V \longrightarrow W$ the element $F(\varphi)(z)$ of $F(W)$ is orthogonal to the subspace $i_W(W)$. To prove the claim we write any given element of W as $\varphi(v) + y$ for some $v \in V$ and $y \in W$ orthogonal to $\varphi(V)$. Then on the one hand we have

$$\langle F(\varphi)(z), i_W(\varphi(v))\rangle \;=\; \langle F(\varphi)(z), F(\varphi)(i_V(v))\rangle \;=\; \langle z, i_V(v)\rangle \;=\; 0$$

by the hypotheses on z.

Now we define $A \in O(W)$ as the linear isometry that is the identity on $\varphi(V)$ and the negative of the identity on the orthogonal complement of $\varphi(V)$. Then $A \circ \varphi = \varphi$ and

$$\langle F(\varphi)(z), i_W(y)\rangle \;=\; \langle F(A)(F(\varphi)(z)), F(A)(i_W(y))\rangle \;=\; \langle F(A\varphi)(z), i_W(A(y))\rangle \;=\; -\langle F(\varphi)(z), i_W(y)\rangle$$

and hence $\langle F(\varphi)(z), i_W(y)\rangle = 0$. Altogether this shows that $\langle F(\varphi)(z), i_W(\varphi(v)+y)\rangle = 0$, which establishes the claim.

Now we consider a compact Lie group G, a G-representation V, a finite G-CW-pair (B, A) and a lifting problem $\alpha : A \longrightarrow Y(V)$ and $\beta : B \longrightarrow Y(F(V))$ for $(Y \circ i)(V)$. Then $\beta|_A = Y(i_V) \circ \alpha$ by hypothesis, and we claim that $Y(i_{F(V)}) \circ \beta$ is G-homotopic to $Y(F(i_V)) \circ \beta = (Y \circ F)(i_V) \circ \beta$, relative A; granting this for the moment, we conclude that the pair $(i_V : V \longrightarrow F(V), \beta)$ solves the lifting problem.

It remains to construct the relative homotopy. The two embeddings

$$F(i_V)\ ,\ i_{F(V)}\ :\ F(V)\ \longrightarrow\ F(F(V))$$

are homotopic, relative to $i_V : V \longrightarrow F(V)$, through G-equivariant isometric embeddings, via

$$\begin{aligned} H\ :\ F(V)\times[0,1]\ &\longrightarrow\qquad F(F(V)) \\ (v,w,t)\qquad &\longmapsto F(i_V)(v)\ +\ t\cdot i_{F(V)}(w)\ +\ \sqrt{1-t^2}\cdot F(i_V)(w)\ , \end{aligned}$$

where $v \in i_V(V)$ and w is orthogonal to $i_V(V)$. The verification that $H(-,t) : F(V) \longrightarrow F(F(V))$ is indeed a linear isometric embedding for every $t \in [0,1]$ uses that $i_{F(V)} = F(i_V)$ on the subspace $i_V(V)$ of $F(V)$, and that $i_{F(V)}(w)$ is orthogonal to $F(i_V)(w)$, by the claim proved above. The continuous functor Y takes this homotopy of equivariant linear isometric embeddings to a G-equivariant homotopy $Y(H(-,t))$ from $Y(F(i_V))$ to $Y(i_{F(V)})$, relative to $Y(i_V)$. Composing with β gives the required relative G-homotopy from $Y(F(i_V))\circ\beta$ to $Y(i_{F(V)})\circ\beta$. □

Example 1.12 (Additive and multiplicative shift). Here are some typical examples where the previous proposition applies. Every inner product space W defines an 'additive shift functor' and a 'multiplicative shift functor' on the category of orthogonal spaces, defined by respective precomposition with the continuous endofunctors

$$-\oplus W\ :\ \mathbf{L}\ \longrightarrow\ \mathbf{L}\qquad\text{and}\qquad -\otimes W\ :\ \mathbf{L}\ \longrightarrow\ \mathbf{L}.$$

In other words, the additive respectively multiplicative W-shift of an orthogonal space Y have values

$$(\mathrm{sh}^W_\oplus Y)(V)\ =\ Y(V\oplus W)\qquad\text{respectively}\qquad(\mathrm{sh}^W_\otimes Y)(V)\ =\ Y(V\otimes W)\ .$$

Here, and in the rest of the book, we endow the tensor product $V\otimes W$ of two inner product spaces V and W with the inner product characterized by

$$\langle v\otimes w,\bar v\otimes\bar w\rangle\ =\ \langle v,\bar v\rangle\cdot\langle w,\bar w\rangle$$

for all $v,\bar v\in V$ and $w,\bar w\in W$. Another way to say this is that for every orthonormal basis $\{b_i\}_{i\in I}$ of V and every orthonormal basis $\{d_j\}_{j\in J}$ of W the family $\{b_i\otimes d_j\}_{(i,j)\in I\times J}$ forms an orthonormal basis of $V\otimes W$. Theorem 1.11 then shows that the morphism $Y \longrightarrow \mathrm{sh}^W_\oplus Y$ given by applying Y to the first summand embedding $V \longrightarrow V\oplus W$ is a global equivalence. To get a similar statement for the multiplicative shift we have to assume that $W \neq 0$; then for every vector $w \in W$ of length 1 the map

$$V\ \longrightarrow\ V\otimes W\ ,\quad v\ \longmapsto\ v\otimes w$$

is a natural linear isometric embedding. So Theorem 1.11 shows that the morphism $Y(-\otimes w) : Y \longrightarrow \mathrm{sh}^W_\otimes Y$ is a global equivalence.

Now we consider a *bi-orthogonal space*, i.e., a continuous space valued functor $Z : \mathbf{L}\times\mathbf{L} \longrightarrow \mathbf{T}$ from the product of two copies of the linear isometries category. There are two very different ways to turn Z into an orthogonal space, by restriction to the diagonal and by enriched Kan extension along orthogonal direct sum. We show now that these two orthogonal spaces are related by a natural global equivalence. We denote by $\mathrm{diag}\, Z$ the *diagonal* of Z, i.e., the composite with the diagonal functor $\mathbf{L} \longrightarrow \mathbf{L}\times\mathbf{L}$. We denote by $Z_!$ the enriched left Kan extension of Z along the functor

$$\oplus\ :\ \mathbf{L}\times\mathbf{L}\ \longrightarrow\ \mathbf{L}\ .$$

So $Z_!$ comes with a morphism of bi-orthogonal spaces $i : Z \longrightarrow Z_!\circ\oplus$ that is universal in the sense that for every orthogonal space Y the map

$$spc(Z_!,Y)\ \longrightarrow\ bispc(Z,Y\circ\oplus)\ ,\quad f\longmapsto(f\circ\oplus)\circ i$$

is bijective. As V and W vary through all inner product spaces, the maps

$$Z(V,W)\ \xrightarrow{Z(i_1,i_2)}\ Z(V\oplus W,V\oplus W)\ =\ (\mathrm{diag}\, Z)(V\oplus W)$$

form a morphism of bi-orthogonal spaces $Z \longrightarrow (\operatorname{diag} Z) \circ \oplus$; here $i_1 : V \longrightarrow V \oplus W$ and $i_2 : W \longrightarrow V \oplus W$ are the two direct summand embeddings. The universal property of $(Z_!, i)$ then provides a unique morphism of orthogonal spaces

$$\rho_Z \ : \ Z_! \ \longrightarrow \ \operatorname{diag} Z$$

such that $(\rho_Z \circ \oplus) \circ i$ is the morphism above.

THEOREM 1.13. *For every bi-orthogonal space Z the morphism $\rho_Z : Z_! \longrightarrow \operatorname{diag} Z$ is a global equivalence.*

PROOF. For an orthogonal space A we denote by $\operatorname{sh}(A)$ the orthogonal space defined by

$$(\operatorname{sh} A)(V) \ = \ A(V \oplus V) \qquad \text{and} \qquad (\operatorname{sh} A)(\varphi) \ = \ A(\varphi \oplus \varphi) \ ;$$

thus $\operatorname{sh}(A)$ is isomorphic to $\operatorname{sh}_{\otimes}^{\mathbb{R}^2}(A)$, the multiplicative shift of A by $\mathbb{R}^2$ as defined in Example 1.12. We define a morphism of orthogonal spaces

$$\lambda \ : \ \operatorname{diag} Z \ \longrightarrow \ \operatorname{sh}(Z_!)$$

at an inner product space V as the composite

$$(\operatorname{diag} Z)(V) \ = \ Z(V,V) \ \xrightarrow{i_{V,V}} \ Z_!(V \oplus V) \ = \ (\operatorname{sh} Z_!)(V) \ .$$

Now we consider the two composites $\lambda \circ \rho_Z$ and $\operatorname{sh}(\rho_Z) \circ \lambda$:

$$\begin{tikzcd} Z_! \arrow[r, "\rho_Z"'] \arrow[rr, bend left] \arrow[rrr, bend left=50] & \operatorname{diag} Z \arrow[r, "\lambda"] \arrow[rr, bend right] & \operatorname{sh}(Z_!) \arrow[r, "\operatorname{sh}(\rho_Z)"] & \operatorname{sh}(\operatorname{diag} Z) \end{tikzcd}$$

We claim that the composite $\lambda \circ \rho_Z : Z_! \longrightarrow \operatorname{sh}(Z_!)$ is homotopic to the morphism $Z_! \circ i_1$, where i_1 is the natural linear isometric embedding $V \longrightarrow V \oplus V$ as the first summand. Indeed, for every $t \in [0,1]$ we define a natural linear isometric embedding

$$j_t \ : \ V \oplus W \ \longrightarrow \ V \oplus W \oplus V \oplus W \qquad \text{by} \qquad j_t(v,w) \ = \ (v, \ t \cdot w, \ 0, \ \sqrt{1-t^2} \cdot w) \ .$$

Then the maps

$$Z(V,W) \ \xrightarrow{i_{V,W}} \ Z_!(V \oplus W) \ \xrightarrow{Z_!(j_t)} \ Z_!(V \oplus W \oplus V \oplus W) \ = \ (\operatorname{sh} Z_!)(V \oplus W)$$

form a morphism of bi-orthogonal spaces as V and W vary; this corresponds to a morphism

$$f_t \ : \ Z_! \ \longrightarrow \ \operatorname{sh} Z_!$$

of orthogonal spaces, by the universal property of the left Kan extension. The linear isometric embeddings j_t vary continuously with t, hence so do the morphisms f_t. Moreover, $f_0 = \lambda \circ \rho_Z$ and $f_1 = Z_! \circ i_1$, so this is the desired homotopy. The morphism $Z_! \circ i_1$ is a global equivalence by Theorem 1.11, hence so is the morphism $\lambda \circ \rho_Z$.

The value of the morphism $\operatorname{sh}(\rho_Z) \circ \lambda$ at an inner product space V is

$$Z(i_1, i_2) \ : \ (\operatorname{diag} Z)(V) \ = \ Z(V,V) \ \longrightarrow \ Z(V \oplus V, V \oplus V) \ = \ (\operatorname{sh}(\operatorname{diag} Z))(V) \ .$$

For $t \in [0,1]$ the linear isometric embeddings

$$j'_t \ : \ V \ \longrightarrow \ V \oplus V \qquad \text{defined by} \qquad j'_t(v) \ = \ (t \cdot v, \sqrt{1-t^2} \cdot v)$$

form a homotopy from i_2 to i_1. So

$$Z(\mathrm{Id}_V, j'_t) \ : \ Z(V,V) \ \longrightarrow \ Z(V \oplus V \oplus V \oplus V)$$

form a homotopy from $Z(i_1, i_2)$ to $Z(i_1, i_1)$. These homotopies are natural for linear isometric embeddings in V; so as V varies, they provide a homotopy between $\operatorname{sh}(\rho_Z) \circ \lambda$ and $(\operatorname{diag} Z) \circ i_1$. The morphism $(\operatorname{diag} Z) \circ i_1$ is a global equivalence by Theorem 1.11, hence so is the morphism $\operatorname{sh}(\rho_Z) \circ \lambda$. Since $\lambda \circ \rho_Z$ and $\operatorname{sh}(\rho_Z) \circ \lambda$ are global equivalences, so is the morphism ρ_Z, by Proposition 1.9 (iii). $\square$

Definition 1.14. Let G be a compact Lie group. A *G-universe* is an orthogonal G-representation $\mathcal{U}$ of countably infinite dimension with the following two properties:

- the representation $\mathcal{U}$ has non-zero G-fixed points,
- if a finite dimensional G-representation V embeds into $\mathcal{U}$, then a countable infinite direct sum of copies of V also embeds into $\mathcal{U}$.

A G-universe is *complete* if every finite dimensional G-representation embeds into it.

A G-universe is characterized, up to equivariant isometry, by the set of irreducible G-representations that can be embedded into it. We let $\Lambda = \{\lambda\}$ be a complete set of pairwise non-isomorphic irreducible G-representations that embed into $\mathcal{U}$. The first condition says that Λ contains a trivial 1-dimensional representation, and the second condition is equivalent to the requirement that

$$\mathcal{U} \;\cong\; \bigoplus_{\lambda\in\Lambda}\bigoplus_{\mathbb{N}} \lambda \;.$$

Moreover, $\mathcal{U}$ is complete if and only if Λ contains (representatives of) *all* irreducible G-representations. In the following we fix, for every compact Lie group G, a complete G-universe $\mathcal{U}_G$. We let $s(\mathcal{U}_G)$ denote the poset, under inclusion, of finite dimensional G-subrepresentations of $\mathcal{U}_G$.

Definition 1.15. For an orthogonal space Y and a compact Lie group G we define the *underlying G-space* as

$$Y(\mathcal{U}_G) \;=\; \mathrm{colim}_{V\in s(\mathcal{U}_G)}\, Y(V) \;,$$

the colimit, in the category of compactly generated G-spaces, of the G-spaces $Y(V)$.

Remark 1.16. The underlying G-space $Y(\mathcal{U}_G)$ can always be rewritten as a sequential colimit of values of Y. Indeed, we can choose a nested sequence of finite dimensional G-subrepresentations

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$$

whose union is all of $\mathcal{U}_G$. This is then in particular an exhaustive sequence in the sense of Definition 1.6. Since the subposet $\{V_n\}_{n\geq 0}$ is cofinal in $s(\mathcal{U}_G)$, the colimit of the functor $V \mapsto Y(V)$ over $s(\mathcal{U}_G)$ is also a colimit over the subsequence $Y(V_n)$.

If the group G is finite, then we can define a complete universe as

$$\mathcal{U}_G \;=\; \bigoplus_{\mathbb{N}} \rho_G \;,$$

a countably infinite sum of copies of the regular representation $\rho_G = \mathbb{R}[G]$ (with G as orthonormal basis). Then $\mathcal{U}_G$ is filtered by the finite sums $n\cdot\rho_G$, and we get

$$Y(\mathcal{U}_G) \;=\; \mathrm{colim}_n\, Y(n\cdot\rho_G) \;,$$

where the colimit is taken along the inclusions $n\cdot\rho_G \longrightarrow (n+1)\cdot\rho_G$ that miss the last summand.

Definition 1.17. An orthogonal space Y is *closed* if it takes every linear isometric embedding $\varphi : V \longrightarrow W$ of inner product spaces to a closed embedding $Y(\varphi) : Y(V) \longrightarrow Y(W)$.

Proposition 1.18. *Let $f : X \longrightarrow Y$ be a morphism between closed orthogonal spaces. Then f is a global equivalence if and only if for every compact Lie group G the map*

$$f(\mathcal{U}_G)^G \;:\; X(\mathcal{U}_G)^G \;\longrightarrow\; Y(\mathcal{U}_G)^G$$

is a weak equivalence.

PROOF. The poset $s(\mathcal{U}_G)$ has a cofinal subsequence, so all colimits over $s(\mathcal{U}_G)$ can be realized as sequential colimits. The claim is then a straightforward consequence of the fact that compact spaces such as D^k and ∂D^k are finite with respect to sequences of closed embeddings, compare [**79**, Prop. 2.4.2]. We should recall here that the weak Hausdorff condition guarantees that points in compactly generated spaces are always closed, so the T_1-separation property holds. □

2. Global classifying spaces

In this section we discuss free orthogonal spaces. Important special cases of this construction are the 'global classifying spaces' of compact Lie groups. These global classifying spaces are the basic building blocks of global homotopy theory; we will show in Proposition 5.14 below that the global classifying space $B_{\mathrm{gl}}G$ represents the equivariant homotopy set functor π_0^G (to be defined in Definition 5.5 below) in the unstable global homotopy category.

Free orthogonal spaces are made from spaces of linear isometric embeddings, so we start by establishing various properties of certain spaces of linear isometric embeddings. We consider two compact Lie groups G and K, a finite dimensional G-representation V, and a K-representation U, possibly of countably infinite dimension. The space $\mathbf{L}(V,U)$ of linear isometric embeddings inherits a continuous left K-action and a compatible continuous right G-action from the actions on the target and source, respectively. We turn these two actions into a single left action of the group $K \times G$ by defining

$$((k,g)\cdot\varphi)(v) \;=\; k\cdot\varphi(g^{-1}\cdot v) \tag{2.1}$$

for $\varphi \in \mathbf{L}(V,U)$ and $(k,g) \in K \times G$. We recall that a continuous $(K \times G)$-equivariant map is a $(K \times G)$-*cofibration* if it has the right lifting property with respect to all morphisms of $(K \times G)$-spaces $f : X \longrightarrow Y$ such that the map $f^H : X^H \longrightarrow Y^H$ is a weak equivalence and Serre fibration for every closed subgroup H of $K \times G$.

PROPOSITION 2.2. *Let G and K be compact Lie groups, V a finite-dimensional G-representation, and $\mathcal{U}$ a K-representation of finite or countably infinite dimension.*

(i) *For every finite-dimensional K-subrepresentation U of $\mathcal{U}$, the morphism of $(K \times G)$-spaces*

$$\mathbf{L}(V,U) \;\longrightarrow\; \mathbf{L}(V,\mathcal{U})$$

induced by the inclusion is a $(K \times G)$-cofibration. The induced map of orbit spaces

$$\mathbf{L}(V,U)/G \;\longrightarrow\; \mathbf{L}(V,\mathcal{U})/G$$

is a K-cofibration.

(ii) *The $(K \times G)$-space $\mathbf{L}(V,\mathcal{U})$ is $(K \times G)$-cofibrant. The K-space $\mathbf{L}(V,\mathcal{U})/G$ is K-cofibrant.*

PROOF. (i) We consider two natural numbers $m, n \geq 0$. The space $\mathbf{L}(V,\mathbb{R}^{m+n})$ is homeomorphic to the Stiefel manifold of $\dim(V)$-frames in $\mathbb{R}^{m+n}$, hence a compact smooth manifold, and the action of $O(m) \times O(n) \times G$ is smooth. Illman's theorem [**82**, Cor. 7.2] thus provides an $(O(m) \times O(n) \times G)$-CW-structure on $\mathbf{L}(V,\mathbb{R}^{m+n})$. In particular, $\mathbf{L}(V,\mathbb{R}^{m+n})$ is cofibrant as an $(O(m) \times O(n) \times G)$-space. The group $N = e \times O(n) \times e$ is a closed normal subgroup of $O(m) \times O(n) \times G$, so the inclusion of the N-fixed points into $\mathbf{L}(V,\mathbb{R}^{m+n})$ is an $(O(m) \times O(n) \times G)$-cofibration (compare Proposition A.2.15). The map (2.3)

$$\mathbf{L}(V,\mathbb{R}^m) \;\longrightarrow\; \mathbf{L}(V,\mathbb{R}^{m+n})\;, \tag{2.3}$$

induced by the embedding $\mathbb{R}^m \longrightarrow \mathbb{R}^{m+n}$ as the first m coordinates, is a homeomorphism from $\mathbf{L}(V,\mathbb{R}^m)$ to the N-fixed points of $\mathbf{L}(V,\mathbb{R}^{m+n})$; so the map (2.3) is a $(O(m) \times O(n) \times G)$-cofibration.

Now we can prove the proposition when $\mathcal{U}$ (and hence U) is finite dimensional. We can assume that the underlying inner product space of $\mathcal{U}$ is $\mathbb{R}^{m+n}$ with the standard scalar product, and that U is the subspace of $\mathcal{U} = \mathbb{R}^{m+n}$ where the last n coordinates vanish. The K-action on $\mathcal{U}$ is given by a continuous homomorphism $\psi : K \longrightarrow O(m+n)$. Since U is a K-subrepresentation, the image of ψ must be contained

in the subgroup $O(m) \times O(n)$. The $(K \times G)$-action on the map (2.3) is then obtained by restriction of the $(O(m) \times O(n) \times G)$-action along the homomorphism

$$\psi \times \mathrm{Id} \ : \ K \times G \ \longrightarrow \ O(m) \times O(n) \times G \ .$$

Restriction along any continuous homomorphism between compact Lie groups preserves cofibrations by Proposition A.2.14 (i), so the map (2.3) is a $(K \times G)$-cofibration by the first part.

Now we treat the case when the dimension of $\mathcal{U}$ is infinite. We choose an exhausting nested sequence of K-subrepresentations

$$U = U_0 \subset U_1 \subset U_2 \subset \dots$$

Then all the morphisms $\mathbf{L}(V, U_{n-1}) \longrightarrow \mathbf{L}(V, U_n)$ are $(K \times G)$-cofibrations by the above. Since cofibrations are closed under sequential composites, the morphism

$$\mathbf{L}(V, U_0) \ \longrightarrow \ \mathrm{colim}_n \ \mathbf{L}(V, U_n) \ = \ \mathbf{L}(V, \mathcal{U})$$

is also a $(K \times G)$-cofibration.

Applying Proposition A.2.14 (iii) to the normal subgroup $e \times G$ of $K \times G$ shows that the functor

$$(e \times G)\backslash - \ : \ (K \times G)\mathbf{T} \ \longrightarrow \ \mathbf{KT}$$

takes $(K \times G)$-cofibrations to K-cofibrations. This proves the second claim.

(ii) This is the special case $U = \{0\}$. The space $\mathbf{L}(V, 0)$ is either empty or consists of a single point; in any case $\mathbf{L}(V, \{0\})$ is $(K \times G)$-cofibrant. Part (i) then implies that $\mathbf{L}(V, \mathcal{U})$ is $(K \times G)$-cofibrant and $\mathbf{L}(V, \mathcal{U})/G$ is K-cofibrant. □

The following fundamental contractibility property goes back, at least, to Boardman and Vogt [**19**]. The equivariant version that we need can be found in [**97**, Lemma II 1.5].

PROPOSITION 2.4. *Let G be a compact Lie group, V a G-representation and $\mathcal{U}$ a G-universe such that V embeds into $\mathcal{U}$. Then the space $\mathbf{L}(V, \mathcal{U})$, equipped with the conjugation action by G, is G-equivariantly contractible.*

PROOF. We start by showing that the space $\mathbf{L}^G(V, \mathcal{U})$ of G-equivariant linear isometric embeddings is weakly contractible. We let $\mathcal{U}$ be a G-representation of finite or countably infinite dimension. Then the map

$$H \ : \ [0,1] \times \mathbf{L}^G(V, \mathcal{U}) \ \longrightarrow \ \mathbf{L}^G(V, \mathcal{U} \oplus V)$$

defined by

$$H(t, \varphi)(v) \ = \ (t \cdot \varphi(v), \ \sqrt{1 - t^2} \cdot v)$$

is a homotopy from the constant map with value $i_2 : V \longrightarrow \mathcal{U} \oplus V$ to the map $i_1 \circ -$ (postcomposition with $i_1 : \mathcal{U} \longrightarrow \mathcal{U} \oplus V$).

Since V embeds into $\mathcal{U}$ and $\mathcal{U}$ is a G-universe, it contains infinitely many orthogonal copies of V. In other words, we can assume that

$$\mathcal{U} \ = \ \mathcal{U}' \oplus V^\infty$$

for some G-representation $\mathcal{U}'$. Then

$$\mathbf{L}^G(V, \mathcal{U}) \ = \ \mathbf{L}^G(V, \mathcal{U}' \oplus V^\infty) \ = \ \mathrm{colim}_{n \geq 0} \mathbf{L}^G(V, \mathcal{U}' \oplus V^n) \ ;$$

the colimit is formed along the postcomposition maps with the direct sum embedding $\mathcal{U}' \oplus V^n \longrightarrow \mathcal{U}' \oplus V^{n+1}$. Every map in the colimit system is a closed embedding and homotopic to a constant map, by the previous paragraph. So the colimit is weakly contractible.

Applying the previous paragraph to a closed subgroup H of G shows that the fixed point space $\mathbf{L}^H(V, \mathcal{U})$ is weakly contractible; in other words, $\mathbf{L}(V, \mathcal{U})$ is G-weakly contractible. The space $\mathbf{L}(V, \mathcal{U})$ comes with a $(G \times G)$-action as in (2.1), and it is $(G \times G)$-cofibrant by Proposition 2.2 (ii). Then $\mathbf{L}(V, \mathcal{U})$ is also cofibrant as a G-space for the diagonal action, by Proposition A.2.14 (i). Since $\mathbf{L}(V, \mathcal{U})$ is G-cofibrant and weakly G-contractible, it is actually equivariantly contractible. □

Now we turn to free orthogonal spaces.

Construction 2.5. Given a compact Lie group G and a G-representation V, the functor

$$\mathrm{ev}_{G,V} \ : \ spc \ \longrightarrow \ G\mathbf{T}$$

that sends an orthogonal space Y to the G-space $Y(V)$ has a left adjoint

$$\mathbf{L}_{G,V} \ : \ G\mathbf{T} \ \longrightarrow \ spc \ . \tag{2.6}$$

To construct the left adjoint we observe that G-acts on the right on $\mathbf{L}(V,W)$ by

$$(\varphi \cdot g)(v) \ = \ \varphi(gv)$$

for $\varphi \in \mathbf{L}(V,W)$, $g \in G$ and $v \in V$. Given a G-space A, the value of the free orthogonal space $\mathbf{L}_{G,V}A$ at an inner product space W is

$$(\mathbf{L}_{G,V}A)(W) \ = \ \mathbf{L}(V,W) \times_G A \ = \ (\mathbf{L}(V,W) \times A)\,/(\varphi g, a) \sim (\varphi, ga) \ .$$

We refer to $\mathbf{L}_{G,V}A$ as the *free orthogonal space* generated by A at (G,V). By a slight abuse of notation, we also denote by $\mathbf{L}_{G,V}$ the orthogonal space with

$$\mathbf{L}_{G,V}(W) \ = \ \mathbf{L}(V,W)/G \ .$$

So $\mathbf{L}_{G,V}$ is isomorphic to $\mathbf{L}_{G,V}*$, the free orthogonal space generated at (G,V) by a one-point G-space.

The 'freeness' property of $\mathbf{L}_{G,V}A$ means: for every orthogonal space Y and every continuous G-map $f : A \longrightarrow Y(V)$ there is a unique morphism $\hat{f} : \mathbf{L}_{G,V}A \longrightarrow Y$ of orthogonal spaces such that the composite

$$A \xrightarrow{[\mathrm{Id},-]} \mathbf{L}(V,V) \times_G A = (\mathbf{L}_{G,V}A)(V) \xrightarrow{\hat{f}(V)} Y(V)$$

is f. Indeed, the morphism $\hat{f}$ is given at W as the composite

$$\mathbf{L}(V,W) \times_G A \xrightarrow{\mathrm{Id} \times_G f} \mathbf{L}(V,W) \times_G Y(V) \xrightarrow{\circ} Y(W) \ .$$

Example 2.7. For every compact Lie group G, every G-representation V and every G-space A the free orthogonal space $\mathbf{L}_{G,V}A$ is closed. To see that we let $\varphi : W \longrightarrow U$ be a linear isometric embedding. By Proposition 2.2 the map of G-spaces

$$\mathbf{L}(V,\varphi) \ : \ \mathbf{L}(V,W) \ \longrightarrow \ \mathbf{L}(V,U)$$

is a G-cofibration. Every G-cofibration is in particular an h-cofibration of G-spaces, and the functor $- \times_G A$ takes h-cofibrations of G-spaces to h-cofibrations of spaces. So the map

$$\mathbf{L}(V,\varphi) \times_G A \ : \ \mathbf{L}(V,W) \times_G A \ \longrightarrow \ \mathbf{L}(V,U) \times_G A$$

is an h-cofibration, hence a closed embedding.

The next proposition identifies the fixed point spaces of a free orthogonal space $\mathbf{L}_{G,V}$. A certain family $\mathcal{F}(K;G)$ of subgroups of $K \times G$ arises naturally, which we call 'graph subgroups'.

Definition 2.8. Let K and G be compact Lie groups. The family $\mathcal{F}(K;G)$ of *graph subgroups* consists of those closed subgroups Γ of $K \times G$ that intersect $1 \times G$ only in the neutral element $(1,1)$.

The name 'graph subgroup' is justified by the observation that $\mathcal{F}(K;G)$ consists precisely of the graphs of all 'subhomomorphisms', i.e., continuous homomorphisms $\alpha : L \longrightarrow G$ from a closed subgroup L of K. Clearly, the graph $\Gamma(\alpha) = \{(l, \alpha(l)) \mid l \in L\}$ of every such homomorphism belongs to $\mathcal{F}(K;G)$. Conversely, for $\Gamma \in \mathcal{F}(K;G)$ we let $L \leq K$ be the image of Γ under the projection $K \times G \longrightarrow K$. Since $\Gamma \cap (1 \times G) = \{(1,1)\}$, every element $l \in L$ then has a unique preimage $(l, \alpha(l))$ under the projection, and the assignment $l \mapsto \alpha(l)$ is a continuous homomorphism from L to G whose graph is Γ.

If V and W are G-representations, then restriction of a linear isometry from $V \oplus W$ to V defines a G-equivariant morphism of orthogonal spaces

$$\rho_{V,W} \ : \ \mathbf{L}_{V\oplus W} \ \longrightarrow \ \mathbf{L}_V \ . \tag{2.9}$$

If U is a K-representation, then we combine the left K-action and the right G-action on $\mathbf{L}(V,U)$ into a left action of $K \times G$ as in (2.1).

Proposition 2.10. *Let K and G be compact Lie groups and V a faithful G-representation.*

(i) *The $(K\times G)$-space $\mathbf{L}_V(\mathcal{U}_K) = \mathbf{L}(V,\mathcal{U}_K)$ is a universal space for the family $\mathcal{F}(K;G)$ of graph subgroups.*

(ii) *If W is another G-representation, then the restriction map*

$$\rho_{V,W}(\mathcal{U}_K) \ : \ \mathbf{L}(V\oplus W,\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(V,\mathcal{U}_K) \ ,$$

is a $(K \times G)$-homotopy equivalence. For every G-space A, the map

$$(\rho_{V,W} \times_G A)(\mathcal{U}_K) \ : \ (\mathbf{L}_{G,V\oplus W}A)(\mathcal{U}_K) \ \longrightarrow \ (\mathbf{L}_{G,V}A)(\mathcal{U}_K)$$

is a K-homotopy equivalence and the morphism of orthogonal spaces

$$\rho_{V,W} \times_G A \ : \ \mathbf{L}_{G,V\oplus W}A \ \longrightarrow \ \mathbf{L}_{G,V}A$$

is a global equivalence.

Proof. (i) We let Γ be any closed subgroup of $K\times G$. Since the G-action on V is faithful, the induced right G-action on $\mathbf{L}(V,\mathcal{U}_K)$ is free. So if Γ intersects $1 \times G$ non-trivially, then $\mathbf{L}(V,\mathcal{U}_K)^\Gamma$ is empty. On the other hand, if $\Gamma\cap(1\times G) = \{(1,1)\}$, then Γ is the graph of a unique continuous homomorphism $\alpha : L \longrightarrow G$, where L is the projection of Γ to K. Then

$$\mathbf{L}(V,\mathcal{U}_K)^\Gamma \ = \ \mathbf{L}^L(\alpha^*V,\mathcal{U}_K)$$

is the space of L-equivariant linear isometric embeddings from the L-representation α^*V to the underlying L-universe of $\mathcal{U}_K$. Since $\mathcal{U}_K$ is a complete K-universe, the underlying L-universe is also complete, and so the space $\mathbf{L}^L(\alpha^*V,\mathcal{U}_K)$ is contractible by Proposition 2.4. The space $\mathbf{L}(V,\mathcal{U}_K)$ is cofibrant as a $(K \times G)$-space by Proposition 2.2 (ii).

(ii) Every equivariant map between universal spaces for the same family of subgroups is an equivariant homotopy equivalence. Since G acts faithfully on V, and hence also on $V \oplus W$, the $(K \times G)$-spaces $\mathbf{L}(V \oplus W,\mathcal{U}_K)$ and $\mathbf{L}(V,\mathcal{U}_K)$ are universal spaces for the same family $\mathcal{F}(K;G)$, by part (i). So the map $\rho_{V,W}(\mathcal{U}_K) : \mathbf{L}(V\oplus W,\mathcal{U}_K) \longrightarrow \mathbf{L}(V,\mathcal{U}_K)$ is a $(K\times G)$-equivariant homotopy equivalence. The functor $-\times_G A$ preserves homotopies, so the restriction map $(\rho_{V,W} \times_G A)(\mathcal{U}_K)$ is a K-homotopy equivalence.

The orthogonal spaces $\mathbf{L}_{G,V\oplus W}A$ and $\mathbf{L}_{G,V}A$ are closed by Example 2.7, so Proposition 1.18 applies and shows that $\rho_{V,W} \times_G A$ is a global equivalence. □

Definition 2.11. The *global classifying space* $B_{\mathrm{gl}}G$ of a compact Lie group G is the free orthogonal space

$$B_{\mathrm{gl}}G \ = \ \mathbf{L}_{G,V} \ = \ \mathbf{L}(V,-)/G \ ,$$

where V is any faithful G-representation.

The global classifying space $B_{\mathrm{gl}}G$ is well-defined up to preferred zigzag of global equivalences of orthogonal spaces. Indeed, if V and $\bar{V}$ are two faithful G-representations, then $V \oplus \bar{V}$ is yet another one, and the two restriction morphisms

$$\mathbf{L}_{G,V} \ \longleftarrow \ \mathbf{L}_{G,V\oplus\bar{V}} \ \longrightarrow \ \mathbf{L}_{G,\bar{V}}$$

are global equivalences by Proposition 2.10 (ii).

Example 2.12. We make the global classifying space more explicit in the smallest non-trivial example, i.e., for the cyclic group C_2 of order 2. The sign representation σ of C_2 is faithful, so we can take $B_{\mathrm{gl}}C_2$ to be the free orthogonal space generated by (C_2, σ); its value at an inner product space W is

$$(B_{\mathrm{gl}}C_2)(W) \;=\; \mathbf{L}_{C_2,\sigma}(W) \;=\; \mathbf{L}(\sigma, W)/C_2 \;.$$

Evaluation at any of the two unit vectors in σ is a homeomorphism $\mathbf{L}(\sigma, W) \cong S(W)$ to the unit sphere of W, and the C_2-action on the left becomes the antipodal action on $S(W)$. So the map descends to a homeomorphism between $\mathbf{L}(\sigma, W)/C_2$ and $P(W)$, the projective space of W, and hence

$$(B_{\mathrm{gl}}C_2)(W) \;\cong\; P(W) \;.$$

So for a compact Lie group K, the K-space represented by $B_{\mathrm{gl}}C_2$ is $P(\mathcal{U}_K)$, the projective space of a complete K-universe. In particular, the underlying non-equivariant space is homeomorphic to $\mathbb{R}\mathrm{P}^\infty$.

Remark 2.13 ($B_{\mathrm{gl}}G$ globally classifies principal G-bundles). The term 'global classifying space' is justified by the fact that $B_{\mathrm{gl}}G$ 'globally classifies principal G-bundles'. We recall that a (K, G)-*bundle*, also called a K-*equivariant* G-*principal bundle*, is a principal G-bundle in the category of K-spaces, i.e., a G-principal bundle $p : E \longrightarrow B$ that is also a morphism of K-spaces and such that the actions of G and K on the total space E commute (see for example [**166**, Ch. I (8.7)]). For every compact Lie group K, the quotient map

$$q \;:\; \mathbf{L}(V, \mathcal{U}_K) \;\longrightarrow\; \mathbf{L}(V, \mathcal{U}_K)/G \;=\; \mathbf{L}_{G,V}(\mathcal{U}_K) \;=\; (B_{\mathrm{gl}}G)(\mathcal{U}_K)$$

is a principal (K, G)-bundle. Indeed, the total space $\mathbf{L}(V, \mathcal{U}_K)$ is homeomorphic to the Stiefel manifold of $\dim(V)$-frames in $\mathbb{R}^\infty$, and hence it admits a CW-structure. Every CW-complex is a normal Hausdorff space (see for example [**68**, Prop. A.3] or [**55**, Prop. 1.2.1]), hence completely regular. So $\mathbf{L}(V, \mathcal{U}_K)$ is completely regular. Since the G-action on $\mathbf{L}(V, \mathcal{U}_K)$ is free, the quotient map q is a G-principal bundle by [**123**, Prop. 1.7.35] or [**27**, II Thm. 5.8]. Moreover, this bundle is universal in the following proposition. Every G-space that admits a G-CW-structure is paracompact, see [**116**, Thm. 3.2] (this reference is rather sketchy, but one can follow the non-equivariant argument spelled out in more detail in [**55**, Thm. 1.3.5]). So the next proposition applies in particular to all G-CW-complexes.

Proposition 2.14. *Let V be a faithful representation of a compact Lie group G. Then for every paracompact K-space B the map*

$$[B, \mathbf{L}(V, \mathcal{U}_K)/G]^K \;\longrightarrow\; \mathrm{Prin}_{(K,G)}(B) \;, \quad [f] \;\longmapsto\; [f^*(q)]$$

from the set of equivariant homotopy classes of K-maps to the set of isomorphism classes of principal (K, G)-bundles is bijective.

Proof. I do not know a reference for the result in precisely this form, so I sketch how to deduce it from various results in the literature about equivariant fiber bundles. A principal (K, G)-bundle $p : E \longrightarrow B$ is *equivariantly trivializable* if there is a closed subgroup L of K, a continuous homomorphism $\alpha : L \longrightarrow G$, an L-space U and an isomorphism of (K, G)-bundles between p and the projection

$$(K \times G) \times_L U \;\longrightarrow\; K \times_L U \;;$$

here the source is the quotient space of $K \times G \times U$ by the equivalence relation $(k, g, lu) \sim (kl, g\alpha(l), u)$ for all $(k, g, l, u) \in K \times G \times L \times U$. A principal (K, G)-bundle $p : E \longrightarrow B$ is *numerable* if B has a trivializing (in the above sense) open cover $\{U_i\}_{i \in I}$ by K-invariant subset, such that moreover the cover admits a subordinate partition of unity by G-invariant functions.

A *universal* (K, G)-*principal bundle* is a numerable (K, G)-bundle $p : E(K, G) \longrightarrow B(K, G)$ such that for every K-space B the map

$$\begin{array}{ccc} [B, B(K, G)]^K & \longrightarrow & \mathrm{Prin}^{\mathrm{num}}_{(K,G)}(B) \\ [f] & \longmapsto & [f^*(q)] \end{array}$$

is bijective, where now the target is the set of *numerable* principal (K,G)-bundles. Over a paracompact base, every principal (K,G)-bundle is numerable by [**92**, Cor. 1.5]. So we are done if we can show that $q : \mathbf{L}(V,\mathcal{U}_K) \longrightarrow \mathbf{L}(V,\mathcal{U}_K)/G$ is a universal (K,G)-principal bundle in the above sense.

Universal (K,G)-principal bundles can be constructed in different way; the most commonly used construction is Milnor's infinite join construction, see for example [**161**, 3.1 Satz] or [**166**, I Theorem (8.12)]. Another construction is via bar construction, compare [**171**]. I do not know of a reference that explicitly identifies the projection $q : \mathbf{L}(V,\mathcal{U}_K) \longrightarrow \mathbf{L}(V,\mathcal{U}_K)/G$ as a universal bundle in the present sense, so we use Lashof's criterion [**92**, Thm. 2.14]. The base space $\mathbf{L}(V,\mathcal{U}_K)/G$ is the union, along h-cofibrations, of the compact spaces $\mathbf{L}(V,W_i)/G$, where $\{W_i\}_{i\geq 1}$ is any exhausting sequence of subrepresentations of $\mathcal{U}_K$. Since compact spaces are paracompact, the union $\mathbf{L}(V,\mathcal{U}_K)/G$ is paracompact, see for example [**55**, Prop. A.5.1 (v)]. Since $\mathbf{L}(V,\mathcal{U}_K)/G$ is also normal, hence completely regular, the bundle q is numerable Cor. 1.5 and Cor. 1.13 of [**92**]. Moreover, the fixed points of $\mathbf{L}(V,\mathcal{U}_K)$ under any graph subgroup of $(K \times G)$ are contractible by Proposition 2.10 (i), so Theorem 2.14 of [**92**] applies and shows that $q : \mathbf{L}(V,\mathcal{U}_K) \longrightarrow \mathbf{L}(V,\mathcal{U}_K)/G$ is strongly universal, hence universal, principal (K,G)-bundle. □

In other words, the K-space $(B_{\mathrm{gl}}G)(\mathcal{U}_K)$ is a classifying space for principal (K,G)-bundles. In particular, the underlying non-equivariant homotopy type of $B_{\mathrm{gl}}G$ is that of the ordinary classifying space for G.

The total space of any universal principal (K,G)-bundle can be characterized up to $(K \times G)$-homotopy equivalence as a universal $(K \times G)$-space for the family $\mathcal{F}(K;G)$ in the sense of Definition 2.8. In other words, if E is any cofibrant $(K \times G)$-space with free G-action and such that the fixed point set $E(K,G)^H$ is contractible for every subgroup Γ of $K \times G$ with $\Gamma \cap (1 \times G) = \{(1,1)\}$, then the projection

$$E \ \longrightarrow \ G\backslash E$$

is a universal principal (K,G)-bundle, and $G\backslash E$ is a classifying space for principal (K,G)-bundle, and hence K-homotopy equivalent to $(B_{\mathrm{gl}}G)(\mathcal{U}_K)$.

As an example we look at the case $G = O(n)$, the n-th orthogonal group. The category of principal $O(n)$-bundles is equivalent to the category of euclidean vector bundles of rank n, via the associated frame bundle. By the same construction, principal $(K,O(n))$-bundles can be identified with K-equivariant euclidean vector bundles of rank n over K-spaces. The space $\mathbf{L}(\mathbb{R}^n,\mathcal{U}_K)/O(n)$ is homeomorphic to $Gr_n(\mathcal{U}_K)$, the Grassmannian of n-planes in $\mathcal{U}_K$. When K is a trivial group, the fact that $Gr_n(\mathbb{R}^\infty)$ is a classifying space for rank n vector vector bundles over paracompact spaces is proved in various text books, for example [**69**, Thm. 1.16]. Since $O(1)$ is a cyclic group of order 2, this gives another perspective on Example 2.12.

Global classifying spaces are defined from faithful orthogonal representations of compact Lie groups. As we shall now explain, we can also use faithful *unitary* representations instead.

Construction 2.15. To define the next example we introduce some notation for going back and forth between euclidean inner product spaces of $\mathbb{R}$ and hermitian inner product spaces over $\mathbb{C}$. For a real inner product space V we let

$$V_{\mathbb{C}} \ = \ \mathbb{C} \otimes_{\mathbb{R}} V$$

be the complexification; the euclidean inner product $\langle -,- \rangle$ on V induces a hermitian inner product $(-,-)$ on the complexification $V_{\mathbb{C}}$ defined as the unique hermitian inner product that satisfies

$$(1 \otimes v, 1 \otimes w) \ = \ \langle v,w \rangle$$

for all $v,w \in V$. For a hermitian inner product space W we let uW denote the underlying $\mathbb{R}$-vector space of W, equipped with the euclidean inner product

$$\langle v,w \rangle \ = \ \mathrm{Re}(v,w) \ ,$$

the real part of the given hermitian inner product. Every $\mathbb{C}$-linear isometric embedding is in particular an $\mathbb{R}$-linear isometric embedding of underlying euclidean vector spaces; so $U(W) \subseteq O(uW)$, i.e., the unitary

group of W is a subgroup of the orthogonal group of uW. We thus view a unitary representation on W as an orthogonal representation on uW. If V and W are two finite dimensional $\mathbb{C}$-vector spaces equipped with hermitian inner products, we denote by $\mathbf{L}^{\mathbb{C}}(V,W)$ the space of $\mathbb{C}$-linear isometric embeddings. We topologize this as a complex Stiefel manifold, i.e., homeomorphic to the space of $\dim_{\mathbb{C}}(V)$-frames in W.

Now we can define complex versions of free orthogonal spaces. We let G be a compact Lie group and W a finite dimensional unitary G-representation. We define an orthogonal space $\mathbf{L}^{\mathbb{C}}_{G,W}$ by

$$\mathbf{L}^{\mathbb{C}}_{G,W}(V) \;=\; \mathbf{L}^{\mathbb{C}}(W,V_{\mathbb{C}})/G\ .$$

We define a morphism of orthogonal spaces

$$f\ :\ \mathbf{L}_{G,uW}\ \longrightarrow\ \mathbf{L}^{\mathbb{C}}_{G,W}$$

as follows. The map

$$j\ :\ W\ \longrightarrow\ \mathbb{C}\otimes_{\mathbb{R}} W\ =\ (uW)_{\mathbb{C}}\ ,\quad j(w)\ =\ i\otimes w\ +\ 1\otimes(iw)$$

is a $U(W)$-equivariant (and hence G-equivariant) $\mathbb{C}$-linear isometric embedding. At a real inner product space V, we can thus define

$$f(V)\ :\ \mathbf{L}(uW,V)/G\ \longrightarrow\ \mathbf{L}^{\mathbb{C}}(W,V_{\mathbb{C}})/G \qquad \text{by} \qquad f(V)(\varphi\cdot G)\ =\ (\varphi_{\mathbb{C}}\circ j)\cdot G\ .$$

Proposition 2.16. *Let G be a compact Lie group and W a unitary G-representation. Then the morphism $f:\mathbf{L}_{G,uW}\longrightarrow\mathbf{L}^{\mathbb{C}}_{G,W}$ is a global equivalence. So if G acts faithfully on W, then $\mathbf{L}^{\mathbb{C}}_{G,W}$ is a global classifying space for G.*

Proof. By dividing by the kernel of the representation we can assume without loss of generality that G acts faithfully on W. Both source and target of f are closed orthogonal spaces, so it suffices (by Proposition 1.18) to show that for every compact Lie group K the map

$$f(\mathcal{U}_K)\ :\ \mathbf{L}(uW,\mathcal{U}_K)/G\ \longrightarrow\ \mathbf{L}^{\mathbb{C}}(W,\mathbb{C}\otimes_{\mathbb{R}}\mathcal{U}_K)/G$$

is a K-weak equivalence. We consider the $(K\times G)$-equivariant continuous map

$$\tilde f\ :\ \mathbf{L}(uW,\mathcal{U}_K)\ \longrightarrow\ \mathbf{L}^{\mathbb{C}}(W,\mathbb{C}\otimes_{\mathbb{R}}\mathcal{U}_K)\ ,\quad \varphi\ \longmapsto\ \varphi_{\mathbb{C}}\circ j$$

that 'covers' $f(\mathcal{U}_K)$. The source of $\tilde f$ is a universal $(K\times G)$-space for the family $\mathcal{F}(K;G)$ of graph subgroups, by Proposition 2.10 (i). Since $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{U}_K$ is a complete complex G-universe, the complex analog of Proposition 2.10 (i), proved in much the same way, shows that the target of $\tilde f$ is also such a universal space for the same family of subgroups of $K\times G$. So $\tilde f$ is a $(K\times G)$-equivariant homotopy equivalence. The map $f(\mathcal{U}_K)=\tilde f/G$ induced on G-orbit spaces is thus a K-equivariant homotopy equivalence. □

3. Global model structure for orthogonal spaces

In this section we establish the global model structure on the category of orthogonal spaces, see Theorem 3.22. Towards this aim we first discuss a 'strong level model structure', which we then localize. In Proposition 3.26 we use the global model structure to compare unstable global homotopy theory to the homotopy theory of K-spaces for a fixed compact Lie group K. We also show that global classifying spaces of *finite* groups and of *abelian* compact Lie groups are 'cofree', i.e., right induced from non-equivariant classifying spaces, see Example 3.32 and Theorem 3.33. We also discuss monoidal properties from a global perspective: we prove the invariance of the box product of orthogonal spaces under global equivalences in Theorem 3.38, and we check the compatibility of the box product of orthogonal spaces with the global model structure in Proposition 3.42.

There is a functorial way to write an orthogonal space as a sequential colimit of orthogonal spaces which are made from the information below a fixed level. We refer to this as the *skeleton filtration* of an

orthogonal space. The word 'filtration' should be used with caution because the maps from the skeleta to the orthogonal space need not be injective.

The skeleton filtration is in fact a special case of a more general skeleton filtration on certain enriched functor categories that we discuss in Appendix A.3. Indeed, if we specialize the base category to $\mathcal{V} = \mathbf{T}$, the category of spaces under cartesian product, and the index category to $\mathcal{D} = \mathbf{L}$, then the functor category $\mathcal{D}^*$ becomes the category *spc* of orthogonal spaces. The dimension function needed in the construction and analysis of skeleta is the vector space dimension.

We denote by $\mathbf{L}^{\leq m}$ the full topological subcategory of the linear isometries category $\mathbf{L}$ with objects all inner product spaces of dimension at most m. We denote by $spc^{\leq m}$ the category of continuous functors from $\mathbf{L}^{\leq m}$ to $\mathbf{T}$. The restriction functor

$$spc \longrightarrow spc^{\leq m}\ , \quad Y \longmapsto Y^{\leq m} = Y|_{\mathbf{L}^{\leq m}}$$

has a left adjoint

$$l_m : spc^{\leq m} \longrightarrow spc$$

given by an enriched Kan extension as follows. The extension $l_m(Z)$ of a continuous functor $Z : \mathbf{L}^{\leq m} \longrightarrow \mathbf{T}$ is a coequalizer of the two morphisms of orthogonal spaces

$$\coprod_{0\leq j\leq k\leq m} \mathbf{L}(\mathbb{R}^k, -) \times \mathbf{L}(\mathbb{R}^j, \mathbb{R}^k) \times Z(\mathbb{R}^j) \underset{V}{\overset{U}{\rightrightarrows}} \coprod_{0\leq i\leq m} \mathbf{L}(\mathbb{R}^i, -) \times Z(\mathbb{R}^i) \longrightarrow l_m(Z) \tag{3.1}$$

The morphism U arises from the composition morphisms

$$\mathbf{L}(\mathbb{R}^k, -) \times \mathbf{L}(\mathbb{R}^j, \mathbb{R}^k) \longrightarrow \mathbf{L}(\mathbb{R}^j, -)$$

and the identity on $Z(\mathbb{R}^j)$; the morphism V arises from the action maps

$$\mathbf{L}(\mathbb{R}^j, \mathbb{R}^k) \times Z(\mathbb{R}^j) \longrightarrow Z(\mathbb{R}^k)$$

and the identity on the free orthogonal space $\mathbf{L}(\mathbb{R}^k, -)$. Colimits in the category of orthogonal spaces are created objectwise, so the value $l_m(Z)(V)$ at an inner product space V can be calculated by plugging V into the variable slot in the coequalizer diagram.

It is a general property of Kan extensions along a fully faithful functor (such as the inclusion $\mathbf{L}^{\leq m} \longrightarrow \mathbf{L}$) that the values do not change on the given subcategory. More precisely, the adjunction unit

$$Z \longrightarrow (l_m(Z))^{\leq m}$$

is an isomorphism for every continuous functor $Z : \mathbf{L}^{\leq m} \longrightarrow \mathbf{T}$.

Definition 3.2. The *m-skeleton*, for $m \geq 0$, of an orthogonal space Y is the orthogonal space

$$\mathrm{sk}^m Y = l_m(Y^{\leq m})\ ,$$

the extension of the restriction of Y to $\mathbf{L}^{\leq m}$. It comes with a natural morphism $i_m : \mathrm{sk}^m Y \longrightarrow Y$, the counit of the adjunction $(l_m, (-)^{\leq m})$. The *m-th latching space* of Y is the $O(m)$-space

$$L_m Y = (\mathrm{sk}^{m-1} Y)(\mathbb{R}^m)\ ;$$

it comes with a natural $O(m)$-equivariant map

$$\nu_m = i_{m-1}(\mathbb{R}^m) : L_m Y \longrightarrow Y(\mathbb{R}^m)\ ,$$

the *m-th latching map*.

We also agree to set $\mathrm{sk}^{-1} Y = \emptyset$, the empty orthogonal space, and $L_0 Y = \emptyset$, the empty space. The value

$$i_m(V) : (\mathrm{sk}^m Y)(V) \longrightarrow Y(V)$$

of this morphism is an isomorphism for all inner product spaces V of dimension at most m.

Now we consider $0 \le l \le m$. The two morphisms $i_l : \mathrm{sk}^l\, Y \longrightarrow Y$ and $i_m : \mathrm{sk}^m\, Y \longrightarrow Y$ both restrict to isomorphisms on $\mathbf{L}^{\le l}$, so there is a unique morphism $j_{l,m} : \mathrm{sk}^l\, Y \longrightarrow \mathrm{sk}^m\, Y$ such that $i_m \circ j_{l,m} = i_l$. These morphisms satisfy

$$j_{l,m} \circ j_{k,l} \;=\; j_{k,m}$$

for all $0 \le k \le l \le m$. The sequence of skeleta stabilizes to Y in a very strong sense. For every inner product space V, the maps $j_{m,m+1}(V)$ and $i_m(V)$ are isomorphisms as soon as $m \ge \dim(V)$. In particular, $Y(V)$ is a colimit, with respect to the morphisms $i_m(V)$, of the sequence of maps $j_{m,m+1}(V)$. Since colimits in the category of orthogonal spaces are created objectwise, we deduce that the orthogonal space Y is a colimit, with respect to the morphisms i_m, of the sequence of morphisms $j_{m,m+1}$.

We denote by

$$G_m \;:\; O(m)\mathbf{T} \;\longrightarrow\; spc$$

the left adjoint to the functor $Y \mapsto Y(\mathbb{R}^m)$. So G_m is a shorthand notation for $\mathbf{L}_{O(m),\mathbb{R}^m}$, the free functor (2.6) indexed by the tautological $O(m)$-representation. Proposition A.3.19 specializes to:

Proposition 3.3. *For every orthogonal space Y and every $m \ge 0$ the commutative square*

$$\begin{tikzcd} G_m L_m Y \arrow[r, "G_m \nu_m"] \arrow[d] & G_m Y(\mathbb{R}^m) \arrow[d] \\ \mathrm{sk}^{m-1}\, Y \arrow[r, "j_{m-1,m}"'] & \mathrm{sk}^m\, Y \end{tikzcd} \tag{3.4}$$

is a pushout of orthogonal spaces. The two vertical morphisms are adjoint to the identity of $L_m Y$ respectively $Y(\mathbb{R}^m)$.

Example 3.5. As an illustration of the definition, we describe the skeleta and latching objects for small values of m. We have

$$\mathrm{sk}^0\, Y \;=\; \mathrm{const}(Y(0))\ ,$$

the constant orthogonal space with value $Y(0)$ with trivial $O(1)$-action; the latching map

$$\nu_1 \;:\; L_1 Y \;=\; (\mathrm{sk}^0\, Y)(\mathbb{R}) \;=\; Y(0) \;\xrightarrow{Y(u)}\; Y(\mathbb{R})$$

is the map induced by the unique linear isometric embedding $u : 0 \longrightarrow \mathbb{R}$. Now we evaluate the pushout square (3.4) for $m = 1$ at an inner product space V; the results is a pushout square of $O(1)$-spaces

$$\begin{tikzcd} P(V) \times Y(0) \arrow[r] \arrow[d, "\mathrm{proj}"'] & \mathbf{L}(\mathbb{R}, V) \times_{O(1)} Y(\mathbb{R}) \arrow[d] \\ Y(0) \arrow[r] & (\mathrm{sk}^1\, Y)(V) \end{tikzcd}$$

where $P(V)$ is the projective space of V. The upper horizontal map sends $(\varphi(\mathbb{R}), y)$ to $[\varphi, Y(u)(y)]$. Here we exploited that $O(1)$ acts trivially on $L_1 Y = Y(0)$ and we can thus identify

$$(G_1 L_1 Y)(V) \;=\; \mathbf{L}(\mathbb{R}, V) \times_{O(1)} Y(0) \;\cong\; P(V) \times Y(0)\ , \quad [\varphi, y] \;\longmapsto\; (\varphi(\mathbb{R}), y)$$

Example 3.6 (Latching objects of free orthogonal spaces). Let G be a compact Lie group, V a G-representation of dimension n and A a G-space. Then the free orthogonal space (2.6) generated by A in level V is 'purely n-dimensional' in the following sense. The evaluation functor

$$\mathrm{ev}_{G,V} \;:\; spc \;\longrightarrow\; G\mathbf{T}$$

factors through the category $\mathbf{L}^{\le n}$ as the composite

$$spc \;\longrightarrow\; spc^{\le n} \;\xrightarrow{\mathrm{ev}_{G,V}}\; G\mathbf{T}\ .$$

So the left adjoint free functor $\mathbf{L}_{G,V}$ can be chosen as the composite of the two individual left adjoints

$$\mathbf{L}_{G,V} \;=\; l_n \circ l_{G,V}\ .$$

Here $l_{G,V} = (-)^{\leq n} \circ \mathbf{L}_{G,V} : G\mathbf{T} \longrightarrow spc^{\leq n}$ is given at a G-space A and an inner product space W of dimension at most n by

$$(l_{G,V}A)(W) \;=\; \begin{cases} \mathbf{L}(V,W) \times_G A & \text{if } \dim(W) = n, \\ \emptyset & \text{if } \dim(W) < n. \end{cases}$$

The space $(\mathbf{L}_{G,V}A)_m$ is trivial for $m < n$, and hence the latching space $L_m(\mathbf{L}_{G,V}A)$ is trivial for $m \leq n$. For $m > n$ the latching map $\nu_m : L_m(\mathbf{L}_{G,V}A) \longrightarrow (\mathbf{L}_{G,V}A)_m$ is an isomorphism. So the skeleton $\mathrm{sk}^m(\mathbf{L}_{G,V}A)$ is trivial for $m < n$ and $\mathrm{sk}^m(\mathbf{L}_{G,V}A) = \mathbf{L}_{G,V}A$ is the entire orthogonal space for $m \geq n$.

Now we work our way towards the strong level model structure of orthogonal spaces. We recall that strong level equivalences were defined in Definition 1.8.

Proposition 3.7. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spaces such that f is a strong level equivalence. If in addition f or g is an h-cofibration, then the morphism k is a strong level equivalence.

Proof. We let G be a compact Lie group and V a G-representation. Then the square

$$\begin{tikzcd} A(V) \arrow[r, "f(V)"] \arrow[d, "g(V)"'] & B(V) \arrow[d, "h(V)"] \\ C(V) \arrow[r, "k(V)"'] & D(V) \end{tikzcd}$$

is a pushout square of G-spaces such that $f(V)$ or $g(V)$ is an h-cofibration of G-spaces (by Corollary A.1.18 (ii)). Since h-cofibrations are in particular closed embeddings (compare Proposition A.1.19 (ii)), the square yields a pushout square of spaces

$$\begin{tikzcd} A(V)^G \arrow[r, "f(V)^G"] \arrow[d, "g(V)^G"'] & B(V)^G \arrow[d, "h(V)^G"] \\ C(V)^G \arrow[r, "k(V)^G"'] & D(V)^G \end{tikzcd}$$

upon taking G-fixed points. The map $(f(V))^G$ or $(g(V))^G$ is an h-cofibration of spaces (again by Corollary A.1.18 (ii)), and $f(V)^G$ is a weak equivalence. The gluing lemma for weak equivalences and pushout along h-cofibrations (see for example [**20**, Appendix, Prop. 4.8 (b)]) shows that then $k(V)^G$ is also a weak equivalence. Hence the morphism k is a strong level equivalence. □

Proposition A.3.27 is a fairly general recipe for constructing level model structures on a category such as orthogonal spaces. We specialize the general construction to the situation at hand.

Lemma 3.8. *For every morphism $f : X \longrightarrow Y$ of orthogonal spaces, the following are equivalent.*

(i) *The morphism f is a strong level equivalence.*

(ii) *For every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is a G-weak equivalence.*
(iii) *The map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-weak equivalence for every $m \geq 0$.*

Proof. Clearly, condition (i) implies condition (ii), and that implies condition (iii) (because the tautological action of $O(m)$ on $\mathbb{R}^m$ is faithful). So we suppose that $f(\mathbb{R}^m)$ is an $O(m)$-weak equivalence for every $m \geq 0$, and we show that f is a strong level equivalence. Given a G-representation V of dimension m, we choose a linear isometry $\varphi : V \cong \mathbb{R}^m$; conjugation by φ turns the G-action on V into a homomorphism $\rho : G \longrightarrow O(m)$, i.e.,

$$\rho(g) \;=\; \varphi \circ l_g \circ \varphi^{-1} \;,$$

and the homeomorphism $X(\varphi) : X(V) \longrightarrow X(\mathbb{R}^m)$ restricts to a homeomorphism

$$X(V)^G \;\cong\; X(\mathbb{R}^m)^{\rho(G)} \;.$$

This homeomorphism is natural for morphisms of orthogonal spaces, so the hypothesis that $f(\mathbb{R}^m)^{\rho(G)} : X(\mathbb{R}^m)^{\rho(G)} \longrightarrow Y(\mathbb{R}^m)^{\rho(G)}$ is a weak equivalence implies that also the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence. □

Definition 3.9. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *strong level fibration* if for every compact Lie group G and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a Serre fibration. The morphism f is a *flat cofibration* if the latching morphism

$$\nu_m f \;=\; f(\mathbb{R}^m) \cup \nu_m^Y \;:\; X(\mathbb{R}^m) \cup_{L_m X} L_m Y \;\longrightarrow Y(\mathbb{R}^m)$$

is an $O(m)$-cofibration for all $m \geq 0$. An orthogonal space Y is *flat* if the unique morphism from the empty orthogonal space to Y is a flat cofibration. Equivalently, for every $m \geq 0$ the latching map $\nu_m : L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-cofibration.

The same kind of reasoning as in Lemma 3.8 shows:

Lemma 3.10. *For every morphism $f : X \longrightarrow Y$ of orthogonal spaces, the following are equivalent.*
(i) *The morphism f is a strong level fibration.*
(ii) *For every compact Lie group G and every faithful G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a Serre fibration*
(iii) *The map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-fibration for every $m \geq 0$.*

We are ready to establish the strong level model structure.

Proposition 3.11. *The strong level equivalences, strong level fibrations and flat cofibrations form a model structure, the* strong level model structure*, on the category of orthogonal spaces. The strong level model structures is proper, topological and cofibrantly generated.*

Proof. We apply Proposition A.3.27 as follows. We let $\mathcal{C}(m)$ be the projective model structure on the category of $O(m)$-spaces (with respect to the family of all closed subgroups of $O(m)$), compare Proposition A.2.10. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition A.3.27 precisely become the strong level equivalences, strong level fibrations and flat cofibrations.

In this situation the consistency condition (see Definition A.3.26) is a consequence of the stronger property, namely that the functor

$$\mathbf{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times_{O(m)} - \;:\; O(m)\mathbf{T} \;\longrightarrow\; O(m+n)\mathbf{T}$$

takes acyclic cofibrations to acyclic cofibrations (in the two relevant projective model structures). Since the functor is a left adjoint, it suffices to prove the claim for the generating acyclic cofibrations, i.e., the maps

$$O(m)/H \times j^k$$

for all $k \geq 0$ and all closed subgroups H of $O(m)$, where $j^k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion. The functor under consideration takes this generator to the map $j^k \times \mathbf{L}(\mathbb{R}^m, \mathbb{R}^{m+n})/H$, so it suffices to show that $\mathbf{L}(\mathbb{R}^m, \mathbb{R}^{m+n})/H$ is cofibrant as an $O(m+n)$-space. Since $\mathbf{L}(\mathbb{R}^m, \mathbb{R}^{m+n})/H$ admits the structure of a smooth $O(m+n)$-manifold, Illman's theorem [**82**, Cor. 7.2] provides an $O(m+n)$-equivariant CW-structure, so this space is indeed equivariantly cofibrant.

We describe explicit sets of generating cofibrations and generating acyclic cofibrations. We let I^{str} be the set of all morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (2.11) of Section A.2. Then the set I^{str} detects the acyclic fibrations in the strong level model structure by Proposition A.3.27 (iii). Similarly, we let J^{str} be the set of all morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (2.12) of Section A.2. Again by Proposition A.3.27 (iii), J^{str} detects the fibrations in the strong level model structure.

The model structure is topological by Proposition A.2.8, where we take $\mathcal{G}$ as the set of orthogonal spaces $L_{H,\mathbb{R}^m}$ for all $m \geq 0$ and all closed subgroups H of $O(m)$. Limits in the category of orthogonal spaces are constructed objectwise (i.e., evaluation at V preserves limits). Since weak equivalences and fibrations are also defined levelwise, right properness is inherited levelwise. The projective model structure on the category of $O(m)$-spaces is right proper for all $m \geq 0$, so right properness of the strong level model structure follows.

We already know that the strong level model structure is topological, and clearly every orthogonal space is fibrant. So flat cofibrations are h-cofibrations (by Corollary A.1.18 (iii)). Left properness is then a special case of Proposition 3.7. □

For easier reference we make the generating (acyclic) cofibrations of the strong level model structure even more explicit. Using the isomorphism

$$G_m(O(m)/H) \;=\; \mathbf{L}(\mathbb{R}^m, -) \times_{O(m)} (O(m)/H) \;\cong\; \mathbf{L}(\mathbb{R}^m, -)/H \;=\mathbf{L}_{H,\mathbb{R}^m}\;.$$

for a closed subgroup H of $O(m)$, we can identify I^{str} with the set of all morphisms

$$i_k \times \mathbf{L}_{H,\mathbb{R}^m} \;:\; \partial D^k \times \mathbf{L}_{H,\mathbb{R}^m} \;\longrightarrow\; D^k \times \mathbf{L}_{H,\mathbb{R}^m}$$

for all $k, m \geq 0$ and all closed subgroups H of $O(m)$. The tautological action of H on $\mathbb{R}^m$ is faithful; conversely every pair (G, V) consisting of a compact Lie group and a faithful representation is isomorphic to a pair $(H, \mathbb{R}^m)$ for some closed subgroup H of $\mathbb{R}^m$. We conclude that I^{str} is a set of representatives of the isomorphism classes of morphisms

$$i_k \times \mathbf{L}_{G,V} \;:\; \partial D^k \times \mathbf{L}_{G,V} \;\longrightarrow\; D^k \times \mathbf{L}_{G,V}$$

for G a compact Lie group, V a faithful G-representation and $k \geq 0$. Similarly, J^{str} is a set of representatives of the isomorphism classes of morphisms

$$j_k \;:\; D^k \times \{0\} \times \mathbf{L}_{G,V} \;\longrightarrow\; D^k \times [0,1] \times \mathbf{L}_{G,V}$$

for G a compact Lie group, V a faithful G-representation and $k \geq 0$.

Proposition 3.12. *Let K a compact Lie group and $\varphi : W \longrightarrow U$ a linear isometric embedding of K-representations, where W is finite dimensional, and U is finite dimensional or countably infinite dimensional.*

(i) *For every flat cofibration of orthogonal spaces $i : A \longrightarrow B$ the maps*

$$i(U) \;:\; A(U) \;\longrightarrow\; B(U) \qquad \text{and} \qquad i(U) \cup B(\varphi) \;:\; A(U) \cup_{A(W)} B(W) \;\longrightarrow\; B(U)$$

are K-cofibrations of K-spaces.

(ii) *For every flat orthogonal space B the map $B(\varphi) : B(W) \longrightarrow B(U)$ is a K-cofibration of K-spaces and the K-space $B(U)$ is K-cofibrant.*

(iii) *Every flat orthogonal space is closed.*

Proof. (i) The class of those morphisms of orthogonal spaces i such that the map $i(U) \cup B(\varphi)$ is a K-cofibration of K-spaces is closed under coproducts, cobase change, composition and retract. Similarly, the class of those morphisms of orthogonal spaces i such that the map $i(U)$ is a K-cofibration of K-spaces is closed under coproducts, cobase change, composition and retract. So it suffices to show each of the two claims for a set of generating cofibrations. We do this for the morphisms $i_k \times \mathbf{L}_{G,V}$ for all $k \geq 0$, all compact Lie groups G and all G-representations V, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. In this case the first map specializes to $i_k \times \mathbf{L}(V,U)/G$. The map i_k is a cofibration and $\mathbf{L}(V,U)/G$ is cofibrant as a K-space by Proposition 2.2 (ii). So $i_k \times \mathbf{L}(V,U)/G$ is a K-cofibration of K-spaces.

The second map in question becomes the pushout product of the sphere inclusion $i_k : \partial D^k \longrightarrow D^k$ with the map

$$\mathbf{L}(V,\varphi)/G \ : \ \mathbf{L}(V,W)/G \ \longrightarrow \ \mathbf{L}(V,U)/G \ .$$

The map i_k is a cofibration and $\mathbf{L}(V,\varphi)/G$ is a K-cofibration by Proposition 2.2 (i). So their pushout product is again a K-cofibration.

Part (ii) is the special case of part (i) where $A = \emptyset$ is the empty orthogonal space. Part (iii) is the special case of (ii) where K is a trivial group, using that cofibrations of spaces are in particular closed embeddings. □

Now we proceed towards the *global model structure* on the category of orthogonal spaces, see Theorem 3.22. The weak equivalences in this model structure are the global equivalences and the cofibrations are the flat cofibrations. The fibrations in the global model structure are defined as follows.

Definition 3.13. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *global fibration* if it is a strong level fibration and for every compact Lie group G, every faithful G-representation V and every equivariant linear isometric embedding $\varphi : V \longrightarrow W$ of G-representations, the map

$$(f(V)^G, X(\varphi)^G) \ : \ X(V)^G \ \longrightarrow \ Y(V)^G \times_{Y(W)^G} X(W)^G$$

is a weak equivalence.

An orthogonal space X is *static* if for every compact Lie group G, every faithful G-representation V, and every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the structure map

$$X(\varphi) \ : \ X(V) \ \longrightarrow \ X(W)$$

is a G-weak equivalence.

Equivalently, a morphism f is a global fibration if and only if f is a strong level fibration and for every compact Lie group G, every faithful G-representation V and equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the square of G-fixed point spaces

$$\begin{tikzcd} X(V)^G \arrow[r, "X(\varphi)^G"] \arrow[d, "f(V)^G"'] & X(W)^G \arrow[d, "f(W)^G"] \\ Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G \end{tikzcd} \tag{3.14}$$

is homotopy cartesian.

Clearly, an orthogonal space X is static if and only if the unique morphism from X to a terminal orthogonal space is a global fibration; the static orthogonal spaces will thus turn out to be the fibrant objects in the global model structure. The static orthogonal spaces are those which, roughly speaking, don't change the equivariant homotopy type once a faithful representation has been reached.

Proposition 3.15. (i) *Every global equivalence that is also a global fibration is a strong level equivalence.*

(ii) *Every global equivalence between static orthogonal spaces is a strong level equivalence.*

PROOF. (i) We let $f : X \longrightarrow Y$ be a morphism of orthogonal spaces that is both a global fibration and a global equivalence. We consider a compact Lie group G, a faithful G-representation V, a finite G-CW-pair (B, A) and a commutative square:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V) \arrow[d, "f(V)"] \\ B \arrow[r, "\beta"'] & Y(V) \end{tikzcd}$$

We will exhibit a continuous G-map $\mu : B \longrightarrow X(V)$ such that $\mu|_A = \alpha$ and such that $f(V) \circ \mu$ is homotopic, relative A, to β. This shows that the map $f(V)$ is a G-weak equivalence, so f is a strong level equivalence.

Since f is a global equivalence, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ \alpha : A \longrightarrow X(W)$ and such that $f(W) \circ \lambda : B \longrightarrow Y(W)$ is G-homotopic, relative to A, to $Y(\varphi) \circ \beta$. Since f is a strong level fibration, we can improve λ into a continuous G-map $\lambda' : B \longrightarrow X(W)$ such that $\lambda'|_A = \lambda|_A = X(\varphi) \circ \alpha$ and such that $f(W) \circ \lambda'$ is equal to $Y(\varphi) \circ \beta$.

Since f is a global fibration the G-map

$$(f(V), X(\varphi)) \ : \ X(V) \ \longrightarrow \ Y(V) \times_{Y(W)} X(W)$$

is a G-weak equivalence. So we can find a continuous G-map $\mu : B \longrightarrow X(V)$ such that $\mu|_A = \alpha$ and $(f(V), X(\varphi)) \circ \mu$ is G-homotopic, relative A, to $(\beta, \lambda') : B \longrightarrow Y(V) \times_{Y(W)} X(W)$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V) \arrow[d, "(f(V),X(\varphi))"] \\ B \arrow[r, "(\beta,\lambda')"'] \arrow[ur, dashed, "\mu"] & Y(V) \times_{Y(W)} X(W) \end{tikzcd}$$

This is the desired map.

(ii) We let $f : X \longrightarrow Y$ be a global equivalence between static orthogonal spaces. We let G be a compact Lie group, V a faithful G-representation, (B, A) a finite G-CW-pair and $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ continuous G-maps such that $f(V) \circ \alpha = \beta|_A$. Since f is a global equivalence, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ \alpha$ and $f(W) \circ \lambda$ is G-homotopic to $Y(\varphi) \circ \beta$ relative A. Since X is static, the map $X(\varphi) : X(V) \longrightarrow X(W)$ is a G-weak equivalence, so there is a continuous map $\bar{\lambda} : B \longrightarrow X(V)$ such that $\bar{\lambda}|_A = \alpha$ and $X(\varphi) \circ \bar{\lambda}$ is G-homotopic to λ relative A. The two G-maps $f(V) \circ \bar{\lambda}$ and $\beta : B \longrightarrow Y(V)$ then agree on A and become G-homotopic, relative A, after composition with $Y(\varphi) : Y(V) \longrightarrow Y(W)$. Since Y is static, the map $Y(\varphi)$ is a G-weak equivalence, so $f(V) \circ \bar{\lambda}$ and $\beta : B \longrightarrow Y(V)$ are already G-homotopic relative A. This shows that $f(V) : X(V) \longrightarrow Y(V)$ is a G-weak equivalence, and hence f is a strong level equivalence. □

CONSTRUCTION 3.16. We let $j : A \longrightarrow B$ be a morphism in a topological model category. We factor j through the mapping cylinder as the composite

$$A \ \xrightarrow{c(j)} \ Z(j) = ([0,1] \times A) \cup_j B \ \xrightarrow{r(j)} \ B \ ,$$

where $c(j)$ is the 'front' mapping cylinder inclusion and $r(j)$ is the projection, which is a homotopy equivalence. In our applications we will assume that both A and B are cofibrant, and then the morphism $c(j)$ is a cofibration by the pushout product property. We then define $\mathcal{Z}(j)$ as the set of all pushout product maps

$$i_k \square c(j) \ : \ D^k \times A \cup_{\partial D^k \times A} \partial D^k \times Z(j) \ \longrightarrow \ D^k \times Z(j)$$

for $k \geq 0$, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion.

PROPOSITION 3.17. *Let $\mathcal{C}$ be a topological model category, $j : A \longrightarrow B$ a morphism between cofibrant objects and $f : X \longrightarrow Y$ a fibration. Then the following two conditions are equivalent:*

(i) *The square of spaces*

$$\begin{tikzcd} \mathrm{map}(B,X) \arrow[r, "\mathrm{map}(j,X)"] \arrow[d, "\mathrm{map}(B,f)"'] & \mathrm{map}(A,X) \arrow[d, "\mathrm{map}(A,f)"] \\ \mathrm{map}(B,Y) \arrow[r, "\mathrm{map}(j,Y)"'] & \mathrm{map}(A,Y) \end{tikzcd} \tag{3.18}$$

is homotopy cartesian.

(ii) *The morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$.*

PROOF. The square (3.18) maps to the square

$$\begin{tikzcd} \mathrm{map}(Z(j),X) \arrow[r, "\mathrm{map}(c(j),X)"] \arrow[d, "\mathrm{map}(Z(j),f)"'] & \mathrm{map}(A,X) \arrow[d, "\mathrm{map}(A,f)"] \\ \mathrm{map}(Z(j),Y) \arrow[r, "\mathrm{map}(c(j),Y)"'] & \mathrm{map}(A,Y) \end{tikzcd} \tag{3.19}$$

via the map induced by $r(j) : Z(j) \longrightarrow B$ on the left part and the identity on the right part. Since $r(j)$ is a homotopy equivalence, the map of squares is a weak equivalence at all four corners. So the square (3.18) is homotopy cartesian if and only if the square (3.19) is homotopy cartesian.

Since A is cofibrant and f a fibration, $\mathrm{map}(A,f)$ is a Serre fibration. So the square (3.19) is homotopy cartesian if and only if the map

$$(\mathrm{map}(Z(j),f), \mathrm{map}(c(j),X)) \ : \ \mathrm{map}(Z(j),X) \ \longrightarrow \ \mathrm{map}(Z(j),Y) \times_{\mathrm{map}(A,Y)} \mathrm{map}(A,X) \tag{3.20}$$

is a weak equivalence. Since $c(j)$ is a cofibration and f is a fibration, the map (3.20) is always a Serre fibration. So (3.20) is a weak equivalence if and only if it is an acyclic fibration, which is equivalent to the right lifting property for the inclusions $i_k : \partial D^k \longrightarrow D^k$ for all $k \geq 0$. By adjointness, the map (3.20) has the right lifting property with respect to the maps i_k if and only if the morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$. □

The set J^{str} was defined in the proof of Proposition 3.11 as the set of morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (2.12) of Section A.2. The set J^{str} detects the fibrations in the strong level model structure. We add another set of morphisms K that detects when the squares (3.14) are homotopy cartesian. Given any compact Lie group G and G-representations V and W, the restriction morphism

$$\rho_{G,V,W} \ = \ \rho_{V,W}/G \ : \ \mathbf{L}_{G,V\oplus W} \ \longrightarrow \ \mathbf{L}_{G,V}$$

restricts (the G-orbit of) a linear isometric embedding from $V \oplus W$ to V. If the representation V is faithful, then this morphism is a global equivalence by Proposition 2.10 (ii). We set

$$K \ = \ \bigcup_{G,V,W} \mathcal{Z}(\rho_{G,V,W}) \ ,$$

the set of all pushout products of boundary inclusions $\partial D^k \longrightarrow D^k$ with the mapping cylinder inclusions of the morphisms $\rho_{G,V,W}$; here the union is over a set of representatives of the isomorphism classes of triples (G,V,W) consisting of a compact Lie group G, a faithful G-representation V and an arbitrary G-representation W. The morphism $\rho_{G,V,W}$ represents the map of G-fixed point spaces $X(i_{V,W})^G : X(V)^G \longrightarrow$

$X(V \oplus W)^G$; every G-equivariant linear isometric embedding is isomorphic to a direct summand inclusion $i_{V,W}$, so by Proposition 3.17, the right lifting property with respect to the union $J^{\mathrm{str}} \cup K$ characterizes the global fibrations, i.e., we have shown:

PROPOSITION 3.21. *A morphism of orthogonal spaces is a global fibration if and only if it has the right lifting property with respect to the set $J^{\mathrm{str}} \cup K$.*

Now we are ready for the main result of this section.

THEOREM 3.22 (Global model structure). *The global equivalences, global fibrations and flat cofibrations form a model structure, the* global model structure *on the category of orthogonal spaces. The fibrant objects in the global model structure are the static orthogonal spaces. The global model structure is proper, topological and compactly generated.*

PROOF. We refer the reader to [**47**, 3.3] for the numbering of the model category axioms. The category of orthogonal spaces is complete and cocomplete, so axiom MC1 holds. Global equivalences satisfy the 2-out-of-6 property by Proposition 1.9 (iii); the 2-out-of-6 property specializes to the model category axiom MC2 (2-out-of-3) by letting one of the respective maps f, g or h be an identity. Global equivalences are closed under retracts by Proposition 1.9 (iv); it is straightforward that cofibrations and global fibrations are closed under retracts, so axiom MC3 (closure properties under retracts) holds.

The strong level model structure shows that every morphism of orthogonal spaces can be factored as a flat cofibration followed by a strong level equivalence. Since strong level equivalences are in particular global equivalences, this provides one of the factorizations as required by MC5. For the other half of the factorization axiom MC5 we apply the small object argument (see for example [**47**, 7.12] or [**79**, Thm. 2.1.14]) to the set $J^{\mathrm{str}} \cup K$. All morphisms in J^{str} are flat cofibrations and strong level equivalences. Since $\mathbf{L}_{G,V\oplus W}$ and $\mathbf{L}_{G,V}$ are flat, the morphisms in K are also flat cofibrations, and they are global equivalences because the morphisms $\rho_{G,V,W}$ are (Proposition 2.10 (ii)). The small object argument provides a functorial factorization of every morphism $\varphi : X \longrightarrow Y$ of orthogonal spaces as a composite

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in K, and q has the right lifting property with respect to $J^{\mathrm{str}} \cup K$. Since all morphisms in K are flat cofibrations and global equivalences, the morphism i is a flat cofibration and a global equivalence by the closure properties of Proposition 1.9. Moreover, q is a global fibration by Proposition 3.21.

Now we show the lifting properties MC4. By Proposition 3.15 (i) a morphism that is both a global equivalence and a global fibration is a strong level equivalence, and hence an acyclic fibration in the strong level model structure. So every morphism that is simultaneously a global equivalence and a global fibration has the right lifting property with respect to flat cofibrations. Now we let $j : A \longrightarrow B$ be a flat cofibration that is also a global equivalence and we show that it has the left lifting property with respect to all global fibrations. We factor $j = q \circ i$, via the small object argument for $J^{\mathrm{str}} \cup K$, where $i : A \longrightarrow W$ is an $(J^{\mathrm{str}} \cup K)$-cell complex and $q : W \longrightarrow B$ a global fibration. Then q is a global equivalence since j and i are, and hence an acyclic fibration in the strong level model structure, again by Proposition 3.15 (i). Since j is a flat cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting property with respect to global fibrations. But then j itself has this lifting property. This finishes the verification of the model category axioms. Alongside we have also specified sets of generating flat cofibrations I^{str} and generating acyclic

cofibrations $J^{\mathrm{str}} \cup K$. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of flat cofibrations. So the global model structure is compactly generated.

Left properness of the global model structure follows from Proposition 1.9 (x) and the fact that flat cofibrations are h-cofibrations (Corollary A.1.18 (iii)). Right properness follows from Proposition 1.9 (xi) because global fibrations are in particular strong level fibrations.

It remains to show that the global model structure is topological. The cofibrations in the global model structure coincide with the cofibrations in the strong level model structure, so the pushout product of a cofibration of spaces with a flat cofibration is a flat cofibration by Proposition 3.11. Similarly, the pushout product of an acyclic cofibration of spaces with a flat cofibration is a strong level equivalence by Proposition 3.11, hence a global equivalence. Finally, we have to show that pushout products of cofibrations of spaces with flat cofibrations that are also global equivalences are again global equivalences. It suffices to consider a generating cofibration $i_k : \partial D^k \longrightarrow D^k$ of spaces and a generating acyclic cofibration in the set $J^{\mathrm{str}} \cup K$. The morphisms in J^{str} are strong level equivalences, hence taken care of by Proposition 3.11 again. The pushout product of i_k and a morphism $i_m \Box c(j)$ in $\mathcal{Z}(\rho_{G,V,W})$ is isomorphic to $i_{k+m} \Box c(j)$, hence again a flat cofibration and global equivalence. $\Box$

Remark 3.23. We can relate the unstable global homotopy theory of orthogonal spaces to the homotopy theory of G-spaces for a fixed compact Lie group G. We fix a faithful G-representation V. Then evaluation at V and the free functor at (G, V) are a pair of adjoint functors

$$G\mathbf{T} \underset{\mathrm{ev}_V}{\overset{\mathbf{L}_{G,V}}{\rightleftarrows}} spc$$

between the categories of G-spaces and orthogonal spaces. This adjoint pair is a Quillen pair with respect to the global model structure of orthogonal spaces and the 'genuine' model structure of G-spaces (i.e., the projective model structure with respect to the family of all subgroups, compare Proposition A.2.10). The adjoint total derived functors

$$\mathrm{Ho}(G\mathbf{T}) \underset{R(\mathrm{ev}_V)}{\overset{L(\mathbf{L}_{G,V})}{\rightleftarrows}} \mathrm{Ho}(spc)$$

are independent of the faithful representation V up to preferred natural isomorphism, by Proposition 2.10 (ii).

Every G-space is G-weakly equivalent to a G-CW-complex, and these are built from the orbits G/H. So the derived left adjoint $L(\mathbf{L}_{G,V}) : G\mathbf{T} \longrightarrow spc$ is essentially determined by its values on the coset spaces G/H. Since $\mathbf{L}_{G,V}(G/H)$ is isomorphic to $\mathbf{L}_{H,V} = B_{\mathrm{gl}}H$, the derived left adjoint takes the orbit space G/H to a global classifying space of H.

The derived right adjoint also has a more explicit description, at least for closed orthogonal spaces Y, as $Y(\mathcal{U}_G)$, the evaluation at the chosen complete G-universe. Indeed, we can choose a fibrant replacement of Y in the global model structure, i.e., a flat cofibration $j : Y \longrightarrow Z$ that is also a global equivalence, and such that Z is globally fibrant (i.e., static). Then Z is also closed, and so the induced map $j(\mathcal{U}_G) : Y(\mathcal{U}_G) \longrightarrow Z(\mathcal{U}_G)$ is a G-weak equivalence by Proposition 1.18. We may assume that V is a subrepresentation of $\mathcal{U}_G$; we choose a nested sequence

$$V = V_1 \ \subset \ V_2 \ \subset \ \dots \ \subset \ V_n \ \subset \ \dots$$

of finite dimensional G-subrepresentations that exhaust $\mathcal{U}_G$. Since V is faithful and Z is closed and static, the induced maps

$$Z(V) = Z(V_1) \ \longrightarrow \ Z(V_2) \ \longrightarrow \ \dots \ \longrightarrow \ Z(V_n) \ \longrightarrow \ \dots$$

are all closed embeddings and G-weak equivalences, So the canonical map

$$Z(V) \ \longrightarrow \ \mathrm{colim}_{n \geq 1} \, Z(V_n) \ = \ Z(\mathcal{U}_G)$$

is also a G-weak equivalences. Since Z is a globally fibrant replacement of Y, the G-space $Z(V)$ calculates the right derived functor of ev_V at Y. This exhibits a chain of two G-weak equivalences

$$R(\mathrm{ev}_V)(Y) \;=\; Z(V) \;\xrightarrow{\simeq}\; Z(\mathcal{U}_G) \;\xleftarrow{\simeq}\; Y(\mathcal{U}_G)\ .$$

Construction 3.24 (Cofree orthogonal spaces). For every compact Lie group K, we will now define and study a right adjoint to the functor that takes an orthogonal space Y to the 'underlying' K-space $Y(\mathcal{U}_K)$. We refer to the right adjoint R_K as the *cofree functor*. We consider the continuous functor

$$\mathbf{L}(-,\mathcal{U}_K)\ :\ \mathbf{L}^{\mathrm{op}} \;\longrightarrow\; K\mathbf{T}\ ,\quad V \;\longmapsto\; \mathbf{L}(V,\mathcal{U}_K)\ ,$$

with functoriality by precomposition with linear isometric embeddings. The group K acts on the values of this functor through the action on the complete universe $\mathcal{U}_K$. The cofree orthogonal space $R_K(A)$ associated to a K-space A is then the composite

$$\mathbf{L} \;\xrightarrow{\mathbf{L}(-,\mathcal{U}_K)^{\mathrm{op}}}\; K\mathbf{T}^{\mathrm{op}} \;\xrightarrow{\mathrm{map}^K(-,A)}\; \mathbf{T}\ .$$

A left adjoint to R_K is given by an enriched coend, sending an orthogonal space Y to the K-space

$$\mathbf{L}(-,\mathcal{U}_K)\otimes_{\mathbf{L}} Y \;=\; \int_{V\in\mathbf{L}} \mathbf{L}(V,\mathcal{U}_K)\times Y(V)\ .$$

We have already seen this functor under a different name. Indeed, the continuous evaluation maps

$$\mathbf{L}(V,\mathcal{U}_K)\times Y(V) \;\longrightarrow\; Y(\mathcal{U}_K)\ ,\quad (\varphi,y)\;\longmapsto\; Y(\varphi)(y)$$

are compatible as V varies through the objects of the category $\mathbf{L}$. The universal property of a coend produces a natural continuous map

$$\mathbf{L}(-,\mathcal{U}_K)\otimes_{\mathbf{L}} Y \;\longrightarrow\; Y(\mathcal{U}_K)\ .$$

Both sides of this map preserve colimits in Y and commute with products with spaces. Moreover, the map is a homeomorphism when $Y=\mathbf{L}_V$ is a free orthogonal space. Since every orthogonal space is a coend of orthogonal spaces of the form $\mathbf{L}_V\times Y(V)$, this map is a homeomorphism in general. In other words, the functors

$$spc \;\underset{R_K}{\overset{(-)(\mathcal{U}_K)}{\rightleftarrows}}\; \mathbf{T}$$

form an adjoint pair. The unit of the adjunction is the morphism of orthogonal spaces

$$\eta_Y\ :\ Y \;\longrightarrow\; R_K(Y(\mathcal{U}_K))$$

whose value at an inner product space V is the adjoint of the action map

$$\mathbf{L}(V,\mathcal{U}_K)\times Y(V) \;\longrightarrow\; Y(\mathcal{U}_K)\ ,\quad (\varphi,y)\;\longmapsto\; Y(\varphi)(y)\ .$$

We can endow the cofree functor with a lax symmetric monoidal transformation

$$\mu_{A,B}\ :\ R_K(A)\boxtimes R_K(B) \;\longrightarrow\; R_K(A\times B)\ .$$

To construct $\mu_{A,B}$ we start from the continuous maps

$$\begin{aligned}\mathrm{map}^K(\mathbf{L}(V,\mathcal{U}_K),A)\times\mathrm{map}^K(\mathbf{L}(W,\mathcal{U}_K),B) \;&\xrightarrow{\times}\; \mathrm{map}^K(\mathbf{L}(V,\mathcal{U}_K)\times\mathbf{L}(W,\mathcal{U}_K),A\times B)\\ &\xrightarrow{(\mathrm{res}_{V,W})^*}\; \mathrm{map}^K(\mathbf{L}(V\oplus W,\mathcal{U}_K),A\times B)\end{aligned}\tag{3.25}$$

that constitute a bimorphism from $(R_K(A),R_K(B))$ to $R_K(A\times B)$. Here

$$\mathrm{res}_{V,W}\ :\ \mathbf{L}(V\oplus W,\mathcal{U}_K) \;\longrightarrow\; \mathbf{L}(V,\mathcal{U}_K)\times\mathbf{L}(W,\mathcal{U}_K)$$

is the map that takes an embedding of $V\oplus W$ to the pair of its restrictions to V and W. The morphism $\mu_{A,B}$ is associated to this bimorphism via the universal property of the box product. Since the cofree

functor R_K is lax symmetric monoidal with respect to the maps $\mu_{A.B}$, it takes topological G-space monoids to orthogonal monoid spaces, in a way preserving commutativity.

PROPOSITION 3.26. *Let K be a compact Lie group.*

(i) *The adjoint functor pair $((-)(\mathcal{U}_K), R_K)$ is a Quillen pair for the global model structure of orthogonal spaces and the projective model structure of K-spaces.*
(ii) *For every K-space A the orthogonal space $R_K(A)$ is static.*
(iii) *For every closed orthogonal space Y the map*

$$\mathrm{Ho}(K\mathbf{T})(Y(\mathcal{U}_K), A) \xrightarrow{(\eta_Y)^* \circ R_K} \mathrm{Ho}(spc)(Y, R_K(A))$$

is bijective.

PROOF. (i) We let $f : X \longrightarrow Y$ be a fibration of K-spaces. We let G be another compact Lie group and V a faithful G-representation. Then the K-space $\mathbf{L}(V,\mathcal{U}_K)/G$ is K-cofibrant by Proposition 2.2 (ii). The projective model structure on K-spaces is topological, so $\mathrm{map}^K(\mathbf{L}(V,\mathcal{U}_K)/G, -)$ takes fibrations of K-spaces to fibrations of spaces. Because

$$\mathrm{map}^K(\mathbf{L}(V,\mathcal{U}_K)/G, X) \ = \ (R_K(X)(V))^G \ ,$$

this means that R_K takes fibrations of K-spaces to strong level fibrations of orthogonal spaces. By the same argument, R_K takes acyclic fibrations of K-spaces to acyclic fibrations in the strong level model structure, which coincide with the acyclic fibrations in the global model structure of orthogonal spaces.

Now we let $\varphi : V \longrightarrow W$ be a G-equivariant linear isometric embedding. Then the map

$$\rho_{V,W}(\mathcal{U}_K)/G \ : \ \mathbf{L}(V \oplus W,\mathcal{U}_K)/G \ \longrightarrow \ \mathbf{L}(V,\mathcal{U}_K)/G$$

is a K-homotopy equivalence by Proposition 2.10 (ii). So the induced map

$$(R_K(X)(\varphi))^G \ : \ (R_K(X)(V))^G \ \longrightarrow \ (R_K(X)(V \oplus W))^G$$

is a homotopy equivalence of non-equivariant spaces. So in the commutative square

$$\begin{array}{ccc} (R_K(X)(V))^G & \xrightarrow{R_K(X)(\varphi)^G} & (R_K(X)(V \oplus W))^G \\ \downarrow & & \downarrow \\ (R_K(Y)(V))^G & \xrightarrow[R_K(Y)(\varphi)^G]{} & (R_K(Y)(V \oplus W))^G \end{array}$$

both vertical maps are Serre fibrations and both horizontal maps are weak equivalences. The square is then homotopy cartesian, and so the morphism $R_K(f) : R_K(X) \longrightarrow R_K(Y)$ is a global fibration of orthogonal spaces. Altogether this shows that the right adjoint R_K preserves fibrations and acyclic fibrations, so $((-)(\mathcal{U}_K), R_K)$ is a Quillen pair.

(ii) In the projective model structure of K-spaces every object A is fibrant. So $R_K(A)$ is fibrant in the global model structure of orthogonal spaces; by Theorem 3.22 these fibrant object are precisely the static orthogonal spaces.

(iii) We choose a global equivalence $f : Y^c \longrightarrow Y$ with flat source. Then Y^c and Y are both closed, the former by Proposition 3.12 (iii). So the map $f(\mathcal{U}_K) : Y^c(\mathcal{U}_K) \longrightarrow Y(\mathcal{U}_K)$ is a K-weak equivalence by Proposition 1.18. So the morphism f induces bijections on both sides of the map in question, hence it suffices to prove the claim for Y^c instead of Y. But Y^c is cofibrant and A is fibrant, so in this case the claim is just the derived adjunction isomorphism. □

For $K = e$ the trivial group we drop the decoration and abbreviate the cofree functor to R.

DEFINITION 3.27. An orthogonal space Y is *cofree* if it is globally equivalent to an orthogonal space of the form RA for some space A.

We will now develop criteria for detecting cofree orthogonal spaces, and then recall some non-tautological examples. One criterion involves the unit of the adjunction, the morphism of orthogonal spaces

$$\eta_Y \ : \ Y \ \longrightarrow \ R(Y(\mathbb{R}^\infty))$$

whose value at an inner product space V is the adjoint of the action map

$$\mathbf{L}(V,\mathbb{R}^\infty) \times Y(V) \ \longrightarrow \ Y(\mathbb{R}^\infty) \ , \quad (\varphi, y) \ \longmapsto \ Y(\varphi)(y) \ .$$

The next proposition shows that the morphism η_Y is always a non-equivariant equivalence, at least if Y is closed.

Proposition 3.28. *For every closed orthogonal space Y the morphism $\eta_Y : Y \longrightarrow R(Y(\mathbb{R}^\infty))$ induces a weak equivalence*

$$\eta_Y(\mathbb{R}^\infty) \ : Y(\mathbb{R}^\infty) \ \longrightarrow \ R(Y(\mathbb{R}^\infty))(\mathbb{R}^\infty)$$

on underlying non-equivariant spaces.

Proof. We start with a general observation about cofree orthogonal spaces. Since RA is static and closed, the map $(RA)(\varphi) : (RA)(V) \longrightarrow (RA)(W)$ induced by any linear isometric embedding $\varphi : V \longrightarrow W$. is a weak equivalence and a closed embedding. So the canonical map

$$A \ \cong \ \mathrm{map}(\mathbf{L}(0,\mathbb{R}^\infty), A) \ = \ (RA)(0) \ \longrightarrow \ \mathrm{colim}_{W \in s(\mathbb{R}^\infty)}(RA)(W) \ = \ (RA)(\mathbb{R}^\infty) \tag{3.29}$$

is a weak equivalence as well. This canonical map has a preferred retraction: for every $W \in s(\mathbb{R}^\infty)$, evaluation at the inclusion $i_W : W \longrightarrow \mathbb{R}^\infty$ is a continuous map

$$(RA)(W) \ = \ \mathrm{map}(\mathbf{L}(W,\mathbb{R}^\infty), A) \ \longrightarrow \ A \ , \quad f \ \longmapsto \ f(i_W) \ .$$

These maps are compatible as W runs over the poset $s(\mathbb{R}^\infty)$, so they combine into a continuous map

$$s \ : \ (RA)(\mathbb{R}^\infty) \ \longrightarrow \ A$$

on the colimit that is left inverse to (3.29). Since s is a retraction to a weak equivalence, it is a weak equivalence itself.

Now we turn to the proof of the proposition. Even though the map $\eta_Y(\mathbb{R}^\infty)$ under consideration is *not* the same as the canonical map (3.29) for $A = Y(\mathbb{R}^\infty)$, the retraction $s : R(Y(\mathbb{R}^\infty))(\mathbb{R}^\infty) \longrightarrow Y(\mathbb{R}^\infty)$ is also a retraction to $\eta_Y(\mathbb{R}^\infty)$. Since s is a weak equivalence, so is $\eta_Y(\mathbb{R}^\infty)$. □

While the morphism $\eta_Y : Y \longrightarrow R(Y(\mathbb{R}^\infty))$ tends to be a non-equivariant equivalence, it is typically *not* a global equivalence. However, we will now see that for a closed orthogonal space Y the morphism η_Y is a global equivalence if and only if Y is cofree.

We recall that a *universal free K-space*, for a compact Lie group K, is a free K-space that is K-homotopy equivalent to a cofibrant K-space and whose underlying space in non-equivariantly contractible. Any two universal free K-spaces are K-homotopy equivalent. We call a K-space A *cofree* if for some (hence any) universal free K-space EK the map

$$\mathrm{const} \ : \ A \ \longrightarrow \ \mathrm{map}(EK, A)$$

that sends a point to the corresponding constant map is a K-weak equivalence.

Proposition 3.30. *For a closed orthogonal space Y the following three conditions are equivalent.*

(i) *The orthogonal space Y is cofree.*
(ii) *For every compact Lie group K the K-space $Y(\mathcal{U}_K)$ is cofree.*
(iii) *The adjunction unit $\epsilon_Y : Y \longrightarrow R(Y(\mathbb{R}^\infty))$ is a global equivalence.*

Proof. In a first step we show that for every space A and every compact Lie group K, the K-space $(RA)(\mathcal{U}_K)$ is cofree. We choose a faithful K-representation W. Then $\mathbf{L}(W,\mathbb{R}^\infty)$ is a universal free K-space by Proposition 2.10 (i). So the projection

$$EK \times \mathbf{L}(W,\mathbb{R}^\infty) \;\longrightarrow\; \mathbf{L}(W,\mathbb{R}^\infty)$$

is a K-weak equivalence between cofibrant K-spaces, hence a K-homotopy equivalence. So the induced map

$$\text{const} \;:\; (RA)(W) \;=\; \text{map}(\mathbf{L}(W,\mathbb{R}^\infty),A) \;\longrightarrow\; \text{map}(EK\times\mathbf{L}(W,\mathbb{R}^\infty),A) \;\cong\; \text{map}(EK,(RA)(W))$$

is a K-homotopy equivalence. Hence the K-space $(RA)(W)$ is cofree as soon as K acts faithfully on W. Since RA is static (by Proposition 3.26 (ii)) and closed, the canonical map

$$(RA)(W) \;\longrightarrow\; (RA)(\mathcal{U}_K)$$

is a K-weak equivalence. So $(RA)(\mathcal{U}_K)$ is K-cofree. Now we prove the equivalence of conditions (i), (ii) and (iii).

(i)$\Longrightarrow$(ii) The global equivalences are part of the global model structure on the category of orthogonal spaces, compare Theorem 3.22. Moreover, the orthogonal space RA is static, hence fibrant in the global model structure. So if Y is globally equivalent to RA, then for some (hence any) global equivalence $p: Y^c \longrightarrow Y$ with cofibrant (i.e., flat) source, there is a global equivalence $f: Y^c \longrightarrow RA$.

Now we let K be any compact Lie group. The orthogonal space Y^c is closed by Proposition 3.12 (iii). Since Y and RA are also closed, the global equivalences induce K-weak equivalences

$$Y(\mathcal{U}_K) \;\xleftarrow[\sim]{p(\mathcal{U}_K)}\; Y^c(\mathcal{U}_K) \;\xrightarrow[\sim]{f(\mathcal{U}_K)}\; (RA)(\mathcal{U}_K)$$

by Proposition 1.18. Since $(RA)(\mathcal{U}_K)$ is K-cofree by the introductory remark, so is $Y(\mathcal{U}_K)$.

(ii)$\Longrightarrow$(iii) We start with a preliminary observation. We let Y and Z be two closed orthogonal spaces such that the K-spaces $Y(\mathcal{U}_K)$ and $Z(\mathcal{U}_K)$ are cofree for all compact Lie groups K. We claim that every morphism $f: Y \longrightarrow Z$ of orthogonal spaces such that $f(\mathbb{R}^\infty): Y(\mathbb{R}^\infty) \longrightarrow Z(\mathbb{R}^\infty)$ is a weak equivalence is already a global equivalence. Indeed, for every compact Lie group K the two vertical maps in the commutative square of K-spaces

$$\begin{tikzcd}
Y(\mathcal{U}_K) \arrow[r, "f(\mathcal{U}_K)"] \arrow[d, "\text{const}"'] & Z(\mathcal{U}_K) \arrow[d, "\text{const}"] \\
\text{map}(EK, Y(\mathcal{U}_K)) \arrow[r, "\simeq", "\text{map}(EK,f(\mathcal{U}_K))"'] & \text{map}(EK, Z(\mathcal{U}_K))
\end{tikzcd}$$

are K-weak equivalences by hypothesis. Since $\mathcal{U}_K$ is non-equivariantly isometrically isomorphic to $\mathbb{R}^\infty$, the K-map $f(\mathcal{U}_K): Y(\mathcal{U}_K) \longrightarrow Z(\mathcal{U}_K)$ is a non-equivariant weak equivalence by hypothesis. So the lower horizontal map is a K-weak equivalence. We conclude that the upper horizontal map is a K-weak equivalence. Since Y and Z are closed, the criterion of Proposition 1.18 shows that f is a global equivalence.

Now we applying the criterion to the morphism $\eta_Y: Y \longrightarrow R(Y(\mathbb{R}^\infty))$. The map $\eta_Y(\mathbb{R}^\infty)$ is a weak equivalence by Proposition 3.28. Moreover, for every compact Lie group K the space $Y(\mathcal{U}_K)$ is K-cofree by hypothesis (ii), and $R(Y(\mathbb{R}^\infty))(\mathcal{U}_L)$ is K-cofree by the introductory remark. The criterion of the previous paragraph thus applies and shows that the morphism $\eta_Y: Y \longrightarrow R(Y(\mathbb{R}^\infty))$ is a global equivalence.

Condition (i) is a special case of (iii). □

Remark 3.31. We rewrite the criterion (ii) of Proposition 3.30 a little bit. We let Y be a closed orthogonal space and K a compact Lie group. We let K be another compact Lie group; to see that the space $Y(\mathcal{U}_K)$ is K-cofree we may check that for every closed subgroup L of K the map on L-fixed points

$$(Y(\mathcal{U}_K))^L \;\longrightarrow\; \text{map}^L(EK, Y(\mathcal{U}_K))$$

is a weak equivalence. Since the underlying L-representation of $\mathcal{U}_K$ is a complete L-universe and the underlying L-space of EK is a universal free L-space, it suffices to check the case $L = K$. Now we choose a linear isometry $\psi : \mathbb{R}^\infty \cong (\mathcal{U}_K)^K$. Then the induced map

$$Y(\psi) \ : \ Y(\mathbb{R}^\infty) \ \longrightarrow \ Y(\mathcal{U}_K)$$

is K-equivariant and a non-equivariant weak equivalence. So it induces a weak equivalence

$$\mathrm{map}^K(EK, Y(\psi)) \ : \ \mathrm{map}^K(EK, Y(\mathbb{R}^\infty)) \ \xrightarrow{\sim} \ \mathrm{map}^K(EK, Y(\mathcal{U}_K)) \ .$$

Since K acts trivially on $Y(\mathbb{R}^\infty)$, the source of the latter map is homeomorphic to $\mathrm{map}(BK, Y(\mathbb{R}^\infty))$, the space of maps from a non-equivariant classifying space BK to the underlying space $Y(\mathbb{R}^\infty)$. We conclude that a closed orthogonal space Y is cofree if any only if for every compact Lie group K a certain weak map

$$Y(\mathcal{U}_K)^K \ \dashrightarrow \ \mathrm{map}(BK, Y(\mathbb{R}^\infty))$$

is a weak equivalence.

Example 3.32 (Global classifying spaces of finite groups). We recall that the global classifying space of every finite group G is cofree. To see this we choose a faithful G-representation V and verify criterion (ii) of Proposition 3.30, or rather its reinterpretation given in the previous Remark 3.31, for the closed orthogonal space $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$. We need to check that for every compact Lie group K a specific weak map

$$(\mathbf{L}(V, \mathcal{U}_K)/G)^K \ \dashrightarrow \ \mathrm{map}(BK, BG)$$

is a weak equivalence, where we exploited that $BG = \mathbf{L}_{G,V}(\mathbb{R}^\infty)$ is a non-equivariant classifying space for the group G.

By Proposition 5.16 (ii) below the path components of the left hand side biject with the set $\mathrm{Rep}(K, G)$ of conjugacy classes of continuous group homomorphisms $\alpha : K \longrightarrow G$. The path components of the right hand side is the set of homotopy classes of continuous maps from BK to BG. Since G is finite, this set also bijects with $\mathrm{Rep}(K, G)$; this goes back to Hurewicz [**81**] (who proved it under certain finiteness hypotheses). Now we fix a continuous homomorphism $\alpha : K \longrightarrow G$. The path component of $(\mathbf{L}(V, \mathcal{U}_K)/G)^K$ indexed by α is a classifying spaces of $C\alpha$, the centralizer of the image of α, by Proposition 5.16 (i). But the path component of $B\alpha : BK \longrightarrow BG$ in the space $\mathrm{map}(BK, BG)$ also is a classifying spaces of $C\alpha$, see for example [**6**, III Prop. 2.10] So the (weak) map in question induces a bijection on path components.

Lashof, May and Segal show in [**93**] that for *abelian* compact Lie groups G and any compact Lie group K the mapping space $\mathrm{map}(EK, BG)$ is a classifying space for principal G-bundles over K-spaces. A corollary is that for all K-spaces X the isomorphism classes of principal (K, G)-bundles over X biject with isomorphism classes of principal G-bundles over $EK \times_K X$. In our present language, their result has the following reformulation:

Theorem 3.33. *The global classifying space $B_{\mathrm{gl}}G$ of every abelian compact Lie group G is cofree.*

Proof. We let V be any faithful G-representation. Then $\mathbf{L}_{G,V}(\mathcal{U}_K) = \mathbf{L}(V, \mathcal{U}_K)/G$ is a classifying space for principal (K, G)-bundles, by Proposition 2.10. Since G is abelian, the K-space $\mathbf{L}_{G,V}(\mathcal{U}_K)$ is cofree by the main result [**93**, Thm. 2] of Lashof, May and Segal. So criterion (ii) of Proposition 3.30 is satisfies; since the orthogonal space $\mathbf{L}_{G,V}$ is closed, we have thus shows that it is cofree. □

Remark 3.34. The global classifying space $B_{\mathrm{gl}}G$ is *not* cofree in general, i.e., when G is neither finite nor abelian. Indeed, there are continuous maps $BK \longrightarrow BG$, for compact Lie groups K and G, that are not homotopic to $B\alpha$ for any continuous homomorphism $\alpha : K \longrightarrow G$. Then the adjunction unit $\eta : B_{\mathrm{gl}}G \longrightarrow R(BG)$ is not surjective on the path components of the K-fixed points.

The first examples of such 'exotic' maps between classifying spaces of compact Lie groups were constructed by Sullivan and appeared in his widely circulated and highly influential MIT lecture notes [**155**];

an edited version of Sullivan's notes was eventually published in [**156**]. Indeed, Corollary 5.11 of [**156**] constructs 'unstable Adams operations' $\psi^p : BU(n) \longrightarrow BU(n)$ and $\psi^p : BSU(n) \longrightarrow BSU(n)$ for a prime p and all $n < p$; for $n > 1$ these maps are not induced by any continuous homomorphism.

We give a specific example with $G = U(2)$; as we shall explain, there is a continuous map $f : B\Sigma_3 \longrightarrow BU(2)$ that is not homotopic to $B\alpha$ for any continuous homomorphism $\alpha : \Sigma_3 \longrightarrow U(2)$. To construct f we use the arithmetic square:

$$\begin{array}{ccc} BU(2) & \longrightarrow & \prod_{p \text{ prime}} (BU(2))^\wedge_p \\ \downarrow & & \downarrow \\ (BU(2))_{\mathbb{Q}} & \longrightarrow & \left(\prod_{p \text{ prime}} (BU(2))^\wedge_p\right)_{\mathbb{Q}} \end{array}$$

Here $(-)^\wedge_p$ is p-completion, the horizontal maps are induced by the p-completion maps $BU(2) \longrightarrow (BU(2))^\wedge_p$, and $(-)_{\mathbb{Q}}$ and the two vertical maps are rationalization. Since $BU(2)$ is simply connected, the arithmetic square is homotopy cartesian, see for example [**44**, Thm. 4.1]. For every finite group K the space BK is rationally equivalent to a one-point space, and so is the p-completion $(BK)^\wedge_p$ for all primes p that do not divide the order of G. So postcomposition with the completion map is a bijection

$$[BK, BU(2)] \;\cong\; \prod_{p \text{ divides } |G|} [BK, (BU(2))^\wedge_p] \;. \tag{3.35}$$

We let

$$\beta \;:\; \Sigma_3 \;\longrightarrow\; U(2) \qquad \text{and} \qquad \gamma \;:\; \Sigma_3 \;\longrightarrow\; U(2)$$

be the group homomorphisms that classify the 2-dimensional complex sign representation respectively the complex reduced natural representations of Σ_3. In terms of matrices, these are given by

$$\beta(g) \;=\; \begin{pmatrix} \mathrm{sgn}(g) & 0 \\ 0 & \mathrm{sgn}(g) \end{pmatrix}$$

respectively

$$\gamma((12)) \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;, \quad \gamma((123)) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \;.$$

For $K = \Sigma_3$ the bijection (3.35) shows that there is a continuous map $f : B\Sigma_3 \longrightarrow BU(2)$, unique up to homotopy, such that

- after 2-completion, f becomes homotopic to $B\beta : B\Sigma_3 \longrightarrow BU(2)$, and
- after 3-completion, f becomes homotopic to $B\gamma : B\Sigma_3 \longrightarrow BU(2)$.

By (3.35) for $K = \Sigma_2$, the restriction of f to $B\Sigma_2$ is homotopic to $B(\beta|_{\Sigma_2}) : B\Sigma_3 \longrightarrow BU(2)$; similarly, by (3.35) for $K = A_3$, the restriction of f to BA_3 is homotopic to $B(\gamma|_{A_3}) : BA_3 \longrightarrow BU(2)$. However, because the image of β is central in $U(2)$, there is no group homomorphism $\alpha : \Sigma_3 \longrightarrow U(2)$ whose restriction to Σ_2 is conjugate to $\beta|_{\Sigma_2}$ and whose restriction to A_3 is conjugate to $\gamma|_{A_3}$. So f is not homotopic to $B\alpha$ for any homomorphism α.

We conclude this section with a brief discussion of the box product of orthogonal spaces, with emphasis on global homotopical features. We prove the invariance of the box product under global equivalences (Theorem 3.38) and check the compatibility of the strong level and the global model structure with the box product of orthogonal spaces.

We define a *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of orthogonal spaces (X, Y) to another orthogonal space Z as a collection of continuous maps

$$b_{V,W} \;:\; X(V) \times Y(W) \;\longrightarrow\; Z(V \oplus W),$$

for all inner product spaces V and W, such that for all linear isometric embeddings $\varphi : V \longrightarrow V'$ and $\psi : W \longrightarrow W'$ the following square commutes:

$$\begin{tikzcd} X(V) \times Y(W) \arrow[r, "b_{V,W}"] \arrow[d, "X(\varphi)\times Y(\psi)"'] & Z(V \oplus W) \arrow[d, "Z(\varphi\oplus\psi)"] \\ X(V') \times Y(W') \arrow[r, "b_{V',W'}"'] & Z(V' \oplus W') \end{tikzcd}$$

We define a box product of X and Y as a universal example of an orthogonal space with a bimorphism from X and Y. More precisely, a box product for X and Y is a pair $(X \boxtimes Y, i)$ consisting of an orthogonal space $X \boxtimes Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \boxtimes Y$, i.e., a bimorphism such that for every orthogonal space Z the map

$$spc(X \boxtimes Y, Z) \;\longrightarrow\; \mathrm{Bimor}((X, Y), Z)\ , \quad f \longmapsto fi = \{f(V \oplus W) \circ i_{V,W}\}_{V,W} \tag{3.36}$$

is bijective. Very often only the object $X \boxtimes Y$ will be referred to as the box product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (3.36) as the *universal property* of the box product of orthogonal spaces.

The existence of a universal bimorphism out of any pair of orthogonal spaces X and Y, and thus of a box product $X \boxtimes Y$, is a special case of the existence of Day type convolution products on certain functor categories; the construction is an enriched Kan extension of the 'pointwise' cartesian product of X and Y along the direct sum functor $\oplus : \mathbf{L} \times \mathbf{L} \longrightarrow \mathbf{L}$ (see Proposition A.3.5), or more explicitly an enriched coend (see Remark A.3.6).

Also by the general theory of convolution products, the box product $X \boxtimes Y$ is a functor in both variables (Construction A.3.8) and it supports a preferred symmetric monoidal structure (see Theorem A.3.12); so there are specific natural associativity respectively symmetry isomorphisms

$$(X \boxtimes Y) \boxtimes Z \;\longrightarrow\; X \boxtimes (Y \boxtimes Z) \qquad \text{respectively} \qquad X \boxtimes Y \;\longrightarrow\; Y \boxtimes X$$

and a strict unit, i.e., such that $\mathbf{1} \boxtimes X = X = X \boxtimes \mathbf{1}$. The upshot is that the associativity and symmetry isomorphisms make the box product of orthogonal spaces into a symmetric monoidal product with the terminal orthogonal space $\mathbf{1}$ as unit object. The box product of orthogonal spaces is *closed* symmetric monoidal in the sense that the box product is adjoint to an internal Hom orthogonal space. The construction is a special case of Construction A.3.14 for enriched functor categories.

The next result proves a key feature, namely that up to global equivalence, the box product of orthogonal spaces coincides with the categorical product. Given two orthogonal spaces X and Y, the maps

$$X(V) \times Y(W) \xrightarrow{X(i_1)\times Y(i_2)} X(V \oplus W) \times Y(V \oplus W) \;=\; (X \times Y)(V \oplus W)$$

form a bimorphism $(X, Y) \longrightarrow X \times Y$ as V and W vary through all inner product spaces; here $i_1 : V \longrightarrow V \oplus W$ and $i_2 : W \longrightarrow V \oplus W$ are the two direct summand embeddings. This bimorphism is represented by a morphism

$$\rho_{X,Y} \;:\; X \boxtimes Y \;\longrightarrow\; X \times Y \tag{3.37}$$

of orthogonal spaces that is natural in both variables.

THEOREM 3.38. *Let X and Y be orthogonal spaces.*

(i) *The morphism $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ is a global equivalence.*
(ii) *The functor $X \boxtimes -$ preserves global equivalences.*

Proof. Part (i) is the special case of Theorem 1.13 for the external cartesian product of X and Y, i.e., for the bi-orthogonal space

$$\mathbf{L}\times\mathbf{L} \xrightarrow{X\times Y} \mathbf{T}\times\mathbf{T} \xrightarrow{\times} \mathbf{T}\ .$$

The cartesian product $X\times -$ preserves global equivalences by Proposition 1.9 (vi). Together with part (i) this implies part (ii). □

Example 3.39 (Box product of free orthogonal spaces). We show that the box product of two free orthogonal spaces is another free orthogonal space. This can be deduced from the general fact that a convolution product of two representable functors is again representable (see Propositions A.3.13); however, the argument is simple enough that we make it explicit for orthogonal spaces.

We consider two compact Lie groups G and K, representations V and W of G respectively K, a G-space A and a K-space B. Then $V\oplus W$ is a $(G\times K)$-representation via

$$(g,k)\cdot(v,w) \ = \ (gv,\, kw)\ ,$$

and $A\times B$ is a $(G\times K)$-space in much the same way. The map

$$A\times B \xrightarrow{(\mathrm{Id}_V\cdot G,-)\times(\mathrm{Id}_W\cdot K,-)} (\mathbf{L}_{G,V}A)(V)\times(\mathbf{L}_{K,W}B)(W) \xrightarrow{i_{V,W}} ((\mathbf{L}_{G,V}A)\boxtimes(\mathbf{L}_{K,W}B))(V\oplus W)$$

is $(G\times K)$-equivariant, so it extends freely to a morphism

$$\mathbf{L}_{G\times K,V\oplus W}(A\times B) \ \longrightarrow\ (\mathbf{L}_{G,V}A)\boxtimes(\mathbf{L}_{K,W}B)\ . \tag{3.40}$$

The maps

$$\begin{array}{ccc} (\mathbf{L}(V,U)\times_G A)\times(\mathbf{L}(W,U')\times_K B) & \longrightarrow & \mathbf{L}(V\oplus W,U\oplus U')\times_{G\times K}(A\times B) \\ ([\varphi,a],\,[\psi,b]) & \longmapsto & [\varphi\oplus\psi,(a,b)] \end{array}$$

form a bimorphism from $(\mathbf{L}_{G,V}A,\mathbf{L}_{K,W}B)$ to $\mathbf{L}_{G\times K,V\oplus W}(A\times B)$ as the inner product spaces U and U' vary. The universal property of the box product translates this into a morphism

$$(\mathbf{L}_{G,V}A)\boxtimes(\mathbf{L}_{K,W}B) \ \longrightarrow\ \mathbf{L}_{G\times K,V\oplus W}(A\times B)\ .$$

These two morphisms are mutually inverse isomorphisms, i.e., the box product $(\mathbf{L}_{G,V}A)\boxtimes(\mathbf{L}_{K,W}B)$ is isomorphic to $\mathbf{L}_{G\times K,V\times W}(A\times B)$. A special case of this shows that the box product of two global classifying spaces is another global classifying space. Indeed, if G acts faithfully on V and K acts faithfully on W, then the $(G\times K)$-action on $V\oplus W$ is also faithful, and hence

$$(B_{\mathrm{gl}}G)\boxtimes(B_{\mathrm{gl}}K) \ = \ \mathbf{L}_{G,V}\boxtimes\mathbf{L}_{K,W} \ \cong\ \mathbf{L}_{G\times K,V\oplus W} \ = \ B_{\mathrm{gl}}(G\times K)\ . \tag{3.41}$$

If we compose the isomorphism (3.40) with the global equivalence $\rho_{\mathbf{L}_{G,V}A,\mathbf{L}_{K,W}B}$, we obtain a global equivalence of orthogonal spaces

$$\mathbf{L}_{G\times K,V\oplus W}(A\times B) \ \longrightarrow\ (\mathbf{L}_{G,V}A)\times(\mathbf{L}_{K,W}B)\ .$$

Again, if G acts faithfully on V and K acts faithfully on W, then the special case $A=B=*$ becomes a global equivalence

$$B_{\mathrm{gl}}(G\times K) \ \longrightarrow\ (B_{\mathrm{gl}}G)\times(B_{\mathrm{gl}}K)\ .$$

Given two morphisms $f:A\longrightarrow B$ and $g:X\longrightarrow Y$ of orthogonal spaces we denote by $f\Box g$ the *pushout product* morphism defined as

$$f\Box g=(f\boxtimes Y)\cup(A\boxtimes g)\ :\ A\boxtimes Y\cup_{A\boxtimes X}B\boxtimes X \ \longrightarrow\ B\boxtimes Y\ .$$

We recall that a model structure on a symmetric monoidal category satisfies the *pushout product property* if the following two conditions hold:

- for every pair of cofibrations $f:A\longrightarrow B$ and $g:X\longrightarrow Y$ the pushout product morphism $f\Box g$ is also a cofibration;
- if in addition f or g is a weak equivalence, then so is the pushout product morphism $f\Box g$.

PROPOSITION 3.42. (i) *The strong level model structure of orthogonal spaces satisfies the pushout product property with respect to the box product.*

(ii) *The pushout product of a flat cofibration that is also a global equivalence with any morphism of orthogonal spaces is a global equivalence.*

(iii) *The global model structure of orthogonal spaces satisfies the pushout product property with respect to the box product.*

PROOF. (i) We start by showing that the pushout product of two flat cofibrations is a flat cofibration. It suffices to show the claim for a set of generating flat cofibrations, for example the morphisms

$$i_k \times \mathbf{L}_{G,V} \ : \ \partial D^k \times \mathbf{L}_{G,V} \ \longrightarrow \ D^k \times \mathbf{L}_{G,V}$$

for G a compact Lie group, V a faithful G-representation and $k \geq 0$. The pushout product $i_k \Box i_m$ of two sphere inclusions is isomorphic to the inclusion $i_{k+m} : \partial D^{k+m} \longrightarrow D^{k+m}$, so the pushout product of two such generators is isomorphic to the map

$$i_{k+m} \times \mathbf{L}_{G\times K, V\oplus W} \ ,$$

by Example 3.39. This pushout product morphism is another generating flat cofibration.

Now we show that the pushout product of a flat cofibration with a flat cofibration that is also a strong level equivalence is again a strong level equivalence. Again it suffices to check the pushout product with a generating acyclic cofibration J^{str}, i.e., a morphism of the form

$$j_m \times \mathbf{L}_{K,W} \ : \ D^m \times \{0\} \times \mathbf{L}_{K,W} \ \longrightarrow \ D^m \times [0,1] \times \mathbf{L}_{K,W}$$

for K a compact Lie group, W a faithful K-representation and $m \geq 0$. The pushout product $i_k \Box j_m$ is isomorphic to the inclusion j_{k+m}, so the pushout product of two such generators is isomorphic to the map

$$j_{k+m} \times \mathbf{L}_{G\times K, V\oplus W} \ ,$$

by Example 3.39. This pushout product morphism is a flat cofibration and strong level equivalence.

(ii) We let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ be morphisms of orthogonal spaces such that f is a flat cofibration and a global equivalence. Then $f \boxtimes X$ and $f \boxtimes Y$ are global equivalences by Theorem 3.38. Moreover, i is an h-cofibration by Corollary A.1.18 (iii), hence so is $i \boxtimes K : A \boxtimes K \longrightarrow B \boxtimes K$. Thus its cobase change, the canonical morphism

$$A \boxtimes L \ \longrightarrow \ A \boxtimes L \cup_{A\boxtimes K} B \boxtimes K$$

is a global equivalence by Proposition 3.15 (i). Since $i \boxtimes L : A \boxtimes L \longrightarrow B \boxtimes L$ is also a global equivalence, so is the pushout product map $f \Box g$, by 2-out-of-6, compare Proposition 1.9 (iii).

(iii) The pushout product of two flat cofibrations is a flat cofibration by part (i). The pushout product of two flat cofibrations one of which is also a global equivalence is another global equivalence by part (ii). □

The unit object for the box product is the constant one-point orthogonal space $\mathbf{1}$, which is flat. So with respect to the box product, the global model structure is a symmetric monoidal model category in the sense of [**79**, Def. 4.2.6]. A corollary is that the unstable global homotopy category, i.e., the localization of the category of orthogonal spaces at the class of global equivalences, inherits a closed symmetric monoidal structure, compare [**79**, Thm. 4.3.3]. This 'derived box product' is nothing new, though: since the morphism $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ is a global equivalence for all orthogonal space X and Y, the derived box product is just a categorical product in $\mathrm{Ho}(spc)$.

4. Global families

In this section we explain a variant of unstable global homotopy theory based on a *global family*, i.e., a class $\mathcal{F}$ of compact Lie groups with certain closure properties. We introduce $\mathcal{F}$-equivalences, a relative version of global equivalences, and establish $\mathcal{F}$-relative versions of the strong level and the global model structure in Proposition 4.3 and Theorem 4.8.

Definition 4.1. A *global family* is a non-empty class of compact Lie groups that is closed under isomorphism, closed subgroups and quotient groups.

Some relevant examples of global families are: all compact Lie groups; all finite groups; all abelian compact Lie groups; all finite abelian groups; all finite cyclic groups; all finite p-groups. Another example is the global family $\langle G\rangle$ generated by a compact Lie group G, i.e., the class of all compact Lie groups isomorphic to a quotient of a closed subgroup of G. A degenerate case is the global family $\langle e\rangle$ of all trivial groups. In this case our theory specializes to the non-equivariant homotopy theory of orthogonal spaces.

For a global family $\mathcal{F}$ and a compact Lie group G we write $\mathcal{F}\cap G$ for the family of those closed subgroups of G that belong to $\mathcal{F}$. We also write $\mathcal{F}(m)$ for $\mathcal{F}\cap O(m)$, the family of closed subgroups of $O(m)$ that belong to $\mathcal{F}$. We recall that an equivariant continuous map of $O(m)$-spaces is an $\mathcal{F}(m)$-*cofibration* if it has the right lifting property with respect to all morphisms $q : A \longrightarrow B$ of $O(m)$-spaces such that the map $q^H : A^H \longrightarrow B^H$ is a weak equivalence and Serre fibration for all $H \in \mathcal{F}(m)$.

The following definitions of $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations are direct relativizations of the corresponding concepts in the strong level model structure of orthogonal spaces.

Definition 4.2. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is

- an $\mathcal{F}$-*level equivalence* if for every compact Lie group G in $\mathcal{F}$ and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence;
- an $\mathcal{F}$-*level fibration* if for every compact Lie group G in $\mathcal{F}$ and every G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a Serre fibration;
- an $\mathcal{F}$-cofibration if the latching morphism $\nu_m f : X(\mathbb{R}^m) \cup_{L_m X} L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-cofibration for all $m \geq 0$.

Every inner product space V is isometrically isomorphic to $\mathbb{R}^m$ with the standard scalar product, where m is the dimension of V. So a morphism $f : X \longrightarrow Y$ of orthogonal spaces is an $\mathcal{F}$-level equivalence (respectively $\mathcal{F}$-level fibration) precisely if for every $m \geq 0$ the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-equivalence (respectively $\mathcal{F}(m)$-projective fibration). The formal argument is analogous to Lemmas 3.8 and 3.10 which treat the case $\mathcal{F} = \mathcal{A}ll$. Clearly, the classes of $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations are closed under composition, retracts and coproducts.

For the minimal global family $\langle e\rangle$ of trivial groups, the notion of $\langle e\rangle$-level equivalence specializes to the non-equivariant level equivalences and the $\langle e\rangle$-level fibrations are the non-equivariant level fibrations. Thus we have the following implications for the various kinds of cofibrations:

$$\langle e\rangle\text{-cofibration} \implies \mathcal{F}\text{-cofibration} \implies \text{flat cofibration} \implies \text{h-cofibration}$$

When $\mathcal{F}$ is not the minimal or the maximal global family, then the first two containments are strict.

Now we discuss the $\mathcal{F}$-level model structures on orthogonal spaces. When $\mathcal{F} = \mathcal{A}ll$ is the global family of all compact Lie groups, then $\mathcal{A}ll(m)$ is the family of all closed subgroups of $O(m)$. For this maximal global family, an $\mathcal{A}ll$-level equivalence is just a strong level equivalence in the sense of Definition 1.8. Moreover, the $\mathcal{A}ll$-level fibrations coincide with the strong level fibrations in the sense of Definition 3.9. The $\mathcal{A}ll$-cofibrations coincide with the flat cofibrations. So for the global family of all compact Lie groups the $\mathcal{A}ll$-level model structure on orthogonal spaces specializes to the strong level model structure of Proposition 3.11.

Proposition 4.3. *Let $\mathcal{F}$ be a global family. The $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations form a model structure, the $\mathcal{F}$-*level model structure*, on the category of orthogonal spaces. The $\mathcal{F}$-level model structure is proper, topological and cofibrantly generated.*

Proof. We specialize Proposition A.3.27 by letting $\mathcal{C}(m)$ be the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces, compare Proposition A.2.10. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition A.3.27 precisely become the $\mathcal{F}$-level equivalences, $\mathcal{F}$-fibrations and $\mathcal{F}$-cofibrations. Every acyclic

cofibration in the $\mathcal{F}(m)$-projective model structure of $O(m)$-spaces is also an acyclic cofibration in the $\mathcal{A}ll$-projective model structure of $O(m)$-spaces. So the consistency condition (see Definition A.3.26) in the present situation is a special case of the consistency condition for the strong level model structure that we established in the proof of Proposition 3.11.

We describe explicit sets of generating cofibrations and generating acyclic cofibrations for the $\mathcal{F}$-level model structure. We let $I_{\mathcal{F}}$ be the set of all morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (2.11) of Section A.2. Then the set $I_{\mathcal{F}}$ detects the acyclic fibrations in the $\mathcal{F}$-level model structure by Proposition A.3.27 (iii). Similarly, we let $J_{\mathcal{F}}$ be the set of all morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (2.12) of Section A.2. Again by Proposition A.3.27 (iii), $J_{\mathcal{F}}$ detects the fibrations in the $\mathcal{F}$-level model structure.

The $\mathcal{F}$-level model structure is topological by Proposition A.2.8, where we take $\mathcal{G}$ as the set of orthogonal spaces $L_{H,\mathbb{R}^m}$ for all $m \geq 0$ and all $H \in \mathcal{F}(m)$.

Limits in the category of orthogonal spaces are constructed levelwise (i.e., evaluation at $\mathbb{R}^m$ preserves limits). Since weak equivalences and fibrations are also defined levelwise, right properness is inherited levelwise. The $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces is right proper for all $m \geq 0$, so right properness of the $\mathcal{F}$-level model structure follows.

Since the $\mathcal{F}$-level model structure is topological and every orthogonal space is fibrant, the $\mathcal{F}$-cofibrations are h-cofibrations (by Corollary A.1.18 (iii)). Left properness is then a special case of Proposition 3.7, or rather its analog for $\mathcal{F}$-level equivalences, which is proved in the same way. □

Now we proceed towards the construction of the *$\mathcal{F}$-global model structure*, see Theorem 4.8 below. The weak equivalences in this model structures are the $\mathcal{F}$-equivalences of the following definition, the direct generalization of global equivalences in the presence of a global family.

Definition 4.4. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is an *$\mathcal{F}$-equivalence* if the following condition holds: for every compact Lie group G in $\mathcal{F}$, every G-representation V, every $k \geq 0$ and all maps $\alpha : \partial D^k \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $f(V)^G \circ \alpha = \beta|_{\partial D^k}$ there is a G-representation W, a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a continuous map $\lambda : D^k \longrightarrow X(W)^G$ such that $\lambda|_{\partial D^k} = X(\varphi)^G \circ \alpha$ and such that $f(W)^G \circ \lambda$ is homotopic, relative to ∂D^k, to $Y(\varphi)^G \circ \beta$.

When $\mathcal{F} = \mathcal{A}ll$ is the maximal global family of all compact Lie groups, then $\mathcal{A}ll$-equivalences are precisely the global equivalences. The following diagram collects various notions of equivalences and their implications:

$$\begin{tikzcd}
\text{homotopy equivalence} \arrow[r, Rightarrow] & \text{strong level equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{global equivalence} \arrow[d, Rightarrow] \\
& \mathcal{F}\text{-level equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathcal{F}\text{-equivalence} \arrow[d, Rightarrow] \\
& \text{level equivalence} \arrow[r, Rightarrow] & \text{weak equivalence}
\end{tikzcd}$$

The following proposition generalizes Proposition 1.7, and it proved in much the same way.

Proposition 4.5. *Let $\mathcal{F}$ be a global family. For every morphism of orthogonal spaces $f : X \longrightarrow Y$, the following four conditions are equivalent.*

(i) *The morphism f is an $\mathcal{F}$-equivalence.*
(ii) *Let G be a compact Lie group G, V a G-representation and (B, A) a finite G-CW-pair all of whose isotropy groups belong to $\mathcal{F}$: Then for all continuous G-maps $\alpha : A \longrightarrow X(V)$ and $\beta : B \longrightarrow Y(V)$ such that $\beta|_A = f(V) \circ \alpha$, there is a G-representation W, a G-equivariant linear isometric embedding*

$\varphi : V \longrightarrow W$ and a continuous G-map $\lambda : B \longrightarrow X(W)$ such that $\lambda|_A = X(\varphi) \circ \alpha$ and such that $f(W) \circ \lambda$ is G-homotopic, relative to A, to $Y(\varphi) \circ \beta$.

(iii) *For every compact Lie group G in the family $\mathcal{F}$ and every exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations the induced map*

$$\mathrm{tel}_i\, f(V_i) \ : \ \mathrm{tel}_i\, X(V_i) \ \longrightarrow \ \mathrm{tel}_i\, Y(V_i)$$

is a G-weak equivalence.

(iv) *For every compact Lie group G in the family $\mathcal{F}$ there is an exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations such that the induced map*

$$\mathrm{tel}_i\, f(V_i) \ : \ \mathrm{tel}_i\, X(V_i) \ \longrightarrow \ \mathrm{tel}_i\, Y(V_i)$$

is a G-weak equivalence.

Definition 4.6. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is an *$\mathcal{F}$-global fibration* if it is an $\mathcal{F}$-level fibration and for every compact Lie group G in the family $\mathcal{F}$, every faithful G-representation V and every equivariant linear isometric embedding $\varphi : V \longrightarrow W$ of G-representations, the map

$$(f(V)^G, X(\varphi)^G) \ : \ X(V)^G \ \longrightarrow \ Y(V)^G \times_{Y(W)^G} X(W)^G$$

is a weak equivalence.

The next proposition contains various properties of $\mathcal{F}$-equivalences that generalize Proposition 1.9 and certain parts of Proposition 3.15.

Proposition 4.7. *Let $\mathcal{F}$ be a global family.*

(i) *Every $\mathcal{F}$-level equivalence is an $\mathcal{F}$-equivalence.*
(ii) *The composite of two $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence.*
(iii) *If f, g and h are composable morphisms of orthogonal spaces such that gf and hg are $\mathcal{F}$-equivalences, then f, g, h and hgf are also $\mathcal{F}$-equivalences.*
(iv) *Every retract of an $\mathcal{F}$-equivalence is an $\mathcal{F}$-equivalence.*
(v) *A coproduct of any set of $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence.*
(vi) *A finite product of $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence.*
(vii) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of orthogonal spaces that are objectwise closed embeddings, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be $\mathcal{F}$-equivalences of orthogonal spaces that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is an $\mathcal{F}$-equivalence.*
(viii) *Let $f_n : Y_n \longrightarrow Y_{n+1}$ be an $\mathcal{F}$-equivalence and a closed embedding of orthogonal spaces, for $n \geq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to the colimit of the sequence $\{f_n\}_{n\geq 0}$ is an $\mathcal{F}$-equivalence.*
(ix) *Let*

$$\begin{tikzcd} C \arrow[d, "\gamma"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ C' & A' \arrow[l, "g'"] \arrow[r, "f'"'] & B' \end{tikzcd}$$

be a commutative diagram of orthogonal spaces such that g and g' are h-cofibrations. If the morphisms α, β and γ are $\mathcal{F}$-equivalences, then so is the induced morphism of pushouts

$$\gamma \cup \beta \ : \ C \cup_A B \ \longrightarrow \ C' \cup_{A'} B' \ .$$

(x) *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spaces such that f is an $\mathcal{F}$-equivalence. If in addition f or g is an h-cofibration, then the morphism k is an $\mathcal{F}$-equivalence.

(xi) *Let*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"'] & Y \end{tikzcd}$$

be a pullback square of orthogonal spaces in which f is an $\mathcal{F}$-equivalence. If in addition one of the morphisms f or h is an $\mathcal{F}$-level fibration, then the morphism g is also an $\mathcal{F}$-equivalence.

(xii) *Every $\mathcal{F}$-equivalence that is also an $\mathcal{F}$-global fibration is an $\mathcal{F}$-level equivalence.*

(xiii) *The box product of two $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence.*

Proof. The proofs of (i) through (xi) are almost verbatim the same as the corresponding parts of Proposition 1.9, and we omit them. Part (xii) is proved in the same way as Proposition 3.15 (i).

(xii) The product of orthogonal spaces preserves $\mathcal{F}$-equivalences in both variables by part (vi). The morphism $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ is a global equivalence, hence an $\mathcal{F}$-equivalence, for all orthogonal spaces X and Y, by Theorem 3.38 (i); this implies the claim. □

Now we establish the $\mathcal{F}$-global model structures on the category of orthogonal spaces. We spell out sets of generating cofibrations and generating acyclic cofibrations for the $\mathcal{F}$-global model structures. In Proposition 4.3 we introduced $I_{\mathcal{F}}$ as the set of all morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (2.11) of Section A.2. The set $I_{\mathcal{F}}$ detects the acyclic fibrations in the $\mathcal{F}$-level model structure, which coincide with the acyclic fibrations in the $\mathcal{F}$-global model structure. In particular, the set $I_{\mathcal{A}ll}$, which was denoted I^{str} in Proposition 3.11, detects the acyclic fibrations in the strong level model structure, which are also the acyclic fibrations in the flat $\mathcal{F}$-level model structure.

Also in Proposition 4.3 we defined $J_{\mathcal{F}}$ as the set of all morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (2.12) of Section A.2. The set $J_{\mathcal{F}}$ detects the fibrations in the $\mathcal{F}$-level model structure.

We add another set of morphisms $K_{\mathcal{F}}$ that detects when the squares (3.14) are homotopy cartesian for $G \in \mathcal{F}$. We set

$$K_{\mathcal{F}} \;=\; \bigcup_{G,V,W\,:\,G\in\mathcal{F}} \mathcal{Z}(\rho_{G,V,W})\ ,$$

the set of all pushout products of sphere inclusions $i_k : \partial D^k \longrightarrow D^k$ with the mapping cylinder inclusions of the morphisms $\rho_{G,V,W}$; here the union is over a set of representatives of the isomorphism classes of triples (G, V, W) consisting of a compact Lie group G in $\mathcal{F}$, a faithful G-representation V and an arbitrary G-representation W. By Proposition 3.17, the right lifting property with respect to the union $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ characterizes the $\mathcal{F}$-global fibrations.

The proof of the following theorem proceeds by mimicking the proof in the special case $\mathcal{F} = \mathcal{A}ll$, and all arguments in the proof of Theorem 3.22 go through almost verbatim. Whenever the small object argument is used, it now has to be taken with respect to the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ (as opposed to the set $J^{\mathrm{str}} \cup K$).

Theorem 4.8 ($\mathcal{F}$-global model structure). *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-equivalences, $\mathcal{F}$-global fibrations and $\mathcal{F}$-cofibrations form a model structure, the $\mathcal{F}$-*global model structure*, on the category of orthogonal spaces.*
(ii) *The fibrant objects in the $\mathcal{F}$-global model structure are the $\mathcal{F}$-*static orthogonal spaces*, i.e., those orthogonal spaces X such that for every compact Lie group G in $\mathcal{F}$, every faithful G-representation V and every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the map of G-fixed point spaces*

$$X(\varphi)^G \ : \ X(V)^G \ \longrightarrow \ X(W)^G$$

is a weak equivalence.
(iii) *A morphism of orthogonal spaces is:*
- *an acyclic fibration in the $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $I_{\mathcal{F}}$;*
- *a fibration in the $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$.*

(iv) *The $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

Example 4.9. In the case $\mathcal{F} = \langle e \rangle$ of the minimal global family of trivial groups, the $\langle e \rangle$-global homotopy theory of orthogonal spaces just another model for the (non-equivariant) homotopy theory of spaces. Indeed, the evaluation functor $\mathrm{ev}_0 : spc \longrightarrow \mathbf{T}$ is a right Quillen equivalence with respect to the $\langle e \rangle$-global model structure. So the total derived functor

$$\mathrm{Ho}^{\langle e \rangle}(\mathrm{ev}_0) \ : \ \mathrm{Ho}(spc) \ \longrightarrow \mathrm{Ho}(\mathbf{T})$$

is an equivalence of homotopy categories.

In fact, for the global family $\mathcal{F} = \langle e \rangle$, most of what we do here has already been studied before: The $\langle e \rangle$-global model structure and the fact that it is Quillen equivalent to the model category of spaces were established by Lind [**98**, Thm. 1.1]; in [**98**], orthogonal spaces are called '$\mathcal{I}$-spaces' and $\langle e \rangle$-global equivalences are called 'weak homotopy equivalences' and are defined as those morphisms that induce weak equivalences on homotopy colimits.

Corollary 4.10. *Let $f : A \longrightarrow B$ be a morphism of orthogonal spaces and $\mathcal{F}$ a global family. Then the following conditions are equivalent.*
(i) *The morphism f is an $\mathcal{F}$-equivalence.*
(ii) *The morphism can be written as $f = w_2 \circ w_1$ for an $\mathcal{F}$-level equivalence w_2 and a global equivalence w_1.*
(iii) *For some (hence any) $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ in the $\mathcal{F}$-level model structure and every $\mathcal{F}$-static orthogonal space Y the induced map*

$$[f^c, Y] \ : \ [B^c, Y] \ \longrightarrow \ [A^c, Y]$$

on homotopy classes of morphisms is bijective.

Proof. (i)$\Longleftrightarrow$(ii) The $\mathcal{F}$-equivalences contain the global equivalences by definition and the $\mathcal{F}$-level equivalences by Proposition 4.7 (i), and are closed under composition by Proposition 4.7 (ii), so all composites $w_2 \circ w_1$ as in the claim are $\mathcal{F}$-equivalences. On the other hand, every $\mathcal{F}$-equivalence f can be factored in the global model structure of Theorem 3.22 as $f = qj$ where j is a global equivalence and q is a global fibration. Since f and j are $\mathcal{F}$-equivalences, so is q by Proposition 4.7 (iii); so q is an $\mathcal{F}$-equivalence and a global fibration, hence an $\mathcal{F}$-level equivalence by Proposition 4.7 (xii).

(i)$\Longleftrightarrow$(iii) The morphism f is an $\mathcal{F}$-equivalence if and only if the $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ is an $\mathcal{F}$-equivalence. Since A^c and B^c are $\mathcal{F}$-cofibrant, they are cofibrant in the $\mathcal{F}$-global model structure. So by general model category theory, f^c is an $\mathcal{F}$-equivalence if and only if the induced map $[f^c, X]$ is bijective for every fibrant object in the $\mathcal{F}$-global model structure. By Theorem 4.8 (ii) these fibrant objects are precisely the $\mathcal{F}$-static orthogonal spaces. $\square$

Remark 4.11 (Mixed global model structures). Cole's 'mixing theorem' for model structures [**37**, Thm. 2.1] allows to construct many more $\mathcal{F}$-model structures on the category of orthogonal spaces. We will concentrate on the 'mixed' $\mathcal{F}$-global model structures, but the same kind of mixing can also be performed with the $\mathcal{F}$-level model structures.

We consider two global families such that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is an $\mathcal{F}$-equivalence and every fibration in the $\mathcal{E}$-global model structure is a fibration in the $\mathcal{F}$-global model structure. By Cole's theorem [**37**, Thm. 2.1] the $\mathcal{F}$-equivalences and the fibrations of the $\mathcal{E}$-global model structure are part of a model structure, the *$\mathcal{E}$-mixed $\mathcal{F}$-global model structure* on the category of orthogonal spaces. By [**37**, Prop. 3.2] the cofibrations in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure are precisely the retracts of all composites $h \circ g$ in which g is an $\mathcal{F}$-cofibration and h is simultaneously an $\mathcal{E}$-equivalence and an $\mathcal{E}$-cofibration. In particular, an orthogonal space is cofibrant in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure if it is $\mathcal{E}$-cofibrant and $\mathcal{E}$-equivalent to an $\mathcal{F}$-cofibrant orthogonal space [**37**, Cor. 3.7]. The $\mathcal{E}$-mixed $\mathcal{F}$-global model structure is again proper (Propositions 4.1 and 4.2 of [**37**]).

When $\mathcal{F} = \langle e \rangle$ is the minimal family of trivial groups, this provides infinitely many $\mathcal{E}$-mixed model structures on the category of orthogonal spaces that are all Quillen equivalent to the model category of (non-equivariant) spaces, with respect to weak equivalences.

The next topic is the compatibility of the $\mathcal{F}$-global model structure with the box product of orthogonal spaces. We let $\mathcal{E}$ and $\mathcal{F}$ be two global families. We denote by $\mathcal{E} \times \mathcal{F}$ the smallest global family that contains all groups of the form $G \times K$ for $G \in \mathcal{E}$ and $K \in \mathcal{F}$. So a compact Lie group H belongs to $\mathcal{E} \times \mathcal{F}$ if and only if H is isomorphic to a closed subgroup of a group of the form $(G \times K)/N$ for some groups $G \in \mathcal{E}$ and $K \in \mathcal{F}$, and some closed normal subgroup N of $G \times K$.

Proposition 4.12. *Let $\mathcal{E}$ and $\mathcal{F}$ be two global families.*

(i) *The pushout product of an $\mathcal{E}$-cofibration with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-cofibration.*
(ii) *The pushout product of a flat cofibration that is also an $\mathcal{F}$-equivalence with any morphism of orthogonal spaces is an $\mathcal{F}$-equivalence.*
(iii) *Let $\mathcal{F}$ be a multiplicative global family, i.e., $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. Then the $\mathcal{F}$-global model structure satisfies the pushout product property with respect to the box product of orthogonal spaces.*

Proof. (i) It suffices to show the claim for sets of generating cofibrations. The $\mathcal{E}$-cofibrations are generated by the morphisms

$$i_k \times \mathbf{L}_{G,V} \ : \ \partial D^k \times \mathbf{L}_{G,V} \ \longrightarrow \ D^k \times \mathbf{L}_{G,V}$$

for $G \in \mathcal{E}$, V a G-representation and $k \geq 0$. Similarly, the $\mathcal{F}$-cofibrations are generated by the morphisms

$$i_m \times \mathbf{L}_{K,W} \ : \ \partial D^m \times \mathbf{L}_{K,W} \ \longrightarrow \ D^m \times \mathbf{L}_{K,W} \tag{4.13}$$

for $K \in \mathcal{F}$, W a K-representation and $m \geq 0$. The pushout product of two such generators is isomorphic to the map

$$i_{k+m} \times \mathbf{L}_{G\times K, V\oplus W} \ : \ \partial D^{k+m} \times \mathbf{L}_{G\times K, V\oplus W} \ \longrightarrow \ D^{k+m} \times \mathbf{L}_{G\times K, V\oplus W} \ ,$$

compare Example 3.39. Since $G \times K$ belongs to the family $\mathcal{E} \times \mathcal{F}$, this pushout product morphism is an $(\mathcal{E} \times \mathcal{F})$-cofibration.

(ii) We let $i : A \longrightarrow B$ and $j : K \longrightarrow L$ be morphisms of orthogonal spaces such that i is a flat cofibration and an $\mathcal{F}$-equivalence. Then $i \boxtimes K$ and $i \boxtimes L$ are $\mathcal{F}$-equivalences by Proposition 4.7 (xiii). Moreover, i is an h-cofibration by Corollary A.1.18 (iii), hence so is $i \boxtimes K : A \boxtimes K \longrightarrow B \boxtimes K$. Thus its cobase change, the canonical morphism

$$A \boxtimes L \ \longrightarrow \ A \boxtimes L \cup_{A \boxtimes K} B \boxtimes K$$

is an $\mathcal{F}$-equivalence by Proposition 4.7 (x). Since $i \boxtimes L : A \boxtimes L \longrightarrow B \boxtimes L$ is also an $\mathcal{F}$-equivalence, so is the pushout product map, by 2-out-of-6 (Proposition 4.7 (iii)).

(iii) The part of the pushout product property that refers only to cofibrations is true by part (i) with $\mathcal{E} = \mathcal{F}$ and the hypothesis that $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. Every cofibration in the $\mathcal{F}$-global model structure is in particular a flat cofibration, so the part of the pushout product property that also refers to acyclic cofibrations in the $\mathcal{F}$-global model structure is a special case of (ii). □

We let $\mathcal{F}$ be a multiplicative global family, i.e., $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. The constant one-point orthogonal space $\mathbf{1}$ is the unit object for the box product of orthogonal spaces, and it is 'free', i.e., $\langle e \rangle$-cofibrant. So $\mathbf{1}$ is cofibrant in the $\mathcal{F}$-global model structure. So with respect to the box product, the $\mathcal{F}$-level model structure and the $\mathcal{F}$-global model structure are symmetric monoidal model categories in the sense of [**79**, Def. 4.2.6].

Finally, we will discuss another important relationship between the $\mathcal{F}$-global model structures and the box product, namely the *monoid axiom* [**133**, Def. 3.3]. We only discuss a slightly weaker form of the monoid axiom in the sense that we only cover sequential (as opposed to more general transfinite) compositions.

Proposition 4.14 (Monoid axiom). *We let $\mathcal{F}$ be a global family. For every flat cofibration $i : A \longrightarrow B$ that is also an $\mathcal{F}$-equivalence and every orthogonal space Y the morphism*

$$i \boxtimes Y \ : \ A \boxtimes Y \ \longrightarrow \ B \boxtimes Y$$

is an h-cofibration and an $\mathcal{F}$-equivalence. Moreover, the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential compositions.

Proof. Every flat cofibration is an h-cofibration (Corollary A.1.18 (iii) applied to the flat $\mathcal{A}ll$-level model structure), and h-cofibrations are closed under box product with any orthogonal space (Corollary A.1.18 (ii)), so $i \boxtimes Y$ is an h-cofibration. Since i is an $\mathcal{F}$-equivalence, so is $i \boxtimes Y$ by Proposition 4.7 (xiii).

Proposition 4.7 shows that the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential compositions. □

Definition 4.15. An *orthogonal monoid space* is an orthogonal space R equipped with unit morphism $\eta : \mathbf{1} \longrightarrow R$ and a multiplication morphism $\mu : R \boxtimes R \longrightarrow R$ that is unital and associative in the sense that the square

$$\begin{tikzcd}
(R \boxtimes R) \boxtimes R \arrow[r, "\text{associativity}", "\cong"'] \arrow[d, "\mu \boxtimes R"'] & R \boxtimes (R \boxtimes R) \arrow[r, "R \boxtimes \mu"] & R \boxtimes R \arrow[d, "\mu"] \\
R \boxtimes R \arrow[rr, "\mu"'] & & R
\end{tikzcd}$$

commutes. An orthogonal monoid space R is *commutative* if moreover $\mu \circ \tau_{R,R} = \mu$, where $\tau_{R,R} : R \boxtimes R \longrightarrow R \boxtimes R$ is the symmetry isomorphism of the box product.

A *morphism* of orthogonal monoid spaces is a morphism of orthogonal spaces $f : R \longrightarrow S$ such that $f \circ \mu^R = \mu^S \circ (f \boxtimes f)$ and $f \circ \eta_R = \eta_S$.

We will later refer to commutative orthogonal monoid spaces as *ultra-commutative monoids*. One can expand the data of an orthogonal monoid space into an 'external' form as follows. The unit morphism $\eta : \mathbf{1} \longrightarrow R$ is determined by a unit element $0 \in R(0)$, the image of the map $\eta(0) : \mathbf{1}(0) \longrightarrow R(0)$. The multiplication map corresponds to continuous maps $\mu_{V,W} : R(V) \times R(W) \longrightarrow R(V \oplus W)$ for all inner product spaces V and W that form a bimorphism as (V, W) varies and such that

$$\mu_{V,0}(x, 0) \ = \ x \qquad \text{and} \qquad \mu_{0,W}(0, y) \ = \ y \ .$$

Put another way, the data of an orthogonal monoid space in external form is that of a lax monoidal functor. The commutativity condition can be expressed in terms of the external multiplication as the commutativity

of the diagrams

$$\begin{tikzcd} R(V) \times R(W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\text{twist}"'] & R(V \oplus W) \arrow[d, "R(\tau_{V,W})"] \\ R(W) \times R(V) \arrow[r, "\mu_{W,V}"'] & R(W \oplus V) \end{tikzcd}$$

where $\tau_{V,W} : V \oplus W \longrightarrow W \oplus V$ interchanges the summands. So commutative orthogonal monoid spaces in external form are lax symmetric monoidal functors.

Every $\mathcal{F}$-cofibration is in particular a flat cofibration, so the monoid axiom in the $\mathcal{F}$-global model structure holds. If the global family $\mathcal{F}$ is closed under products, Theorem 4.1 of [**133**] applies to the $\mathcal{F}$-global model structure of Theorem 4.8 and shows:

Corollary 4.16. *Let R be an orthogonal monoid space and $\mathcal{F}$ a multiplicative global family.*

(i) *The category of R-modules admits the* $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spaces is an $\mathcal{F}$-equivalence (respectively $\mathcal{F}$-global fibration). This model structure is cofibrantly generated. Every cofibration in this $\mathcal{F}$-global model structure is an h-cofibration of underlying orthogonal spaces. If the underlying orthogonal space of R is $\mathcal{F}$-cofibrant, then every cofibration of R-modules is a $\mathcal{F}$-cofibration of underlying orthogonal spaces.*

(ii) *If R is commutative, then with respect to $\boxtimes_R$ the $\mathcal{F}$-global model structure of R-modules is a monoidal model category that satisfies the monoid axiom.*

(iii) *If R is commutative, then the category of R-algebras admits the* $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spaces is an $\mathcal{F}$-equivalence (respectively $\mathcal{F}$-global fibration). Every cofibrant R-algebra is also cofibrant as an R-module.*

Proof. Almost of the statements are in Theorem 4.1 of [**133**]. The only additional claims that require justification are the two statements in part (i) that concern the behavior of the forgetful functor on the cofibrations in the $\mathcal{F}$-global model structure.

Since the forgetful functor from R-modules to orthogonal spaces preserves all colimits and the classes of h-cofibrations and of $\mathcal{F}$-cofibrations of orthogonal spaces are both closed under coproducts, cobase change, sequential colimits and retracts, it suffices to show each claim for the generating cofibrations in the $\mathcal{F}$-global model structure on the category of R-modules. These are of the form

$$i_k \times (R \boxtimes \mathbf{L}_{H,\mathbb{R}^m})$$

for some $k, m \geq 0$ and H a closed subgroup of $O(m)$ that belongs to the global family $\mathcal{F}$; as usual $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. Since i_k is an h-cofibration of spaces, the morphisms $i_k \times (R \boxtimes \mathbf{L}_{H,\mathbb{R}^m})$ are h-cofibrations of orthogonal spaces. This concludes the proof that every cofibration of R-modules is an h-cofibration of underlying orthogonal spaces.

Now we suppose that the underlying orthogonal space of R is $\mathcal{F}$-cofibrant. Because H belongs to $\mathcal{F}$, the orthogonal space $\mathbf{L}_{H,\mathbb{R}^m}$ is $\mathcal{F}$-cofibrant. Hence the orthogonal space $R \boxtimes \mathbf{L}_{H,\mathbb{R}^m}$ is $\mathcal{F}$-cofibrant by Proposition 4.12 (iii). So $i_k \times (R \boxtimes \mathbf{L}_{H,\mathbb{R}^m})$ is an $\mathcal{F}$-cofibration of orthogonal spaces. This concludes the proof that every cofibration of R-modules is an $\mathcal{F}$-cofibration of underlying orthogonal spaces. □

Strictly speaking, Theorem 4.1 of [**133**] does not apply verbatim to the $\mathcal{F}$-global model structure because the hypothesis that every object is small (with respect to some regular cardinal) is not satisfied and our version of the monoid axiom in Proposition 4.14 is weaker than Theorem 3.3 of [**133**] in that we do not close under *transfinite* compositions. However, in our situation the sources of the generating cofibrations and generating acyclic cofibrations are small with respect to sequential compositions of flat cofibrations, and this suffices to run the countable small object argument (compare also Remark 2.4 of [**133**]).

PROPOSITION 4.17. *Let R be an orthogonal monoid space and N a right R-module that is cofibrant in the $\mathcal{A}ll$-global model structure of Corollary* 4.16 (i)*. Then for every global family $\mathcal{F}$, the functor $N \boxtimes_R -$ takes $\mathcal{F}$-equivalences of left R-modules to $\mathcal{F}$-equivalences of orthogonal spaces.*

PROOF. For the course of this proof we call an R-module N *homotopical* if the functor $N \boxtimes_R -$ takes $\mathcal{F}$-equivalences of left R-modules to $\mathcal{F}$-equivalences of orthogonal spaces. Since the $\mathcal{A}ll$-global model structure on the category of right R-modules is obtained by lifting the global model structure of orthogonal spaces along the free and forgetful adjoint functor pair, every cofibrant right R-module is a retract of an R-module that arises as the colimit of a sequence

$$\emptyset = M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_k \longrightarrow \dots \tag{4.18}$$

in which each M_k is obtained from M_{k-1} as a pushout

$$\begin{tikzcd} A_k \boxtimes R \arrow[r, "f_k \boxtimes R"] \arrow[d] & B_k \boxtimes R \arrow[d] \\ M_{k-1} \arrow[r] & M_k \end{tikzcd}$$

for some flat cofibration $f_k : A_k \longrightarrow B_k$ of orthogonal spaces. For example, f_k can be chosen as a disjoint union of morphisms in the set I^{str} of generating flat cofibrations. We show by induction on k that each module M_k is homotopical. The induction starts with the empty R-module, where there is nothing to show. Now we suppose that M_{k-1} is homotopical, and we claim that then M_k is homotopical as well. To see this we consider an $\mathcal{F}$-equivalence of left R-modules $\varphi : X \longrightarrow Y$. Then $M_k \boxtimes_R \varphi$ is obtained by passing to horizontal pushouts in the following commutative diagram of orthogonal spaces:

$$\begin{tikzcd} M_{k-1} \boxtimes_R X \arrow[d, "M_{k-1} \boxtimes_R \varphi"'] & A_k \boxtimes X \arrow[l] \arrow[r, "f_k \boxtimes X"] \arrow[d, "A_k \boxtimes \varphi"] & B_k \boxtimes X \arrow[d, "B_k \boxtimes \varphi"] \\ M_{k-1} \boxtimes_R Y & A_k \boxtimes Y \arrow[l] \arrow[r, "f_k \boxtimes Y"'] & B_k \boxtimes Y \end{tikzcd}$$

Here we have exploited that $(A_k \boxtimes R) \boxtimes_R X$ is naturally isomorphic to $A_k \boxtimes X$. In the diagram, the left vertical morphism is an $\mathcal{F}$-equivalence by hypothesis. The middle and right vertical morphisms are $\mathcal{F}$-equivalences because box product is homotopical for $\mathcal{F}$-equivalences (Proposition 4.7 (xiii)). Moreover, since the morphism f_k is a flat cofibration, it is an h-cofibration (by Corollary A.1.18 (iii)), and so the morphisms $f_k \boxtimes X$ and $f_k \boxtimes Y$ are h-cofibrations. Proposition 4.7 (ix) then shows that the induced morphism on horizontal pushouts $M_k \boxtimes_R \varphi$ is again an $\mathcal{F}$-equivalence.

Now we let M be a colimit of the sequence (4.18). Then $M \boxtimes_R X$ is a colimit of the sequence $M_k \boxtimes_R X$. Moreover, since $f_k : A_k \longrightarrow B_k$ is an h-cofibration, so is the morphism $f_k \boxtimes R$, and hence also its cobase change $M_{k-1} \longrightarrow M_k$. So the sequence whose colimit is $M \boxtimes_R X$ consists of h-cofibrations. The same is true for $M \boxtimes_R Y$. Since each M_k is homotopical and colimits of orthogonal spaces along closed embeddings are homotopical (by Proposition 4.7 (vii)), we conclude that the morphism $M \boxtimes_R \varphi : M \boxtimes_R X \longrightarrow M \boxtimes_R Y$ is an $\mathcal{F}$-equivalence, so that M is homotopical. Since $\mathcal{F}$-equivalences are closed under retracts, the class of homotopical R-modules is closed under retracts, and so every cofibrant right R-module is homotopical. □

5. Equivariant homotopy sets

In this section we define the equivariant homotopy sets $\pi_0^G(Y)$ of orthogonal spaces and relate them by restriction maps defined from continuous homomorphisms between compact Lie groups. We show that the global classifying space of a compact Lie group G represents the functor π_0^G (see Proposition 5.14) and use that to identify the category of all natural operations with the category Rep of conjugacy classes

of continuous homomorphisms, compare, Proposition 5.16. We show in Proposition 5.21 that every Rep-functor can be realized by a global homotopy type; more precisely, the functor $\underline{\pi}_0$ that assigns to an orthogonal space the collection of all equivariant homotopy sets has a right adjoint, right inverse.

We recall that for every compact Lie group G, $\mathcal{U}_G$ is a chosen complete G-universe and $s(\mathcal{U}_G)$ denotes the poset, under inclusion, of finite dimensional G-subrepresentations of $\mathcal{U}_G$.

Definition 5.1. Let Y be an orthogonal space, G be a compact Lie group and A a G-space. We define

$$[A,Y]^G \;=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)}\,[A,Y(V)]^G\ ,$$

the colimit over the poset $s(\mathcal{U}_G)$ of the sets of G-homotopy classes of G-maps from A to $Y(V)$.

The canonical G-maps $Y(V)\longrightarrow Y(\mathcal{U}_G)$ induce maps $[A,Y(V)]^G\longrightarrow[A,Y(\mathcal{U}_G)]^G$ and hence a canonical map

$$[A,Y]^G \;\longrightarrow\; [A,Y(\mathcal{U}_G)]^G\ .$$

In general there is no reason for this map to be injective or surjective. If Y is closed and A is compact, the situation improves:

Proposition 5.2. *Let G be a compact Lie group.*

(i) *For every closed orthogonal space Y and every compact G-space A the canonical map*

$$[A,Y]^G \;\longrightarrow\; [A,Y(\mathcal{U}_G)]^G$$

is bijective.

(ii) *Let $\mathcal{F}$ be a global family and $f:X\longrightarrow Y$ an $\mathcal{F}$-equivalence of orthogonal spaces. Then for every finite G-CW-complex A all of whose isotropy groups belong to $\mathcal{F}$, the induced map*

$$[A,f]^G\ :\ [A,X]^G \;\longrightarrow\; [A,Y]^G$$

is bijective.

Proof. (i) Since the poset $s(\mathcal{U}_G)$ contains a cofinal subsequence, $Y(\mathcal{U}_G)$ is a sequential colimit of values of Y along closed embeddings. The colimit in the category of compactly generated spaces is thus given by the colimit of the underlying sequence of sets endowed with the weak topology. Since compactly generated spaces have the T_1-separation property, every continuous G-map $A\longrightarrow Y(\mathcal{U}_G)$ thus factors through $Y(V)$ for some finite dimensional $V\in s(\mathcal{U}_G)$, which shows surjectivity. Injectivity follows by the same argument applied to the compact G-space $A\times[0,1]$.

(ii) We let $\beta:A\longrightarrow Y(V)$ be a continuous G-map, for some $V\in s(\mathcal{U}_G)$, that represents an element of $[A,Y]^G$. Together with the unique map from the empty space this specifies an equivariant lifting problem on the left:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X(V) \arrow[d, "f(V)"] \\
A \arrow[r, "\beta"'] & Y(V)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X(V) \arrow[r, "X(\varphi)"] & X(W) \arrow[d, "f(W)"] \\
A \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V) \arrow[r, "Y(\varphi)"'] & Y(W)
\end{tikzcd}$$

Since $(A,\emptyset)$ is a finite G-CW-pair with isotropy in $\mathcal{F}$ and f an $\mathcal{F}$-equivalence, Proposition 4.5 (ii) provides a G-equivariant linear isometric embedding $\varphi:V\longrightarrow W$ and a continuous G-map λ on the right hand side such that $f(W)\circ\lambda$ is G-homotopic to $Y(\varphi)\circ\beta$. We choose a G-equivariant linear isometric embedding $j:W\longrightarrow\mathcal{U}_G$ extending the inclusion of V. Then the class in $[A,X]^G$ represented by the G-map

$$X(j)\circ\lambda\ :\ A\ \longrightarrow\ X(j(W))$$

is taken to $[\beta]$ by the map $[A,f]^G$. This shows that $[A,f]^G$ is surjective.

For injectivity we consider two G-maps $g, g' : A \longrightarrow X(V)$, for some $V \in s(\mathcal{U}_G)$, such that $[A, f]^G[g] = [A, f]^G[g']$. By enlarging V, if necessary, we can assume that the two composites $f(V) \circ g$ and $f(V) \circ g'$ are G-homotopic. A choice of such a homotopy specifies an equivariant lifting problem on the left:

$$\begin{tikzcd} A \times \{0,1\} \arrow[r, "g,g'"] \arrow[d] & X(V) \arrow[d, "f(V)"] \\ A \times [0,1] \arrow[r, "\beta"'] & Y(V) \end{tikzcd} \qquad \begin{tikzcd} A \times \{0,1\} \arrow[r, "g,g'"] \arrow[d] & X(V) \arrow[r, "X(\varphi)"] & X(W) \arrow[d, "f(W)"] \\ A \times [0,1] \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V) \arrow[r, "Y(\varphi)"'] & Y(W) \end{tikzcd}$$

Proposition 4.5 (ii) provides a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ and a lift λ on the right hand side such that $\lambda(-, 0) = g$, $\lambda(-, 1) = g'$ and $f(W) \circ \lambda$ is G-homotopic, relative $A \times \{0, 1\}$, to $Y(\varphi) \circ \beta$. As in the first part, we use a G-equivariant linear isometric embedding $j : W \longrightarrow \mathcal{U}_G$, extending the inclusion of V, to transform λ into the G-homotopy

$$X(j) \circ \lambda \ : \ A \times [0,1] \ \longrightarrow \ X(j(W))$$

that connects the images of g and g' in $X(j(W))$. This shows that $[g] = [g']$ in $[A, X]^G$, so $[A, f]^G$ is also injective. □

Example 5.3. We specialize to the case where $Y = B_{\mathrm{gl}}G$ is the global classifying space of a compact Lie group G. Proposition 2.14 above already gave an explanation for the name 'global classifying space' by exhibiting $(B_{\mathrm{gl}}G)(\mathcal{U}_K)$ as a classifying space for principal (K, G)-bundle over compact K-spaces. Now we slightly reinterpret this result as a natural bijection

$$[A, B_{\mathrm{gl}}G]^K \ \cong \ \mathrm{Prin}_{(K,G)}(A)$$

for compact K-spaces A.

We let V be a faithful G-representation and set $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$. Then for every K-representation W the projection $q : \mathbf{L}(V, W) \longrightarrow \mathbf{L}(V, W)/G$ is a principal (K, G)-bundle. Since the space $\mathbf{L}(V, W)/G$ is compact, hence paracompact and completely regular, this principal (K, G)-bundle is numerable by Cor. 1.5 and Cor. 1.13 of [**92**]. So given a continuous K-equivariant map $f : A \longrightarrow \mathbf{L}(V, W)/G$ we obtain a principal (K, G)-bundle f^*q over A by pullback, and equivariantly homotopic maps give rise to isomorphic bundles by [**92**, Cor. 2.11]. So pullback give a well-defined map

$$[A, \mathbf{L}(V, W)/G]^K \ \longrightarrow \ \mathrm{Prin}_{(K,G)}(A) \ .$$

These maps are compatible as W varies over the poset of finite dimensional K-subrepresentations of $\mathcal{U}_K$, so they assemble into a map

$$[A, B_{\mathrm{gl}}G]^K \ = \ [A, \mathbf{L}_{G,V}]^K \ \longrightarrow \ \mathrm{Prin}_{(K,G)}(A) \ . \tag{5.4}$$

We claim that this map is bijective. Indeed, it factors as the composite

$$[A, \mathbf{L}_{G,V}]^K \ \longrightarrow \ [A, \mathbf{L}_{G,V}(\mathcal{U}_K)]^K \ \longrightarrow \ \mathrm{Prin}_{(K,G)}(A) \ ,$$

where the first map is the bijection of Proposition 5.2 (i), exploiting that the orthogonal space $\mathbf{L}_{G,V}$ is closed. The second map is bijective by Proposition 2.14.

Now we specialize the equivariant homotopy sets $[A, Y]^G$ to the case $A = \{*\}$ of a one-point G-space, and then give it a new name.

Definition 5.5. Let G be a compact Lie group. The *G-equivariant homotopy set* of an orthogonal space Y is the set

$$\pi_0^G(Y) \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} \ \pi_0(Y(V)^G) \ . \tag{5.6}$$

Specializing Proposition 5.2 to a one-point G-space yields:

COROLLARY 5.7. *Let G be a compact Lie group.*

(i) *For every closed orthogonal space Y the canonical map*

$$\pi_0^G(Y) \;\longrightarrow\; \pi_0(Y(\mathcal{U}_G)^G)$$

is bijective.

(ii) *Let $f : X \longrightarrow Y$ be a global equivalence of orthogonal spaces. Then the induced map*

$$\pi_0^G(f) \;:\; \pi_0^G(X) \;\longrightarrow\; \pi_0^G(Y)$$

of equivariant homotopy sets is bijective.

As the group varies, the homotopy sets $\pi_0^G(Y)$ have contravariant functoriality in G: every continuous group homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups induces a restriction map $\alpha^* : \pi_0^G(Y) \longrightarrow \pi_0^K(Y)$, as we shall now explain. We denote by α^* the restriction functor from G-spaces to K-spaces (or from G-representations to K-representations) along α, i.e., $\alpha^* Z$ (respectively $\alpha^* V$) is the same topological space as Z (respectively the same inner product space as V) endowed with K-action via

$$k \cdot z \;=\; \alpha(k) \cdot z \;.$$

Given an orthogonal space Y, we note that for every G-representation V, the K-spaces $\alpha^*(Y(V))$ and $Y(\alpha^* V)$ are equal (not just isomorphic).

The restriction $\alpha^*(\mathcal{U}_G)$ is a K-universe, but if α has a non-trivial kernel, then this K-universe is not complete. When α is injective, then $\alpha^*(\mathcal{U}_G)$ is a complete K-universe, but typically different from the chosen complete K-universe $\mathcal{U}_K$. To deal with this we explain how a G-fixed point $y \in Y(V)^G$, for an arbitrary G-representation V, gives rise to an unambiguously defined element $\langle y \rangle$ in $\pi_0^G(Y)$. The point here is that V need not be a subrepresentation of the chosen universe $\mathcal{U}_G$ and the resulting class does not depend on any additional choices.

To construct $\langle y \rangle$ we choose a G-equivariant linear isometry $j : V \longrightarrow \bar{V}$ onto a G-subrepresentation $\bar{V}$ of $\mathcal{U}_G$. Then $Y(j)(y)$ is a G-fixed point of $Y(\bar{V})$, so we obtain an element

$$\langle y \rangle \;=\; [Y(j)(y)] \;\in\; \pi_0^G(Y) \;.$$

It is crucial, but not completely obvious, that $\langle f \rangle$ does not depend on the choice of isometry j. To show this we need an auxiliary lemma:

LEMMA 5.8. *Let G be a compact Lie group, V and W two G-representations and $j, j' : V \longrightarrow W$ two G-equivariant linear isometric embeddings. If the images $j(V)$ and $j'(V)$ are orthogonal, then j and j' are homotopic through G-equivariant linear isometric embeddings.*

PROOF. The desired homotopy $H : V \times [0,1] \longrightarrow W$ from j to j' is given by

$$H(v,t) \;=\; \sqrt{1-t^2}\;\cdot j(v) \;+\; t \cdot j'(v) \;. \qquad \square$$

PROPOSITION 5.9. *Let Y be an orthogonal space, G a compact Lie group, V a G-representation and $y \in Y(V)^G$ a G-fixed point.*

(i) *The class $\langle y \rangle$ in $\pi_0^G(Y)$ is independent of the choice of linear isometry from V to a subrepresentation of $\mathcal{U}_G$.*

(ii) *For every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the relation*

$$\langle Y(\varphi)(y) \rangle \;=\; \langle y \rangle \qquad \textit{holds in} \quad \pi_0^G(Y).$$

PROOF. (i) We let $j : V \longrightarrow \bar{V}$ and $j' : V \longrightarrow \bar{V}'$ be two G-equivariant linear isometries, with $\bar{V}, \bar{V}' \in s(\mathcal{U}_G)$. We choose a third G-equivariant linear isometry $\bar{j} : V \longrightarrow U$ such that $U \in s(\mathcal{U}_G)$ and U is orthogonal to both $\bar{V}$ and $\bar{V}'$. We let W be the span of $\bar{V}, \bar{V}'$ and U inside $\mathcal{U}_G$. We can then view j, j' and $\bar{j}$ as equivariant linear isometric embedding from V to W. Then j and j' are homotopic to $\bar{j}$ through G-equivariant linear isometric embeddings into W, by Lemma 5.8. In particular, j and j' are homotopic

to each other; if $H(-,t) : V \longrightarrow W$ is a continuous 1-parameter family of G-equivariant linear isometric embeddings from j to j', then

$$t \;\longmapsto\; Y(H(-,t))(y)$$

is a path in $Y(W)^G$ from $Y(j)(y)$ to $Y(j')(y)$, so $[Y(j)(y)] = [Y(j')(y)]$ in $\pi_0^G(Y)$.

(ii) If $j : W \longrightarrow \bar{W}$ is an equivariant linear isometry with $\bar{W} \in s(\mathcal{U}_G)$, we define $\bar{V} = j(\varphi(V))$ and we let $k : V \longrightarrow \bar{V}$ be the equivariant linear isometry that is defined by $k(v) = j(\varphi(v))$ (i.e., k is essentially $j \circ \varphi$, but with range $\bar{V}$ instead of $\bar{W}$). Then

$$\langle Y(\varphi)(y)\rangle \;=\; [Y(j)(Y(\varphi)(y))] \;=\; [Y(j\varphi)(y)] \;=\; [Y(k)(y)] \;=\; \langle y\rangle \;. \qquad \square$$

We can now define the *restriction map* associated to a continuous group homomorphism $\alpha : K \longrightarrow G$ by

$$\alpha^* \;:\; \pi_0^G(Y) \;\longrightarrow\; \pi_0^K(Y)\;, \quad [y] \;\longmapsto\; \langle y\rangle \;. \tag{5.10}$$

This makes sense because every G-fixed point of $Y(V)$ is also a K-fixed point of $\alpha^*(Y(V)) = Y(\alpha^* V)$. For a second continuous group homomorphism $\beta : L \longrightarrow K$ we have

$$\beta^* \circ \alpha^* \;=\; (\alpha\beta)^* \;:\; \pi_0^G(Y) \;\longrightarrow\; \pi_0^L(Y)\;.$$

Clearly, restriction along the identity homomorphism is the identity, so we have made the collection of equivariant homotopy sets $\pi_0^G(Y)$ into a contravariant functor in the group variable.

An important special case of the restriction homomorphisms are conjugation maps. Here we consider a closed subgroup H of G, an element $g \in G$ and denote by

$$c_g \;:\; {}^gH \;\longrightarrow\; H\;, \quad c_g(h) \;=\; g^{-1}hg$$

the conjugation homomorphism. As any group homomorphism, c_g induces a restriction map

$$g_\star \;=\; (c_g)^* \;:\; \pi_0^H(Y) \;\longrightarrow\; \pi_0^{{}^gH}(Y) \tag{5.11}$$

of equivariant homotopy sets. For $g, \bar{g} \in G$ we have $c_{g\bar{g}} = c_{\bar{g}} \circ c_g : {}^{g\bar{g}}H \longrightarrow H$ and thus

$$(g\bar{g})_\star \;=\; (c_{g\bar{g}})^* \;=\; (c_g)^* \circ (c_{\bar{g}})^* \;=\; g_\star \circ \bar{g}_\star \;:\; \pi_0^H(Y) \;\longrightarrow\; \pi_0^{{}^{g\bar{g}}H}(Y)\;.$$

A key fact is that inner automorphisms act trivially, i.e., the restriction map c_g^* is the identity on $\pi_0^G(Y)$. So the action, by the restriction maps, of the automorphism group of G on $\pi_0^G(Y)$ factors through the outer automorphism group.

Proposition 5.12. *For every orthogonal space Y, every compact Lie group G, and every $g \in G$, the conjugation map $g_\star : \pi_0^G(Y) \longrightarrow \pi_0^G(Y)$ is the identity.*

Proof. We consider a finite dimensional G-subrepresentation V of $\mathcal{U}_G$ and a G-fixed point $y \in Y(V)^G$ that represents an element in $\pi_0^G(Y)$. Then the map $l_g : c_g^*(V) \longrightarrow V$ given by left multiplication by g is a G-equivariant linear isometry. So

$$g_\star[y] \;=\; (c_g)^*[y] \;=\; [Y(l_g)(y)] \;=\; [g \cdot y] \;=\; [y]\;,$$

by the very definition of the restriction map. The third equation is the definition of the G-action on $Y(V)$ through the G-action on V. The fourth equation is the hypothesis that y is G-fixed. $\square$

We denote by Rep the category whose objects are the compact Lie groups and whose morphisms are conjugacy classes of continuous group homomorphisms. We can summarize the discussion thus far by saying that for every orthogonal space Y the restriction maps make the equivariant homotopy sets $\{\pi_0^G(Y)\}$ into a functor

$$\underline{\pi}_0(Y) \;:\; \mathrm{Rep}^{\mathrm{op}} \;\longrightarrow (\text{sets})\;.$$

We will refer to such a contravariant functor as a *Rep-functor*.

In the following, $\mathrm{Ho}(spc)$ denotes the homotopy category of orthogonal spaces, taken with respect to global equivalences. Since the functor π_0^G takes global equivalences of orthogonal spaces to bijections, it factors uniquely through the localization functor $spc \longrightarrow \mathrm{Ho}(spc)$; we abuse notation slightly and denote the resulting functor on the unstable global homotopy category by the same symbol

$$\pi_0^G \ : \ \mathrm{Ho}(spc) \ \longrightarrow \ (\mathrm{sets}) \ .$$

For every representation V of a compact Lie group G we define the *tautological class*

$$u_{G,V} \ \in \ \pi_0^G(\mathbf{L}_{G,V}) \tag{5.13}$$

as the path component of the G-fixed point

$$\mathrm{Id}_V \cdot G \ \in \ (\mathbf{L}(V,V)/G)^G = (\mathbf{L}_{G,V}(V))^G \ ,$$

the G-orbit of the identity of V.

Proposition 5.14. *For every compact Lie group G and every faithful G-representation V, the pair $(\mathbf{L}_{G,V}, u_{G,V})$ represents the functor $\pi_0^G : \mathrm{Ho}(spc) \longrightarrow (\mathrm{sets})$.*

Proof. We need to show that for every orthogonal space Y the map

$$\mathrm{Ho}(spc)(\mathbf{L}_{G,V}, Y) \ \longrightarrow \ \pi_0^G(Y) \ , \quad f \longmapsto f_*(u_G)$$

is bijective. Since both sides take global equivalences in Y to bijections, we can assume that Y is fibrant in the global model structure. The free orthogonal space $\mathbf{L}_{G,V}$ is flat, and hence cofibrant in the global model structure. So the localization functor induces a bijection

$$spc(\mathbf{L}_{G,V}, Y)/\mathrm{homotopy} \ \longrightarrow \ \mathrm{Ho}(spc)(\mathbf{L}_{G,V}, Y)$$

from the set of homotopy classes of morphisms of orthogonal spaces to the set of morphisms in $\mathrm{Ho}(spc)$. By the freeness property, morphisms from $\mathbf{L}_{G,V}$ to Y biject with G-fixed points of $Y(V)$, and homotopies of morphisms biject with paths between fixed points. The composite

$$\pi_0(Y(V)^G) \ \xrightarrow{\cong} \ \mathrm{Ho}(spc)(\mathbf{L}_{G,V}, Y) \ \xrightarrow{f \mapsto f_*(u_{G,V})} \ \pi_0^G(Y)$$

is bijective because Y is static. Since the left map and the composite are bijective, so is the evaluation map at the unstable tautological class. □

Remark 5.15. The global classifying space $B_{\mathrm{gl}}G$ represents the functor π_0^G, and it is also characterized by this property up to global equivalence. More precisely, we suppose that Y is an orthogonal space and $u \in \pi_0^G(Y)$ a class such that (Y, u) also represents the functor $\pi_0^G : \mathrm{Ho}(spc) \longrightarrow (\mathrm{sets})$. Then in the homotopy category $\mathrm{Ho}(spc)$ there is a unique morphism $f : B_{\mathrm{gl}}G \longrightarrow Y$ such that $f_*(u_G) = u$, and f is an isomorphism. Since $B_{\mathrm{gl}}G$ is cofibrant, there is a chain of two global equivalences

$$B_{\mathrm{gl}}G \ \xrightarrow{\sim} \ Y^{\mathrm{f}} \ \xleftarrow[\sim]{q} \ Y \ ,$$

that realizes f, where $q : Y \longrightarrow Y^{\mathrm{f}}$ is any global equivalence with static (i.e., globally fibrant) target.

Since the functor π_0^G is representable, the calculation of the algebra of natural operations reduces to the calculation of the equivariant homotopy sets of the representing object. The next proposition carries this out, and shows in particular that the restriction maps along continuous group homomorphisms are the only natural operations between equivariant homotopy sets of orthogonal spaces.

Proposition 5.16. *Let G and K be compact Lie groups and V a faithful G-representation.*

(i) *The K-fixed point space $(\mathbf{L}_{G,V}(\mathcal{U}_K))^K$ is a disjoint union, indexed by conjugacy classes of continuous group homomorphisms $\alpha : K \longrightarrow G$, of classifying spaces of the centralizer of the image of α.*

(ii) *The map*

$$\mathrm{Rep}(K,G) \;\longrightarrow\; \pi_0^K(\mathbf{L}_{G,V})\ , \quad [\alpha : K \longrightarrow G] \;\longmapsto\; \alpha^*(u_{G,V})$$

is bijective.

(iii) *Every natural transformation $\pi_0^G \longrightarrow \pi_0^K$ of set valued functors on the category of orthogonal spaces is of the form α^* for a unique conjugacy class of continuous group homomorphism $\alpha : K \longrightarrow G$.*

Proof. Part (i) works for any universal $(K \times G)$-space E for the family $\mathcal{F}(K;G)$, for example for $E = \mathbf{L}(V,\mathcal{U}_K)$. The argument can be found in Theorem 2.17 of [**92**], Proposition 5 of [**96**] and – in a more general context – in [**101**, Thm. 13.1]. We repeat the proof for the convenience of the reader. For a continuous homomorphism $\alpha : K \longrightarrow G$, we let $C(\alpha)$ denote the centralizer, in G, of the image of α, and we set

$$E^\alpha \;=\; \{x \in E \mid (k,\alpha(k)) \cdot x = x \text{ for all } k \in K\}\ ,$$

the space of fixed points of the graph of α. Since the G-action on this universal space E is free, Proposition A.2.27 provides a homeomorphism

$$\coprod \alpha^\flat \;:\; \coprod_{\langle\alpha\rangle} E^\alpha/C(\alpha) \;\longrightarrow\; (E/G)^K\ ,$$

where the coproduct is indexed by conjugacy classes of continuous homomorphisms. The graph of α belongs to the family $\mathcal{F}(K;G)$, so E^α is a contractible space. The action of $C(\alpha)$ on E^α is a restriction of the G-action on E, hence free. Since E is $(K \times G)$-cofibrant, the fixed point space E^α is cofibrant for the action of the normalizer (inside $K \times G$) of the graph of α, by Proposition A.2.15. Hence E^α is also cofibrant as a $C(\alpha)$-space, by Proposition A.2.14 (i). So for every homomorphism α the space $E^\alpha/C(\alpha)$ is a classifying space for the group $C(\alpha)$. This shows part (i).

(ii) Since the classifying space of a topological group is connected, part (i) identifies the path components of $(\mathbf{L}_{G,V}(\mathcal{U}_K))^K$ with the conjugacy classes of continuous homomorphisms $\alpha : K \longrightarrow G$. The bijection sends the class of α to $\alpha^*(u_{G,V})$. The claim then follows by applying Corollary 5.7 (i).

(iii) We let V be any faithful G-representation. The composite

$$\mathrm{Rep}(K,G) \;\xrightarrow{[\alpha]\mapsto\alpha^*}\; \mathrm{Nat}(\pi_0^G,\pi_0^K) \;\xrightarrow{\mathrm{ev}}\; \pi_0^K(\mathbf{L}_{G,V})$$

is bijective by part (ii), where the second map is evaluation at the tautological class $u_{G,V}$. Because we use the notation π_0^G for the functor on the category of orthogonal spaces and also for the induced functor on the homotopy category, there is a deliberate ambiguity here as to which kind of natural transformations we consider. Since π_0^G takes all global equivalences to bijections, the map

$$-\circ\gamma \;:\; \mathrm{Nat}_{\mathrm{Ho}(spc)\to(\mathrm{sets})}(\pi_0^G,\pi_0^K) \;\longrightarrow\; \mathrm{Nat}_{spc\to(\mathrm{sets})}(\pi_0^G\circ\gamma,\pi_0^K\circ\gamma)$$

given by precomposition with the localization functor $\gamma : spc \longrightarrow \mathrm{Ho}(spc)$ is bijective; so the ambiguity does not cause any harm. The evaluation map is bijective by Proposition 5.14 and the Yoneda Lemma. So the first map is bijective as well. □

Our next aim is to show that every Rep-functor is realized by an orthogonal space. More is true: the next theorem effectively constructs a right adjoint functor

$$(-)^\sharp \;:\; (\text{Rep-functors}) \;\longrightarrow\; \mathrm{Ho}(spc)$$

to the functor $\underline{\pi}_0$ such that the adjunction counit is an isomorphism $\underline{\pi}_0(F^\sharp) \cong F$ of Rep-functors.

The next proposition abstracts certain formal arguments that we use several times in this book.

Proposition 5.17. *Let*

$$\pi \;:\; \mathcal{C} \;\longrightarrow\; \mathcal{A}$$

be a functor and Y an object of $\mathcal{A}$. Let $\mathcal{X}$ denote the class of $\mathcal{C}$-morphisms $i : A \longrightarrow B$ with the following property: for every $\mathcal{C}$-morphism $f : A \longrightarrow Y$ and every $\mathcal{A}$-morphism $\psi : \pi(B) \longrightarrow \pi(Y)$ such that $\psi \circ \pi(i) = \pi(f)$, there is a $\mathcal{C}$-morphism $g : B \longrightarrow Y$ such that $\pi(g) = \psi$ and $gi = f$.

(i) *The class $\mathcal{X}$ is closed under retracts.*

(ii) *Let*

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "i'"'] & A \arrow[d, "i"] \\ B' \arrow[r, "\beta"'] & B \end{tikzcd}$$

be a pushout in $\mathcal{C}$ such that i' belongs to $\mathcal{X}$ and the morphism

$$\pi(\beta) + \pi(i) \ : \ \pi(B') \amalg \pi(A) \ \longrightarrow \ \pi(B)$$

is an epimorphism in $\mathcal{A}$. Then i belongs to the class $\mathcal{X}$.

(iii) *If π preserves coproducts, then the class $\mathcal{X}$ is closed under coproducts.*

(iv) *Let i_k be a composable sequence of morphisms in $\mathcal{X}$ whose colimit exists and is preserved by π. Then the sequential composite belongs to $\mathcal{X}$.*

Proof. (i) We consider a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "i"'] & A' \arrow[r, "\alpha"] \arrow[d, "i'"'] & A \arrow[d, "i"] \\ B \arrow[r, "t"'] & B' \arrow[r, "\beta"'] & B \end{tikzcd}$$

that expresses i as a retract of i', i.e., $\alpha s = \mathrm{Id}_A$ and $\beta t = \mathrm{Id}_B$. We suppose that i' belongs to $\mathcal{X}$. We consider a morphism $f : A \longrightarrow Y$ and $\psi : \pi(B) \longrightarrow \pi(Y)$ such that $\psi \circ \pi(i) = \pi(f)$. Then

$$\psi \circ \pi(\beta) \circ \pi(i') \ = \ \psi \circ \pi(i) \circ \pi(\alpha) \ = \ \pi(f\alpha) \ ,$$

so the hypothesis on i' provides a morphism $g' : B' \longrightarrow Y$ such that $\pi(g') = \psi \circ \pi(\beta)$ and $g'i' = f\alpha$. The morphism $g = g't : B \longrightarrow Y$ then satisfies

$$\pi(g) \ = \ \pi(g') \circ \pi(t) \ = \ \psi \circ \pi(\beta) \circ \pi(t) \ = \ \psi$$

and $gi = g'ti = g'i's = f\alpha s = f$.

(ii) We let $f : A \longrightarrow Y$ and $\psi : \pi(B) \longrightarrow \pi(Y)$ be morphisms such that $\psi \circ \pi(i) = \pi(f)$. Then

$$\psi \circ \pi(\beta) \circ \pi(i') \ = \ \psi \circ \pi(i) \circ \pi(\alpha) \ = \ \pi(f\alpha) \ ,$$

so by the hypothesis on i' there is a morphism $g' : B' \longrightarrow Y$ such that $\pi(g') = \psi \circ \pi(\beta)$ and $g'i' = f\alpha$. The universal property of a pushout provides a morphism $g : B \longrightarrow Y$ such that $g\beta = g'$ and $gi = f$. This implies that

$$\pi(g) \circ \pi(\beta) \ = \ \psi \circ \pi(\beta) \qquad \text{and} \qquad \pi(g) \circ \pi(i) \ = \ \psi \circ \pi(i) \ .$$

The epimorphism assumption on $\pi(i) + \pi(\beta)$ then implies that $\pi(g) = \psi$.

Parts (iii) and (iv) are straightforward. □

We recall that a topological space T is weakly equivalent to a discrete topological space (namely to $\pi_0(T)$ with discrete topology) if and only if every continuous map $\partial D^k \longrightarrow T$ for $k \geq 2$ has a continuous extension to D^k.

Definition 5.18. An orthogonal space Y is *globally discrete* if the following condition holds: for every compact Lie group G, every G-representation V, every $k \geq 2$ and all continuous maps $\alpha : \partial D^k \longrightarrow Y(V)^G$, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ such that the composite $Y(\varphi)^G \circ \alpha : \partial D^k \longrightarrow Y(W)^G$ admits a continuous extension to D^k.

Proposition 5.19. *Let Y be a globally discrete, static orthogonal space. Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spaces, $f : A \longrightarrow Y$ a morphism of orthogonal spaces and $\psi : \underline{\pi}_0(B) \longrightarrow \underline{\pi}_0(Y)$ a morphism of* Rep*-functors such that $\psi \circ \underline{\pi}_0(i) = \underline{\pi}_0(f)$. Then there is a morphism of orthogonal spaces $g : B \longrightarrow Y$ such that $\underline{\pi}_0(g) = \psi$ and $gi = f$.*

Proof. Flat cofibrations of orthogonal spaces are in particular h-cofibrations by Corollary A.1.18 (iii), and hence levelwise closed embeddings by Proposition A.1.19. So the functor

$$\underline{\pi}_0 \ : \ spc \ \longrightarrow \ (\text{sets})$$

preserves coproducts and sequential colimits along flat cofibrations. Moreover, for every pushout square of orthogonal spaces on the left

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "i'"'] & A \arrow[d, "i"] \\ B' \arrow[r, "\beta"'] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} A'(V)^G \arrow[r, "\alpha(V)^G"] \arrow[d, "i'(V)^G"'] & A(V)^G \arrow[d, "i(V)^G"] \\ B'(V)^G \arrow[r, "\beta(V)^G"'] & B(V)^G \end{tikzcd}$$

where i' is a flat cofibration, then $\underline{\pi}_0(B)$ is the union of the images of $\underline{\pi}_0(\beta)$ and $\underline{\pi}_0(i)$. Said differently, the morphism $\underline{\pi}_0(\beta) + \underline{\pi}_0(i) : \underline{\pi}_0(B') \amalg \underline{\pi}_0(A) \longrightarrow \underline{\pi}_0(B)$ is an epimorphism of Rep-functors. Indeed, we let $b \in B(V)^G$ represent any given element of $\pi_0^G(Y)$, where V is some G-representation. Since flat cofibrations are levelwise closed embeddings, the square of fixed point spaces on the right is also a pushout. In particular, b is in the image of $\beta(V)^G$ or in the image of $i(V)^G$, which shows the claim.

Proposition 5.17 then shows that the class of cofibrations $i : A \longrightarrow B$ of orthogonal spaces that satisfy the claim of the proposition is closed under coproducts, cobase change, retract and countable composites. These closure properties reduce the proof to showing the conclusion for the generating flat cofibrations

$$i_k \times \mathbf{L}_{G,V} \ : \ \partial D^k \times \mathbf{L}_{G,V} \ \longrightarrow \ D^k \times \mathbf{L}_{G,V} \ ,$$

where G is a compact Lie group and V a faithful G-representation.

We start with the case $k = 0$. Since Y is static and G acts faithfully on V, the class $\psi(u_{G,V}) \in \pi_0^G(Y)$ is represented by a fixed point $y \in Y(V)^G$. The freeness property of $\mathbf{L}_{G,V}$ provides a unique morphism $g : \mathbf{L}_{G,V} \longrightarrow Y$ such that $g(V)(\mathrm{Id}_V \cdot G) = y$. Then $\underline{\pi}_0(g)$ and ψ agree on the universal element $u_{G,V} \in \pi_0^G(\mathbf{L}_{G,V})$, hence $\underline{\pi}_0(g) = \psi$. The condition $gi = f$ is automatically satisfied because the orthogonal space $\partial D^0 \times \mathbf{L}_{G,V}$ is empty.

Now we consider the case $k = 1$. Since ∂D^1 is a discrete space with two points, the orthogonal space $\partial D^1 \times \mathbf{L}_{G,V}$ is freely generated by the two G-fixed points

$$a_- = (-1, \mathrm{Id}_V \cdot G) \ , \quad a_+ = (+1, \mathrm{Id}_V \cdot G) \ \in \ \partial D^1 \times \mathbf{L}(V,V)/G \ .$$

Since $\underline{\pi}_0$ commutes with coproducts, $\underline{\pi}_0(\partial D^1 \times \mathbf{L}_{G,V})$ is then freely generated, as a Rep-functor, by the classes of a_- and a_+ in $\pi_0^G(\partial D^1 \times \mathbf{L}_{G,V})$. These two classes have the same image in $\pi_0^G(D^1 \times \mathbf{L}_{G,V})$; so the hypothesis that $\underline{\pi}_0(f) = \psi \circ \underline{\pi}_0(i_1 \times \mathbf{L}_{G,V})$ implies that

$$\pi_0^G(f)(\langle a_- \rangle) \ = \ \pi_0^G(f)(\langle a_+ \rangle)$$

in $\pi_0^G(Y)$. This means that for some G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the fixed points $Y(\varphi)(f(V)(a_-))$ and $Y(\varphi)(f(V)(a_+))$ lie in the same path component of $Y(W)^G$. Since Y is static and G acts faithfully on V, the map $Y(\varphi)^G : Y(V)^G \longrightarrow Y(W)^G$ is a weak equivalence. So the fixed

points $f(V)(a_+)$ and $f(V)(a_-)$ already lie in the same path component of $Y(V)^G$. A choice of path $D^1 \longrightarrow Y(V)^G$ from $f(V)(a_-)$ to $f(V)(a_+)$ is adjoint to a morphism

$$g \ : \ D^1 \times \mathbf{L}_{G,V} \ \longrightarrow \ Y$$

such that $g \circ (i_1 \times \mathbf{L}_{G,V}) = f$. Since the morphism $\underline{\pi}_0(i_1 \times \mathbf{L}_{G,V})$ is an epimorphism of Rep-functors, the relation $\underline{\pi}_0(g) \circ \underline{\pi}_0(i_1 \times \mathbf{L}_{G,V}) = \underline{\pi}_0(f) = \psi \circ \underline{\pi}_0(i_1 \times \mathbf{L}_{G,V})$ forces $\underline{\pi}_0(g) = \psi$.

For $k \geq 2$, finally, the morphism $f : \partial D^k \times \mathbf{L}_{G,V} \longrightarrow Y$ is adjoint to a continuous map $\tilde{f} : \partial D^k \longrightarrow Y(V)^G$. Since Y is globally discrete, there is a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ such that $Y(\varphi)^G \circ \tilde{f}$ is null-homotopic. Since Y is static, the map $Y(\varphi)^G : Y(V)^G \longrightarrow Y(W)^G$ is a weak equivalence, so $\tilde{f}$ itself is already null-homotopic. We can thus choose a continuous extension of $\tilde{f}$ to the disc D^k, and the adjoint of that extension is a morphism $g : D^k \times \mathbf{L}_{G,V} \longrightarrow Y$ such that $g \circ (i_k \times \mathbf{L}_{G,V}) = f$. For $k \geq 2$ the morphism $\underline{\pi}_0(i_k \times \mathbf{L}_{G,V})$ is an isomorphism of Rep-functors, so again the relation $\underline{\pi}_0(g) \circ \underline{\pi}_0(i_k \times \mathbf{L}_{G,V}) = \underline{\pi}_0(f) = \psi \circ \underline{\pi}_0(i_k \times \mathbf{L}_{G,V})$ forces $\underline{\pi}_0(g) = \psi$. □

PROPOSITION 5.20. *For every orthogonal space X and every globally discrete orthogonal space Y the map*

$$\underline{\pi}_0 \ : \ \mathrm{Ho}(spc)(X,Y) \ \longrightarrow \ (\text{Rep-functors})(\underline{\pi}_0(X), \underline{\pi}_0(Y))$$

is bijective.

PROOF. Both sides of the map in question take global equivalences in either variable to bijections. Moreover, if $Y \longrightarrow Y'$ is a global equivalence and Y is globally discrete, then Y' is also globally discrete. So we can assume without loss of generality that X is cofibrant and Y is fibrant in the global model structure, i.e., static. Then the set $\mathrm{Ho}(spc)(X,Y)$ can be calculated as the set of morphisms of orthogonal spaces from X to Y, modulo homotopy.

For surjectivity we let $\psi : \underline{\pi}_0(X) \longrightarrow \underline{\pi}_0(Y)$ be a morphism of Rep-functors and apply Proposition 5.19 to the case $A = \emptyset$ and $B = X$ (and the unique morphism $f : \emptyset \longrightarrow Y$). The proposition provides a morphism of orthogonal spaces $g : X \longrightarrow Y$ with $\underline{\pi}_0(g) = \psi$. For injectivity we consider two morphisms of orthogonal spaces $f_0, f_1 : X \longrightarrow Y$ such that $\underline{\pi}_0(f_0) = \underline{\pi}_0(f_1)$. We apply Proposition 5.19 to $A = \{0,1\} \times X$, $B = [0,1] \times X$, for i the inclusion and $f : \{0,1\} \times X \longrightarrow Y$ the disjoint union of f_0 and f_1. We define $\psi : \underline{\pi}_0([0,1] \times X) \longrightarrow \underline{\pi}_0(Y)$ as the composite

$$\underline{\pi}_0([0,1] \times X) \ \xrightarrow{\underline{\pi}_0(i_0)^{-1}} \ \underline{\pi}_0(Y) \ \xrightarrow{\underline{\pi}_0(f_0)} \ \underline{\pi}_0(Y) \ .$$

Because $\underline{\pi}_0(i_0) = \underline{\pi}_0(i_1) : \underline{\pi}_0(X) \longrightarrow \underline{\pi}_0([0,1] \times X)$ and $\underline{\pi}_0(f_0) = \underline{\pi}_0(f_1)$, the relation $\psi \circ \underline{\pi}_0(i) = \underline{\pi}_0(f)$ holds. The proposition provides a morphism of orthogonal spaces $g : [0,1] \times X \longrightarrow Y$ that is a homotopy from f_0 to f_1. □

Since the functor $\underline{\pi}_0 : spc \longrightarrow$ (Rep-functors) takes global equivalences to isomorphisms, it descends to a functor on the homotopy category $\mathrm{Ho}(spc)$ for which we use the same name.

PROPOSITION 5.21. (i) *Every orthogonal space Y admits a morphism $\kappa : Y \longrightarrow Y_{\mathrm{dis}}$ such that $\underline{\pi}_0(\kappa)$ is an isomorphism of* Rep-*functors and Y_{dis} is globally discrete.*

(ii) *For every* Rep-*functor F there is a globally discrete orthogonal space $F^\sharp$ and an isomorphism of* Rep-*functors*

$$\underline{\pi}_0(F^\sharp) \ \cong \ F \ .$$

(iii) *The functor*

$$\underline{\pi}_0 \ : \ \mathrm{Ho}(spc) \ \longrightarrow \ (\text{Rep-functors})$$

has a right adjoint which is also right inverse.

Proof. (i) We 'kill all higher homotopy groups' in a global fashion. We choose a set J of representatives of the isomorphism classes of pairs (G,V) consisting of compact Lie groups G and faithful G-representations V. For example, all pairs $(H,\mathbb{R}^n)$ for all $n\geq 0$ and all closed subgroups H of $O(n)$ would do the job. We apply the small object argument (see for example [**47**, 7.12] or [**79**, Thm. 2.1.14]), in the category of orthogonal spaces, to the morphism $Y\longrightarrow *$ to the terminal orthogonal space, with respect to the set I of closed embeddings

$$i_k\times \mathbf{L}_{G,V}\ :\ \partial D^k\times \mathbf{L}_{G,V}\ \longrightarrow\ D^k\times \mathbf{L}_{G,V}$$

for all $k\geq 2$ and all $(G,V)\in J$. The result is a morphism $\kappa: Y\longrightarrow Y_{\mathrm{dis}}$ that is an I-cell complex and whose target Y_{dis} has the right lifting property with respect to the morphisms in I. By adjointness, this means that every continuous map $\partial D^k\longrightarrow (Y_{\mathrm{dis}}(V))^G$, for $k\geq 2$, admits a continuous extension to D^k. So the orthogonal space Y_{dis} is in particular globally discrete. Colimits of orthogonal spaces are formed objectwise, and fixed points commute with pushouts and sequential colimits along closed embeddings. So for every compact Lie group K and every K-representation W, the map $\kappa(W)^K: Y(W)^K\longrightarrow (Y_{\mathrm{dis}}(W))^K$ is an $I_{W,K}$-cell complex, where $I_{W,K}$ is the set of maps $i_k\times(\mathbf{L}(V,W)/G)^K$ for all $k\geq 2$ and all $(G,V)\in J$. So $\kappa(W)^K$ induces a bijection on path components; hence $\underline{\pi}_0(\kappa)$ is an isomorphism of Rep-functors.

(ii) We choose an index set I, compact Lie groups G_i and elements $x_i\in F(G_i)$, for $i\in I$, that altogether generate F as a Rep-functor. We start with the orthogonal space

$$T\ =\ \amalg_{i\in I}\, B_{\mathrm{gl}}G_i\ ,$$

the disjoint union of global classifying spaces. The functor $\underline{\pi}_0$ takes disjoint union of orthogonal spaces to disjoint union of Rep-functors. So by Proposition 5.16 (ii), $\underline{\pi}_0(T)$ is the free Rep-functor generated by the classes $e_i\in \pi_0^{G_i}(T)$, defined as the images of the unstable tautological classes $u_{G_i}\in \pi_0^{G_i}(B_{\mathrm{gl}}G_i)$. So there is a unique morphism of Rep-functors

$$\epsilon\ :\ \underline{\pi}_0(T)\ \longrightarrow\ F$$

sending e_i to x_i, and this morphism is an epimorphism.

Now we 'kill the kernel of ϵ'. We consider the set D of all triples (G,x,y) where G runs through the set of all closed subgroups of $O(n)$ for all $n\geq 1$, and (x,y) runs through all pairs in $\pi_0^G(T)\times\pi_0^G(T)$ such that $\epsilon(x)=\epsilon(y)$ in $F(G)$. For every triple (G,x,y) in D we choose a G-representation V such that both x and y are represented by a G-fixed point of $T(V)$. Then we form the pushout

$$\begin{tikzcd}
\amalg_{(G,x,y)\in D}\{0,1\}\times \mathbf{L}_{G,V} \arrow[r, "\mathrm{incl}"] \arrow[d] & \amalg_{(G,x,y)\in D}[0,1]\times \mathbf{L}_{G,V} \arrow[d] \\
T \arrow[r, "g"'] & T'
\end{tikzcd}$$

On the summand indexed by (G,x,y), the left vertical map is adjoint to choices of points in $T(V)^G$ that represent x respectively y. For every K-representation W, the map $g(W)^K: T(W)^K\longrightarrow T'(W)^K$ is then surjective on path components. So $\underline{\pi}_0(g):\underline{\pi}_0(T)\longrightarrow\underline{\pi}_0(T')$ is an epimorphism of Rep-functors. On the other hand, for all (G,x,y) in D the classes x and y have the same image under the map $\pi_0^G(g):\pi_0^G(T)\longrightarrow \pi_0^G(T')$, by construction. So $\epsilon:\underline{\pi}_0(T)\longrightarrow F$ descends to an isomorphism of Rep-functors $\underline{\pi}_0(T')\cong F$. We can then take $F^\sharp = T'_{\mathrm{dis}}$, where $\kappa: T'\longrightarrow T'_{\mathrm{dis}}$ is as in part (i).

(iii) To show the existence of a right adjoint it suffices to prove that for every Rep-functor F the functor

$$\underline{\pi}_0\ :\ \mathrm{Ho}(spc)\ \longrightarrow\ (\mathrm{sets})\ ,\quad X\ \longmapsto\ (\text{Rep-functors})(\underline{\pi}_0(X),F)$$

is representable. But that follows by combining part (ii) and Proposition 5.20. □

CONSTRUCTION 5.22. Given two orthogonal spaces X and Y, we endow the equivariant homotopy sets with a pairing

$$\times \ : \ \pi_0^G(X) \ \times \ \pi_0^G(Y) \ \longrightarrow \ \pi_0^G(X \boxtimes Y) \ , \tag{5.23}$$

where G is any compact Lie group. We suppose that V and W are G-representations and $x \in X(V)^G$ and $y \in Y(W)^G$ are fixed points that represent classes in $\pi_0^G(X)$ respectively $\pi_0^G(Y)$. We denote by $x \times y$ the image of the G-fixed point (x, y) under the G-map

$$i_{V,W} \ : \ X(V) \times Y(W) \ \longrightarrow \ (X \boxtimes Y)(V \oplus W)$$

that is part of the universal bimorphism. If $\varphi : V \longrightarrow V'$ and $\psi : W \longrightarrow W'$ are equivariant linear isometric embeddings, then

$$\begin{aligned} X(\varphi)(x) \times Y(\psi)(y) \ &= \ i_{V',W'}(X(\varphi)(x), Y(\psi)(y)) \\ &= \ (X \boxtimes Y)(\varphi \oplus \psi)(i_{V,W}(x, y)) \ = \ (X \boxtimes Y)(\varphi \oplus \psi)(x \times y) \ . \end{aligned}$$

So by Proposition 5.9 the classes $\langle x \times y \rangle$ and $\langle X(\varphi)(x) \times Y(\psi)(y) \rangle$ coincide in $\pi_0^G(X \boxtimes Y)$. The upshot is that the assignment

$$[x] \times [y] \ = \ \langle x \times y \rangle \ \in \ \pi_0^G(X \boxtimes Y)$$

is well-defined.

The pairings of equivariant homotopy sets have several expected properties that we summarize in the next proposition.

PROPOSITION 5.24. *Let G be a compact Lie groups and X, Y and Z orthogonal spaces.*

(i) (Unitality) *Let $1 \in \pi_0^G(\mathbf{1})$ be the unique element. Then $1 \times x = x = x \times 1$ for all $x \in \pi_0^G(X)$.*

(ii) (Associativity) *For all classes $x \in \pi_0^G(X)$, $y \in \pi_0^G(Y)$ and $z \in \pi_0^G(Z)$ the relation*

$$a_*((x \times y) \times z) \ = \ x \times (y \times z)$$

holds in $\pi_0^G(X \boxtimes (Y \boxtimes Z))$, where $a : (X \boxtimes Y) \boxtimes Z \cong X \boxtimes (Y \boxtimes Z)$ is the associativity isomorphism.

(iii) (Commutativity) *For all classes $x \in \pi_0^G(X)$ and $y \in \pi_0^G(Y)$ the relation*

$$\tau_*^{X,Y}(x \times y) \ = \ y \times x$$

holds in $\pi_0^G(Y \boxtimes X)$, where $\tau^{X,Y} : X \boxtimes Y \longrightarrow Y \boxtimes X$ is the symmetry isomorphism of the box product.

(iv) (Restriction) *For all classes $x \in \pi_0^G(X)$ and $y \in \pi_0^G(Y)$ and all continuous homomorphisms $\alpha : K \longrightarrow G$ the relation*

$$\alpha^*(x) \times \alpha^*(y) \ = \ \alpha^*(x \times y)$$

holds in $\pi_0^K(X \boxtimes Y)$.

PROOF. The unitality property (i), the associativity property (ii) and compatibility with restriction (iv) are straightforward from the definitions. Part (iii) exploits that the square

$$\begin{tikzcd} X(V) \times Y(W) \arrow[r, "i_{V,W}"] \arrow[d, "\text{twist}"'] & (X \boxtimes Y)(V \oplus W) \arrow[d, "\tau_{X,Y} \circ (\chi_{V,W})_*"] \\ Y(W) \times X(V) \arrow[r, "i_{W,V}"'] & (Y \boxtimes X)(W \oplus V) \end{tikzcd}$$

commutes. The image of (x, y) under the upper right composite represents $x \times y$, whereas the image of (y, x) under the lower left composite represents $y \times x$, so $\tau_*^{X,Y}(x \times y) = y \times x$. $\square$

Remark 5.25 (External versus internal products). If G and K are two compact Lie groups, we can define an 'external' form of the pairing (5.23) as the composite

$$\pi_0^G(X) \times \pi_0^K(Y) \xrightarrow{p_1^* \times p_2^*} \pi_0^{G\times K}(X) \times \pi_0^{G\times K}(Y) \xrightarrow{\times} \pi_0^{G\times K}(X \boxtimes Y) \ , \tag{5.26}$$

where $p_1 : G \times K \longrightarrow G$ and $p_2 : G \times K \longrightarrow K$ are the two projections. These external pairings also satisfy various naturality, unitality, associativity and commutativity properties that we do not spell out. On the other hand, the internal pairing (5.23) can be recovered from the external products (5.26) by taking $G = K$ and restricting along the diagonal embedding $\Delta_G : G \longrightarrow G \times G$. Indeed, the $p_1 \circ \Delta_G = p_2 \circ \Delta_G = \mathrm{Id}_H$, and hence

$$\Delta_G^*(p_1^*(x) \times p_2^*(y)) \ = \ \Delta_G^*(p_1^*(x)) \times \Delta_G^*(p_2^*(y)) \ = \ x \times y \ .$$

Theorem 3.38 (i) and the fact that the functor π_0^G commutes with finite products imply:

Corollary 5.27. *For every compact Lie group G, and all orthogonal spaces X and Y the three maps*

$$\pi_0^G(X) \times \pi_0^G(Y) \xrightarrow{\times} \pi_0^G(X \boxtimes Y) \xrightarrow{(\rho_{X,Y})_*} \pi_0^G(X \times Y) \xrightarrow{((p_1)_*,(p_2)_*)} \pi_0^G(X) \times \pi_0^G(Y)$$

are bijections, where $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ are the projections. Moreover, the composite is the identity.

Construction 5.28 (Infinite box products). We close by proving a generalization of the previous corollary to 'infinite box products' of based orthogonal spaces, but we first have to clarify what we mean by that. We let I be an indexing set and $\{X_i\}_{i\in I}$ a family of based orthogonal spaces, i.e., each equipped with a distinguished point $x_i \in X_i(0)$. If $K \subset J$ are two nested, *finite* subsets of I, then the basepoints of X_k for $k \in J - K$ provide a morphism

$$\boxtimes_{k\in K} X_k \ \longrightarrow \ \boxtimes_{j\in J} X_j \ . \tag{5.29}$$

In terms of the universal property of the box product, this morphism arises from the maps

$$\prod_{k\in K} X_k(V_k) \ \longrightarrow \ \prod_{k\in K} X_k(V_k) \times \prod_{j\in J-K} X_j(0) \ \longrightarrow \ (\boxtimes_{j\in J} X_j)\left(\bigoplus_{k\in K} V_k\right) \ ,$$

where the second map is part of the universal multimorphism. We can thus define the *infinite box product* as the colimit of the finite box products over the filtered poset of finite subsets of I:

$$\boxtimes'_{i\in I} X_i \ = \ \mathrm{colim}_{J\subset I, |J|<\infty} \ , \boxtimes_{j\in J} X_j \ .$$

If I happens to be finite, then this recovers the iterated box product.

The distinguished basepoint of X_i represents a distinguished basepoint in the equivariant homotopy set $\pi_0^G(X_i)$ for every compact Lie group G. In fact, these point all arise from the basepoint in $\pi_0^e(X_i)$ by restriction along the unique homomorphism $G \longrightarrow e$. The *weak product* $\prod'_{i\in I} \pi_0^G(X_i)$ is the subset of the product consisting of all tuples $(x_i)_{i\in I}$ with the property that almost all x_i are the distinguished basepoint. Equivalently, the weak product is the filtered colimits, over the poset of finite subsets of I, of the finite products.

If we iterate the pairing (5.23) it provides a multi-pairing

$$\prod_{j\in J} \pi_0^G(X_j) \ \longrightarrow \ \pi_0^G(\boxtimes_{j\in J} X_j)$$

for every finite set J. Passing to colimits over finite subsets of I on both sides yields a map

$$\prod\nolimits'_{i\in I} \pi_0^G(X_i) \ \longrightarrow \ \pi_0^G(\boxtimes'_{i\in I} X_i) \ . \tag{5.30}$$

PROPOSITION 5.31. *Let I be an indexing set and $\{X_i\}_{i\in I}$ a family of based orthogonal spaces. Then for every compact Lie group G the map* (5.30) *is bijective.*

PROOF. For every $k \in I$ we define a 'projection'

$$\Pi_k \ : \ \boxtimes'_{i\in I} X_i \ \longrightarrow \ X_k$$

as follows. Since the infinite box product is defined as a colimit, we must specify the 'restriction' of Π_k to $\boxtimes_{j\in J} X_j$ for every finite subset J of I, compatibly as J increases. For $k \notin J$ we defined this restriction as the constant morphism factoring through the basepoint of X_k. For $k \in J$ we define the restriction

$$\boxtimes_{j\in J} X_j \ \longrightarrow \ X_k$$

as the morphism corresponding, under the universal property of the box product, to the multi-morphism with components

$$\prod_{j\in J} X_j(V_j) \ \xrightarrow{\mathrm{proj}_k} \ X_k(V_k) \ \xrightarrow{X_k(\mathrm{incl})} \ X_k(\bigoplus_{j\in J} V_j) \ .$$

Then the composite

$$\prod\nolimits'_{i\in I} \pi_0^G(X_i) \ \xrightarrow{(5.30)} \ \pi_0^G(\boxtimes'_{i\in I} X_i) \ \xrightarrow{\pi_0^G(\Pi_k)} \ \pi_0^G(X_k)$$

is the projection onto the k-th factor. So if two tuples in the weak product have the same image under the map (5.30), they coincide. This shows injectivity.

Now we show surjectivity. Every element of $\pi_0^G(\boxtimes'_{i\in I} X_i)$ is represented by a G-fixed point of $(\boxtimes'_{i\in I} X_i)(V)$ for some G-representation V. Colimits of orthogonal space are formed objectwise, so $(\boxtimes'_{i\in I} X_i)(V)$ is a colimit, over finite subsets J of I, of the spaces $(\boxtimes_{j\in J} X_j)(V)$. For every nested pair of finite subsets $K \subset J$ of I the morphism (5.29) has a retraction, by 'projection'. So at every inner product space V, the map

$$(\boxtimes_{k\in K} X_k)(V) \ \longrightarrow \ (\boxtimes_{j\in J} X_j)(V)$$

is a closed embedding by Proposition A.1.19 (i). For fixed V, the colimit $(\boxtimes'_{i\in I} X_i)(V)$ in the category $\mathbf{T}$ of compactly generated spaces can thus be calculated in the ambient category of all topological spaces, by Proposition A.1.9 (ii). In particular, every G-fixed point of $(\boxtimes'_{i\in I} X_i)(V)$ arises from a G-fixed point of $(\boxtimes_{j\in J} X_j)(V)$ for some finite subset J of I. In other words, the canonical map

$$\mathrm{colim}_{J\subset I, |J|<\infty} \pi_0^G(X_j) \ \longrightarrow \ \pi_0^G(\boxtimes'_{i\in I} X_i)$$

is surjective. For finite sets J the map $\prod_{j\in J} \pi_0^G(X_j) \longrightarrow \pi_0^G(\boxtimes_{j\in J} X_j)$ is bijective by Corollary 5.27, so this shows surjectivity. □

We let R be an orthogonal monoid space, N a right R-module and X a left R-module. We recall that the 'relative' box product $N \boxtimes_R X$ is the orthogonal space defines as a coequalizer of the two morphisms

$$N \boxtimes R \boxtimes X \quad \underset{N\boxtimes \mathrm{act}}{\overset{\mathrm{act}\boxtimes X}{\rightrightarrows}} \quad N \boxtimes X$$

In particular, the relative box product comes with a canonical morphism $N \boxtimes X \longrightarrow N \boxtimes_R X$ which we call the 'projection'.

PROPOSITION 5.32. *Let R be an orthogonal monoid space and N a right R-module that is cofibrant in the $\mathcal{A}ll$-global model structure of Corollary* 4.16 (i)*. For every left R-module X and every compact Lie group G the composite*

$$\pi_0^G(N) \times \pi_0^G(X) \ \xrightarrow{\times} \ \pi_0^G(N \boxtimes X) \ \xrightarrow{\mathrm{proj}_*} \ \pi_0^G(N \boxtimes_R X)$$

is surjective.

Proof. In a first step we show an auxiliary statement. We consider a pushout square of orthogonal spaces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

such that f is objectwise a closed embedding. We claim that then for every compact Lie group G the map

$$\pi_0^G(k) + \pi_0^G(h) \ : \ \pi_0^G(C) \amalg \pi_0^G(B) \ \longrightarrow \ \pi_0^G(D)$$

is surjective. To see this we represent any given element of $\pi_0^G(D)$ by a G-fixed point $D(V)^G$, for some G-representation V. The square

$$\begin{tikzcd} A(V) \arrow[r, "f(V)"] \arrow[d, "g(V)"'] & B(V) \arrow[d, "h(V)"] \\ C(V) \arrow[r, "k(V)"'] & D(V) \end{tikzcd}$$

is a pushout square of G-spaces, and the map $f(V)$ is a closed embedding by hypothesis. The pushout can thus be calculated in the category of all topological spaces (not necessarily compactly generated), see Proposition A.1.9 (i). In particular, the pushout is created by the forgetful functor to sets. So the given G-fixed in $D(V)^G$ is in the image of $B(V)^G$ or $C(V)^G$. This proves the claim.

Now we proceed to prove the proposition. Since the $\mathcal{A}ll$-global model structure on the category of right R-modules is obtained by lifting the global model structure of orthogonal spaces along the free and forgetful adjoint functor pair, every cofibrant right R-module is a retract of an R-module that arises as the colimit of a sequence

$$\emptyset = M_0 \ \longrightarrow \ M_1 \ \longrightarrow \dots \ \longrightarrow \ M_k \ \longrightarrow \ \dots \tag{5.33}$$

in which each M_k is obtained from M_{k-1} as a pushout

$$\begin{tikzcd} A_k \boxtimes R \arrow[r, "f_k \boxtimes R"] \arrow[d] & B_k \boxtimes R \arrow[d] \\ M_{k-1} \arrow[r] & M_k \end{tikzcd}$$

for some flat cofibration $f_k : A_k \longrightarrow B_k$ of orthogonal spaces. For example, f_k can be chosen as a disjoint union of morphisms in the set I^{str} of generating flat cofibrations. We show by induction on k that each module M_k satisfies the conclusion of the proposition. The induction starts with the empty R-module, where there is nothing to show. Now we suppose that M_{k-1} satisfies the proposition, and we claim that the same is true for M_k. To see this we exploit that $M_k \boxtimes_R X$ is a pushouts of the diagram of orthogonal spaces:

$$M_{k-1} \boxtimes_R X \ \longleftarrow \ A_k \boxtimes X \ \xrightarrow{f_k \boxtimes X} \ B_k \boxtimes X$$

Here we have exploited that $(A_k \boxtimes R) \boxtimes_R X$ is naturally isomorphic to $A_k \boxtimes X$. Since the morphism f_k is a flat cofibration, it is an h-cofibration (by Corollary A.1.18 (iii)), and so the morphisms $f_k \boxtimes R$ and $f_k \boxtimes X$ are h-cofibrations.

Since $f_k \boxtimes R$ and $f_k \boxtimes X$ are h-cofibrations, they are objectwise h-cofibrations of spaces, hence objectwise closed embeddings by Proposition A.1.19 (ii). So the two maps

$$\begin{array}{ccccc} \pi_0^G(M_{k-1}) & \amalg & \pi_0^G(B_k \boxtimes R) & \longrightarrow & \pi_0^G(M_k) \qquad \text{and} \\ \pi_0^G(M_{k-1} \boxtimes_R X) & \amalg & \pi_0^G(B_k \boxtimes X) & \longrightarrow & \pi_0^G(M_k \boxtimes_R X) \end{array}$$

are surjective by the initial auxiliary remark. The composite

$$\pi_0^G(M_{k-1}) \times \pi_0^G(X) \xrightarrow{\times} \pi_0^G(M_{k-1} \boxtimes X) \xrightarrow{\mathrm{proj}_*} \pi_0^G(M_{k-1} \boxtimes_R X)$$

is surjective by the inductive hypothesis and the map

$$\pi_0^G(B_k) \times \pi_0^G(X) \xrightarrow{\times} \pi_0^G(B_k \boxtimes X)$$

is bijective by Corollary 5.27. We arrive at a commutative square

$$\begin{array}{ccc} \pi_0^G(M_{k-1}) \times \pi_0^G(X) \amalg \pi_0^G(B_k) \times \pi_0^G(X) & \longrightarrow & \pi_0^G(M_k) \times \pi_0^G(X) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{proj}_* \circ \times} \\ \pi_0^G(M_{k-1} \boxtimes_R X) \amalg \pi_0^G(B_k \boxtimes X) & \longrightarrow & \pi_0^G(M_k \boxtimes_R X) \end{array}$$

in which the left vertical and lower horizontal maps are surjective. So the right vertical map is surjective as well, and this proves that M_k satisfies the conclusion of the proposition.

Now we let M be a colimit of the sequence (5.33). Then $M \boxtimes_R X$ is a colimit of the sequence $M_k \boxtimes_R X$. Moreover, since $f_k \boxtimes R$ is an h-cofibration, so is its cobase change $M_{k-1} \longrightarrow M_k$. So the sequence whose colimit is $M \boxtimes_R X$ consists of h-cofibrations of orthogonal spaces, which are in particular objectwise h-cofibrations of spaces, hence objectwise closed embeddings by Proposition A.1.19 (ii). So the canonical map

$$\mathrm{colim}_k\ \pi_0^G(M_k \boxtimes_R X) \longrightarrow \pi_0^G(M \boxtimes_R X)$$

is bijective. In particular, every element of $\pi_0^G(M \boxtimes_R X)$ arises from $\pi_0^G(M_k \boxtimes_R X)$ for some k, and hence from $\pi_0^G(M_k) \times \pi_0^G(X)$. So M satisfies the conclusion of the proposition. The class of right R-modules that satisfy the proposition is closed under retracts, so it contains all cofibrant right R-modules. $\square$

CHAPTER II

Ultra-commutative monoids

This chapter is devoted to the study of ultra-commutative monoids, including a global model structure, an algebraic study of their homotopy operations, and many examples. Orthogonal monoid spaces are the lax monoidal continuous functors from the category $\mathbf{L}$ to the category of spaces, compare Definition I.4.15. Orthogonal monoid spaces with strictly commutative multiplication (i.e., the lax *symmetric* monoidal continuous functors) play a special role, and we honor this by special terminology, referring to them as *ultra-commutative monoids*.

I want to motivate the adjective 'ultra-commutative'. In various contexts of homotopy theory, highly structured multiplications that are only associative or commutative up to higher coherence homotopies can in fact be rigidified to multiplications that are strictly associative or possibly strictly commutative. One example of this is the fact that E_∞ spaces can be rigidified to strictly commutative $\mathcal{I}$-space monoids [**130**, Thm. 1.3]; another example is the fact that E_∞ ring objects internal to symmetric spectra can be rigidified to strictly commutative symmetric ring spectra, see for example Theorem 1.4 and the following paragraph of [**48**]. More to the point of our present discussion, in [**98**, Thm. 1.3] Lind establishes a Quillen equivalence between the non-equivariant homotopy theory of commutative orthogonal monoid spaces (there called 'commutative $\mathcal{I}$-FCPs') and the non-equivariant homotopy theory of E_∞-spaces, i.e., spaces with an action of the linear isometries operad.

Our use of the word 'ultra-commutative' is intended as a warning that the slogan 'E_∞=commutative' is no longer true in equivariant or global contexts. More specifically, one can consider orthogonal monoid spaces with an action of an E_∞-operad; up to non-equivariant equivalence, these objects model E_∞-spaces, and they can be replaced by equivalent strictly commutative orthogonal monoid spaces. The analogous statement for global equivalences is false, i.e., E_∞-orthogonal spaces cannot in general be replaced by globally equivalent strictly commutative orthogonal monoid spaces. In fact, the definition of power operations and transfer maps requires strict commutativity of the multiplication, and Remark 4.27 illustrates how the lack of transfers obstructs ultra-commutativity.

The study of orthogonal monoid spaces goes back to Boardman and Vogt [**19**], who introduce them as a 'delooping machine' (in a non-equivariant context). More precisely, they show that for every ultra-commutative monoid R the space $R(\mathbb{R}^\infty)$ has the structure of an 'E-space' (nowadays called an E_∞-space) and, if in addition $\pi_0(R(\mathbb{R}^\infty))$ is a group, then $R(\mathbb{R}^\infty)$ is an infinite loop space. Ultra-commutative monoids also appear, with an extra pointset topological hypothesis and under the name $\mathscr{I}_*$-prefunctor, in [**107**, IV Def. 2.1]; in [**98**], they are studied under the name 'commutative $\mathcal{I}$-FCPs'.

In Section 1 we formally define ultra-commutative monoids and establish the global model structure. Section 2 is devoted to the algebraic structure on the homotopy group Rep-functor $\underline{\pi}_0(R)$ of an ultra-commutative monoid. We refer to this structure as a 'global power monoids'; it consist of an abelian monoid structure on the set $\pi_0^G(R)$ for every compact Lie group G, natural for restriction along continuous homomorphisms, and an additional structure that can equivalently be encoded as power operations (see Definition 2.8) or as transfer maps (see Construction 2.32). A third way to encode the additional operations is via coalgebras over a certain cotriple of 'exponential sequences. In this section we also show that these operations are the entire natural structure, and that every global power monoid can be realized by an ultra-commutative monoid. Section 3 collects various examples of ultra-commutative monoids: among these are

examples made from the infinite families of classical Lie groups (orthogonal, special orthogonal, unitary, special unitary, symplectic, spin and spin^c); examples consisting of Grassmannians under direct sum of subspaces (in real, oriented, complex and quaternionic flavors); examples made from Grassmannians under tensor product of subspaces (in a real and complex version); and ultra-commutative multiplicative models for global classifying spaces of abelian compact Lie groups.

Section 4 is a case study of how non-equivariant homotopy types can 'fold up' into many different global homotopy types. We define, discuss and compare different ultra-commutative and E_∞-orthogonal monoid spaces whose underlying non-equivariant homotopy type is BO, a classifying space for the infinite orthogonal group; in all examples we also identify the associated global power monoids and fixed point spaces. We end the section with a global, highly structured version of Bott periodicity. Section 5 discusses 'units' and 'group completion' of ultra-commutative monoids. The two constructions are dual to each other, and they are homotopically right adjoint respectively left adjoint to the inclusion of group-like ultra-commutative monoids. On the algebraic level of global power monoids, the topological constructions pick out the invertible elements and perform the algebraic group completion respectively. A naturally occurring example of a global group completion is the morphism from the additive Grassmannians to the periodic global version of BO.

1. Global model structure

In this section we formally define ultra-commutative monoids and construct a model structure on the category of ultra-commutative monoids with global equivalences as the weak equivalences, see Theorem 1.13. As a step along the way we build the positive global model structure on the category of orthogonal space in Proposition 1.4, and we introduce free ultra-commutative monoids in Example 1.7.

Definition 1.1. An *ultra-commutative monoid* is a commutative orthogonal monoid space.

As we explained after Definition I.4.15, the data of an ultra-commutative monoid is the same as that of a lax symmetric monoidal continuous functor from the category $\mathbf{L}$ (under orthogonal direct sum) to the category $\mathbf{T}$ (under cartesian product).

Remark 1.2. One can think about an ultra-commutative monoid as encoding a collection of E_∞-G-spaces, one for every compact Lie group G, compatible under restriction. If R is a closed orthogonal space and G a compact Lie group, then the G-equivariant homotopy type encoded in R can be accessed as the 'underlying G-space'

$$R(\mathcal{U}_G) \;=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)}\, R(V)\ .$$

The additional structure of an ultra-commutative monoid on R gives rise to an action of a specific E_∞-G-operad on this G-space, namely the linear isometries operad $\mathcal{L}(\mathcal{U}_G)$ of the complete G-universe $\mathcal{U}_G$. The n-th space of this operad is the space $\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)$ of linear isometric embeddings (not necessarily equivariant) of $\mathcal{U}_G^n$ into $\mathcal{U}_G$. The group G acts on $\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)$ by conjugation and the operad structure is by direct sum and composition of linear isometric embeddings. The symmetric group Σ_n permutes the summands in the source. The space $\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)$ is G-equivariantly contractible by [**97**, II Lemma 1.5], and the Σ_n-action is free; in fact, $\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)$ even has the $(G\times\Sigma_n)$-equivariant homotopy type of a universal space for (G,Σ_n)-bundles.

By simultaneous passage to colimit over $s(\mathcal{U}_G)$ in all n variables, the iterated multiplication maps

$$R(V_1)\times\cdots\times R(V_n) \;\longrightarrow\; R(V_1\oplus\cdots\oplus V_n)$$

give rise to a multiplication map $\mu_{(n)} : R(\mathcal{U}_G)^n \longrightarrow R(\mathcal{U}_G^n)$. A linear isometric embedding $\psi:\mathcal{U}\longrightarrow\mathcal{U}'$ between countably infinite dimensional inner product spaces induces a map $R(\psi):R(\mathcal{U})\longrightarrow R(\mathcal{U}')$; the resulting 'action map'

$$\mathbf{L}(\mathcal{U},\mathcal{U}')\times R(\mathcal{U}) \;\longrightarrow\; R(\mathcal{U}')\ ,\quad (\psi,y)\;\longmapsto\; R(\psi)(y)$$

is continuous. The operadic action map is then simply the composite

$$\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)\times R(\mathcal{U}_G)^n \xrightarrow{\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)\times\mu_{(n)}} \mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)\times R(\mathcal{U}_G^n) \xrightarrow{\text{act}} R(\mathcal{U}_G)\ .$$

As is well-known from similar contexts (for example, the stable model structure for commutative orthogonal ring spectra), model structures cannot be lifted naively to multiplicative objects with strictly commutative products. The solution, as usual, is to lift a 'positive' version of the global model structure in which the values at the trivial inner product space are homotopically meaningless and where the fibrant objects are the 'positive static' orthogonal spaces. There is a positive version of the $\mathcal{F}$-global model structure for every global family $\mathcal{F}$, and it lifts to an *$\mathcal{F}$-global model structure* on the category of ultra-commutative monoids. To simplify the exposition we only discuss the case $\mathcal{F}=\mathcal{A}ll$ of the maximal global family.

Definition 1.3. A morphism $f:A\longrightarrow B$ of orthogonal spaces is a *positive cofibration* if it is a flat cofibration and the map $f(0):A(0)\longrightarrow B(0)$ is a homeomorphism. An orthogonal space Y is *positively static* if for every compact Lie group G, every faithful G-representation V with $V\neq 0$ and an arbitrary G-representation W the structure map

$$Y(i_{V,W})\ :\ Y(V)\ \longrightarrow\ Y(V\oplus W)$$

is a G-weak equivalence.

If G is a non-trivial compact Lie group, then any faithful G-representation is automatically non-trivial. So a positively static orthogonal space is static (in the absolute sense) if the structure map $Y(0)\longrightarrow Y(\mathbb{R})$ is a non-equivariant weak equivalence.

Proposition 1.4 (Positive global model structure). *The global equivalences and positive cofibrations are part of a proper topological model structure, the* positive global model structure *on the category of orthogonal spaces. A morphism $f:X\longrightarrow Y$ of orthogonal spaces is a fibration in the positive global model structure if and only if for every compact Lie group G, every faithful G-representation V with $V\neq 0$ and every equivariant linear isometric embedding $\varphi:V\longrightarrow W$ the map $f(V)^G:X(V)^G\longrightarrow Y(V)^G$ is a Serre fibration and the square of G-fixed point spaces*

$$\begin{tikzcd} X(V)^G \arrow[r, "X(\varphi)^G"] \arrow[d, "f(V)^G"'] & X(W)^G \arrow[d, "f(W)^G"] \\ Y(V)^G \arrow[r, "Y(\varphi)^G"'] & Y(W)^G \end{tikzcd}$$

is homotopy cartesian. The fibrant objects in the positive global model structure are the positively static orthogonal spaces. The model structure is monoidal with respect to the box product of orthogonal spaces.

Proof. We start by establishing a *positive strong level model structure*. We call a morphism $f:X\longrightarrow Y$ of orthogonal spaces a *positive strong level equivalence* (respectively *positive strong level fibration*) if for every inner product space V with $V\neq 0$ the map $f(V):X(V)\longrightarrow Y(V)$ is an $O(V)$-weak equivalence (respectively an $O(V)$-fibration). Then we claim that the positive strong level equivalences, positive strong level fibrations and positive cofibrations form a model structure on the category of orthogonal spaces.

The proof is another application of the general construction method for level model structures in Proposition A.3.27. Indeed, we let $\mathcal{C}(0)$ be the degenerate model structure on the category $\mathbf{T}$ of spaces in which every morphism is a weak equivalence and a fibration, but only the isomorphisms are cofibrations. For $m\geq 1$ we let $\mathcal{C}(m)$ be the projective model structure (for the family of all closed subgroups) on the category of $O(m)$-spaces, compare Proposition A.2.10. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition A.3.27 become the positive strong level equivalences, positive strong level fibrations and positive cofibrations. The consistency

condition (Definition A.3.26) is now strictly weaker than for the strong level model structure, so it holds. The verification that the model structure is proper and topological is the same as for the strong level model structure in Proposition I.3.11. The positive strong level model structure is cofibrantly generated; we can simply take the same sets of generating cofibrations and generating acyclic cofibrations as for the strong level model structure, *except* that we omit all morphisms freely generated in level 0.

We obtain the positive global model structure for orthogonal spaces by 'mixing' the positive strong level model structure with the global model structure of Theorem I.3.22. Every positive strong level equivalence is a global equivalence and every positive cofibration is a flat cofibration. The global equivalences and the positive cofibrations are part of a model structure by Cole's mixing theorem [**37**, Thm. 2.1], which is our first claim. By [**37**, Cor. 3.7] (or rather its dual formulation), an orthogonal space is fibrant in the positive global model structure if and only if it is weakly equivalent in the positive strong level model structure to a static orthogonal space; this is equivalent to being positively static. The positive global model structure is again proper by Propositions 4.1 and 4.2 of [**37**]. The proof that this model structure is topological and monoidal is similar as for the global model structure. □

Now we work towards the main result of this section, the global model structure for ultra-commutative monoids. Before we start with the homotopical considerations, we get some of the necessary category theory out of the way. For a moment we consider more generally any symmetric monoidal model category $\mathcal{C}$ with monoidal product $\boxtimes$ and unit object I. We can then study operads in $\mathcal{C}$ and algebras over a fixed operad. The following (co-)completeness and preservation results can be found in [**128**, Prop. 2.3.5] or [**53**, Prop. 3.3.1].

Proposition 1.5. *Let $(\mathcal{C}, \boxtimes, I)$ be a complete and cocomplete symmetric monoidal category such that the monoidal product preserves colimits in each variable. Let $\mathcal{P}$ be an operad in $\mathcal{C}$. Then the category of $\mathcal{P}$-algebras is complete and cocomplete. Moreover, the forgetful functor from the category of $\mathcal{P}$-algebras to the underlying category $\mathcal{C}$ creates all limits, all filtered colimits and those coequalizers that are reflexive in the underlying category $\mathcal{C}$.*

We let $\mathcal{C}om$ denote the incarnation of the commutative operad internal to the category of orthogonal spaces, under box product. So for every $n \geq 0$ the orthogonal space $\mathcal{C}om(n)$ of n-ary operations is constant with value a one-point space. Equivalently, $\mathcal{C}om$ is a terminal operad in orthogonal spaces. Endowing an orthogonal space with an ultra-commutative multiplication is the same as giving it an algebra structure over the commutative operad $\mathcal{C}om$. More formally, the category of ultra-commutative monoids is isomorphic to the category of $\mathcal{C}om$-algebras. So Proposition 1.5 has the following special case:

Corollary 1.6. *The category of ultra-commutative monoids is complete and cocomplete. The forgetful functor from the category of ultra-commutative monoids to the category of orthogonal spaces creates all limits, all filtered colimits and those coequalizers that are reflexive in the category of orthogonal spaces.*

We will establish this model structure as a special case of a lifting theorem for model structures to categories of commutative monoids that was formulated by White [**175**, Thm. 3.2]. Like its predecessor for associative monoids [**133**, Thm. 4.1 (3)], the input is a cofibrantly generated symmetric monoidal model category that satisfies the monoid axiom. However, lifting a model structure to *commutative* monoids is more subtle and needs extra hypotheses; the essence of the additional condition is that, loosely speaking, 'symmetric powers must be homotopy invariant'. Before White, Gorchinskiy and Guletskii [**60**] have also studied symmetric power constructions in a symmetric monoidal model category, and there is a substantial overlap in the arguments of [**60**] and [**175**].

To simplify the exposition we follow the common abuse to suppress the associativity and unit isomorphisms from the notation, i.e., we pretend that the underlying monoidal structure is strict (i.e., a *permutative* structure). We let $i : A \longrightarrow B$ be a $\mathcal{C}$-morphism and arrange the n-fold smash power of i into

an n-dimensional cube $K^n(i)$ in $\mathcal{C}$, i.e., a functor

$$K^n(i) \ : \ \mathcal{P}(\{1,2,\dots,n\}) \ \longrightarrow \ \mathcal{C}$$

from the poset category of subsets of $\{1,2,\dots,n\}$ and inclusions to $\mathcal{C}$. More explicitly, if $S \subseteq \{1,2,\dots,n\}$ is a subset, then the vertex of the cube at S is defined to be

$$K^n(i)(S) \ = C_1 \boxtimes C_2 \boxtimes \cdots \boxtimes C_n$$

with

$$C_i \ = \begin{cases} A & \text{if } i \notin S \\ B & \text{if } i \in S. \end{cases}$$

All morphisms in the cube $K^n(i)$ are $\boxtimes$-products of identities and copies of the morphism $i : A \longrightarrow B$. The initial vertex of the cube is $K^n(i)(\emptyset) = A^{\boxtimes n}$ and the terminal vertex is $K^n(i)(\{1,\dots,n\}) = B^{\boxtimes n}$.

We denote by $Q^n(i)$ the colimit of the punctured cube, i.e., the cube $K^n(i)$ with the terminal vertex removed, and $i^{\square n} : Q^n(i) \longrightarrow K^n(i)(\{1,\dots,n\}) = B^{\boxtimes n}$ is the canonical map. The morphism $i^{\square n}$ is an iterated pushout product morphism. Indeed, for $n = 2$ the cube $K^2(i)$ is a square and looks like

$$\begin{tikzcd} A \boxtimes A \arrow[r, "A\boxtimes i"] \arrow[d, "i\boxtimes A"'] & A \boxtimes B \arrow[d, "i\boxtimes B"] \\ B \boxtimes A \arrow[r, "B\boxtimes i"'] & B \boxtimes B \end{tikzcd}$$

Hence

$$i^{\square 2} \ = \ i \square i \ = \ (B \boxtimes i) \cup (i \boxtimes B) \ : \ B \boxtimes A \cup_{A\boxtimes A} A \boxtimes B \ \longrightarrow \ B \boxtimes B \ .$$

Similarly, $i^{\square 3}$ is the morphism from the colimit of the punctured cube to the terminal vertex of the following cube:

$$\begin{tikzcd} A \boxtimes A \boxtimes A \arrow[rr, "A\boxtimes A\boxtimes i"] \arrow[dr, "A\boxtimes i\boxtimes A"] \arrow[dd, "i\boxtimes A\boxtimes A"'] & & A \boxtimes A \boxtimes B \arrow[dr, "A\boxtimes i\boxtimes B"] \arrow[dd, "i\boxtimes A\boxtimes B"] & \\ & A \boxtimes B \boxtimes A \arrow[rr, "A\boxtimes B\boxtimes i"'] \arrow[dd, "i\boxtimes B\boxtimes A"] & & A \boxtimes B \boxtimes B \arrow[dd, "i\boxtimes B\boxtimes B"] \\ B \boxtimes A \boxtimes A \arrow[rr, "B\boxtimes A\boxtimes i"] \arrow[dr, "B\boxtimes i\boxtimes A"'] & & B \boxtimes A \boxtimes B \arrow[dr, "B\boxtimes i\boxtimes B"] & \\ & B \boxtimes B \boxtimes A \arrow[rr, "B\boxtimes B\boxtimes i"'] & & B \boxtimes B \boxtimes B \end{tikzcd}$$

We observe that the symmetric group Σ_n acts on $Q^n(i)$ and $B^{\boxtimes n}$ by permuting the factors, and the iterated pushout product morphism $i^{\square n} : Q^n(i) \longrightarrow B^{\boxtimes n}$ is Σ_n-equivariant. We recall from [**60**] the notions of *symmetrizable cofibration* and *symmetrizable acyclic cofibration* and from [**175**] the definition of the *strong commutative monoid axiom*.

Example 1.7 (Free ultra-commutative monoids). For every orthogonal space Y and $m \geq 0$ we denote by

$$\mathbb{P}^m(Y) \ = \ Y^{\boxtimes m}/\Sigma_m$$

the m-symmetric power, with respect to the box product, of Y. Then – as for commutative monoids in any cocomplete symmetric monoidal category – the orthogonal space

$$\mathbb{P}(Y) \ = \ \coprod_{m\geq 0} \mathbb{P}^m(Y) \ = \ \coprod_{m\geq 0} Y^{\boxtimes m}/\Sigma_m$$

is a ultra-commutative monoid under the concatenation product, and it is in fact the free ultra-commutative monoid generated by Y.

DEFINITION 1.8. [**60**, Def. 3] Let $\mathcal{C}$ be a symmetric monoidal model category. A morphism $i : A \longrightarrow B$ is a *symmetrizable cofibration* (respectively a *symmetrizable acyclic cofibration*) if the morphism

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\boxtimes n}/\Sigma_n = \mathbb{P}^n(B)$$

is a cofibration (respectively an acyclic cofibration) for every $n \geq 1$.

Since the morphism $i^{\square 1}/\Sigma_1$ is the original morphism i, every symmetrizable cofibration is in particular a cofibration and every symmetrizable acyclic cofibration is in particular an acyclic cofibration.

DEFINITION 1.9. [**175**, Def. 3.4] A symmetric monoidal model category $\mathcal{C}$ satisfies the *strong commutative monoid axiom* if every cofibration is a symmetrizable cofibration and every acyclic cofibration is a symmetrizable acyclic cofibration.

We will now proceed to prove the strong commutative monoid axiom for the positive global model structure of orthogonal spaces with respect to the box product. In other words, in the category of orthogonal spaces, all cofibrations and acyclic cofibrations in the positive global model structure are symmetrizable with respect to the monoidal structure given by the box product.

The next proposition will be used to verify that the generating acyclic cofibrations of the positive global model structure are symmetrizable. We recall from Construction I.3.16 that given a morphism $j : A \longrightarrow B$, the set $\mathcal{Z}(j)$ consists of all pushout product maps

$$i_k \square c(j) \ : \ D^k \times A \cup_{\partial D^k \times A} \partial D^k \times Z(j) \ \longrightarrow \ D^k \times Z(j)$$

for $k \geq 0$ of the sphere inclusions with the mapping cylinder inclusion $c(j) : A \longrightarrow Z(j)$.

PROPOSITION 1.10. *Let $\mathcal{C}$ be a symmetric monoidal topological model category.*

(i) *For every $n \geq 1$ the functor $\mathbb{P}^n$ preserves the homotopy relation on morphisms and it preserves homotopy equivalences.*
(ii) *Let $j : A \longrightarrow B$ be a symmetrizable acyclic cofibration between cofibrant objects. Then for every $k \geq 0$, the pushout product map*

$$i_k \square j \ : \ D^k \times A \cup_{\partial D^k \times A} \partial D^k \times B \ \longrightarrow \ D^k \times B$$

is a symmetrizable acyclic cofibration.
(iii) *Let $j : A \longrightarrow B$ be a morphism between cofibrant objects such that the morphism $\mathbb{P}^n(j) : \mathbb{P}^n(A) \longrightarrow \mathbb{P}^n(B)$ is a weak equivalence for every $n \geq 1$. Then every morphism in the set $\mathcal{Z}(j)$ is a symmetrizable acyclic cofibration.*

PROOF. (i) This is the topological version of [**60**, Lemma 1]. For every space K and every object X of $\mathcal{C}$ the morphism

$$K \times X^{\boxtimes n} \ \xrightarrow{\Delta \times X^{\boxtimes n}} \ K^n \times X^{\boxtimes n} \ \cong \ (K \times X)^{\boxtimes n}$$

is Σ_n-equivariant (with respect to the trivial Σ_n-action on K in the source) and factors over a natural morphism

$$\tilde{\Delta} \ : \ K \times \mathbb{P}^n(X) \ = \ (K \times X^{\boxtimes n})/\Sigma_n \ \longrightarrow \ (K \times X)^{\boxtimes n}/\Sigma_n \ = \ \mathbb{P}^n(K \times X) \ .$$

If $H : [0,1] \times X \longrightarrow Y$ if a homotopy from a morphism $f = H(0,-)$ to another morphism $g = H(1,-)$, then the composite

$$[0,1] \times \mathbb{P}^n(X) \ \xrightarrow{\tilde{\Delta}} \ \mathbb{P}^n([0,1] \times X) \ \xrightarrow{\mathbb{P}^n(H)} \ \mathbb{P}^n(Y)$$

is a homotopy from the morphism $\mathbb{P}^n(f)$ to $\mathbb{P}^n(g)$. So $\mathbb{P}^n$ preserves the homotopy relation, and hence also homotopy equivalences.

(ii) We argue by induction on k. For $k = 0$ the pushout product map $i_0 \Box j$ is isomorphic to j, hence a symmetrizable acyclic cofibration by hypothesis. Now we assume the claim for some k, and deduce it for $k+1$. Since j is a symmetrizable acyclic cofibration between cofibrant objects, the morphism $\mathbb{P}^n(j)$ is a weak equivalence for every $n \geq 1$ by [**60**, Cor. 23]. Since the functors $\mathbb{P}^n$ preserve the homotopy relation and the projections $D^k \times A \longrightarrow A$ and $D^k \times B \longrightarrow B$ are homotopy equivalences, the morphism $\mathbb{P}^n(D^k \times j)$ is a weak equivalence for every $n \geq 1$. So $D^k \times j : D^k \times A \longrightarrow D^k \times B$ is a symmetrizable acyclic cofibration, again by [**60**, Cor. 23]. We write $\partial D^{k+1} = D^k_+ \cup_{\partial D^k} D^k_-$ as the union of the upper and lower hemisphere along the equator. The upper morphism in the pushout square

$$\begin{tikzcd} D^k_+ \times A \arrow[r, "D^k_+ \times j"] \arrow[d] & D^k_+ \times B \arrow[d] \\ \partial D^{k+1} \times A \arrow[r] & \partial D^{k+1} \times A \cup_{D^k_+ \times A} D^k_+ \times B \end{tikzcd}$$

is a symmetrizable acyclic cofibration by the previous paragraph. The class of symmetrizable acyclic cofibrations is closed under cobase change by [**60**, Thm. 7 (A)]; the lower morphism is thus a symmetrizable acyclic cofibration.

The square

$$\begin{tikzcd} D^k_- \times A \cup_{\partial D^k \times A} \partial D^k \times B \arrow[r, "i_k \Box j"] \arrow[d] & D^k_- \times B \arrow[d] \\ \partial D^{k+1} \times A \cup_{D^k_+ \times A} D^k_+ \times B \arrow[r, "i' \Box j"'] & \partial D^{k+1} \times B \end{tikzcd}$$

is a pushout. The upper morphism is a symmetrizable acyclic cofibration by the inductive hypothesis, hence so is the lower morphism, again by stability under cobase change. The morphism $\partial D^{k+1} \times j : \partial D^{k+1} \times A \longrightarrow \partial D^{k+1} \times B$ is thus the composite of two symmetrizable acyclic cofibrations, hence is a symmetrizable acyclic cofibration itself, by [**60**, Thm. 7 (C)]. As a cobase change, the morphism

$$D^{k+1} \times A \ \longrightarrow \ D^{k+1} \times_{\partial D^{k+1} \times A} \partial D^{k+1} \times B$$

is then a symmetrizable acyclic cofibration by [**60**, Thm. 7 (A)]. The induced morphism

$$\mathbb{P}^n(D^{k+1} \times A) \ \longrightarrow \ \mathbb{P}^n(D^{k+1} \times_{\partial D^{k+1} \times A} \partial D^{k+1} \times B)$$

is then a weak equivalence by [**60**, Cor. 23]. Since $\mathbb{P}^n(D^{k+1} \times j) : \mathbb{P}^n(D^{k+1} \times A) \longrightarrow \mathbb{P}^n(D^{k+1} \times B)$ is a weak equivalence, so is the morphism

$$\mathbb{P}^n(i_{k+1} \Box j) \ : \ \mathbb{P}^n(D^{k+1} \times_{\partial D^{k+1} \times A} \partial D^{k+1} \times B) \ \longrightarrow \ \mathbb{P}^n(D^{k+1} \times B) \ .$$

One more time by [**60**, Cor. 23], this shows that $i_{k+1} \Box j$ is a symmetrizable acyclic cofibration. This completes the induction step.

(iii) Since A and B are cofibrant, the mapping cylinder inclusion

$$c(j) \ : \ A \ \longrightarrow \ [0,1] \times A \cup_j B = Z(j)$$

is a cofibration. Moreover, the projection $Z(j) \longrightarrow B$ is a homotopy equivalence, hence so is $\mathbb{P}^n(Z(j)) \longrightarrow \mathbb{P}^n(B)$ for every $n \geq 1$. Since $\mathbb{P}^n(j)$ is a weak equivalence by hypothesis, the morphism $\mathbb{P}^n(c(j)) : \mathbb{P}^n(A) \longrightarrow \mathbb{P}^n(Z(j))$ is a weak equivalence for every $n \geq 1$. So $c(j)$ is a symmetrizable acyclic cofibration by [**60**, Cor. 23]. The claim now follows by applying (ii) to the morphism $c(j)$. $\Box$

Now we can verify the strong commutative monoid axiom for the positive global model structure of orthogonal spaces. The cofibration part (i) is in fact slightly stronger in that it does not need any positivity hypothesis.

THEOREM 1.11. (i) *Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spaces. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\boxtimes n}/\Sigma_n$$

is a flat cofibration. In other words, all cofibrations in the global model structure of orthogonal spaces are symmetrizable.

(ii) *Let $i : A \longrightarrow B$ be a positive flat cofibration of orthogonal spaces that is also a global equivalence. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\boxtimes n}/\Sigma_n$$

is a global equivalence. In other words, all acyclic cofibrations in the positive global model structure of orthogonal spaces are symmetrizable.

PROOF. (i) We recall from the proof of Proposition I.3.11 the set

$$I^{\mathrm{str}} \ = \ \{ \ G_m(i_k \times O(m)/H) \mid m, k \geq 0, H \leq O(m)\}$$

of generating flat cofibrations of orthogonal spaces, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. The set I^{str} detects the acyclic fibrations in the strong level model structure of orthogonal spaces. In particular, every flat cofibration is a retract of an I^{str}-cell complex. By [**60**, Cor. 9] it suffices to show that the generating flat cofibrations in I^{str} are symmetrizable.

The orthogonal space $G_m(K \times O(m)/H)$ is isomorphic to $K \times \mathbf{L}_{H,\mathbb{R}^m}$, so we show more generally that every morphism of the form

$$i_k \times \mathbf{L}_{G,V} \ : \ \partial D^k \times \mathbf{L}_{G,V} \ \longrightarrow \ D^k \times \mathbf{L}_{G,V}$$

is a symmetrizable cofibration, where V is any representation of a compact Lie group G. The symmetrized iterated pushout product

$$(i_k \times \mathbf{L}_{G,V})^{\square n}/\Sigma_n \ : \ Q^n(i_k \times \mathbf{L}_{G,V})/\Sigma_n \ \longrightarrow \ (D^k \times \mathbf{L}_{G,V})^{\boxtimes n}/\Sigma_n \tag{1.12}$$

is isomorphic to

$$\mathbf{L}_{\Sigma_n \wr G, V^n}(i_k^{\square n}) \ : \ \mathbf{L}_{\Sigma_n \wr G, V^n}(Q^n(i_k)) \ \longrightarrow \ \mathbf{L}_{\Sigma_n \wr G, V^n}((D^k)^n) \ ,$$

where

$$i_k^{\square n} \ : \ Q^n(i_k) \ \longrightarrow \ (D^k)^n$$

is the n-fold pushout product of the inclusion $i_k : \partial D^k \longrightarrow D^k$, with respect to the cartesian product of spaces. Here the wreath product $\Sigma_n \wr G$ acts on V^n by

$$(\sigma; g_1, \dots, g_n) \cdot (v_1, \dots, v_n) \ = \ (g_{\sigma^{-1}(1)} v_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(n)} v_{\sigma^{-1}(n)}) \ .$$

The map $i_k^{\square n}$ is Σ_n-equivariant, and we claim that $i_k^{\square n}$ is a cofibration of Σ_n-spaces. One way to see this is to exploit that i_k is homeomorphic to the geometric realization of the inclusion $\iota_k : \partial\Delta[k] \longrightarrow \Delta[k]$ of the boundary of the simplicial k-simplex. So $i_k^{\square n}$ is Σ_n-homeomorphic to the geometric realization of the inclusion $\iota_k^{\square n} : Q^n(\iota_k) \longrightarrow \Delta[k]^n$ of Σ_n-simplicial sets. The geometric realization of an equivariant embedding of simplicial sets is always an equivariant cofibration of spaces, so altogether this shows that $i_k^{\square n}$ is a cofibration of Σ_n-spaces. Proposition A.2.14 (i) then shows that $i_k^{\square n}$ is also a cofibration of $(\Sigma_n \wr G)$-spaces, with respect to the action by restriction along the projection $(\Sigma_n \wr G) \longrightarrow \Sigma_n$. So the morphism (1.12) is a flat cofibration.

(ii) Proposition I.4.8 (iii) describes a set $J_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$ of generating acyclic cofibrations for the global model structure on the category of orthogonal spaces. From this we obtain a set $J^+ \cup K^+$ of generating acyclic cofibration for the *positive* global model structure of Proposition 1.4 by restricting to those morphisms in $J_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$ that are positive cofibrations, i.e., homeomorphisms in level 0; so explicitly, we set

$$J^+ \ = \ \{ \ G_m(j_k \times O(m)/H) \mid m \geq 1, k \geq 0, H \leq O(m)\} \ ,$$

where $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion, and

$$K^+ \;=\; \bigcup_{G,V,W\,:\,V\neq 0} \mathcal{Z}(\rho_{G,V,W})\ ,$$

the set of all pushout products of sphere inclusions i_k with the mapping cylinder inclusions of the global equivalences $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,V}$. Here (G,V,W) runs through a set of representatives of the isomorphism classes of triples consisting of a compact Lie group G, a non-zero faithful G-representation V and an arbitrary G-representation W. By [**60**, Cor. 9] it suffices to show that all morphisms in $J^+ \cup K^+$ are symmetrizable acyclic cofibrations.

We start with a morphism $G_m(j_k \times O(m)/H)$ in J^+. For every $n \geq 1$, the morphism

$$(G_m(j_k \times O(m)/H))^{\Box n}/\Sigma_n$$

is a flat cofibration by part (i), and a homeomorphism in level 0 because $m \geq 1$. Moreover, the morphism j_k is a homotopy equivalence of spaces, so $G_m(j_k \times O(m)/H)$ is a homotopy equivalence of orthogonal spaces; the morphism $\mathbb{P}^n(G_m(j_k \times O(m)/H))$ is then again a homotopy equivalence for every $n \geq 1$, by Proposition 1.10 (i). Then [**60**, Cor. 23] shows that $G_m(j_k \times O(m)/H)$ is a symmetrizable acyclic cofibration. This takes care of the set J^+.

Now we consider the morphisms in the set K^+. Since G acts faithfully on the non-zero inner product space V, the action of the wreath product $\Sigma_n \wr G$ on V^n is again faithful. So the morphism

$$\rho_{\Sigma_n \wr G, V^n, W^n} \;:\; \mathbf{L}_{\Sigma_n \wr G, V^n \oplus W^n} \;\longrightarrow\; \mathbf{L}_{\Sigma_n \wr G, V^n}$$

is a global equivalence by Proposition I.2.10 (ii). By the natural isomorphism

$$\mathbb{P}^n(\mathbf{L}_{G,V}) \;=\; \mathbf{L}_{G,V}^{\boxtimes n}/\Sigma_n \;\cong\; \mathbf{L}_{\Sigma_n \wr G, V^n}\ ,$$

the morphism $\rho_{\Sigma_n \wr G, V^n, W^n}$ is isomorphic to $\mathbb{P}^n(\rho_{G,V,W}) : \mathbb{P}^n(\mathbf{L}_{G,V\oplus W}) \longrightarrow \mathbb{P}^n(\mathbf{L}_{G,V})$, which is thus a global equivalence. Proposition 1.10 (iii) then shows that all morphisms in $\mathcal{Z}(\rho_{G,V,W})$ are symmetrizable acyclic cofibrations. $\Box$

The hypothesis in Theorem 1.11 (ii) that i is a *positive* flat cofibration is really necessary. Indeed, the unique morphism $\rho : \mathbf{L}_{\mathbb{R}} \longrightarrow *$ to the terminal orthogonal space is a global equivalence, and source and target of ρ are flat, but only the source is reduced. Then the mapping cylinder inclusion $c(\rho) : \mathbf{L}_{\mathbb{R}} \longrightarrow C(\mathbf{L}_{\mathbb{R}})$ is a global equivalence between flat orthogonal space, but it is *not* a homeomorphism at 0. And indeed, for no $m \geq 2$ is the morphism $\mathbb{P}^m(\mathbf{L}_{\mathbb{R}}) \longrightarrow \mathbb{P}^m(C(\mathbf{L}_{\mathbb{R}}))$ a global equivalence, because the source is isomorphic to $\mathbf{L}_{\Sigma_n, \mathbb{R}^m} = B_{\mathrm{gl}}\Sigma_m$, whereas the target is homotopy equivalent to the terminal orthogonal space.

Now we put all the pieces together and prove the global model structure for ultra-commutative monoids. We call a morphism of ultra-commutative monoids a *global equivalence* (respectively *positive global fibration*) if the underlying morphism of orthogonal spaces is a global equivalence (respectively fibration in the positive global model structure).

THEOREM 1.13 (Global model structure for ultra-commutative monoids).

(i) *The global equivalences and positive global fibrations are part of a cofibrantly generated, topological model structure on the category of ultra-commutative monoids, the* global model structure.
(ii) *Let $j : R \longrightarrow S$ be a cofibration in the global model structure of ultra-commutative monoids.*
  (a) *The morphism of R-modules underlying j is a cofibration in the global model structure of R-modules of Corollary* I.4.16 (i).
  (b) *The morphism of orthogonal spaces underlying j is an h-cofibration, and hence a closed embedding.*
  (c) *If the underlying orthogonal space of R is flat, then j is a flat cofibration of orthogonal spaces.*
(iii) *The global model structure on ultra-commutative monoids is proper.*

Proof. (i) The positive global model structure of orthogonal spaces (Proposition 1.4) is monoidal and cofibrantly generated. The 'unit axiom' also holds: we let $f : I \longrightarrow *$ be any positive flat replacement of the monoidal unit, the constant one-point orthogonal space. Then for every orthogonal space Y the induced morphism $f \boxtimes X : I \boxtimes Y \longrightarrow * \boxtimes Y$ is a global equivalence by Theorem I.3.38 (ii). The monoid axiom holds by Proposition I.4.14. Cofibrations and acyclic cofibrations are symmetrizable by Theorem 1.11, so the model structure satisfies the commutative monoid axiom. Theorem 3.2 of [**175**] thus shows that the positive global model structure of orthogonal spaces lifts to the category of ultra-commutative monoids.

The global model structure is topological by Proposition A.2.8, where we take $\mathcal{G}$ as the set of free ultra-commutative monoids $\mathbb{P}(L_{H,\mathbb{R}^m})$ for all $m \geq 1$ and all closed subgroups H of $O(m)$.

(ii) For (a) we recall that the global model structure on the category of R-modules is lifted, via the free and forgetful adjoint functor pair, from the absolute global model structure of Theorem I.3.22. By Corollary I.4.16 (i) and (ii) this model structure of R-modules is a cofibrantly generated monoidal model category that satisfies the monoid axiom. Moreover, the unit object R is cofibrant; for this it is relevant that we have lifted the *absolute* model structure (as opposed to the *positive model structure*). We claim that all cofibrations in this model structure are symmetrizable with respect to the box product of R-modules. By [**60**, Cor. 9] it suffices to show this for a set of generating cofibrations, which can be taken of the form $R \boxtimes i$ for i in a set of flat cofibrations of orthogonal spaces (for example the set I^{str} defined in the proof of the strong level model structure, Proposition I.3.11). A box product, over R, of free R-modules induced from orthogonal spaces is isomorphic to the free R-module generated by the box product of underlying orthogonal spaces:

$$(R \boxtimes X) \boxtimes_R (R \boxtimes Y) \;\cong\; R \boxtimes (X \boxtimes Y) \;.$$

Since $R \boxtimes -$ is a left adjoint, it commutes with pushouts and orbits by Σ_n-actions. Hence the analogous statement carries over to symmetrized iterated box products. In other words, for every morphism $i : A \longrightarrow B$ of orthogonal spaces there is a natural isomorphism in the arrow category of R-modules between

$$(R \boxtimes i)^{\square_R n}/\Sigma_n \;:\; Q^n_R(R \boxtimes i)/\Sigma_n \;\longrightarrow\; \mathbb{P}^n_R(R \boxtimes B)$$

and

$$R \boxtimes (i^{\square n}/\Sigma_n) \;:\; R \boxtimes (Q^n(i)/\Sigma_n) \;\longrightarrow\; R \boxtimes \mathbb{P}^n(B) \;.$$

If i is a flat cofibration of orthogonal spaces, then so is the morphism $i^{\square n}/\Sigma_n$, by Theorem 1.11 (i). So the morphism $R \boxtimes (i^{\square n}/\Sigma_n)$ is a cofibration of R-modules, hence so is the morphism $(R \boxtimes i)^{\square_R n}/\Sigma_n$. This completes the proof the all cofibrations in the global model structure for R-modules of Corollary I.4.16 (i) are symmetrizable with respect to $\boxtimes_R$.

Now we apply Corollary 3.6 of [**175**]; there is a slight caveat here, because the hypotheses ask for the validity of the 'strong monoid axiom', which requires the symmetrizability of both the cofibrations and the acyclic cofibrations. Since the model structure on R-modules was lifted from an absolute model structure, it is *not* the case that all acyclic cofibrations are symmetrizable. However, [**175**, Cor. 3.6] and its proof are only about cofibrations, but don't involve the weak equivalences at all. So the proof of [**175**, Cor. 3.6] only needs the symmetrizability of the cofibrations, which we just established for the global model category of R-modules of Corollary I.4.16 (i). Since R is cofibrant as an R-module, [**175**, Cor. 3.6] shows that for every cofibrant commutative R-algebra S, the structure morphism $i : R \longrightarrow S$ is a cofibration of R-modules. Part (i) now follows because commutative R-algebras 'are' morphisms of ultra-commutative monoids with source R. More precisely, the category of commutative R-algebras is equivalent to the category of ultra-commutative monoids under R. Moreover, an R-algebra S is cofibrant if and only if the structure morphism $i : R \longrightarrow S$ is a cofibration of ultra-commutative monoids.

(b) This is a combination of part (a) and the fact, proved in Corollary I.4.16 (i), that all cofibrations of R-modules are h-cofibrations of orthogonal spaces.

(c) This is a combination of part (a) and the fact, also proved in Corollary I.4.16 (i), that if R itself is flat, then all cofibrations of R-modules are flat cofibrations of orthogonal spaces.

(iii) Since weak equivalences and fibrations of ultra-commutative monoid are defined on underlying orthogonal spaces, and since pullbacks of ultra-commutative monoids are created on underlying orthogonal spaces, right properness is inherited from the positive global model structure of orthogonal spaces (Proposition 1.4).

Pushouts in a category of commutative algebras are given by the relative monoidal product. For ultra-commutative monoids this means that a pushout square has the form

$$\begin{tikzcd} R \arrow[r, "f", "\simeq"'] \arrow[d, "j"'] & T \arrow[d, "j\boxtimes_R T"] \\ S \arrow[r, "S\boxtimes_R f"'] & S\boxtimes_R T \end{tikzcd}$$

where S and T are considered as R-modules by restriction along i respectively f. For left properness we now suppose that j is a cofibration and f is a global equivalence. By part (a) of (ii), the morphism j is then a cofibration of R-modules in the global model structure of Corollary I.4.16 (i). Since R is cofibrant in that model structure, also S is cofibrant as an R-module. Proposition I.4.17 then shows that the functor $S\boxtimes_R -$ preserves global equivalences. So the cobase change $S\boxtimes_R f$ of f is a global equivalence. This shows that the global model structure of ultra-commutative monoids is left proper. □

2. Global power monoids

In this section we investigate the algebraic structure that an ultra-commutative multiplication produces on the Rep-functor $\underline{\pi}_0(R)$. Besides an abelian monoid structure on $\pi_0^G(R)$ for every compact Lie group G, this structure includes power operations and transfer maps. We formalize this algebraic structure under the name 'global power monoid', see Definition 2.8. Theorem 2.26 then says that global power monoids are precisely the natural algebraic structure, i.e., they parametrize all natural operations on $\underline{\pi}_0(R)$ for ultra-commutative monoids. In Construction 2.32 we introduce the transfer maps, which are an equivalent way of packaging the power operations in a global power monoid; the main properties of the transfers are summarized in Proposition 2.33. Proposition 2.38 shows that every global power monoid is realized by an ultra-commutative monoid. Even better, the realization can be chosen to be 'globally discrete', and then it defines a right adjoint to $\underline{\pi}_0$, considered as a functor from the homotopy category of ultra-commutative monoids to the category of global power monoids. Construction 2.40 defines the comonad of 'exponential sequences', which yields yet another way to encode the structure of global power monoids: Theorem 2.44 identifies them with the coalgebras over the exponential sequence comonad.

Given an orthogonal monoid space R (not necessarily commutative) with multiplication morphism $\mu : R\boxtimes R \longrightarrow R$ and a compact Lie group G, we define a binary operation

$$+ \ : \ \pi_0^G(R)\times\pi_0^G(R) \ \longrightarrow \ \pi_0^G(R) \tag{2.1}$$

on the G-equivariant homotopy set of R as the composite

$$\pi_0^G(R)\times\pi_0^G(R) \ \xrightarrow{\times} \ \pi_0^G(R\boxtimes R) \ \xrightarrow{\mu_*} \ \pi_0^G(R) \ .$$

The pairing $\times$ was defined in Construction I.5.22. If we expand the definition, it boils down to the following explicit recipe: if V and W are G-representations and $x\in R(V)^G$ and $y\in R(W)^G$ are G-fixed points that represent two classes in $\pi_0^G(R)$, then $[x]+[y]$ is represented by the G-fixed point

$$\mu_{V,W}(x,y) \ \in \ R(V\oplus W) \ .$$

We write the internal pairing on the equivariant homotopy sets of R additively because we will mostly be interested in *commutative* orthogonal monoid spaces. Another reason for using the symbol '+' is that we will later consider orthogonal ring spectra, and it will then be convenient to still have the symbol '·' available for their multiplicative structure. Obviously, the additive notation is slightly dangerous for non-commutative orthogonal monoid spaces, because there the internal pairing need not by commutative.

The following properties of the operation '+' are direct consequences of the corresponding properties of the pairings '×', compare Proposition I.5.24; a direct proof from the explicit definition of the operation '+' above is also straightforward.

Corollary 2.2. *Let R be an orthogonal monoid space.*

(i) *For every compact Lie group G the binary operation $+$ makes the set $\underline{\pi}_0^G(R)$ into a monoid.*
(ii) *If the multiplication of R is commutative, then so is the operation $+$.*
(iii) *For every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups the associated restriction map $\alpha^* : \pi_0^G(R) \longrightarrow \pi_0^K(R)$ is a monoid homomorphism.*

Now we turn to special features that happen for ultra-commutative monoids. If the multiplication on an orthogonal monoid space R is commutative, then this does not only imply commutativity of the monoids $\pi_0^G(R)$; strict commutativity of the multiplication also gives rise to additional *power operations* that we discuss now. An important special case will later be the multiplicative ultra-commutative monoid $\Omega^\bullet R$ arising from an ultra-commutative ring spectrum R. In this situation the power operations satisfy further compatibility conditions with respect to the addition and the transfer maps on $\underline{\pi}_0(\Omega^\bullet R) = \underline{\pi}_0(R)$; altogether this structure makes altogether makes the 0-th equivariant homotopy groups of an ultra-commutative ring spectrum into a *global power functor*.

Construction 2.3. We let R be an ultra-commutative monoid, G a compact Lie group and $m \geq 0$. We construct a natural *power operation*

$$[m] \ : \ \pi_0^G(R) \ \longrightarrow \ \pi_0^{\Sigma_m \wr G}(R) \tag{2.4}$$

that is an equivariant refinement of the map $x \longmapsto m \cdot x$.

We recall that the wreath product $\Sigma_m \wr G$ of a symmetric group Σ_m and a group G is the semidirect product

$$\Sigma_m \wr G \ = \ \Sigma_m \ltimes G^m$$

formed with respect to the action of Σ_m by permuting the factors of G^m. So the multiplication in $\Sigma_m \wr G$ is given by

$$(\sigma;\, g_1, \dots, g_m) \cdot (\tau;\, k_1, \dots, k_m) \ = \ (\sigma\tau;\, g_{\tau(1)}k_1, \dots, g_{\tau(m)}k_m) \ .$$

For every G-space E, the wreath product $\Sigma_m \wr G$ acts on the space E^m by

$$(\sigma;\, g_1, \dots, g_m) \cdot (e_1, \dots, e_m) \ = \ (g_{\sigma^{-1}(1)}e_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(m)}e_{\sigma^{-1}(m)}) \ .$$

For every G-representation V, this action even makes V^m into a $(\Sigma_m \wr G)$-representation. We let

$$\mu_{V,\dots,V} \ : \ R(V) \times \cdots \times R(V) \ \longrightarrow \ R(V \oplus \cdots \oplus V)$$

denote the $(V, \dots, V)$-component of the multiplication map of R, and we observe that this map is $(\Sigma_m \wr G)$-equivariant because the multiplication of R is commutative. If $x \in R(V)^G$ is a G-fixed point representing a class in $\pi_0^G(R)$, then $(x, \dots, x) \in R(V)^m$ is a $(\Sigma_m \wr G)$-fixed point. So its image under the map $\mu_{V,\dots,V}$ is a $(\Sigma_m \wr G)$-fixed point of $R(V^m)$, representing an element

$$[m]([x]) \ = \ \langle \mu_{V,\dots,V}(x, \dots, x) \rangle \ \in \ \pi_0^{\Sigma_m \wr G}(R) \ .$$

If we stabilize x along a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ to $R(\varphi)(x) \in R(W)^G$, then $\mu_{V,\dots,V}(x, \dots, x)$ changes into

$$\mu_{W,\dots,W}(R(\varphi)(x), \dots, R(\varphi)(x)) \ = \ R(\varphi^m)(\mu_{V,\dots,V}(x, \dots, x)) \ \in \ R(W^m)^{\Sigma_m \wr G} \ .$$

Since $\varphi^m : V^m \longrightarrow W^m$ is a $(\Sigma_m \wr G)$-equivariant linear isometric embedding, this element represents the same class in $\pi_0^{\Sigma_m \wr G}(R)$ as $\mu_{V,\dots,V}(x,\dots,x)$, so the class $[m]([x])$ only depends on the class of x in $\pi_0^G(R)$. So we have constructed a well-defined power operation (2.4).

The power operations are clearly natural for homomorphisms $\varphi : R \longrightarrow S$ of ultra-commutative monoids, i.e., for every compact Lie group G, every $m \geq 0$ and all $x \in \pi_0^G(R)$ the relation

$$[m](\varphi_*(x)) \;=\; \varphi_*([m](x))$$

holds in $\pi_0^{\Sigma_m \wr G}(S)$.

The power operations $[m]$ satisfy various properties reminiscent of the map $x \mapsto m \cdot x$ in an abelian monoid. We formalizes these properties into the concept of a *global power monoid*. In the definition we need certain homomorphisms between different wreath products, so we fix notation for these now. We use the plus symbol for the 'concatenation' group monomorphism

$$+ \;:\; \Sigma_i \times \Sigma_j \;\longrightarrow\; \Sigma_{i+j}$$

defined by

$$(\sigma + \sigma')(k) \;=\; \begin{cases} \sigma(k) & \text{for } 1 \leq k \leq i, \text{ and} \\ \sigma'(k-i) + i & \text{for } i+1 \leq k \leq i+j. \end{cases}$$

This operation is strictly associative, so we will leave out parentheses. The operation $+$ is *not* commutative, but the permutations $\sigma + \sigma'$ and $\sigma' + \sigma$ differ by conjugation with the (i,j)-shuffle. An embedding of a product of wreath products is now defined by

$$\begin{aligned} \Phi_{i,j} \;:\; (\Sigma_i \wr G) \;\times\; (\Sigma_j \wr G) \;&\longrightarrow\; \Sigma_{i+j} \wr G \\ ((\sigma;\, g_1,\dots,g_i),\, (\sigma';\, g_{i+1},\dots,g_{i+j})) \;&\longmapsto\; (\sigma+\sigma';\, g_1,\dots,g_{i+j})\ . \end{aligned} \tag{2.5}$$

Another group monomorphism

$$\natural \;:\; \Sigma_k \wr \Sigma_m \;\longrightarrow\; \Sigma_{km}$$

is defined by

$$(\sigma\natural(\tau_1,\dots,\tau_k))((i-1)m+j) \;=\; (\sigma(i)-1)m + \tau_i(j)\ ,$$

for $1 \leq i \leq k$ and $1 \leq j \leq m$. This yields an embedding of an iterated wreath product

$$\begin{aligned} \Psi_{k,m} \;:\; \Sigma_k \wr (\Sigma_m \wr G) \;&\longrightarrow\; \Sigma_{km} \wr G \\ (\sigma;\, (\tau_1;\, g^1),\dots,(\tau_k;\, g^k)) \;&\longmapsto\; (\sigma\natural(\tau_1,\dots,\tau_k);\, g^1 + \dots + g^k)\ . \end{aligned} \tag{2.6}$$

Here each $g^k = (g_1^k,\dots,g_m^k)$ is an m-tuple of elements of G, and

$$g^1 + \dots + g^k \;=\; (g_1^1,\dots,g_m^1, g_1^2,\dots,g_m^2,\dots,g_1^k,\dots,g_m^k)$$

denotes the concatenation of the tuples.

Remark 2.7. The formula for the homomorphism $\natural : \Sigma_k \wr \Sigma_m \longrightarrow \Sigma_{km}$ may seems slightly ad hoc, but it can be motivated in a more conceptual way as a composite

$$\Sigma_k \wr \Sigma_m \;\longrightarrow\; \Sigma_{\{1,\dots,k\}\times\{1,\dots,m\}} \;\cong\; \Sigma_{km}\ .$$

The first monomorphism sends $(\sigma;\, \tau_1,\dots,\tau_k)$ to the permutation of the product set $\{1,\dots,k\}\times\{1,\dots,m\}$ defined by

$$(i,j) \;\longmapsto\; (\sigma(i), \tau_i(j))\ .$$

The second isomorphism is conjugation by the lexicographic ordering

$$\{1,\dots,k\}\times\{1,\dots,m\} \;\cong\; \{1,\dots,km\}\ , \quad (i,j) \;\longmapsto\; (i-1)m + j\ .$$

The use of the lexicographic ordering (and hence the precise formula for the homomorphism $\natural$) is not essential here: if we use a different bijection between the sets $\{1,\dots,k\}\times\{1,\dots,m\}$ and $\{1,\dots,km\}$, then

the homomorphisms $\natural$ and $\Psi_{k,m}$ change by inner automorphisms. So the conjugacy classes of $\natural$ and $\Psi_{k,m}$ (but not the actual homomorphisms) are canonical. Since we will always hit $\Psi_{k,m}$ with functors that are invariant under conjugation, this should motivate that the construction is reasonably natural.

Definition 2.8. A *global power monoid* is a functor

$$M \ : \ \mathrm{Rep}^{\mathrm{op}} \ \longrightarrow \ (\text{abelian monoids})$$

from the opposite of the category Rep of compact Lie groups and conjugacy classes of homomorphisms to the category of abelian monoids, equipped with monoid homomorphisms

$$[m] \ : \ M(G) \ \longrightarrow \ M(\Sigma_m \wr G)$$

for all compact Lie groups G and $m \geq 1$, called *power operations*, that satisfy the following relations.

(i) (Identity) The operation [1] is restriction along the preferred isomorphism $\Sigma_1 \wr G \cong G$, $(1; g) \mapsto g$.

(ii) (Naturality) For every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups and every $m \geq 1$ the relation

$$[m] \circ \alpha^* \ = \ (\Sigma_m \wr \alpha)^* \circ [m]$$

holds as homomorphism $M(G) \longrightarrow M(\Sigma_m \wr K)$.

(iii) (Transitivity) For all compact Lie groups G, all $k, m \geq 1$ the relation

$$\Psi_{k,m}^* \circ [km] \ = \ [k] \circ [m]$$

holds as homomorphisms $M(G) \longrightarrow M(\Sigma_k \wr (\Sigma_m \wr G))$, where $\Psi_{k,m}$ is the monomorphism (2.6).

(iv) (Additivity) For all compact Lie groups G, all $m > i > 0$ and all $x \in M(G)$ the relation

$$\Phi_{i,m-i}^*([m](x)) \ = \ p_1^*([i](x)) \ + \ p_2^*([m-i](x))$$

holds in $M((\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G))$ where $\Phi_{i,m-i}$ is the monomorphism (2.5) and

$$p_1 : (\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G) \longrightarrow \Sigma_i \wr G \quad \text{and} \quad p_2 : (\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G) \longrightarrow \Sigma_{m-i} \wr G$$

are the two projections.

A *morphism* of global power monoids is a natural transformation of abelian monoid valued functors that also commutes with the power operations $[m]$ for all $m \geq 1$.

Remark 2.9. In any abelian Rep-monoid M we can define an external pairings

$$\oplus \ : \ M(G) \times M(K) \ \longrightarrow \ M(G \times K) \qquad \text{by} \qquad x \oplus y \ = \ p_G^*(x) + p_K^*(y) \ , \tag{2.10}$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the two projections. In this notation, the additivity requirement in Definition 2.8 becomes the relation

$$\Phi_{i,m-i}^*([m](x)) \ = \ [i](x) \ \oplus [m-i](x) \ . \tag{2.11}$$

In a global power monoid, the power operations are also additive with respect to the external addition: for all compact Lie groups G and K and all $m \geq 1$, and all classes $x \in M(G)$ and $y \in M(K)$ the relation

$$[m](x \oplus y) \ = \ \Delta^*([m](x) \oplus [m](y))$$

holds in $M(\Sigma_m \wr (G \times K))$, where Δ is the 'diagonal' monomorphism

$$\begin{aligned} \Delta \ : \ \Sigma_m \wr (G \times K) \ &\longrightarrow \ (\Sigma_m \wr G) \times (\Sigma_m \wr K) \\ (\sigma; (g_1, k_1), \dots, (g_m, k_m)) \ &\longmapsto \ ((\sigma; g_1, \dots, g_m), (\sigma; k_1, \dots, k_m)) \ . \end{aligned} \tag{2.12}$$

Indeed,

$$\begin{aligned}
[m](x \oplus y) &= [m](p_G^*(x) + p_K^*(y)) \;=\; [m](p_G^*(x)) + [m](p_K^*(y)) \\
&= (\Sigma_m \wr p_G)^*([m](x)) + (\Sigma_m \wr p_K)^*([m](y)) \\
&= \Delta^*_{\Sigma_m \wr (G\times K)}((\Sigma_m \wr p_G)^*([m](x)) \oplus (\Sigma_m \wr p_K)^*([m](y))) \\
&= \Delta^*_{\Sigma_m \wr (G\times K)}(((\Sigma_m \wr p_G) \times (\Sigma_m \wr p_K))^*([m](x) \oplus [m](y))) \;=\; \Delta^*([m](x) \oplus [m](y)) \;.
\end{aligned}$$

Here we exploit that Δ factors as the composite

$$\begin{aligned}
\Sigma_m \wr (G \times K) \xrightarrow{\Delta_{\Sigma_m \wr (G\times K)}} & \quad (\Sigma_m \wr (G \times K)) \times (\Sigma_m \wr (G \times K)) \\
\xrightarrow{(\Sigma_m \wr p_G)\times(\Sigma_m \wr p_K)} & (\Sigma_m \wr G) \times (\Sigma_m \wr K) \;.
\end{aligned}$$

The class $[m](x)$ is an equivariant refinement of $m \cdot x = x + \ldots + x$ (m summands) in the following sense. Applying the relation (2.11) repeatedly shows that $[m](x)$ restricts to the external m-fold sum

$$x \oplus \ldots \oplus x \;\in\; M(G^m)$$

on the normal subgroup G^m of $\Sigma_m \wr G$. Restricting further to the diagonal takes the m-fold external sum to $m \cdot x$ in $M(G)$.

We will soon discuss that the power operations of an ultra-commutative monoid define a global power monoid. One aspect of this is the additivity of the power operations, which could be proved directly from the definition. However, we will use this opportunity to establish a very general (and rather formal) additivity result that we will use several times in this book. We let $\mathcal{C}$ be a category with a zero object and finite coproducts. We let $X \vee Y$ be a coproduct of two objects X and Y with universal morphisms $i : X \longrightarrow X \vee Y$ and $j : Y \longrightarrow X \vee Y$. If $f : X \longrightarrow A$ and $g : Y \longrightarrow A$ are two morphisms with common target, we denote by $f + g : X \vee Y \longrightarrow A$ the unique morphism that satisfies $(f + g)i = y$ and $(f + g)j = g$.

We call a functor F from $\mathcal{C}$ to the category of abelian monoids *reduced* if it takes every zero object in $\mathcal{C}$ to the trivial monoid. We call the functor F *additive* if for every pair of objects X, Y of $\mathcal{C}$ the map

$$(F(\mathrm{Id}_X + 0), F(0 + \mathrm{Id}_Y)) \;:\; F(X \vee Y) \;\longrightarrow\; F(X) \times F(Y)$$

is bijective (and hence an isomorphism of monoids).

Proposition 2.13. *Let $\mathcal{C}$ be a category with a zero object and finite coproducts and*

$$F, G \;:\; \mathcal{C} \;\longrightarrow\; AbMon$$

two reduced functors to the category of abelian monoids. Suppose that the functor G is additive. Then every natural transformation of set valued functors from F to G is automatically additive.

Proof. We let $\tau : F \longrightarrow G$ be a natural transformation of set valued functors. We consider two classes $x, y \in F(X)$. We let $i, j : X \longrightarrow X \vee X$ be the two inclusions into the coproduct. We claim that

$$\tau_{X\vee X}(F(i)(x) + F(j)(y)) \;=\; G(i)(\tau_X(x)) + G(j)(\tau_X(y)) \tag{2.14}$$

in the abelian monoid $G(X \vee X)$. Indeed,

$$\begin{aligned}
G(\mathrm{Id}_X + 0)(\tau_{X\vee X}(F(i)(x) + F(j)(y))) &= \tau_X(F(\mathrm{Id}_X + 0)(F(i)(x) + F(j)(y))) \\
&= \tau_X(F(\mathrm{Id}_X)(x) + F(0)(y)) \;=\; \tau_X(x) \\
&= G(\mathrm{Id}_X)(\tau_X(x)) + G(0)(\tau_X(x)) \\
&= G(\mathrm{Id}_X + 0)(G(i)(\tau_X(x)) + G(j)(\tau_X(y)))
\end{aligned}$$

in $G(X)$. Similarly,

$$G(0 + \mathrm{Id}_X)(\tau_{X\vee X}(F(i)(x) + F(j)(y))) \;=\; G(0 + \mathrm{Id}_X)(G(i)(\tau_X(x)) + G(j)(\tau_X(y))) \;.$$

Since G is additive, this shows the relation (2.14). We let $\nabla = (\mathrm{Id} + \mathrm{Id}) : X \vee X \longrightarrow X$ denote the fold morphism, so that

$$F(\nabla)(F(i)(x) + F(j)(y)) \;=\; F(\nabla i)(x) + F(\nabla j)(y) \;=\; x + y \;.$$

Then

$$\begin{aligned} \tau_X(x+y) \;&=\; \tau_X(F(\nabla)(F(i)(x) + F(j)(y))) \\ &=\; G(\nabla)(\tau_{X\vee X}(F(i)(x) + F(j)(y))) \\ &\overset{(2.14)}{=}\; G(\nabla)(G(i)(\tau_X(x)) + G(j)(\tau_X(y))) \\ &=\; G(\nabla i)(\tau_X(x)) + G(\nabla j)(\tau_X(y)) \;=\; \tau_X(x) + \tau_X(y) \;. \end{aligned}$$

This completes the proof. □

PROPOSITION 2.15. *Let R be an ultra-commutative monoid. Then the binary operations* (2.10) *and the power operations* (2.4) *make the functor $\underline{\pi}_0(R)$ into a global power monoid.*

PROOF. Corollary 2.2 shows that the binary operations (2.10) make the Rep functor $\underline{\pi}_0(R)$ into a functor to the category of abelian monoids. The coproduct of ultra-commutative monoids is given by the box product, so the two reduced functors

$$\pi_0^G \;,\; \pi_0^{\Sigma_m \wr G} \;:\; umon \;\longrightarrow\; AbMon \;.$$

are additive by Corollary I.5.27. Since the power operation $[m] : \pi_0^G(R) \longrightarrow \pi_0^{\Sigma_m \wr G}(R)$ is natural in R, Proposition 2.13, applied to the category of ultra-commutative monoids, shows that $[m]$ is additive. The identity condition (i), the naturality (ii), the transitivity condition (iii) and the additivity property (iv) are straightforward from the definition, and we omit the proofs. □

EXAMPLE 2.16. We let M be a commutative topological monoid. Then the constant orthogonal space $\underline{M}$ is naturally an ultra-commutative monoid. Moreover, the equivariant homotopy functor $\underline{\pi}_0(\underline{M})$ is constant with value $\pi_0(M)$, and monoid structure induced from the multiplication of M. The power operation

$$[m] \;:\; \pi_0(M) = \pi_0^G(\underline{M}) \longrightarrow \pi_0^{\Sigma_m \wr G}(\underline{M}) = \pi_0(M)$$

then sends an element x to $m \cdot x$.

EXAMPLE 2.17 (Naive units of an orthogonal monoid space). Every orthogonal monoid space R contains an interesting orthogonal monoid subspace $R^{n\times}$, the *naive units* of R. The value of $R^{n\times}$ at an inner product space V is the union of those path components of $R(V)$ that are taken to invertible elements, with respect to the monoid structure on $\pi_0(R)$, under the map

$$R(V) \;\longrightarrow\; \pi_0(R(V)) \;\longrightarrow\; \pi_0(R) \;.$$

In other words, a point $x \in R(V)$ belongs to $R^{n\times}(V)$ if and only if there is an inner product space W and a point $y \in R(W)$ such that

$$\mu_{V,W}(x,y) \;\in\; R(V \oplus W) \qquad \text{and} \qquad \mu_{W,V}(y,x) \;\in\; R(W \oplus V)$$

are in the same path component as the respective unit elements. We omit the verification that the subspaces $R^{n\times}(V)$ indeed form an orthogonal monoid subspace of R as V varies. The induced map

$$\underline{\pi}_0(R^{n\times}) \;\longrightarrow\; \underline{\pi}_0(R)$$

is also an inclusion, and the value $\pi_0^e(R^{n\times})$ at the trivial group is, by construction, the set of invertible elements of $\pi_0^e(R)$. For a general compact Lie group G,

$$\pi_0^G(R^{n\times}) \;=\; \{\, x \in \pi_0^G(R) \mid \mathrm{res}_e^G(x) \text{ is invertible in } \pi_0^e(R) \}$$

is the submonoid of $\pi_0^G(R)$ of elements that become invertible when restricted to the trivial group. So contrary to what one could suspect at first sight, $\pi_0^G(R^{n\times})$ may contain non-invertible elements and the orthogonal monoid space $R^{n\times}$ is not necessarily group-like; this is why we use the adjective 'naive'.

Example 2.18 (Units of a global power monoid). We call a functor to the category of abelian monoids *group-like* if all its values are abelian groups. We call a global power monoid M *group-like* if the underlying abelian Rep-monoid is group-like, i.e., if the abelian monoid $M(G)$ is a group for every compact Lie group G.

Every global power monoid M has a global power submonoid $M^\times$ of *units*. The value $M^\times(G)$ at a compact Lie group G consists of the set of invertible elements of $M(G)$. Since the restriction maps are homomorphisms, the sets $M^\times(G)$ are closed under restriction maps, so the subsets $M^\times(G)$ indeed form an abelian Rep-submonoid of M. Since the power operations are additive, the map $[m]$ takes $M^\times(G)$ into $M^\times(\Sigma_m \wr G)$, so $M^\times$ is even a global power submonoid of M. If $f : N \longrightarrow M$ is a homomorphism of global power monoids and N is group-like, then the image of f is contained in $M^\times$. So the functor $M \mapsto M^\times$ is right adjoint to the inclusion of the full subcategory of group-like global power monoids.

If R is an ultra-commutative monoid, then we introduce a topological version $R^\times$ of the units in Construction 5.19 below. This construction comes with a homomorphism of ultra-commutative monoids $R^\times \longrightarrow R$ that realizes the inclusion of the units of $\underline{\pi}_0(R)$, compare Proposition 5.20.

Example 2.19 (Group completion of a global power monoid). A morphism $j : M \longrightarrow M^\star$ of global power monoids is a *group completion* if for every group-like global power monoid N the map

$$j^* \ : \ (\text{global power monoids})(M^\star, N) \ \longrightarrow \ (\text{global power monoids})(M, N)$$

is bijective. Every global power monoid M has a group completion, and that can be constructed 'group-wise'. Since the pair $(M^\star, j)$ represents a functor, it is unique up to preferred isomorphism under M. We define a global power monoid $M^\star$ at a compact Lie group G by letting $M^\star(G)$ be a group completion (Grothendieck construction) of the abelian monoid $M(G)$, with $j(G) : M(G) \longrightarrow M^\star(G)$ the universal homomorphism. Since the restriction maps $\alpha^* : M(G) \longrightarrow M(K)$ and the power operations $[m] : M(G) \longrightarrow M(\Sigma_m \wr G)$ are monoid homomorphisms, the universal property provides unique homomorphisms $\alpha^* : M^\star(G) \longrightarrow M^\star(K)$ and $[m] : M^\star(G) \longrightarrow M^\star(\Sigma_m \wr G)$ such that

$$\alpha^* \circ j(G) = j(K) \circ \alpha^* \qquad \text{and} \qquad [m] \circ j(G) = j(\Sigma_m \wr G) \circ [m] \ .$$

The functoriality of the restriction maps α^* and the additional relations required of a global power monoid are relations between monoid homomorphism; so they are inherited by $M^\star$ via the universal property of group completion of abelian monoids.

If R is an ultra-commutative monoid, then in Construction 5.24 below we introduce a global topological version $R^\star$ of the group completion. This construction comes with a homomorphism of ultra-commutative monoids $R \longrightarrow R^\star$ that realizes the algebraic group completion, compare Proposition 5.23.

Example 2.20 (Free ultra-commutative monoid of a global classifying space). We look in more detail at the free ultra-commutative monoid generated by the global classifying space $B_{\mathrm{gl}}G$ of a compact Lie group G. For every G-representation V, Example I.3.39 provides an isomorphism of orthogonal spaces

$$\mathbf{L}_{G,V}^{\boxtimes m} \ \cong \ \mathbf{L}_{G^m,V^m} \ .$$

At an inner product space W, the permutation action of Σ_m on the left hand side becomes the action on

$$\mathbf{L}_{G^m,V^m}(W) \ = \ \mathbf{L}(V^m, W)/G^m$$

by permuting the summands in V^m. So passage to Σ_m-orbits gives an isomorphism

$$\mathbf{L}_{G,V}^{\boxtimes m}/\Sigma_m \ \cong \ \mathbf{L}_{G^m,V^m}/\Sigma_m \ \cong \ \mathbf{L}_{\Sigma_m \wr G,V^m} \ .$$

Thus

$$\mathbb{P}(\mathbf{L}_{G,V}) \ = \ \coprod_{m\geq 0} \mathbf{L}_{\Sigma_m \wr G,V^m} \ .$$

If G acts faithfully on V and $V \neq 0$, then the action of $\Sigma_m \wr G$ on V^m is again faithful. So in terms of global classifying spaces the free ultra-commutative monoid generated by $B_{\mathrm{gl}}G$ is given by

$$\mathbb{P}(B_{\mathrm{gl}}G) \;=\; \coprod_{m\geq 0} B_{\mathrm{gl}}(\Sigma_m \wr G)\ . \tag{2.21}$$

The tautological class $u_G \in \pi_0^G(B_{\mathrm{gl}}G)$ is represented by the orbit of the identity of V in

$$(\mathbf{L}_{G,V}(V))^G \;=\; (\mathbf{L}(V,V)/G)^G\ ,$$

compare (5.13) of Chapter I. So the class $[m](u_G) \in \pi_0^{\Sigma_m \wr G}(\mathbb{P}(B_{\mathrm{gl}}G))$ is represented by the orbit of the identity of V^m in

$$(\mathbf{L}_{\Sigma_m \wr G, V^m}(V^m))^G \;=\; (\mathbf{L}(V^m,V^m)/\Sigma_m \wr G)^{\Sigma_m \wr G}\ ;$$

so with respect to the identification (2.21) we have

$$[m](u_G) \;=\; u_{\Sigma_m \wr G}\ . \tag{2.22}$$

Example 2.23 (Coproducts of ultra-commutative monoids). The category of ultra-commutative monoids is cocomplete; in particular, every family $\{R_i\}_{i\in I}$ of ultra-commutative monoids has a coproduct that we denote

$$\boxtimes'_{i\in I} R_i\ .$$

We claim that the functor

$$\underline{\pi}_0\ :\ (umon) \;\longrightarrow\; (\text{global power monoids})$$

preserves coproducts. Indeed, if the indexing set I is finite, then the underlying orthogonal space of the coproduct $\boxtimes'_{i\in I} R_i$ is simply the iterated box product of the underlying orthogonal spaces. The functor $\underline{\pi}_0$ takes box products of orthogonal spaces to the objectwise product of Rep-functors, by Corollary I.5.27. In the category of abelian monoids, finite products are also finite coproducts. Coproducts of global power functors are formed objectwise, so a finite product of global power monoids is also a coproduct. This proves the claim whenever the indexing set I is finite.

In any category, an infinite coproduct is the filtered colimit of the finite coproducts. Moreover, filtered colimits of ultra-commutative monoids are formed on underlying orthogonal spaces, compare Corollary 1.6. So if the set I is infinite, then the underlying orthogonal space of the coproduct is the 'infinite box product' in the sense of Construction I.5.28, i.e., the filtered colimit, formed over the poset of finite subsets of I, of the finite coproducts,

$$\boxtimes'_{i\in I} R_i \;\cong\; \mathrm{colim}_{J\subset I, J \text{ finite}} \boxtimes_{j\in J} R_j\ .$$

Proposition I.5.31 thus provides a bijection

$$\prod\nolimits'_{i\in I} \pi_0^G(R_i) \;\longrightarrow\; \pi_0^G(\boxtimes'_{i\in I} R_i)$$

from the weak product of the abelian monoids $\pi_0^G(R_i)$ to the abelian monoid $\pi_0^G(\boxtimes'_{i\in I} R_i)$. For abelian monoids, the weak product is also the direct sum, i.e., the categorical coproduct. Since colimits of global power monoids are calculated objectwise, this proves the claim in general.

Now we are going to show that the restriction maps along a group homomorphism and the power operations give all natural operations between equivariant homotopy sets of ultra-commutative monoids. The strategy is the same as in Proposition I.5.16: the functor π_0^G is representable, this time by the free ultra-commutative monoid $\mathbb{P}(B_{\mathrm{gl}}G)$ generated by a global classifying space of G, so we have to determine the equivariant homotopy sets $\pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))$ of these representing objects.

We will want to use similar representability arguments several other times in this book, so we generalize Proposition I.5.14 to a more general context of a model category $\mathcal{C}$ related to the category of orthogonal spaces by a Quillen adjoint functor pair. Proposition I.5.14 is the degenerate case $\mathcal{C} = spc$ and the identity functors. Right now we are interested in $\mathcal{C} = umon$, the category of ultra-commutative monoids with the

free and forgetful functor pair $(\mathbb{P}, U)$; later we will also consider the case $\mathcal{C} = \mathcal{S}p$ of orthogonal spectra with the adjoint functor pair $(\Sigma^\infty_+, \Omega^\bullet)$, and the combination of these two cases, where $\mathcal{C}$ is the category of ultra-commutative ring spectra.

PROPOSITION 2.24. *Let $\mathcal{C}$ be a model category and*

$$spc \underset{U}{\overset{\Lambda}{\rightleftarrows}} \mathcal{C}$$

a Quillen adjoint functor pair with respect to the positive global model structure on the category of orthogonal spaces. Suppose that the right adjoint U takes all weak equivalences in $\mathcal{C}$ to global equivalences of orthogonal spaces. Let G be a compact Lie group, V a non-zero faithful G-representation and

$$u_G^{\mathcal{C}} \;=\; \eta_*(u_{G,V}) \;\in\; \pi_0^G(U(\Lambda(\mathbf{L}_{G,V}))) \;,$$

where $\eta : \mathbf{L}_{G,V} \longrightarrow U(\Lambda(\mathbf{L}_{G,V}))$ is the adjunction unit.

(i) *The pair $(\Lambda(\mathbf{L}_{G,V}), u_G^{\mathcal{C}})$ represents the composite functor*

$$\mathrm{Ho}(\mathcal{C}) \xrightarrow{\mathrm{Ho}(U)} \mathrm{Ho}(spc) \xrightarrow{\pi_0^G} (\mathrm{sets}) \;.$$

(ii) *Let $\Psi : \mathcal{C} \longrightarrow (\mathrm{sets})$ be a functor that takes all weak equivalences to bijections. Then evaluation at the class $u_G^{\mathcal{C}}$ is a bijection*

$$\mathrm{Nat}_{\mathcal{C}\to(\mathrm{sets})}(\pi_0^G \circ U,\, \Psi) \;\longrightarrow\; \Psi(\Lambda(\mathbf{L}_{G,V})) \;,\quad \tau \;\longmapsto\; \tau(u_G^{\mathcal{C}})$$

between the set of natural transformations from the functor $\pi_0^G \circ U$ to Ψ, and the set $\Psi(\Lambda(\mathbf{L}_{G,V}))$.

PROOF. (i) We need to show that for every object X of $\mathcal{C}$ the map

$$\mathrm{Ho}(\mathcal{C})(\Lambda(\mathbf{L}_{G,V}), X) \;\longrightarrow\; \pi_0^G(UX) \;,\quad f \;\longmapsto\; (\mathrm{Ho}(U)(f))_*(u_G^{\mathcal{C}}) \tag{2.25}$$

is bijective. Since both sides take weak equivalences in X to bijections, we can assume that X is fibrant in the given model structure of $\mathcal{C}$.

Since (Λ, U) is a Quillen adjoint pair, the total left derived functor $L\Lambda$ and the total right derived functor RU exist and form an adjoint functor pair

$$\mathrm{Ho}(spc) \underset{RU}{\overset{L\Lambda}{\rightleftarrows}} \mathrm{Ho}(\mathcal{C})$$

between the homotopy categories. Since U takes all weak equivalences to global equivalence, we can in fact take $RU = \mathrm{Ho}(U)$, and this is given on objects by U. The free orthogonal space $\mathbf{L}_{G,V}$ is flat and its value at 0 is empty because V is non-zero; so $\mathbf{L}_{G,V}$ is cofibrant in the positive global model structure. Hence the object $\Lambda(\mathbf{L}_{G,V})$ calculates the value of the total left derived functor of Λ on $\mathbf{L}_{G,V}$. By definition of the class $u_G^{\mathcal{C}}$, the map (2.25) factors as the composite

$$\mathrm{Ho}(\mathcal{C})(\Lambda(\mathbf{L}_{G,V}), X) \xrightarrow{\cong} \mathrm{Ho}(spc)(\mathbf{L}_{G,V}, UX) \xrightarrow{f \mapsto f_*(u_G)} \pi_0^G(UX)$$

where the first map is the derived adjunction, hence bijective. The second map is bijective by Proposition I.5.14, hence the map (2.25) is bijective.

(ii) We let $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ denote the localization functor. Since Ψ takes all weak equivalences to bijections, there is a unique functor $\bar{\Psi} : \mathrm{Ho}(\mathcal{C}) \longrightarrow (\mathrm{sets})$ such that $\Psi = \bar{\Psi} \circ \gamma$. Moreover, the map

$$- \circ \gamma \;:\; \mathrm{Nat}_{\mathrm{Ho}(\mathcal{C})\to(\mathrm{sets})}(\pi_0^G \circ \mathrm{Ho}(U),\, \bar{\Psi}) \;\longrightarrow\; \mathrm{Nat}_{\mathcal{C}\to(\mathrm{sets})}(\pi_0^G \circ U,\, \Psi)$$

given by precomposition with γ is bijective; since the functor $\pi_0^G \circ \mathrm{Ho}(U)$ is representable by part (i), the claim is then a special case of the Yoneda lemma. $\square$

We specialize to the category of ultra-commutative monoids. For a compact Lie group G we set

$$u_G^{umon} \;=\; \eta_*(u_G) \;\in\; \pi_0^G(\mathbb{P}(B_{\mathrm{gl}}G)) \;,$$

where $\eta : B_{\mathrm{gl}}G \longrightarrow \mathbb{P}(B_{\mathrm{gl}}G)$ is the adjunction unit, i.e., the inclusion of the homogeneous summand for $m = 1$. The next theorem says in particular that the global power monoid $\underline{\pi}_0(\mathbb{P}(B_{\mathrm{gl}}G))$ is freely generated by the element u_G^{umon}.

THEOREM 2.26. *Let G and K be compact Lie groups. Let V be a non-zero faithful G-representation, and set $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$.*

(i) *For every compact Lie group K, every class $x \in \pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))$ is of the form $\alpha^*([m](u_G^{umon}))$ for a unique $m \geq 0$ and a unique conjugacy class of continuous homomorphism $\alpha : K \longrightarrow \Sigma_m \wr G$.*

(ii) *Let M be a global power monoid. Then for every element $x \in M(G)$ there is a unique morphism of global power monoids $f : \underline{\pi}_0(\mathbb{P}(B_{\mathrm{gl}}G)) \longrightarrow M$ such that*

$$f(G)(u_G^{umon}) \;=\; x \;.$$

(iii) *Every natural transformation $\pi_0^G \longrightarrow \pi_0^K$ of set valued functors on the category of ultra-commutative monoids is of the form $\alpha^* \circ [m]$ for a unique $m \geq 0$ and a unique conjugacy class of continuous group homomorphism $\alpha : K \longrightarrow \Sigma_m \wr G$.*

PROOF. For the course of the proof we abbreviate $u = u_G^{umon}$.

(i) By (2.21) the underlying orthogonal space of $\mathbb{P}(B_{\mathrm{gl}}G)$ is the disjoint union of global classifying spaces for the wreath product groups $\Sigma_m \wr G$; moreover, the class $[m](u)$ lies in the m-th summand of $\mathbb{P}(B_{\mathrm{gl}}G)$ and is a universal element for $B_{\mathrm{gl}}(\Sigma_m \wr G)$. So part (i) follows from Proposition I.5.16 (ii) and the fact that π_0^K commutes with disjoint unions.

(ii) By (i) every element of $\pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))$ is of the form $\alpha^*([m](u))$; every morphism of global power monoids $f : \underline{\pi}_0(\mathbb{P}(B_{\mathrm{gl}}G)) \longrightarrow M$ satisfies

$$f(K)(\alpha^*([m](u))) \;=\; \alpha^*([m](f(G)(u))) \;,$$

so f is determined by its value on the class u. This show uniqueness.

Conversely, if $x \in M(G)$ is given we define $f(K) : \pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G)) \longrightarrow M(K)$ by the formula

$$f(K)(\alpha^*([m](u))) \;=\; \alpha^*([m](x)) \;.$$

Then $f(G)(u) = x$. It remains to show that f is indeed a morphism of global power monoids. This is a routine – but somewhat lengthy – calculation as follows. Given $\alpha : K \longrightarrow \Sigma_m \wr G$ and $\bar{\alpha} : K \longrightarrow \Sigma_n \wr G$, we have

$$\begin{aligned}
\alpha^*([m](u)) \;+\; \bar{\alpha}^*([n](u)) \;&=\; \Delta_K^*(\alpha^*([m](u)) \oplus (\bar{\alpha})^*([n](u))) \\
&=\; (\alpha, \bar{\alpha})^*([m](u) \oplus [n](u)) \\
&=\; (\alpha, \bar{\alpha})^*(\Phi_{m,n}^*([m+n](u))) \\
&=\; (\Phi_{m,n} \circ (\alpha, \bar{\alpha}))^*([m+n](u)) \;.
\end{aligned}$$

So $f(K)$ is additive because

$$\begin{aligned}
f(K)(\alpha^*([m](u)) \;+\; \bar{\alpha}^*([n](u))) \;&=\; f(K)((\Phi_{m,n} \circ (\alpha, \bar{\alpha}))^*([m+n](u))) \\
&=\; (\Phi_{m,n} \circ (\alpha, \bar{\alpha}))^*([m+n](x))) \\
&=\; (\alpha, \bar{\alpha})^*(\Phi_{m,n}^*([m+n](x))) \\
&=\; (\alpha, \bar{\alpha})^*([m](x) \oplus [n](x)) \\
&=\; \alpha^*([m](x)) \;+\; \bar{\alpha}^*([n](x)) \\
&=\; f(K)(\alpha^*([m](u))) \;+\; f(K)(\bar{\alpha}^*([n](u))) \;.
\end{aligned}$$

For a continuous homomorphism $\beta : L \longrightarrow K$ we have

$$\begin{aligned}(\beta^* \circ f(K))(\alpha^*([m](u))) &= \beta^*(\alpha^*([m](x))) = (\alpha \circ \beta)^*([m](x)) \\ &= f(L)((\alpha \circ \beta)^*([m](u))) = (f(L) \circ \beta^*)(\alpha^*([m](u)))\ ;\end{aligned}$$

so the homomorphisms $f(K)$ form a natural transformation of Rep-functors. For $n \geq 0$ we have

$$\begin{aligned}[n](\alpha^*([m](u))) &= (\Sigma_n \wr \alpha^*)([n]([m](u))) \\ &= (\Sigma_n \wr \alpha^*)(\Psi_{nm}^*([nm](u))) = (\Psi_{n,m} \circ (\Sigma_n \wr \alpha))^*([nm](u))\ ;\end{aligned}$$

hence

$$\begin{aligned}f(\Sigma_n \wr K)([n](\alpha^*([m](u)))) &= (\Psi_{n,m} \circ (\Sigma_n \wr \alpha))^*([nm](x)) \\ &= (\Sigma_n \wr \alpha)^*(\Psi_{n,m}^*([nm](x))) \\ &= (\Sigma_n \wr \alpha)^*([n]([m](x))) \\ &= [n](\alpha^*([m](x))) \\ &= [n](f(K))(\alpha^*([m](u)))\ .\end{aligned}$$

So the homomorphisms $f(K)$ are compatible with power operations.

(iii) We apply the representability result of Proposition 2.24 (ii) to the category of ultra-commutative monoids, the free and forgetful adjoint functor pair

$$spc \underset{U}{\overset{\mathbb{P}}{\rightleftarrows}} umon$$

and the functor $\pi_0^K \circ U$. We conclude that evaluation at the tautological class is a bijection

$$\mathrm{Nat}^{\mathrm{umon}}(\pi_0^G, \pi_0^K) \longrightarrow \pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))\ , \quad \tau \longmapsto \tau(u)\ .$$

Part (i) thus completes the argument. □

Remark 2.27 (Natural n-ary operations). By similar arguments as in the previous proposition we can also identify the natural n-ary operations on equivariant homotopy sets of ultra-commutative monoids. For every n-tuple $G_1, \dots, G_n$ of compact Lie groups the functor

$$\mathrm{Ho}(umon) \longrightarrow (\mathrm{sets})\ , \quad X \longmapsto \pi_0^{G_1}(X) \times \cdots \times \pi_0^{G_n}(X)$$

is represented by the free ultra-commutative monoid $\mathbb{P}(B_{\mathrm{gl}}G_1 \amalg \dots \amalg B_{\mathrm{gl}}G_n)$. So the set of natural transformations from the functor $\pi_0^{G_1} \times \cdots \times \pi_0^{G_n}$ to the functor π_0^K, for another compact Lie group K, bijects with the K-equivariant homotopy set of this representing object. Because

$$\begin{aligned}\mathbb{P}(B_{\mathrm{gl}}G_1 \amalg \dots \amalg B_{\mathrm{gl}}G_n) &\cong \mathbb{P}(B_{\mathrm{gl}}G_1) \boxtimes \cdots \boxtimes \mathbb{P}(B_{\mathrm{gl}}G_n) \\ &\cong \coprod_{j_1,\dots,j_n \geq 0} B_{\mathrm{gl}}(\Sigma_{j_1} \wr G_1) \boxtimes \cdots \boxtimes B_{\mathrm{gl}}(\Sigma_{j_n} \wr G_n) \\ &\cong \coprod_{j_1,\dots,j_n \geq 0} B_{\mathrm{gl}}((\Sigma_{j_1} \wr G_1) \times \cdots \times (\Sigma_{j_n} \wr G_n))\ ,\end{aligned}$$

the group $\pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G_1 \amalg \dots \amalg B_{\mathrm{gl}}G_n))$ bijects with the disjoint union of the sets

$$\pi_0^K(B_{\mathrm{gl}}((\Sigma_{j_1} \wr G_1) \times \cdots \times (\Sigma_{j_n} \wr G_n))) \cong \mathrm{Rep}(K, (\Sigma_{j_1} \wr G_1) \times \cdots \times (\Sigma_{j_n} \wr G_n))\ .$$

So every natural operation from the functor $\pi_0^{G_1} \times \cdots \times \pi_0^{G_n}$ to the functor π_0^K is of the form

$$(x_1, \dots, x_n) \longmapsto \alpha^*([j_1](x_1) \oplus \cdots \oplus [j_n](x_n))$$

for a unique tuple $(j_1, \dots, j_n)$ of non-negative integers and a unique conjugacy class of continuous homomorphisms $\alpha : K \longrightarrow (\Sigma_{j_1} \wr G_1) \times \cdots \times (\Sigma_{j_n} \wr G_n)$. In particular, all n-ary operations are generated by unary operations and external sum.

Construction 2.28. We describe an alternative (but isomorphic) way to organize the book keeping of the natural operations between the 0-th equivariant homotopy sets of ultra-commutative monoids. We denote by Nat^{umon} the category whose objects are all compact Lie groups and where the morphism set $\mathrm{Nat}^{umon}(G, K)$ is the set of all natural transformations, of functors from the ultra-commutative monoids to sets, from π_0^G to π_0^K. We define an isomorphic algebraic category $\mathbb{A}^+$, the 'effective Burnside category'. Both $\mathrm{Nat}^{umon}$and $\mathbb{A}^+$ are 'pre-preadditive' in the sense that all morphism sets are abelian monoids and composition is biadditive. In Nat^{umon}, the monoid structure is objectwise addition of natural transformations.

The category $\mathbb{A}^+$ has the same objects as Nat^{umon}, namely all compact Lie groups. In the effective Burnside category, the morphism set $\mathbb{A}^+(G, K)$ is the set of isomorphism classes of those K-G-spaces that are a disjoint unions of finitely many free right G-orbits. This set is an abelian monoid via disjoint union of K-G-spaces. Composition

$$\circ \ : \ \mathbb{A}^+(K, L) \times \mathbb{A}^+(G, K) \ \longrightarrow \ \mathbb{A}^+(G, L)$$

is induced by the balanced product over K:

$$[T] \circ [S] \ = \ [T \times_K S] \ .$$

Here T has a left L-action and a commuting free right K-action, whereas S has a left K-action and a commuting free right G-action. The balanced product $T \times_K S$ then inherits a left L-action from T and a free right G-action from S. We define a functor

$$B \ : \ \mathrm{Nat}^{umon} \ \longrightarrow \ \mathbb{A}^+ \tag{2.29}$$

as the identity on objects; on morphisms, the functor is given by

$$B \ : \ \mathrm{Nat}^{umon}(G, K) \ \longrightarrow \ \mathbb{A}^+(G, K) \ , \quad B(\alpha^* \circ [m]) \ = \ [\alpha^*(\{1, \dots, m\} \times G)_G] \ .$$

In the definition we use the characterization of the natural operations given by Theorem 2.26 (iii); also, we consider $\{1, \dots, m\} \times G$ as a right G-space by translation; the wreath product $\Sigma_m \wr G$ acts from the left on $\{1, \dots, m\} \times G$ by

$$(\sigma;\, g_1, \dots, g_m) \cdot (i, \gamma) \ = \ (\sigma(i),\, g_i \cdot \gamma) \ . \tag{2.30}$$

Then we let K act by restriction of the $(\Sigma_m \wr G)$-action along α.

Proposition 2.31. *The functor B of* (2.29) *is additive and an isomorphisms of categories.*

Proof. We start by showing that the map $B : \mathrm{Nat}^{umon}(G, K) \longrightarrow \mathbb{A}^+(G, K)$ is additive. The map

$$(\{1, \dots, k\} \times G) \ \amalg \ (\{1, \dots, m\} \times G) \ \longrightarrow \ \Phi_{k,m}^*(\{1, \dots, k+m\} \times G)$$

that is the inclusion on the first summand and given by $(j, g) \mapsto (k + j, g)$ on the second summand is an isomorphism of $((\Sigma_k \wr G) \times (\Sigma_m \wr G))$-$G$-bispaces. Restriction along the homomorphism $(\beta, \alpha) : K \longrightarrow (\Sigma_k \wr G) \times (\Sigma_m \wr G)$ provides an isomorphism of K-G-spaces between

$$\beta^*(\{1, \dots, k\} \times G)_G \ \amalg \ \alpha^*(\{1, \dots, m\} \times G)_G \qquad \text{and} \qquad (\Phi_{k,m} \circ (\beta, \alpha))^*(\{1, \dots, k+m\} \times G)_G \ .$$

This shows that

$$B((\beta^* \circ [k]) + (\alpha^* \circ [m])) \ = \ B((\Phi_{k,m} \circ (\beta, \alpha))^* \circ [k+m]) \ = \ B(\beta^* \circ [k]) \ + \ B(\alpha^* \circ [m]) \ .$$

The identity operation in $\mathrm{Nat}^{umon}(G, G)$ can be written as $\mathrm{Id}_G^* \circ [1]$; on the other hand, the G-bispace $\mathrm{Id}_G^*(\{1\} \times G)_G$ is isomorphic to G under left and right translation; since the isomorphism class of ${}_G G_G$ is the identity of G in $\mathbb{A}^+$, the construction B preserves identities.

For the compatibility of B with composition we consider another operation $\beta^* \circ [k] \in \mathrm{Nat}^{umon}(K.M)$ and observe that

$$(\beta^* \circ [k]) \circ (\alpha^* \circ [m]) \;=\; (\Psi_{k,m} \circ (\Sigma_k \wr \alpha) \circ \beta)^* \circ [km] \;.$$

Moreover, an isomorphism of M-G-spaces

$$\beta^*(\{1,\dots,k\} \times K) \times_K \alpha^*(\{1,\dots,m\} \times G) \;\cong\; (\Psi_{k,m} \circ (\Sigma_k \wr \alpha) \circ \beta)^*(\{1,\dots,km\} \times G)$$

is given by

$$[(j,\kappa),(i,\gamma)] \;\longmapsto\; ((j-1)m + \sigma(i) - 1,\; g_i \cdot \gamma) \;,$$

where

$$\alpha(\kappa) \;=\; (\sigma;\, g_1,\dots,g_m) \;\in\; \Sigma_m \wr G \;.$$

So B is a functor, which is bijective on objects by definition.

To see that B is full we let S be any K-G-space that is a disjoint union of m free right G orbits. We choose a G-equivariant homeomorphism

$$\psi \;:\; S \;\longrightarrow\; \{1,\dots,m\} \times G \;.$$

We transport the left K-action from S to $\{1,\dots,m\} \times G$ along ψ, so that ψ becomes an isomorphism of K-G-spaces. The $(\Sigma_n \wr G)$-action on $\{1,\dots,m\} \times G$ defined in (2.30) identifies the wreath product with the group of G-equivariant automorphism of $\{1,\dots,m\} \times G$; so the K-action on $\{1,\dots,m\} \times G$ corresponds to a continuous homomorphism $\alpha : K \longrightarrow \Sigma_m \wr G$. Altogether, S is isomorphic to $\alpha^*(\{1,\dots,m\} \times G)_G$.

If the K-G-spaces constructed from $\alpha^* \circ [m]$ and $\beta^* \circ [n]$ are isomorphic, then we must have $m = n$. Moreover, a K-G-isomorphism

$$\alpha^*(\{1,\dots,m\} \times G)_G \;\cong\; \beta^*(\{1,\dots,m\} \times G)_G$$

is given by the action of a unique element $\omega \in \Sigma_m \wr G$, and then the homomorphisms $\alpha, \beta : K \longrightarrow \Sigma_m \wr G$ are conjugate by ω. So the functor B is faithful. □

Now we define *transfer maps* $\mathrm{tr}_H^G : M(H) \longrightarrow M(G)$ in global power monoids, for every subgroup H of finite index in a compact Lie group G. As we will see in Proposition 2.33 below, the set of operations from π_0^G to π_0^K is a free abelian monoid with an explicit basis involving transfers.

Construction 2.32 (Transfer maps). In the following we let M be a global power monoid, G a compact Lie group and H a closed subgroup of G of finite index m. We choose an 'H-basis' of G, i.e., an ordered m-tuple $\bar{g} = (g_1,\dots,g_m)$ of elements in disjoint H-orbits such that

$$G \;=\; \bigcup_{i=1}^{m} g_i H \;.$$

The wreath product $\Sigma_m \wr H$ acts freely and transitively from the right on the set of all such H-bases of G, by the formula

$$(g_1,\dots,g_m) \cdot (\sigma;\, h_1,\dots,h_m) \;=\; (g_{\sigma(1)}h_1,\dots,g_{\sigma(m)}h_m) \;.$$

We obtain a continuous homomorphism $\Psi_{\bar{g}} : G \longrightarrow \Sigma_m \wr H$ by requiring that

$$\gamma \cdot \bar{g} \;=\; \bar{g} \cdot \Psi_{\bar{g}}(\gamma) \;.$$

We define the transfer $\mathrm{tr}_H^G : M(H) \longrightarrow M(G)$ as the composite

$$M(H) \;\xrightarrow{[m]}\; M(\Sigma_m \wr H) \;\xrightarrow{\Psi_{\bar{g}}^*}\; M(G) \;.$$

Any other H-basis is of the form $\bar{g}\omega$ for a unique $\omega \in \Sigma_m \wr H$. We have $\Psi_{\bar{g}\omega} = c_\omega \circ \Psi_{\bar{g}}$ as maps $G \longrightarrow \Sigma_m \wr H$, where $c_\omega(\gamma) = \omega^{-1}\gamma\omega$. Since inner automorphisms induce the identity in any Rep-functor, we have

$$\Psi_{\bar{g}}^* = \Psi_{\bar{g}\omega}^* \;:\; M(\Sigma_m \wr H) \;\longrightarrow\; M(G) \;.$$

So the transfer tr_H^G does not depend on the choice of basis $\bar{g}$.

The various properties of the power operations imply certain properties of the transfer maps. Moreover, the last item of the following proposition shows that power operations in a global power monoid are determined by the transfer and restriction maps.

PROPOSITION 2.33. *The transfer homomorphisms of a global power monoid M satisfy the following relations, where H is any subgroup of finite index in a compact Lie group G.*

(i) (Transitivity) *We have* $\mathrm{tr}_G^G = \mathrm{Id}_{M(G)}$ *and for nested subgroups $H \subseteq G \subseteq F$ of finite index the relation*

$$\mathrm{tr}_G^F \circ \mathrm{tr}_H^G \;=\; \mathrm{tr}_H^F$$

holds as maps $M(H) \longrightarrow M(F)$.

(ii) (Double coset formula) *For every subgroup K of G (not necessarily of finite index) the relation*

$$\mathrm{res}_K^G \circ \mathrm{tr}_H^G \;=\; \sum_{[g]\in K\backslash G/H} \mathrm{tr}_{K\cap {}^gH}^K \circ c_g^* \circ \mathrm{res}_{K^g\cap H}^H$$

holds as homomorphisms $M(H) \longrightarrow M(K)$. Here $[g]$ runs over a set of representatives of the finite set of K-H-double cosets.

(iii) (Epimorphic restriction) *For every continuous epimorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation*

$$\alpha^* \circ \mathrm{tr}_H^G \;=\; \mathrm{tr}_L^K \circ (\alpha|_L)^*$$

holds as maps from $M(H)$ to $M(K)$, where $L = \alpha^{-1}(H)$.

(iv) *For every $m \geq 1$ the power m-th power operation can be recovered as*

$$[m] \;=\; \mathrm{tr}_K^{\Sigma_m \wr G} \circ q^* \;,$$

where K is the subgroup of $\Sigma_m \wr G$ consisting of all $(\sigma;\, g_1, \ldots, g_m)$ such that $\sigma(m) = m$ and $q : K \longrightarrow G$ is defined by $q(\sigma;\, g_1, \ldots, g_m) = g_m$.

PROOF. (i) For $G = H$ we can choose the unit 1 as the G-basis, and with this choice $\Psi_1 : G \longrightarrow \Sigma_1 \wr G$ is the preferred isomorphism that sends g to $(1; g)$. The restriction of $[1](x)$ along this isomorphism is x, so we get $\mathrm{tr}_G^G(x) = x$.

For the second claim we choose a G-basis $\bar f = (f_1, \ldots, f_k)$ of F and an H-basis $\bar g = (g_1, \ldots, g_m)$ of G. Then

$$\bar f \bar g \;=\; (f_1 g_1, \ldots, f_1 g_m,\; f_2 g_1, \ldots, f_2 g_m,\; \ldots,\; f_k g_1, \ldots, f_k g_m)$$

is an H-basis of F. With respect to this basis, the homomorphism $\Psi_{\bar f \bar g} : F \longrightarrow \Sigma_{km} \wr H$ equals the composite

$$F \;\xrightarrow{\Psi_{\bar f}}\; \Sigma_k \wr G \;\xrightarrow{\Sigma_k \wr \Psi_{\bar g}}\; \Sigma_k \wr (\Sigma_m \wr H) \;\xrightarrow{\Psi_{k,m}}\; \Sigma_{km} \wr H$$

where the monomorphism $\Psi_{k,m}$ was defined in (2.6). So

$$\begin{aligned} \mathrm{tr}_H^F \;&=\; \Psi_{\bar f \bar g}^* \circ [km] \;=\; \Psi_{\bar f}^* \circ (\Sigma_k \wr \Psi_{\bar g})^* \circ \Psi_{k,m}^* \circ [km] \\ &=\; \Psi_{\bar f}^* \circ (\Sigma_k \wr \Psi_{\bar g})^* \circ [k] \circ [m] \;=\; \Psi_{\bar f}^* \circ [k] \circ \Psi_{\bar g}^* \circ [m] \;=\; \mathrm{tr}_G^F \circ \mathrm{tr}_H^G \;. \end{aligned}$$

(ii) We generalize the transfer construction slightly. We let S be a K-H-bispace that consists of finitely many free H-orbits. We choose an 'H-basis', i.e., an ordered m-tuple $\bar s = (s_1, \ldots, s_m)$ of elements in disjoint H-orbits such that

$$S \;=\; \bigcup_{i=1}^m s_i H \;.$$

Again the wreath product $\Sigma_m \wr H$ acts freely and transitively from the right on the set of all such H-bases of S. We obtain a continuous homomorphism $\Psi_{\bar s} : K \longrightarrow \Sigma_m \wr H$ by requiring that

$$k \cdot \bar s \;=\; \bar s \cdot \Psi_{\bar s}(k) \;.$$

In this generality, $\Psi_{\bar s}$ need not be injective anymore. We define a generalized transfer map $\langle S|-\rangle : M(H) \longrightarrow M(K)$ as the composite

$$M(H) \xrightarrow{[m]} M(\Sigma_m \wr H) \xrightarrow{\Psi_{\bar s}^*} M(K) \ .$$

When $S = G$ with left G-action by translation, then $\langle G|-\rangle$ specializes to the transfer map tr_H^G. As in this special case, the homomorphism $\Psi_{\bar s}$ is independent up to conjugacy of the choice of H-basis, so the map $\langle S|-\rangle$ does not depend on the choice. Moreover, $\langle S|-\rangle$ depends only on the isomorphism class of S as a K-H-bispace.

We let T be another K-H-bispace that consists of finitely many free H-orbits, and $\bar t = (t_1, \dots, t_n)$ an H-basis of T. Then $\bar s + \bar t = (s_1, \dots, s_m, t_1, \dots, t_n)$ is an H-basis of the disjoint union $S \amalg T$. Moreover, the associated homomorphism $\Psi_{\bar s + \bar t}$ is the composite

$$K \xrightarrow{(\Psi_{\bar s}, \Psi_{\bar t})} (\Sigma_m \wr H) \times (\Sigma_n \wr H) \xrightarrow{\Phi_{m,n}} \Sigma_{m+n} \wr H \ ,$$

where the monomorphism $\Phi_{m,n}$ was defined in (2.5). So we conclude that

$$\begin{aligned} \langle S \amalg T|-\rangle \ &= \ \Psi_{\bar s + \bar t}^* \circ [m+n] \ = \ (\Psi_{\bar s}, \Psi_{\bar t})^* \circ \Phi_{m,n}^* \circ [m+n] \\ &= \ (\Psi_{\bar s}, \Psi_{\bar t})^* \circ \oplus \circ ([m], [n]) \ = \ (\Psi_{\bar s}^* \circ [m]) + (\Psi_{\bar t}^* \circ [n]) \ = \ \langle S|-\rangle + \langle T|-\rangle \ . \end{aligned} \tag{2.34}$$

Now we let L be a subgroup of K of finite index $m = [K : L]$, and $\bar k = (k_1, \dots, k_m)$ an L-basis of K. We let $\alpha : L \longrightarrow H$ be a continuous homomorphism. Then the K-H-bispace

$$K \times_\alpha H \ = \ (K \times H)/(kl, h) \sim (k, \alpha(l)h)$$

consists of m free H-orbits, and $\bar s = ([k_1, 1], \dots, [k_m, 1])$ is an H-basis of $K \times_\alpha H$. Moreover, the associated homomorphism $\Psi_{\bar s}$ is the composite

$$K \xrightarrow{\Psi_{\bar k}} \Sigma_m \wr L \xrightarrow{\Sigma_m \wr \alpha} \Sigma_m \wr H \ .$$

So we conclude that

$$\langle K \times_\alpha H|-\rangle \ = \ \Psi_{\bar s}^* \circ [m] \ = \ \Psi_{\bar k}^* \circ (\Sigma_m \wr \alpha)^* \circ [m] \ = \ \Psi_{\bar k}^* \circ [m] \circ \alpha^* \ = \ \mathrm{tr}_L^K \circ \alpha^* \ . \tag{2.35}$$

Now we can prove the double coset formula. We let $R \subset G$ be a set of double coset representatives, so that G is the disjoint union of the subspaces KgH for $g \in R$. Moreover, KgH is isomorphic, as a K-H-space, to $K \times_{c_g} H$, where $c_g : K \cap {}^gH \longrightarrow H$ is conjugation by g. So we deduce

$$\begin{aligned} \mathrm{res}_K^G \circ \mathrm{tr}_H^G \ &= \ \mathrm{res}_K^G \circ \langle {}_G G_H|-\rangle \ = \ \langle {}_K G_H|-\rangle \\ &\overset{(2.34)}{=} \ \sum\nolimits_{g \in R} \langle KgH|-\rangle \ = \ \sum\nolimits_{g \in R} \langle K \times_{c_g} H|-\rangle \\ &\overset{(2.35)}{=} \ \sum\nolimits_{g \in R} \mathrm{tr}_{K \cap {}^gH}^K \circ c_g^* \circ \mathrm{res}_{K^g \cap H}^H \ . \end{aligned}$$

(iii) If $\bar k = (k_1, \dots, k_m)$ is an L-basis of K, then $\alpha(\bar k) = (\alpha(k_1), \dots, \alpha(k_m))$ is an H-basis of G. With respect to these bases we have

$$\Psi_{\alpha(\bar k)} \circ \alpha \ = \ (\Sigma_m \wr (\alpha|_L)) \circ \Psi_{\bar k} \ : \ K \ \longrightarrow \ \Sigma_m \wr H \ .$$

So

$$\alpha^* \circ \mathrm{tr}_H^G \ = \ \alpha^* \circ \Psi_{\alpha(\bar k)}^* \circ [m] \ = \ \Psi_{\bar k}^* \circ (\Sigma_m \wr (\alpha|_L))^* \circ [m] \ = \ \Psi_{\bar k}^* \circ [m] \circ (\alpha|_L)^* \ = \ \mathrm{tr}_L^K \circ (\alpha|_L)^* \ .$$

(iv) The subgroup K has index m in $\Sigma_m \wr G$ and a K-basis of $\Sigma_m \wr G$ is given by the elements $\tau_j = ((j, m); 1, \dots, 1)$ for $j = 1, \dots, m$. In order to determine the monomorphism $\Psi_{\bar\tau} : \Sigma_m \wr G \longrightarrow \Sigma_m \wr K$ associated to this K-basis, we consider any element $(\sigma; g_1, \dots, g_m)$ of $\Sigma_m \wr G$. The permutation $(\sigma(j), m) \cdot \sigma \cdot (j, m) \in \Sigma_m$ fixes m, so the element

$$l_j \ = \ \tau_{\sigma(j)} \cdot (\sigma; g_1, \dots, g_m) \cdot \tau_j \ \in \ \Sigma_m \wr G$$

in fact belongs to the subgroup K. Then

$$(\sigma;\, g_1, \dots, g_m) \cdot \tau_j \;=\; \tau_{\sigma(j)} \cdot l_j$$

in the group $\Sigma_m \wr G$, by definition. This means that

$$(\sigma;\, g_1, \dots, g_m) \cdot (\tau_1, \dots, \tau_m) \;=\; (\tau_1, \dots, \tau_m) \cdot (\sigma;\, l_1, \dots, l_m)\ ,$$

and hence

$$\Psi_{\bar\tau}(\sigma;\, g_1, \dots, g_m) \;=\; (\sigma;\, l_1, \dots, l_m)\ .$$

Because

$$(\Sigma_m \wr q)(\Psi_{\bar\tau}(\sigma;\, g_1, \dots, g_m)) \;=\; (\Sigma_m \wr q)(\sigma;\, l_1, \dots, l_m) \;=\; (\sigma;\, g_1, \dots, g_m)\ ,$$

we conclude that the composite $(\Sigma_m \wr q) \circ \Psi_{\bar\tau}$ is the identity of $\Sigma_m \wr G$. So

$$\mathrm{tr}_K^{\Sigma_m \wr G} \circ q^* \;=\; \Psi_{\bar\tau}^* \circ [m] \circ q^* \;=\; \Psi_{\bar\tau}^* \circ (\Sigma_m \wr q)^* \circ [m] \;=\; ((\Sigma_m \wr q) \circ \Psi_{\bar\tau})^* \circ [m] \;=\; [m]\ . \qquad \square$$

Theorem 2.26 (iii) gives a description of the set of natural operations, on the category of ultra-commutative monoids, from π_0^G to π_0^K. The next proposition gives an alternative description that also captures the monoid structure given by pointwise addition of operations.

PROPOSITION 2.36. *Let G and K be compact Lie groups. The monoid* $\mathrm{Nat}^{umon}(\pi_0^G, \pi_0^K)$ *is a free abelian monoid generated by the operations* $\mathrm{tr}_L^K \circ \alpha^*$ *where (L, α) runs over all $(K \times G)$-conjugacy classes of pairs consisting of*

- *a subgroup $L \le K$ of finite index, and*
- *a continuous group homomorphism $\alpha : L \longrightarrow G$.*

PROOF. This a straightforward algebraic consequence of the calculation of the category Nat^{umon} given in Theorem 2.26 (iii). Every K-G-space with finitely many free G-orbits is the disjoint union of transitive K-G-spaces with the same property. So $\mathbb{A}^+(G, K)$ is a free abelian monoid with basis the isomorphism classes of the transitive K-G-spaces. A transitive K-G-space with finitely many free G-orbits is isomorphic to one of the form

$$K \times_\alpha G \;=\; (K \times G)/(kl, g) \sim (k, \alpha(l)g)$$

for a pair $(L, \alpha : L \longrightarrow G)$, with L of finite index in K. Moreover, $K \times_\alpha G$ is isomorphic to $K \times_{\alpha'} G$ if and only if (L, α) is conjugate to (L', α') by an element of $K \times G$. So is $\mathbb{A}^+(G, K)$ freely generated by the classes of the K-G-spaces $K \times_\alpha G$, where (L, α) runs through the $(K \times G)$-conjugacy classes of the relevant pairs.

The claim then follows from the verification that the isomorphism of categories $B : \mathrm{Nat}^{umon} \longrightarrow \mathbb{A}^+$ established in Proposition 2.31 takes the operation $\mathrm{tr}_L^K \circ \alpha^*$ to the class of $K \times_\alpha G$. Indeed, if $\bar k = (k_1, \dots, k_m)$ is an L-basis of K and $\Psi_{\bar k} : K \longrightarrow \Sigma_m \wr L$ the classifying homomorphism, then

$$B(\mathrm{tr}_L^K \circ \alpha^*) \;=\; B(\Psi_{\bar k}^* \circ [m] \circ \alpha^*) \;=\; B(\Psi_{\bar k}^* \circ (\Sigma_m \wr \alpha)^* \circ [m])\ .$$

On the other hand, the map

$$((\Sigma_m \wr \alpha) \circ \Psi_{\bar k})^*(\{1, \dots, m\} \times G)_G \;\longrightarrow\; K \times_\alpha G\ , \quad (i, \gamma) \;\longmapsto\; [k_i, \gamma]$$

is an isomorphism of K-G-bispaces, so $B(\mathrm{tr}_K^K \circ \alpha^*) = [K \times_\alpha G]$ in $\mathbb{A}^+(G, K)$. $\square$

Our next major result is the fact that every global power monoid is realized by an ultra-commutative monoid. More is true: Proposition 2.38 below effectively constructs a right adjoint functor

$$(-)^\sharp \;:\; (\text{global power monoids}) \;\longrightarrow\; \mathrm{Ho}(umon)$$

to the functor $\underline{\pi}_0$ such that the adjunction counit is an isomorphism $\underline{\pi}_0(M^\sharp) \cong M$ of global power monoids.

The next proposition is an analog for ultra-commutative monoids of Proposition I.5.19. Globally discrete orthogonal spaces were defined in Definition I.5.18.

PROPOSITION 2.37. *Let Y be a globally discrete, positively static ultra-commutative monoid. Let $i : A \longrightarrow B$ be a cofibration of ultra-commutative monoids, $f : A \longrightarrow Y$ a morphism of ultra-commutative monoids and $\psi : \underline{\pi}_0(B) \longrightarrow \underline{\pi}_0(Y)$ a morphism of global power monoids such that $\psi \circ \underline{\pi}_0(i) = \underline{\pi}_0(f)$. Then there is a morphism of ultra-commutative monoids $g : B \longrightarrow Y$ such that $\underline{\pi}_0(g) = \psi$ and $gi = f$.*

PROOF. The argument is very similar as in the analogous Proposition I.5.19 for orthogonal spaces. Every cofibration of ultra-commutative monoids is an h-cofibration, hence a closed embedding, of underlying orthogonal spaces, by Theorem 1.13 (ii). Moreover, filtered colimits of ultra-commutative monoids are created on the underlying orthogonal spaces, compare Corollary 1.6. So the functor

$$\underline{\pi}_0 \ : \ umon \ \longrightarrow \ (\text{global power monoids})$$

preserves sequential colimits along cofibrations, since these are levelwise closed embeddings. The same functor $\underline{\pi}_0$ preserves coproducts by Example 2.23.

Now we consider a pushout square of ultra-commutative monoids

$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[d, "j"'] & A \arrow[d, "i"] \\ B' \arrow[r, "\beta"'] & B \end{tikzcd}$$

where j is a cofibration. Theorem 1.13 (ii) shows that the morphism of A'-modules underlying i is a cofibration in the global model structure of A'-modules of Corollary I.4.16 (i). Since A' is cofibrant in that model structure, A is cofibrant when considered as an A'-module by restriction of scalars along i. But then proposition I.5.32 shows that the map

$$\pi_0^G(B') \times \pi_0^G(A) \ \longrightarrow \ \pi_0^G(B' \boxtimes_{A'} A)$$

is surjective for every compact Lie group G. Since the relative box product $B' \boxtimes_{A'} A$ calculates the underlying orthogonal space of the pushout B, this shows that the morphism $\underline{\pi}_0(\beta) + \underline{\pi}_0(i) : \underline{\pi}_0(B') \times \underline{\pi}_0(A) \longrightarrow \underline{\pi}_0(B)$ is surjective, hence an epimorphism of global power monoids.

Proposition I.5.17 now shows that the class of cofibrations $i : A \longrightarrow B$ of ultra-commutative monoids that satisfy the claim of the proposition is closed under coproducts, cobase change, retract and countable composites. These closure properties reduce the proof to showing the conclusion for the generating cofibrations of the global model structure (see Theorem 1.13)

$$\mathbb{P}(i_k \times \mathbf{L}_{G,V}) \ : \ \mathbb{P}(\partial D^k \times \mathbf{L}_{G,V}) \ \longrightarrow \ \mathbb{P}(D^k \times \mathbf{L}_{G,V}) \ ,$$

where G is a compact Lie group, V a non-zero faithful G-representation and $k \geq 0$. The argument for the generators is the same as in Proposition I.5.19, using that the free functor $\mathbb{P}$ is left adjoint to the forgetful functor from ultra-commutative monoids to orthogonal spaces. □

PROPOSITION 2.38. (i) *For every ultra-commutative monoid S and every globally discrete ultra-commutative monoid R the map*

$$\underline{\pi}_0 \ : \ \mathrm{Ho}(umon)(S, R) \ \longrightarrow \ (\text{global power monoids})(\underline{\pi}_0(S), \underline{\pi}_0(R))$$

is bijective.

(ii) *Every ultra-commutative monoid R admits a morphism $\kappa : R \longrightarrow R_{\mathrm{dis}}$ of ultra-commutative monoids such that $\underline{\pi}_0(\kappa)$ is an isomorphism of global power monoids and R_{dis} is globally discrete.*

(iii) *For every global power monoid M there is a globally discrete ultra-commutative monoid $M^\sharp$ and an isomorphism of global power monoids*

$$\underline{\pi}_0(M^\sharp) \ \cong \ M \ .$$

(iv) *The functor*

$$\underline{\pi}_0 \ : \ \mathrm{Ho}(umon) \ \longrightarrow \ (\text{global power monoids})$$

has a right adjoint which is also right inverse.

Proof. (i) This argument is completely analogous to the proof of Proposition I.5.20, the analogous statement for orthogonal spaces, but using Proposition 2.37 instead of Proposition I.5.19.

(ii) We 'kill all higher homotopy groups' in a global, ultra-commutative fashion. We choose a set J of representatives of the isomorphism classes of pairs (G, V) consisting of compact Lie groups G and non-zero faithful G-representations V. For example, all pairs $(H, \mathbb{R}^m)$ for all $m \geq 1$ and all closed subgroups H of $O(m)$ would do the job. We apply the countable small object argument (see for example [**47**, 7.12] or [**79**, Thm. 2.1.14]), in the category of ultra-commutative monoids, to the morphism $R \longrightarrow *$ to the terminal ultra-commutative monoid, with respect to the set of closed embeddings

$$\mathbb{P}(i_k \times \mathbf{L}_{G,V}) \ : \ \mathbb{P}(\partial D^k \times \mathbf{L}_{G,V}) \ \longrightarrow \ \mathbb{P}(D^k \times \mathbf{L}_{G,V}) \tag{2.39}$$

for all $k \geq 2$ and all $(G, V) \in J$. The resulting morphism $\kappa : R \longrightarrow R_{\mathrm{dis}}$ of ultra-commutative monoids has the desired properties. Indeed, the small object argument insures that the map from R_{dis} to the terminal ultra-commutative monoid has the right lifting property with respect to the morphisms (2.39). By adjointness this means that for every compact Lie group G, every faithful G-representation V, and every $k \geq 2$ all continuous maps $\partial D^k \longrightarrow Y(V)^G$, can be extended continuously to D^k. So in particular, the underlying orthogonal space of R_{dis} is globally discrete.

On the other hand, the induced morphism $\underline{\pi}_0(\kappa) : \underline{\pi}_0(R) \longrightarrow \underline{\pi}_0(R_{\mathrm{dis}})$ of global power monoids is an isomorphism. Indeed, the morphism $\kappa : R \longrightarrow R_{\mathrm{dis}}$ is the colimit of a sequence of cofibrations of ultra-commutative monoids

$$R \ = \ T_0 \ \longrightarrow \ T_1 \ \longrightarrow \ \dots \ \longrightarrow \ T_m \ \longrightarrow \dots \ ,$$

where each step is obtained from the previous one as a pushout of ultra-commutative monoids

$$\begin{array}{ccc} \boxtimes'_{i\in I}\, \mathbb{P}(\partial D^{k_i} \times \mathbf{L}_{G_i,V_i}) & \xrightarrow{\boxtimes \mathbb{P}(i_{k_i} \times \mathbf{L}_{G_i,V_i})} & \boxtimes'_{i\in I}\, \mathbb{P}(D^{k_i} \times \mathbf{L}_{G_i,V_i}) \\ \downarrow & & \downarrow \\ T_m & \longrightarrow & T_{m+1} \end{array}$$

Every cofibration of ultra-commutative monoids is in particular an h-cofibration of underlying orthogonal spaces (Theorem 1.13 (ii)) and sequential colimits of ultra-commutative monoids are formed in underlying orthogonal spaces (Corollary 1.6). So the canonical map $\mathrm{colim}_{m\geq 0}\, \underline{\pi}_0(T_m) \longrightarrow \underline{\pi}_0(R_{\mathrm{dis}})$ is an isomorphism. So it suffices to show that all the morphisms $\underline{\pi}_0(T_m) \longrightarrow \underline{\pi}_0(T_{m+1})$ are isomorphisms of global power monoids. [...expand...].

(iii) We choose compact Lie groups G_i and elements $x_i \in M(G_i)$, for i in some indexing set I, that together generate M as a global power monoid. Then the coproduct

$$T \ = \ \boxtimes'_{i\in I}\, \mathbb{P}(B_{\mathrm{gl}} G_i)$$

of the free ultra-commutative monoids is another ultra-commutative monoid. The global power monoid $\underline{\pi}_0(\mathbb{P}(B_{\mathrm{gl}} G))$ is freely generated by the element u_G^{umon}, by Theorem 2.26 (ii). On the other hand, the functor $\underline{\pi}_0$ preserves coproducts of ultra-commutative monoids, compare Example 2.23. So the global power monoid $\underline{\pi}_0(T)$ is freely generated by the family of elements u_i in $\pi_0^{G_i}(T)$. There is thus a unique morphism of global power monoids

$$\epsilon \ : \ \underline{\pi}_0(T) \ \longrightarrow \ M$$

such that $\epsilon(G_i)(u_i) = (x_i)$; since the classes x_i generate M, the morphism ϵ is an epimorphism. Now we kill the kernel of ϵ and then make the resulting ultra-commutative monoid globally discrete by attaching free ultra-commutative monoid cells. In more detail, we let J be the set of all pairs

$$(y,z) \ \in \ \pi_0^G(T) \times \pi_0^G(T)$$

such that $\epsilon(y) = \epsilon(z)$, where G runs over a set of representatives of isomorphism classes of compact Lie groups. We represent the classes as morphisms of orthogonal spaces

$$f_{(y,z)} \ : \ \{0,1\} \times B_{\mathrm{gl}} G_{(y,z)} \ \longrightarrow \ T$$

such that

$$(f_{(y,z)}(0,-))_*(e_{G_{(y,z)}}) \ = \ y \qquad \text{and} \qquad (f_{(y,z)}(1,-))_*(e_{G_{(y,z)}}) \ = \ z \ ;$$

this involves an implicit choice of faithful representation of the group $G_{(y,z)}$. We form the disjoint union of all these morphisms and freely extend that to a morphism of ultra-commutative monoids

$$F \ : \ \mathbb{P}\left(\{0,1\} \times \coprod\nolimits_{(y,z)\in J} B_{\mathrm{gl}} G_{(y,z)}\right) \ \longrightarrow \ T \ .$$

We let T' be a pushout, in the category of ultra-commutative monoids, of the diagram

$$\mathbb{P}\left([0,1] \times \coprod\nolimits_{(y,z)\in J} B_{\mathrm{gl}} G_{(y,z)}\right) \ \longleftarrow \ \mathbb{P}\left(\{0,1\} \times \coprod\nolimits_{(y,z)\in J} B_{\mathrm{gl}} G_{(y,z)}\right) \ \xrightarrow{F} \ T \ ,$$

where the left map is induced by the inclusion of the endpoints of the interval $[0,1]$. The resulting morphism of ultra-commutative monoids $T \longrightarrow T'$ induces an epimorphism of global power monoids

$$\underline{\pi}_0(T) \ \longrightarrow \ \underline{\pi}_0(T')$$

that sends y and z to the same class for all $(y,z) \in J$. So it factors over an isomorphism of global power monoids $\underline{\pi}_0(T') \cong M$. We can then take $M^\sharp = T'_{\mathrm{dis}}$, where $\kappa : T' \longrightarrow T'_{\mathrm{dis}}$ is as in part (ii).

Part (iv) is again proved in much the same way as its orthogonal space analog in Proposition I.5.21 (iii). The existence of a right adjoint is a formal consequence of the fact that for every global power monoid M the functor

$$\underline{\pi}_0 \ : \ \mathrm{Ho}(umon) \ \longrightarrow \ (\text{sets}) \ , \quad R \ \longmapsto \ (\text{global power monoids})(\underline{\pi}_0(R), M)$$

is representable (by combining parts (i) and (iii)). □

We end with another 'comonadic' perspective on the algebra of global power monoids. We introduce the functor exp of *exponential sequences* that takes an abelian Rep-monoid M to another abelian Rep-monoid $\exp(M)$. We give exp the structure of a comonad and show in Theorem 2.44 that global power monoids are precisely the coalgebras over this comonad.

Construction 2.40. We let M be an abelian Rep-monoid and G a compact Lie group. For consistency with the analogous construction for global power functors in Section V.2 we write the monoid structure multiplicatively. We let

$$\exp(M;G) \ \subset \ \prod\nolimits_{n\geq 0} M(\Sigma_n \wr G)$$

be the set of *exponential sequences*, i.e., of those families $(x_n)_n$ that satisfy $x_0 = 1$ in $M(\Sigma_0 \wr G) = M(e)$ and

$$\Phi^*_{k,n-k}(x_n) \ = \ x_k \times x_{n-k}$$

in $M((\Sigma_k \wr G) \times (\Sigma_{n-k} \wr G))$ for all $0 < k < n$, where $\Phi_{k,n-k}$ is the monomorphism defined in (2.5). Given a continuous group homomorphism $\alpha : K \longrightarrow G$ and an exponential sequence $x \in \exp(M; G)$, then

$$\begin{aligned} \Phi^*_{k,n-k}((\Sigma_n \wr \alpha)^*(x_n)) &= ((\Sigma_k \wr \alpha) \times (\Sigma_{n-k} \wr \alpha))^*(\Phi^*_{k,n-k}(x_n)) \\ &= ((\Sigma_k \wr \alpha) \times (\Sigma_{n-k} \wr \alpha))^*(x_k \times x_{n-k}) \\ &= (\Sigma_k \wr \alpha)^*(x_k) \times (\Sigma_{n-k} \wr \alpha)^*(x_{n-k}) \ . \end{aligned}$$

The first step uses that

$$(\Sigma_n \wr \alpha) \circ \Phi_{k,n-k} \ = \ \Phi_{k,n-k} \circ ((\Sigma_k \wr \alpha) \times (\Sigma_{n-k} \wr \alpha))$$

as homomorphisms from $(\Sigma_k \wr K) \times (\Sigma_{n-k} \wr K)$ to $\Sigma_n \wr G$. So the sequence

$$\alpha^*(x) \ = \ ((\Sigma_n \wr \alpha)^*(x_n))_{n \geq 0}$$

is again exponential, and this defines a map

$$\alpha^* = \exp(M; \alpha) \ : \ \exp(M; G) \ \longrightarrow \ \exp(M; K) \ .$$

We define a multiplication on the set $\exp(M; G)$ by coordinatewise multiplication in the monoid $M(\Sigma_n \wr G)$, i.e.,

$$(x \cdot y)_n \ = \ x_n \cdot y_n \ .$$

Proposition 2.41. *Let M be an abelian Rep-monoid. For every compact Lie group G, the set* $\exp(M; G)$ *of exponential sequences is an abelian monoid under componentwise multiplication. For varying G, the restriction maps α^* make* $\exp(M) = \exp(M; -)$ *into an abelian Rep-monoid.*

Proof. If x and y are exponential sequences, then the relation

$$\begin{aligned} \Phi^*_{k,n-k}(x_n \cdot y_n) &= \Phi^*_{k,n-k}(x_n) \cdot \Phi^*_{k,n-k}(y_n) \\ &= (x_k \times x_{n-k}) \cdot (y_k \times y_{n-k}) \ = \ (x_k \cdot y_k) \times (x_{n-k} \cdot y_{n-k}) \end{aligned}$$

holds in $M((\Sigma_k \wr G) \times (\Sigma_{n-k} \wr G))$; so the product $x \cdot y$ is again exponential. The product is associative and commutative since all the monoids $M(\Sigma_n \wr G)$ have this property. The exponential sequence $(1)_{n \geq 0}$ is a multiplicative unit.

The restriction map $\alpha^* : \exp(M; G) \longrightarrow \exp(M; K)$ is multiplicative and unital because each of the maps $(\Sigma_n \wr \alpha)^* : M(\Sigma_n \wr G) \longrightarrow M(\Sigma_n \wr K)$ is multiplicative and unital. Functoriality of the restriction maps is a consequence of functoriality in M and the relation

$$(\Sigma_n \wr \alpha) \circ (\Sigma_n \wr \beta) \ = \ \Sigma_n \wr (\alpha\beta) \ . \qquad \square$$

The previous proposition establishes the functor exp of exponential sequences as an endofunctor of the category of abelian Rep-monoids; now we make this endofunctor into a comonad. A natural transformation of abelian Rep-monoids

$$\eta_M \ : \ \exp(M) \ \longrightarrow \ M$$

is given by $\eta(x) = x_1$, using the identification $G \cong \Sigma_1 \wr G$ via $g \mapsto (1; g)$. A natural transformation

$$\kappa_M \ : \ \exp(M) \ \longrightarrow \ \exp(\exp(M))$$

is given at a compact Lie group G by

$$(\kappa(x)_n)_k \ = \ \Psi^*_{k,n}(x_{kn}) \ \in \ M(\Sigma_k \wr (\Sigma_n \wr G)) \ ;$$

here the restriction is along the monomorphism (2.6)

$$\begin{aligned} \Psi_{k,n} \ : \ \Sigma_k \wr (\Sigma_n \wr G) \qquad &\longrightarrow \qquad \Sigma_{kn} \wr G \\ (\sigma; (\tau_1; g^1), \ldots, (\tau_k; g^k)) \ &\longmapsto \ (\sigma \natural (\tau_1, \ldots, \tau_k); g^1 + \cdots + g^k) \ . \end{aligned}$$

Proposition 2.42. *Let M be an abelian Rep-monoid.*

(i) *For every compact Lie group G and for every exponential sequence $x \in \exp(M;G)$, the sequence $\kappa(x)$ is an element of* $\exp(\exp(M);G)$.

(ii) *As the group varies, the maps κ form a morphism of abelian Rep-monoids $\kappa_M : \exp(M) \longrightarrow \exp(\exp(M))$, natural in M.*

(iii) *The natural transformations*

$$\eta \ : \ \exp \ \longrightarrow \ \mathrm{Id} \qquad \textit{and} \qquad \kappa \ : \ \exp \ \longrightarrow \ \exp \circ \exp$$

make the functor exp *into a comonad on the category of abelian Rep-monoids.*

Proof. (i) Because the square of group homomorphisms

$$\begin{array}{ccc} (\Sigma_j \wr (\Sigma_n \wr G)) \times (\Sigma_{k-j} \wr (\Sigma_n \wr G)) & \xrightarrow{\Phi_{j,k-j}} & \Sigma_k \wr (\Sigma_n \wr G) \\ \downarrow{\scriptstyle \Psi_{j,n} \times \Psi_{k-j,n}} & & \downarrow{\scriptstyle \Psi_{k,n}} \\ (\Sigma_{jn} \wr G) \times (\Sigma_{(k-j)n} \wr G) & \xrightarrow[\Phi_{jn,(k-j)n}]{} & \Sigma_{kn} \wr G \end{array}$$

commutes, the elements $\Psi^*_{k,n}(x_{kn})$ satisfy

$$\begin{aligned} \Phi^*_{j,k-j}(\Psi^*_{k,n}(x_{kn})) \ &= \ (\Psi_{j,n} \times \Psi_{k-j,n})^*(\Phi^*_{jn,(k-j)n}(x_{kn})) \\ &= \ (\Psi_{j,n} \times \Psi_{k-j,n})^*(x_{jn} \times x_{(k-j)n}) \ = \ \Psi^*_{j,n}(x_{jn}) \times \Psi^*_{k-j,n}(x_{(k-j)n}) \ . \end{aligned}$$

This shows that for fixed $n \geq 0$, the sequence $\kappa(x)_n = (\Psi^*_{k,n}(x_{kn}))_{k \geq 0}$ is exponential, i.e., an element of the ring $\exp(M; \Sigma_n \wr G)$.

The square of group homomorphisms

$$\begin{array}{ccc} \Sigma_k \wr ((\Sigma_i \wr G) \times (\Sigma_{n-i} \wr G)) & \xrightarrow{\Sigma_k \wr \Phi_{i,n-i}} & \Sigma_k \wr (\Sigma_n \wr G) \\ \downarrow{\scriptstyle \Delta} & & \\ (\Sigma_k \wr (\Sigma_i \wr G)) \times (\Sigma_k \wr (\Sigma_{n-i} \wr G)) & & \downarrow{\scriptstyle \Psi_{k,n}} \\ \downarrow{\scriptstyle \Psi_{k,i} \times \Psi_{k,n-i}} & & \\ (\Sigma_{ki} \wr G) \times (\Sigma_{k(n-i)} \wr G) & \xrightarrow[\Phi_{ki,k(n-i)}]{} & \Sigma_{kn} \wr G \end{array} \tag{2.43}$$

does *not* commute; we invite the reader to check the case $k = n = 2$, $i = 1$ and $G = e$, where the phenomenon is already visible in the fact that the square

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\Sigma_2 \wr \Phi_{1,1}} & \Sigma_2 \wr \Sigma_2 \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \natural} \\ \Sigma_2 \times \Sigma_2 & \xrightarrow[+]{} & \Sigma_4 \end{array}$$

does *not* commute. However, the square (2.43) does commute up to conjugation by an element of $\Sigma_{kn} \wr G$. Since inner automorphisms are invisible through the eyes of a Rep-functor, we conclude that the relation

$$\begin{aligned} \left(\Phi^*_{i,n-i}(\kappa(x)_n)\right)_k \ &= \ (\Sigma_k \wr \Phi_{i,n-i})^*(\Psi^*_{k,n}(x_{kn})) \\ &= \ \Delta^*((\Psi_{k,i} \times \Psi_{k,n-i})^*(\Phi^*_{ki,k(n-i)}(x_{kn}))) \\ &= \ \Delta^*(\Psi^*_{k,i}(x_{ki}) \times \Psi^*_{k,n-i}(x_{k(n-i)})) \\ &= \ \Delta^*((\kappa(x)_i)_k \times (\kappa(x)_{n-i})_k) \ = \ (\kappa(x)_i \times \kappa(x)_{n-i})_k \end{aligned}$$

holds in $M(\Sigma_k \wr ((\Sigma_i \wr G) \times (\Sigma_{n-i} \wr G)))$. For varying $k \geq 0$, this shows that

$$\Phi^*_{i,n-i}(\kappa(x)_n) \;=\; \kappa(x)_i \times \kappa(x)_{n-i} \qquad \text{in } \exp(M; (\Sigma_i \wr G) \times (\Sigma_{n-i} \wr G)).$$

In other words, the sequence $\kappa(x) = (\kappa(x)_n)_{n\geq 0}$ is itself exponential.

(ii) The relations $\kappa(1) = 1$ and $\kappa(x \cdot y) = \kappa(x) \cdot \kappa(y)$ are straightforward from the definitions, using that multiplication is defined coordinatewise and that the restriction map $\Psi^*_{k,n}$ is multiplicative and unital. For every homomorphism $\alpha : K \longrightarrow G$ the relations

$$\begin{aligned}(\kappa(\alpha^*(x))_n)_k \;&=\; \Psi^*_{k,n}((\Sigma_{kn} \wr \alpha)^*(x_{kn})) \;=\; (\Sigma_k \wr \Sigma_n \wr \alpha)^*(\Psi^*_{k,n}(x_{kn})) \\ &=\; (\Sigma_k \wr \Sigma_n \wr \alpha)^*((\kappa(x)_n)_k) \;=\; ((\alpha^*(\kappa(x)))_n)_k\end{aligned}$$

holds in $M(\Sigma_k \wr \Sigma_n \wr K)$. So $\kappa \circ \alpha^* = \alpha^* \circ \kappa$. Altogether this shows that the maps $\kappa_M(G)$ are monoid homomorphisms and compatible with restriction, so they form a morphism of abelian Rep-monoids.

(iii) We have to show that the transformation κ is coassociative, and counital with respect to η, and these are all straightforward from the definitions. The counitality relations

$$\eta_{\exp(M)} \circ \kappa_M \;=\; \mathrm{Id}_{\exp(M)} \;=\; \exp(\eta_M) \circ \kappa_M$$

come down to the facts that the homomorphism $\Psi_{k,1}$ is the result of applying $\Sigma_k \wr -$ to the preferred isomorphism $\Sigma_1 \wr G \cong G$, and that the homomorphism $\Psi_{1,n}$ is the preferred isomorphism $\Sigma_1 \wr \Sigma_n \wr G \cong \Sigma_n \wr G$. The coassociativity relation

$$\exp(\kappa_M) \circ \kappa_M \;=\; \kappa_{\exp(M)} \circ \kappa_M$$

ultimately boils down to the observation that the following square of monomorphisms commutes:

$$\begin{tikzcd}
\Sigma_l \wr \Sigma_k \wr \Sigma_n \wr G \arrow[r, "\Sigma_l \wr \Psi_{k,n}"] \arrow[d, "\Psi_{l,k}"'] & \Sigma_l \wr \Sigma_{kn} \wr G \arrow[d, "\Psi_{l,kn}"] \\
\Sigma_{lk} \wr \Sigma_n \wr G \arrow[r, "\Psi_{lk,n}"'] & \Sigma_{lkn} \wr G
\end{tikzcd}$$

□

Now we can finally get to the main result of this section, identifying global power functors with coalgebras over the comonad of exponential sequences. We suppose that M is an abelian Rep-monoid and $P : M \longrightarrow \exp(M)$ a natural transformation of abelian Rep-monoids. For every compact Lie group G, a sequence of operations $P^n : M(G) \longrightarrow M(\Sigma_n \wr G)$ is then defined by

$$P^n(x) \;=\; (P(x))_n \;,$$

i.e., $P^n(x)$ is the n-th component of the exponential sequence $P(x)$.

Theorem 2.44 (Comonadic description of global power monoids).

(i) *Let M be an abelian Rep-monoid and $P : M \longrightarrow \exp(M)$ a morphism of abelian Rep-monoids that makes M into a coalgebra over the comonad $(\exp, \eta, \kappa)$. Then the operations $P^n : M(G) \longrightarrow M(\Sigma_n \wr G)$ make M into a global power monoid.*

(ii) *The functor*

$$(\exp\text{-coalgebras}) \;\longrightarrow\; (\text{global power monoids}) \;, \quad (M, P : M \longrightarrow \exp(M)) \;\longmapsto\; (M, \{P^n\}_{n\geq 0})$$

is an isomorphism of categories.

Proof. (i) The fact that $P : M \longrightarrow \exp(M)$ takes values in exponential sequences is equivalent to the additivity condition in Definition 2.8 (but written in multiplicative notation). The fact that $P : M \longrightarrow \exp(M)$ is a transformation of abelian Rep-monoids encodes simultaneously that the power operations

are monoid homomorphisms and the naturality condition of a global power monoid. The (multiplicatively written) identity relation $P^1 = \mathrm{Id}$ is equivalent to the counit condition of a coalgebra, i.e., that the composite

$$M \xrightarrow{P} \exp(M) \xrightarrow{\eta_M} M$$

is the identity. The transitivity relation is equivalent to

$$\exp(P) \circ P \;=\; \kappa_M \circ P \;,$$

the coassociativity condition of a coalgebra.

Part (ii) is essentially reading part (i) backwards, and we omit the details. □

Here is another family of global power monoids, with underlying Rep-functor the one represented by an abelian compact Lie groups. In fact, the next proposition shows that $\mathrm{Rep}(-, A)$ is the free global power monoid subject to a specific set of explicit 'power relations'.

Proposition 2.45. *For every abelian compact Lie group A, the functor $\mathrm{Rep}(-, A)$ has a unique structure of global power monoid. The monoid structure of $\mathrm{Rep}(G, A)$ is given by pointwise multiplication of homomorphisms. The power operation*

$$[m] \;:\; \mathrm{Rep}(G, A) \;\longrightarrow\; \mathrm{Rep}(\Sigma_m \wr G, A)$$

is given by

$$([m](\alpha))(\sigma;\, g_1, \dots, g_m) \;=\; \alpha(g_1) \cdot \ldots \cdot \alpha(g_m) \;.$$

Moreover, for every global power monoid M the map

$$(\text{global power monoids})(\mathrm{Rep}(-, A), M) \;\longrightarrow\; \{x \in M(A) \mid\; [m](x) = p_m^*(x) \text{ for all } m \geq 1\}$$

sending a morphism $f : \mathrm{Rep}(-, A) \longrightarrow M$ to the class $f(A)(1_A) \in M(A)$ is bijective.

Proof. Since A is abelian, conjugate homomorphisms into A are already equal, i.e., we can ignore the difference between homomorphisms and their conjugacy classes. We establish the description of the external products and power operations first, which also shows the uniqueness. Since $\mathrm{Rep}(e, A)$ has only one element, it is the additive unit. Since restriction maps are monoid homomorphisms, the trivial homomorphism is the neutral element of $\mathrm{Rep}(G, A)$. The sum

$$p_1 + p_2 \;\in\; \mathrm{Rep}(A \times A, A)$$

of the two projections is a homomorphism from $A \times A$ to A whose restriction along the two maps $(-, 1), (1, -) : A \longrightarrow A \times A$ is the identity. The only such homomorphism is the multiplication $\mu : A \times A \longrightarrow A$ of A, so we conclude that

$$p_1 + p_2 \;=\; \mu \;.$$

Naturality now gives

$$\alpha + \beta \;=\; (\alpha, \beta)^*(p_1) + (\alpha, \beta)^*(p_2) \;=\; (\alpha, \beta)^*(p_1 + p_2) \;=\; (\alpha, \beta)^*(\mu) \;=\; \mu \circ (\alpha, \beta) \;.$$

Since power operations refine power maps, the element

$$[m](\mathrm{Id}_A) \;\in\; \mathrm{Rep}(\Sigma_m \wr A, A)$$

restricts to the sum of the m projections on $A^m \leq \Sigma_m \wr A$. We define a homomorphism φ_m as the composite

$$\Sigma_m \wr e \xrightarrow{\Sigma_m \wr 0} \Sigma_m \wr A \xrightarrow{[m](\mathrm{Id}_A)} A \;,$$

where $0 : e \longrightarrow A$ is the unique homomorphism. Then

$$\varphi_m \;=\; [m](\mathrm{Id}_A) \circ (\Sigma_m \wr 0) \;=\; (\Sigma_m \wr 0)^*([m](\mathrm{Id}_A)) \;=\; [m](0^*(\mathrm{Id}_A)) \;=\; [m](0) \;=\; 0 \;,$$

using that the operation $[m]$ is a monoid homomorphism. Thus φ_m is the trivial homomorphism, and thus

$$([m](\mathrm{Id}_A))(\sigma;\, a_1, \dots, a_m) \;=\; ([m](\mathrm{Id}_A))(\sigma;\, 1, \dots, 1) \cdot ([m](\mathrm{Id}_A))(1;\, a_1, \dots, a_m) \;=\; a_1 \cdots a_m \;.$$

Naturality now gives

$$[m](\alpha) \;=\; [m](\alpha^*(\mathrm{Id}_A)) \;=\; (\Sigma_m \wr \alpha)([m](\mathrm{Id}_A)) \;.$$

Evaluating this on elements of $\Sigma_m \wr G$ yields the desired formula for the homomorphism $[m](\alpha)$.

It remains to show the existence of the global power monoid structure. Clearly, pointwise multiplication of homomorphisms makes $\mathrm{Rep}(G, A)$ into an abelian monoid (even an abelian group), and the monoid structure is contravariantly functorial in G. When we define $[m](\alpha)$ by the formula of the proposition, then the remaining axioms of a global power monoid (compare Definition 2.8) are similarly straightforward. Identity property (i) is clear, and naturality (ii) follows from the relation

$$[m](\alpha) \;=\; (\Sigma_m \wr \alpha)^*([m](\mathrm{Id}_A)) \;.$$

The transitivity relation (iii) holds by inspection:

$$\begin{aligned}
\Psi^*_{k,m}([km](\alpha))(\sigma; (\tau_1; h^1), \dots, (\tau_k, h^k)) \;&=\; ([km](\alpha))(\Psi_{k,m}(\sigma; (\tau_1; h^1), \dots, (\tau_k, h^k))) \\
&=\; \prod_{i=1}^{k} \left(\prod_{j=1}^{m} h^i_j \right) \;=\; \prod_{i=1}^{k} ([m](\alpha))(\tau_1; h^i)) \\
&=\; ([k]([m](\alpha)))(\sigma; (\tau_1; h^1), \dots, (\tau_k, h^k)) \;,
\end{aligned}$$

and so does the additivity relation (iv):

$$\begin{aligned}
\Phi^*_{i,m-i}([m](\alpha))((\sigma;\, g_1, \dots, g_i), (\sigma';\, g_{i+1}, \dots, g_m)) \;&=\; ([m](\alpha))(\sigma + \sigma';\, g_1, \dots, g_m) \\
&=\; \alpha(g_1) \cdot \dots \cdot \alpha(g_m) \\
&=\; ([i](\alpha))(\sigma;\, g_1, \dots, g_i) \cdot ([m-i](\alpha))(\sigma';\, g_i, \dots, g_m) \\
&=\; ([i](\alpha) \oplus [m-i](\alpha))((\sigma;\, g_1, \dots, g_i), (\sigma';\, g_{i+1}, \dots, g_m)) \;.
\end{aligned}$$

It remains to prove the formula for global power morphisms out of $\mathrm{Rep}(-, A)$. Since the class 1_A generates $\mathrm{Rep}(-, A)$ as a Rep-functor, the evaluation map in injective. For surjectivity we consider a class $x \in M(A)$ such that $P^m(x) = p_m^*(x)$ for all $m \geq 1$. Then there is a unique morphism of Rep-functors

$$f \;:\; \mathrm{Rep}(-, A) \;\longrightarrow\; M$$

such that $f_A(1_A) = x$. We need to show that f is even a morphism of global power monoids, i.e., multiplicative and compatible with power operations. The multiplicativity means that the following square of global power monoids commutes:

$$\begin{tikzcd}
\mathrm{Rep}(-, A) \times \mathrm{Rep}(-, A) \arrow[r, "f \times f"] \arrow[d] & M \times M \arrow[d] \\
\mathrm{Rep}(-, A) \arrow[r, "f"'] & M
\end{tikzcd}$$

where the vertical maps are the respective multiplications. The upper left corner of the square is isomorphic to the free Rep-functor $\mathrm{Rep}(-, A \times A)$. The commutativity of the square is thus equivalent to the relation

$$\mu^*(x) \;=\; x \oplus x$$

in $M(A \times A)$, where $\mu : A \times A \longrightarrow A$ is the multiplication map of the Lie group. We let $i : A \times A \longrightarrow \Sigma_2 \wr A$ denote the inclusion; then this relation follows by

$$\mu^*(x) \;=\; (p_2 \circ i)^*(x) \;=\; \mathrm{res}^{\Sigma_2 \wr A}_{A \times A}(p_2^*(x)) \;=\; \mathrm{res}^{\Sigma_2 \wr A}_{A \times A}(P^2(x)) \;=\; P^1(x) \oplus P^1(x) \;=\; x \oplus x \;.$$

This shows that f is a morphism of abelian Rep-monoids. Finally, we show that f is compatible with power operations. By the comonadic description of global power monoids (Theorem II.2.44), this is equivalent to the commutativity of the following square of abelian Rep-monoids:

$$\begin{tikzcd} \mathrm{Rep}(-,A) \arrow[r, "f"] \arrow[d, "P"'] & M \arrow[d, "P"] \\ \exp(\mathrm{Rep}(-,A)) \arrow[r, "\exp(f)"'] & \exp(M) \end{tikzcd}$$

where the vertical morphisms are the total power operations. Since $\mathrm{Rep}(-,A)$ is generated as a Rep-functor by the element 1_A, it suffices to check the commutativity on this one element, where it amounts to the relations $P^m(x) = p_m^*(x)$ that the class x was assumed to satisfy. □

3. Examples of ultra-commutative monoids

In this section we discuss various examples of ultra-commutative monoids, mostly of a geometric nature, and various geometrically defined morphisms between them. We start with the ultra-commutative monoids **O** and **SO** (Example 3.6), **U** and **SU** (Example 3.7), **Sp** (Example 3.8) and **Spin** and $\mathbf{Spin}^c$ (Example 3.9), all made from the corresponding families of classical Lie groups. All of these are examples of 'symmetric monoid valued orthogonal spaces' in the sense of Definition 3.4, a more general source of examples of ultra-commutative monoids. The additive Grassmannian **Gr** (Example 3.11), the oriented variant $\mathbf{Gr}^{\mathrm{or}}$ (Example 3.15) and the complex and quaternionic analogs $\mathbf{Gr}^{\mathbb{C}}$ and $\mathbf{Gr}^{\mathbb{H}}$ (Example 3.16) consist – as the names suggest – of Grassmannians with monoid structure arising from direct sum of subspaces. The multiplicative Grassmannian $\mathbf{Gr}_{\otimes}$ (Example 3.18) is globally equivalent as an orthogonal space to **Gr**, but the monoid structure arises from tensor product of subspaces; the global projective space **P** is the ultra-commutative submonoid of $\mathbf{Gr}_{\otimes}$ consisting of lines (1-dimensional subspaces). The global projective space **P** is a multiplicative model of a global classifying space for the cyclic group of order 2, and Example 3.22 describes multiplicative models of global classifying spaces for all *abelian* compact Lie groups. Example 3.23 introduces the ultra-commutative monoid **F** of unordered frames, with monoid structure arising from disjoint union. The ultra-commutative 'multiplicative monoid of the sphere spectrum' $\Omega^\bullet\mathbb{S}$ is introduced in Example 3.27; this a special case of the multiplicative monoid of an ultra-commutative ring spectrum, and we return to the more general construction in Example IV.1.16.

Construction 3.1 (Orthogonal monoid spaces from monoid valued orthogonal spaces). Our first series of examples involves orthogonal spaces made from the infinite families of classical Lie groups, namely the orthogonal, unitary and symplectic groups, the special orthogonal and unitary groups, and the pin, pin^c, spin and spin^c groups. These orthogonal spaces have the special feature that they are *group valued*; we will now explain that a group valued (or even just a monoid valued) orthogonal space automatically leads to an orthogonal monoid space. In the cases of the orthogonal, unitary, spin, spin^c and symplectic groups, these multiplications are commutative, so those examples yield ultra-commutative monoids.

Definition 3.2. A *monoid valued orthogonal space* is a monoid object in the category of orthogonal spaces.

Since orthogonal spaces are an enriched functor category, monoid valued orthogonal spaces are the same thing as continuous functors from the category **L** to the category of topological monoids and continuous monoid homomorphisms (i.e., monoid objects in the category **T** of compactly generated spaces).

Now we let M be a monoid valued orthogonal space. In (3.37) of Chapter I we introduced a lax symmetric monoidal natural transformation

$$\rho_{X,Y} \ : \ X \boxtimes Y \ \longrightarrow \ X \times Y$$

from the box product to the cartesian product of orthogonal spaces. For $X = Y = M$ we can form the composite

$$M \boxtimes M \xrightarrow{\rho_{M,M}} M \times M \xrightarrow{\text{multiplication}} M \tag{3.3}$$

with the objectwise multiplication of M. Since $\rho_{X,Y}$ is lax monoidal, this composite makes M into an orthogonal monoid space with unit the multiplicative unit $1 \in M(0)$. If we unpack this definition we see that the above composite corresponds, via the universal property of $\boxtimes$, to the bimorphism whose (V, W)-component is the composite

$$\mu_{V,W} \ : \ M(V) \times M(W) \xrightarrow{M(i_V)\times M(i_W)} M(V \oplus W) \times M(V \oplus W) \xrightarrow{\text{multiply}} M(V \oplus W) \ ,$$

where $i_V : V \longrightarrow V \oplus W$ and $i_W : W \longrightarrow V \oplus W$ are the direct summand embeddings. If $f : M \longrightarrow M'$ is a morphism of monoid valued orthogonal spaces (i.e., a morphism of orthogonal spaces that is objectwise a monoid homomorphism), then f is also a homomorphism of orthogonal monoid spaces with respect to the multiplications (3.3).

If M is a commutative monoid valued orthogonal space, then the associated $\boxtimes$-multiplication is also commutative, simply because the transformation $\rho_{X,Y}$ is *symmetric* monoidal. However, there is a more general condition on M that provides M with the structure of an ultra-commutative monoid.

Definition 3.4. A monoid valued orthogonal space M is *symmetric* if for all inner product spaces V and W the images of the two homomorphisms

$$M(i_V) \ : \ M(V) \ \longrightarrow \ M(V \oplus W) \qquad \text{and} \qquad M(i_W) \ : \ M(W) \ \longrightarrow \ M(V \oplus W)$$

commute.

We emphasize that the objectwise multiplications in a symmetric monoid valued orthogonal space need not be commutative – we'll discuss many interesting examples of this kind below. The proof of the following proposition is straightforward from the definitions, and we omit it.

Proposition 3.5. *Let M be a symmetric monoid valued orthogonal space. Then the multiplication* (3.3) *makes M into an ultra-commutative monoid.*

Example 3.6 (Orthogonal group ultra-commutative monoid). We denote by $\mathbf{O}$ the orthogonal space that sends an inner product space V to its orthogonal group $O(V)$. A linear isometric embedding $\varphi : V \longrightarrow W$ induces a continuous group homomorphism $\mathbf{O}(\varphi) : \mathbf{O}(V) \longrightarrow \mathbf{O}(W)$ by conjugation (and the identity on the orthogonal complement of the image of φ). Construction 3.1 then gives $\mathbf{O}$ the structure of an orthogonal monoid space. The (V, W)-component of the bimorphism

$$\mu_{V,W} \ : \ \mathbf{O}(V) \times \mathbf{O}(W) \longrightarrow \ \mathbf{O}(V \oplus W)$$

is direct sum of orthogonal transformations. The unit element is the identity of the trivial vector space, the only element of $O(0)$. So $\mathbf{O}$ is a symmetric group valued orthogonal space, and hence it becomes an ultra-commutative monoid.

If G is a compact Lie group and V a G-representation, then the G-action on the group $\mathbf{O}(V) = O(V)$ is by conjugation, so the fixed points $\mathbf{O}(V)^G$ is the group of G-equivariant orthogonal automorphisms of V. Moreover, $\mathbf{O}(\mathcal{U}_G)$ is the orthogonal group of $\mathcal{U}_G$, i.e., $\mathbb{R}$-linear isometries of $\mathcal{U}_G$ (not necessarily G-equivariant) that are the identity on the orthogonal complement of some finite dimensional subspace; the G-action is again by conjugation. Any G-equivariant isometry preserves the decomposition of $\mathcal{U}_G$ into isotypical summands, and the restriction to almost all of these isotypical summands must be the identity. The G-fixed subgroup is thus given by

$$\mathbf{O}(\mathcal{U}_G)^G \ = \ O^G(\mathcal{U}_G) \ = \ \prod_{[\lambda]}{}' \, O^G(\mathcal{U}_\lambda) \ ,$$

where the weak product is indexed by the isomorphism classes of irreducible orthogonal G-representations λ, and $\mathcal{U}_\lambda$ is the λ-isotypical summand. If the compact Lie group G is finite, then there are only finitely many isomorphism classes of irreducible G-representations, so in that case the weak product coincides with the product.

Irreducible orthogonal representations come in three different flavors, and the group $O^G(\mathcal{U}_\lambda)$ has one of three different forms. If λ is an irreducible orthogonal G-representation, then the endomorphism ring $\mathrm{Hom}_{\mathbb{R}G}(\lambda,\lambda)$ is a finite dimensional skew field extension of $\mathbb{R}$, so it is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; the representation λ is accordingly called 'real', 'complex' respectively 'quaternionic'. We have

$$O^G(\mathcal{U}_\lambda) \;\cong\; O^G(\lambda^\infty) \;\cong\; \begin{cases} O & \text{if } \lambda \text{ is real,} \\ U & \text{if } \lambda \text{ is complex, and} \\ Sp & \text{if } \lambda \text{ is quaternionic.} \end{cases}$$

So we conclude that the G-fixed point space $\mathbf{O}(\mathcal{U}_G)^G$ is a weak product of copies of the infinite orthogonal, unitary and symplectic groups, indexed by the different types of irreducible orthogonal representations of G. Since the infinite unitary and symplectic groups are connected, only the 'real' factors contribute to $\pi_0(\mathbf{O}(\mathcal{U}_G)^G) = \pi_0^G(\mathbf{O})$, which is a weak product of copies of $\pi_0(O) = \mathbb{Z}/2$ indexed by the irreducible G-representations of real type.

There is a straightforward 'special orthogonal' analog: the property of having determinant 1 is preserved under conjugation by linear isometric embeddings and under direct sum of linear isometries, so the spaces $SO(V)$ form an ultra-commutative submonoid $\mathbf{SO}$ of $\mathbf{O}$. Here, $\mathbf{SO}(\mathcal{U}_G)$ is the group of $\mathbb{R}$-linear isometries of $\mathcal{U}_G$ (not necessarily G-equivariant) that have determinant 1 on some finite dimensional subspace V of $\mathcal{U}_G$ and are the identity on the orthogonal complement of V.

Example 3.7 (Unitary group ultra-commutative monoid). There is a straightforward unitary analog $\mathbf{U}$ of $\mathbf{O}$, defined as follows. Given an inner product space V, we denote by

$$V_{\mathbb{C}} \;=\; \mathbb{C} \otimes_{\mathbb{R}} V$$

its complexification. The euclidean inner product $\langle -,-\rangle$ on V induces a hermitian inner product $(-,-)$ on the complexification $V_{\mathbb{C}}$, defined as the unique hermitian inner product that satisfies

$$(1\otimes v, 1\otimes w) \;=\; \langle v,w\rangle$$

for all $v,w\in V$. We now define an orthogonal space $\mathbf{U}$ by

$$\mathbf{U}(V) \;=\; U(V_{\mathbb{C}})\ ,$$

the unitary group of the complexification of V. The complexification of every $\mathbb{R}$-linear isometric embedding $\varphi : V \longrightarrow W$ preserves the hermitian inner products, so we can define a continuous group

$$\mathbf{U}(\varphi) \;:\; \mathbf{U}(V) \;\longrightarrow\; \mathbf{U}(W)$$

by conjugation with $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow W_{\mathbb{C}}$ and the identity on the orthogonal complement of the image of $\varphi_{\mathbb{C}}$. The $\boxtimes$ multiplication on $\mathbf{U}$ produced by Construction 3.1 is by direct sum of unitary transformations; this multiplication is also symmetric, and hence ultra-commutative. There is a straightforward 'special unitary' ultra-commutative submonoid $\mathbf{SU}$ of $\mathbf{U}$; the value $\mathbf{SU}(V)$ is the group of unitary automorphisms of $V_{\mathbb{C}}$ of determinant 1.

If G is a compact Lie group, then the identification of the G-fixed points of $\mathbf{U}$ also works much like the orthogonal analog. The outcome is in an isomorphism between $\mathbf{U}(\mathcal{U}_G)^G$ and $U^G(\mathcal{U}_G^{\mathbb{C}})$. Here $\mathcal{U}_G = \mathbb{C}\otimes_{\mathbb{R}}\mathcal{U}_G$ is the complexified complete universe for G, which happens to be a 'complete complex G-universe' in the sense that every finite dimensional complex G-representation embeds into it. The complete complex G-universe breaks up into (unitary) isotypical summands $\mathcal{U}_\lambda^{\mathbb{C}}$, indexed by the isomorphism classes of irreducible unitary G-representations λ, and the group $U^G(\mathcal{U}_G^{\mathbb{C}})$ breaks up accordingly as a weak product. In contrast to the orthogonal situation above, there is only one 'type' of irreducible unitary representation, and the

group $U^G(\mathcal{U}_\lambda^{\mathbb{C}})$ is always isomorphic to the infinite unitary group U, independent of λ. So in the unitary context, we get a decomposition

$$\mathbf{U}(\mathcal{U}_G)^G \;=\; U^G(\mathcal{U}_G^{\mathbb{C}}) \;=\; \prod_{[\lambda]}{}' \, U^G(\mathcal{U}_\lambda^{\mathbb{C}}) \;\cong\; \prod_{[\lambda]}{}' \, U \;.$$

This weak product is indexed by the isomorphism classes of irreducible unitary G-representations. Since the unitary group U is connected, the set $\pi_0(\mathbf{U}(\mathcal{U}_G)^G) = \pi_0^G(\mathbf{U})$ has only one element, and so $\mathbf{U}$ is globally connected.

The orthogonal monoid space $\mathbf{U}$ comes with an involution

$$\psi \;:\; \mathbf{U} \;\longrightarrow\; \mathbf{U}$$

by *complex conjugation* that is an automorphism of ultra-commutative monoids. The value of ψ at an inner product space V is the map

$$\psi(V) \;:\; U(V_{\mathbb{C}}) \;\longrightarrow\; U(V_{\mathbb{C}}) \;, \quad A \;\longmapsto\; \psi_V \circ A \circ \psi_V \;,$$

where

$$\psi_V \;:\; V_{\mathbb{C}} \;\longrightarrow\; V_{\mathbb{C}} \;, \quad \lambda \otimes v \;\longmapsto\; \bar{\lambda} \otimes v$$

is the canonical $\mathbb{C}$-semilinear 'conjugation' map on $V_{\mathbb{C}}$.

The *complexification morphism*

$$c \;:\; \mathbf{O} \;\longrightarrow\; \mathbf{U}$$

is the homomorphism of ultra-commutative monoids given by complexification

$$c(V) \;:\; O(V) \;\longrightarrow\; U(V_{\mathbb{C}}) \;, \quad \varphi \;\longmapsto\; \varphi_{\mathbb{C}} \;.$$

Complexification is an isomorphism onto the ψ-invariant ultra-commutative submonoid of $\mathbf{U}$, and it takes $\mathbf{SO}$ to $\mathbf{SU}$.

Every hermitian inner product space W has an underlying $\mathbb{R}$-vector space equipped with the euclidean inner product defined by

$$\langle v, w \rangle \;=\; \mathrm{Re}(v, w) \;,$$

the real part of the given hermitian inner product. Every $\mathbb{C}$-linear isometric embedding is in particular an $\mathbb{R}$-linear isometric embedding of underlying euclidean vector spaces. In particular, the unitary group $U(W)$ is a subgroup of the orthogonal group of the underlying euclidean vector space of W: We can thus define the *realification morphism*

$$r \;:\; \mathbf{U} \;\longrightarrow\; \mathrm{sh}_{\mathbb{C}}^{\otimes}(\mathbf{O})$$

at V as the inclusion

$$r(V) \;:\; U(V_{\mathbb{C}}) \;\longrightarrow\; O(\mathbb{C} \otimes V) \;.$$

Here $\mathrm{sh}_{\mathbb{C}}^{\otimes}$ denotes the multiplicative shift by $\mathbb{C}$ as defined in Example 1.12. The realification morphism actually takes values in the submonoid $\mathrm{sh}_{\mathbb{C}}^{\otimes}(\mathbf{SO})$, and it is a homomorphism of ultra-commutative monoids.

Example 3.8 (Symplectic group ultra-commutative monoid). There is also a quaternionic analog of $\mathbf{O}$ and $\mathbf{U}$, the ultra-commutative monoid $\mathbf{Sp}$ made from symplectic groups. Given an $\mathbb{R}$-inner product space V, we denote by

$$V_{\mathbb{H}} \;=\; \mathbb{H} \otimes_{\mathbb{R}} V$$

the extension of scalars to the skew field $\mathbb{H}$ of quaternions. The extension comes with a $\mathbb{H}$-sesquilinear form

$$[-,-] \;:\; V_{\mathbb{H}} \times V_{\mathbb{H}} \;\longrightarrow\; \mathbb{H}$$

defined by

$$[\lambda \otimes v, \mu \otimes w] \;=\; \lambda \bar{\mu} \langle v, w \rangle$$

for $\lambda, \mu \in \mathbb{H}$ and $v, w \in V$. The symplectic group

$$\mathbf{Sp}(V) \;=\; Sp(V_{\mathbb{H}})$$

is the compact Lie group of $\mathbb{H}$-linear automorphisms $A : V_{\mathbb{H}} \longrightarrow V_{\mathbb{H}}$ that leave the form invariant, i.e., such that

$$[Ax, Ay] \;=\; [x, y]$$

for all $x, y \in V_{\mathbb{H}}$. The $\mathbb{H}$-linear extension $\varphi_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} \varphi : V_{\mathbb{H}} \longrightarrow W_{\mathbb{H}}$ of an $\mathbb{R}$-linear isometric embedding $\varphi : V \longrightarrow W$ preserves the new inner products, so we can define a continuous group

$$\mathbf{Sp}(\varphi) \;:\; \mathbf{Sp}(V) \;\longrightarrow\; \mathbf{Sp}(W)$$

by conjugation with $\varphi_{\mathbb{H}} : V_{\mathbb{H}} \longrightarrow W_{\mathbb{H}}$ and the identity on the orthogonal complement of the image of $\varphi_{\mathbb{H}}$. As for $\mathbf{O}$ and $\mathbf{U}$, the $\boxtimes$ multiplication on $\mathbf{Sp}$ produced by Construction 3.1 is by direct sum of symplectic automorphisms; so $\mathbf{Sp}$ is symmetric, hence ultra-commutative.

If G is a compact Lie group, then the identification of the G-fixed points of $\mathbf{Sp}$ also works much like the orthogonal case. Quaternionic representations decompose canonically into isotypical summands, and this results in a product decomposition for the G-fixed subgroup

$$\mathbf{Sp}(\mathcal{U}_G)^G \;=\; (Sp(\mathcal{U}_G^{\mathbb{H}}))^G \;=\; \prod_{[\lambda]}{}' (Sp(\mathcal{U}_\lambda^{\mathbb{H}}))^G \;,$$

where the weak product is indexed by the isomorphism classes of irreducible quaternionic G-representations λ, and $\mathcal{U}_\lambda^{\mathbb{H}}$ is the λ-isotypical summand in $\mathcal{U}_G^{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} \mathcal{U}_G$. As in the real case, irreducible quaternionic representations λ come in three different flavors, depending on whether the endomorphism ring $\mathrm{Hom}_{\mathbb{H}G}(\lambda, \lambda)$ – again a finite dimensional skew field extension of $\mathbb{R}$ – is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Moreover,

$$(Sp(\mathcal{U}_\lambda^{\mathbb{H}}))^G \;\cong\; (Sp(\lambda^\infty))^G \;\cong\; \begin{cases} O & \text{if } \lambda \text{ is real,} \\ U & \text{if } \lambda \text{ is complex, and} \\ Sp & \text{if } \lambda \text{ is quaternionic.} \end{cases}$$

So the G-fixed point space $\mathbf{Sp}(\mathcal{U}_G)^G$ is a weak product of copies of the infinite orthogonal, unitary and symplectic groups, indexed by the different types of irreducible quaternionic representations of G.

There is a symplectic analog of complex conjugation on $\mathbf{U}$, as follows. We let $\psi : \mathbb{H} \longrightarrow \mathbb{H}$ be any $\mathbb{R}$-algebra automorphism. Then for every $\mathbb{R}$-linear inner product space V the map

$$\psi_V \;=\; \psi \otimes_{\mathbb{R}} V \;:\; V_{\mathbb{H}} \;\longrightarrow\; V_{\mathbb{H}}$$

is ψ-linear in the sense that

$$\psi_V(\lambda \cdot x) \;=\; \psi(\lambda) \cdot \psi_V(x)$$

for all $\lambda \in \mathbb{H}$ and $x \in V_{\mathbb{H}}$. So for every $A \in Sp(V_{\mathbb{H}})$ the conjugate

$${}^{\psi}A \;=\; \psi_V \circ A \circ \psi_V^{-1} \;:\; V_{\mathbb{H}} \;\longrightarrow\; V_{\mathbb{H}}$$

is again $\mathbb{H}$-linear, and it preserves the inner product $[-,-]$ on $V_{\mathbb{H}}$. So conjugation by ψ_V is a continuous group homomorphism

$$\psi(V) \;:\; Sp(V_{\mathbb{H}}) \;\longrightarrow\; Sp(V_{\mathbb{H}}) \;, \quad A \;\longmapsto\; \psi_V \circ A \circ \psi_V \;.$$

We let $\mathrm{Aut}(\mathbb{H})$ be the Lie group of $\mathbb{R}$-algebra automorphism of the quaternions. In fact, every $\mathbb{R}$-automorphism of $\mathbb{H}$ is inner, and $\mathrm{Aut}(\mathbb{H})$ is abstractly isomorphism to the group $SO(3)$. The conjugation maps $\psi(V)$ also depend continuously on ψ, so altogether we obtain a continuous action of the Lie group $\mathrm{Aut}(\mathbb{H})$ on $\mathbf{Sp}$ through homomorphisms of ultra-commutative monoids.

The symplectic analog of the complexification morphism is the homomorphism of ultra-commutative monoids $c_{\mathbb{H}} : \mathbf{O} \longrightarrow \mathbf{Sp}$ whose value at V is the map

$$c_{\mathbb{H}}(V) \;:\; O(V) \;\longrightarrow\; Sp(V_{\mathbb{H}}) \;, \quad \varphi \;\longmapsto\; \varphi_{\mathbb{H}} \;.$$

This morphism is an isomorphism onto the $\mathrm{Aut}(\mathbb{H})$-invariant ultra-commutative submonoid of $\mathbf{Sp}$.

Example 3.9 (Pin and Spin group orthogonal spaces). Given a real inner product space V we denote by $\mathrm{Cl}(V)$ the Clifford algebra of the negative definite quadratic form on V, i.e., the quotient of the $\mathbb{R}$-tensor algebra of V by the ideal generated by $v \otimes v + \langle v, v\rangle \cdot 1$ for all $v \in V$; the Clifford algebra is $\mathbb{Z}/2$-graded with the even (respectively odd) part generated by an even (respectively odd) number of vectors from V. The composite

$$V \xrightarrow{\text{linear summand}} TV \xrightarrow{\text{proj}} \mathrm{Cl}(V)$$

is $\mathbb{R}$-linear and injective, and we denote it by $v \mapsto [v]$.

We recall that orthogonal vectors of V anti-commute in the Clifford algebra: given $v, \bar{v} \in V$ with $\langle v, \bar{v}\rangle = 0$, then

$$\begin{aligned} [v][\bar{v}] + [\bar{v}][v] &= (|v|^2 \cdot 1 + [v]^2) + [v][\bar{v}] + [\bar{v}][v] + (|\bar{v}^2| \cdot 1 + [\bar{v}]^2) \\ &= |v|^2 \cdot 1 + |\bar{v}^2| \cdot 1 + [v + \bar{v}]^2 \;=\; \big(|v|^2 + |\bar{v}^2| - |v + \bar{v}|^2\big) \cdot 1 \;=\; 0\ . \end{aligned}$$

In $\mathrm{Cl}(V)$ every unit vector $v \in S(V)$ satisfies $[v]^2 = -1$, so all unit vectors of V become units in $\mathrm{Cl}(V)$. The *pin group* of V is the subgroup

$$\mathrm{Pin}(V) \;\subset\; \mathrm{Cl}(V)^\times$$

generated inside the multiplicative group of $\mathrm{Cl}(V)$ by ± 1 and all unit vectors of V. There is a straightforward way to make the pin groups into an ultra-commutative monoid: a linear isometric embedding $\varphi : V \longrightarrow W$ induces a morphism of $\mathbb{Z}/2$-graded $\mathbb{R}$-algebras $\mathrm{Cl}(\varphi) : \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(W)$ that restricts to φ on V. So $\mathrm{Cl}(\varphi)$ restricts to a continuous homomorphism

$$\mathrm{Pin}(\varphi) \;:\; \mathrm{Cl}(\varphi)|_{\mathrm{Pin}(V)} \;:\; \mathrm{Pin}(V) \;\longrightarrow\; \mathrm{Pin}(W)$$

between the pin groups. The map $\mathrm{Pin}(\varphi)$ depends continuously on φ and satisfies $\mathrm{Pin}(\psi) \circ \mathrm{Pin}(\varphi) = \mathrm{Pin}(\psi \circ \varphi)$, so we have defined a group-valued orthogonal space **Pin**. Construction 3.1 then gives **Pin** the structure of an orthogonal monoid space.

Since the group $\mathrm{Pin}(V)$ is generated by homogeneous elements of the Clifford algebra, all of its elements are homogeneous. The $\mathbb{Z}/2$-grading of $\mathrm{Cl}(V)$ provides a continuous homomorphism

$$\mathrm{Pin}(V) \;\longrightarrow\; \mathbb{Z}/2$$

whose kernel

$$\mathrm{Spin}(V) \;=\; \mathrm{Cl}(V)_{\mathrm{ev}} \cap \mathrm{Pin}(V)$$

is the *spin group* of V. The map $\mathrm{Pin}(\varphi)$ induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is homogeneous, so it restricts to a homomorphism

$$\mathrm{Spin}(\varphi) \;=\; \mathrm{Pin}(\varphi)|_{\mathrm{Spin}(V)} \;:\; \mathrm{Spin}(V) \;\longrightarrow\; \mathrm{Spin}(W)$$

between the spin groups. So the spin groups form an orthogonal monoid subspace **Spin** of **Pin**. We claim that **Spin** is symmetric, i.e., for all inner product spaces V and W the images of the two group homomorphisms

$$\mathrm{Spin}(V) \xrightarrow{\mathrm{Spin}(i_V)} \mathrm{Spin}(V \oplus W) \xleftarrow{\mathrm{Spin}(i_W)} \mathrm{Spin}(W)$$

commute, i.e.,

$$\mathrm{Spin}(i_V)(x) \cdot \mathrm{Spin}(i_W)(y) \;=\; \mathrm{Spin}(i_W)(y) \cdot \mathrm{Spin}(i_V)(x)$$

for all $x \in \mathrm{Spin}(V)$ and $y \in \mathrm{Spin}(W)$. Indeed, $\mathrm{Spin}(V)$ is generated by -1 and the elements $[v][v']$ for $v, v' \in S(V)$, and similarly for W. The elements -1 map to the central element -1 in $\mathrm{Spin}(V \oplus W)$, and

$$[v, 0] \cdot [v', 0] \cdot [0, w] \cdot [0, w'] \;=\; [0, w] \cdot [0, w'] \cdot [v, 0] \cdot [v', 0]$$

because $[v, 0]$ and $[v', 0]$ anti-commute with $[0, w]$ and $[0, w']$. So **Spin** is an ultra-commutative monoid with respect to the $\boxtimes$ multiplication of Construction 3.1.

In contrast to **Spin**, the group valued orthogonal space **Pin** is *not* symmetric; equivalently, the continuous map

$$\mu_{V,W} \ : \ \mathrm{Pin}(V) \times P(W) \longrightarrow \ \mathrm{Pin}(V \oplus W)$$

is *not* a group homomorphism. The issue is that for $v \in S(V)$ and $w \in S(W)$ the elements $[v, 0]$ and $[0, w]$ *anti-commute* in $\mathrm{Pin}(V \oplus W)$.

Now we turn to the groups pin^c and spin^c, the complex variations on the pin and spin groups. For a euclidean inner product space V we denote by $\mathrm{Cl}_{\mathbb{C}}(V) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V)$ the complexification of the Clifford algebra discussed in the previous example. So $\mathrm{Cl}_{\mathbb{C}}(V)$ is a $\mathbb{Z}/2$-graded $\mathbb{C}$-algebra with the even (respectively odd) part generated by an even (respectively odd) number of vectors from V.

The *pin^c group* of V is the subgroup

$$\mathrm{Pin}^c(V) \ \subset \ \mathrm{Cl}_{\mathbb{C}}(V)^{\times}$$

generated inside the multiplicative group of $\mathrm{Cl}_{\mathbb{C}}(V)$ by the unit scalars $\lambda \cdot 1$ for all $\lambda \in U(1)$ and all unit vectors of V. The pin^c-groups form a group values orthogonal space $\mathbf{Pin}^c$, and hence an orthogonal monoid space, in much the same way as for the pin groups above. As for **Pin**, the monoid valued orthogonal space $\mathbf{Pin}^c$ is *not* symmetry, so the associated $\boxtimes$ multiplication of $\mathbf{Pin}^c$ is *not* commutative.

Since the group $\mathrm{Pin}^c(V)$ is generated by homogeneous elements of the complexified Clifford algebra, all of its elements are homogeneous. So the $\mathbb{Z}/2$-grading of $\mathrm{Cl}_{\mathbb{C}}(V)$ provides a continuous homomorphism

$$\mathrm{Pin}^c(V) \ \longrightarrow \ \mathbb{Z}/2$$

whose kernel

$$\mathrm{Spin}^c(V) \ = \ \mathrm{Cl}_{\mathbb{C}}(V)_{\mathrm{ev}} \cap \mathrm{Pin}^c(V)$$

is the *spin^c group* of V. As V varies, the spin^c groups from a group valued orthogonal subspace $\mathbf{Spin}^c$ of $\mathbf{Pin}^c$. As for **Spin**, the images of the homomorphisms $\mathrm{Spin}^c(i_V) : \mathrm{Spin}^c(V) \longrightarrow \mathrm{Spin}^c(V \oplus W)$ and $\mathrm{Spin}^c(i_W) : \mathrm{Spin}^c(W) \longrightarrow \mathrm{Spin}^c(V \oplus W)$ commute, so $\mathbf{Spin}^c$ is even an ultra-commutative monoid.

We view the real Clifford algebra as an $\mathbb{R}$-subalgebra of the complexified Clifford algebra, via the embedding

$$1 \otimes - \ : \mathrm{Cl}(V) \ \longrightarrow \ \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V) \ = \ \mathrm{Cl}_{\mathbb{C}}(V) \ .$$

This embedding restricts to embeddings

$$\mathrm{Pin}(V) \ \longrightarrow \ \mathrm{Pin}^c(V) \qquad \text{and} \qquad \mathrm{Spin}(V) \ \longrightarrow \ \mathrm{Spin}^c(V) \ ;$$

as V varies through all inner product spaces, these map form morphisms of group valued orthogonal spaces $\mathbf{Pin} \longrightarrow \mathbf{Pin}^c$ and $\mathbf{Spin} \longrightarrow \mathbf{Spin}^c$, and hence morphisms of orthogonal monoid spaces. These maps extend to isomorphism

$$\mathbf{Pin} \times_{\{\pm 1\}} U(1) \ \cong \ \mathbf{Pin}^c \qquad \text{and} \qquad \mathbf{Spin} \times_{\{\pm 1\}} U(1) \ \cong \ \mathbf{Spin}^c$$

of orthogonal monoid spaces.

We let $\alpha : \mathrm{Cl}_{\mathbb{C}}(V) \longrightarrow \mathrm{Cl}_{\mathbb{C}}(V)$ denote the unique $\mathbb{C}$-algebra automorphism of the complex Clifford algebra that restricts to multiplication by -1 on V. The map α is the grading involution, i.e., it is the identity on the even part and -1 on the odd part of the Clifford algebra. For every element $x \in \mathrm{Pin}^c(V)$ the twisted conjugation map

$$c_x \ : \ \mathrm{Cl}_{\mathbb{C}}(V) \ \longrightarrow \ \mathrm{Cl}_{\mathbb{C}}(V) \ , \quad c_x(y) \ = \ \alpha(x) y x^{-1}$$

is an automorphism of $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras. Twisted conjugation by an element $v \in S(V)$ takes v to $-v$ and fixes the orthogonal complement of v in V. So twisted conjugation by $v \in S(V)$ is reflection in the hyperplane orthogonal to v, hence a linear isometry of V of determinant -1. The complex scalars are central in $\mathrm{Cl}_{\mathbb{C}}(V)$, so conjugation by $U(1) \cdot 1$ is the identity. So all the conjugation maps c_x for elements $x \in \mathrm{Pin}^c(V)$ restrict to linear isometries on V. We thus obtain a continuous group homomorphism

$$\mathrm{ad}(V) \ : \ \mathrm{Pin}^c(V) \ \longrightarrow \ O(V) \ , \quad x \ \longmapsto \ \mathrm{ad}(x) \ ,$$

the *twisted adjoint representation*, characterized by

$$[\mathrm{ad}(x)(v)] \;=\; \alpha(x)[v]x^{-1}\ .$$

The kernel of the twisted adjoint representation is the subgroup $U(1)\cdot 1$, compare [**8**, Thm. 3.17]. This homomorphism takes the spin^c group to the special orthogonal group, and restricts to a continuous homomorphism

$$\mathrm{ad}(V)\ :\ \mathrm{Spin}^c(V)\ \longrightarrow\ SO(V)\ ,$$

the *adjoint representation*. As V varies, these homomorphisms form morphisms of group valued orthogonal spaces

$$\mathrm{ad}\ :\ \mathbf{Pin}^c\ \longrightarrow\ \mathbf{O} \qquad \text{and} \qquad \mathrm{ad}\ :\ \mathbf{Spin}^c\ \longrightarrow\ \mathbf{SO}.$$

Via Construction 3.1 we can interpret these as morphisms of orthogonal monoid spaces.

The Clifford algebras come with certain anti-automorphisms

$$(-)^!\ :\ \mathrm{Cl}(V)\ \longrightarrow\ \mathrm{Cl}(V) \qquad \text{and} \qquad (-)^!\ :\ \mathrm{Cl}_{\mathbb{C}}(V)\ \longrightarrow\ \mathrm{Cl}_{\mathbb{C}}(V)\ .$$

To define these anti-automorphisms, we apply the universal property of the Clifford algebra to the $\mathbb{R}$-linear embedding $[-] : V \longrightarrow \mathrm{Cl}(V)^{\mathrm{op}}$, but with the *opposite* of the Clifford algebra as target. This map extends to a unique homomorphism of $\mathbb{C}$-algebras $(-)^! : \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(V)^{\mathrm{op}}$ such that $(x^!)^! = x$ for all $x \in \mathrm{Cl}(V)$. An isomorphism to the opposite algebra is the same as an anti-automorphism. The anti-automorphism $(-)^!$ of the complex Clifford algebra is defined by complexifying the real version.

We claim that for every element $x \in \mathrm{Pin}^c(V)$ the element $x^! \cdot x$ is a unit scalar multiple of the unit of $\mathrm{Cl}(V)$. Because $(xy)^! \cdot (xy) = y^! \cdot x^! \cdot x \cdot y$, the class of elements $x \in \mathrm{Cl}_{\mathbb{C}}(V)$ such that $x^! \cdot x \in U(1)\cdot 1$ is closed under multiplication. For $\lambda \in U(1)$ we have $(\lambda\cdot 1)^! \cdot (\lambda \cdot 1) = \lambda^2 \cdot 1$; on the other hand, the elements $v \in S(V)$ satisfy $[v]^! \cdot [v] = [v]\cdot[v] = -1$. Since $\mathrm{Pin}^c(V)$ is generated by $U(1)\cdot 1$ and $S(V)$, this proves the claim. We can now define a homomorphism

$$\nu\ :\ \mathrm{Pin}^c(V)\ \longrightarrow\ U(1) \qquad \text{by} \qquad x^! \cdot x\ =\ \nu(x)\cdot 1\ .$$

We observe that the combined homomorphism

$$(\mathrm{ad}, \nu)\ :\ \mathrm{Spin}^c(V)\ \longrightarrow\ SO(V)\times U(1)$$

is surjective with kernel $\{\pm 1\}$. Indeed, for surjectivity we consider $(A,\lambda) \in SO(V)\times U(1)$, and we choose an $x \in \mathrm{Spin}^c(V)$ with $\mathrm{ad}(x) = A$ and a square root $\mu \in U(1)$ of $\lambda\nu(x)^{-1}$. Then $\mathrm{ad}(x\lambda) = \mathrm{ad}(x) = A$ and

$$\nu(\mu\cdot x)\ =\ \nu(\mu)\cdot\nu(x)\ =\ \mu^2\cdot\nu(x)\ =\ \lambda\ .$$

Now we recall another morphism of group valued orthogonal spaces

$$l\ :\ \mathbf{U}\ \longrightarrow\ \mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Spin}^c)$$

that lifts the forgetful realification morphism. To define the morphism l we consider a hermitian inner product space W (which will later be the complexification of an euclidean inner product space), and we define a continuous homomorphism

$$l(W)\ :\ U(W)\ \longrightarrow\ \mathrm{Spin}^c(uW) \tag{3.10}$$

from the unitary group of W to the Spin^c group of the underlying euclidean inner product space of W. The definition of $l(W)$ is as follows. Given a unitary automorphism $A \in U(W)$, we choose an orthonormal basis $b_1,\dots,b_n$ of W consisting of eigenvectors of A. Let $\lambda_j \in U(1)$ be the eigenvalues of b_j. We set

$$e_k\ =\ -\sqrt{\lambda_k}\cdot[i\sqrt{\lambda_k}\cdot b_k]\cdot[i\cdot b_k]\ \in\ \mathrm{Spin}^c(uW)\ .$$

Here $\sqrt{\lambda}$ is one of the two complex square roots of λ; the element e_k is independent of which of the roots of λ_k we use. For $j \neq k$ the vectors $i\sqrt{\lambda_j}\cdot b_j$ and $i\cdot b_j$ are orthogonal in uW to the vectors $i\sqrt{\lambda_k}\cdot b_k$

and $i \cdot b_k$. Orthogonal vectors in uW become anti-commuting elements in $\mathrm{Cl}_{\mathbb{C}}(uW)$, so the elements e_j and e_k commute in $\mathrm{Cl}_{\mathbb{C}}(uW)$. We now set

$$l(W)(A) \;=\; e_1 \cdot \ldots \cdot e_n \;.$$

If we write $\sqrt{\lambda_k} = a + bi$ with $a, b \in \mathbb{R}$, then

$$[i\sqrt{\lambda_k} \cdot b_k] \cdot [ib_k] \;=\; [(ai - b) \cdot b_k] \cdot [ib_k] \;=\; -(b[b_k] - a[ib_k]) \cdot [ib_k] \;=\; -(a \cdot 1 \;+\; b \cdot [b_k] \cdot [ib_k]) \;,$$

and hence

$$e_k \;=\; \sqrt{\lambda_k} \cdot (a \cdot 1 \;+\; b \cdot [b_k] \cdot [ib_k])$$

in $\mathrm{Cl}_{\mathbb{C}}(u\mathbb{C})$. So our formula is the same as in [**8**, §3]. For the time being we omit the verification that $l(W)(A)$ is independent of the choice of orthonormal basis and continuous in A.

We claim that the composite

$$\mathbf{U}(W) \;\xrightarrow{\;l(W)\;}\; \mathbf{Spin}^c(uW) \;\xrightarrow{\;(\mathrm{ad},\nu)\;}\; SO(uW) \times U(1)$$

is the inclusion of the unitary group of W into the special orthogonal group of the underlying euclidean vector space of W in the first coordinate, and the determinant in the second coordinate. We observe that for every unit vector $w \in S(W)$ and all $\lambda \in U(1)$,

$$\mathrm{ad}(i\lambda \cdot w) \circ \mathrm{ad}(i \cdot w) \;\in\; SO(uW)$$

is scalar multiplication by λ^2 on the $\mathbb{C}$-span of w, and the identity on the orthogonal complement of $\mathbb{C} \cdot w$. So

$$\mathrm{ad}(e_k) \;=\; \mathrm{ad}(i\sqrt{\lambda_k} \cdot b_k) \circ \mathrm{ad}(i \cdot b_k)$$

is scalar multiplication by λ_k on the $\mathbb{C}$-span of b_k, and the identity on the orthogonal complement of $\mathbb{C} \cdot b_k$. Thus

$$\mathrm{ad}(l(W)(A)) \;=\; \mathrm{ad}(e_1) \circ \cdots \circ \mathrm{ad}(e_n)$$

is precisely A, but now considered as an element of $SO(uW)$.. On the other hand, $\nu(e_k) = \lambda_k$ and so

$$\nu(l(W)(A)) \;=\; \nu(e_1 \cdot \ldots \cdot e_n) \;=\; \lambda_1 \cdot \ldots \cdot \lambda_n \;=\; \det(A) \;.$$

Since the map $(\mathrm{ad}, \nu) : \mathrm{Spin}^c(uW) \longrightarrow SO(uW) \times U(1)$ is a covering space projection, a continuous map $f : X \longrightarrow SO(uW) \times U(1)$ from a path connected based space (X, x) has at most one continuous lift $\tilde{f} : X \longrightarrow \mathrm{Spin}^c(uW)$ such that $\tilde{f}(x) = 1$. The two continuous maps

$$\mathrm{mult} \circ (l(W) \times l(W)) \;,\; l(W) \circ \mathrm{mult} \;:\; U(W) \times U(W) \;\longrightarrow\; \mathrm{Spin}^c(uW)$$

agree on the point $(1, 1)$ and coincide after postcomposition with (ad, ν). So the two maps coincide, which shows that $l(W)$ is a group homomorphisms. The same kind of argument shows that the square

$$\begin{array}{ccc} U(V) \times U(W) & \xrightarrow{\;l(V)\times l(W)\;} & \mathrm{Spin}^c(uV) \times \mathrm{Spin}^c(uW) \\ \Big\downarrow{\scriptstyle \mu_{V,W}} & & \Big\downarrow{\scriptstyle \mu_{V,W}} \\ U(V \oplus W) & \xrightarrow[\;l(V\oplus W)\;]{} & \mathrm{Spin}^c(uV \oplus uW) \end{array}$$

commutes for all hermitian inner product spaces V and W. By taking $W = V_{\mathbb{C}}$ as the complexification of a euclidean vector space V, the homomorphisms $l(V_{\mathbb{C}})$ form a morphism of ultra-commutative monoids $l : \mathbf{U} \longrightarrow \mathrm{sh}^{\otimes}_{\mathbb{C}}\, \mathbf{Spin}^c$.

The kernel of the homomorphism $\nu : \mathrm{Spin}^c(V) \longrightarrow U(1)$ is the group $\mathrm{Spin}(V)$. Since $\nu(uW) \circ l(W) = \det$, the homomorphism $l(W) : U(W) \longrightarrow \mathrm{Spin}^c(uW)$ takes the special unitary group $SU(W)$ to the group $\mathrm{Spin}(uW)$. So l restricts to a morphism of ultra-commutative monoids $l : \mathbf{SU} \longrightarrow \mathrm{sh}^{\otimes}_{\mathbb{C}}\, \mathbf{Spin}$.

All the examples discussed so far in this section can be summarized in the commutative diagram of orthogonal monoid spaces:

$$\begin{tikzcd}
\mathbf{SU} \arrow[r, "\mathrm{incl}"] \arrow[d, dotted, "l"'] & \mathbf{U} \arrow[d, dotted, "l"'] \arrow[dr, dashed, "r"] & \\
\mathbf{Spin} \arrow[r, "\mathrm{incl}"'] \arrow[d, "\mathrm{incl}"'] & \mathbf{Spin}^c \arrow[r, "\mathrm{ad}"'] \arrow[d, "\mathrm{incl}"] & \mathbf{SO} \arrow[d, "\mathrm{incl}"] \\
\mathbf{Pin} \arrow[r, "\mathrm{incl}"'] & \mathbf{Pin}^c \arrow[r, "\mathrm{ad}"'] & \mathbf{O}
\end{tikzcd}$$

The two dotted arrows mean that the actual morphism goes to the multiplicative shift $\mathrm{sh}_{\mathbb{C}}^{\otimes}$ of the target. With the exception of **Pin** and $\mathbf{Pin}^c$, all the orthogonal monoid spaces are ultra-commutative.

Example 3.11 (Additive Grassmannian). We define an ultra-commutative monoid **Gr**, the *additive Grassmannian*. The value of **Gr** at an inner product space V is

$$\mathbf{Gr}(V) \;=\; \coprod_{n\geq 0} Gr_n(V)\ ,$$

the disjoint union of all Grassmannians in V. The structure map induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is given by $\mathbf{Gr}(\varphi)(L) = \varphi(L)$. A commutative multiplication on **Gr** is given by direct sum:

$$\mu_{V,W}\ :\ \mathbf{Gr}(V)\times\mathbf{Gr}(W)\ \longrightarrow\ \mathbf{Gr}(V\oplus W)\ ,\quad (L,L')\ \longmapsto\ L\oplus L'\ .$$

The unit is the only point $\{0\}$ in $\mathbf{Gr}(0)$. The orthogonal space **Gr** is naturally $\mathbb{N}$-graded, with degree n part given by $\mathbf{Gr}^{[n]}(V) = Gr_n(V)$. The multiplication is graded in that it sends $\mathbf{Gr}^{[m]}(V)\times\mathbf{Gr}^{[n]}(W)$ to $\mathbf{Gr}^{[m+n]}(V\oplus W)$.

As an orthogonal space, **Gr** is the disjoint union of global classifying spaces of the orthogonal groups. Indeed, the homeomorphism

$$\mathbf{L}(\mathbb{R}^n,V)/O(n)\ \cong\ \mathbf{Gr}^{[n]}(V)\ ,\quad \varphi\cdot O(n)\ \longmapsto\ \varphi(\mathbb{R}^n)$$

shows that the homogeneous summand $\mathbf{Gr}^{[n]}$ is isomorphic to the free orthogonal space $\mathbf{L}_{O(n),\mathbb{R}^n}$. Since the tautological action of $O(n)$ on $\mathbb{R}^n$ is faithful, this is a global classifying space $B_{\mathrm{gl}}O(n)$ for the orthogonal group. So as orthogonal spaces,

$$\mathbf{Gr}\ =\ \coprod_{n\geq 0} B_{\mathrm{gl}}O(n)\ . \tag{3.12}$$

Proposition I.5.16 (ii) identifies the equivariant homotopy set $\pi_0^G(B_{\mathrm{gl}}O(n))$ with the set of conjugacy classes of continuous homomorphisms from G to $O(n)$; by restricting the tautological $O(n)$-representation on $\mathbb{R}^n$, this set bijects with the set of isomorphism classes of n-dimensional G-representations. An explicit isomorphism of monoids is given as follows. We let V be a finite dimensional orthogonal G-representation. The G-fixed points of $\mathbf{Gr}(V)$ are the G-invariant subspaces of V, i.e., the G-subrepresentations. We define a map

$$\mathbf{Gr}(V)^G\ =\ \coprod_{n\geq 0}\left(Gr_n(V)\right)^G\ \longrightarrow\ \mathbf{RO}^+(G)$$

from this fixed point space to the monoid of isomorphism classes of G-representations by sending $W\in\mathbf{Gr}(V)^G$ to its isomorphism class. Representations of compact Lie groups are discrete (see the example after [**141**, Prop. 1.3]), so the isomorphism class of W only depends on the path component of W in $\mathbf{Gr}(V)^G$, and the resulting maps $\pi_0(\mathbf{Gr}(V)^G)\longrightarrow\mathbf{RO}^+(G)$ are compatible as V runs through the finite dimensional G-subrepresentations of $\mathcal{U}_G$. So they assemble into a map

$$\pi_0^G(\mathbf{Gr})\ =\ \mathrm{colim}_{V\in s(\mathcal{U}_G)}\ \pi_0(\mathbf{Gr}(V)^G)\ \longrightarrow\ \mathbf{RO}^+(G)\ , \tag{3.13}$$

and this map is an isomorphism of monoids with respect to the direct sum of representations on the target. Moreover, the isomorphism is compatible with restriction maps, and it takes the transfer maps induced by the commutative multiplication of $\mathbf{Gr}$ to induction of representations on the right hand side; so as G varies, the maps (3.13) form a morphism of global power monoids.

We mention an interesting morphism of ultra-commutative monoids:

$$\tau \ : \ \mathbf{Gr} \ \longrightarrow \ \mathbf{O} \ , \tag{3.14}$$

defined at an inner product space V as the map

$$\tau(V) \ : \ \mathbf{Gr}(V) \ \longrightarrow \ \mathbf{O}(V) \ , \quad L \ \longmapsto \ p_{L^\perp} - p_L \ ,$$

sending a subspace $L \subset V$ to the difference of the orthogonal projection onto $L^\perp = V - L$ and the orthogonal projection onto L. Put differently, $\tau(V)(L)$ is the linear isometry that is multiplication by -1 on L and the identity on the orthogonal complement $L^\perp$. We omit the straightforward verification that these maps do define a morphism of ultra-commutative monoids. The induced monoid homomorphism

$$\pi_0^G(\tau) \ : \ \pi_0^G(\mathbf{Gr}) \ \longrightarrow \ \pi_0^G(\mathbf{O})$$

is easily calculated. The isomorphism (3.13) identifies the source with the monoid $\mathbf{RO}^+(G)$ of isomorphism classes of orthogonal G-representations, under direct sum. By Example 3.6 the group $\pi_0^G(\mathbf{O})$ is a weak product (i.e., a direct sum) of copies of $\mathbb{Z}/2$, indexed by the isomorphism classes of irreducible orthogonal G-representations of real type. If λ is any irreducible orthogonal G-representation, then $\pi_0^G(\tau)$ sends its class to the automorphism $-\operatorname{Id}_\lambda$. The group $\mathbf{O}(\lambda)^G$ is isomorphic to $O(1)$, $U(1)$ or $Sp(1)$ depending on whether λ is of real, complex or quaternionic type. In the real case, the map $-\operatorname{Id}_\lambda$ lies in the non-identity path component; in the complex and quaternionic cases, the group $\mathbf{O}(\lambda)^G$ is path connected. So under the previous isomorphisms, $\pi_0^G(\tau)$ becomes the homomorphism

$$\mathbf{RO}^+(G) \ \longrightarrow \ \bigoplus_{[\lambda] \text{ real}} \mathbb{Z}/2$$

that sends the class of λ to the generator of the λ-summand if λ is of real type, and to the trivial element if λ is of complex or quaternionic type. Since the classes of irreducible representations freely generate $\mathbf{RO}^+(G)$ as an abelian monoid, this determines the morphism $\pi_0^G(\tau)$. Moreover, this also shows that $\pi_0^G(\mathbf{Gr}) \longrightarrow \pi_0^G(\mathbf{O})$ is surjective.

Example 3.15 (Oriented Grassmannian). There is a variation of the previous example, the orthogonal monoid space $\mathbf{Gr}^{\mathrm{or}}$ of *oriented Grassmannians*. There is caveat, however, namely that $\mathbf{Gr}^{\mathrm{or}}$ is *not* commutative. The value of $\mathbf{Gr}^{\mathrm{or}}$ at an inner product space V is

$$\mathbf{Gr}^{\mathrm{or}}(V) \ = \ \coprod_{n \geq 0} Gr_n^{\mathrm{or}}(V) \ ,$$

the disjoint union of all oriented Grassmannians in V, where a point in $Gr_n^{\mathrm{or}}(V)$ is a pair $(L, [b_1, \dots, b_n])$ consisting of an $L \in Gr_n(V)$ and an orientation $[b_1, \dots, b_n]$ of L (an $SO(L)$-equivalence class of bases). The structure map induced by $\varphi : V \longrightarrow W$ is given by $\mathbf{Gr}(\varphi)(L, [b_1, \dots, b_n]) = (\varphi(L), [\varphi(b_1), \dots \varphi(b_n)])$. A multiplication on $\mathbf{Gr}$ is given by direct sum:

$$\begin{aligned} \mu_{V,W} \ : \ \mathbf{Gr}(V) \times \mathbf{Gr}(W) \ &\longrightarrow \ \mathbf{Gr}(V \oplus W) \\ ((L, [b_1, \dots, b_n]), \, (L', [b'_1, \dots, b'_m])) \ &\longmapsto \ (L \oplus L', [(b_1, 0), \dots, (b_n, 0), (0, b'_1), \dots, (0, b'_m)]) \ . \end{aligned}$$

The unit is the only point $(\{0\}, \emptyset)$ in $\mathbf{Gr}^{\mathrm{or}}(0)$.

As an orthogonal space, $\mathbf{Gr}^{\mathrm{or}}$ is the disjoint union of global classifying spaces of the special orthogonal groups, via the homeomorphisms

$$(B_{\mathrm{gl}} SO(n))(V) \ = \ \mathbf{L}(\mathbb{R}^n, V)/SO(n) \ \cong \ Gr_n^{\mathrm{or}}(V) \ , \quad \varphi \cdot SO(n) \ \longmapsto \ (\varphi(\mathbb{R}^n), [\varphi(e_1), \dots \varphi(e_n)]) \ .$$

The multiplication of $\mathbf{Gr}^{\mathrm{or}}$ is *not* commutative. The issue is that when pushing a pair around the two ways of the square

$$\begin{tikzcd} Gr^{\mathrm{or}}_m(V) \times Gr^{\mathrm{or}}_n(W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\mathrm{twist}"'] & Gr^{\mathrm{or}}_{m+n}(V \oplus W) \arrow[d, "Gr^{\mathrm{or}}_{m+n}(\tau_{V,W})"] \\ Gr^{\mathrm{or}}_n(W) \times Gr^{\mathrm{or}}_m(V) \arrow[r, "\mu_{W,V}"'] & Gr^{\mathrm{or}}_{n+m}(W \oplus V) \end{tikzcd}$$

then we end up with the same subspaces of $W \oplus V$, but they come with *different* orientations if m and n are both odd.

We can arrange commutativity of the multiplication by passing to the orthogonal submonoid $\mathbf{Gr}^{\mathrm{or,ev}}$ of *even-dimensional* oriented oriented Grassmannians, defined as

$$\mathbf{Gr}^{\mathrm{or,ev}}(V) \;=\; \coprod_{m \geq 0} Gr^{\mathrm{or}}_{2m}(V) \;;$$

the multiplication of $\mathbf{Gr}^{\mathrm{or,ev}}$ is then commutative. Moreover, the forgetful map $\mathbf{Gr}^{\mathrm{or,ev}} \longrightarrow \mathbf{Gr}$ to the additive Grassmannian is a homomorphism of ultra-commutative monoids.

Example 3.16 (Complex and quaternionic Grassmannians). The *complex additive Grassmannian* $\mathbf{Gr}^{\mathbb{C}}$ and the *quaternionic additive Grassmannian* $\mathbf{Gr}^{\mathbb{H}}$ are two more ultra-commutative monoids, the complex respectively quaternionic analogs of the real additive Grassmannian of Example 3.11. The underlying orthogonal spaces send an inner product space V to

$$\mathbf{Gr}^{\mathbb{C}}(V) \;=\; \coprod_{n \geq 0} Gr^{\mathbb{C}}_n(V_{\mathbb{C}}) \qquad \text{respectively} \qquad \mathbf{Gr}^{\mathbb{H}}(V) \;=\; \coprod_{n \geq 0} Gr^{\mathbb{H}}_n(V_{\mathbb{H}}) \;,$$

the disjoint union of all complex (respectively quaternionic) Grassmannians in the complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ (respectively in $V_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} V$). As in the real analog, the structure maps are given by taking images under (complexified respectively quaternified) linear isometric embeddings; direct sum of subspace, plus identification along the isomorphism $V_{\mathbb{C}} \oplus W_{\mathbb{C}} \cong (V \oplus W)_{\mathbb{C}}$, (respectively $V_{\mathbb{H}} \oplus W_{\mathbb{H}} \cong (V \oplus W)_{\mathbb{H}}$,) provides an ultra-commutative multiplication on $\mathbf{Gr}^{\mathbb{C}}$ and on $\mathbf{Gr}^{\mathbb{H}}$.

The homogeneous summand $\mathbf{Gr}^{\mathbb{C},[n]}$ is a global classifying space for the unitary group $U(n)$. Indeed, $\mathbf{Gr}^{\mathbb{C},[n]}$ is isomorphic to the 'complex free' orthogonal space $\mathbf{L}^{\mathbb{C}}_{U(n),\mathbb{C}^n}$ associated to the tautological $U(n)$-representation on $\mathbb{C}^n$, by

$$\mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, V_{\mathbb{C}})/U(n) \;\cong\; \mathbf{Gr}^{\mathbb{C},[n]}(V) \;, \quad \varphi \cdot U(n) \;\longmapsto\; \varphi(\mathbb{C}^n) \;.$$

Proposition I.2.16 then exhibits a global equivalence

$$B_{\mathrm{gl}} U(n) \;=\; \mathbf{L}_{U(n),u(\mathbb{C}^n)} \;\xrightarrow{\sim}\; \mathbf{Gr}^{\mathbb{C},[n]} \;.$$

Similarly, the homogeneous summand $\mathbf{Gr}^{\mathbb{H},[n]}$ is a global classifying space for the symplectic group $Sp(n)$.

The ultra-commutative monoid $\mathbf{Gr}^{\mathbb{C}}$ comes with an involutive automorphism

$$\psi \;:\; \mathbf{Gr}^{\mathbb{C}} \;\longrightarrow\; \mathbf{Gr}^{\mathbb{C}}$$

given by complex conjugation. Here we exploit that the complexification of an $\mathbb{R}$-vector space V comes with a preferred anti-automorphism

$$\psi_V \;:\; V_{\mathbb{C}} \;\longrightarrow\; V_{\mathbb{C}} \;, \quad \lambda \otimes v \;\longmapsto\; \bar{\lambda} \otimes v \;.$$

The value of ψ at V takes a $\mathbb{C}$-subspace $L \subset V_{\mathbb{C}}$ to the conjugate subspace $\bar{L} = \psi_V(L)$. Complexification of subspaces

$$\mathbf{Gr}(V) \;\longrightarrow\; \mathbf{Gr}^{\mathbb{C}}(V) \;, \quad L \;\longmapsto\; L_{\mathbb{C}}$$

defines a morphism of ultra-commutative monoids

$$c \ : \ \mathbf{Gr} \ \longrightarrow \ \mathbf{Gr}^{\mathbb{C}}$$

from the real to the complex additive Grassmannian. A complex subspace of $V_{\mathbb{C}}$ is invariant under ψ_V if and only if it is the complexification of an $\mathbb{R}$-subspace of V (namely the ψ_V-fixed subspace of V). So the morphism c is an isomorphism of $\mathbf{Gr}$ onto the ψ-invariant ultra-commutative submonoid $(\mathbf{Gr}^{\mathbb{C}})^{\psi}$.

Realification defines a morphism of ultra-commutative monoids

$$r \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \mathrm{sh}^{\mathbb{C}}_{\otimes}(\mathbf{Gr}^{\mathrm{or,ev}})$$

to the multiplicative shift (see Example I.1.12) of the even part of the oriented Grassmannian of the previous Example 3.15. The value $r(V) : \mathbf{Gr}^{\mathbb{C}}(V) \longrightarrow \mathbf{Gr}^{\mathrm{or,ev}}(V_{\mathbb{C}})$ takes a complex subspace of $V_{\mathbb{C}}$ to the underlying real vector space, endowed with the preferred orientation $[x_1, ix_1, \dots, x_n, ix_n]$, where $(x_1, \dots, x_n)$ is any complex basis.

The isomorphism (3.13) between $\pi_0^G(\mathbf{Gr})$ and $\mathbf{RO}^+(G)$ has an obvious complex analog. For every compact Lie group G an isomorphism of commutative monoids

$$\pi_0^G(\mathbf{Gr}^{\mathbb{C}}) \ = \ \mathrm{colim}_{V \in \bar{s}(\mathcal{U}_G)} \ \pi_0(\mathbf{Gr}^{\mathbb{C}}(V)^G) \ \cong \ \mathbf{RU}^+(G) \tag{3.17}$$

is given by sending the class of a G-fixed point in $\mathbf{Gr}^{\mathbb{C}}(V)^G$, i.e., a complex G-subrepresentation of $V_{\mathbb{C}}$, to its isomorphism class. The isomorphism is compatible with restriction maps, takes the involution $\pi_0^G(\psi)$ of $\pi_0^G(\mathbf{Gr}^{\mathbb{C}})$ to the complex conjugation involution of $\mathbf{RU}^+(G)$, and it takes the transfer maps induced by the commutative multiplication of $\mathbf{Gr}^{\mathbb{C}}$ to induction of representations. So as G varies, the maps form an isomorphism of global power monoids $\underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \cong \mathbf{RU}^+$. The isomorphisms are also compatible with realification, in the sense of the commutative diagram

$$\begin{tikzcd}
\underline{\pi}_0(\mathbf{Gr}) \arrow[r, "\underline{\pi}_0(c)"] \arrow[d, "(3.13)"', "\cong"] & \underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \arrow[r, "\underline{\pi}_0(r)"] \arrow[d, "\cong"', "(3.17)"] & \underline{\pi}_0(\mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Gr}^{\mathrm{or,ev}})) \arrow[r, "\cong"] & \underline{\pi}_0(\mathbf{Gr}^{\mathrm{or,ev}}) \arrow[r, "\text{forget}"] & \underline{\pi}_0(\mathbf{Gr}) \arrow[d, "(3.13)"', "\cong"] \\
\mathbf{RO}^+ \arrow[r, "c"'] & \mathbf{RU}^+ \arrow[rrr, "r"'] & & & \mathbf{RO}^+
\end{tikzcd}$$

commutes, where the lower horizontal maps are complexification and realification of representations. The isomorphism in the upper row is inverse to the one induced by the homomorphism $\mathbf{Gr}^{\mathrm{or,ev}}(i) : \mathbf{Gr}^{\mathrm{or,ev}} \longrightarrow \mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Gr}^{\mathrm{or,ev}})$ induced by precomposition with the natural linear isometric embedding

$$i_V \ : \ V \ \longrightarrow \ V_{\mathbb{C}} \ , \quad v \ \longmapsto \ 1 \otimes v \ ;$$

the morphism $\mathbf{Gr}^{\mathrm{or,ev}}(i)$ is a global equivalence by Theorem I.1.11.

In the complex reign, there is also an analog of the real oriented Grassmannian of Example 3.15, the orthogonal monoid space $\mathbf{Gr}^{\mathbb{C},SU}$. The value of $\mathbf{Gr}^{\mathbb{C},SU}$ at an inner product space V is

$$\mathbf{Gr}^{\mathbb{C},SU}(V) \ = \ \coprod_{n \geq 0} \ Gr_n^{\mathbb{C},SU}(V_{\mathbb{C}}) \ ,$$

where a point in $Gr_n^{\mathbb{C},SU}(V_{\mathbb{C}})$ is a pair $(L, [x_1, \dots, x_n])$ consisting of n-dimensional complex subspace L of $V_{\mathbb{C}}$ and an $SU(L)$-equivalence class of $\mathbb{C}$-basis $(x_1, \dots, x_n)$ of L. The structure map induced by $\varphi : V \longrightarrow W$ is given by $\mathbf{Gr}^{\mathbb{C},SU}(\varphi)((L, [b_1, \dots, b_n]) = (\varphi_{\mathbb{C}}(L), [\varphi_{\mathbb{C}}(b_1), \dots \varphi_{\mathbb{C}}(b_n)])$, where $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow W_{\mathbb{C}}$ is the complexification of φ. The multiplication of $\mathbf{Gr}^{\mathbb{C},SU}$ is by direct sum and concatenation of bases, much like in $\mathbf{Gr}^{\mathrm{or}}$.

As an orthogonal space, $\mathbf{Gr}^{\mathbb{C},SU}$ is the disjoint union of global classifying spaces of the special unitary groups. Indeed, by the same argument as for $\mathbf{Gr}^{\mathbb{C},[n]}$ above, Proposition I.2.16 exhibits a global equivalence

$$B_{\mathrm{gl}}SU(n) \ = \ \mathbf{L}_{SU(n),u(\mathbb{C}^n)} \ \xrightarrow{\sim} \ \mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, V_{\mathbb{C}})/SU(n) \ \cong \ Gr_n^{\mathbb{C},SU}(V_{\mathbb{C}}) \ .$$

As with its real cousin, the multiplication of $\mathbf{Gr}^{\mathbb{C},SU}$ is *not* commutative, but the orthogonal monoid subspace consisting of the even summands is ultra-commutative. The various kinds of additive Grassmannians fit into a commutative diagram of ultra-commutative monoids:

$$\begin{tikzcd}
\mathbf{Gr}^{\mathbb{C},SU,\mathrm{ev}} \arrow[r, "r"] \arrow[d, "\mathrm{forget}"'] & \mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Gr}^{\mathrm{or,ev}}) \arrow[d, "\mathrm{forget}"] & \mathbf{Gr}^{\mathrm{or,ev}} \arrow[l, "\mathbf{Gr}^{\mathrm{or,ev}}(i)", "\simeq"'] \arrow[d, "\mathrm{forget}"] \\
\mathbf{Gr}^{\mathbb{C}} \arrow[r, "r"'] & \mathrm{sh}^{\otimes}_{\mathbb{C}}(\mathbf{Gr}) & \mathbf{Gr} \arrow[l, "\simeq", "\mathbf{Gr}(i)"']
\end{tikzcd}$$

The vertical maps forget orientations, the left horizontal maps pass to underlying $\mathbb{R}$-vector spaces.

Example 3.18 (Multiplicative Grassmannians). We define an ultra-commutative monoid $\mathbf{Gr}_{\otimes}$, the *multiplicative Grassmannian*. Given an inner product space V we let

$$\mathrm{Sym}(V) \;=\; \bigoplus_{i\geq 0} \mathrm{Sym}^i(V) \;=\; \bigoplus_{i\geq 0} V^{\otimes i}/\Sigma_i$$

denote the symmetric algebra of V. If W is another inner product space, then the two summand inclusions of V and W into $V \oplus W$ induce algebra homomorphisms

$$\mathrm{Sym}(V) \;\longrightarrow\; \mathrm{Sym}(V\oplus W) \;\longleftarrow\; \mathrm{Sym}(W)$$

and we use the commutative multiplication on $\mathrm{Sym}(V\oplus W)$ to combine these into an $\mathbb{R}$-algebra isomorphism

$$\mathrm{Sym}(V)\otimes\mathrm{Sym}(W) \;\cong\; \mathrm{Sym}(V\oplus W)\ . \tag{3.19}$$

These isomorphisms are natural for linear isometric embeddings in V and W. The value of $\mathbf{Gr}_{\otimes}$ at an inner product space V is then

$$\mathbf{Gr}_{\otimes}(V) \;=\; \coprod_{n\geq 0} Gr_n(\mathrm{Sym}(V))\ ,$$

the disjoint union of all Grassmannians in the symmetric algebra of V. The structure map $\mathbf{Gr}_{\otimes}(\varphi) : \mathbf{Gr}_{\otimes}(V) \longrightarrow \mathbf{Gr}_{\otimes}(W)$ induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is given by

$$\mathbf{Gr}_{\otimes}(\varphi)(L) \;=\; \mathrm{Sym}(\varphi)(L)\ ,$$

where $\mathrm{Sym}(\varphi) : \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(W)$ is the induced map of symmetric algebras. A commutative multiplication on $\mathbf{Gr}_{\otimes}$ is given by tensor product, i.e.,

$$\mu_{V,W} \;:\; \mathbf{Gr}_{\otimes}(V)\times\mathbf{Gr}_{\otimes}(W) \;\longrightarrow\; \mathbf{Gr}_{\otimes}(V\oplus W)$$

sends $(L,L') \in \mathbf{Gr}_{\otimes}(V)\times\mathbf{Gr}_{\otimes}(W)$ to the image of $L\otimes L'$ under the isomorphism (3.19). The multiplicative unit is the point $\mathbb{R}$ in $\mathbf{Gr}_{\otimes}(0) = \mathbb{R}$. As the additive Grassmannian $\mathbf{Gr}$, the multiplicative Grassmannian $\mathbf{Gr}_{\otimes}$ is $\mathbb{N}$-graded, with degree n part given by $\mathbf{Gr}^{[n]}_{\otimes}(V) = Gr_n(\mathrm{Sym}(V))$. The multiplication sends $\mathbf{Gr}^{[n]}_{\otimes}(V) \times \mathbf{Gr}^{[m]}_{\otimes}(W)$ to $\mathbf{Gr}^{[n\cdot m]}_{\otimes}(V\oplus W)$.

As orthogonal spaces, the additive and multiplicative Grassmannians are globally equivalent. Indeed, for an inner product space V we let $i : V \longrightarrow \mathrm{Sym}(V)$ be the embedding as the linear summand of the symmetric algebra. Then as V varies, the maps

$$\mathbf{Gr}(V) \;=\; \coprod_{n\geq 0} Gr_n(V) \;\longrightarrow\; \coprod_{n\geq 0} Gr_n(\mathrm{Sym}(V)) \;=\; \mathbf{Gr}_{\otimes}(V)\ , \quad L \;\longmapsto\; i(L)$$

form a global equivalence $\mathbf{Gr} \longrightarrow \mathbf{Gr}_{\otimes}$. Indeed, for each $n \geq 0$, $\mathbf{Gr}^{[n]}_{\otimes}(V)$ is a sequential colimit, along closed embeddings, of a sequence of orthogonal spaces

$$\mathbf{Gr}^{[n]} \;\longrightarrow\; \mathbf{Gr}^{[n]}_{\leq 1} \;\longrightarrow\; \mathbf{Gr}^{[n]}_{\leq 2} \;\longrightarrow\; \dots \;\longrightarrow\; \mathbf{Gr}^{[n]}_{\leq k} \;\longrightarrow\; \dots\ ,$$

where $\mathbf{Gr}^{[n]}_{\leq k}(V) = Gr_n(\bigoplus_{i=0}^k \mathrm{Sym}^i(V))$. Each of the morphisms $\mathbf{Gr}^{[n]} \longrightarrow \mathbf{Gr}^{[n]}_{\leq k}$ is a global equivalence by Theorem I.1.11; so all morphisms in the sequence are global equivalences, hence so is the map from $\mathbf{Gr}^{[n]}$ to the colimit $\mathbf{Gr}^{[n]}_{\otimes}$, by Proposition I.1.9 (viii). This global equivalence induces a bijection

$$\pi_0^G(\mathbf{Gr}) \;\cong\; \pi_0^G(\mathbf{Gr}_\otimes)$$

for every compact Lie group G, hence both are isomorphic to the set $\mathbf{RO}^+(G)$ of isomorphism classes of orthogonal G-representations. The commutative monoid structures and transfer maps induced by the products of $\mathbf{Gr}$ respectively $\mathbf{Gr}_\otimes$ are quite different though: the monoid structure of $\pi_0^G(\mathbf{Gr})$ corresponds to direct sum of representations, and the transfer maps are additive transfers; the monoid structure of $\pi_0^G(\mathbf{Gr}_\otimes)$ corresponds to tensor product of representations, and the transfer maps are multiplicative transfers, also called norm maps.

The orthogonal subspace $\mathbf{P} = \mathbf{Gr}^{[1]}_{\otimes}$ of the multiplicative Grassmannian $\mathbf{Gr}_\otimes$ is closed under the product and contains the multiplicative unit, hence $\mathbf{P}$ is an ultra-commutative monoid in its own right. Because

$$\mathbf{P}(V) \;=\; \mathbf{Gr}^{[1]}_{\otimes}(V) \;=\; P(\mathrm{Sym}(V))$$

is the projective space of the symmetric algebra of V, we use the symbol $\mathbf{P}$ and refer to it as the *global projective space*. The multiplication is given by tensor product of lines, and application of the isomorphism (3.19). Since $\mathbf{P} = \mathbf{Gr}^{[1]}_{\otimes}$ is globally equivalent to the additive variant $\mathbf{Gr}^{[1]}$, it is a global classifying space for the group $O(1)$, a cyclic group of order 2,

$$\mathbf{P} \;\simeq\; \mathbf{Gr}^{[1]} \;\simeq\; B_{\mathrm{gl}}O(1) \;=\; B_{\mathrm{gl}}C_2 \;.$$

In other words, $\mathbf{P}$ is an ultra-commutative multiplicative model for $B_{\mathrm{gl}}C_2$.

There is a straightforward complex analog of the multiplicative Grassmannian, the ultra-commutative monoid $\mathbf{Gr}^{\mathbb{C}}_{\otimes}$ with value at an inner product space V given by

$$\mathbf{Gr}^{\mathbb{C}}_{\otimes}(V) \;=\; \coprod_{n\geq 0} Gr^{\mathbb{C}}_n(\mathrm{Sym}(V)_{\mathbb{C}}) \;,$$

the disjoint union of all Grassmannians in the complexified symmetric algebra of V. The structure maps and multiplication (by tensor product) are as in the real case. The underlying orthogonal spaces of $\mathbf{Gr}^{\mathbb{C}}_{\otimes}$ and the additive version $\mathbf{Gr}^{\mathbb{C}}$ are globally equivalent, but the multiplications are essentially different. The orthogonal subspace $\mathbf{P}^{\mathbb{C}} = \mathbf{Gr}^{\mathbb{C},[1]}_{\otimes}$ of $\mathbf{Gr}_\otimes$ consisting of 1-dimensional subspaces is closed under the product and contains the multiplicative unit; hence $\mathbf{P}^{\mathbb{C}}$ is an ultra-commutative monoid in its own right, the *complex global projective space*. As an orthogonal space, $\mathbf{P}^{\mathbb{C}}$ is globally equivalent to the additive variant $\mathbf{Gr}^{\mathbb{C},[1]}$, and hence a global classifying space for the group $U(1)$,

$$\mathbf{P}^{\mathbb{C}} \;\simeq\; \mathbf{Gr}^{\mathbb{C},[1]} \;\simeq\; B_{\mathrm{gl}}U(1) \;.$$

Since the multiplication in the skew field of quaternions $\mathbb{H}$ is not commutative, there is no tensor product of $\mathbb{H}$-vector spaces; so there is no multiplicative version of the quaternionic Grassmannian $\mathbf{Gr}^{\mathbb{H}}$.

Construction 3.20 (Bar construction). For the next class examples we quickly recall the *bar construction* of a topological monoid M. This is the geometric realization of the simplicial space whose space of n-simplices is M^n, the n-fold cartesian power of M. For $n \geq 1$ and $0 \leq i \leq n$, the face map $d_i : M^n \longrightarrow M^{n-1}$ is given by

$$d_i(x_1, \dots x_n) \;=\; \begin{cases} (x_2, \dots, x_n) & \text{for } i = 0, \\ (x_1, \dots, x_{i-1}, x_i \cdot x_{i+1}, x_{i+2}, \dots, x_n) & \text{for } 0 < i < n, \\ (x_1, \dots, x_{n-1}) & \text{for } i = n. \end{cases}$$

For $n \geq 1$ and $0 \leq i \leq n-1$ the degeneracy map $s_i : M^{n-1} \longrightarrow M^n$ is given by

$$s_i(x_1, \dots, x_{n-1}) \;=\; (x_1, \dots, x_i, 1, x_{i+1}, \dots, x_{n-1}) \;.$$

The bar construction is the geometric realization

$$BM \;=\; |[n] \mapsto M^n|$$

of this simplicial space; the construction $M \mapsto BM$ is functorial in continuous monoid homomorphisms. The bar construction commutes with products in the sense that for a pair of topological monoids M and N, the canonical map

$$B(M \times N) \;\longrightarrow\; BM \times BN \tag{3.21}$$

is a homeomorphism.

Construction 3.22 (Multiplicative global classifying spaces). We discuss multiplicative models of global classifying spaces for abelian compact Lie groups. We use the bar construction (see Construction 3.20), giving a non-equivariant classifying space, followed by the cofree functor R (see Construction I.3.24). The bar construction preserves products in the sense that for every pair of compact Lie groups G and K the natural map

$$B(G \times K) \;\longrightarrow\; BG \times BK$$

is a homeomorphism. So the composite $A \mapsto R(BA)$ is a lax symmetric monoidal functor via the morphism of orthogonal spaces

$$R(BG) \boxtimes R(BK) \;\xrightarrow{\mu_{A,A'}}\; R(BG \times BK) \;\cong\; R(B(G \times K))\ ,$$

where the first morphism was defined in (3.25) of Chapter I. The bar construction is functorial in group homomorphisms, so for an abelian compact Lie group A the composite

$$R(BA) \boxtimes R(BA) \;\longrightarrow\; R(B(A \times A)) \;\xrightarrow{R(B\mu_A)}\; R(BA)$$

is an ultra-commutative and associative multiplication on the orthogonal space $R(BA)$, where $\mu_A : A \times A \longrightarrow A$ is the multiplication of A. Theorem I.3.33 shows that for abelian A the cofree orthogonal space $R(BA)$ is a global classifying space for A. In particular, the Rep-functor $\underline{\pi}_0(R(BA))$ is representable by A. We saw in Proposition 2.45 that there is then a *unique* structure of global power monoid on $\underline{\pi}_0(R(BA))$, and the power operations are characterized by naturality and the relation

$$[m](u_A) \;=\; p_m^*(u_A)$$

where $u_A \in \underline{\pi}_0^A(R(BA))$ is a tautological class and $p_m : \Sigma_m \wr A \longrightarrow A$ is the homomorphism defined by

$$p_m(\sigma;\, a_1, \ldots, a_m) \;=\; a_1 \cdot \ldots \cdot a_m\ .$$

Example 3.23 (Unordered frames). The ultra-commutative monoid $\mathbf{F}$ of *unordered frames* sends an inner product space V to

$$\mathbf{F}(V) \;=\; \{A \subset V \mid A \text{ is orthonormal}\}\ ,$$

the space of all unordered frames in V, i.e., subsets of V that consist of pairwise orthogonal unit vectors. Since V is finite dimensional, such a subset is necessarily finite. The topology on $\mathbf{F}(V)$ is as the disjoint union, over the cardinality of the sets, of quotient spaces of Stiefel manifolds. The structure map induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is given by $\mathbf{F}(\varphi)(A) = \varphi(A)$. A commutative multiplication on $\mathbf{F}$ is given, essentially, by disjoint union:

$$\mu_{V,W} \;:\; \mathbf{F}(V) \times \mathbf{F}(W) \;\longrightarrow\; \mathbf{F}(V \oplus W)\ ,\quad (A, A') \;\longmapsto\; i_V(A) \cup i_W(A')\ ; \tag{3.24}$$

here $i_V : V \longrightarrow V \oplus W$ and $i_W : W \longrightarrow V \oplus W$ are the direct summand embeddings. The unit is the empty set, the only point in $\mathbf{F}(0)$. The orthogonal space $\mathbf{F}$ is naturally $\mathbb{N}$-graded, with degree n part $\mathbf{F}^{[n]}$ given by the unordered frames of cardinality n; the multiplication sends $\mathbf{F}^{[m]}(V) \times \mathbf{F}^{[n]}(W)$ to $\mathbf{F}^{[m+n]}(V \oplus W)$.

As an orthogonal space, $\mathbf{F}$ is the disjoint union of global classifying spaces of the symmetric groups. We let Σ_n acts on $\mathbb{R}^n$ by permuting the coordinates, which is also the permutation representation of the

tautological Σ_n-action on $\{1,\dots,n\}$. This Σ_n-action is faithful, so the free orthogonal space $\mathbf{L}_{\Sigma_n,\mathbb{R}^n}$ is a global classifying space $B_{\mathrm{gl}}\Sigma_n$ for the symmetric group. The homeomorphism

$$\mathbf{L}(\mathbb{R}^n,V)/\Sigma_n \;\cong\; \mathbf{F}^{[n]}(V)\ , \quad \varphi\cdot\Sigma_n \;\longmapsto\; \{\varphi(e_1),\dots,\varphi(e_n)\}$$

shows that the homogeneous summand $\mathbf{F}^{[n]}$ is isomorphic to $\mathbf{L}_{\Sigma_n\mathbb{R}^n} = B_{\mathrm{gl}}\Sigma_n$; here $e_1,\dots,e_n$ is the canonical basis of $\mathbb{R}^n$. So as orthogonal spaces,

$$\mathbf{F} \;=\; \coprod_{n\geq 0} B_{\mathrm{gl}}\Sigma_n\ .$$

Proposition I.5.16 (ii) identifies the equivariant homotopy set $\pi_0^G(B_{\mathrm{gl}}\Sigma_n)$ with the set of conjugacy classes of continuous homomorphisms from G to Σ_n; by restricting the tautological Σ_n-representation on $\{1,\dots,n\}$, this set bijects with the set of isomorphism classes of finite G-sets of cardinality n.

As n varies, this gives an isomorphism of monoids from $\pi_0^G(\mathbf{F})$ to the set $\mathbb{A}^+(G)$ of isomorphism classes of finite G-sets that we make explicit now. We let V be a finite dimensional orthogonal G-representation. An unordered frame in $\mathbf{F}(V)$ is a G-fixed point if and only if it is G-invariant. So for such frames $A\in\mathbf{F}(V)^G$, the G-action restricts to an action on A, making it a finite G-set. We define a map

$$\mathbf{F}(V)^G \;\longrightarrow\; \mathbb{A}^+(G)\ , \quad A \;\longmapsto\; [A]$$

from this fixed point space to the monoid of isomorphism classes of finite G-sets by sending $A\in\mathbf{F}(V)^G$ to its isomorphism class. The isomorphism class of A as a G-set only depends on the path component of A in $\mathbf{F}(V)^G$, and the resulting maps $\pi_0(\mathbf{F}(V)^G)\longrightarrow\mathbb{A}^+(G)$ are compatible as V runs through the finite dimensional G-subrepresentations of $\mathcal{U}_G$. So they assemble into a map

$$\pi_0^G(\mathbf{F}) \;=\; \mathrm{colim}_{V\in s(\mathcal{U}_G)}\ \pi_0(\mathbf{F}(V)^G) \;\longrightarrow\; \mathbb{A}^+(G)\ , \tag{3.25}$$

and this map is a monoid isomorphism with respect to the disjoint union of G-sets on the target. Moreover, the isomorphism is compatible with restriction maps, and it takes the transfer maps induced by the commutative multiplication of $\mathbf{F}$ to induction of equivariant sets on the right hand side.

It goes without saying that actions of compact Lie groups are required to be *continuous* and that the use of the term 'set' (as opposed to 'space') implies the discrete topology on the set; so the identity path component G° acts trivially on every G-set. Hence the monoids $\pi_0^G(\mathbf{F})$ and $\mathbb{A}^+(G)$ only see the finite group $\pi_0(G)=G/G^\circ=\bar{G}$ of path components, i.e., for every compact Lie group G, the restriction maps

$$p^*\ :\ \pi_0^{\bar{G}}(\mathbf{F}) \;\longrightarrow\; \pi_0^G(\mathbf{F}) \qquad\text{and}\qquad p^*\ :\ \mathbb{A}^+(\bar{G}) \;\longrightarrow\; \mathbb{A}^+(G)$$

along the projection $p\ :\ G\longrightarrow\bar{G}$ are isomorphisms. So if G has positive dimension, then the group completion of the monoid $\mathbb{A}^+(G)$ need not be isomorphic to what is sometimes called the Burnside ring of G (such as the 0-th G-equivariant stable stem).

A morphism of $\mathbb{N}$-graded ultra-commutative monoids

$$\mathrm{span}\ :\ \mathbf{F}\ \longrightarrow\ \mathbf{Gr} \qquad\text{is defined by}\qquad \mathrm{span}(V)(A)\ =\ \mathrm{span}(A)\ ,$$

i.e., a frame is sent to its linear span. The induced morphism of global power monoids is linearization: the square of monoid homomorphisms

$$\begin{tikzcd}
\pi_0^G(\mathbf{F}) \arrow[r, "\pi_0^G(\mathrm{span})"] \arrow[d, "(3.25)"', "\cong"] & \pi_0^G(\mathbf{Gr}) \arrow[d, "\cong"', "(3.13)"] \\
\mathbb{A}^+(G) \arrow[r, "{[S]\longmapsto[\mathbb{R}S]}"'] & \mathbf{RO}^+(G)
\end{tikzcd} \tag{3.26}$$

commutes, where the lower map sends the class of a G-set to the class of its permutation representation.

Example 3.27 (Multiplicative monoid of the sphere spectrum). We define an ultra-commutative monoid $\Omega^\bullet\mathbb{S}$, the 'multiplicative monoid of the sphere spectrum'. The notation and terminology indicate that this is a special case of a more general construction that associates to an ultra-commutative ring spectrum R its 'multiplicative monoid' $\Omega^\bullet R$, see Example IV.1.16 below.

The values of the orthogonal space $\Omega^\bullet\mathbb{S}$ are given by

$$(\Omega^\bullet\mathbb{S})(V) \;=\; \mathrm{map}(S^V, S^V)\ ,$$

the space of continuous based self-maps of the sphere S^V. A linear isometric embedding $\varphi : V \longrightarrow W$ acts by conjugation and extension by the identity, i.e., the map

$$(\Omega^\bullet\mathbb{S})(\varphi)\ :\ \mathrm{map}(S^V, S^V) \;\longrightarrow\; \mathrm{map}(S^W, S^W)$$

sends a continuous based map $f : S^V \longrightarrow S^V$ to the composite

$$S^W \;\cong\; S^V \wedge S^{W-\varphi(V)} \;\xrightarrow{f\wedge S^{W-\varphi(V)}}\; S^V \wedge S^{W-\varphi(V)} \;\cong\; S^W\ .$$

The two unnamed homeomorphisms between $S^V \wedge S^{W-\varphi(V)}$ and S^W use the map φ on the factor S^V. In particular, the orthogonal group $O(V)$ acts on $\mathrm{map}(S^V, S^V)$ by conjugation.

The multiplication of the orthogonal space $\Omega^\bullet\mathbb{S}$ is by smash product, i.e., the map

$$\mu_{V,W}\ :\ (\Omega^\bullet\mathbb{S})(V) \times (\Omega^\bullet\mathbb{S})(W) \;\longrightarrow\; (\Omega^\bullet\mathbb{S})(V \oplus W)$$

smashes a self-map of S^V with a self-map of S^W and conjugates with the canonical homeomorphism between $S^V \wedge S^W$ and $S^{V\oplus W}$. The unit is the identity of S^V.

The equivariant homotopy set $\pi_0^G(\Omega^\bullet\mathbb{S})$ is equal to the stable G-equivariant 0-stem $\pi_0^G(\mathbb{S})$, compare Construction IV.1.6 below. The monoid structure on $\pi_0^G(\Omega^\bullet\mathbb{S})$ arising from the commutative multiplication on $\Omega^\bullet\mathbb{S}$ is the *multiplicative* (rather than the additive) monoid structure of $\pi_0^G(\mathbb{S})$. The set $\pi_0^G(\Omega^\bullet\mathbb{S})$ thus bijects with the underlying set of the Burnside ring $\mathbb{A}(G)$ of the group G (compare Example IV.2.7), which is additively a free abelian group with basis the conjugacy classes of subgroups of G with finite Weyl group. The multiplication on $\pi_0^G(\Omega^\bullet\mathbb{S})$ corresponds to the multiplication (not the addition !) in the Burnside ring $\mathbb{A}(G)$. When G is finite, $\pi_0^G(\Omega^\bullet\mathbb{S})$ thus bijects with the underlying set of the Grothendieck group of finite G-sets, and the multiplication corresponds to the product of G-sets. The power operations in $\underline{\pi}_0(\Omega^\bullet\mathbb{S})$ are thus represented by 'raising a G-set to the cartesian power', and the transfer maps are known as 'norm maps' or 'multiplicative induction'.

Example 3.28 (Exponential homomorphisms). The classical J-homomorphism fits in nicely here, in the form of a global refinement to a morphism of ultra-commutative monoids

$$J\ :\ \mathbf{O} \;\longrightarrow\; \Omega^\bullet\mathbb{S}$$

defined at an inner product space V as the map

$$J(V)\ :\ \mathbf{O}(V) \;\longrightarrow\; \mathrm{map}(S^V, S^V)$$

sending a linear isometry $\varphi : V \longrightarrow V$ to its one-point compactification $S^\varphi : S^V \longrightarrow S^V$. The fact that these maps are multiplicative and compatible with the structure maps is straightforward. The induced map

$$\pi_0^G(J)\ :\ \pi_0^G(\mathbf{O}) \;\longrightarrow\; \pi_0^G(\Omega^\bullet\mathbb{S}) \;=\; \pi_0^G(\mathbb{S})\ ,$$

for G a compact Lie group, can be described as follows. By Example 3.6, the group $\pi_0^G(\mathbf{O})$ is a direct sum of copies of $\mathbb{Z}/2$, indexed by the isomorphism classes of irreducible orthogonal G-representations of real type. If λ is such an irreducible G-representation, then the image of the λ-indexed copy of $\mathbb{Z}/2$ is the represented by the one-point compactification of the involution $-\mathrm{Id}_\lambda \in \mathbf{O}(\lambda)$.

In (3.14) we defined a morphism of ultra-commutative monoids $\tau : \mathbf{Gr} \longrightarrow \mathbf{O}$. The composite morphism of ultra-commutative monoids

$$\mathbf{Gr} \;\xrightarrow{\tau}\; \mathbf{O} \;\xrightarrow{J}\; \Omega^\bullet\mathbb{S} \tag{3.29}$$

realizes an 'exponential' homomorphism from the real representation ring to the multiplicative group of the Burnside ring of a compact Lie group G. The exponential homomorphism was first studied by tom Dieck and is therefore sometimes called the 'tom Dieck exponential map'. Tom Dieck's definition of the exponential homomorphism (see [**165**, 5.5.9]) is completely algebraic: we start from the homomorphism

$$s \ : \ \mathbf{RO}^+(G) \ \longrightarrow \ \mathrm{Cl}(G,\mathbb{Z})^\times \ , \quad s[V](H) \ = \ (-1)^{\dim(V^H)}$$

that sends an orthogonal representation V to the class function $s(V)$ that records the parities of the dimensions of the fixed point spaces. The Burnside ring embeds into the ring of class functions by 'fixed point counting':

$$\Phi \ : \ \mathbb{A}(G) \ \longrightarrow \ \mathrm{Cl}(G,\mathbb{Z}) \ , \ \Phi[S](H) \ = \ |S^H| \ ,$$

i.e., a (virtual) G-set S is sent to the class function that counts the number of fixed points. This map is injective, and for finite groups the image can be characterized by an explicit system of congruences [**165**, Prop. 1.3.5]. The image of the homomorphism s satisfies the congruences, so there is a unique monoid homomorphism $\overline{\exp} : \mathbf{RO}^+(G) \longrightarrow \mathbb{A}(G)^\times$ such that $\Phi \circ \overline{\exp} = s$. The map s sends the direct sum of representations to the product of the parity functions, so $\overline{\exp}$ is a homomorphism of abelian monoids, and extends to a homomorphism

$$\exp \ : \ \mathbf{RO}(G) \ \longrightarrow \ \mathbb{A}(G)^\times$$

from the orthogonal representation ring. The morphism of ultra-commutative monoids (3.29) realizes the exponential morphism in the sense that the following diagram of monoid homomorphisms commutes:

$$\begin{tikzcd}
\pi_0^G(\mathbf{Gr}) \arrow[r, "\pi_0^G(J\circ\tau)"] \arrow[d, "(3.13)"', "\cong"] & \pi_0^G(\Omega^\bullet\mathbb{S}) \arrow[r, equal] & (\pi_0^G(\mathbb{S}))^\times \\
\mathbf{RO}^+(G) \arrow[rr, "\overline{\exp}"'] & & \mathbb{A}(G)^\times \arrow[u, "\alpha"', "\cong"]
\end{tikzcd}$$

To show the commutativity of this diagram we may compose with the degree monomorphism

$$\deg \ : \ (\pi_0^G(\mathbb{S}))^\times \longrightarrow \mathrm{Cl}(G,\mathbb{Z}) \ , \quad \deg[f](H) \ = \ \deg(f^H : S^{V^H} \ \longrightarrow S^{V^H})$$

that takes the class of an equivariant self-map $f : S^V \longrightarrow S^V$ of a representation sphere to the class function that records the fixed point dimensions. The composite $\deg \circ \alpha : \mathbb{A}(G) \longrightarrow \mathrm{Cl}(G,\mathbb{Z})$ coincides with the fixed point counting map Φ, so

$$\deg \circ \alpha \circ \overline{\exp} \ = \ \Phi \circ \overline{\exp} \ = \ s \ : \mathbf{RO}^+(G) \ \longrightarrow \ \mathrm{Cl}(G,\mathbb{Z}) \ .$$

On the other hand, the map $\pi_0^G(J \circ \tau)$ sends the class of a G-representation V to the involution $S^{-\operatorname{Id}_V} : S^V \longrightarrow S^V$. The diagram thus commutes because

$$\deg\big((S^{-\operatorname{Id}_V})^H\big) \ = \ \deg(S^{-\operatorname{Id}_{V^H}}) \ = \ (-1)^{\dim(V^H)} \ = \ s[V](H) \ .$$

Finally, the composite morphism of ultra-commutative monoids

$$\mathbf{F} \ \xrightarrow{\text{span}} \ \mathbf{Gr} \ \xrightarrow{\tau} \ \mathbf{O} \ \xrightarrow{J} \ \Omega^\bullet\mathbb{S}$$

realizes another exponential homomorphism that has received some attention in algebra. By the above and the commutative square (3.26), this composite realizes the morphism of global power monoids

$$\exp \ : \ \mathbb{A}^+(G) \ \longrightarrow \ \mathbb{A}(G)^\times$$

that is characterized, via the fixed point class function, by

$$\Phi(\exp[S])(H) \ = \ (-1)^{|H\backslash S|} \ ,$$

the parity of the number of H-orbits on S, for every G-set S and subgroup $H \leq G$. Since the source $\mathbb{A}^+(G)$ only depends on the finite group $\bar{G} = G/G^\circ$ of path components, this last exponential map is only interesting for finite groups.

4. Global forms of BO

In this section we discuss different orthogonal spaces whose underlying non-equivariant homotopy type is BO, a classifying space for the infinite orthogonal group. Each example is interesting in its own right, and as a whole, the global forms of BO are a great illustration of how non-equivariant homotopy types 'fold up' into many different global homotopy types. The different forms of BO have associated orthogonal Thom spectra with underlying non-equivariant stable homotopy type MO; we will return to these Thom spectra in Section VI.2. The examples we discuss here all come with multiplications, some ultra-commutative, but some only E_∞-commutative, so our case study also serves to illustrate the difference in the degrees of commutativity that arise 'in nature'.

We summarize our discussion of 'global forms of BO'. We can name five different global homotopy types that all have the same underlying non-equivariant homotopy type, namely that of a classifying space of the infinite orthogonal group:

- the constant orthogonal space $c(BO)$ with value a classifying space of the infinite orthogonal group;
- the bar construction model $\mathbf{BO}^\circ$ (Construction 4.15);
- the 'full Grassmannian model' $\mathbf{BO}$, the degree 0 part of the periodic global Grassmannian $\mathbf{BOP}$ (Example 4.2);
- the 'restricted Grassmannian model' $\mathbf{bO}$ that is also a sequential homotopy colimit of the global classifying spaces $B_{\mathrm{gl}}O(n)$ (Example 4.20);
- the cofree orthogonal space $R(BO)$ associated to a classifying space of the infinite orthogonal group (Construction I.3.24).

These global homotopy types are related by (weak) morphisms of orthogonal spaces:

$$\begin{tikzcd} c(BO) \arrow[r] \arrow[d] & \mathbf{bO} \arrow[d] & \\ \mathbf{BO}^\circ \arrow[r] & \mathbf{BO} \arrow[r] & R(BO) \end{tikzcd} \tag{4.1}$$

The orthogonal spaces $\mathbf{BO}^\circ$ and $\mathbf{BO}$ come with ultra-commutative multiplications. The global homotopy type of $R(BO)$ also admits an ultra-commutative multiplication; we will not expand on this for $R(BO)$, but one way to see this is to extend the cofree functor R to a lax symmetric monoidal functor on the category of orthogonal spaces, so that $R(\mathbf{BO})$ is an ultra-commutative monoid within this global homotopy type. The orthogonal spaces $c(BO)$ and $\mathbf{bO}$ admit E_∞-multiplications; for $c(BO)$ this is a consequence of the non-equivariant E_∞-structure of BO. All (weak) morphisms above can be arranged to preserve the E_∞-multiplication, so they induce additive maps of abelian monoids on π_0^G for every compact Lie group G. We denote by $\mathbf{RO}^\sharp(G)$ the abelian submonoid of $\mathbf{RO}^+(G)$ consisting of the isomorphism classes of G-representations with trivial G-fixed points. Then up to isomorphisms that we specify below, the induced diagram of abelian monoids is:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbf{RO}^\sharp(G) \arrow[d, "{[U]\mapsto \dim(U)\cdot 1-[U]}"] & \\ 0 \arrow[r] & \mathbf{IO}(G) \arrow[r] & \widetilde{KO}^0(BG) \end{tikzcd}$$

The last group is the reduced real K-theory group of the non-equivariant classifying space BG, by Propositions I.3.26 (iii) and I.5.14, and the lower right horizontal map takes a virtual G-representation to the associated virtual vector bundle over BG.

Example 4.37 introduces the complex and quaternionic analogs of the above object, the ultra-commutative monoids $\mathbf{BU}$ and $\mathbf{BSp}$ and the E_∞-orthogonal monoid spaces $\mathbf{bU}$ and $\mathbf{bSp}$. The last result in this section is global, highly structured version of Bott periodicity: Theorem 4.39 shows that $\mathbf{BUP}$ and $\Omega\mathbf{U}$ are globally equivalent as ultra-commutative monoids.

Example 4.2 (Periodic Grassmannian). We define an ultra-commutative monoid $\mathbf{BOP}$ that is a global refinement of the non-equivariant homotopy type $\mathbb{Z}\times BO$, and at the same time a global group completion of the additive Grassmannian $\mathbf{Gr}$ introduced in Example 3.11. The orthogonal space $\mathbf{BOP}$ comes with tautological vector bundles whose Thom spaces form the periodic unoriented bordism spectrum $\mathbf{MOP}$, compare Example VI.2.7 below.

For an inner product space V we set

$$\mathbf{BOP}(V) \;=\; \coprod_{n\geq 0} Gr_n(V^2)\ ,$$

the disjoint union of the Grassmannians of n-dimensional subspaces in $V^2 = V\oplus V$. The structure map associated to a linear isometric embedding $\varphi : V \longrightarrow W$ is given by

$$\mathbf{BOP}(\varphi)(L) \;=\; \varphi^2(L)\oplus((W-\varphi(V))\oplus 0)\ ,$$

the internal orthogonal sum of the image of L under $\varphi^2 : V^2 \longrightarrow W^2$ and the orthogonal complement of the image of $\varphi : V \longrightarrow W$, viewed as sitting in the first summand of $W^2 = W\oplus W$. In particular, the orthogonal group $O(V)$ acts on $\mathbf{BOP}(V)$ through its diagonal action on V^2. We have thus defined an orthogonal space $\mathbf{BOP}$.

We make $\mathbf{BOP}$ into an ultra-commutative monoid by endowing it with multiplication maps

$$\mu_{V,W}\ :\ \mathbf{BOP}(V)\times\mathbf{BOP}(W)\ \longrightarrow\ \mathbf{BOP}(V\oplus W)\ ,\quad (L,L')\ \longmapsto\ \kappa_{V,W}(L\oplus L')\ ,$$

where

$$\kappa_{V,W}\ :\ V^2\oplus W^2\ \cong\ (V\oplus W)^2 \qquad \text{is defined by} \qquad \kappa_{V,W}(v,v',w,w') = (v,w,v',w')\ .$$

The unit is the unique element $\{0\}$ of $\mathbf{BOP}(0)$.

The orthogonal space $\mathbf{BOP}$ is naturally $\mathbb{Z}$-graded: for $k\in\mathbb{Z}$ we let

$$\mathbf{BOP}^{[k]}(V)\ \subset\ \mathbf{BOP}(V)$$

be the path component consisting of all subspaces $L\subset V^2$ such that $\dim(L)-\dim(V)=k$. For fixed k the spaces $\mathbf{BOP}^{[k]}(V)$ form a subfunctor of $\mathbf{BOP}$, i.e., $\mathbf{BOP}^{[k]}$ is an orthogonal subspace of $\mathbf{BOP}$. The multiplication is graded in the sense that $\mu_{V,W}$ takes $\mathbf{BOP}^{[k]}(V)\times\mathbf{BOP}^{[l]}(W)$ to $\mathbf{BOP}^{[k+l]}(V\oplus W)$. We write $\mathbf{BO}=\mathbf{BOP}^{[0]}$ for the homogeneous summand of $\mathbf{BOP}$ of degree 0, which is thus an ultra-commutative monoid in its own right. The underlying non-equivariant homotopy type of $\mathbf{BO}=\mathbf{BOP}^{[0]}$ is that of a classifying space of the infinite orthogonal group; similarly, $\mathbf{BOP}$ has underlying non-equivariant homotopy type $\mathbb{Z}\times BO$, a $\mathbb{Z}$-indexed union of copies of BO.

While $\mathbf{BOP}$ and the additive Grassmannian $\mathbf{Gr}$ are both made from Grassmannians, one should beware of the different nature of their structure maps. There is a variation $\mathbf{Gr}'$ of the additive Grassmannian with values $\mathbf{Gr}'(V)=\coprod_{n\geq 0}Gr_n(V^2)$ and structure maps $\mathbf{Gr}'(\varphi)(L)=\varphi^2(L)$. This orthogonal space is a 'multiplicative shift' of $\mathbf{Gr}$ in the sense of Example I.1.12, it admits a commutative multiplication in much the same way as $\mathbf{Gr}$, and the maps

$$\mathbf{Gr}(V)\ \longrightarrow\ \mathbf{Gr}'(V)\ ,\quad L\ \longmapsto\ L\oplus 0$$

form a global equivalence of ultra-commutative monoids (compare Theorem I.1.11). A source of possible confusion is the fact that $\mathbf{Gr}'(V)$ and $\mathbf{BOP}(V)$ are *equal* as spaces, but they come with very different structure maps making them into two different global homotopy types.

Example 4.3 ($\mathbf{Gr}$ versus $\mathbf{BOP}$). In Example 3.11 we explained that for every compact Lie group G, the monoid $\pi_0^G(\mathbf{Gr})$ is isomorphic to the monoid $\mathbf{RO}^+(G)$, under direct sum, of isomorphism classes of orthogonal G-representations. In Theorem 4.14 we will identify the monoid $\pi_0^G(\mathbf{BOP})$ with the orthogonal representation ring $\mathbf{RO}(G)$. The latter is the algebraic group completion of the former, and this group completion is realized by a morphism of ultra-commutative monoids

$$i \ : \ \mathbf{Gr} \ \longrightarrow \ \mathbf{BOP} \ . \tag{4.4}$$

The morphism i is given at an inner product space V by the map

$$\mathbf{Gr}(V) \ = \ \coprod_n Gr_n(V) \ \longrightarrow \ \coprod_m Gr_m(V^2) = \mathbf{BOP}(V) \ , \quad L \ \longmapsto \ V \oplus L \ .$$

The morphism is homogeneous in that it takes $\mathbf{Gr}^{[n]}$ to $\mathbf{BOP}^{[n]}$.

As we will now show, the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ induces an algebraic group completion of abelian monoids upon taking equivariant homotopy set from any equivariant space. This fact is the algebraic shadow of a more refined relationship: as we will show in Theorem 5.35 below, the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ is a group completion in the world of ultra-commutative monoids, i.e., 'homotopy-initial', in the category of ultra-commutative monoids, among morphisms from $\mathbf{Gr}$ to group-like ultra-commutative monoids.

We recall from Definition I.5.1 the equivariant homotopy set

$$[A, R]^G \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} [A, R(V)]^G \ ,$$

where R is an orthogonal space, G is a compact Lie group and A a G-space. If R is an ultra-commutative monoid, then this set inherits an abelian monoid structure defined as follows. We let $\alpha : A \longrightarrow R(V)$ and $\beta : A \longrightarrow R(W)$ be two G-maps that represent classes in $[A, R]^G$. Then their sum is defined as

$$[\alpha] + [\beta] \ = \ [\mu^R_{V,W}(\alpha, \beta)] \ , \tag{4.5}$$

where $\mu^R_{V,W} \ : \ R(V) \times R(W) \ \longrightarrow \ R(V \oplus W)$ is the (V, W)-component of the multiplication of R. The monoid structure is contravariantly functorial for G-maps in A, and covariantly functorial for morphisms of ultra-commutative monoids in R.

Proposition 4.6. *For every compact Lie group G and every G-space A, the homomorphism*

$$[A, i]^G \ : \ [A, \mathbf{Gr}]^G \ \longrightarrow \ [A, \mathbf{BOP}]^G$$

is a group completion of abelian monoids.

Proof. We start by showing that the abelian monoid $[A, \mathbf{BOP}]^G$ is a group. We consider a G-representation V. For a linear subspace $L \subseteq V^2$ we consider the 1-parameter family of linear isometric embeddings

$$H_L \ : \ [0, 1] \times L^\perp \ \longrightarrow \ V^2 \oplus V^2 \ , \quad (t, x) \ \longmapsto \ (t \cdot x, \sqrt{1 - t^2} \cdot x) \ .$$

For every $t \in [0, 1]$, the image of $H_L(t, -)$ is isomorphic to $L^\perp$ and orthogonal to the space $L \oplus 0 \oplus 0$. We can thus define a G-equivariant homotopy

$$K \ : \ [0, 1] \times Gr(V^2) \ \longrightarrow \ Gr(V^2 \oplus V^2) \qquad \text{by} \qquad K(t, L) \ = \ (L \oplus 0 \oplus 0) + H_L(t, L^\perp) \ .$$

Then

$$K(0, L) \ = \ (L \oplus 0 \oplus 0) + H_L(0, L^\perp) \ = \ (L \oplus 0) + (0 \oplus L^\perp) \ = \ L \oplus L^\perp$$

and

$$K(1, L) \ = \ (L \oplus 0 \oplus 0) + H_L(1, L^\perp) \ = \ (L \oplus 0 \oplus 0) + (L^\perp \oplus 0 \oplus 0) \ = \ V \oplus V \oplus 0 \oplus 0 \ .$$

We recall that the multiplication of **BOP** is given by

$$\mu^{\mathbf{BOP}}_{V,V} \ : \ \mathbf{BOP}(V) \times \mathbf{BOP}(V) \ \longrightarrow \mathbf{BOP}(V \oplus V) \ , \quad \mu^{\mathbf{BOP}}_{V,V}(L, L') \ = \ \kappa_{V,V}(L \oplus L') \ ,$$

where $\kappa_{V,V}(v, v', w, w') = (v, w, v', w')$. So the equivariant homotopy $Gr(\kappa_{V,V}) \circ K$ interpolates between the composite

$$\mathbf{BOP}(V) \ \xrightarrow{(\mathrm{Id},(-)^\perp)} \ \mathbf{BOP}(V) \times \mathbf{BOP}(V) \ \xrightarrow{\mu^{\mathbf{BOP}}_{V,V}} \ \mathbf{BOP}(V \oplus V)$$

and the constant map with value

$$\kappa_{V,V}(V \oplus V \oplus 0 \oplus 0) \ = \ V \oplus 0 \oplus V \oplus 0 \ .$$

The subspace $V \oplus 0 \oplus V \oplus 0$ lies in the same path component of $\mathbf{BOP}(V^2)^G = (Gr(V^2 \oplus V^2))^G$ as the subspace $V \oplus V \oplus 0 \oplus 0$. So altogether this shows that the composite $\mu^{\mathbf{BOP}}_{V,V} \circ (\mathrm{Id}, (-)^\perp)$ is G-equivariantly homotopic to the constant map with value $V \oplus V \oplus 0 \oplus 0$

Now we let $f : A \longrightarrow \mathbf{BOP}(V)$ be a G-map, representing a class in $[A, \mathbf{BOP}]^G$. The composite of f and the orthogonal complement map $(-)^\perp : \mathbf{BOP}(V) \longrightarrow \mathbf{BOP}(V)$ represents another class in $[A, \mathbf{BOP}]^G$, and

$$[f] + [(-)^\perp \circ f] \ = \ [\mu^{\mathbf{BOP}}_{V,V} \circ (\mathrm{Id}, (-)^\perp) \circ f] \ = \ [c_{V \oplus V \oplus 0 \oplus 0} \circ f] \ = \ 0$$

in the monoid structure of $[A, \mathbf{BOP}]^G$, because the subspace $V \oplus V \oplus 0 \oplus 0$ is the neutral element in $\mathbf{BOP}(V \oplus V)$. So the class $[f]$ has an additive inverse, and this concludes the proof that the abelian monoid $[A, \mathbf{BOP}]^G$ is a group.

To show that the homomorphism $i_* = [A, i]^G$ is a group completion we show two separate statements that amount to the surjectivity, respectively injectivity, of the extension of $[A, i]^G$ to a homomorphism on the Grothendieck group of the monoid $[A, \mathbf{Gr}]^G$, compare Example 5.6 below.

(a) We show that every class in $[A, \mathbf{BOP}]^G$ is the difference of two classes in the image of $i_* : [A, \mathbf{Gr}]^G \longrightarrow [A, \mathbf{BOP}]^G$. To see this, we represent a given class $x \in [A, \mathbf{BOP}]^G$ by a G-map $f : A \longrightarrow \mathbf{BOP}(V)$, for some G-representation V. Because $\mathbf{BOP}(V) = \mathbf{Gr}(V \oplus V)$, the same map f also represents a class in $[A, \mathbf{Gr}]^G$; to emphasize the different role, we write this map as $f^\sharp : A \longrightarrow \mathbf{Gr}(V \oplus V)$. We let $c_V : A \longrightarrow \mathbf{Gr}(V)$ denote the constant map with value V and $\chi : V^4 \longrightarrow V^4$ the linear isometry defined by

$$\chi(v_1, v_2, v_3, v_4) \ = \ (v_2, v_3, v_1, v_4) \ .$$

We observe that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{BOP}(V) \arrow[rr, "(i(V)\circ c_V, \mathrm{Id})"] \arrow[d, equal] & & \mathbf{BOP}(V) \times \mathbf{BOP}(V) \arrow[d, "\mu^{\mathbf{BOP}}_{V,V}"] \\
\mathbf{Gr}(V \oplus V) \arrow[r, "i(V \oplus V)"'] & \mathbf{BOP}(V \oplus V) \arrow[r, "Gr(\chi)"'] & \mathbf{BOP}(V \oplus V)
\end{tikzcd}$$

Since χ is equivariantly homotopic, through linear isometries, to the identity, this shows that the composite $\mu^{\mathbf{BOP}}_{V,V} \circ (i(V) \circ c_V, \mathrm{Id})$ and G-equivariantly homotopic to $i(V \oplus V)$. Thus

$$\begin{aligned} i_*[c_V] + x \ = \ [\mu^{\mathbf{BOP}}_{V,V} \circ (i(V) \circ c_V, f)] \ &= \ [\mu^{\mathbf{BOP}}_{V,V} \circ (i(V) \circ c_V, \mathrm{Id}) \circ f] \\ &= \ [i(V \oplus V) \circ f^\sharp] \ = \ i_*[f^\sharp] \ . \end{aligned}$$

Thus $x = i_*[f^\sharp] - i_*[c_V]$, which shows the claim.

(b) Now we consider two classes $a, b \in [A, \mathbf{Gr}]^G$ such that $i_*(a) = i_*(b)$ in $[A, \mathbf{BOP}]^G$. We show that there exist another class $c \in [A, \mathbf{Gr}]^G$ such that $c + a = c + b$. We can represent a and b by two G-maps $\alpha : A \longrightarrow \mathbf{Gr}(V)$ and $\beta : A \longrightarrow \mathbf{Gr}(V)$ such that the two composites

$$i(V) \circ \alpha \ , \quad i(V) \circ \beta \ : \ A \ \longrightarrow \ \mathbf{BOP}(V)$$

are equivariantly homotopic. As before we let $c_V : A \longrightarrow \mathbf{Gr}(V)$ be the constant map with value V. The map $i(V) : \mathbf{Gr}(V) \longrightarrow \mathbf{BOP}(V) = \mathbf{Gr}(V \oplus V)$ factors as the composite

$$\mathbf{Gr}(V) \xrightarrow{(c_V,\mathrm{Id})} \mathbf{Gr}(V) \times \mathbf{Gr}(V) \xrightarrow{\mu^{\mathbf{Gr}}_{V,V}} \mathbf{Gr}(V \oplus V) \ ,$$

so

$$[c_V] \ + a \ = \ [c_V] + [\alpha] \ = \ [\mu^{\mathbf{Gr}}_{V,V} \circ (c_V, \alpha)] \ = \ [\mu_{V,V} \circ (c_V, , \mathrm{Id}) \circ \alpha] \ = \ [i(V) \circ \alpha] \ .$$

Similarly, $[c_V] + b = [i(V) \circ \beta]$. So $[c_V] + a = [c_V] + b$ in $[A, \mathbf{Gr}]^G$, as claimed. □

Now we are ready to show that the ultra-commutative monoid $\mathbf{Gr}$ represents equivariant vector bundles, and $\mathbf{BOP}$ represents equivariant K-theory, at least for compact G-spaces.

Construction 4.7. We let G be a compact Lie group and A a G-space. We recall that a *G-vector bundle* over A consists of a vector bundle $\xi : E \longrightarrow A$ and a continuous G-action on the total space E such that

- the bundle projection $\xi : E \longrightarrow A$ is a G-map,
- for every $g \in G$ and $b \in A$ the map $g \cdot - : E_b \longrightarrow E_{gb}$ is $\mathbb{R}$-linear.

We let $\mathrm{Vect}_G(A)$ be the commutative monoid, under direct sum, of isomorphism classes of G-vector bundles over A. We define a homomorphism of monoids

$$\langle - \rangle \ : \ [A, \mathbf{Gr}]^G \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} [A, \ \mathbf{Gr}(V)]^G \ \longrightarrow \ \mathrm{Vect}_G(A) \tag{4.8}$$

that will turn out to be an isomorphism for compact A and that specializes to the isomorphism (3.13) from $\pi_0^G(\mathbf{Gr})$ to $\mathbf{RO}^+(G)$ when A is a one-point G-space. We let $f : A \longrightarrow \mathbf{Gr}(V)$ be a continuous G-map, for some G-representation V. We pull back the tautological G-vector bundle γ_V over $\mathbf{Gr}(V)$ and obtain a G-vector bundle $f^\star(\gamma_V) : E \longrightarrow A$ over A with total space

$$E \ = \ \{(v, b) \in V \times A \mid v \in f(b)\} \ .$$

The G-action and bundle structure are all as a G-subbundle of the trivial bundle $V \times A$. Since the base $\mathbf{Gr}(V)$ of the tautological bundle is a disjoint union of compact spaces, the isomorphism class of the bundle $f^\star(\gamma_V)$ depends only on the G-homotopy class of f (see e.g. [**141**, Prop. 1.3]), so the construction yields a well-defined map

$$[A, \ \mathbf{Gr}(V)]^G \ \longrightarrow \ \mathrm{Vect}_G(A) \ , \quad [f] \ \longmapsto \ [f^\star(\gamma_V)] \ .$$

If $\varphi : V \longrightarrow W$ is a linear isometric embedding of G-representations, then the restriction along $\mathbf{Gr}(\varphi) : \mathbf{Gr}(V) \longrightarrow \mathbf{Gr}(W)$ of the tautological G-vector bundle over $\mathbf{Gr}(W)$ is isomorphic to the tautological G-vector bundle over $\mathbf{Gr}(V)$. So the two G-vector bundles $f^\star$ and $(\mathbf{Gr}(\varphi) \circ f)^*$ over A are isomorphic. So we can pass to the colimit over the poset $s(\mathcal{U}_G)$ and get a well-defined map (4.8).

Now we 'group complete' the picture. We denote by $\mathbf{KO}_G(A)$ the G-equivariant K-group of A, i.e., the group completion (Grothendieck group) of the abelian monoid $\mathrm{Vect}_G(A)$. We define a homomorphism of monoids

$$\langle - \rangle \ : \ [A, \mathbf{BOP}]^G \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} [A, \ \mathbf{BOP}(V)]^G \ \longrightarrow \ \mathbf{KO}_G(A) \tag{4.9}$$

for compact G-spaces A. We let $f : A \longrightarrow \mathbf{BOP}(V)$ be a G-map for some G-representation V. We pull back the tautological G-vector bundle over $\mathbf{BOP}(V) = Gr(V^2)$ and obtain a G-vector bundle $f^\star(\gamma_{V^2}) : E \longrightarrow A$ over A with total space

$$E \ = \ \{(v, b) \in V^2 \times A \mid v \in f(b)\} \ .$$

Again the G-action and bundle structure are all as a G-subbundle of the trivial bundle $V^2 \times A$. As before, the isomorphism class of the bundle $f^\star(\gamma_{V^2})$ depends only on the G-homotopy class of f, so the construction yields a well-defined map

$$[A, \ \mathbf{BOP}(V)]^G \ \longrightarrow \ \mathbf{KO}_G(A) \ , \quad [f] \ \longmapsto \ [f^\star(\gamma_{V^2})] - [V \times A] \ .$$

We emphasize that now, in contrast to the case of $\mathbf{Gr}$, we subtract the class of the trivial G-vector bundle $V \times A$ over A. If $\varphi : V \longrightarrow W$ is a linear isometric embedding of G-representations, then the restriction along $\mathbf{BOP}(\varphi) : \mathbf{BOP}(V) \longrightarrow \mathbf{BOP}(W)$ of the tautological G-vector bundle over $\mathbf{BOP}(W)$ is isomorphic to the direct sum of the tautological G-vector bundle over $\mathbf{BOP}(V)$ and a copy of the trivial bundle $(W - \varphi(V)) \times \mathbf{BOP}(V)$. So the bundle $(\mathbf{BOP}(\varphi) \circ f)^\star(\gamma_{W^2})$ is isomorphic to the direct sum of $f^\star(\gamma_{V^2})$ and a copy of the trivial bundle $(W - \varphi(V)) \times A$. Thus

$$\begin{aligned}[(\mathbf{BOP}(\varphi) \circ f)^\star(\gamma_{W^2})] - [W \times A] \\ = [f^\star(\gamma_{V^2})] + [(W - \varphi(V)) \times A] - [W \times A] \ = \ [f^\star(\gamma_{V^2})] - [V \times A]\end{aligned}$$

in the Grothendieck group $\mathbf{KO}_G(A)$. So we can pass to the colimit over the poset $s(\mathcal{U}_G)$ and get a well-defined map (4.9), and the right vertical map is algebraic group completion.

The two maps (4.8) and (4.9) are clearly natural for G-maps in A, and they are additive, ultimately because all additions in sight arise from direct sum of inner product spaces. As is immediate from the definitions, the two pullback bundle homomorphisms are compatible, in the sense that the square

$$\begin{tikzcd} {[A, \mathbf{Gr}]^G} \arrow[r, "\langle - \rangle"] \arrow[d, "{[A,i]^G}"'] & \mathrm{Vect}_G(A) \arrow[d] \\ {[A, \mathbf{BOP}]^G} \arrow[r, "\langle - \rangle"'] & \mathbf{KO}_G(A) \end{tikzcd} \tag{4.10}$$

commutes, where $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ is the morphism of ultra-commutative monoids defined in (4.4), and the right vertical map is algebraic group completion.

THEOREM 4.11. *For every compact Lie group G and every compact G-space B the homomorphism* (4.8)

$$\langle - \rangle \ : \ [A, \mathbf{Gr}]^G \ \longrightarrow \ \mathrm{Vect}_G(A)$$

and the homomorphism (4.9)

$$\langle - \rangle \ : \ [A, \mathbf{BOP}]^G \ \longrightarrow \ \mathbf{KO}_G(A)$$

are isomorphisms. As G varies the isomorphisms are compatible with restriction along continuous homomorphism.

PROOF. The Grassmannian $\mathbf{Gr}$ is the disjoint union of the homogeneous pieces $\mathbf{Gr}^{[n]}$, and the latter is isomorphic to the free orthogonal space $\mathbf{L}_{O(n),\mathbb{R}^n}$, via

$$\mathbf{L}(\mathbb{R}^n, W)/O(n) \ \longrightarrow \ \mathbf{Gr}^{[n]}(W) \ , \quad \varphi \cdot O(n) \ \longmapsto \ \varphi(\mathbb{R}^n) \ .$$

Since the tautological action of $O(n)$ on $\mathbb{R}^n$ is faithful, $\mathbf{L}_{O(n),\mathbb{R}^n}$ is a global classifying space for $O(n)$; Example I.5.3 thus provides a bijection

$$[A, \mathbf{Gr}^{[n]}]^G \ \longrightarrow \ \mathrm{Prin}_{(G,O(n))}(A)$$

to the set of isomorphism classes of G-equivariant principal $O(n)$-bundles over A by pulling back the $(G, O(n))$-principal bundles $\mathbf{L}(\mathbb{R}^n, W) \longrightarrow \mathbf{Gr}^{[n]}(W)$ (which is the frame bundle of the tautological vector bundle over $\mathbf{Gr}^{[n]}(W)$). On the other hand, we can consider the map

$$\mathrm{Prin}_{(G,O(n))}(A) \ \longrightarrow \ \mathrm{Vect}_G^{[n]}(A)$$

to the set of isomorphism classes of rank n G-vector bundles over A, given by sending a $(G, O(n))$-bundle $\gamma : E \longrightarrow A$ to the associated G-vector bundle with total space $E \times_{O(n)} \mathbb{R}^n$. Since A is compact, every G-vector bundle admits a G-invariant euclidean inner product, so it arises from a $(G, O(n))$-bundle; hence the latter map is bijective as well.

Altogether this shows that map

$$[A, \mathbf{Gr}^{[n]}]^G \;\longrightarrow\; \mathrm{Vect}_G^{[n]}(A) \tag{4.12}$$

given by pulling back the tautological vector bundles is bijective.

A general G-vector bundle need not have constant rank, so it remains to assemble the results for different $n \geq 0$. We let ξ be any G-vector bundle over A, not necessarily of constant rank. Then the subset

$$A_{(n)} \;=\; \{a \in A \mid \dim(\xi_a) = n\}$$

of points over which ξ is n-dimensional is open by local triviality of vector bundles. So A is the disjoint union of the sets $A_{(n)}$ for $n \geq 0$, and each subset $A_{(n)}$ is also closed and hence hence compact in the subspace topology. Moreover, each $A_{(n)}$ is G-invariant, so the restriction of $\xi_{(n)}$ of the bundle to $A_{(n)}$ is classified by a G-map $f_{(n)} : A_{(n)} \longrightarrow \mathbf{Gr}^{[n]}(V_n)$ for some finite dimensional G-representation V_n. Since A is compact, almost all $A_{(n)}$ are empty, so by increasing the representations, if necessary, we can assume that the classifying maps have target $\mathbf{Gr}^{[n]}(V)$ for a fixed finite dimensional G-representation V, independent of n. Then

$$\coprod_{n\geq 0} f_{(n)} \;:\; \coprod_{n\geq 0} A_{(n)} \;=\; A \;\longrightarrow\; \coprod_{n\geq 0} \mathbf{Gr}^{[n]}(V) \;=\; \mathbf{Gr}(V)$$

is a classifying G-map for the original bundle ξ. This shows that the map (4.8) is surjective.

The argument for injectivity is similar. Any pair of classes in $[A, \mathbf{Gr}]^G$ can be represented by G-maps $f, \bar{f} : A \longrightarrow \mathbf{Gr}(V)$ for some finite dimensional G-representation V. Since $\mathbf{Gr}(V)$ is the disjoint union of the subspaces $\mathbf{Gr}^{[n]}(V)$ for $n \geq 0$, their inverse images under f and $\bar{f}$ provide disjoint union decompositions of A. If the bundles $f^\star(\gamma_V)$ and $\bar{f}^\star(\gamma_V)$ are isomorphic, the decompositions of A induced by f and $\bar{f}$ must be the same. The rank n summands $f_{(n)}, \bar{f}_{(n)} : A_{(n)} \longrightarrow \mathbf{Gr}^{[n]}(V)$ become equivariantly homotopic after increasing the representation V, because the map (4.12) is injective. Moreover, almost all summands are empty, one more time by compactness. So there is a single finite dimensional representation W and a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ such that $f, \bar{f} : A \longrightarrow \mathbf{Gr}(V)$ become equivariantly homotopic after composition with $\mathbf{Gr}(\varphi) : \mathbf{Gr}(V) \longrightarrow \mathbf{Gr}(W)$. Hence f and $\bar{f}$ represent the same class in $[A, \mathbf{Gr}]^G$, and so the map (4.8) is injective. This completes the proof that the map is an isomorphism for compact A.

The left vertical map in the commutative square (4.10) is a group completion by Proposition 4.6, and the right vertical map is a group completion by definition. So the lower horizontal map (4.9) is also an isomorphism. □

We take the time to specialize Theorem 4.11 to the one-point G-space. This special case identifies the global power monoid $\underline{\pi}_0(\mathbf{BOP})$ with the global power monoid $\mathbf{RO}$ of orthogonal representation rings. For every compact Lie group G the abelian monoid $\mathbf{RO}(G)$ is the Grothendieck group, under direct sum, of finite dimensional G-representations. The restriction maps are induced by restriction of representations, and the power operation $[m] : \mathbf{RO}(G) \longrightarrow \mathbf{RO}(\Sigma_m \wr G)$ takes the class of a G-representation V to the class of the $(\Sigma_m \wr G)$-representation V^m. The resulting transfer $\mathrm{tr}_H^G : \mathbf{RO}(H) \longrightarrow \mathbf{RO}(G)$ of Construction 2.32, for H of finite index in G, is then the transfer (or induction), sending the class of an H-representation V to the class of the induced G-representation $\mathrm{map}^H(G, V)$. A G-vector bundle over a one-point space 'is' a G-representation and the map

$$\mathbf{RO}(G) \;\longrightarrow\; \mathbf{KO}_G(*)$$

that considers a (virtual) representation as a (virtual) vector bundle is an isomorphism of groups and compatible with restriction along continuous homomorphisms of compact Lie groups.

For easier reference we spell out the isomorphism (4.9) in the special case $B = *$ more explicitly. We let V be a finite dimensional orthogonal G-representation. The G-fixed points of $\mathbf{BOP}(V)$ are the G-invariant subspaces of V^2, i.e., the G-subrepresentations W of V^2. Representations of compact Lie groups

are discrete (compare the example after [**141**, Prop. 1.3]), so two fixed points in the same path component of $\mathbf{BOP}(V)^G$ are isomorphic as G-representations. So we obtain a well-defined map

$$\pi_0(\mathbf{BOP}(V)^G) \;\longrightarrow\; \mathbf{RO}(G)$$

by sending $W \in \mathbf{BOP}(V)^G$ to $[W]-[V]$, the formal difference in $\mathbf{RO}(G)$ of the classes of W and V. These maps are compatible as V runs through the finite dimensional G-subrepresentations of $\mathcal{U}_G$, so they assemble into a map

$$\pi_0^G(\mathbf{BOP}) \;=\; \operatorname{colim}_{V\in s(\mathcal{U})}\, \pi_0(\mathbf{BOP}(V)^G) \;\longrightarrow\; \mathbf{RO}(G)\ . \tag{4.13}$$

THEOREM 4.14. *For every compact Lie group G the map* (4.13) *is an isomorphism of group. As G varies, these bijections form an isomorphism of global power monoids*

$$\underline{\pi}_0(\mathbf{BOP}) \;\cong\; \mathbf{RO}\ .$$

PROOF. The special case $B = *$ of Theorem 4.11 shows that the map (4.13) is an isomorphism and compatible with restriction along continuous homomorphisms. We have to observe that in addition, the (4.13) are also compatible with transfers (or equivalently, with power operations). The compatibility with transfer can either be deduced directly from the definitions; equivalently it can be formally deduced from the compatibility of the isomorphisms $\underline{\pi}_0(\mathbf{Gr}) \cong \mathbf{RO}^+$ with transfers by the universal property of a group completion. □

The bijection (4.13) sends elements of $\pi_0^G(\mathbf{BOP}^{[k]})$ to virtual representations of dimension k, so we can also identify the global power monoid of the homogeneous degree 0 part $\mathbf{BO} = \mathbf{BOP}^{[0]}$. Indeed, the map (4.13) restricts to an isomorphism of abelian groups

$$\pi_0^G(\mathbf{BO}) \;\cong\; \mathbf{IO}(G)$$

to the augmentation ideal $\mathbf{IO}(G) \subset \mathbf{RO}(G)$ of the orthogonal representation ring, compatible with restriction maps, power operations and transfer maps.

EXAMPLE 4.15 (Bar construction model $\mathbf{BO}^\circ$). Using a functorial bar construction we define yet another global refinement $\mathbf{BO}^\circ$ of the classifying space of the infinite orthogonal group. The ultra-commutative monoid $\mathbf{BO}^\circ$ is globally connected, and it comes with a weak homomorphism to $\mathbf{BO}$ that 'picks out' the path components of the neutral element in the G-fixed point spaces $\mathbf{BO}(\mathcal{U}_G)^G$. In a sense that we will make precise in Corollary 4.18 below, the ultra-commutative monoids $\mathbf{BO}^\circ$ and $\mathbf{BO}$ are to $\mathbf{BOP}$ what the trivial global power monoid respectively the augmentation ideal global power monoid are to $\mathbf{RO}$.

We define an orthogonal space $\mathbf{BO}^\circ$ by applying the bar construction (see Construction 3.20) objectwise to the monoid valued orthogonal space $\mathbf{O}$ of Example 3.6. So the value at an inner product space V is

$$(\mathbf{BO}^\circ)(V) \;=\; B(O(V))\ ,$$

the bar construction of the orthogonal group of V. The structure map of a linear isometric embedding $\varphi : V \longrightarrow W$ is obtained by applying the bar construction to the continuous homomorphism $\mathbf{O}(\varphi) : \mathbf{O}(V) \longrightarrow \mathbf{O}(W)$. We make $\mathbf{BO}^\circ$ into an ultra-commutative monoid by endowing it with multiplication maps

$$\mu_{V,W} \;:\; (\mathbf{BO}^\circ)(V) \times (B\mathbf{O})(W) \;\longrightarrow\; (\mathbf{BO}^\circ)(V \oplus W)$$

defined as the composite

$$B(O(V)) \times B(O(W)) \;\xrightarrow{\cong}\; B\big(O(V) \times O(W)\big) \;\xrightarrow{B\oplus}\; B(O((V \oplus W)))\ ,$$

where the first map is inverse to the homeomorphism (3.21).

Now we let G be a compact Lie group and V an orthogonal G-representation. Then

$$((\mathbf{BO}^\circ)(V))^G \;=\; (B(O(V)))^G \;=\; B(O^G(V))\ .$$

Taking colimit over the poset $s(\mathcal{U}_G)$ gives

$$((\mathbf{BO}^\circ)(\mathcal{U}_G))^G \;\cong\; \mathrm{colim}_{V\in s(\mathcal{U}_G)}\; B(O^G(V)) \;\cong\; B(O^G(\mathcal{U}_G)) \;\cong\; \prod_{[\lambda]}{}' B(O^G(\mathcal{U}_\lambda))\ .$$

Here the last weak product is indexed by isomorphism classes of irreducible G-representations, and each of the groups $O^G(\mathcal{U}_\lambda)$ is either an infinite orthogonal, unitary or symplectic group, depending on the type of the irreducible representation, compare Example 3.6. In particular, the space $((\mathbf{BO}^\circ)(\mathcal{U}_G))^G$ is connected, so the equivariant homotopy set $\pi_0^G(\mathbf{BO}^\circ)$ has one element for every compact Lie group G; the global power monoid structure is then necessarily trivial. In particular, $\mathbf{BO}^\circ$ is *not* globally equivalent to $\mathbf{BO}$.

However, the difference seen by $\underline{\pi}_0$ is the only difference between $\mathbf{BO}^\circ$ and $\mathbf{BO}$, as shall now explain. We construct a weak morphism of ultra-commutative monoids that exhibits $\mathbf{BO}^\circ$ as the 'globally connected component' of $\mathbf{BO}$. We define an ultra-commutative monoid $\mathbf{B'O}^\circ$ by combining the constructions of $\mathbf{BO}^\circ$ (bar construction) and $\mathbf{BO}$ (Grassmannians) into one definition. The value of $\mathbf{B'O}^\circ$ at an inner product space V is

$$(\mathbf{B'O}^\circ)(V) \;=\; B(\mathbf{L}(V,V^2),\mathbf{O}(V),*)\ ,$$

the two-sided bar construction (homotopy orbit construction) of the right $O(V)$-action on the space $\mathbf{L}(V,V^2)$ by precomposition. To define the structure map associated to a linear isometric embedding $\varphi : V \longrightarrow W$ we recall that the structure map $\mathbf{O}(\varphi) : \mathbf{O}(V) \longrightarrow \mathbf{O}(W)$ of the orthogonal space $\mathbf{O}$ is given by conjugation by φ, direct sum with the identity on $W-\varphi(V)$. We define a continuous map

$$\varphi_\sharp \;:\; \mathbf{L}(V,V^2) \;\longrightarrow\; \mathbf{L}(W,W^2)$$

by

$$(\varphi_\sharp\psi)(\varphi(v)+w) \;=\; \varphi^2(\psi(v)+(w,0))\ ;$$

here $v\in V$ and $w\in W-\varphi(V)$ is orthogonal to $\varphi(V)$. The map $\varphi_\sharp$ is compatible with the actions of the orthogonal groups, i.e., the following square commutes:

$$\begin{tikzcd}
\mathbf{L}(V,V^2)\times\mathbf{O}(V) \arrow[r, "\varphi_\sharp\times\mathbf{O}(\varphi)"] \arrow[d, "\circ"'] & \mathbf{L}(W,W^2)\times\mathbf{O}(W) \arrow[d, "\circ"] \\
\mathbf{L}(V,V^2) \arrow[r, "\varphi_\sharp"'] & \mathbf{L}(W,W^2)
\end{tikzcd}$$

This equivariance property of $\varphi_\sharp$ ensures that it passes to the two-sided bar construction, so that we can define the structure map as

$$(\mathbf{B'O}^\circ)(\varphi) \;=\; B(\varphi_\sharp,\mathbf{O}(\varphi),*) \;:\; B(\mathbf{L}(V,V^2),\mathbf{O}(V),*) \;\longrightarrow\; B(\mathbf{L}(W,W^2),\mathbf{O}(W),*)\ .$$

A commutative multiplication

$$\mu_{V,W} \;:\; (\mathbf{B'O}^\circ)(V)\times(\mathbf{B'O}^\circ)(W) \;\longrightarrow\; (\mathbf{B'O}^\circ)(V\oplus W)$$

is obtained by combining the multiplications of $\mathbf{BO}^\circ$ and $\mathbf{BO}$. The construction comes with two collections of continuous maps:

$$(\mathbf{BO}^\circ)(V) \xleftarrow{\alpha(V)} B(\mathbf{L}(V,V^2),\mathbf{O}(V),*) = (\mathbf{B'O}^\circ)(V) \xrightarrow{\beta(V)} \mathbf{L}(V,V^2)/\mathbf{O}(V) \;=\; \mathbf{BO}(V)$$

The left map $\alpha(V)$ is defined by applying the bar construction to the unique map from $\mathbf{L}(V,V^2)$ to the one-point space. The right map $\beta(V)$ is the canonical map from homotopy orbits to strict orbits. As V varies, the α and β maps form morphisms of ultra-commutative monoids

$$\mathbf{BO}^\circ \;\underset{\sim}{\xleftarrow{\alpha}}\; \mathbf{B'O}^\circ \;\xrightarrow{\beta}\; \mathbf{BO}\ , \tag{4.16}$$

essentially by construction. As we shall now see, the morphism α is a global equivalence; so we can view the chain as a weak morphism of ultra-commutative monoids from $\mathbf{BO}^\circ$ to $\mathbf{BO}$. The ultra-commutative monoid $\mathbf{BO}^\circ$ is globally connected, whereas $\mathbf{BO}$ is not, so the morphism β cannot be a global equivalence. However, the second part of the next proposition shows that it is as close to a global equivalence as it can be.

PROPOSITION 4.17. (i) *The morphism $\alpha : \mathbf{B'O}^\circ \longrightarrow \mathbf{BO}^\circ$ is a global equivalence of ultra-commutative monoids.*

(ii) *For every compact Lie group G, the morphism $\beta : \mathbf{B'O}^\circ \longrightarrow \mathbf{BO}$ of ultra-commutative monoids induces a weak equivalence from the G-fixed point space $(\mathbf{B'O}^\circ(\mathcal{U}_G))^G$ to the path component of the unit element in the G-fixed point space $(\mathbf{BO}(\mathcal{U}_G))^G$.*

PROOF. (i) We let V be a representation of a compact Lie group G and compare the two-sided bar constructions for the $O(V)$-equivariant map from $\mathbf{L}(V, V^2)$ to the one-point space

$$\tilde{\alpha}(V) \ : \ B(\mathbf{L}(V, V^2), O(V), O(V)) \ \longrightarrow \ B(*, O(V), O(V)) = EO(V) \ .$$

The group G acts by conjugation on $\mathbf{L}(V, V^2)$ and on $O(V)$. The group $O(V)$ acts freely from the right on the last factor in the bar construction, and this right $O(V)$-action then commutes with the G-action. The map $\tilde{\alpha}(V)$ is $(G \times O(V))$-equivariant. The map $\alpha(V) : \mathbf{B'O}^\circ(V) \longrightarrow \mathbf{BO}(V)$ is obtained from $\tilde{\alpha}(V)$ by passage to $O(V)$-orbits. So Proposition A.2.27 allows us to analyze and compare the G-fixed points of $\alpha(V)$. Indeed, the proposition shows that the G-fixed points of $\mathbf{B'O}^\circ(V) = B(\mathbf{L}(V, V^2), O(V), O(V))/O(V)$ are a disjoint union, indexed by conjugacy classes of continuous homomorphisms $\gamma : G \longrightarrow O(V)$ of the spaces

$$\big(B(\mathbf{L}(V, V^2), O(V), O(V))\big)^{\Gamma(\gamma)} / C(\gamma) \ ,$$

where $\Gamma(\gamma)$ is the graph of γ. Since fixed points commute with geometric realization, we have

$$\big(B(\mathbf{L}(V, V^2), O(V), O(V))\big)^{\Gamma(\gamma)} \ = \ B(\mathbf{L}^G(V, V^2), O^G(V), \mathbf{L}^G(\gamma^*(V), V)) \ .$$

If $\gamma^*(V)$ and V are not isomorphic as G-representations, then the space $\mathbf{L}^G(\gamma^*(V), V)$ and hence the bar construction is empty. So there is in fact only one summand in the disjoint union decomposition of $(\mathbf{B'O}^\circ(V))^G$, namely the one indexed by the representation homomorphism $G \longrightarrow O(V)$ that specifies the given G-action on V. We conclude that the inclusions of G-fixed points

$$\mathbf{L}^G(V, V^2) \ \longrightarrow \ \mathbf{L}(V, V^2) \qquad \text{and} \qquad O^G(V) \ \longrightarrow \ O(V)$$

induce a homeomorphism

$$B(\mathbf{L}^G(V, V^2), O^G(V), *) \ \xrightarrow{\cong} \ B(\mathbf{L}(V, V^2), O(V), *)^G \ = \ (\mathbf{B'O}^\circ(V))^G \ .$$

The same argument identifies the G-fixed points of the bar construction $B(O(V)) = \mathbf{BO}^\circ(V)$, and we arrive at a commutative square

$$\begin{array}{ccc} B(\mathbf{L}^G(V, V^2), O^G(V), *) & \longrightarrow & B(O^G(V)) \\ \cong \big\downarrow & & \big\downarrow \cong \\ (\mathbf{B'O}^\circ(V))^G & \xrightarrow[\alpha(V)^G]{} & (\mathbf{BO}^\circ(V))^G \end{array}$$

in which both vertical maps are homeomorphisms. The space $\mathbf{L}^G(V, V^2)$ becomes arbitrarily highly connected as V exhausts the complete G-universe $\mathcal{U}_G$. So the upper horizontal quotient map also becomes arbitrarily highly connected as V grows. Hence the map $\alpha(V)^G$ becomes an equivalence

$$\mathrm{tel}_i \ \alpha(V_i)^G \ : \ \mathrm{tel}_i \, (\mathbf{B'O}^\circ(V_i))^G \ \longrightarrow \ \mathrm{tel}_i \, (\mathbf{BO}^\circ(V_i))^G$$

on the mapping telescope over an exhaustive sequence $\{V_i\}_{i\geq 1}$ of G-representations. Fixed points commute with mapping telescopes, so we can conclude that the map

$$\mathrm{tel}_i\ \alpha(V_i)\ :\ \mathrm{tel}_i\ \mathbf{B'O}^\circ(V_i)\ \longrightarrow\ \mathrm{tel}_i\ \mathbf{BO}^\circ(V_i)$$

is a G-weak equivalence. The mapping telescope criterion of Proposition I.1.7 thus shows that the morphism $\alpha : \mathbf{B'O}^\circ \longrightarrow \mathbf{BO}^\circ$ is a global equivalence.

(ii) We let V be a G-representation, specified by a continuous homomorphism $\rho : G \longrightarrow O(V)$. We show that the map

$$\beta(V)^G\ :\ (\mathbf{B'O}^\circ(V))^G\ =\ \big(B(\mathbf{L}(V,V^2),O(V),*)\big)^G\ \longrightarrow\ \mathbf{BO}(V)^G$$

is a weak equivalence of the source onto the path component of $\mathbf{BO}(V)^G$ that contains the neutral element of the addition. The claim then follows by passing to colimit over V in $s(\mathcal{U}_G)$.

We showed in part (i) that the inclusions of G-fixed points $\mathbf{L}^G(V,V^2) \longrightarrow \mathbf{L}(V,V^2)$ and $O^G(V) \longrightarrow O(V)$ induce a homeomorphism

$$B(\mathbf{L}^G(V,V^2),O^G(V),*)\ \xrightarrow{\cong}\ B(\mathbf{L}(V,V^2),O(V),*)^G\ =\ (\mathbf{B'O}^\circ(V))^G\ .$$

Since the precomposition action of $O^G(V)$ on $\mathbf{L}^G(V,V^2)$ is free, the homotopy orbits map by a weak equivalence to the strict orbits,

$$(\mathbf{B'O}^\circ(V))^G\ =\ \big(B(\mathbf{L}(V,V^2),O(V),*)\big)^G\ \xrightarrow{\simeq}\ \mathbf{L}^G(V,V^2)/O^G(V)\ .$$

The map $\beta(V)^G$ factors as the composite

$$(\mathbf{B'O}^\circ(V))^G\ \xrightarrow{\simeq}\ \mathbf{L}^G(V,V^2)/O^G(V)\ \longrightarrow\ \big(\mathbf{L}(V,V^2)/O(V)\big)^G\ =\ \mathbf{BO}(V)^G\ ,$$

where the second map is induced by the inclusion $\mathbf{L}^G(V,V^2) \longrightarrow \mathbf{L}(V,V^2)$. The space $\mathbf{BO}(V)^G$ is the Grassmannian of G-invariant subspaces of V^2 of the same dimension as V, and the space $\mathbf{L}^G(V,V^2)/O^G(V)$ consists of those subspaces that are G-isomorphic to V. This is precisely the path component of $\mathbf{BO}(V)^G$ containing the neutral element. □

We have now identified the G-equivariant path components of the three ultra-commutative monoids $\mathbf{BO}^\circ$, $\mathbf{BO}$ and $\mathbf{BOP}$, and they are isomorphic to the trivial group, the augmentation ideal $\mathbf{IO}(G)$ respectively the real representation ring $\mathbf{RO}(G)$. Now we can identify the entire homotopy types of the G-fixed point spaces of the three ultra-commutative monoids $\mathbf{BO}^\circ$, $\mathbf{BO}$ and $\mathbf{BOP}$.

Corollary 4.18. *Let G be a compact Lie group.*

(i) *The G-fixed point space of $\mathbf{BO}^\circ$ is a classifying spaces of the group $O^G(\mathcal{U}_G)$ of G-equivariant orthogonal isometries of the complete G-universe:*

$$(\mathbf{BO}^\circ(\mathcal{U}_G))^G\ \simeq\ B(O^G(\mathcal{U}_G))\ .$$

(ii) *The G-fixed point space of $\mathbf{BO}$ is a disjoint union, indexed by the augmentation ideal $\mathbf{IO}(G)$, of classifying spaces of the group $O^G(\mathcal{U}_G)$:*

$$(\mathbf{BO}(\mathcal{U}_G))^G\ \simeq\ \mathbf{IO}(G)\times B(O^G(\mathcal{U}_G))\ .$$

(iii) *The G-fixed point space of $\mathbf{BOP}$ is a disjoint union, indexed by $\mathbf{RO}(G)$, of classifying spaces of the group $O^G(\mathcal{U}_G)$:*

$$(\mathbf{BOP}(\mathcal{U}_G))^G\ \simeq\ \mathbf{RO}(G)\times B(O^G(\mathcal{U}_G))\ .$$

Proof. (i) This is almost a tautology. Since G-fixed points commute with the bar construction, the space $(\mathbf{BO}^\circ(V))^G$ is homeomorphic to $B(O^G(V))$ for every finite dimensional G-representation V. Since G-fixed points commute with the filtered colimit at hand, we have

$$\begin{aligned}(\mathbf{BO}^\circ(\mathcal{U}_G))^G &= \left(\operatorname{colim}_{V\in s(\mathcal{U}_G)} \mathbf{BO}^\circ(V)\right)^G \cong \operatorname{colim}_{V\in s(\mathcal{U}_G)} (\mathbf{BO}^\circ(V))^G \\ &\cong \operatorname{colim}_{V\in s(\mathcal{U}_G)} B\left(O^G(V)\right) \cong B\left(\operatorname{colim}_{V\in s(\mathcal{U}_G)} O^G(V)\right) = B\left(O^G(\mathcal{U}_G)\right) .\end{aligned}$$

(ii) As we explained in Remark 1.2, the commutative multiplication of $\mathbf{BO}$ makes the G-space $\mathbf{BO}(\mathcal{U}_G)$ into an E_∞-G-space; so the fixed points $\mathbf{BO}(\mathcal{U}_G)^G$ come with the structure of a non-equivariant E_∞-space. The abelian monoid of path components $\pi_0(\mathbf{BO}(\mathcal{U}_G)^G)$ is isomorphic to $\pi_0^G(\mathbf{BO}) \cong \mathbf{IO}(G)$, hence an abelian group. So all path components of the space $\mathbf{BO}(\mathcal{U}_G)^G$ are homotopy equivalent. Proposition 4.17 identifies the zero path component of $\mathbf{BO}(\mathcal{U}_G)^G$ with $\mathbf{BO}^\circ(\mathcal{U}_G)^G$, so part (i) finishes the proof.

The proof of part (iii) is the same as for part (ii), the only difference being that $\underline{\pi}_0(\mathbf{BOP}(\mathcal{U}_G)^G)$ is isomorphic to the underlying abelian group of $\mathbf{RO}(G)$. □

The fixed point spaces described in Corollary 4.18 can be decomposed even further. As we explained in Example 3.6, the group $O^G(\mathcal{U}_G)$ is a weak product of infinite orthogonal, unitary and symplectic groups, indexed by the isomorphism classes of irreducible G-representations. The classifying space construction commutes with weak products, which gives a weak equivalence

$$B(O^G(\mathcal{U}_G)) \simeq \prod{}' B(O^G(\mathcal{U}_\lambda)) ,$$

where the weak product is indexed by the isomorphism classes of irreducible orthogonal G-representations λ. Moreover, the group $O^G(\mathcal{U}_\lambda)$ is isomorphic to an infinite orthogonal, unitary or symplectic group, depending on the type of the irreducible representation λ.

Example 4.19 (More bar construction models). Since the bar construction is functorial and continuous for continuous homomorphisms between topological monoids, we can apply it objectwise to every monoid valued orthogonal space M in the sense of Definition 3.2; the result is an orthogonal space $\mathbf{B}^\circ M$. The bar construction is symmetric monoidal, so if M is symmetric (and hence comes with the structure of ultra-commutative monoid), then $\mathbf{B}^\circ M$ inherits an ultra-commutative multiplication. By the same argument as for $\mathbf{BO}^\circ$, the orthogonal space $\mathbf{B}^\circ M$ is globally connected.

So the ultra-commutative monoid $\mathbf{BO}^\circ = \mathbf{B}^\circ\mathbf{O}$ has variations with $\mathbf{O}$ replaced by $\mathbf{SO}$, $\mathbf{U}$, $\mathbf{SU}$, $\mathbf{Sp}$, $\mathbf{Pin}$, $\mathbf{Spin}$, $\mathbf{Pin}^c$ and $\mathbf{Spin}^c$. We can also apply the bar construction to morphisms of monoid valued orthogonal spaces, i.e., morphism of orthogonal spaces that are objectwise monoid homomorphisms. So hitting all the previous examples with the bar construction yields a commutative diagram of globally connected orthogonal spaces:

$$\begin{tikzcd}
\mathbf{B}^\circ\mathbf{SU} \arrow[r, "\mathbf{B}^\circ\mathrm{incl}"] \arrow[d, dashed, "\mathbf{B}^\circ l"'] & \mathbf{B}^\circ\mathbf{U} \arrow[d, dashed, "\mathbf{B}^\circ l"'] \arrow[dr, dashed, "\mathbf{B}^\circ r"] & \\
\mathbf{B}^\circ\mathbf{Spin} \arrow[r, "\mathbf{B}^\circ\mathrm{incl}"'] \arrow[d, "\mathbf{B}^\circ\mathrm{incl}"'] & \mathbf{B}^\circ\mathbf{Spin}^c \arrow[r, "\mathbf{B}^\circ\,\mathrm{ad}"'] \arrow[d, "\mathbf{B}^\circ\mathrm{incl}"] & \mathbf{B}^\circ\mathbf{SO} \arrow[d, "\mathbf{B}^\circ\mathrm{incl}"] \\
\mathbf{B}^\circ\mathbf{Pin} \arrow[r, "\mathbf{B}^\circ\mathrm{incl}"'] & \mathbf{B}^\circ\mathbf{Pin}^c \arrow[r, "\mathbf{B}^\circ\,\mathrm{ad}"'] & \mathbf{B}^\circ\mathbf{O}
\end{tikzcd}$$

As before, the two dotted arrows mean that the actual morphism goes to the multiplicative shift $\mathrm{sh}_{\mathbb{C}}^{\otimes}$ of the target. With the exception of $\mathbf{B}^\circ\mathbf{Pin}$ and $\mathbf{B}^\circ\mathbf{Pin}^c$, all these orthogonal spaces inherit ultra-commutative multiplications.

Example 4.20. We define an orthogonal space $\mathbf{bO}$; it values come with tautological vector bundles whose Thom spaces form the global Thom spectrum $\mathbf{mO}$, compare Example VI.2.26 below. For finite and

abelian compact Lie groups G, the equivariant homotopy groups of $\mathbf{mO}$ are isomorphic to the bordism groups of smooth closed G-manifold, compare Theorem VI.3.40, so $\mathbf{bO}$ is also geometrically relevant. In Remark 4.27 we define an E_∞-multiplication on $\mathbf{bO}$ and show, using power operations, that this E_∞-multiplication cannot be refined to an ultra-commutative multiplication.

For an inner product space V of dimension n we set

$$\mathbf{bO}(V) \;=\; Gr_n(V \oplus \mathbb{R}^\infty)\ ,$$

the Grassmannian of n-dimensional subspaces in $V \oplus \mathbb{R}^\infty$. The structure map $\mathbf{bO}(\varphi) : \mathbf{bO}(V) \longrightarrow \mathbf{bO}(W)$ is given by

$$\mathbf{bO}(\varphi)(L) \;=\; (\varphi \oplus \mathbb{R}^\infty)(L) + ((W - \varphi(V)) \oplus 0)\ ,$$

the internal orthogonal sum of the image of L under $\varphi \oplus \mathbb{R}^\infty : V \oplus \mathbb{R}^\infty \longrightarrow W \oplus \mathbb{R}^\infty$ and the orthogonal complement of the image of $\varphi : V \longrightarrow W$, viewed as sitting in the first summand of $W \oplus \mathbb{R}^\infty$.

Our next task is to describe the equivariant homotopy sets $\pi_0^G(\mathbf{bO})$ and the homotopy types of the fixed point spaces $\mathbf{bO}(\mathcal{U}_G)^G$, for every compact Lie group G. We denote by $\mathbf{RO}^\sharp(G)$ the abelian submonoid of $\mathbf{RO}^+(G)$ consisting of the isomorphism classes of G-representations with trivial G-fixed points. We let V be a G-representation. The G-fixed points of $\mathbf{bO}(V)$ are the G-subrepresentations L of $V \oplus \mathbb{R}^\infty$ of the same dimension as V. Since G acts trivially on $\mathbb{R}^\infty$, the 'non-trivial summand' $L^\perp = L - L^G$ is contained in $V^\perp = V - V^G$. So $V^\perp - L^\perp$ is a G-representation with trivial fixed points. We can thus define a map

$$(\mathbf{bO}(V))^G \;=\; \big(Gr_{|V|}(V \oplus \mathbb{R}^\infty)\big)^G \;\longrightarrow\; \mathbf{RO}^\sharp(G)$$

from this fixed point space by sending $L \in \mathbf{bO}(V)^G$ to $[V^\perp - L^\perp]$. As before, the isomorphism type of L only depends on the path component of L in $\mathbf{bO}(V)^G$. Moreover, for every linear isometric embedding $\varphi : V \longrightarrow W$ the relation

$$\begin{aligned}(\mathbf{bO}(\varphi)(L))^\perp \;&=\; [(\varphi \oplus \mathbb{R}^\infty)(L) + (W - \varphi(V)) \oplus 0]^\perp \\ &=\; \big[\varphi(L^\perp) + (W^\perp - \varphi(V^\perp))\big] \oplus 0 \;=\; \big[W^\perp - \varphi(V^\perp - L^\perp)\big] \oplus 0\end{aligned}$$

shows that

$$[W^\perp - (\mathbf{bO}(\varphi)(L))^\perp] \;=\; [\varphi(V^\perp - L^\perp)] \;=\; [V^\perp - L^\perp]\ .$$

So the class in $\mathbf{RO}^\sharp(G)$ depends only on the class of L in $\pi_0^G(\mathbf{bO})$, and the assignments assemble into a well-defined map

$$\pi_0^G(\mathbf{bO}) \;=\; \mathrm{colim}_{V \in s(\mathcal{U})}\ \pi_0\left(\mathbf{bO}(V)^G\right) \;\longrightarrow\; \mathbf{RO}^\sharp(G)\ . \tag{4.21}$$

For the description of the G-fixed points of $\mathbf{bO}$ we introduce the abbreviation

$$Gr_j^{G,\perp} \;=\; \big(Gr_j(\mathcal{U}_G^\perp)\big)^G$$

for the space of j-dimensional G-invariant subspaces of $\mathcal{U}_G^\perp = \mathcal{U}_G - (\mathcal{U}_G)^G$. The space $Gr_j^{G,\perp}$ can be decomposed further. Indeed, $Gr_j^{G,\perp}$ is a disjoint union of path connected spaces indexed by the set $\mathbf{RO}^\sharp(G)$ of isomorphism class of j-dimensional G-representations with trivial fixed points. Indeed, before taking G-fixed points, the space $Gr_j(\mathcal{U}_G^\perp)$ is G-equivariantly homeomorphic to the space $\mathbf{L}(\mathbb{R}^j, \mathcal{U}_G^\perp)/O(j)$. So Proposition A.2.27 provides a decomposition of $Gr_j^{G,\perp}$ as the disjoint union, indexed over conjugacy classes of continuous homomorphisms $\alpha : G \longrightarrow O(j)$, of the spaces

$$\mathbf{L}^G(\alpha^*(\mathbb{R}^j), \mathcal{U}_G^\perp)/C(\alpha)\ ,$$

where $C(\alpha)$ is the centralizer, in $O(j)$, of the image of α. Conjugacy classes of homomorphism from G to $O(j)$ biject – by restriction of the tautological $O(j)$-representation – with isomorphism classes of n-dimensional G-representation. If $V = \alpha^*(\mathbb{R}^j)$ is such an n-dimensional G-representation, then the space $\mathbf{L}^G(V, \mathcal{U}_G^\perp)$ is empty if V has non-trivial G-fixed points, and contractible otherwise. Moreover, the centralizer $C(\alpha)$ is precisely the group of G-equivariant linear self-isometries of V, which acts freely on $\mathbf{L}^G(V, \mathcal{U}_G^\perp)$. So

if $V^G = 0$, then the orbit space $\mathbf{L}^G(V, \mathcal{U}_G^\perp)/C(\alpha)$ is a classifying space for the group $\mathbf{L}^G(V,V) = O^G(V)$. So altogether,

$$Gr_j^{G,\perp} \simeq \coprod_{[V]\in \mathbf{RO}^\sharp(G), |V|=j} B(O^G(V)) \ . \tag{4.22}$$

Every G-representation V is the direct sum of its isotypical components V_λ, indexed by the isomorphism classes of irreducible orthogonal G-representations. If $V^G = 0$, then only the non-trivial irreducibles occur, and the group $O^G(V)$ decomposes accordingly as a product

$$O^G(V) \cong \prod_{[\lambda]} O^G(V_\lambda) \ ,$$

indexed by non-trivial irreducible G-representations. The irreducibles come in three flavors (real, complex or quaternionic), and so the group $O^G(V_\lambda)$ is isomorphic to one of the groups $O(n)$, $U(n)$, and $Sp(n)$, where n is the multiplicity of λ in V. So altogether, $Gr_j^{G,\perp}$ is a disjoint union of products of classifying spaces of orthogonal, unitary and symplectic groups.

PROPOSITION 4.23. *Let G be a compact Lie group.*

(i) *The G-fixed point space $\mathbf{bO}(\mathcal{U}_G)^G$ is weakly equivalent to the space*

$$\coprod_{j\geq 0} Gr_j^{G,\perp} \times BO \ .$$

(ii) *The map* (4.21) *is a bijection from $\pi_0^G(\mathbf{bO})$ to $\mathbf{RO}^\sharp(G)$.*

(iii) *If U is a G-representation with trivial fixed points, then the path component of $\mathbf{bO}(\mathcal{U}_G)^G$ indexed by U is a classifying space for the group $O^G(U) \times O$.*

PROOF. (i) We let V be a G-representation with $V^G = 0$ and W a trivial G-representation. Then every G-invariant subspace L of $V \oplus W \oplus \mathbb{R}^\infty$ is the internal direct sum of the fixed part L^G (which is contained in $W \oplus \mathbb{R}^\infty$) and its orthogonal complement $L^\perp = L - L^G$ (which is contained in the summand V). This canonical decomposition provides a homeomorphism

$$\mathbf{bO}(V\oplus W)^G = (Gr_{|V|+|W|}(V \oplus W \oplus \mathbb{R}^\infty))^G \cong \coprod_{j=0,\dots,|V|} (Gr_j(V))^G \times Gr_{j+|W|}(W \oplus \mathbb{R}^\infty)$$

sending L to the pair $(V - L^\perp,\ L^G)$. Every G-invariant subspace of $\mathcal{U}_G$ is the direct sum of its fixed points and their complement, so the poset $s(\mathcal{U}_G)$ is the product of the two posets $s(\mathcal{U}_G^\perp)$ and $s((\mathcal{U}_G)^G)$. We can thus calculate the colimit over $s(\mathcal{U}_G)$ is two steps. For fixed W, passing to the colimit over $s(\mathcal{U}_G^\perp)$ gives a homeomorphism

$$\operatorname{colim}_{V\in s(\mathcal{U}_G^\perp)} \mathbf{bO}(V\oplus W)^G \cong \coprod_{j\geq 0} Gr_j^{G,\perp} \times Gr_{j+|W|}(W \oplus \mathbb{R}^\infty) \ .$$

The factor $Gr_{j+|W|}(W \oplus \mathbb{R}^\infty)$ is a classifying space for the group $O(j + |W|)$. Passing to the colimit over $s((\mathcal{U}_G)^G)$ then provides a weak equivalence

$$\begin{aligned} \mathbf{bO}(\mathcal{U}_G)^G &= \operatorname{colim}_{W\in s((\mathcal{U}_G)^G)} \operatorname{colim}_{V\in s(\mathcal{U}_G^\perp)} \mathbf{bO}(V\oplus W)^G \\ &\simeq \operatorname{colim}_{W\in s((\mathcal{U}_G)^G)} \left(\coprod_{j\geq 0} Gr_j^{G,\perp} \times BO(j+|W|) \right) \simeq \coprod_{j\geq 0} Gr_j^{G,\perp} \times BO \ . \end{aligned}$$

(ii) Since the orthogonal space $\mathbf{bO}$ is closed, we can calculate $\pi_0^G(\mathbf{bO})$ as the set of path components of the space $\mathbf{bO}(\mathcal{U}_G)^G$, by Corollary I.5.7 (i). Since the space BO is path connected, part (i) allows us to identify $\pi_0(\mathbf{bO}(\mathcal{U}_G)^G)$ with the disjoint union, over $j \geq 0$, of the path components of $Gr_j^{G,\perp}$.

As we explained in the weak equivalence (4.22), the set $\pi_0(Gr_j^{G,\perp})$ bijects with the set of isomorphism classes of j-dimensional G-representations with trivial fixed points. Altogether this identifies $\pi_0(\mathbf{bO}(\mathcal{U}_G)^G)$ with $\mathbf{RO}^\sharp(G)$, and unraveling all definition shows that combined bijection between $\pi_0^G(\mathbf{bO})$ and $\mathbf{RO}^\sharp(G)$, is the map (4.21).

(iii) This is a direct consequence of part (i) and the fact, used already in (ii), that $Gr_j^{G,\perp}$ is a disjoint union, indexed by the set of isomorphism classes of G-representations with trivial fixed points, and that the component corresponding to U is a classifying space for the group $O^G(U)$ of G-equivariant linear self-isometries of U. □

If H is a closed subgroup of a compact Lie group G and V a G-representation with $V^G = 0$, then V may have non-trivial H-fixed points. So the restriction homomorphism $\mathrm{res}_H^G : \mathbf{RO}^+(G) \longrightarrow \mathbf{RO}^+(H)$ does not in general take $\mathbf{RO}^\sharp(G)$ to $\mathbf{RO}^\sharp(H)$. So the monoids $\mathbf{RO}^\sharp(G)$ do *not* form a sub Rep-functor of $\mathbf{RO}$, and Proposition 4.23 does not describe $\underline{\pi}_0(\mathbf{bO})$ as a Rep functor. We will give description of $\underline{\pi}_0(\mathbf{bO})$ as sub-Rep monoid of the augmentation ideal global power monoid $\mathbf{IO}$ in Proposition 4.31 below.

As we shall now explain, the global homotopy type of the orthogonal space $\mathbf{bO}$ is that of a sequential homotopy colimit, in the category of orthogonal spaces, of global classifying spaces of the orthogonal groups $O(m)$:

$$\mathbf{bO} \ \simeq\ \mathrm{hocolim}_{m\geq 1}\ B_{\mathrm{gl}}O(m)$$

The homotopy colimit is taken over morphisms $B_{\mathrm{gl}}O(m) \longrightarrow B_{\mathrm{gl}}O(m+1)$ that classify the homomorphisms $O(m) \longrightarrow O(m+1)$ given by $A \mapsto A \oplus \mathbb{R}$. To make this relation rigorous, we define a filtration

$$* \ \cong\ \mathbf{bO}_{(0)} \ \subset\ \mathbf{bO}_{(1)} \ \subset\ \dots \ \subset\ \mathbf{bO}_{(m)} \ \subset\ \dots \tag{4.24}$$

of $\mathbf{bO}$ by orthogonal subspaces. At an inner product space V we define

$$\mathbf{bO}_{(m)}(V) \ =\ Gr_{|V|}(V \oplus \mathbb{R}^m)\ ; \tag{4.25}$$

here we consider $\mathbb{R}^m$ as the subspace of $\mathbb{R}^\infty$ of all vectors of the form $(x_1, \dots, x_m, 0, 0, \dots)$. The inclusion of $\mathbf{bO}_{(m)}$ into $\mathbf{bO}_{(m+1)}$ is a closed embedding, so the global invariance property of Proposition I.1.9 (vii) entitles us to view the union $\mathbf{bO}$ as a global homotopy colimit of the filtration. The inclusion $\mathbf{bO}_{(m)} \longrightarrow \mathbf{bO}_{(m+1)}$ is even a flat cofibration, but we will not show this.

The tautological action of $O(m)$ on $\mathbb{R}^m$ is faithful, so the free orthogonal space

$$B_{\mathrm{gl}}O(m) \ =\ \mathbf{L}_{O(m),\mathbb{R}^m}$$

is a global classifying space for $O(m)$. We define a morphism $\gamma_m : B_{\mathrm{gl}}O(m) \longrightarrow \mathbf{bO}_{(m)}$ by

$$\begin{aligned} \gamma_m(V)\ :\ \mathbf{L}(\mathbb{R}^m, V)/O(m) \ &\longrightarrow\ \ Gr_{|V|}(V \oplus \mathbb{R}^m)\ =\ \mathbf{bO}_{(m)}(V)\ , \\ \varphi \cdot O(m) \ &\longmapsto\ (V - \varphi(\mathbb{R}^m)) \oplus \mathbb{R}^m\ . \end{aligned}$$

We omit the straightforward verification that these maps indeed form a morphism of orthogonal spaces. The following proposition justifies the claim that $\mathbf{bO}$ is a homotopy colimit of the orthogonal spaces $B_{\mathrm{gl}}O(m)$.

The free orthogonal space $B_{\mathrm{gl}}O(m) = \mathbf{L}_{O(m),\mathbb{R}^m}$ comes with a tautological class $u_{O(m),\mathbb{R}^m}$, defined in (5.13) of Chapter I, which freely generates the Rep functor $\underline{\pi}_0(B_{\mathrm{gl}}O(m))$. We denote by

$$u_m \ =\ (\gamma_m)_*(u_{O(m),\mathbb{R}^m}) \ \in\ \pi_0^{O(m)}\left(\mathbf{bO}_{(m)}\right)$$

the image in $\mathbf{bO}_{(m)}$ of the tautological class.

Proposition 4.26. *For every $m \geq 0$ the morphism*

$$\gamma_m \ :\ B_{\mathrm{gl}}O(m) \ \longrightarrow\ \mathbf{bO}_{(m)}$$

is a global equivalence of orthogonal spaces. The inclusion $\mathbf{bO}_{(m)} \longrightarrow \mathbf{bO}_{(m+1)}$ takes the class u_m to the class

$$\mathrm{res}_{O(m)}^{O(m+1)}(u_{m+1}) \ \in\ \pi_0^{O(m)}\left(\mathbf{bO}_{(m+1)}\right)\ .$$

PROOF. The morphism γ_m factors as the composite of two morphisms of orthogonal spaces

$$\mathbf{L}_{O(m),\mathbb{R}^m} \xrightarrow{\mathbf{L}_{O(m),\mathbb{R}^m}\circ i} \mathrm{sh}_{\oplus}^{\mathbb{R}^m}(\mathbf{L}_{O(m),\mathbb{R}^m}) \xrightarrow[\cong]{(-)^{\perp}} \mathbf{bO}_{(m)} \ .$$

For the first morphism we let $i : V \longrightarrow V \oplus \mathbb{R}^m$ denote the embedding of the first summand, and $\mathrm{sh}_{\oplus}^{\mathbb{R}^m}$ is the additive shift by $\mathbb{R}^m$ as defined in Example I.1.12; the first morphism is a global equivalence by Theorem I.1.11. At an inner product space V, the second morphism is the map

$$\begin{aligned}(\mathrm{sh}_{\oplus}^{\mathbb{R}^m}\mathbf{L}_{O(m),\mathbb{R}^m})(V) \ = \ \mathbf{L}(\mathbb{R}^m, V\oplus\mathbb{R}^m)/O(m) \ &\longrightarrow \ Gr_{|V|}(V\oplus\mathbb{R}^m) \ , \\ \varphi\cdot O(m) \quad &\longmapsto \quad \varphi(\mathbb{R}^m)^{\perp} \ ,\end{aligned}$$

the orthogonal complement of the image. This is a homeomorphism, so the second morphism is an isomorphism. Altogether this shows that γ_m is a global equivalence.

The second claim is also reasonably straightforward from the definitions, but it needs one homotopy. We let $j : \mathbb{R}^m \longrightarrow \mathbb{R}^{m+1}$ denote the linear isometric embedding defined by $j(x_1,\dots,x_m) = (x_1,\dots,x_m,0)$. The tautological class $u_{O(m),\mathbb{R}^m}$ is defined as the path component of the point

$$\mathrm{Id}_{\mathbb{R}^m}\cdot O(m) \ \in \ (\mathbf{L}(\mathbb{R}^m,\mathbb{R}^m)/O(m))^{O(m)} \ ,$$

the $O(m)$-orbit of the identity of $\mathbb{R}^m$. If we unravel all definitions, we find that the classes

$$\mathrm{incl}_*(u_m) \qquad \text{and} \qquad \mathrm{res}^{O(m)}_{O(m+1)}(u_{m+1}) \quad \text{in} \quad \pi_0^{O(m)}(\mathbf{bO}_{(m+1)})$$

are represented by the two subspaces

$$(\mathbb{R}^{m+1} - j(\mathbb{R}^m)) \oplus j(\mathbb{R}^m) \qquad \text{respectively} \qquad 0\oplus\mathbb{R}^{m+1}$$

of $\mathbb{R}^{m+1}\oplus\mathbb{R}^{m+1}$. These two representatives are *not* the same. However, the $O(m+1)$-action on $\mathbf{bO}_{(m+1)}(\mathbb{R}^{m+1})$, and hence also the restricted $O(m)$-action, is through the first copy of $\mathbb{R}^{m+1}$ (and *not* diagonally!). So there is a path of $O(m)$-invariant subspaces of $\mathbb{R}^{m+1}\oplus\mathbb{R}^{m+1}$ connecting the two representatives. The two points thus represent the same class in $\pi_0^{O(m)}(\mathbf{bO}_{(m+1)})$, and this proves the second claim. □

REMARK 4.27 (Commutative versus E_∞-orthogonal monoid spaces). Non-equivariantly, every E_∞-multiplication on an orthogonal monoid space can be rigidified to a strictly commutative multiplication. We will now see that this is not the case globally, at least not if 'E_∞-multiplication' is interpreted as an action of an E_∞-operad of spaces. The example of $\mathbf{bO}$ is not only interesting because it is a global refinement of the non-equivariant homotopy type of BO, but also because it illustrates the difference between a strictly commutative multiplication and an E_∞-multiplication on an orthogonal space. The difference is detected by power operations, so if we care about these, then we cannot relax 'commutative' to 'E_∞'.

If we try to define a multiplication on $\mathbf{bO}$ in a similar way as for $\mathbf{BO}$, we run into the problem that $\mathbb{R}^\infty\oplus\mathbb{R}^\infty$ is different from $\mathbb{R}^\infty$; even worse, although $\mathbb{R}^\infty\oplus\mathbb{R}^\infty$ and $\mathbb{R}^\infty$ are isometrically isomorphic, there is no preferred isomorphism. The standard way out is to use *all* isomorphism at once, i.e., to parametrize the multiplications by the E_∞-operad of linear isometric embeddings of $\mathbb{R}^\infty$. We recall that the n-th space of the linear isometries operad is

$$\mathcal{L}(n) \ = \ \mathbf{L}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty) \ ,$$

with operad structure by direct sum and composition of linear isometric embeddings (see for example [**107**, Def. 1.2] for details). For all $n \geq 0$ and all inner product spaces $V_1,\dots,V_n$ we define a linear isometry

$$\kappa \ : \ (V_1\oplus\mathbb{R}^\infty)\oplus\cdots\oplus(V_n\oplus\mathbb{R}^\infty) \ \cong \ V_1\oplus\cdots\oplus V_n\oplus(\mathbb{R}^\infty)^n$$

by shuffling the summands, i.e.,

$$\kappa(v_1,x_1,\dots,v_n,x_n) \ = \ (v_1,\dots,v_n,x_1,\dots,x_n) \ .$$

We can then define a continuous map

$$\mu_n \ : \ \mathcal{L}(n) \times \mathbf{bO}(V_1) \times \cdots \times \mathbf{bO}(V_n) \ \longrightarrow \ \mathbf{bO}(V_1 \oplus \cdots \oplus V_n)$$

by

$$\mu_n(\varphi, L_1, \ldots, L_n) \ = \ ((V_1 \oplus \cdots \oplus V_n) \oplus \varphi) \circ \kappa)(L_1 \oplus \cdots \oplus L_n) \ .$$

For fixed φ these maps form a multimorphism, so the universal property of the box product produces a morphism of orthogonal spaces

$$\mu_n(\varphi, -) \ : \ \mathbf{bO} \boxtimes \cdots \boxtimes \mathbf{bO} \ \longrightarrow \ \mathbf{bO} \ .$$

For varying φ, these maps define a morphism of orthogonal spaces

$$\mu_n \ : \ \mathcal{L}(n) \times (\mathbf{bO} \boxtimes \cdots \boxtimes \mathbf{bO}) \ \longrightarrow \ \mathbf{bO} \ .$$

As n varies, all these morphism together make the orthogonal space $\mathbf{bO}$ into an algebra (with respect to the box product) over the linear isometries operad $\mathcal{L}$. Since the linear isometries operad is an E_∞-operad, we call $\mathbf{bO}$, endowed with this $\mathcal{L}$-action, an E_∞-orthogonal monoid space.

Now we explain why the E_∞-structure on $\mathbf{bO}$ *cannot* be refined to an ultra-commutative multiplication. The obstruction is that the abelian Rep-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be endowed with compatible transfer maps, see Proposition 4.31 below. We start by observing that an E_∞-structure gives rise to abelian monoid structures on the set equivariant homotopy sets. In more detail, we let R be any E_∞-orthogonal monoid space, such as for example $\mathbf{bO}$. We obtain binary pairings

$$\pi_0^G(R) \times \pi_0^G(R) \ \xrightarrow{\times} \ \pi_0^G(R \boxtimes R) \ \xrightarrow{\mu_2(\varphi,-)_*} \ \pi_0^G(R) \ ,$$

where $\varphi \in \mathcal{L}(2)$ is any linear isometric embedding of $(\mathbb{R}^\infty)^2$ into $\mathbb{R}^\infty$. The second map (and hence the composite) is independent of φ because the space $\mathcal{L}(2)$ is contractible. In the same way as for strict multiplications in (2.1), this binary operation makes $\pi_0^G(R)$ into an abelian monoid for every compact Lie group G, such that all restriction maps are homomorphisms. In other words, the E_∞-structure provides a lift of the Rep-functor $\underline{\pi}_0(R)$ to an abelian Rep-monoid, i.e. a functor

$$\underline{\pi}_0(R) \ : \ \mathrm{Rep}^{\mathrm{op}} \ \longrightarrow \ (\text{abelian monoids}) \ .$$

This entire structure is clearly natural for homomorphisms of E_∞-orthogonal monoid spaces.

An ultra-commutative monoid can be viewed as an E_∞-orthogonal monoid space by letting every element of $\mathcal{L}(n)$ act as the iterated multiplication. Equivalently: we let the linear isometries operad act along the unique homomorphism to the terminal operad (whose algebras, with respect to the box product, are the ultra-commutative monoids). For E_∞-orthogonal monoid spaces arising in this way from ultra-commutative monoids, the products on $\underline{\pi}_0$ defined here coincide with those originally defined in (2.1). For ultra-commutative monoids R, the abelian Rep-monoid $\underline{\pi}_0(R)$ is underlying a global power monoid, i.e., it comes with power operations and transfer maps that satisfy various relations. We show below that the abelian Rep-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be extended to a global power monoid whatsoever; hence $\mathbf{bO}$ is not globally equivalent, as an E_∞-orthogonal monoid spaces, to any ultra-commutative monoid. A curious fact, however, is that *after global completion* the E_∞-multiplication of $\mathbf{bO}$ can be refined to an ultra-commutative multiplication, compare Remark 5.38.

Now we compare the E_∞-orthogonal monoid space $\mathbf{bO}$ to the ultra-commutative monoid $\mathbf{BO}$ in the most highly structured way possible. Every ultra-commutative monoid can be viewed as an E_∞-orthogonal monoid space, and we now define a 'weak E_∞-morphism' from $\mathbf{bO}$ to $\mathbf{BO}$. The zigzag of morphisms passes through the orthogonal space $\mathbf{BO}'$ with values

$$\mathbf{BO}'(V) \ = \ Gr_{|V|}(V^2 \oplus \mathbb{R}^\infty) \ . \tag{4.28}$$

The structure maps of $\mathbf{BO}'$ are a mixture of the structure maps for $\mathbf{bO}$ and $\mathbf{BO}$, i.e.,

$$\mathbf{BO}'(\varphi)(L) \ = \ (\varphi^2 \oplus \mathbb{R}^\infty)(L) \oplus ((W - \varphi(V)) \oplus 0 \oplus 0) \ ,$$

where now the orthogonal complement of the image of φ is viewed as sitting in the first summand of $W \oplus W \oplus \mathbb{R}^\infty$. The linear isometries operad acts on $\mathbf{BO}'$ in much the same way as for $\mathbf{bO}$, making it an E_∞-orthogonal monoid space. Postcomposition with the direct summand embeddings

$$V \oplus \mathbb{R}^\infty \xrightarrow{(v,x)\mapsto(v,0,x)} V^2 \oplus \mathbb{R}^\infty \xleftarrow{(v,v',0)\leftarrow\!\shortmid(v,v')} V^2$$

induces maps of Grassmannians

$$\mathbf{bO}(V) \xrightarrow{a(V)} \mathbf{BO}'(V) \xleftarrow{b(V)} \mathbf{BO}(V)$$

that are morphisms of E_∞-orthogonal monoid spaces

$$\mathbf{bO} \xrightarrow{a} \mathbf{BO}' \xleftarrow{b} \mathbf{BO} \tag{4.29}$$

as the inner product space V varies.

PROPOSITION 4.30. *The morphism $b : \mathbf{BO} \longrightarrow \mathbf{BO}'$ is a global equivalence of orthogonal spaces.*

PROOF. We define a exhaustive filtration

$$\mathbf{BO} \;=\; \mathbf{BO}'_{(0)} \;\subset\; \mathbf{BO}'_{(1)} \;\subset\; \dots \;\subset\; \mathbf{BO}'_{(m)} \;\subset\; \dots$$

of $\mathbf{BO}'$ by orthogonal subspaces by setting

$$\mathbf{BO}'_{(m)}(V) \;=\; Gr_{|V|}(V^2 \oplus \mathbb{R}^m)\ .$$

We denote by $\mathrm{sh} = \mathrm{sh}^{\oplus}_{\mathbb{R}}$ the additive shift functor defined in Example I.1.12, and by $i_X : X \longrightarrow \mathrm{sh}\, X$ the morphism of orthogonal spaces given by applying X to the direct summand embeddings $V \longrightarrow V \oplus \mathbb{R}$. The morphism i_X is a global equivalence for every orthogonal space X, by Theorem I.1.11. We define a morphism

$$j \;:\; \mathbf{BO}'_{(m+1)} \;\longrightarrow\; \mathrm{sh}(\mathbf{BO}'_{(m)})$$

at an inner product space V by

$$j(V) \;:\; \mathbf{BO}'_{(m+1)}(V) \;=\; Gr_{|V|}(V \oplus V \oplus \mathbb{R}^{m+1}) \;\longrightarrow\; Gr_{|V|+1}(V \oplus \mathbb{R} \oplus V \oplus \mathbb{R} \oplus \mathbb{R}^m) = \mathrm{sh}(\mathbf{BO}'_{(m)})(V)$$

by applying the linear isometric embedding

$$V \oplus V \oplus \mathbb{R}^{m+1} \;\longrightarrow\; V \oplus \mathbb{R} \oplus V \oplus \mathbb{R} \oplus \mathbb{R}^m\ , \quad (v, v', (x_1, \dots, x_{m+1})) \;\longmapsto\; (v, 0, v', x_1, (x_2, \dots, x_{m+1}))$$

and adding the first copy of $\mathbb{R}$ (the orthogonal complement of this last embedding). Then the following diagram of morphisms of orthogonal spaces commutes:

$$\begin{tikzcd}
\mathbf{BO}'_{(m)} \arrow[r, "\mathrm{incl}"] \arrow[dr, "i_{\mathbf{BO}'_{(m)}}"'] & \mathbf{BO}'_{(m+1)} \arrow[d, "j"'] \arrow[dr, "i_{\mathbf{BO}'_{(m+1)}}"] & \\
& \mathrm{sh}(\mathbf{BO}'_{(m)}) \arrow[r, "\mathrm{sh}(\mathrm{incl})"'] & \mathrm{sh}(\mathbf{BO}'_{(m+1)})
\end{tikzcd}$$

The two diagonal morphisms are global equivalences, hence so is the inclusion $\mathbf{BO}'_{(m)} \longrightarrow \mathbf{BO}'_{(m+1)}$, by the 2-out-of-6 property for global equivalences (Proposition I.1.9 (iii)). The inclusion of $\mathbf{BO}'_{(m)}$ into $\mathbf{BO}'_{(m+1)}$ is also objectwise a closed embedding, so the inclusion

$$\mathbf{BO} \;=\; \mathbf{BO}'_{(0)} \;\longrightarrow\; \bigcup_{m\geq 0} \mathbf{BO}'_{(m)} \;=\; \mathbf{BO}'$$

is a global equivalence, by Proposition I.1.9 (viii). □

We recall that $\mathbf{IO}(G)$ denotes the augmentation ideal in the real representation ring $\mathbf{RO}(G)$ of a compact Lie group G. In the same way as for $\mathbf{BO}$ we define a monoid homomorphism

$$\gamma \;:\; \pi_0^G(\mathbf{BO}') \;\longrightarrow\; \mathbf{IO}(G)$$

by sending the path component of $W \in \mathbf{BO}'(V)^G$ to the class $[W]-[V]$. Then the triangle of monoid homomorphisms on the right of following diagram commutes:

$$\begin{tikzcd}
\pi_0^G(\mathbf{bO}) \arrow[r, "a_*"] \arrow[dr] & \pi_0^G(\mathbf{BO}') \arrow[d, "\gamma", "\cong"'] & \pi_0^G(\mathbf{BO}) \arrow[l, "b_*"', "\cong"] \arrow[dl, "\cong", "(4.13)"'] \\
& \mathbf{IO}(G) &
\end{tikzcd}$$

The two maps on the right are isomorphisms, hence so is the map γ.

The following says that $\pi_0^G(\mathbf{bO})$ is isomorphic to the free abelian submonoid of $\mathbf{IO}(G)$ generated by $\dim(\lambda)\cdot 1-[\lambda]$ as λ runs over all isomorphism classes of non-trivial irreducible G-representations. We emphasize that for non-trivial groups G, the monoid $\pi_0^G(\mathbf{bO})$ does not have inverses, so $\mathbf{bO}$ is *not* group-like.

Proposition 4.31. *The composite morphism of abelian* Rep*-monoids*

$$\underline{\pi}_0(\mathbf{bO}) \;\xrightarrow{a_*}\; \underline{\pi}_0(\mathbf{BO}') \;\xrightarrow{\gamma}\; \mathbf{IO}$$

is a monomorphism. For every compact Lie group G the image of $\pi_0^G(\mathbf{bO})$ in the augmentation ideal $\mathbf{IO}(G)$ consists of the submonoid of elements of the form $\dim(U)\cdot 1-[U]$, for G-representations U. The abelian Rep*-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be extended to a global power monoid.*

Proof. If $L\subset V\oplus\mathbb{R}^\infty$ is a G-invariant subspace of the same dimension as V, then

$$\begin{aligned}
[L]-[V] \;&=\; (\dim(L)-\dim(L^\perp))\cdot 1+[L^\perp]-(\dim(V)-\dim(V^\perp))\cdot 1-[V^\perp] \\
&=\; \dim(V^\perp-L^\perp)\cdot 1-[V^\perp-L^\perp]
\end{aligned}$$

in the group $\mathbf{IO}(G)$. This show that the following square commutes:

$$\begin{tikzcd}
\pi_0^G(\mathbf{bO}) \arrow[r, "a_*"] \arrow[d, "(4.21)"'] & \pi_0^G(\mathbf{BO}') \arrow[d, "\gamma"] \\
\mathbf{RO}^\sharp(G) \arrow[r, "{[U]\,\mapsto\,\dim(U)\cdot 1-[U]}"'] & \mathbf{IO}(G)
\end{tikzcd}$$

The left vertical morphism is an isomorphism by Proposition 4.23 (ii). The right vertical morphism is an isomorphism as explained above. This shows the first two claims because the lower horizontal map is injective and has the correct image.

Now we argue, by contradiction, that the abelian Rep-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be extended to a global power monoid. The additional structure would in particular specify a transfer map

$$\mathrm{tr}_{A_3}^{\Sigma_3} \;:\; \pi_0^{A_3}(\mathbf{bO}) \;\longrightarrow\; \pi_0^{\Sigma_3}(\mathbf{bO}) \tag{4.32}$$

from the alternating group A_3 to the symmetric group Σ_3. Since the monoid $\pi_0^e(\mathbf{bO})$ has only one element, the double coset formula shows that

$$\mathrm{res}_{(12)}^{\Sigma_3}\circ\mathrm{tr}_{A_3}^{\Sigma_3} \;=\; \mathrm{tr}_e^{(12)}\circ\mathrm{res}_e^{A_3} \;=\; 0 \;:\; \pi_0^{A_3}(\mathbf{bO}) \;\longrightarrow\; \pi_0^{(12)}(\mathbf{bO})\;.$$

The group Σ_3 has two non-trivial irreducible orthogonal representations, the 1-dimension sign representation σ and the 2-dimensional reduced natural representation ν. So $\pi_0^{\Sigma_3}(\mathbf{bO})$ 'is' (via $\gamma\circ a_*$) the free abelian submonoid of $\mathbf{IO}(\Sigma_3)$ generated by

$$1-\sigma \qquad\text{and}\qquad 2-\nu\;.$$

We abuse notation and also write σ for the 1-dimensional sign representation of the cyclic subgroup of Σ_3 generated by the transposition (12). Then

$$\mathrm{res}^{\Sigma_3}_{(12)}(1-\sigma) \;=\; 1-\sigma \qquad \text{and} \qquad \mathrm{res}^{\Sigma_3}_{(12)}(2-\nu) \;=\; 2-(1+\sigma) \;=\; 1-\sigma \;,$$

so the only element of $\pi_0^{\Sigma_3}(\mathbf{bO})$ that restricts to 0 in $\pi_0^{(12)}(\mathbf{bO})$ is the zero element. Hence the transfer map (4.32) must be the zero map. However, another instance of the double coset formula is

$$\mathrm{res}^{\Sigma_3}_{A_3} \circ \mathrm{tr}^{\Sigma_3}_{A_3} \;=\; \mathrm{Id} + c^*_{(12)} \;=\; 2\cdot \mathrm{Id} \;:\; \pi_0^{A_3}(\mathbf{bO}) \;\longrightarrow\; \pi_0^{A_3}(\mathbf{bO}) \;.$$

The second equality uses that conjugation by the transposition (12) is the non-trivial automorphism of A_3, which acts trivially on $\mathbf{RO}(A_3)$. Since $\mathbf{bO}(A_3)$ is a non-trivial free abelian monoid, this contradicts the relation $\mathrm{tr}^{\Sigma_3}_{A_3} = 0$. So the abelian Rep-monoid $\underline{\pi}_0(\mathbf{bO})$ cannot be endowed with transfer maps that satisfy the double coset formulas. □

Example 4.33 (Periodic restricted Grassmannian). The E_∞-orthogonal monoid space $\mathbf{bO}$ also has a straightforward 'periodic' variant $\mathbf{bOP}$ that we briefly discuss. For an inner product space V we set

$$\mathbf{bOP}(V) \;=\; \coprod_{n\geq 0} Gr_n(V\oplus\mathbb{R}^\infty) \;,$$

the disjoint unions of all the Grassmannians in $V\oplus\mathbb{R}^\infty$. The structure maps of $\mathbf{bOP}$ are defined in exactly the same way as for $\mathbf{bO}$. The orthogonal space $\mathbf{bOP}$ is naturally $\mathbb{Z}$-graded: for $k\in\mathbb{Z}$ we let

$$\mathbf{bOP}^{[k]}(V) \;\subset\; \mathbf{BOP}(V)$$

be the path component consisting of all subspaces $L\subset V\oplus\mathbb{R}^\infty$ such that $\dim(L)-\dim(V)=k$. For fixed k these spaces form an orthogonal subspace $\mathbf{bOP}^{[k]}$ of $\mathbf{bOP}$. The E_∞-multiplication of $\mathbf{bO}$ extends naturally to a $\mathbb{Z}$-graded E_∞-multiplication on $\mathbf{bOP}$, taking $\mathbf{bOP}^{[k]}\boxtimes\mathbf{bOP}^{[l]}$ to $\mathbf{bOP}^{[k+l]}$. Moreover, $\mathbf{bO}=\mathbf{bOP}^{[0]}$, the homogeneous summand of $\mathbf{BOP}$ of degree 0.

We offer two descriptions of the global homotopy type of the orthogonal space $\mathbf{bOP}$ in terms of other global homotopy types previously discussed. As we show in Proposition 4.34 below, each of the homogeneous summands $\mathbf{bOP}^{[k]}$ is in fact globally equivalent to the degree 0 summand $\mathbf{bO}$, and hence

$$\mathbf{bOP} \;\simeq\; \mathbb{Z}\times\mathbf{bO}$$

globally as orthogonal spaces. When combined with Proposition 4.23, this yields as description of the homotopy types of the fixed point spaces $\mathbf{bOP}(\mathcal{U}_G)^G$.

To identify the different summands of $\mathbf{bOP}$ we choose a linear isometric embedding $\psi:\mathbb{R}^\infty\oplus\mathbb{R}\longrightarrow\mathbb{R}^\infty$ and define an endomorphism $\psi_\sharp:\mathbf{bOP}\longrightarrow\mathbf{bOP}$ of the orthogonal space $\mathbf{bOP}$ at an inner product space V as the map

$$\psi_\sharp(V) \;:\; \mathbf{bOP}(V) \;\longrightarrow \mathbf{bOP}(V) \;,\quad L \;\longmapsto\; (V\oplus\psi)(L\oplus\mathbb{R}) \;.$$

The morphism $\psi_\sharp$ increases the dimension of subspaces by 1, so it takes the summand $\mathbf{bOP}^{[k]}$ to the summand $\mathbf{bOP}^{[k+1]}$. Any two linear isometric embeddings from $\mathbb{R}^\infty\oplus\mathbb{R}$ to $\mathbb{R}^\infty$ are homotopic through linear isometric embeddings, so the homotopy class of $\psi_\sharp$ is independent of the choice of ψ.

Proposition 4.34. *For every linear isometric embedding $\psi:\mathbb{R}^\infty\oplus\mathbb{R}\longrightarrow\mathbb{R}^\infty$ the morphism of orthogonal spaces $\psi_\sharp:\mathbf{bOP}\longrightarrow\mathbf{bOP}$ is a global equivalence. Hence for every $k\in\mathbb{Z}$ the restriction is a global equivalence*

$$\psi_\sharp \;:\; \mathbf{bOP}^{[k]} \;\longrightarrow \mathbf{bOP}^{[k+1]} \;.$$

Proof. We let $\mathrm{sh} = \mathrm{sh}_{\oplus}^{\mathbb{R}}$ denote the additive shift of an orthogonal space as defined in Example I.1.12, and $i : \mathrm{Id} \longrightarrow - \oplus \mathbb{R}$ is the natural transformation given at an inner product space V by the direct summand embedding $V \longrightarrow V \oplus \mathbb{R}$. Then the morphism $\psi_\sharp$ factors as the composite

$$\mathbf{bOP} \xrightarrow{\mathbf{bOP} \circ i} \mathrm{sh}\,\mathbf{bOP} \xrightarrow{\psi_!} \mathbf{bOP} \ .$$

The second morphism is defined at V as the map

$$(\mathrm{sh}\,\mathbf{bOP})(V) \ = \ \mathbf{bOP}(V \oplus \mathbb{R}) \ \longrightarrow \ \mathbf{bOP}(V)$$

that sends a subspace of $V \oplus \mathbb{R} \oplus \mathbb{R}^\infty$ to its image under the linear isometric embedding

$$V \oplus \mathbb{R} \oplus \mathbb{R}^\infty \ \longrightarrow \ V \oplus \mathbb{R}^\infty \ , \quad (v, x, y) \ \longmapsto \ (v, \psi(y, x)) \ .$$

The morphism $\mathbf{bOP} \circ i : \mathbf{bOP} \longrightarrow \mathrm{sh}\,\mathbf{bOP}$ is a global equivalence by Theorem I.1.11. Any two linear isometric embeddings from $\mathbb{R}^\infty \oplus \mathbb{R}$ to $\mathbb{R}^\infty$ are homotopic through linear isometric embeddings; in particular, and linear isometric embedding ψ is homotopic to a linear isometric isomorphism ψ'. Thus $\psi_!$ is homotopic to the isomorphism $\psi'_!$, hence a global equivalence. Since both $\mathbf{bOP} \circ i$ and $\psi_!$ are global equivalences, so is the composite $\psi_\sharp$. □

We have another description of the global homotopy type of **bOP**, as a homotopy colimit, in the category of orthogonal spaces, of a sequence of self maps of **Gr**:

$$\mathbf{bOP} \ \simeq \ \mathrm{hocolim}_{m \geq 1}\, \mathbf{Gr} \ .$$

The homotopy colimit is taken over iterated instances of a morphism $\mathbf{Gr} \longrightarrow \mathbf{Gr}$ that classifies the map 'add a trivial summand $\mathbb{R}$'. To make this relation rigorous, we use the filtration

$$* \ \cong \ \mathbf{bOP}_{(0)} \ \subset \ \mathbf{bOP}_{(1)} \ \subset \ \dots \ \subset \ \mathbf{bOP}_{(m)} \ \subset \ \dots \tag{4.35}$$

of **bOP** by orthogonal subspaces, analogous to the filtration (4.24) for **bO**. In other words,

$$\mathbf{bOP}_{(m)}(V) \ = \ \coprod_{n \geq 0} Gr_n(V \oplus \mathbb{R}^m) \ ;$$

as before we consider $\mathbb{R}^m$ as the subspace of $\mathbb{R}^\infty$ of all vectors of the form $(x_1, \dots, x_m, 0, 0, \dots)$.

We write sh^m for $\mathrm{sh}_{\oplus}^{\mathbb{R}^m}$, the additive shift by $\mathbb{R}^m$ of an orthogonal space, in the sense of Example I.1.12. We define a morphism $\gamma_m : \mathrm{sh}^m\,\mathbf{Gr} \longrightarrow \mathbf{bOP}_{(m)}$ by

$$\gamma_m(V) \ : \ (\mathrm{sh}^m\,\mathbf{Gr})(V) \ = \ \mathbf{Gr}(V \oplus \mathbb{R}^m) \ \longrightarrow \ \mathbf{bOP}_{(m)}(V) \ , \quad L \ \longmapsto \ L^\perp = (V \oplus \mathbb{R}^m) - L \ ,$$

the orthogonal complement of L inside $V \oplus \mathbb{R}^m$. We omit the straightforward verification that these maps indeed form a morphism of orthogonal spaces. Since each map $\gamma_m(V)$ is a homeomorphism, the morphism γ_m is in fact an isomorphism of orthogonal spaces.

The morphism

$$i_m \ : \ \mathbf{Gr} \ \longrightarrow \ \mathrm{sh}^m\,\mathbf{Gr}$$

induced by the direct summand embedding $V \longrightarrow V \oplus \mathbb{R}^m$ is a global equivalence by Theorem I.1.11. The inclusion of $\mathbf{bOP}_{(m)}$ into $\mathbf{bOP}_{(m+1)}$ is a closed embedding, so the global invariance property of Proposition I.1.9 (viii) entitles us to view the union **bOP** as a global homotopy colimit of the filtration. This justifies the interpretation of **bOP** as a global homotopy colimit of a sequence of copies of **Gr**. For this description to be useful we should identify the global homotopy classes of the morphisms in the sequence, i.e., the weak morphisms

$$\mathbf{Gr} \ \xrightarrow[\sim]{\gamma_m \circ i_m} \ \mathbf{bOP}_{(m)} \ \xrightarrow{\mathrm{incl}} \ \mathbf{bOP}_{(m+1)} \ \xleftarrow[\sim]{\gamma_{m+1} \circ i_{m+1}} \ \mathbf{Gr} \ .$$

As is straightforward from the definition, this weak morphism models 'adding a summand $\mathbb{R}$ with trivial action'.

Finally, we compare the E_∞-orthogonal monoid space **bOP** to the ultra-commutative monoid **BOP**. In analogy with the non-periodic version in (4.28), we introduce the orthogonal space $\mathbf{BOP}'$ with values

$$\mathbf{BOP}'(V) \;=\; \coprod_{n\geq 0} Gr_n(V^2 \oplus \mathbb{R}^\infty)\ . \tag{4.36}$$

The structure maps of $\mathbf{BOP}'$ are defined in the same way as for $\mathbf{BO}'$, and they mix the structure maps for **bOP** and **BOP**. Postcomposition with the direct summand embeddings

$$V \oplus \mathbb{R}^\infty \xrightarrow{(v,x)\mapsto(v,0,x)} V^2 \oplus \mathbb{R}^\infty \xleftarrow{(v,v',0)\leftarrow\!\shortmid(v,v')} V^2$$

induces morphisms of E_∞-orthogonal monoid spaces

$$\mathbf{bOP} \xrightarrow{a} \mathbf{BOP}' \xleftarrow{b} \mathbf{BOP}\ ;$$

these morphisms preserve the $\mathbb{Z}$-grading; the restrictions to the homogeneous degree 0 summand are precisely the morphisms with the same names introduced in (4.29). The same argument as in Proposition 4.30 also shows that the morphism $b : \mathbf{BOP} \longrightarrow \mathbf{BOP}'$ is a global equivalence of orthogonal spaces.

We define a monoid homomorphism

$$\gamma \;:\; \pi_0^G(\mathbf{BOP}') \;\longrightarrow\; \mathbf{RO}(G)$$

by sending the path component of $W \in \mathbf{BO}'(V)^G$ to the class $[W]-[V]$. Then the triangle of monoid homomorphisms on the right of following diagram commutes:

$$\begin{tikzcd}
\pi_0^G(\mathbf{bOP}) \arrow[r, "a_*"] \arrow[dr] & \pi_0^G(\mathbf{BOP}') \arrow[d, "\gamma"', "\cong"] & \pi_0^G(\mathbf{BOP}) \arrow[l, "b_*"', "\cong"] \arrow[dl, "\cong", "(4.13)"'] \\
& \mathbf{RO}(G) &
\end{tikzcd}$$

The two maps on the right are isomorphisms, hence so is the map γ. The same argument as in Proposition 4.31 shows that the composite morphism of abelian Rep-monoids

$$\underline{\pi}_0(\mathbf{bOP}) \xrightarrow{a_*} \underline{\pi}_0(\mathbf{BOP}') \xrightarrow{\gamma} \mathbf{RO}$$

is a monomorphism, and for every compact Lie group G the image of $\pi_0^G(\mathbf{bOP})$ in the representation ring $\mathbf{RO}(G)$ consists of the submonoid of elements of the form $n\cdot 1-[U]$, for $n\in\mathbb{Z}$ and U any G-representation.

Example 4.37 (Complex and quaternionic periodic Grassmannians). The ultra-commutative monoids **BO** and **BOP** and the E_∞-orthogonal monoid spaces **bO** and **bOP** have straightforward complex and quaternionic analogs; we quickly give the relevant definitions in the sake of completeness. We define the periodic Grassmannians **BUP** and **BSpP** by

$$\mathbf{BUP}(V) \;=\; \coprod_{n\geq 0} Gr_n^{\mathbb{C}}(V^2_{\mathbb{C}}) \qquad \text{respectively} \qquad \mathbf{BSpP}(V) \;=\; \coprod_{n\geq 0} Gr_n^{\mathbb{H}}(V^2_{\mathbb{H}})\ ,$$

the disjoint union of the respective Grassmannians, with structure maps as for **BOP**. External direct sum of subspaces defines a $\mathbb{Z}$-graded ultra-commutative multiplication on **BUP** and on **BSpP**, again in much the same way as for **BOP** The homogeneous degree summands are closed under the multiplication and form ultra-commutative monoids $\mathbf{BU}=\mathbf{BUP}^{[0]}$ respectively $\mathbf{BSp}=\mathbf{BSpP}^{[0]}$ in their own right. The complex and quaternionic analogues of Theorem 4.14 provide isomorphisms of global power monoids

$$\underline{\pi}_0(\mathbf{BUP}) \;\cong\; \mathbf{RU} \qquad \text{and} \qquad \underline{\pi}_0(\mathbf{BSpP}) \;\cong\; \mathbf{RSp}\ ;$$

here $\mathbf{RU}(G)$ and $\mathbf{RSp}(G)$ are the Grothendieck groups, under direct sum, of isomorphism classes of unitary respectively symplectic G-representations. The isomorphisms above match the $\mathbb{Z}$-grading of **BUP** and **BSpP** with the grading by virtual dimension of representations, so they restrict to isomorphisms of

global power monoids from $\underline{\pi}_0(\mathbf{BU})$ respectively $\underline{\pi}_0(\mathbf{BSp})$ to the augmentation ideal global power monoids inside $\mathbf{RU}$ respectively $\mathbf{RSp}$.

Theorem 4.11 also generalizes to natural group isomorphisms, compatible with restrictions,

$$\langle - \rangle \ : \ \mathbf{BUP}_G(B) \ \longrightarrow \ \mathbf{KU}_G(B) \qquad \text{and} \qquad \langle - \rangle \ : \ \mathbf{BSpP}_G(B) \ \longrightarrow \ \mathbf{KSp}_G(B)$$

to the equivariant unitary respectively symplectic K-groups, where G is any compact Lie groups G and B a compact G-space.

Example 4.20 can be modified to define E_∞-orthogonal monoid spaces $\mathbf{bU}$ and $\mathbf{bSp}$; with values

$$\mathbf{bU}(V) \ = \ Gr^{\mathbb{C}}_{|V|}(V_{\mathbb{C}} \oplus \mathbb{C}^\infty) \qquad \text{respectively} \qquad \mathbf{bSp}(V) \ = \ Gr^{\mathbb{H}}_{|V|}(V_{\mathbb{H}} \oplus \mathbb{H}^\infty) \ .$$

The structure maps and E_∞-multiplication are defined as for $\mathbf{bO}$. As orthogonal spaces, $\mathbf{bU}$ and $\mathbf{bSp}$ are global homotopy colimits of the sequence of global classifying spaces $B_{\mathrm{gl}}U(m)$ respectively $B_{\mathrm{gl}}Sp(m)$. Periodic versions $\mathbf{bUP}$ and $\mathbf{bSpP}$ are defined by taking the full Grassmannian inside $V_{\mathbb{C}} \oplus \mathbb{C}^\infty$ respectively $V_{\mathbb{H}} \oplus \mathbb{H}^\infty$, as in the real case in Example 4.33. As orthogonal spaces, Periodic versions $\mathbf{bUP}$ and $\mathbf{bSpP}$ are global homotopy colimits of iterated instances of the self maps of $\mathbf{Gr}^{\mathbb{C}}$ respectively $\mathbf{Gr}^{\mathbb{H}}$ that represents 'additive a trivial 1-dimensional representation'.

Construction 4.38 (Global Bott periodicity). The Bott periodicity map is traditionally seen as a homotopy equivalence between the space $\mathbb{Z} \times BU$ and the loop space of the infinite unitary group. We explain a highly structured, global form of Bott periodicity, namely that $\mathbf{BUP}$ and $\Omega\mathbf{U}$ are globally equivalent as ultra-commutative monoids.

We define a morphism of ultra-commutative monoids

$$\beta \ : \ \mathbf{BUP} \ \longrightarrow \ \Omega(\mathrm{sh}_\otimes \mathbf{U}) \ .$$

Here $\mathbf{U}$ is the ultra-commutative monoid of unitary groups (compare Example 3.7), and Ω means objectwise continuous based maps from S^1. Moreover $\mathrm{sh}_\otimes = \mathrm{sh}_\otimes^{\mathbb{R}^n}$ is the multiplicative shift by $\mathbb{R}^2$ defined in Example I.1.12. The target of β is globally equivalent, as an ultra-commutative monoid, to $\Omega\mathbf{U}$, the objectwise loops of the unitary group monoid. The morphism β is a modification of Behrens' coordinate free description [**15**] of the non-equivariant Bott map, which is based on ideas of MacDuff [**102**] and Aguilar and Prieto [**3**].

We define the map

$$\beta(V) \ : \ \mathbf{BUP}(V) \ \longrightarrow \ \Omega(\mathrm{sh}_\otimes \mathbf{U})(V) \ = \ \mathrm{map}(S^1, \mathbf{U}(V^2_{\mathbb{C}})) \ .$$

An element of $\mathbf{BUP}(V)$ is a complex subspace L of $V^2_{\mathbb{C}}$; we denote by p_L and $p_{L^\perp}$ the orthogonal projections to L respectively to its orthogonal complement. We define the loop

$$\beta(V)(L) \ : \ S^1 \ \longrightarrow \ U(V^2_{\mathbb{C}})$$

by

$$\beta(V)(L)(t) \ = \ ((c(t) \cdot p_L) \ + \ p_{L^\perp}) \circ ((c(-t) \cdot p_{V_{\mathbb{C}} \oplus 0}) + p_{0 \oplus V_{\mathbb{C}}}) \ .$$

Here we identify S^1 with $U(1)$ via the homeomorphism

$$c \ : \ S^1 \ \longrightarrow \ U(1) \ , \quad t \ \longmapsto \ \frac{t+i}{t-i} \ ,$$

the *Cayley transform*. The map $\beta(V)$ is continuous in L.

The orthogonal space $\mathbf{U}$ has a commutative multiplication by direct sum of unitary automorphisms. So the orthogonal space $\Omega(\mathrm{sh}_\otimes \mathbf{U})$ inherits a commutative multiplication by pointwise multiplication of loops. For every inner product space V we have

$$\beta(V)(V_{\mathbb{C}} \oplus 0)(t) \ = \ ((c(t) \cdot p_{V_{\mathbb{C}} \oplus 0}) \ + \ p_{0 \oplus V_{\mathbb{C}}}) \circ ((c(-t) \cdot p_{V_{\mathbb{C}} \oplus 0}) + p_{0 \oplus V_{\mathbb{C}}}) \ = \ \mathrm{Id}_V \ ;$$

so $\beta(V)(V_{\mathbb{C}} \oplus 0)$ is the constant loop at the identity, which is the unit element of $\Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})(V)$. Now we consider subspaces $L \in \mathbf{BUP}(V)$ and $L' \in \mathbf{BUP}(W)$. Then

$$\begin{aligned}
\beta(V \oplus W)(L \oplus L')(t) \;&=\; ((c(t) \cdot p_{L \oplus L'}) \;+\; p_{L^\perp \oplus (L')^\perp}) \circ ((c(-t) \cdot p_{(V \oplus W)_{\mathbb{C}} \oplus 0}) + p_{0 \oplus (V \oplus W)_{\mathbb{C}}}) \\
&=\; \kappa_*^{V,W}((((c(t) \cdot p_L) \;+\; p_{L^\perp}) \oplus ((c(t) \cdot p_{L'}) \;+\; p_{(L')^\perp}))) \\
&\qquad\qquad \circ\, \kappa_*^{V,W}((((c(-t) \cdot p_{V_{\mathbb{C}} \oplus 0}) + p_{0 \oplus V_{\mathbb{C}}}) \oplus ((c(-t) \cdot p_{W_{\mathbb{C}} \oplus 0}) + p_{0 \oplus W_{\mathbb{C}}}))) \\
&=\; \kappa_*^{V,W}((((c(t) \cdot p_L) \;+\; p_{L^\perp}) \oplus ((c(t) \cdot p_{L'}) \;+\; p_{(L')^\perp})) \circ (((c(-t) \cdot p_{V_{\mathbb{C}} \oplus 0}) + p_{0 \oplus V_{\mathbb{C}}}) \oplus ((c(-t) \cdot p_{W_{\mathbb{C}} \oplus 0}) + p_{0 \oplus W_{\mathbb{C}}}))) \\
&=\; \kappa_*^{V,W}(((c(t) \cdot p_L) \;+\; p_{L^\perp}) \circ ((c(-t) \cdot p_{V_{\mathbb{C}} \oplus 0}) + p_{0 \oplus V_{\mathbb{C}}})) \oplus (((c(t) \cdot p_{L'}) \;+\; p_{(L')^\perp}) \circ ((c(-t) \cdot p_{W_{\mathbb{C}} \oplus 0}) + p_{0 \oplus W_{\mathbb{C}}}))) \\
&=\; \kappa_*^{V,W}\,(\beta(V)(L)(t) \oplus \beta(W)(L')(t)) \;\;,
\end{aligned}$$

where $\kappa^{V,W} : V_{\mathbb{C}}^2 \oplus W_{\mathbb{C}}^2 \cong (V \oplus W)_{\mathbb{C}}^2$ is the preferred natural isometry. In other words, the square

$$\begin{tikzcd}
\mathbf{BUP}(V) \times \mathbf{BUP}(W) \arrow[r, "\beta(V) \times \beta(W)"] \arrow[d, "\oplus"'] & \Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})(V) \times \Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})(W) \arrow[d, "\mu_{\Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})_{V,W}}"] \\
\mathbf{BUP}(V \oplus W) \arrow[r, "\beta(V \oplus W)"'] & \Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})(V \oplus W)
\end{tikzcd}$$

commutes, i.e., β is compatible with the multiplications on both side. Since β respects multiplication and unit, it also respects the structure maps. The upshot is that β is a morphism of ultra-commutative monoids.

We have

$$\det(\beta(V)(L)(t)) \;=\; \det((c(t) \cdot p_L) + p_{L^\perp}) \cdot \det((c(-t) \cdot p_{V_{\mathbb{C}} \oplus 0}) + p_{0 \oplus V_{\mathbb{C}}}) \;=\; c(t)^{\dim(L) - \dim(V)} \;\;,$$

exploiting that $c(-t) = \overline{c(t)} = c(t)^{-1}$. So the composite

$$\mathbf{BUP}(V) \;\xrightarrow{\beta(V)}\; \Omega U(V_{\mathbb{C}}^2) \;\xrightarrow{\Omega \det}\; \Omega U(1)$$

sends the subspace $\mathbf{BUP}^{[n]}(V)$ to the path component of $\Omega(U(1))$ consisting of loops of degree n. So the morphism β restricts to a morphism of orthogonal spaces

$$\beta^{[0]} \;:\; \mathbf{BU} \;\longrightarrow\; \Omega(\mathrm{sh}_{\otimes}\,\mathbf{SU}) \;.$$

Theorem 4.39 (Global Bott periodicity). *The morphisms of ultra-commutative monoids* $\beta : \mathbf{BUP} \longrightarrow \Omega(\mathrm{sh}_{\otimes}\,\mathbf{U})$ *and* $\beta^{[0]} : \mathbf{BU} \longrightarrow \Omega(\mathrm{sh}_{\otimes}\,\mathbf{SU})$ *are global equivalences.*

Proof. Both source and target of the morphism β admit 'isotypical decompositions', as follows. We let G be a compact Lie group and choose representatives $\{\lambda\}$ for the isomorphism classes of irreducible unitary G-representations. For every orthogonal G-representation V, the fixed points $\mathbf{BUP}(V)^G$ are the G-subrepresentations of $V_{\mathbb{C}}^2$. So we can define

$$\mathbf{BUP}(V)_\lambda^G \;=\; \{L \in Gr^{\mathbb{C}}(V_{\mathbb{C}}^2)^G \mid L \text{ is } \lambda\text{-isotypical}\} \;.$$

Since a G-representation is the internal direct sum of its isotypical summands, this gives a decomposition

$$\mathbf{BUP}(V)^G \;=\; \prod_\lambda \mathbf{BUP}(V)_\lambda^G \;.$$

On the other hand, a G-equivariant linear isometry preserves the isotypical summands, so

$$((\mathrm{sh}_{\otimes}\,\mathbf{U})(V))^G \;=\; U^G(V_{\mathbb{C}}^2) \;=\; \prod_\lambda U^G((V_{\mathbb{C}})_\lambda^2) \;.$$

Moreover, the morphism

$$\beta(V)^G \;:\; \mathbf{BUP}(V)^G \;\longrightarrow\; ((\mathrm{sh}_{\otimes}\,\mathbf{U})(V))^G$$

preserves these isotypical product decompositions. We may thus show that the map $\beta(V)^G_\lambda$ is a weak equivalence for every irreducible G-representation V. But

$$\mathbf{BUP}(\mathcal{U}_G)^G_\lambda \;\simeq\; \mathbb{Z}\times BU \qquad \text{and} \qquad ((\Omega\,\mathrm{sh}_\otimes\,\mathbf{U})(\mathcal{U}_G))^G_\Lambda \;\simeq\; \Omega U\ .$$

Under these identifications, λ-isotypical factor of $\beta(\mathcal{U}_G)^G$ becomes the non-equivariant Bott map, which is a weak equivalence by non-equivariant Bott periodicity [...ref...]. □

The embeddings $V_{\mathbb{C}} \longrightarrow V^2_{\mathbb{C}}$ as the first summand induce a morphism of ultra-commutative monoids $i : \mathbf{U} \longrightarrow \mathrm{sh}_\otimes\,\mathbf{U}$ which is a global equivalence by Theorem I.1.11. So together with Theorem 4.39 this provides a zigzag of global equivalences of ultra-commutative monoids between

$$\mathbf{BUP} \qquad \text{and} \qquad \Omega\mathbf{U}\ .$$

On underlying non-equivariant homotopy types this recovers Bott periodicity in the form of a weak equivalence between $\mathbb{Z}\times BU$ and ΩU.

I suspect that there are similar global forms of real Bott periodicity, taking the form of chains of global equivalences of ultra-commutative monoids

$$\begin{array}{rclcrcl}
\mathbf{BOP} & \simeq_{\mathrm{gl}} & \Omega(\mathbf{U}/\mathbf{O})\ , & & \mathbf{U}/\mathbf{O} & \simeq_{\mathrm{gl}} & \Omega(\mathbf{Sp}/\mathbf{U})\ , \\
\mathbf{Sp}/\mathbf{U} & \simeq_{\mathrm{gl}} & \Omega(\mathbf{Sp})\ , & & \mathbf{Sp} & \simeq_{\mathrm{gl}} & \Omega(\mathbf{BSp})\ , \\
\mathbf{BSpP} & \simeq_{\mathrm{gl}} & \Omega(\mathbf{U}/\mathbf{Sp})\ , & & \mathbf{U}/\mathbf{Sp} & \simeq_{\mathrm{gl}} & \Omega(\mathbf{O}/\mathbf{U})\ , \\
\mathbf{O}/\mathbf{U} & \simeq_{\mathrm{gl}} & \Omega\mathbf{O} & \text{and} & \mathbf{O} & \simeq_{\mathrm{gl}} & \Omega(\mathbf{BO})\ .
\end{array}$$

I have not thought about this in detail yet, though.

5. Global group completion and units

For every orthogonal monoid space R and every compact Lie group G, the operation (2.10) makes the equivariant homotopy set $\pi_0^G(R)$ into a monoid, and this multiplication is natural with respect to restriction maps in G. If the multiplication of R is commutative, then so is the multiplication of $\pi_0^G(R)$. In this section we look more closely at the *group-like* ultra-commutative monoids, i.e., the ones where all these monoid structures have inverses. We will show that there is are two 'universal' ways to make an ultra-commutative monoid group-like: the 'global units' (Construction 5.19) are universal from the left and the 'global group completion' (Construction 5.24 and Corollary 5.32) is universal from the right. In the homotopy category of ultra-commutative monoids, these two constructions are right adjoint respectively left adjoint to the inclusion of group-like objects. On π_0^G the two constructions have the expected effect: the global units pick out the invertible elements of π_0^G (see Proposition 5.20), and the effect of global group completion is group completion of the abelian monoid π_0^G (see Proposition 5.23).

The category of ultra-commutative monoids is pointed, and product respectively box product are the categorical product respectively coproduct in the category of ultra-commutative monoids. These descend to product and respectively coproduct in the homotopy category $\mathrm{Ho}(umon)$ of ultra-commutative monoids, with respect to the global model structure of Theorem 1.13. The morphism $\rho_{R,S} : R\boxtimes S \longrightarrow R\times S$ is a global equivalence by Theorem I.3.38 (i), so in $\mathrm{Ho}(umon)$ the canonical map from a coproduct to a product is an isomorphism. Various features of unit maps and group completions only depend on these formal properties, and work just as well in any pointed model category in which coproducts and products coincide up to weak equivalence. So we develop large parts of the theory in this generality.

Construction 5.1. Let $\mathcal{D}$ be a category which has finite products and a zero object. We write $A\times B$ for any product of the objects A and B and leave the projections $A\times B\longrightarrow A$ and $A\times B\longrightarrow B$ implicit. Given morphisms $f : T\longrightarrow A$ and $g : T\longrightarrow B$ we write $(f,g) : T\longrightarrow A\times B$ for the unique morphism that projects to f respectively g. We write 0 for any morphism that factors through a zero object.

We call the category $\mathcal{D}$ *pre-additive* if 'finite products are coproducts'; more precisely, we require that every product $A \times B$ of two objects A and B is also a *co*-product, with respect to the morphisms

$$i_1 \;=\; (\mathrm{Id}_A, 0) \;:\; A \;\longrightarrow\; A \times B \qquad \text{and} \qquad i_2 \;=\; (0, \mathrm{Id}_B) \;:\; B \;\longrightarrow A \times B \;.$$

In other words, we demand that for every object X the map

$$\mathcal{D}(A \times B, X) \;\longrightarrow\; \mathcal{D}(A, X) \times \mathcal{D}(B, X) \;, \quad f \mapsto (fi_1, fi_2)$$

is bijective. The main example we care about is $\mathcal{D} = \mathrm{Ho}(umon)$, the homotopy category of ultra-commutative monoids.

In this situation we can define a binary operation on the morphism set $\mathcal{D}(A, X)$ for every pair of objects A and X. Given morphisms $a, b : A \longrightarrow X$ we let $a \perp b : A \times A \longrightarrow X$ be the unique morphism such that $(a \perp b)i_1 = a$ and $(a \perp b)i_2 = b$. Then we define

$$a + b \;=\; (a \perp b)\Delta \;:\; A \;\longrightarrow\; X \;,$$

where $\Delta = (\mathrm{Id}_A, \mathrm{Id}_A) : A \longrightarrow A \times A$ is the diagonal morphism.

The next proposition is well-known, but I do not know a convenient reference.

Proposition 5.2. *Let $\mathcal{D}$ be a pre-additive category. For every pair of objects A and X of $\mathcal{D}$ the binary operation $+$ makes the set $\mathcal{D}(A, X)$ of morphisms into an abelian monoid with the zero morphism as neutral element. Moreover, the monoid structure is natural for all morphisms in both variables, or, equivalently, composition is biadditive.*

Proof. The proof is lengthy, but completely formal. For the associativity of '+' we consider three morphisms $a, b, c : A \longrightarrow X$. Then $a + (b + c)$ respectively $(a + b) + c$ are the two outer composites around the diagram

$$\begin{tikzcd}
 & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A \times A \arrow[d, "\mathrm{Id} \times \Delta"'] & & A \times A \arrow[d, "\Delta \times \mathrm{Id}"] \\
A \times (A \times A) \arrow[dr, "a \perp (b \perp c)"'] & & (A \times A) \times A \arrow[dl, "(a \perp b) \perp c"] \\
 & X &
\end{tikzcd}$$

If we fill in the canonical associativity isomorphism $A \times (A \times A) \cong (A \times A) \times A$ then the upper part of the diagram commutes because the diagonal morphism is coassociative. The lower triangle then commutes since the two morphisms $a \perp (b \perp c), (a \perp b) \perp c : A \times (A \times A) \longrightarrow X$ have the same 'restrictions', namely a, b respectively c.

The commutativity is a consequence of two elementary facts: first, $b \perp a = (a \perp b)\tau$ where $\tau : A \times A \longrightarrow A \times A$ is the automorphism which interchanges the two factors; this follows from $\tau i_1 = i_2$ and $\tau i_2 = i_1$. Second, the diagonal morphism is cocommutative, i.e., $\tau\Delta = \Delta : A \longrightarrow A \times A$. Altogether we get

$$a + b \;=\; (a \perp b)\Delta \;=\; (a \perp b)\tau\Delta \;=\; (b \perp a)\Delta \;=\; b + a \;.$$

As before we denote by $0 \in \mathcal{D}(A, X)$ the unique morphism which factors through a zero object. Then we have $a \perp 0 = ap_1$ in $\mathcal{D}(A \times A, X)$ where $p_1 : A \times A \longrightarrow A$ is the projection onto the first factor. Hence $a + 0 = (a \perp 0)\Delta = ap_1\Delta = a$; by commutativity we also have $0 + a = a$.

Now we verify naturality of the addition on $\mathcal{D}(A,X)$ in A and X. To check $(a+b)c = ac+bc$ for $a,b : A \longrightarrow X$ and $c : A' \longrightarrow A$ we consider the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "c"] \arrow[d, "\Delta"'] & A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \times A \arrow[d, "a\perp b"] \\
A' \times A' \arrow[r, "c\times c"] \arrow[rr, bend right, "ac\perp bc"'] & A \times A \arrow[r, "a\perp b"] & X
\end{tikzcd}$$

in which the composite through the upper right corner is $(a+b)c$. We have

$$(a\perp b)(c\times c)i_1 \;=\; (a\perp b)(c,0) \;=\; ac \;=\; (ac\perp bc)i_1$$

and similarly for i_2 instead i_1. So $(a\perp b)(c\times c) = ac\perp bc$ since both sides have the same 'restrictions' to the two factors of $A'\times A'$. Since the composite through the lower left corner is $ac+bc$, we have shown $(a+b)c = ac+bc$. Naturality in X is even easier. For a morphism $d : X \longrightarrow Y$ we have $d(a\perp b) = da\perp db : A\times A \longrightarrow Y$ since both sides have the same 'restrictions' da respectively db to the two factors of $A\times A$. Thus $d(a+b) = da+db$ by the definition of '+'. □

In the next definition and in what follows, we denote by $M^\times$ the subgroup of invertible elements in an abelian monoid M.

Definition 5.3. Let $\mathcal{D}$ be a pre-additive category. A morphism $u : R^\times \longrightarrow R$ is a *unit morphism* if for every object T the map

$$u_* \;:\; \mathcal{D}(T,R^\times) \;\longrightarrow\; \mathcal{D}(T,R)$$

is injective with image the subgroup $\mathcal{D}(T,R)^\times$ of invertible elements. A morphism $i : R \longrightarrow R^\star$ in $\mathcal{D}$ is a *group completion* if for every object T the map

$$i^* \;:\; \mathcal{D}(R^\star,T) \;\longrightarrow\; \mathcal{D}(R,T)$$

is injective with image the subgroup $\mathcal{D}(R,T)^\times$ of invertible elements.

Remark 5.4. If $u : R^\times \longrightarrow R$ is a unit morphism, then by definition the pair $(R^\times, u)$ represents the functor

$$\mathcal{D} \;\longrightarrow\; (\text{sets})\;, \quad T \;\longmapsto\; \mathcal{D}(T,R)^\times\;;$$

so $(R^\times, u)$ is unique up to preferred isomorphism. A formal consequence is that if we choose a unit morphism $u_R : R^\times \longrightarrow R$ for every object R, then this extends canonically to a functor

$$(-)^\times \;:\; \mathcal{D} \;\longrightarrow\; \mathcal{D}$$

and a natural transformation $u : (-)^\times \longrightarrow \mathrm{Id}$. The 'dual' remarks apply to group completion: if we choose a group completion $i_R : R \longrightarrow R^\star$ for every object R, then this extends canonically to a functor

$$(-)^\star \;:\; \mathcal{D} \;\longrightarrow\; \mathcal{D}$$

and a natural transformation $i : \mathrm{Id} \longrightarrow (-)^\star$.

Proposition 5.5. *Let $\mathcal{D}$ be a pre-additive category. For every object A of $\mathcal{D}$ the following two conditions are equivalent:*

(a) *The shearing morphism $(p_1,\nabla) = \Delta\perp i_2 = (\Delta p_1) + i_2p_2 : A\times A \longrightarrow A\times A$ is an isomorphism.*
(b) *The identity of A has an inverse in the abelian monoid $\mathcal{D}(A,A)$.*

We call A group-like *if it satisfies* (a) *and* (b)*. If A is group-like, then moreover for every object X of $\mathcal{D}$ the abelian monoids $\mathcal{D}(A,X)$ and $\mathcal{D}(X,A)$ have inverses.*

PROOF. (a)$\Longrightarrow$(b) Since the shearing map is an isomorphism, there is a morphism $(k,j) : A \longrightarrow A \times A$ such that

$$(\mathrm{Id}_A, 0) \;=\; (p_1, \nabla) \circ (k,j) \;=\; (k, k+j) \;.$$

So $k = \mathrm{Id}_A$ and $\mathrm{Id}_A + j = 0$, i.e., j is an additive inverse of the identity of A.

(b)$\Longrightarrow$(a) If $j \in \mathcal{D}(A,A)$ is an inverse of the identity of A, then the morphism

$$(p_1, j \perp \mathrm{Id}_A) \;=\; (\mathrm{Id}_A, j) \perp i_2 \;:\; A \times A \;\longrightarrow\; A \times A$$

is a two-sided inverse to the shearing morphism, which is thus an isomorphism.

If $j \in \mathcal{D}(A,A)$ is an additive inverse to the identity of A, then for all $f \in \mathcal{D}(X,A)$

$$f + (jf) \;=\; (\mathrm{Id}_A \circ f) + (j \circ f) \;=\; (\mathrm{Id}_A + j) \circ f \;=\; 0 \circ f \;=\; 0 \;;$$

so $j \circ f$ is inverse to f. Similarly, for every $g \in \mathcal{D}(A,X)$ the morphism $g \circ j$ is additively inverse to g. $\square$

EXAMPLE 5.6 (Unit morphisms and group completion for abelian monoids). The category of abelian monoids is the prototypical example of a pre-additive category, and the general theory of units and group completions is an abstraction of this special case. So we take the time to convince ourselves that the concepts of 'unit morphism' and 'group completion' have their familiar meanings in the motivating example.

A basic observation is that in the category of abelian monoids, the abstract addition of morphism as in Proposition 5.2 is simply pointwise addition of homomorphisms. So an abelian monoid is group-like in the abstract sense of Proposition 5.5 if and only if every element has an inverse; so the group-like objects are precisely the abelian groups.

Given homomorphism $f : M \longrightarrow N$ of abelian monoids is invertible if and only if it is pointwise invertible in N, which is the case if any only if the image of f lies in the subgroup $N^\times$ of invertible elements. So the inclusion $u : N^\times \longrightarrow N$ of the subgroup of invertible elements is a unit morphism in the sense of Definition 5.3.

We recall the *Grothendieck group* of an abelian monoid M. A equivalence relation $\sim$ on M^2 is defined by declaring (x,y) equivalent to (x',y') if and only if there is an element $z \in M$ with

$$x + y' + z \;=\; x' + y + z \;.$$

The componentwise addition on M^2 is well-defined on equivalence classes, so the set of equivalence classes

$$M^\star \;=\; M^2/\sim$$

inherits an abelian monoid structure. We use the notation $[x,y]$ for the equivalence class of a pair (x,y) in $M^\star$. The pair $(x+y, y+x)$ is equivalent to $(0,0)$, so

$$[x,y] + [y,x] \;=\; [x+y, y+x] \;=\; 0$$

in the monoid $M^\star$. This shows that every element of $M^\star$ has an inverse, so $M^\star$ is an abelian group. We claim that the monoid homomorphism

$$i \;:\; M \;\longrightarrow\; M^\star \;, \quad i(x) \;=\; [x,0]$$

is a group completion in the sense of Definition 5.3. Indeed, given a a monoid homomorphism $h : M \longrightarrow N$ that is pointwise invertible, then we can define $f : M^\star \longrightarrow N$ by

$$f[x,y] \;=\; h(x) - h(y) \;.$$

A routine verification then shows that f is indeed a well-defined homomorphism and that sending h to f is inverse to the restriction map

$$\mathcal{A}bMon(i,N) \;:\; \mathcal{A}bMon(M^\star, N) \;\longrightarrow\; \mathcal{A}bMon(M,N)^\times \;.$$

We observe that

$$[x,y] \;=\; [x,0] + [0,y] \;=\; i(x) - i(y) \;,$$

so every element in $M^\star$ is the difference of two elements in the image of $i : M \longrightarrow M^\star$. Moreover, if $x, y' \in M$ satisfy $i(x) = i(x')$, then the pairs $(x, 0)$ and $(x', 0)$ are equivalent, which happens if and only if there is an element $z \in M$ such that $x + z = x' + z$. Conversely, these two properties of the Grothendieck construction characterize group completions of abelian monoids: a homomorphism $j : M \longrightarrow N$ of abelian monoids is a group completion if and only if the following three conditions are satisfied:

- the monoid N is a group;
- every element in N is the difference of two elements in the image of j;
- if $x, x' \in M$ satisfy $j(x) = j(x')$, then there is an element $z \in M$ such that $x + z = x' + z$.

Indeed, the first condition guarantees that j extends (necessarily uniquely) to a homomorphism $j' : M^\star \longrightarrow N$, the second condition guarantees that the extension j' is surjective, and the third condition implies that j' is injective, hence an isomorphism.

Remark 5.7. A category $\mathcal{D}$ is pre-additive if and only if its opposite category $\mathcal{D}^{\mathrm{op}}$ is pre-additive. Moreover, in that situation $\mathcal{D}(A, X)$ and $\mathcal{D}^{\mathrm{op}}(X, A)$ are not only the same set (by definition), but they also have the same monoid structure. Thus the concepts of unit morphism and group completion are 'dual' (or 'opposite') to each other: a morphism is a unit morphism in $\mathcal{D}$ if and only if it is a group completion in $\mathcal{D}^{\mathrm{op}}$. This is why many properties of unit maps have a corresponding 'dual' property for group completions, and why most proofs for unit maps have 'dual' proofs for group completions.

Since the identity of any object of $\mathcal{D}$ is also the identity of the same object in $\mathcal{D}^{\mathrm{op}}$, part (b) of Proposition 5.5 shows that 'group-like' is a self-dual property: and object is group-like in $\mathcal{D}$ if and only if it is group-like in $\mathcal{D}^{\mathrm{op}}$.

If $u : R^\times \longrightarrow R$ is a unit morphism then the abelian monoid $\mathcal{D}(R^\times, R^\times)$ is a group, by the defining property. So the object $R^\times$ satisfies condition (b) of the previous proposition, i.e., $R^\times$ is group-like. Moreover, if we make choices of units for all object R, then the resulting functor is right adjoint to the inclusion of the full subcategory of group-like objects.

Dually, if $i : R \longrightarrow R^\star$ is a group completion of an object of a pre-additive category $\mathcal{D}$, then the object $R^\star$ satisfies condition (b) of the previous proposition, and is thus group-like. And if we make choices of group completions for all object R, then the resulting functor is left adjoint to the inclusion of the full subcategory of group-like objects.

We mostly care about the situation where $\mathcal{D} = \mathrm{Ho}(\mathcal{C})$ is the homotopy category of a pointed model category $\mathcal{C}$, such as the category of ultra-commutative monoids. The next proposition will be used to show that in this situation units and group completions always exist.

We consider two composable morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ in a pointed category $\mathcal{D}$. We recall that f is a *kernel* of g if for every morphism $\beta : T \longrightarrow B$ such that $g\beta = 0$, there is a unique morphism $\alpha : T \longrightarrow A$ such that $f\alpha = \beta$. Dually, g is a *cokernel* of f if for every morphism $\beta : B \longrightarrow Y$ such that $\beta f = 0$, there is a unique morphism $\gamma : C \longrightarrow Y$ such that $\gamma g = \beta$.

We recall a construction of the group completion (Grothendieck construction) of an abelian monoid M: we let $d : M \times M \longrightarrow M^\star$ be a cokernel, in the category of commutative monoids, of the diagonal morphism $\Delta : M \longrightarrow M \times M$. Then the composite morphism

$$i \ : \ M \ \xrightarrow{(\mathrm{Id},0)} \ M \times M \ \xrightarrow{d} \ M^\star$$

is a group completion, and $d(x, y) = i(x) - i(y)$. Dually, for every abelian monoid M the preimage of the zero element under the addition map $M^2 \longrightarrow M$ is isomorphic to the subgroup of invertible elements, via

$$M^\times \ \longrightarrow \ \ker(+ : M^2 \longrightarrow M) \ , \quad m \longmapsto \ (m, -m) \ .$$

The following proposition provides abstract analogues of these two facts about abelian monoids.

Proposition 5.8. *Let R be an object of a pre-additive category $\mathcal{D}$.*

(i) *Let* $e : R^\times \longrightarrow R \times R$ *be a kernel of the codiagonal morphism* $\mathrm{Id} \perp \mathrm{Id} : R \times R \longrightarrow R$. *Then the composite*

$$u \;=\; (\mathrm{Id} \perp 0) \circ e \;:\; R^\times \;\longrightarrow\; R$$

is a unit morphism and

$$e \;=\; (u, -u) \;:\; R^\times \;\longrightarrow\; R \times R\ .$$

Conversely, if $u : R^\times \longrightarrow R$ *is a unit morphism, then the morphism* $(u, -u) : R^\times \longrightarrow R \times R$ *is a kernel of the codiagonal morphism* $\mathrm{Id} \perp \mathrm{Id} : R \times R \longrightarrow R$.

(ii) *Let* $d : R \times R \longrightarrow R^\star$ *be a cokernel of the diagonal morphism* $(\mathrm{Id}, \mathrm{Id}) : R \longrightarrow R \times R$. *Then the composite*

$$i \;=\; d \circ (\mathrm{Id}, 0) \;:\; R \;\longrightarrow\; R^\star$$

is a group completion and

$$d \;=\; i \perp (-i) \;:\; R \times R \;\longrightarrow\; R^\star\ .$$

Conversely, if $i : R \longrightarrow R^\star$ *is a group completion, then the morphism* $i \perp (-i) : R \times R \longrightarrow R^\star$ *is a cokernel of the diagonal morphism* $(\mathrm{Id}, \mathrm{Id}) : R \longrightarrow R \times R$.

Proof. We prove part (ii). Part (i) is dual, i.e., equivalent to part (ii) in the opposite category $\mathcal{D}^{\mathrm{op}}$. We let T be any object $\mathcal{D}$. Then the map

$$\{(f, g) \in \mathcal{D}(R, T)^2 \mid f + g = 0\} \;\longrightarrow\; \mathcal{D}(R, T)^\times\ , \quad (f, g) \;\longmapsto\; f$$

is bijective because inverses in abelian monoids, if they exist, are unique. A cokernel of the diagonal morphism is a morphism $d : R \times R \longrightarrow R^\star$ that represents the left hand side of this bijection; a group completion is a morphism $i : R \longrightarrow R^\star$ that represents the right hand side of this bijection. Hence $d : R \times R \longrightarrow R^\star$ is a cokernel of the diagonal if and only if $d \circ (\mathrm{Id}, 0)$ is a group completion.

The relation $d \circ (\mathrm{Id}, 0) = i$ holds by definition. The relation

$$(d \circ (0, \mathrm{Id})) + i \;=\; d \circ ((0, \mathrm{Id}) + (\mathrm{Id}, 0)) \;=\; d \circ (\mathrm{Id}, \mathrm{Id}) \;=\; 0$$

holds in the monoid $\mathcal{D}(R, R^\star)$, and thus $d \circ (0, \mathrm{Id}) = -i$. This shows that $d = i \perp (-i)$. □

The previous characterization of unit morphisms and group completions as certain kernels respectively cokernels formally implies the following corollary.

Corollary 5.9. *Let* $F : \mathcal{D} \longrightarrow \mathcal{E}$ *be a functor between pre-additive categories that preserves products.*

(i) *If* F *preserves kernels of split epimorphisms, then for every unit morphism* $u : R^\times \longrightarrow R$ *in* $\mathcal{D}$, *the morphism* $Fu : F(R^\times) \longrightarrow FR$ *is a unit morphism.*

(ii) *If* F *preserves cokernels of split monomorphisms, then for every group completion* $i : R \longrightarrow R^\star$ *in* $\mathcal{D}$, *the morphism* $Fu : FR \longrightarrow F(R^\star)$ *is a group completion.*

Now we consider a pointed model category $\mathcal{C}$ whose homotopy category is pre-additive. The main example we have in mind is $\mathcal{C} = umon$, the category of ultra-commutative monoids with the global model structure of Theorem 1.13. The homotopy category $\mathrm{Ho}(\mathcal{C})$ then comes with an adjoint functor pair (Σ, Ω) of suspension and loop, compare [**126**, I.2].

Proposition 5.10. *Let* $\mathcal{C}$ *be a pointed model category whose homotopy category is pre-additive.*

(i) *For every object* R *of* $\mathcal{C}$, *the loop object* ΩR *and the suspension* ΣR *are group-like in* $\mathrm{Ho}(\mathcal{C})$.

(ii) *If* $u : R^\times \longrightarrow R$ *is a unit morphism, then its loop* $\Omega u : \Omega(R^\times) \longrightarrow \Omega R$ *is an isomorphism in* $\mathrm{Ho}(\mathcal{C})$.

(iii) *If* $i : R \longrightarrow R^\star$ *is a group completion, then its suspension* $\Sigma i : \Sigma R \longrightarrow \Sigma(R^\star)$ *is an isomorphism in* $\mathrm{Ho}(\mathcal{C})$.

Proof. (i) This is a version of the Eckmann-Hilton argument. For every object T of C, the set $[T, \Omega R]$ has one abelian monoid structure via Construction 5.1, coming from the fact that $\mathrm{Ho}(\mathcal{C})$ is pre-additive. A second binary operation on the set $[T, \Omega R]$ arises from the fact that ΩR is a group object in $\mathrm{Ho}(\mathcal{C})$, compare [**126**, I.2]. This operation makes $[T, \Omega R]$ into a group. This monoid structure of Construction 5.1 in natural for morphisms in the second variable ΩR, in particular for the group structure morphism $\Omega R \times \Omega R \longrightarrow \Omega R$. This means that the two binary operations satisfy the interchange law. Since they also share the same neutral element, they coincide. Since one of the two operations has inverses, so does the other.

The argument that ΣR is group-like is dual, using that ΣR is the loop object of R in $\mathrm{Ho}(\mathcal{C})^{\mathrm{op}} = \mathrm{Ho}(\mathcal{C}^{\mathrm{op}})$, and that 'group-like' is a self-dual property,

(ii) Since the functor $\Omega : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ is right adjoint to Σ, it preserves products and kernels. So $\Omega u : \Omega(R^\times) \longrightarrow \Omega R$ is a unit morphism by Corollary 5.9. Since ΩR is already group-like by part (i), Ωu is an isomorphism. Part (iii) is dual to part (ii); so it admits the dual proof, or can be obtained by applying part (ii) to the opposite model category. □

Proposition 5.11. *Consider a commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

in a pointed model category such that the object C is weakly contractible.

(i) *If the square is homotopy cartesian and g admits a section in* $\mathrm{Ho}(\mathcal{C})$*, then the morphism f is a kernel of g in* $\mathrm{Ho}(\mathcal{C})$*.*

(ii) *If the square is homotopy cocartesian and f admits a retraction in* $\mathrm{Ho}(\mathcal{C})$*, then the morphism g is a cokernel of f in* $\mathrm{Ho}(\mathcal{C})$*.*

Proof. We prove part (i). Part (ii) can be proved by dualizing the argument or by applying part (i) to the opposite category with the opposite model structure. Since the square is homotopy cartesian and C is weakly contractible, the object A is weakly equivalent to the homotopy fiber, in the abstract model category sense, of the morphism g. As Quillen explains in Section I.3 of [**126**], there is an action map (up to homotopy)

$$A \times (\Omega D) \ \longrightarrow \ A \ ,$$

by an abstract version of 'concatenation of paths'. For every other object T of $\mathcal{C}$, Proposition 4 of [**126**, I.3] provides a sequence of based sets

$$[T, \Omega B] \xrightarrow{[T,\Omega g]} [T, \Omega D] \xrightarrow{[T,\partial]} [T, A] \xrightarrow{[T,f]} [T, B] \xrightarrow{[T,g]} [T, D]$$

that is exact in the sense explained in [**126**, I.3 Prop. 4], where $[-,-]$ denotes the morphism sets in the homotopy category of $\mathcal{C}$. In particular, the image of $[T, f]$ is equal to the preimage of the zero morphism under the map $[T, g]$.

So in order to show that f is a kernel of g it remains to check that the map $[T, f]$ is injective. So we consider two morphisms $\alpha_1, \alpha_2 \in [T, A]$ such that $f \circ \alpha_1 = f \circ \alpha_2$. Then by Proposition 4 (ii) of [**126**, I.3], there is an element $\lambda \in [A, \Omega D]$ such that $\alpha_2 = \alpha_1 \cdot \lambda$. Since the morphism $g : B \longrightarrow D$ has a section, so does the morphism $\Omega g : \Omega B \longrightarrow \Omega D$. So there is a morphism $\bar{\lambda} \in [T, \Omega B]$ such that $\lambda = (\Omega g) \circ \bar{\lambda}$. But all elements in the image of $[T, \Omega g]$ act trivially on $[T, A]$, so then

$$\alpha_2 \ = \ \alpha_1 \cdot \lambda \ = \ \alpha_1 \cdot ((\Omega g) \circ \bar{\lambda}) \ = \ \alpha_1 \ . \qquad \square$$

Theorem 5.12. *Let $\mathcal{C}$ be a pointed model category whose homotopy category is pre-additive.*

(i) *Every object of $\mathcal{C}$ has a unit morphism and a group completion in* $\mathrm{Ho}(\mathcal{C})$*.*

(ii) *If $\mathcal{C}$ is right proper, then every object R admits a $\mathcal{C}$-morphism $u : R^\times \longrightarrow R$ that becomes a unit morphism in the homotopy category* $\mathrm{Ho}(\mathcal{C})$.

(iii) *If $\mathcal{C}$ is left proper, then every object R admits a $\mathcal{C}$-morphism $i : R \longrightarrow R^\star$ that becomes a group completion in the homotopy category* $\mathrm{Ho}(\mathcal{C})$.

Proof. (i) We let R be any object of $\mathcal{C}$. It suffices to show, by Proposition 5.8, that the codiagonal morphism $\mathrm{Id} \perp \mathrm{Id} : R \times R \longrightarrow R$ has a kernel in $\mathrm{Ho}(\mathcal{C})$ and the diagonal morphism $(\mathrm{Id}, \mathrm{Id}) : R \longrightarrow R \times R$ has a cokernel in $\mathrm{Ho}(\mathcal{C})$. The arguments are again dual to each other, so we only show the first one.

We can assume without loss of generality that R is cofibrant and fibrant. Then the fold map $\nabla : R \amalg R \longrightarrow R$ in the model category $\mathcal{C}$ becomes the codiagonal morphism of R in $\mathrm{Ho}(\mathcal{C})$. We factor $\nabla = q \circ j$ for some weak equivalence $j : R \amalg R \xrightarrow{\sim} Q$ followed by a fibration $q : Q \longrightarrow R$. Then we choose a pullback, so that we arrive at the homotopy cartesian square:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & Q \arrow[d, "q"] \\ * \arrow[r] & R \end{tikzcd}$$

The morphism q still becomes a codiagonal morphism in $\mathrm{Ho}(\mathcal{C})$, and so it has a section. By Proposition 5.11 (i) the morphism f becomes a kernel of q in $\mathrm{Ho}(\mathcal{C})$. So the codiagonal morphism of R has a kernel.

(ii) We choose a weak equivalence $q : R \longrightarrow \bar{R}$ to a fibrant object. A unit morphism with target $\bar{R}$ exists in $\mathrm{Ho}(\mathcal{C})$ by Theorem 5.12. By replacing the source by a weakly equivalent object, if necessary, we can assume that the source is cofibrant as an object in the model category $\mathcal{C}$. Every morphism in $\mathrm{Ho}(\mathcal{C})$ from a cofibrant to a fibrant object is the image of some $\mathcal{C}$-morphism under the localization functor, i.e., there is a $\mathcal{C}$-morphism $\bar{u} : \bar{R}^\times \longrightarrow \bar{R}$ that becomes a unit morphism in $\mathcal{C}$. By factoring $\bar{u}$ as a weak equivalence followed by a fibration we can moreover assume without loss of generality that $\bar{u}$ is a fibration. We form a pullback

$$\begin{tikzcd} R^\times \arrow[r, "u"] \arrow[d, "p"'] & R \arrow[d, "q", "\sim"'] \\ \bar{R}^\times \arrow[r, "\bar{u}"'] & \bar{R} \end{tikzcd}$$

Since q is a weak equivalence, $\bar{u}$ a fibration and $\mathcal{C}$ is right proper, the base change p of q is also a weak equivalence. So u is isomorphic to $\bar{u}$ in the arrow category in $\mathrm{Ho}(\mathcal{C})$, hence u is also a unit morphism in $\mathrm{Ho}(\mathcal{C})$. Part (iii) is dual to part (ii). □

Remark 5.13. We claim that unit morphisms and group completions also behave nicely on derived mapping spaces. We explain this in detail for unit morphism, the other case being dual, one more time. Model categories have derived mapping spaces (i.e., simplicial sets) $R\,\mathrm{map}(-,-)$, giving well-defined homotopy types such that

$$\pi_0(R\,\mathrm{map}(R,S)) \;\cong\; \mathrm{Ho}(\mathcal{C})(R,S)\ , \tag{5.14}$$

compare [**79**, Sec. 5.4] or [**77**, Ch. 18]. We let $u : R^\times \longrightarrow R$ be a $\mathcal{C}$-morphism that becomes a unit morphism in $\mathrm{Ho}(\mathcal{C})$, and T any other object of $\mathcal{C}$. Because of the bijection (5.14) the map

$$u_* \;:\; R\,\mathrm{map}(T, R^\times) \;\longrightarrow\; R\,\mathrm{map}(T,R)$$

lands in the subspace $R\,\mathrm{map}^\times(T,R)$, defined as the union of those path components that represent invertible elements in the monoid $\mathrm{Ho}(\mathcal{C})(T,R)$. We claim that u_* is a weak equivalence onto the subspace $R\,\mathrm{map}^\times(T,R)$. To see this we exploit that both $R\,\mathrm{map}(T,R^\times)$ and $R\,\mathrm{map}^\times(T,R)$ are group-like H-spaces, the multiplication arising from the fact T is a cogroup object up to homotopy. Moreover, the map u_* is

an H-map and bijection on path components (by the universal property of unit morphisms and the bijection (5.14)). So it suffices to show that the restriction of u to the identity path components is a weak equivalence. For this it suffices in turn to show that the looped map

$$\Omega(u_*) \ : \ \Omega(R\,\mathrm{map}(T, R^\times)) \ \longrightarrow \ \Omega(R\,\mathrm{map}^\times(T, R))$$

is a weak equivalence. But this map is weakly equivalent to

$$(\Omega u)_* \ : \ R\,\mathrm{map}(T, \Omega(R^\times)) \ \longrightarrow \ R\,\mathrm{map}(T, \Omega(R)) \ .$$

Since Ωu is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ (by Proposition 5.10 (ii)) it is a weak equivalence in $\mathcal{C}$, hence so is the induced map on derived mapping spaces.

The next proposition will be used below to show that the loops on the bar construction provide functorial global group completions of ultra-commutative monoids.

Proposition 5.15. *Let $\mathcal{C}$ be a pointed model category whose homotopy category is pre-additive. Suppose that for every group-like object R of $\mathcal{C}$ the adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is an isomorphism in* $\mathrm{Ho}(\mathcal{C})$. *Then for every object R of $\mathcal{C}$ the adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is group completion.*

Proof. We let $i : R \longrightarrow R^\star$ be a group completion, which exists by Theorem 5.12. In the commutative square in $\mathrm{Ho}(\mathcal{C})$

$$\begin{tikzcd} R \arrow[r, "\eta_R"] \arrow[d, "i"'] & \Omega(\Sigma R) \arrow[d, "\Omega\Sigma i", "\cong"'] \\ R^\star \arrow[r, "\cong", "\eta_{R^\star}"'] & \Omega(\Sigma(R^\star)) \end{tikzcd}$$

the lower horizontal morphism is an isomorphism by hypothesis because $R^\star$ is group-like. The morphism $\Sigma i : \Sigma R \longrightarrow \Sigma(R^\star)$ is an isomorphism by Proposition 5.10 (iii), hence the right vertical morphism $\Omega(\Sigma i)$ is also an isomorphism. So η_R is isomorphic, as an object in the comma category $R \downarrow \mathrm{Ho}(\mathcal{C})$, to i, and hence also a group completion. □

Remark 5.16. The previous proposition also has a dual statement (with the dual proof): if for every group-like object R of $\mathcal{C}$ the adjunction counit $\epsilon : \Sigma(\Omega R) \longrightarrow R$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$, then ϵ is a unit morphism. In practice, however, this dual formulation is less useful. In other words: in the important examples that arise 'in nature', for example for ultra-commutative monoids, the adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is an isomorphism for all group-like objects R, whereas the adjunction counit $\epsilon : \Sigma(\Omega R) \longrightarrow R$ is *not* always an isomorphism.

Now we specialize the theory of units and group completions to ultra-commutative monoids. We recall that the category of ultra-commutative monoid has the trivial monoid $*$ as zero object, and the canonical morphism $\rho_{R,S} : R \boxtimes S \longrightarrow R \times S$ from the coproduct to the product of two ultra-commutative monoids is a global equivalence by Theorem I.3.38 (i). So the homotopy category $\mathrm{Ho}(umon)$ is pre-additive.

Definition 5.17. An ultra-commutative monoid R is *group-like* if it is group-like as an object of the pre-additive category $\mathrm{Ho}(umon)$. A morphism $u : R^\times \longrightarrow R$ of ultra-commutative monoids is a *global unit morphism* if it is a unit morphism in the pre-additive category $\mathrm{Ho}(umon)$. A morphism $i : R \longrightarrow R^\star$ of ultra-commutative monoids is a *global group completion* if it is a group completion in the pre-additive category $\mathrm{Ho}(umon)$.

Theorem 5.12 guarantees that every ultra-commutative monoid R is the target of a unit morphism and a source of a group completion in the homotopy category $\mathrm{Ho}(umon)$. As we explained in Remark 5.13, the abstract theory also guarantees that every fibrant ultra-commutative monoid R admits a global unit morphism $u : R^\times \longrightarrow R$ in the model category of ultra-commutative monoids. And every cofibrant

ultra-commutative monoid R admits a global group completion $i : R \longrightarrow R^\star$ in the model category of ultra-commutative monoids.

As a reality check we show that for ultra-commutative monoids R, the abstract definition of 'group-like' is equivalent to the requirement that all the abelian monoids $\pi_0^G(R)$ are groups. This part works more generally for all orthogonal monoid spaces, not necessarily commutative. A monoid M (not necessarily abelian) is a group if and only if the shearing map

$$\chi \ : \ M^2 \ \longrightarrow \ M^2 \ , \quad (x,y) \ \longmapsto \ (x,xy)$$

is bijective. Indeed, if M is a group, then the map $(x,z) \mapsto (x,x^{-1}z)$ is inverse to χ. Conversely, if the shearing map is bijective, then for every $x \in M$ there is a $y \in M$ such that $\chi(x,y) = (x,1)$, i.e., with $xy = 1$. Then $\chi(x,yx) = (x,xyx) = (x,x) = \chi(x,1)$, so $yx = 1$ by injectivity of χ. Thus y is a two-sided inverse for x.

For orthogonal monoid spaces R (not necessarily commutative), the group-like condition has a similar characterization as follows. The *shearing morphism* is the morphism of orthogonal spaces

$$\chi = (\rho_1, \mu) \ : \ R \boxtimes R \ \longrightarrow \ R \times R$$

whose first component is the projection ρ_1 to the first factor and whose second component is the multiplication morphism $\mu : R \boxtimes R \longrightarrow R$.

The multiplication morphism $\mu : R \boxtimes R \longrightarrow R$, and hence the shearing morphism χ, is a homomorphism of orthogonal monoid spaces only if R is commutative.

PROPOSITION 5.18. *Let R be an orthogonal monoid space. Then the following two conditions are equivalent:*

(i) *The shearing morphism $\chi : R \boxtimes R \longrightarrow R \times R$ is a global equivalence of orthogonal spaces.*
(ii) *For every compact Lie group G the monoid $\pi_0^G(R)$ is a group.*

For commutative orthogonal monoid spaces, conditions (i) *and* (ii) *are moreover are equivalent to being group-like in the pre-additive homotopy category* Ho(*umon*) *of ultra-commutative monoids.*

PROOF. (i)$\Longrightarrow$(ii) The vertical maps in the commutative diagram

$$\begin{tikzcd}
\pi_0^G(R \boxtimes R) \arrow[r, "\pi_0^G(\chi)"] \arrow[d, "(\pi_0^G(\rho_1),\pi_0^G(\rho_2))"', "\cong"] & \pi_0^G(R \times R) \arrow[d, "(\pi_0^G(p_1),\pi_0^G(p_2))", "\cong"'] \\
\pi_0^G(R) \times \pi_0^G(R) \arrow[r, "(x,y) \ \longmapsto \ (x,xy)"'] & \pi_0^G(R) \times \pi_0^G(R)
\end{tikzcd}$$

are bijective by Corollary I.5.27. If the shearing morphism is a global equivalence, then the map $\pi_0^G(\chi)$ is bijective, hence so is the algebraic shearing map of the monoid $\pi_0^G(R)$. This monoid is thus a group.

(ii)$\Longrightarrow$(i) Now we assume that all the monoids $\pi_0^G(R)$ are groups. We assume first that R is flat as an orthogonal space; then $R \boxtimes R$ is also flat by Proposition I.3.42 (i). Hence $R \boxtimes R$ is closed as an orthogonal space by Proposition I.3.12 (iii). Since R is flat, hence closed, the product $R \times R$ is also closed as an orthogonal space. Since both $R \boxtimes R$ and $R \times R$ are closed, we may show that for every compact Lie group G the continuous map

$$(\chi(\mathcal{U}_G))^G \ : \ ((R \boxtimes R)(\mathcal{U}_G))^G \ \longrightarrow \ ((R \times R)(\mathcal{U}_G))^G \ = \ R(\mathcal{U}_G)^G \times R(\mathcal{U}_G)^G$$

is a weak equivalence. Since $\pi_0^G(R)$, and hence also $\pi_0^G(R \boxtimes R) \cong \pi_0^G(R) \times \pi_0^G(R)$, has inverses, choices of points in the path components provide a homotopy equivalence

$$\pi_0^G(R \boxtimes R) \times ((R \boxtimes R)(\mathcal{U}_G))_1^G \ \longrightarrow \ ((R \boxtimes R)(\mathcal{U}_G))^G$$

where the subscript $(-)_1$ denotes the path component of the identity element. The same applies to $R \times R$ and yields a homotopy equivalence

$$\pi_0^G(R \times R) \times ((R \times R)(\mathcal{U}_G))_1^G \ \longrightarrow\ ((R \times R)(\mathcal{U}_G))^G \ ;$$

we choose the images of the path component representatives for $R \boxtimes R$ as the representatives for $R \times R$, so that the following diagram commutes:

$$\begin{tikzcd}
\pi_0^G(R \boxtimes R) \times ((R \boxtimes R)(\mathcal{U}_G))_1^G \arrow[r, "\pi_0^G(\chi) \times (\chi(\mathcal{U}_G))_1^G"] \arrow[d, "\simeq"'] & \pi_0^G(R \times R) \times ((R \times R)(\mathcal{U}_G))_1^G \arrow[d, "\simeq"] \\
((R \boxtimes R)(\mathcal{U}_G))^G \arrow[r, "(\chi(\mathcal{U}_G))^G"'] & ((R \times R)(\mathcal{U}_G))^G
\end{tikzcd}$$

Since the map $\pi_0^G(\chi)$ is bijective, it suffices to show that the restriction

$$(\chi(\mathcal{U}_G))_1^G \ :\ ((R \boxtimes R)(\mathcal{U}_G))_1^G \ \longrightarrow\ ((R \times R)(\mathcal{U}_G))_1^G$$

to the identity components is a weak equivalence. This is a map between path connected spaces, so we may show that the map $(\chi(\mathcal{U}_G))_1^G$ induces a bijection on all homotopy groups based at the neutral element 1. For every $n \geq 1$ the map $\pi_n((\chi(\mathcal{U}_G))_1^G)$ is a group homomorphism such that the composite

$$\begin{aligned}
(\pi_n(R(\mathcal{U}_G)_1^G, 1))^2 \ &\xrightarrow{(x,y) \mapsto x \times y} \pi_n(((R \boxtimes R)(\mathcal{U}_G))_1^G, 1) \\
&\xrightarrow{\pi_n((\chi(\mathcal{U}_G))_1^G)} \pi_n(((R \times R)(\mathcal{U}_G))_1^G, 1) \xrightarrow{((\rho_1)_*, (\rho_2)_*)} (\pi_n(R(\mathcal{U}_G)_1^G, 1))^2
\end{aligned}$$

sends (x, y) to $(x, \mu_*(x \times y))$, where $\mu : R \boxtimes R \longrightarrow R$ is the multiplication map. By the Eckmann-Hilton argument, $\mu_*(x \times y) = xy$, the product with respect to the group structure of $\pi_n(R(\mathcal{U}_G)_1^G, 1)$. The first and third map are bijective, and so is the composite (because $\pi_n(R(\mathcal{U}_G)_1^G, 1)$ is a group). So the middle map is bijective, and hence $(\chi(\mathcal{U}_G))_1^G$ is a weak equivalence.

For general R we choose a global equivalence $f : R' \longrightarrow R$ of orthogonal monoid spaces such that R' is flat as an orthogonal space. One way to arrange this is by cofibrant replacement in the global model structure of orthogonal monoid spaces (Corollary I.4.16 (ii) with $R = *$ and $\mathcal{F} = \mathcal{A}ll$). Then $f \boxtimes f$ is a global equivalence by Theorem I.3.38 and $f \times f$ is a global equivalence by Proposition I.1.9 (vi). Since $\chi^{R'}$ is a global equivalence by the previous paragraph and $\chi^R \circ (f \boxtimes f) = (f \times f) \circ \chi^{R'}$, the morphism χ^R is also a global equivalence.

Finally, if R is ultra-commutative, then the point-set level shearing morphism χ becomes the shearing morphism in the sense of Proposition 5.5 (a) in the pre-additive homotopy category $\mathrm{Ho}(umon)$. So χ is a global equivalence if and only if the shearing morphism in $\mathrm{Ho}(umon)$ is an isomorphism, i.e., precisely when R is group-like. $\square$

Now we look more closely at global unit morphisms, and we give an explicit, functorial point set level construction. For elements in an abelian monoid M, left inverses are automatically right inverses, and they are unique (if they exist). So the subgroup of invertible elements of an abelian monoid is isomorphic to the kernel of the multiplication map, by

$$M^\times \ \cong\ \ker(+ : M^2 \longrightarrow M)\ , \quad x \ \longmapsto\ (x, -x)\ .$$

Proposition 5.8 (i) gives an abstract formulation of this and explains how an abstract kernel of the multiplication map gives rise to a unit morphism. The proof of Theorem 5.12 then shows that the homotopy fiber of the multiplication maps, formed at the model category level, constructs such a kernel. If we make all this explicit for the model category of ultra-commutative monoids, we arrive at the following construction.

Construction 5.19 (Units of an ultra-commutative monoid). We introduce a functorial pointset level construction of the global units of an ultra-commutative monoid, as the homotopy fiber, over the additive unit element 0, of the multiplication morphism $\mu : R \boxtimes R \longrightarrow R$, i.e.,

$$R^\times \;=\; F(\mu) \;=\; (R \boxtimes R) \times_\mu R^{[0,1]} \times_R * \;.$$

So at an inner product space V, we have

$$(R^\times)(V) \;=\; F(\mu(V)) \;=\; (R \boxtimes R)(V) \times_{\mu(V)} R(V)^{[0,1]} \times_{R(V)} \{0\} \;,$$

the space of pairs (x, ω) consisting of a point $x \in (R \boxtimes R)(V)$ and a path $\omega : [0,1] \longrightarrow R(V)$ such that $\mu(x) = \omega(0)$ and $\omega(1) = 0$, the unit element in $R(V)$. Since limits and cotensors with topological spaces of ultra-commutative monoids are formed on underlying orthogonal spaces, this homotopy fiber inherits a preferred structure of ultra-commutative monoid.

We claim that the composite

$$u \;:\; R^\times \;\xrightarrow{p}\; R \boxtimes R \;\xrightarrow{\rho_1}\; R$$

is a global unit morphism, where p is the projection onto the first factor. Indeed, the commutative square

$$\begin{array}{ccc} R^\times & \xrightarrow{\;p\;} & R \boxtimes R \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ R^{[0,1]} \times_R \{0\} & \xrightarrow[\mathrm{ev}_0]{} & R \end{array}$$

is a pullback of ultra-commutative monoids, by definition, and both horizontal morphisms are strong level fibrations, where q denotes the projection to the second factor. So the square is homotopy cartesian. The multiplication morphism μ has a section, so Proposition 5.11 (i) shows that the morphism $p : R^\times \longrightarrow R \boxtimes R$ becomes a kernel of the multiplication morphism in the homotopy category $\mathrm{Ho}(umon)$. So u is a unit morphism by Proposition 5.8 (i).

We recall from Example 2.18 that every global power monoid M has a global power submonoid $M^\times$ of *units*; the value $M^\times(G)$ at a compact Lie group G is the monoid of invertible elements of $M(G)$. The next proposition verifies that global unit morphisms have the expected behavior on $\underline{\pi}_0$.

Proposition 5.20. *Let $u : R^\times \longrightarrow R$ be a unit morphism of ultra-commutative monoids. Then the morphism of global power monoids*

$$\underline{\pi}_0(u) \;:\; \underline{\pi}_0(R^\times) \;\longrightarrow\; \underline{\pi}_0(R)$$

is an isomorphism onto the global power submonoid $(\underline{\pi}_0(R))^\times$ of units of $\underline{\pi}_0(R)$.

Proof. This is a formal consequence of the fact that the functor π_0^G from the homotopy category of ultra-commutative monoids is representable. We let G be a compact Lie group and consider the free ultra-commutative monoid $\mathbb{P}(B_{\mathrm{gl}}G)$ generated by a global classifying space of G. By Proposition 2.24 (i), applied to the free and forgetful functor pair between orthogonal spaces and ultra-commutative monoids, for every ultra-commutative monoid T, evaluation at the class

$$u_G^{umon} \;=\; \eta_*(u_{G,V}) \;\in \pi_0^G(\mathbb{P}(B_{\mathrm{gl}}G))$$

is an isomorphism of abelian monoids

$$\mathrm{Ho}(umon)(\mathbb{P}(B_{\mathrm{gl}}G), T) \;\cong\; \pi_0^G(T) \;.$$

This isomorphism is natural in the second variable, so we arrive at a commutative square of abelian monoids

$$\begin{array}{ccc} \mathrm{Ho}(umon)(\mathbb{P}(B_{\mathrm{gl}}G), R^\times) & \xrightarrow{\cong} & \pi_0^G(R^\times) \\ \downarrow{\scriptstyle u_*} & & \downarrow{\scriptstyle u_*} \\ \mathrm{Ho}(umon)(\mathbb{P}(B_{\mathrm{gl}}G), R) & \xrightarrow[\cong]{} & \pi_0^G(R) \end{array}$$

in which both horizontal maps are bijective. The left vertical map is injective with image the subgroup of invertible elements; hence the same is true for the right vertical map. □

Remark 5.21. The previous proposition shows in particular that the image of every global unit morphism $u : R^\times \longrightarrow R$ of ultra-commutative monoids is contained in the ultra-commutative submonoid $R^{n\times}$ of naive units as defined in Example 2.17; so u factors as a composite

$$R^\times \xrightarrow{u'} R^{n\times} \xrightarrow{\mathrm{incl}} R\ .$$

Here $R^{n\times}$ is an orthogonal monoid subspace of R with value at an inner product space V defined as the union of those path components of $R(V)$ that become invertible in $\pi_0(R)$. As we explained in Example 2.17, $R^{n\times}$ is *not* in general group-like; in particular, the inclusion $R^{n\times} \longrightarrow R$ is *not* in general a global unit morphism. The morphism $u' : R^\times \longrightarrow R^{n\times}$ is a non-equivariant weak equivalence, but *not* generally a global equivalence.

Example 5.22. The ultra-commutative monoid $\mathbf{Gr}$, the additive Grassmannian of Example 3.11, has 'trivial units' in the sense that the unit morphism $* \longrightarrow \mathbf{Gr}$ is a global unit morphism. Indeed, the monoid $\pi_0^e(\mathbf{Gr})$ is isomorphic to the additive monoid $\mathbb{N}$ of natural numbers, so 0 is the only invertible element. Thus the naive unit $\mathbf{Gr}^{n\times}$ coincide with the 0-graded part of the Grassmannian, i.e., $\mathbf{Gr}^{n\times} = \mathbf{Gr}^{[0]} = *$. Since the 'true' global units factor through $\mathbf{Gr}^{n\times}$, these must also be globally equivalent to the trivial ultra-commutative monoid.

The same reasoning applies to the ultra-commutative monoid $\mathbf{F}$ of unordered frames (see Example 3.23) and $\mathbb{P}X$, the free ultra-commutative monoid generated by an orthogonal space X (see Example 1.7). So both the naive unit and the 'true' global units are trivial in these cases,

$$\mathbf{F}^\times \simeq_{\mathrm{gl}} \mathbf{F}^{n\times} = * \qquad \text{and} \qquad (\mathbb{P}X)^\times \simeq_{\mathrm{gl}} (\mathbb{P}X)^{n\times} = *\ .$$

Now we turn to global group completions of ultra-commutative monoids. Again we perform a reality check, showing that global group completions have the expected effect on equivariant homotopy sets.

Proposition 5.23. *Let $i : R \longrightarrow R^\star$ be a global group completion of ultra-commutative monoids. Then*

$$\underline{\pi}_0(i) \ : \ \underline{\pi}_0(R) \ \longrightarrow \ \underline{\pi}_0(R^\star)$$

is a group completion of global power monoids, and for every compact Lie group G the map

$$\pi_0^G(i) \ : \ \pi_0^G(R) \ \longrightarrow \ \pi_0^G(R^\star)$$

is an algebraic group completion.

Proof. The homotopy category of ultra-commutative monoids and the category of global power monoids are pre-additive and the functor

$$\underline{\pi}_0 \ : \ \mathrm{Ho}(umon) \ \longrightarrow \ (\text{global power monoids})$$

has a right adjoint by Proposition 2.38, sending a global power monoid M to a globally discrete ultra-commutative monoid that realizes M on $\underline{\pi}_0$. So the functor $\underline{\pi}_0$ preserves products and cokernels. Corollary 5.9 (ii) then applies and shows that $\underline{\pi}_0(i) : \underline{\pi}_0(R) \longrightarrow \underline{\pi}_0(R^\star)$ is a group completion in the pre-additive category of global power monoids. The second claim then follows because group completions of global power monoids are calculated 'group-wise', compare Example 2.19. □

Every ultra-commutative monoid has a group completion in the homotopy category of ultra-commutative monoids, by Theorem 5.12 (i). Even better: since the model category of ultra-commutative monoids is left proper, every ultra-commutative monoid is the source of a global group completion in the model category of ultra-commutative monoids, by Theorem 5.12 (iii). Now we discuss two functorial pointset level constructions of global group completions. The first construction is dual to Construction 5.19 of global units as the homotopy fiber of the multiplication map.

Construction 5.24 (Global group completion of an ultra-commutative monoid). We let R be an ultra-commutative monoid that is cofibrant in the global model structure of Theorem 1.13. We define the cone of R is a pushout in the category of ultra-commutative monoids:

$$\begin{tikzcd} \{0\} \otimes R \arrow[r] \arrow[d, "\mathrm{incl}\otimes R"'] & * \arrow[d] \\ {[0,1]} \otimes R \arrow[r, "q"'] & CR \end{tikzcd}$$

Since R is cofibrant, the left vertical morphism is any acyclic cofibration, and so the cone CR is globally weakly equivalent to the zero monoid. Then we can construct a global group completion as a homotopy cofiber of the diagonal morphism $\Delta : R \longrightarrow R \times R$, i.e., a pushout in the category of ultra-commutative monoids:

$$\begin{tikzcd} R \arrow[r, "\Delta"] \arrow[d] & R \times R \arrow[d, "d"] \\ CR \arrow[r] & R^\star \end{tikzcd}$$

The left vertical map is the composite

$$R \;\cong\; \{1\} \otimes R \;\xrightarrow{\mathrm{incl}\otimes R}\; [0,1] \otimes R \;\xrightarrow{q}\; CR\;;$$

it is a cofibration since R is cofibrant.

We claim that the composite

$$u \;:\; R \;\xrightarrow{(\mathrm{Id},0)}\; R \times R \;\xrightarrow{d}\; R^\star$$

is a global group completion. Indeed, the square above is homotopy cocartesian by construction and the diagonal morphism has a retraction. So Proposition 5.11 (ii) shows that the morphism $d : R \times R \longrightarrow R^\star$ becomes a cokernel of the diagonal morphism in the homotopy category $\mathrm{Ho}(umon)$. So u is a global group completion by Proposition 5.8 (ii).

The homotopy pushout above defining $R^\star$ can be obtained more concretely as the realization of a simplicial ultra-commutative monoid, the two-sided bar construction $B^{\boxtimes}(*, R, R\times R)$ with respect to the box product. In the context of topological monoids, this bar construction of a group completion for 'sufficiently homotopy commutative' monoids is sketched by Segal on p. 305 of [**140**].

For topological monoids, the loop space of the bar construction (see Construction 3.20) provides a functorial group completion. We will now explain that a similar construction also provides global group completion for ultra-commutative monoids, before passing to the homotopy category. Part of this works for arbitrary orthogonal monoid spaces, not necessarily commutative. If R is such an orthogonal monoid space, then the *bar construction* is the simplicial object of orthogonal spaces

$$B_\bullet(R) \;=\; \Big([n] \mapsto\; R^{\boxtimes n}\Big)\;.$$

The simplicial face morphisms are induced by the multiplication in R, and the degeneracy morphisms are induced by the unit morphism of R, much like for the bar construction with respect to cartesian product

(as opposed to box product) in Construction 3.20. The geometric realization in the category of orthogonal spaces is then the orthogonal space

$$B(R) \;=\; |B_\bullet(R)| \;. \tag{5.25}$$

Geometric realization of orthogonal spaces is 'objectwise', i.e., for an inner product space V we have

$$B(R)(V) \;=\; |B_\bullet(R)(V)| \;,$$

the realization of the simplicial space $[n] \mapsto (R^{\boxtimes n})(V)$. The canonical morphism

$$\Delta^1 \times R \;=\; \Delta^1 \times B_1(R) \;\longrightarrow\; |B_\bullet(R)| \;=\; B(R)$$

takes $\partial\Delta^1 \times R$ to the basepoint, so it factors over a morphism of orthogonal spaces

$$S^1 \wedge R \;\cong\; (\Delta^1/\partial\Delta^1) \wedge R \;\longrightarrow\; B(R) \;.$$

Adjoint to this is a morphism of orthogonal spaces

$$\eta_R \;:\; R \;\longrightarrow\; \Omega B(R) \;. \tag{5.26}$$

Proposition 5.27. *Let R be an orthogonal monoid space that is closed as an orthogonal space. If for every compact Lie group G the monoid $\pi_0^G(R)$ is a group, then the morphism $\eta_R : R \longrightarrow \Omega B(R)$ is a global equivalence.*

Proof. If R is closed, then so is $\Omega B(R)$ [...fix this...], so we can detect global equivalences on G-fixed points. So we let G be any compact Lie group. Then

$$((\Omega B(R))(\mathcal{U}_G))^G \;\cong\; \Omega\left((B(R)(\mathcal{U}_G))^G\right) \;\cong\; \Omega(|[n] \mapsto (R^{\boxtimes n}(\mathcal{U}_G))^G|) \;.$$

We define $i_k : [1] \longrightarrow [n]$ by $i_k(0) = k-1$ and $j_k(1) = k$. Then the morphism

$$(i_1^*, \dots, i_n^*) \;:\; R^{\boxtimes n} \;=\; B_n(R) \;\longrightarrow\; (B_1(R))^n \;=\; R^n$$

is precisely the morphism $\rho_{R,\dots,R} : R^{\boxtimes n} \longrightarrow R^n$, and hence a global equivalence by Theorem I.3.38 (i). Since $\pi_0^G(R)$ is a group, the simplicial space

$$[n] \;\mapsto (R^{\boxtimes n}(\mathcal{U}_G))^G$$

satisfies the hypotheses of Segal's theorem [**140**, Prop. 1.5]; so the adjoint of the canonical map

$$S^1 \wedge R(\mathcal{U}_G)^G \;\longrightarrow\; |[n] \mapsto ((R^{\boxtimes n})(\mathcal{U}_G))^G|$$

is a weak equivalence. This adjoint is precisely the underlying map of G-fixed points of the morphism $\eta_R : R \longrightarrow \Omega B(R)$. □

The previous proposition works for general orthogonal monoid spaces, not necessarily commutative; in that generality the bar construction $B(R)$ is an orthogonal space, but it does not have any natural multiplication. When we apply the bar construction to ultra-commutative monoids, then something special happens: since the multiplication morphism $\mu : R \boxtimes R \longrightarrow R$ is then a homomorphism of ultra-commutative monoids, the simplicial object $B_\bullet(R)$ is a simplicial object in the category of ultra-commutative monoids, i.e., a simplicial ultra-commutative monoid.

For a simplicial ultra-commutative monoid $B_\bullet$, the term 'geometric realization' actually has two potentially different interpretations, and we take some time to clarify this issue. On the one hand we can form the geometric realization $|B_\bullet|_{\mathrm{un}}$ in the underlying category of orthogonal spaces; this is, by definition, a coend, in the category of orthogonal spaces, of the functor

$$\Delta \times \Delta^{\mathrm{op}} \;\longrightarrow\; spc \;, \quad ([m],[n]) \;\longmapsto\; \Delta^m \times B_n \;.$$

We call this the *underlying realization* of $B_\bullet$. Coends of orthogonal spaces are calculated objectwise, so $|B_\bullet|_{\mathrm{un}}(V)$ is a realization of the simplicial space $[n] \mapsto B_n(V)$. It is not a priori obvious, however, whether this realization inherits any multiplication.

On the other hand, the category of ultra-commutative monoids is a topological model category by Theorem 1.13. In particular, ultra-commutative monoids are enriched, tensored and cotensored over topological spaces. We shall write $A\Box R$ for the tensor of a space A with an ultra-commutative monoid R, in order to distinguish it from the (objectwise) product of A with the underlying orthogonal space of R. We can also consider the realization $|B_\bullet|_{\mathrm{in}}$ *internal to ultra-commutative monoids*, i.e., a coend, in the category of ultra-commutative monoids, of the functor

$$\Delta \times \Delta^{\mathrm{op}} \longrightarrow umon\ , \quad ([m],[n]) \longmapsto \Delta^m \Box B_n\ .$$

We call this the *internal realization*. The internal realization is, by definition, an ultra-commutative monoid, but it is not immediately clear how it relates to the underlying realization of $|B_\bullet|_{\mathrm{un}}$ as an orthogonal space. As we shall now show, the forgetful functor from a category of ultra-commutative monoids to orthogonal spaces commutes with realization of simplicial objects. We do not claim any originality here, and many results of this kind can be found in the literature, see for example [**108**, Thm. 12.2], [**49**, VII Prop. 3.3], [**111**, Prop. 4.5], [**106**, Prop. 12.4] or [**59**, Thm. 2.2].

PROPOSITION 5.28. *Let $B_\bullet$ be a simplicial object in the category of ultra-commutative monoids. Then the canonical map*

$$|B_\bullet|_{\mathrm{un}} \longrightarrow |B_\bullet|_{\mathrm{in}}$$

from the underlying realization to the internal realization is an isomorphism of orthogonal spaces.

PROOF. We adapt an argument given by Mandell in an unpublished preprint [**106**, Prop. 12.4]. We start by considering two simplicial orthogonal spaces $X_\bullet, Y_\bullet : \Delta^{\mathrm{op}} \longrightarrow spc$. We denote by $X_\bullet \boxtimes Y_\bullet$ the diagonal of the external box product, i.e., the composite simplicial orthogonal space

$$\Delta^{\mathrm{op}} \xrightarrow{\text{diagonal}} \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{X_\bullet \times Y_\bullet} spc \times spc \xrightarrow{\boxtimes} spc\ .$$

For every $n \geq 0$ we consider the composite

$$\Delta_n \times (X_n \boxtimes Y_n) \xrightarrow{\text{diag}\times\text{Id}} (\Delta_n \times \Delta^n) \times (X_n \boxtimes Y_n) \xrightarrow{\text{shuffle}} (\Delta_n \times X_n) \boxtimes (\Delta_n \times Y_n) \longrightarrow |X_\bullet| \boxtimes |Y_\bullet|\ ,$$

where the last morphism is the box product of the two canonical morphisms $\Delta_n \times X_n \longrightarrow |X_\bullet|$ and $\Delta_n \times Y_n \longrightarrow |Y_\bullet|$. These composites are compatible with the coend relations, so they assemble into a morphism of orthogonal spaces

$$|X_\bullet \boxtimes Y_\bullet| \longrightarrow |X_\bullet| \boxtimes |Y_\bullet|\ .$$

We claim that this morphism is an isomorphism. Indeed, since $\boxtimes$ preserves colimits in each variable, the right hand side is a coend of the functor

$$\Delta^2 \times (\Delta^2)^{\mathrm{op}} \longrightarrow spc\ , \quad ([k],[l],[m],[n]) \longmapsto \Delta^k \times \Delta^l \times (X_m \boxtimes Y_n)\ .$$

Implicitly, we have also used the interchange homeomorphism

$$(\Delta^k \times X_m) \boxtimes (\Delta^l \times Y_n) \cong \Delta^k \times \Delta^l \times (X_m \boxtimes Y_n)\ .$$

Coends of orthogonal spaces are calculated objectwise, and for bisimplicial spaces the bi-realization is homeomorphic to the realization of the diagonal (see for example [**127**, p. 94, Lemma]).

By iterating, we obtain a Σ_m-equivariant isomorphism of orthogonal spaces

$$|X_\bullet^{\boxtimes m}| \cong |X_\bullet|^{\boxtimes m}$$

for every $m \geq 0$. Since coends commute with colimits, we can pass to Σ_m-orbits and take coproduct over $m \geq 0$, and obtain an isomorphism of orthogonal spaces

$$|\mathbb{P}(X_\bullet)|_{\mathrm{un}} = |\amalg_{m\geq 0} (X_\bullet^{\boxtimes m})/\Sigma_m| \cong \amalg_{m\geq 0} |(X_\bullet^{\boxtimes m})/\Sigma_m| = \mathbb{P}|X_\bullet|\ . \tag{5.29}$$

On the other hand, the ultra-commutative monoid $\mathbb{P}|X_\bullet|$ has the universal property of the internal realization of the simplicial ultra-commutative monoid $\mathbb{P}(X_\bullet)$. This shows the claim in the special case where $B_\bullet$ is freely generated by a simplicial orthogonal space.

Now we treat the general case. The diagram

$$\mathbb{P}(\mathbb{P}B_\bullet) \underset{\mu}{\overset{\mathbb{P}\alpha}{\rightrightarrows}} \mathbb{P}B_\bullet \xrightarrow{\alpha} B_\bullet$$

is a coequalizer diagram of simplicial ultra-commutative monoids. Here $\alpha : \mathbb{P}R \longrightarrow R$ is the adjunction counit and μ is the monad structure of the free functor. Moreover, the coequalizer is split in the underlying category of orthogonal spaces, by the morphisms

$$\mathbb{P}(\mathbb{P}B_\bullet) \xleftarrow{\eta_{\mathbb{P}B_\bullet}} \mathbb{P}B_\bullet \xleftarrow{\eta_{B_\bullet}} B_\bullet$$

where $\eta : R \longrightarrow \mathbb{P}R$ is the unit of the free-forget adjunction, i.e., the inclusion as the 'linear' summand $\mathbb{P}^1 R$.

Applying the two functors under consideration gives a commutative diagram of orthogonal spaces

$$\begin{tikzcd} |\mathbb{P}(\mathbb{P}B_\bullet)|_{\mathrm{un}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & |\mathbb{P}B_\bullet|_{\mathrm{un}} \arrow[r] \arrow[d] & |B_\bullet|_{\mathrm{un}} \arrow[d] \\ |\mathbb{P}(\mathbb{P}B_\bullet)|_{\mathrm{in}} \arrow[r, shift left] \arrow[r, shift right] & |\mathbb{P}B_\bullet|_{\mathrm{in}} \arrow[r] & |B_\bullet|_{\mathrm{in}} \end{tikzcd} \tag{5.30}$$

We claim that both rows are coequalizer diagrams of orthogonal spaces. For the upper row we argue as follows. For every $n \geq 0$ the diagram

$$\mathbb{P}(\mathbb{P}B_n) \rightrightarrows \mathbb{P}B_n \longrightarrow B_n$$

is a coequalizer in the category of ultra-commutative monoids. Since the diagram splits in the underlying category of orthogonal spaces, it is also a coequalizers diagram there, compare [**103**, IV.6, Lemma]. So the diagram

$$\mathbb{P}(\mathbb{P}B_\bullet) \rightrightarrows \mathbb{P}B_\bullet \longrightarrow B_\bullet$$

is also a coequalizer diagram of simplicial orthogonal spaces. Since product with Δ^n and coends commute with colimits in the category of orthogonal spaces, the diagram stays a coequalizer after (underlying) geometric realization. For the bottom row of (5.30) we use that coends commute with all colimits, so the bottom row is a coequalizer diagram of ultra-commutative monoids. Again the diagram splits in the underlying category of orthogonal spaces, so the lower diagram is also a coequalizer diagram of orthogonal spaces. Since the two left vertical morphisms in the commutative diagram (5.30) are isomorphisms of orthogonal spaces by the special case above, this proves that the morphism $|B_\bullet|_{\mathrm{un}} \longrightarrow |B_\bullet|_{\mathrm{in}}$ is an isomorphism as well. $\square$

Since ultra-commutative monoids are a pointed category (the constant orthogonal monoid space is a zero object), the enrichment, tensors and cotensors over space extend to enrichment, tensors and cotensors over the category of *based* topological spaces. We shall write $A \lhd R$ for the tensor of a based space A with an ultra-commutative monoid R, in order to distinguish it from the (objectwise) smash product of A with the underlying based orthogonal space of R. Thus $A \lhd R$ is a pushout, in the category of ultra-commutative monoids, of the diagram

$$* \longleftarrow \{a_0\}\Box R \xrightarrow{\mathrm{incl}\Box R} A\Box R \ ,$$

where a_0 is the basepoint. As may be familiar from similar contexts, the bar construction $B(R)$ of an ultra-commutative monoid R can be interpreted as $S^1 \lhd R$, the tensor of R with the based space S^1. Another way to say this is that the bar construction is the internal suspension, in the category of an ultra-commutative monoids. We show a more general statement and consider a based simplicial set A. We define a simplicial object of ultra-commutative monoids by

$$B_n(A, R) \ = \ A_n \lhd R \ ,$$

with simplicial structure induced by that of A. Since A_n is a based set, $A_n \lhd R$ is in fact a box product of copies of R, indexed by the non-basepoint elements of A.

The next proposition constructs an isomorphism of ultra-commutative monoids between $|A| \lhd R$ and $|B_\bullet(A,R)|_{\mathrm{in}}$, the internal geometric realization. By Proposition 5.28, we can (and will) confuse the internal realization with the underlying realization of $B_\bullet(A,R)$ in the category of orthogonal spaces.

PROPOSITION 5.31. *Let A be a based simplicial set and R an ultra-commutative monoid. Then $|A| \lhd R$ is an internal realization of the simplicial ultra-commutative monoid $B_\bullet(A,R)$. Moreover, there is an isomorphism of ultra-commutative monoids*

$$S^1 \lhd R \;\cong\; |\Delta^1/\partial\Delta^1| \lhd R \cong\; B(R)$$

whose adjoint $R \longrightarrow \Omega B(R)$ is the morphism η_R of (5.26).

PROOF. The geometric realization $|A|$ is a coend of the functor

$$\Delta \times \Delta^{\mathrm{op}} \;\longrightarrow\; \mathbf{T}_* \;, \quad ([m],[n]) \;\longmapsto\; \Delta^m_+ \wedge A_n \;.$$

Since the functor $-\lhd R$ preserves colimits, $|A|\lhd R$ is a coend, in the category of ultra-commutative monoids, of the functor

$$\Delta \times \Delta^{\mathrm{op}} \;\longrightarrow\; umon \;, \quad ([m],[n]) \;\longmapsto\; (\Delta^m_+ \wedge A_n) \lhd R \;.$$

This isomorphisms

$$(\Delta^m_+ \wedge A_n) \lhd R \;\cong\; \Delta^m_+ \lhd (A_n \lhd R) \;\cong\; \Delta^m \Box (A_n \lhd R) \;=\; \Delta^m \Box B_n(A,R) \;,$$

natural in $([m],[n]) \in \Delta \times \Delta^{\mathrm{op}}$, then show that $|A| \lhd R$ is an internal realization of the simplicial ultra-commutative monoid $B_\bullet(A,R)$.

The second claim is the special case $A = \Delta^1/\partial\Delta^1$ of the simplicial circle, plus an isomorphism of simplicial ultra-commutative monoids

$$p_\bullet \;:\; B_\bullet(R) \;\cong\; B_\bullet(\Delta^1/\partial\Delta^1, R) \;.$$

For the latter we enumerate the n-simplices of Δ^1 as $\kappa_0, \dots, \kappa_{n+1}$, where $\kappa_j : [n] \longrightarrow [1]$ is given by

$$\kappa_j(k) \;=\; \begin{cases} 0 & \text{for } 0 \le k < j, \text{ and} \\ 1 & \text{for } j \le k \le n. \end{cases}$$

Then κ_0 and κ_{n+1} become equal in $\Delta^1/\partial\Delta^1$, and $\kappa_1, \dots, \kappa_n$ represent the non-basepoint n-simplices of $\Delta^1/\partial\Delta^1$. An isomorphism

$$p_n \;:\; R^{\boxtimes n} \;\xrightarrow{\cong}\; (\Delta^1/\partial\Delta^1)_n \lhd R \;=\; B_n(\Delta^1/\partial\Delta^1, R)$$

is then given by multiplying the morphisms

$$R \cong \{k_j\}_+ \lhd R \;\longrightarrow\; (\Delta^1/\partial\Delta^1)_n \lhd R$$

induced by the inclusion $\{\kappa_j\} \longrightarrow (\Delta^1/\partial\Delta^1)_n$, for $j = 1, \dots, n$. The simplices κ_j satisfy

$$d_i(\kappa_j) \;=\; \begin{cases} \kappa_{j-1} & \text{for } i < j, \text{ and} \\ \kappa_j & \text{for } i \ge j, \end{cases} \qquad \text{and} \qquad s_i(\kappa_j) \;=\; \begin{cases} \kappa_{j+1} & \text{for } i < j, \text{ and} \\ \kappa_j & \text{for } i \ge j. \end{cases}$$

So the isomorphisms p_n are compatible with the simplicial structure maps on both sides, i.e., they define an isomorphism of simplicial ultra-commutative monoids $p_\bullet$. □

COROLLARY 5.32. *Let R be an ultra-commutative monoid R for which the unit morphism $* \longrightarrow R$ is a flat cofibration of orthogonal spaces. Then the adjunction unit*

$$\eta_R \;:\; R \;\longrightarrow\; \Omega(S^1 \lhd R)$$

is a global group completion.

Proof. We let R be a group-like cofibrant Since ultra-commutative monoids form a topological model category, $S^1 \lhd R$ is an abstract suspension of R. Because the isomorphism $B(R) \cong S^1 \lhd R$ of the Proposition 5.31 transforms the morphism $\eta_R : R \longrightarrow \Omega B(R)$ into the adjunction unit $R \longrightarrow \Omega(S^1 \lhd R)$, Proposition 5.27 shows that this adjunction unit is a global equivalence for every cofibrant group-like ultra-commutative monoid R. In the homotopy category $\mathrm{Ho}(umon)$ this implies that for every group-like ultra-commutative monoid R the derived adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is an isomorphism. So Proposition 5.15 shows that for every ultra-commutative monoid R the derived adjunction unit $\eta : R \longrightarrow \Omega(\Sigma R)$ is group completion in the pre-additive category $\mathrm{Ho}(umon)$. For cofibrant R the pointset level adjunction unit $R \longrightarrow \Omega(S^1 \lhd R)$ realizes the derived unit, hence the claim follows for every ultra-commutative monoid that is cofibrant in the global model structure of Theorem 1.13.

Now we consider a global equivalence of ultra-commutative monoids $f : T \longrightarrow R$ such that both unit morphisms $* \longrightarrow T$ and $* \longrightarrow R$ are flat cofibrations of orthogonal spaces. We show that then the morphism $S^1 \lhd f : S^1 \lhd T \longrightarrow S^1 \lhd R$ is a global equivalence. [...]

Now we can prove the general case. We let R be an ultra-commutative monoid for which the unit morphism $* \longrightarrow R$ is a flat cofibration of orthogonal spaces. We choose a cofibrant replacement $q : R^c \longrightarrow R$ in the global model structure of Theorem 1.13, i.e., a global equivalence of ultra-commutative monoids with cofibrant source. Since R^c is cofibrant, the unit morphism $* \longrightarrow R^c$ is a flat cofibration of orthogonal spaces by Theorem 1.13 (ii). The morphism $S^1 \lhd q : S^1 \lhd R^c \longrightarrow S^1 \lhd R$ is then a global equivalence by the previous paragraph, hence so is the morphism $\Omega(S^1 \lhd q) : \Omega(S^1 \lhd R^c) \longrightarrow \Omega(S^1 \lhd R)$. The morphism $\eta_{R^c} : R^c \longrightarrow \Omega(S^1 \lhd R^c)$ is a global group completion by the first paragraph. Since $\eta_R : R \longrightarrow \Omega(S^1 \lhd R)$ is isomorphic to the morphism η_{R^c} in the homotopy category $\mathrm{Ho}(umon)$, the morphism η_R is also a global group completion. [... can we weaken 'flat cofibration' to 'h-cofibration'?...] □

Remark 5.33. We let M be a monoid valued orthogonal space in the sense of Definition 3.2. Then we have two bar constructions available: on the one hand we can take the bar construction (Construction II.3.20) objectwise as in Example 4.19, resulting in the orthogonal space $\mathbf{B}^\circ M$. On the other hand, we can first pass to the associated orthogonal monoid space as in (3.3), and then perform the bar construction with respect to the $\boxtimes$-multiplication as in (5.25). There is a natural comparison map: the symmetric monoidal transformation $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ defined in (3.37) of Chapter I has an analog for any finite number of factors, and the morphisms

$$\rho_{M,\dots,M} \ : \ M^{\boxtimes n} \ \longrightarrow \ M^n$$

fit together into a morphism of simplicial orthogonal spaces from the $\boxtimes$-bar construction to the $\times$-bar construction. Geometric realization yields a morphism of orthogonal spaces

$$\rho \ : \ B(M) \ \longrightarrow \ \mathbf{B}^\circ M \ .$$

The transformation $\rho_{X,Y}$ is *symmetric* monoidal; whenever M is a *symmetric* monoid valued orthogonal space, source and target of ρ inherit ultra-commutative multiplications, and ρ is even a morphism of orthogonal monoid spaces. If the unit morphism $* \longrightarrow M$ is a flat cofibration of orthogonal spaces, then the morphism $\rho : B(M) \longrightarrow \mathbf{B}^\circ M$ is a global equivalence of orthogonal space. We refrain from proving this since we have no interesting application.

We close this section with an example of a global group completion that comes up naturally, namely the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ introduced in Example 4.3. The verification of the group completion property will be done by checking a homological criterion that we develop first. For the purpose of the following proposition, we define the homology groups of an orthogonal space Y as

$$H_*(Y^G;\mathbb{Z}) \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} \ H_*(Y(V)^G;\mathbb{Z}) \ .$$

Every global equivalence induces isomorphisms on $H_*((-)^G;\mathbb{Z})$ for all compact Lie groups G. Indeed, the functor $H_*((-)^G;\mathbb{Z})$ takes strong level equivalences to isomorphisms, which reduces the claim (by

cofibrant approximation in the strong level model structure) to global equivalences $f : X \longrightarrow Y$ between flat orthogonal spaces. Flat orthogonal spaces are closed, so the global equivalence induces weak equivalences $f(\mathcal{U}_G)^G : X(\mathcal{U}_G)^G \longrightarrow Y(\mathcal{U}_G)^G$ on G-fixed points. The poset $s(\mathcal{U}_G)$ is filtered, so homology then commutes with this colimit, i.e.,

$$H_*(Y^G;\mathbb{Z}) \;\cong\; H_*(Y(\mathcal{U}_G)^G;\mathbb{Z})\ .$$

Thus the morphism f also induces an isomorphism on $H_*((-)^G;\mathbb{Z})$.

The multiplication of an orthogonal monoid space R induces a graded multiplication on the homology groups $H_*(R^G;\mathbb{Z})$ by simultaneous passage to colimits in both variables, of the maps

$$H_*(R(V)^G;\mathbb{Z}) \otimes H_*(R(W)^G;\mathbb{Z}) \;\xrightarrow{\times}\; H_*(R(V)^G \times R(W)^G;\mathbb{Z}) \;\xrightarrow{H_*(\mu^G_{V,W})}\; H_*(R(V\oplus W)^G;\mathbb{Z})\ .$$

Assigning to a path component its homology class is a map

$$\pi_0(R(V)^G) \;\longrightarrow\; H_0(R(V)^G;\mathbb{Z})$$

compatible with increasing V. On colimits over $s(\mathcal{U}_G)$ this provides a map

$$\pi_0^G(R) \;\longrightarrow\; H_0(R^G;\mathbb{Z})\ .$$

This map takes the addition in R to the multiplication in $H_0(R^G;\mathbb{Z})$, so its image is a multiplicative subset of $H_0(R^G;\mathbb{Z})$. If the multiplication of R is commutative, then the product of $H_*(R^G;\mathbb{Z})$ is commutative in the graded sense. In particular, the multiplicative subset of $\pi_0^G(R)$ is then automatically central.

Proposition 5.34. *A morphism $i : R \longrightarrow R^\star$ of ultra-commutative monoids is a global group completion if and only if the following two conditions are satisfied.*

(i) *The ultra-commutative monoid $R^\star$ is group-like, and*
(ii) *for every compact Lie group G the map of graded commutative rings*

$$H_*(i^G;\mathbb{Z}) \;:\; H_*(R^G;\mathbb{Z}) \;\longrightarrow\; H_*((R^\star)^G;\mathbb{Z})$$

is a localization at the multiplicative subset $\pi_0^G(R)$ of $H_0(R^G;\mathbb{Z})$.

Proof. (i)$\Longrightarrow$(ii) By Corollary 5.32 it suffices to show that for every cofibrant ultra-commutative monoid R the morphism $\eta_R : R \longrightarrow \Omega(BR)$ has property (ii). Since R and $\Omega(BR)$ are closed orthogonal spaces [...justify...], it suffices to show that for every compact Lie group G the map

$$H_*(R(\mathcal{U}_G)^G;\mathbb{Z}) \;\longrightarrow\; H_*((\Omega(BR))(\mathcal{U}_G)^G;\mathbb{Z})$$

is a localization at the multiplicative subset $\pi_0(R(\mathcal{U}_G)^G)$ of the source. Since the H-space structure of $R(\mathcal{U}_G)^G$ comes from the action of an E_∞-operad, the graded ring $H_*(R(\mathcal{U}_G)^G;\mathbb{Z})$ is graded commutative. We can thus apply Quillen's group completion theorem from the unpublished, but widely circulated preprint *On the group completion of a simplicial monoid.* Quillen's manuscript was later published as Appendix Q of the Friedlander-Mazur paper [**54**], where the relevant theorem appears on page 104 in Section Q.9.

(ii)$\Longrightarrow$(i) We contemplate the commutative diagram

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "\eta_R"'] & R^\star \arrow[d, "\eta_{R^\star}"] \\ \Omega(BR) \arrow[r, "\Omega(Bi)"'] & \Omega(BR^\star) \end{tikzcd}$$

Since $R^\star$ is group-like, the morphism $\eta_{R^\star}$ is a global equivalence. We may thus show that the morphism $Bi : BR \longrightarrow BR^\star$ is a global equivalence. [...needs flat replacement...]

For every coefficient system L on $(BR^\star)^G$ we compare the spectral sequence

$$E^2_{p,q} \;=\; \mathrm{Tor}_p^{H_*(R^G;k)}(k,L) \;\Longrightarrow\; H_*((BR)^G;L)$$

with the analogous one for the homology of $(BR^\star)^G$. The localization hypothesis implies that the map of Tor groups

$$\mathrm{Tor}_p^{H_*(R^G;k)}(k,L) \;\longrightarrow\; \mathrm{Tor}_p^{H_*((R^\star)^G;k)}(k,L)$$

is an isomorphism, see for example [**129**, Prop. 7.17] or [**174**, Prop. 3.2.9]. So we have a morphism of first quadrant spectral sequences that is an isomorphism of E^2-terms; so the map on abutments is an isomorphism as well. □

We showed in Theorem 4.14 that the ultra-commutative monoid **BOP** is group-like and that its equivariant homotopy sets $\underline{\pi}_0(\mathbf{BOP})$ realize the orthogonal representation rings additively. In Example 4.3 we introduced a morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ of ultra-commutative monoids from the additive Grassmannian and showed in Proposition 4.6 that for every compact Lie group G and every G-space A, the homomorphism

$$[A,i]^G \;:\; [A,\mathbf{Gr}]^G \;\longrightarrow\; [A,\mathbf{BOP}]^G$$

is a group completion of abelian monoids. In particular, the map $\pi_0^G(i) : \pi_0^G(\mathbf{Gr}) \longrightarrow \pi_0^G(\mathbf{BOP})$ is an algebraic group completion. In much the same way we can define morphisms of ultra-commutative monoids

$$i \;:\; \mathbf{Gr}^{\mathbb{C}} \;\longrightarrow\; \mathbf{BUP} \qquad \text{and} \qquad i \;:\; \mathbf{Gr}^{\mathbb{H}} \;\longrightarrow\; \mathbf{BSpP}$$

by replacing $\mathbb{R}$-subspaces in V by by replacing $\mathbb{C}$-subspaces in $V_{\mathbb{C}}$ respectively by replacing $\mathbb{H}$-subspaces in $V_{\mathbb{H}}$.

THEOREM 5.35. *The morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$, $i : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ and $i : \mathbf{Gr}^{\mathbb{H}} \longrightarrow \mathbf{BSpP}$ are global group completions of ultra-commutative monoids.*

PROOF. We prove the real case in details and leave the complex and quaternionic cases to the reader. We verify the localization criterion of Proposition 5.34. To this end we define a 'bi-orthogonal space', i.e., a functor

$$\mathbf{Gr}^\sharp \;:\; \mathbf{L}\times\mathbf{L} \;\longrightarrow\; \mathbf{T}$$

on objects by

$$\mathbf{Gr}^\sharp(U,V) \;=\; \mathbf{Gr}(U\oplus V)\;.$$

For linear isometric embeddings $\varphi : U \longrightarrow \bar{U}$ and $\psi : V \longrightarrow \bar{V}$, the induced map is

$$\mathbf{Gr}^\sharp(\varphi,\psi) \;:\; \mathbf{Gr}^\sharp(U,V) \;\longrightarrow\; \mathbf{Gr}^\sharp(\bar{U},\bar{V})\;, \quad L \;\longmapsto (\varphi\oplus\psi)(L) \;+\; ((\bar{U}-\varphi(U))\oplus 0)\;.$$

We emphasize that the behavior on objects is *not* symmetric in the two variables, and in the first variable it is not just applying φ.

Now we fix a compact Lie group G and consider the colimit of the bi-orthogonal space $\mathbf{Gr}^\sharp$ over the poset $s(\mathcal{U}_G)\times s(\mathcal{U}_G)$. Since the diagonal is cofinal in the poset $s(\mathcal{U}_G)\times s(\mathcal{U}_G)$, this 'double colimit' is also a colimit over the restriction to the diagonal. But the diagonal of $\mathbf{Gr}^\sharp$ is precisely the orthogonal space **BOP**, and so

$$\mathrm{colim}_{(U,V)\in s(\mathcal{U}_G)^2}\,\mathbf{Gr}^\sharp(U,V) \;=\; \mathrm{colim}_{W\in s(\mathcal{U}_G)}\,\mathbf{BOP}(W) \;=\; \mathbf{BOP}(\mathcal{U}_G)\;.$$

On the other hand, if we fix an inner product space U as the first variable, then $\mathbf{Gr}^\sharp(U,-)$ is isomorphic to the additive U-shift (in the sense of Example I.1.12) of the Grassmannian **Gr**. Hence for fixed U,

$$\mathrm{colim}_{V\in s(\mathcal{U}_G)}\,\mathbf{Gr}^\sharp(U,V) \;=\; \mathbf{Gr}(U\oplus\mathcal{U}_G)\;.$$

A colimit over $s(\mathcal{U}_G)\times s(\mathcal{U}_G)$ can be calculated in two steps, first in one variable and then in the other, so we conclude that

$$\mathbf{BOP}(\mathcal{U}_G) \;=\; \mathrm{colim}^\sharp_{U\in s(\mathcal{U}_G)}\,\mathbf{Gr}(U\oplus\mathcal{U}_G)\;, \tag{5.36}$$

and under this identification, the map $i(\mathcal{U}_G) : \mathbf{Gr}(\mathcal{U}_G) \longrightarrow \mathbf{BOP}(\mathcal{U}_G)$ becomes the canonical morphism

$$i^\sharp \;:\; \mathbf{Gr}(\mathcal{U}_G) \;\longrightarrow\; \mathrm{colim}^\sharp_{U\in s(\mathcal{U}_G)}\,\mathbf{Gr}(U\oplus\mathcal{U}_G)$$

to the colimit, for $U = 0$.

The decoration '$\sharp$' is meant to emphasize that the structure maps in this colimit system come from the functoriality of $\mathbf{Gr}^\sharp$ in the first variable, so they are *not* the maps obtained by applying $\mathbf{Gr}(-\oplus\mathcal{U}_G)$ to an inclusion $U \subset \bar{U}$. For example, the maps in the colimit (5.36) do *not* preserve the $\mathbb{N}$-grading by dimension. So one should not confuse the colimit (5.36) with the space $\mathbf{Gr}(\mathcal{U}_G \oplus \mathcal{U}_G)$ (which is G-homeomorphic to $\mathbf{Gr}(\mathcal{U}_G)$ by a choice is equivariant linear isometry $\mathcal{U}_G \oplus \mathcal{U}_G \cong \mathcal{U}_G$).

We claim that the map

$$H_*((i^\sharp)^G) \ : \ H_*(\mathbf{Gr}(\mathcal{U}_G)^G) \ \longrightarrow \ H_*(\operatorname{colim}^\sharp_{U\in s(\mathcal{U}_G)} \mathbf{Gr}(U\oplus\mathcal{U}_G)^G) \tag{5.37}$$

is a localization at the multiplicative subset $\pi_0^G(\mathbf{Gr})$. Here, and in the rest of this proof, homology stands for singular homology with integer coefficients. To see this we observe that all the maps in the colimit system are closed embeddings; since $s(\mathcal{U}_G)$ also contains a cofinal subsequence, singular homology commutes with this particular colimit.

For $U \in s(\mathcal{U}_G)$ we denote by $j_U : \mathbf{Gr}(\mathcal{U})^G \longrightarrow \mathbf{Gr}(U\oplus\mathcal{U}_G)^G$ the map induced by applying the direct summand inclusion $\mathcal{U}_G \longrightarrow U\oplus\mathcal{U}_G$. The map j_U is a homotopy equivalence because $\mathcal{U}_G$ is a complete G-universe. For all $U \subset V$ in $s(\mathcal{U}_G)$ the following square commutes

$$\begin{array}{ccc} H_*(\mathbf{Gr}(\mathcal{U}_G)^G) & \xrightarrow{\cdot[V-U]} & H_*(\mathbf{Gr}(\mathcal{U}_G)^G) \\ {\scriptstyle H_*(j_U)}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle H_*(j_V)} \\ H_*(\mathbf{Gr}(U\oplus\mathcal{U}_G)^G) & \xrightarrow[H_*(i^\sharp_{U,V})]{} & H_*(\mathbf{Gr}(V\oplus\mathcal{U}_G)^G) \end{array}$$

and the vertical maps are isomorphisms. So the target of (5.37) is the colimit of the functor on $s(\mathcal{U}_G)$ that takes all objects to the ring $H_*(\mathbf{Gr}(\mathcal{U}_G)^G;\mathbb{Z})$ and an inclusion $V \subset W$ to multiplication by the class $[W-V]$ in the multiplicative subset under consideration. Hence the map (5.37) is indeed a localization as claimed. Since the ultra-commutative monoid $\mathbf{BOP}$ is group-like, the criteria of Proposition 5.34 are satisfied, and so the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ is a global group completion. □

Remark 5.38. We had earlier defined an E_∞-orthogonal monoid space $\mathbf{BOP}'$ as a mixture of $\mathbf{bOP}$ and $\mathbf{BOP}$: the value at an inner product space V is

$$\mathbf{BOP}'(V) \ = \ \coprod_{m\geq 0} Gr_m(V^2\oplus\mathbb{R}^\infty) \ .$$

The structure maps and an E_∞-multiplication can be defined in essentially the same way as for $\mathbf{BO}'$, which was defined in (4.28). So $\mathbf{BOP}'$ becomes the $\mathbb{Z}$-graded periodic analog of the orthogonal space $\mathbf{BO}'$. In the same way as for the homogeneous degree 0 summands above, we defined two morphisms of E_∞-orthogonal monoid spaces

$$\mathbf{bOP} \ \xrightarrow{a} \ \mathbf{BOP}' \ \xleftarrow[\simeq]{b} \ \mathbf{BOP} \ .$$

The same arguments as in Proposition 4.24 show that the morphism b is a global equivalence. A very similar argument as in Proposition 5.35 shows that the morphism a is a global group completion in the homotopy category of E_∞-orthogonal monoid spaces. Strictly speaking we would first have to justify that that homotopy category is pre-additive (which we won't do), so that the formalism of group completions applies. In other words, $\mathbf{BOP}'$ is E_∞ globally equivalent to the ultra-commutative monoid $\mathbf{BOP}$.

As we argued in Proposition 4.31, the E_∞-structure on $\mathbf{bO}$ *cannot* be refined to an ultra-commutative multiplication. The argument was based on an algebraic obstruction that exists in the same way in $\underline{\pi}_0(\mathbf{bOP})$,

so **bOP** cannot be refined to an ultra-commutative monoid either. The fact that **bOP** has an ultra-commutative group completion can be interpreted as saying that in this particular case 'global group completion kills to obstruction to ultra-commutativity'.

In Theorem 4.39 we formulated a global equivariant version of Bott periodicity, saying that a specific morphism of ultra-commutative monoids $\beta : \mathbf{BUP} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{U})$ is a global equivalence. We define another morphism of ultra-commutative monoids

$$\beta' \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \Omega\mathbf{U} \tag{5.39}$$

by

$$\beta'(V)(L)(t) \ = \ (c(t) \cdot p_L) \ + \ p_{L^\perp} \ ,$$

where L is a complex subspace of $V_{\mathbb{C}}$ and

$$c \ : \ S^1 \ \longrightarrow \ U(1) \ , \quad t \ \longmapsto \ \frac{t+i}{t-i} \ ,$$

is the Cayley transform. The verification that β' is a morphism of ultra-commutative monoids is very similar (but slightly easier) as for $\beta : \mathbf{BUP} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{U})$, and we omit it. ultra-commutative monoids.

Corollary 5.40. *The morphism $\beta' : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ is a global group completion of ultra-commutative monoids. For every compact Lie group G and every finite G-CW-complex A, the map*

$$[A, \beta']^G \ : \ [A, \mathbf{Gr}^{\mathbb{C}}]^G \ \longrightarrow \ [A, \Omega\mathbf{U}]^G$$

is a group completion of abelian monoids.

Proof. The following diagram of homomorphisms of ultra-commutative monoids on the left commutes by direct inspection:

$$\begin{tikzcd} \mathbf{Gr}^{\mathbb{C}} \arrow[r, "\beta'"] \arrow[d, "i"'] & \Omega\mathbf{U} \arrow[d, "\Omega j", "\simeq"'] \\ \mathbf{BUP} \arrow[r, "\simeq", "\beta"'] & \Omega(\mathrm{sh}_\otimes \mathbf{U}) \end{tikzcd} \qquad \begin{tikzcd} {[A, \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "{[A,\beta']^G}"] \arrow[d, "{[A,i]^G}"'] & {[A, \Omega\mathbf{U}]^G} \arrow[d, "{[A,\Omega j]^G}", "\cong"'] \\ {[A, \mathbf{BUP}]^G} \arrow[r, "\cong", "{[A,\beta]^G}"'] & {[A, \Omega\,\mathrm{sh}_\otimes \mathbf{U}]^G} \end{tikzcd}$$

The left vertical morphism is a global group completion by Theorem 5.35. The lower horizontal morphism β is a global equivalence by global Bott periodicity (Theorem 4.39). The morphism $j : \mathbf{U} \longrightarrow \mathrm{sh}_\otimes \mathbf{U}$ is a global equivalence by Theorem I.1.11, hence so is Ωj because Ω preserves global equivalences. So the morphism $\beta' : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ is also a global group completion of ultra-commutative monoids.

The square on the left induces a commutative diagram of homomorphisms of abelian monoids on the right above, in which $[A, \beta]^G$ and $[A, \Omega j]^G$ are isomorphisms, by Proposition I.5.2 (ii). The morphism $[A, i]^G$ is a group completion of abelian monoids by Proposition 4.6. So $[A, \beta']^G : [A, \mathbf{Gr}]^G \longrightarrow [A, \Omega\mathbf{U}]^G$ is also a group completion of abelian monoids. $\square$

CHAPTER III

Equivariant stable homotopy theory

In this chapter we give a largely self-contained exposition of many basics about equivariant stable homotopy theory for a fixed compact Lie group, modeled by orthogonal G-spectra. In Section 1 we review orthogonal spectra and orthogonal G-spectra; we define equivariant stable homotopy groups and prove their basic properties, such as the suspension isomorphism and long exact sequences of mapping cones and homotopy fibers, and the behavior of equivariant homotopy groups on sums and products. Section 2 discusses the Wirthmüller isomorphism that relates the equivariant homotopy groups of a spectrum over a subgroup to the equivariant homotopy groups of the induced spectrum; intimately related to the Wirthmüller isomorphism are various transfers that we also discuss. In Section 3 we introduce and study geometric fixed point homotopy groups, an alternative invariant to characterize equivariant stable equivalences. We establish the isotropy separation sequence that facilitates inductive arguments, and show that equivariant equivalences can also be detected by geometric fixed points. In Section 3 we also derive a functorial description of the 0-th equivariant stable homotopy groups of a G-space Y in terms of the path components of the fixed points spaces Y^H. We also include a proof of the double coset formula. We show that rationally and for finite groups, geometric fixed point homotopy groups can be obtained from equivariant homotopy groups by dividing out transfers from proper subgroups. Section 5 is devoted to multiplicative aspects of equivariant stable homotopy theory. In our model, all multiplicative features can be phrased in terms of the smash product of orthogonal spectra (or orthogonal G-spectra), another example of a Day type convolution product. The smash product gives rise to various pairings of equivariant and geometric fixed point homotopy groups; when specialized to equivariant ring spectra, these pairings turn the equivariant stable homotopy groups and geometric fixed point homotopy groups into graded rings.

We do not discuss model category structures for orthogonal G-spectra; the interested reader can find different ones in the memoir of Mandell and May [**105**], in the thesis of Stolz [**150**], the article by Brun, Dundas and Stolz [**32**] and (for finite groups) in the paper of Hill, Hopkins and Ravenel [**76**].

1. Equivariant orthogonal spectra

In this section we begin to develop some of the basic features of equivariant stable homotopy theory for compact Lie groups in the context of equivariant orthogonal spectra. After introducing orthogonal G-spectra and defining the equivariant stable homotopy groups we show the loop and suspension isomorphisms (Proposition 1.30) and the long exact homotopy group sequences of homotopy fibers and mapping cones (Proposition 1.37). We discuss shifts by a representation and show that they are equivariantly equivalent to smashing with the representation sphere (Proposition 1.25).

We recall *orthogonal spectra*. These objects are used, at least implicitly, already in [**107**]; the term 'orthogonal spectrum' was introduced by Mandell, May, Shipley and the author in [**104**], where the (non-equivariant) stable model structure for orthogonal spectra was constructed.

Orthogonal spectra are stable versions of orthogonal spaces, and before recalling the formal definition we try to motivate it – already with a view towards the global perspective. An orthogonal space Y assigns values to all finite dimensional inner product spaces. The global homotopy type is encoded in the G-spaces $Y(\mathcal{U}_G)$, where $\mathcal{U}_G$ is a complete G-universe, which we can informally think of as 'the homotopy

colimit of $Y(V)$ over all G-representations V'. So besides the values $Y(V)$, an orthogonal space uses the information about the $O(V)$-action (which is turned into a G-action when G acts on V) and the information about inclusions of inner product spaces (in order to be able to stabilize to the colimit $\mathcal{U}_G$). The information about the $O(V)$-actions and how to stabilize are conveniently encoded together as a continuous functor from the category $\mathbf{L}$ of linear isometric embeddings.

An orthogonal spectrum X is a stable analog of this: it assigns a based space $X(V)$ to every inner product space, and it keeps track of an $O(V)$-action on $X(V)$ (to get G-homotopy types when G acts on V) and of a way to stabilize by suspensions (needed when exhausting a complete universe by its finite dimensional subrepresentations). When doing this in a coordinate free way, the stabilization data assigns to a linear isometric embedding $\varphi : V \longrightarrow W$ a continuous based map

$$\varphi_\star \ : \ S^{W-\varphi(V)} \wedge X(V) \ \longrightarrow \ X(W)$$

where $W - \varphi(V)$ is the orthogonal complement of the image of φ. This structure map should 'vary continuously with φ', but this phrase has no literal meaning because the source of $\varphi_\star$ depends on φ. The way to make the continuous dependence rigorous is to exploit that the complements $W - \varphi(V)$ vary in a locally trivial way, i.e., they are the fibers of a distinguished vector bundle, the 'orthogonal complement bundle', over the space of $\mathbf{L}(V, W)$ of linear isometric embeddings. All the structure maps $\varphi_\star$ together define a map on the smash product of $X(V)$ with the Thom space of this complement bundle, and the continuity of the dependence on φ is formalized by requiring continuity of that map. All these Thom spaces together form the morphism spaces of a based topological category, and the data of an orthogonal spectrum can conveniently be packaged as a continuous based functor on this category.

Construction 1.1. We let V and W be inner product spaces. Over the space $\mathbf{L}(V, W)$ of linear isometric embeddings sits a certain 'orthogonal complement' vector bundle with total space

$$\xi(V, W) \ = \ \{ \, (w, \varphi) \in W \times \mathbf{L}(V, W) \mid w \perp \varphi(V) \, \} \ .$$

The structure map $\xi(V, W) \longrightarrow \mathbf{L}(V, W)$ is the projection to the second factor. The vector bundle structure of $\xi(V, W)$ is as a vector subbundle of the trivial vector bundle $W \times \mathbf{L}(V, W)$, and the fiber over $\varphi : V \longrightarrow W$ is the orthogonal complement $W - \varphi(V)$ of the image of φ.

We let $\mathbf{O}(V, W)$ be the Thom space of the bundle $\xi(V, W)$, i.e., the one-point compactification of the total space of $\xi(V, W)$. Up to non-canonical homeomorphism, we can describe the space $\mathbf{O}(V, W)$ differently as follows. If the dimension of W is smaller than the dimension of V, then the space $\mathbf{L}(V, W)$ is empty and $\mathbf{O}(V, W)$ consists of a single point at infinity. Otherwise we can choose a linear isometric embedding $\varphi : V \longrightarrow W$, and we let $V^\perp = W - \varphi(V)$ denote the orthogonal complement of its image. Then the maps

$$\begin{aligned} O(W)/O(V^\perp) \quad &\longrightarrow \ \mathbf{L}(V, W) \ , \quad A \cdot O(V^\perp) \ \longmapsto \ A\varphi \qquad \text{and} \\ O(W) \ltimes_{O(V^\perp)} S^{V^\perp} \ &\longrightarrow \ \mathbf{O}(V, W) \ , \quad [A, w] \ \longmapsto \ (Aw, A\varphi) \end{aligned}$$

are homeomorphisms. Put yet another way: if $\dim V = n$ and $\dim W = n + m$, then $\mathbf{L}(V, W)$ is homeomorphic to the homogeneous space $O(n+m)/O(m)$ and $\mathbf{O}(V, W)$ is homeomorphic to $O(n+m) \ltimes_{O(m)} S^m$. The vector bundle $\xi(V, W)$ becomes trivial upon product with the trivial bundle V, via the trivialization

$$\xi(V, W) \times V \ \cong \ W \times \mathbf{L}(V, W) \ , \quad ((w, \varphi), v) \ \longmapsto \ (w + \varphi(v), \varphi) \ .$$

When we pass to Thom spaces on both sides this becomes the *untwisting homeomorphism:*

$$\mathbf{O}(V, W) \wedge S^V \ \cong \ S^W \wedge \mathbf{L}(V, W)_+ \ . \tag{1.2}$$

The Thom spaces $\mathbf{O}(V, W)$ are the morphism spaces of a based topological category. Given a third inner product space U, the bundle map

$$\xi(V, W) \times \xi(U, V) \ \longrightarrow \ \xi(U, W) \ , \quad ((w, \varphi), \, (v, \psi)) \ \longmapsto \ (w + \varphi(v), \, \varphi\psi)$$

covers the composition map $\mathbf{L}(V,W)\times\mathbf{L}(U,V)\longrightarrow\mathbf{L}(U,W)$. Passage to Thom spaces gives a based map

$$\circ\ :\ \mathbf{O}(V,W)\wedge\mathbf{O}(U,V)\ \longrightarrow\ \mathbf{O}(U,W)$$

which is clearly associative, and is the composition in the category $\mathbf{O}$. The identity of V is $1_V=(0,\mathrm{Id}_V)$ in $\mathbf{O}(V,V)$.

Definition 1.3. An *orthogonal spectrum* is a based continuous functor from $\mathbf{O}$ to the category $\mathbf{T}_*$ of based spaces. A *morphism* of orthogonal spectra is a natural transformation of functors.

For every linear isometric embedding $\varphi:V\longrightarrow W$ we define a continuous map

$$(-,\varphi)\ :\ S^{W-\varphi(V)}\ \longrightarrow\ \mathbf{O}(V,W)\ ,\quad w\ \longmapsto\ (w,\varphi)\ .$$

This map is the one-point compactification of the inclusion of the fiber over φ of the bundle $\xi(V,W)$. If X is an orthogonal spectrum, we refer to the composite

$$\varphi_\star\ =\ X\circ((-,\varphi)\wedge X(V))\ :\ S^{W-\varphi(V)}\wedge X(V)\ \longrightarrow\ \mathbf{O}(V,W)\wedge X(V)\ \longrightarrow\ X(W)$$

as the *structure map* of X associated to φ. When $\varphi=(0,-):W\longrightarrow V\oplus W$ is a direct summand inclusion, then we identify the orthogonal complement of its image with V by $v\mapsto(v,0)$ and use the notation

$$\sigma_{V,W}\ =\ (0,-)_\star\ :\ S^V\wedge X(W)\ \longrightarrow\ X(V\oplus W) \tag{1.4}$$

for the associated structure map. Often it will be convenient to use the *opposite structure map*

$$\sigma^{\mathrm{op}}_{V,W}\ :\ X(V)\wedge S^W\ \longrightarrow\ X(V\oplus W) \tag{1.5}$$

which we define as the following composite:

$$X(V)\wedge S^W\ \xrightarrow{\text{twist}}\ S^W\wedge X(V)\ \xrightarrow{\sigma_{W,V}}\ X(W\oplus V)\ \xrightarrow{X(\tau_{V,W})}\ X(V\oplus W)\ .$$

Remark 1.6 (Coordinatized orthogonal spectra). Every inner product space is isometrically isomorphic to $\mathbb{R}^n$ with standard inner product, for some $n\geq 0$. So the topological category $\mathbf{O}$ has a small skeleton, and the functor category of orthogonal spectra has 'small' morphism sets. This also leads to the following more explicit coordinatized description of orthogonal spectra in a way that resembles a presentation by 'generators and relations'.

Up to isomorphism, an orthogonal spectrum X is determined by the values $X_n=X(\mathbb{R}^n)$ and the following additional data relating these values:

- a based continuous left action of the orthogonal group $O(n)$ on X_n for each $n\geq 0$,
- based maps $\sigma_n:S^1\wedge X_n\longrightarrow X_{1+n}$ for $n\geq 0$.

This data is subject to the following condition: for all $m,n\geq 0$, the iterated structure map

$$\sigma^m\ :\ S^m\wedge X_n\ \longrightarrow\ X_{m+n}$$

defined as the composition

$$S^m\wedge X_n\xrightarrow{S^{m-1}\wedge\sigma_n}S^{m-1}\wedge X_{1+n}\xrightarrow{S^{m-2}\wedge\sigma_{1+n}}\quad\cdots\quad S^1\wedge X_{m-1+n}\xrightarrow{\sigma_{m-1+n}}X_{m+n}$$

is $(O(m)\times O(n))$-equivariant. Here the orthogonal group $O(m)\times O(n)$ acts on the target by restriction, along orthogonal sum, of the $O(m+n)$-action.

Indeed, the map

$$O(n)_+\ \longrightarrow\ \mathbf{O}(\mathbb{R}^n,\mathbb{R}^n)\ ,\quad A\ \longmapsto\ (0,A)$$

is a homeomorphism, so $O(n)$ 'is' the endomorphism monoid of $\mathbb{R}^n$ as an object of the category $\mathbf{O}$; via the map, $O(n)$ acts on the value at $\mathbb{R}^n$ of any functor on $\mathbf{O}$. The map $\sigma_n=\sigma_{\mathbb{R},\mathbb{R}^n}$ is just one of the structure maps (1.4).

Definition 1.7. Let G be a compact Lie group. An *orthogonal G-spectrum* is a based continuous functor from $\mathbf{O}$ to the category $G\mathbf{T}_*$ of based G-spaces. A *morphism* of orthogonal G-spectra is a natural transformation of functors.

We write $G\mathcal{S}p$ for the category of orthogonal G-spectra and G-equivariant morphisms. A continuous functor to based G-spaces is the same as a G-object of continuous functors. So orthogonal G-spectra could equivalently be defined as orthogonal spectra equipped with a continuous G-action. An orthogonal G-spectrum X can also be evaluated on a G-representation V, and then $X(V)$ is a $(G \times G)$-space by the 'external' G-actions on X and the 'internal' G-action from the G-action on V and the $O(V)$-functoriality of X. We consider $X(V)$ as a G-space via the diagonal G-action. If V and W are G-representations, then the structure map (1.4) and the opposite structure map (1.5) are G-equivariant where the group G also acts on the representation spheres.

Remark 1.8. Our definition of orthogonal G-spectra is not the same as the one used by Mandell and May [**105**] and Hill, Hopkins and Ravenel [**76**], who define orthogonal G-spectra as G-functors on an extension of the category $\mathbf{O}$ that is enriched in G-spaces and contains all G-representations as objects. However, our category of orthogonal G-spectra is equivalent to theirs by [**105**, V Thm. 1.5]. The substance of this equivalence is the fact that for every orthogonal G-spectrum in the sense of Mandell and May, the values at arbitrary G-representations are in fact determined by the values at trivial representations.

Next we recall the equivariant stable homotopy groups $\pi_*^G(X)$ (indexed by the complete G-universe $\mathcal{U}_G$), of an orthogonal G-spectrum X. We introduce a convenient piece of notation. If $\varphi : V \longrightarrow W$ is a linear isometric embedding and $f : S^V \longrightarrow X(V)$ a continuous based map, we define $\varphi_* f : S^W \longrightarrow X(W)$ as the composite

$$\varphi_* f \ : \ S^W \cong S^{W-\varphi(V)} \wedge S^V \xrightarrow{S^{W-\varphi(V)} \wedge f} S^{W-\varphi(V)} \wedge X(V) \xrightarrow{\varphi_*} X(W) \tag{1.9}$$

where the first unnamed homeomorphism uses the linear isometry

$$(W - \varphi(V)) \oplus V \ \cong \ W \ , \quad (w, v) \ \longmapsto \ w + \varphi(v) \ .$$

For example, if φ is bijective (i.e., an equivariant isometry), then $\varphi_* f$ becomes the 'φ-conjugate' of f, i.e., the composite

$$S^W \xrightarrow{S^{\varphi^{-1}}} S^V \xrightarrow{f} X(V) \xrightarrow{X(\varphi)} X(W) \ .$$

The construction is continuous in both variables, i.e., the map

$$\mathbf{L}(V, W) \times \mathrm{map}(S^V, X(V)) \ \longrightarrow \ \mathrm{map}(S^W, X(W)) \ , \quad (\varphi, f) \ \longmapsto \ \varphi_* f$$

is continuous.

As before we let $s(\mathcal{U}_G)$ denote the poset, under inclusion, of finite dimensional G-subrepresentations of the chosen complete G-universe $\mathcal{U}_G$. We obtain a functor from $s(\mathcal{U}_G)$ to sets by sending $V \in s(\mathcal{U}_G)$ to

$$[S^V, X(V)]^G \ ,$$

the set of G-equivariant homotopy classes of based G-maps from S^V to $X(V)$. For $V \subseteq W$ in $s(\mathcal{U}_G)$ the inclusion $i : V \longrightarrow W$ is sent to the map

$$i_* \ : \ [S^V, X(V)]^G \ \longrightarrow \ [S^W, X(W)]^G \ , \quad [f] \ \longmapsto [i_* f] \ .$$

The *0-th equivariant homotopy group* $\pi_0^G(X)$ is then defined as

$$\pi_0^G(X) \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} \, [S^V, X(V)]^G \ ,$$

the colimit of this functor over the poset $s(\mathcal{U}_G)$.

The groups $\pi_0^G(X)$ have a lot of extra structure as G varies. First we recall the abelian group structure on $\pi_0^G(X)$. We consider a finite dimensional G-subrepresentation V of the universe $\mathcal{U}_G$ with non-zero fixed

points. We choose a G-fixed unit vector $v_0 \in V$, and we let $V^\perp$ denote the orthogonal complement of v_0 in V. This induces a decomposition

$$\mathbb{R} \oplus V^\perp \;\cong\; V \;, \quad (t,v) \;\longmapsto\; tv_0 + v$$

that extends to a G-equivariant homeomorphism $S^1 \wedge S^{V^\perp} \cong S^V$ on one-point compactifications. From this we obtain a bijection

$$[S^V, X(V)]^G \;\cong\; [S^1, \mathrm{map}^G(S^{V^\perp}, X(V))]_* = \pi_1(\mathrm{map}^G(S^{V^\perp}, X(V)))\;, \tag{1.10}$$

natural in the orthogonal G-spectrum X. We use the bijection (1.10) to transfer the group structure on the fundamental group into a group structure on the set $[S^V, X(V)]^G$.

Now we suppose that the dimension of the fixed point space V^G is at least 2. Then the space of G-fixed unit vectors in V is connected and similar arguments as for the commutativity of higher homotopy groups show:

- the group structure on the set $[S^V, X(V)]^G$ defined by the bijection (1.10) is commutative and independent of the choice of G-fixed unit vector;
- if W is another finite dimensional G-subrepresentation of $\mathcal{U}_G$ containing V, then the map
$$i_* \;:\; [S^V, X(V)]^G \;\longrightarrow\; [S^W, X(W)]^G$$
is a group homomorphism.

The G-subrepresentations V of $\mathcal{U}_G$ with $\dim(V^G) \geq 2$ are cofinal in the poset $s(\mathcal{U}_G)$, so the two properties above show that the abelian group structures on $[S^V, X(V)]^G$ for $\dim(V^G) \geq 2$ assemble into a well-defined and natural abelian group structure on the colimit $\pi_0^G(X)$.

We generalize the definition of $\pi_0^G(X)$ to integer graded equivariant homotopy groups of an orthogonal G-spectrum X. If k is a positive integer, then we set

$$\begin{aligned} \pi_k^G(X) \;&=\; \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, [S^{V \oplus \mathbb{R}^k}, X(V)]^G \qquad \text{and} \\ \pi_{-k}^G(X) \;&=\; \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, [S^V, X(V \oplus \mathbb{R}^k)]^G\;. \end{aligned} \tag{1.11}$$

The colimits are taken over the analogous stabilization maps as for π_0^G, and they come with natural abelian group structures by the same reasoning as for $\pi_0^G(X)$.

Definition 1.12. A morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is a $\underline{\pi}_*$*-isomorphism* if the induced map $\pi_k^H(f) : \pi_k^H(X) \longrightarrow \pi_k^H(Y)$ is an isomorphism for all closed subgroups H of G and all integers k.

Construction 1.13. While the definition of $\pi_k^G(X)$ involves a case distinction in positive and negative dimensions k, classes in $\pi_k^G(X)$ can always be represented by a G-map

$$f \;:\; S^{V \oplus \mathbb{R}^{n+k}} \;\longrightarrow\; X(V \oplus \mathbb{R}^n)$$

for suitable $n \in \mathbb{N}$ such that $n + k \geq 0$. Moreover, V can be any finite dimensional G-representation, not necessarily a subrepresentation of the chosen complete G-universe. Since we will frequently use this way to represent elements of $\pi_k^G(X)$, we make the construction explicit here.

We start with the case $k \geq 0$. We choose a G-equivariant linear isometry $j : V \oplus \mathbb{R}^n \longrightarrow \bar{V}$ onto a G-subrepresentation $\bar{V}$ of $\mathcal{U}_G$. Then the composite

$$S^{\bar{V} \oplus \mathbb{R}^k} \;\xrightarrow[\cong]{(S^{j \oplus \mathbb{R}^k})^{-1}}\; S^{V \oplus \mathbb{R}^{n+k}} \;\xrightarrow{f}\; X(V \oplus \mathbb{R}^n) \;\xrightarrow[\cong]{X(j)}\; X(\bar{V})$$

represents a class $\langle f \rangle \in \pi_k^G(X)$. For $k \leq 0$, we choose a G-equivariant linear isometry $j : V \oplus \mathbb{R}^{n+k} \longrightarrow \bar{V}$ onto a G-subrepresentation $\bar{V}$ of $\mathcal{U}_G$. Then the composite

$$S^{\bar{V}} \;\xrightarrow[\cong]{(S^j)^{-1}}\; S^{V \oplus \mathbb{R}^{n+k}} \;\xrightarrow{f}\; X(V \oplus \mathbb{R}^n) \;\xrightarrow[\cong]{X(j \oplus \mathbb{R}^{-k})}\; X(\bar{V} \oplus \mathbb{R}^{-k})$$

represents a class $\langle f\rangle \in \pi_k^G(X)$.

We also need a way to recognize that 'stabilization along a linear isometric embedding' does not change the class in $\pi_k^G(X)$. For this we let $\varphi : V \longrightarrow W$ be a G-equivariant linear isometric embedding and $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow X(V\oplus\mathbb{R}^n)$ a continuous based G-map as above. We define $\varphi_*(f) : S^{W\oplus\mathbb{R}^{n+k}} \longrightarrow X(W\oplus\mathbb{R}^n)$ as the composite

$$S^{W\oplus\mathbb{R}^{n+k}} \cong S^{W-\varphi(V)} \wedge S^{V\oplus\mathbb{R}^{n+k}} \xrightarrow{S^{W-\varphi(V)}\wedge f} S^{W-\varphi(V)} \wedge X(V\oplus\mathbb{R}^n) \xrightarrow{(\varphi\oplus\mathbb{R}^n)_\star} X(W\oplus\mathbb{R}^n)$$

where the first unnamed homeomorphism uses the linear isometry

$$(W-\varphi(V))\oplus V\oplus\mathbb{R}^{n+k} \cong W\oplus\mathbb{R}^{n+k}\ , \quad (w,v,x) \longmapsto (w+\varphi(v),x)\ .$$

In the special case $n=k=0$, this construction reduces to (1.9).

The same reasoning as in the unstable situation in Proposition I.5.9 shows the following stable analog:

Proposition 1.14. *Let G be a compact Lie group and X an orthogonal G-spectrum. Let V be a G-representation and $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow X(V\oplus\mathbb{R}^n)$ a based continuous G-map, where $n\in\mathbb{N}$ and $k\in\mathbb{Z}$ are such that $n+k\geq 0$.*

(i) *The class $\langle f\rangle$ in $\pi_k^G(X)$ is independent of the choice of linear isometry from V to a subrepresentation of $\mathcal{U}_G$.*

(ii) *For every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the relation*

$$\langle\varphi_*(f)\rangle = \langle f\rangle \qquad \text{holds in} \quad \pi_k^G(X).$$

Let K and G be two compact Lie groups. Every continuous based functor $F : G\mathbf{T}_* \longrightarrow K\mathbf{T}_*$ from based G-spaces to based K-spaces gives rise to a functor

$$F\circ - \ :\ G\mathcal{S}p \longrightarrow K\mathcal{S}p$$

from orthogonal G-spectra to orthogonal K-spectra by postcomposition: if X is an G-orthogonal spectrum, then the composite

$$\mathbf{O} \xrightarrow{X} G\mathbf{T}_* \xrightarrow{F} K\mathbf{T}_*\ .$$

is an orthogonal K-spectrum. The next two constructions are examples of this.

Construction 1.15 (Restriction maps). We let $\alpha : K \longrightarrow G$ be a continuous homomorphism between compact Lie groups. Given an orthogonal G-spectrum X, we apply restriction of scalars levelwise and obtain an orthogonal K-spectrum $\alpha^*(X)$. We can also define a *restriction homomorphism*

$$\alpha^* \ :\ \pi_0^G(X) \longrightarrow \pi_0^K(\alpha^*(X))$$

as follows. If $f : S^V \longrightarrow X(V)$ is a G-map that represents a class in $\pi_0^G(X)$, then we define

$$\alpha^*[f] = \langle\alpha^*(f)\rangle\ ,$$

i.e., the class represented by the K-map

$$\alpha^*(f)\ :\ S^{\alpha^*(V)} = \alpha^*(S^V) \longrightarrow \alpha^*(X(V)) = X(\alpha^*(V))\ ,$$

appealing to Construction 1.13. The restriction maps α^* are clearly transitive (contravariantly functorial) for composition of group homomorphisms.

We recall that for $g\in G$ the conjugation homomorphism is defined as

$$c_g\ :\ G \longrightarrow G\ , \quad c_g(h) = g^{-1}hg\ .$$

For every G-space A, left multiplication by g is then a G-equivariant homeomorphism $l_g : c_g^*(A) \longrightarrow A$. For an orthogonal G-spectrum X the maps $l_g^{X(V)} : (c_g^*X)(V) = c_g^*(X(V)) \longrightarrow X(V)$ assemble into an isomorphism of orthogonal G-spectra $l_g^X : c_g^*(X) \longrightarrow X$, as V runs over all inner product spaces (with trivial G-action).

PROPOSITION 1.16. *Let G be a compact Lie group, X an orthogonal G-spectrum and $g \in G$. Then the two isomorphisms*

$$c_g^* \ : \ \pi_0^G(X) \ \longrightarrow \ \pi_0^G(c_g^* X) \qquad \text{and} \qquad (l_g^X)_* \ : \ \pi_0^G(c_g^* X) \ \longrightarrow \ \pi_0^G(X)$$

are inverse to each other.

PROOF. We let V be a G-representation, and we recall that the G-action on $X(V)$ is diagonally, from the external G-action on X and the internal G-action on V. Hence the map $l_g^{X(V)} : c_g^*(X(V)) \longrightarrow X(V)$ is the composite of the map $l_g^X(c_g^*V) : (c_g^*X)(c_g^*V) \longrightarrow X(c_g^*V)$ and the map $X(l_g^V) : X(c_g^*V) \longrightarrow X(V)$. Now we let $f : S^V \longrightarrow X(V)$ be a G-map representing a class in $\pi_0^G(X)$. The following diagram of G-space and G-maps commutes because f is G-equivariant:

$$\begin{tikzcd}
S^{c_g^*V} \arrow[r, "c_g^* f"] \arrow[d, "l_g^V"'] & c_g^*(X(V)) = (c_g^*X)(c_g^*V) \arrow[r, "l_g^X(c_g^*V)"] \arrow[dr, bend left, "l_g^{X(V)}"'] & X(c_g^*V) \arrow[d, "X(l_g^V)"] \\
S^V \arrow[rr, "f"'] & & X(V)
\end{tikzcd}$$

The upper horizontal composite represents the class $(l_g^X)_*(c_g^*[f])$. Since it differs from f by conjugation with an equivariant isometry, the upper composite represents the same class as f. Thus we conclude that $(l_g)_*(c_g^*[f]) = [f]$. $\square$

REMARK 1.17 (Weyl group action on equivariant homotopy groups). Now we consider a closed subgroup H of a compact Lie group G and an orthogonal G-spectrum X. Every $g \in G$ gives rise to a conjugation homomorphism $c_g : {}^gH \longrightarrow H$ by $c_g(h) = g^{-1}hg$. One should beware that while $c_g^*(\mathrm{res}_H^G(X))$ and $\mathrm{res}_{{}^gH}^G(X)$ have the same underlying orthogonal spectrum, they come with *different* actions of the group ${}^gH$. However, left translation by g is an isomorphism of orthogonal $({}^gH)$-spectra $l_g^X : c_g^*(X) \longrightarrow X$. So combining the restriction map along c_g with the effect of l_g gives a isomorphism

$$g_\star \ : \ \pi_0^H(X) \ \xrightarrow{c_g^*} \ \pi_0^{{}^gH}(c_g^*(X)) \ \xrightarrow{(l_g^X)_*} \ \pi_0^{{}^gH}(X) \ . \tag{1.18}$$

Moreover,

$$\begin{aligned} g_\star \circ g'_\star \ &= \ (l_g^X)_* \circ c_g^* \circ (l_{g'}^X)_* \circ c_{g'}^* \ = \ (l_g^X)_* \circ (l_{g'}^X)_* \circ c_g^* \circ c_{g'}^* \\ &= (l_g^X \circ l_{g'}^X)_* \circ (c_{g'} \circ c_g)^* \ = \ (l_{gg'}^X)_* \circ (c_{gg'})^* \ = \ (gg')_\star \end{aligned}$$

by naturality of c_g^*. If g normalizes H, then $g_\star$ is a self map of the group $\pi_0^H(X)$. If moreover g belongs to H, then the above map $g_\star$ is the identity by Proposition 1.16; so the maps $g_\star$ define an action of the Weyl group $W_GH = N_GH/H$ on the equivariant homotopy group $\pi_0^H(X)$.

If H has finite index in its normalizer, this is the end of the story concerning Weyl group actions on $\pi_0^H(X)$. In general, however the group H need not have finite index in its normalizer N_GH, and the Weyl group W_GH may have positive dimension, and hence a non-trivial identity path component $(W_GH)^\circ$. We will now show that the entire identity path component acts trivially on $\pi_k^H(X)$ for any orthogonal G-spectrum X. This is a consequence of the fact that every element of $(W_GH)^\circ$ has the form zH for an element z in $(C_GH)^\circ$, the identity component of the centralizer of H in G (compare Proposition A.2.22 for $K = H$, using that $(G/H)^H = W_GH$). But then $c_z : H \longrightarrow H$ is the identity because z centralizes H. On the other hand, any path from z and 1 in C_GH induces a homotopy of morphisms of orthogonal H-spectra from $l_z : X \longrightarrow X$ to the identity of X. So

$$z_\star \ = \ (l_z^X)_* \circ c_z^* \ = \ \mathrm{Id}_{\pi_k^H(X)} \ .$$

This shows that the identity component of the Weyl group $W_G H$ acts trivially on $\pi_k^H(X)$. So the Weyl group action factors over an action of the discrete group

$$\pi_0(W_G H) \;=\; (W_G H)/(W_G H)^\circ \;.$$

Now we recall some important properties of the equivariant homotopy groups, such as stability under suspension and looping and the long exact sequences associated to mapping cones and homotopy fibers.

Construction 1.19. If A is a pointed G-space, then smashing with A and taking based maps out of A are two continuous based endofunctors on the category of based G-spaces. So for every orthogonal G-spectrum X, we can define two new orthogonal G-spectra $A \wedge X$ and $\mathrm{map}(A, X)$ by smashing with A (and letting G act diagonally) or taking maps from A levelwise (and letting G act by conjugation). More explicitly, we have

$$(X \wedge A)(V) \;=\; X(V) \wedge A \qquad \text{respectively} \qquad \mathrm{map}(A, X)(V) \;=\; \mathrm{map}(A, X(V))$$

for an inner product space V. The structure maps and actions of the orthogonal groups do not interact with A: the group $O(V)$ acts through its action on $X(V)$, and the structure maps are given by the composite

$$S^V \wedge (X \wedge A)(W) \;=\; S^V \wedge X(W) \wedge A \;\xrightarrow{\sigma_{V,W} \wedge A}\; X(V \oplus W) \wedge A \;=\; (X \wedge A)(V \oplus W)$$

respectively by the composite

$$S^V \wedge \mathrm{map}(A, X(W)) \;\longrightarrow\; \mathrm{map}(A, S^V \wedge X(W)) \;\xrightarrow{\mathrm{map}(A, \sigma_{V,W})}\; \mathrm{map}(A, X(V \oplus W))$$

where the first is an assembly map that sends $v \wedge \varphi$ in $S^V \wedge \mathrm{map}(A, X(W))$ to the map sending $a \in A$ to $v \wedge \varphi(a)$.

Just as the functors $- \wedge A$ and $\mathrm{map}(A, -)$ are adjoint on the level of based G-spaces, the two functors just introduced are an adjoint pair on the level of orthogonal G-spectra. The adjunction

$$G\mathcal{S}p(X, \mathrm{map}(A, Y)) \;\xrightarrow{\cong}\; G\mathcal{S}p(X \wedge A, Y) \tag{1.20}$$

takes a morphism $f : X \longrightarrow \mathrm{map}(A, Y)$ to the morphism $\hat{f} : X \wedge A \longrightarrow Y$ whose V-th level $\hat{f}(V) : X(V) \wedge A \longrightarrow Y(V)$ is given by $\hat{f}(V)(x \wedge a) = f(V)(x)(a)$.

An important special case of this construction is when $A = S^W$ is a representation sphere, i.e., the one-point compactification of an orthogonal G-representation. The *W-th suspension* $X \wedge S^W$ is defined by

$$(X \wedge S^W)(V) \;=\; X(V) \wedge S^W \;,$$

the smash product of the V-th level of X with the sphere S^W. The *W-th loop spectrum* $\Omega^W X = \mathrm{map}(S^W, X)$, defined by

$$(\Omega^W X)(V) \;=\; \Omega^W X(V) \;=\; \mathrm{map}(S^W, X(V)) \;,$$

the based mapping space from S^W to the V-th level of X. We obtain an adjunction between $- \wedge S^W$ and Ω^W as the special case $A = S^W$ of (1.20).

Construction 1.21 (Shift of an orthogonal spectrum). We introduce a spectrum analog of the additive shift of orthogonal spaces defined in Example I.1.12. We let V be an inner product space and denote by

$$- \oplus V \;:\; \mathbf{O} \;\longrightarrow\; \mathbf{O}$$

the continuous functor given on objects by orthogonal direct sum with V, and on morphism spaces by

$$\mathbf{O}(U, W) \;\longrightarrow\; \mathbf{O}(U \oplus V, W \oplus V) \;, \quad (w, \varphi) \;\longmapsto\; ((w, 0), \varphi \oplus V) \;.$$

The *V-th shift* of an orthogonal spectrum X is the composite

$$\mathrm{sh}^V X \;=\; X \circ (- \oplus V) \;. \tag{1.22}$$

In other words, the value of $\mathrm{sh}^V X$ at an inner product space U is

$$(\mathrm{sh}^V X)(U) \;=\; X(U \oplus V)\ .$$

The orthogonal group $O(U)$ acts through the monomorphism $- \oplus V : O(U) \longrightarrow O(U \oplus V)$. The structure map $\sigma^{\mathrm{sh}^V X}_{U,W}$ of $\mathrm{sh}^V X$ is the structure map $\sigma^X_{U,W\oplus V}$ of X.

Since composition of functors is associative, the shift construction commutes on the nose with all constructions on orthogonal spectra that are given by postcomposition with a continuous based functor as in Construction 1.19. This applies in particular to smashing with and taking mapping space from a based space A, i.e.,

$$(\mathrm{sh}^V X) \wedge A \;=\; \mathrm{sh}^V(X \wedge A) \qquad \text{and} \qquad \mathrm{map}(A, \mathrm{sh}^V X) \;=\; \mathrm{sh}^V(\mathrm{map}(A, X))\ .$$

So we can – and will – omit the parentheses in expressions such as $\mathrm{sh}^V X \wedge A$.

The shift construction is also transitive in the following sense. The values of $\mathrm{sh}^V(\mathrm{sh}^W X)$ and $\mathrm{sh}^{V\oplus W} X$ at an inner product space U are given by

$$(\mathrm{sh}^V(\mathrm{sh}^W X))(U) \;=\; X((U \oplus V) \oplus W) \quad \text{respectively} \quad (\mathrm{sh}^{V\oplus W} X)(U) \;=\; X(U \oplus (V \oplus W))\ .$$

We use the effect of X on the associativity isomorphism $(U \oplus V) \oplus W \cong U \oplus (V \oplus W)$ to identify these two spaces; then we allow ourselves the abuse of notation and write

$$\mathrm{sh}^V(\mathrm{sh}^W X) \;=\; \mathrm{sh}^{V\oplus W} X\ .$$

The suspension and the shift of an orthogonal spectrum X are related by a natural morphism

$$\lambda^V_X \;:\; X \wedge S^V \;\longrightarrow\; \mathrm{sh}^V X\ . \tag{1.23}$$

In level U, this is defined as $\lambda^V_X(U) = \sigma^{\mathrm{op}}_{U,V}$, the opposite structure map (1.5), i.e., the composite

$$X(U) \wedge S^V \xrightarrow{\tau_{X(U),S^V}} S^V \wedge X(U) \xrightarrow{\sigma_{V,U}} X(V \oplus U) \xrightarrow{X(\tau_{V,U})} X(U \oplus V) = (\mathrm{sh}^V X)(U)\ .$$

In the special case $V = \mathbb{R}$ we abbreviate $\lambda^{\mathbb{R}}_X$ to $\lambda_X : X \wedge S^1 \longrightarrow \mathrm{sh}\, X$. The λ-maps are transitive in the sense that for another inner product space W, the morphism $\lambda^{V\oplus W}_X$ coincides with the two composites in the commutative square:

$$\begin{tikzcd}
X \wedge S^{V\oplus W} \arrow[r, "\cong"] & X \wedge S^V \wedge S^W \arrow[r, "\lambda^W_{X\wedge S^V}"] \arrow[d, "\lambda^V_X \wedge S^W"'] & \mathrm{sh}^W X \wedge S^V \arrow[d, "\lambda^V_{\mathrm{sh}^W X}"] & \\
& \mathrm{sh}^V X \wedge S^W \arrow[r, "\mathrm{sh}^V(\lambda^W_X)"'] & \mathrm{sh}^V(\mathrm{sh}^W X) \arrow[r, equal] & \mathrm{sh}^{V\oplus W} X
\end{tikzcd}$$

Now we let G be a compact Lie group, V a G-representation and X an orthogonal G-spectrum. Then the orthogonal spectra $X \wedge S^V$ and $\mathrm{sh}^V X$ become orthogonal G-spectra by letting G act diagonally on X and V. With respect to these diagonal actions, the morphism $\lambda^V_X : X \wedge S^V \longrightarrow \mathrm{sh}^V X$ is a morphism of orthogonal G-spectra. Our next aim is to show that λ^V_X is in fact an equivariant stable equivalence. We define a homomorphism

$$\psi^V_X \;:\; \pi^G_k(\mathrm{sh}^V X) \;\longrightarrow\; \pi^G_k(X \wedge S^V) \tag{1.24}$$

by sending the class represented by a G-map

$$f \;:\; S^{U\oplus\mathbb{R}^{n+k}} \;\longrightarrow\; X(U \oplus \mathbb{R}^n \oplus V) \;=\; (\mathrm{sh}^V X)(U \oplus \mathbb{R}^n)$$

to the class represented by the composite

$$S^{U\oplus V\oplus\mathbb{R}^{n+k}} \xrightarrow{S^U\wedge\tau_{V,\mathbb{R}^{n+k}}} S^{U\oplus\mathbb{R}^{n+k}\oplus V} \xrightarrow{f\wedge S^V} X(U\oplus\mathbb{R}^n\oplus V)\wedge S^V$$
$$\xrightarrow{X(U\oplus\tau_{\mathbb{R}^n,V})\wedge S^V} X(U\oplus V\oplus\mathbb{R}^n)\wedge S^V \ .$$

We omit the straightforward verification that this assignment is compatible with stabilization, and hence well defined. The map ψ_X^V is natural for morphisms of orthogonal G-spectra in X. Finally, we define

$$\epsilon_V \ : \ \pi_k^G(X\wedge S^V) \ \longrightarrow \ \pi_k^G(X\wedge S^V)$$

as the effect of the involution

$$X\wedge S^{-\operatorname{Id}_V} \ : \ X\wedge S^V \ \longrightarrow \ X\wedge S^V$$

induced by the linear isometry $-\operatorname{Id}_V : V \longrightarrow V$ given by multiplication by -1.

Proposition 1.25. *Let G be a compact Lie group, X an orthogonal G-spectrum and V a G-representation.*

(i) *For every integer k, each of the three composites around the triangle*

$$\begin{tikzcd} \pi_k^G(X\wedge S^V) \arrow[rr, "(\lambda_X^V)_*"] & & \pi_k^G(\operatorname{sh}^V X) \arrow[dl, "\psi_X^V"] \\ & \pi_k^G(X\wedge S^V) \arrow[ul, "\epsilon_V"] & \end{tikzcd}$$

is the respective identity.

(ii) *The morphism*

$$\lambda_X^V \ : \ X\wedge S^V \ \longrightarrow \ \operatorname{sh}^V X \ , \qquad \textit{its adjoint} \qquad \tilde{\lambda}_X^V \ : \ X \ \longrightarrow \ \Omega^V \operatorname{sh}^V X \ ,$$

the adjunction unit $\eta_X^V : X \longrightarrow \Omega^V(X\wedge S^V)$ and the adjunction counit $\epsilon_X^V : (\Omega^V X)\wedge S^V \longrightarrow X$ are $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.

Proof. We introduce an auxiliary functor $\pi^G(A;-)$ from orthogonal G-spectra to abelian groups that generalizes equivariant homotopy groups and depends on a based G-space A. We set

$$\pi^G(A;X) \ = \ \operatorname{colim}_{U\in s(\mathcal{U}_G)} [S^U\wedge A, X(U)]^G \ , \tag{1.26}$$

where the colimit is taken over the analogous stabilization maps as for π_0^G; the set $\pi^G(A;X)$ comes with a natural abelian group structure by the same reasoning as for $\pi_0^G(X)$. Then $\pi^G(S^k;X)$ is naturally isomorphic to $\pi_k^G(X)$; more generally, for a G-representation W, the adjunction bijections

$$[S^U\wedge S^{\mathbb{R}^k\oplus W}, X(U)]^G \ \cong \ [S^{U\oplus\mathbb{R}^k}, \Omega^W X(U)]^G$$

assemble into a natural isomorphism of abelian groups between $\pi^G(S^{\mathbb{R}^k\oplus W};X)$ and $\pi_k^G(\Omega^W X)$.

The definition of the map ψ_X^V has a straightforward generalization to a natural homomorphism

$$\psi_X^V \ : \ \pi^G(A;\operatorname{sh}^V X) \ \longrightarrow \ \pi^G(A;X\wedge S^V)$$

by sending the class represented by a G-map

$$f \ : \ S^U\wedge A \ \longrightarrow \ X(U\oplus V) \ = \ (\operatorname{sh}^V X)(U)$$

to the class represented by the composite

$$S^{U\oplus V}\wedge A \xrightarrow{S^U\wedge\tau_{S^V,A}} S^U\wedge A\wedge S^V \xrightarrow{f\wedge S^V} X(U\oplus V)\wedge S^V \ .$$

We omit the straightforward verification that this assignment is compatible with stabilization, and hence well defined.

We show the statement that each of the three composites around the triangle

$$\begin{tikzcd}
\pi^G(A; X \wedge S^V) \arrow[rr, "(\lambda_X^V)_*"] & & \pi^G(A; \mathrm{sh}^V X) \arrow[dl, "\psi_X^V"] \\
& \pi^G(A; X \wedge S^V) \arrow[ul, "\epsilon_V"] &
\end{tikzcd}$$

is the respective identity.

We consider a based continuous G-map $f : S^U \wedge A \longrightarrow X(U) \wedge S^V$ that represents a class in $\pi^G(A; X \wedge S^V)$. Then the class $\epsilon_V(\psi_X^V((\lambda_X^V)_*\langle f\rangle))$ is represented by the composite

$$S^{U\oplus V} \wedge A \;\cong\; S^U \wedge A \wedge S^V \;\xrightarrow{f\wedge S^V}\; X(U) \wedge S^V \wedge S^V \;\xrightarrow{\sigma^{\mathrm{op}}_{U,V}\wedge S^{-\mathrm{Id}_V}}\; X(U \oplus V) \wedge S^V \;.$$

The map $V \oplus (-\mathrm{Id}_V) : V \oplus V \longrightarrow V \oplus V$ is homotopic, through G-equivariant linear isometries, to the twist map $\tau_{V,V} : V \oplus V \longrightarrow V \oplus V$ that interchanges the two summands. So $\epsilon_V(\psi_X^V((\lambda_X^V)_*\langle f\rangle))$ is also represented by the left vertical composite in the following diagram of based continuous G-maps:

$$\begin{tikzcd}
S^U \wedge S^V \wedge A \arrow[r, "\tau_{U,V}\wedge A"] \arrow[d, "S^U\wedge\tau_{A,S^V}"'] & S^V \wedge S^U \wedge A \arrow[dd, "S^V\wedge f"] \arrow[ddd, bend left, "V\diamond f"] \\
S^U \wedge A \wedge S^V \arrow[d, "f\wedge S^V"'] & \\
X(U) \wedge S^V \wedge S^V \arrow[d, "X(U)\wedge\tau_{V,V}"'] \arrow[dr, "\tau_{X(U)\wedge S^V, S^V}"] & \\
X(U) \wedge S^V \wedge S^V \arrow[r, "\tau_{X(U),S^V}\wedge S^V"'] \arrow[d, "\sigma^{\mathrm{op}}_{U,V}\wedge S^V"'] & S^V \wedge X(U) \wedge S^V \arrow[d, "\sigma_{V,U}\wedge S^V"'] \\
X(U \oplus V) \wedge S^V \arrow[r, "X(\tau_{U,V})\wedge S^V"'] & X(V \oplus U) \wedge S^V
\end{tikzcd}$$

The right vertical composite is the stabilization of f, so it represents the same class in $\pi^G(A; X \wedge S^V)$. Since the left and right vertical composites differ by conjugation with an equivariant isometry, they also represent the same class in $\pi^G(A; X \wedge S^V)$. Altogether this shows that the composite $\epsilon_V \circ \psi_X^V \circ (\lambda_X^V)_*$ is the identity. Since ϵ_V^2 is the identity, this also implies that the composite $\psi_X^V \circ (\lambda_X^V)_* \circ \epsilon_V$ is the identity.

The remaining case is similar. We consider a based continuous G-map $g : S^U \wedge A \longrightarrow X(U \oplus V) = (\mathrm{sh}^V X)(U)$ that represents a class in $\pi^G(A; \mathrm{sh}^V X)$. Then the class $(\lambda_X^V)_*(\epsilon_V(\psi_X^V\langle g\rangle))$ is represented by the composite

$$\begin{aligned}
S^U \wedge S^V \wedge A \;&\xrightarrow{S^U\wedge\tau_{S^V,A}}\; S^U \wedge A \wedge S^V \;\xrightarrow{g\wedge S^V}\; X(U \oplus V) \wedge S^V \\
&\xrightarrow{X(U\oplus V)\wedge S^{-\mathrm{Id}_V}}\; X(U \oplus V) \wedge S^V \;\xrightarrow{\sigma^{\mathrm{op}}_{U\oplus V,V}}\; X(U \oplus V \oplus V) \;=\; (\mathrm{sh}^V X)(U \oplus V)\;.
\end{aligned}$$

Since

$$\sigma^{\mathrm{op}}_{U\oplus V,V} \circ (X(U \oplus V) \wedge S^{-\mathrm{Id}}) \;=\; X(U \oplus V \oplus (-\mathrm{Id})) \circ \sigma^{\mathrm{op}}_{U\oplus V,V}$$

and $V \oplus (-\operatorname{Id}_V) : V \oplus V \longrightarrow V \oplus V$ is G-homotopic to the twist $\tau_{V,V}$, the class $(\lambda_X^V)_*(\epsilon_V(\psi_X^V\langle g\rangle))$ is also represented by the left vertical composite in the following diagram:

$$\begin{tikzcd}
S^U \wedge S^V \wedge A \arrow[r, "\tau_{U,V}\wedge A"] \arrow[d, "S^U\wedge\tau_{S^V,A}"'] & S^V \wedge S^U \wedge A \arrow[dd, "S^V\wedge g"] \arrow[ddd, bend left, "V\diamond g"] \\
S^U \wedge A \wedge S^V \arrow[d, "g\wedge S^V"'] & \\
X(U\oplus V)\wedge S^V \arrow[d, "\sigma^{\mathrm{op}}_{U\oplus V,V}"'] \arrow[dr, "\tau_{X(U)\wedge S^V,S^V}"] & \\
X(U\oplus V\oplus V) \arrow[d, "X(U\oplus\tau_{V,V})"'] \arrow[dr, "X(\tau_{U\oplus V,V})"] & S^V\wedge X(U\oplus V) \arrow[d, "\sigma_{V,U\oplus V}"] \\
X(U\oplus V\oplus V) \arrow[r, "X(\tau_{U,V}\oplus V)"'] \arrow[d, equal] & X(U\oplus V\oplus V) \arrow[d, equal] \\
(\mathrm{sh}^V X)(U\oplus V) \arrow[r, "(\mathrm{sh}^V X)(\tau_{U,V})"'] & (\mathrm{sh}^V X)(U\oplus V)
\end{tikzcd}$$

The right vertical composite is the stabilization of g, so it represents the same class in $\pi^G(A;\mathrm{sh}^V X)$. Since the left and right vertical composites differ by conjugation with an equivariant isometry, they represent the same class, so the composite $(\lambda_X^V)_* \circ \epsilon_V \circ \psi_X^V$ is the identity.

(i) For $k > 0$, part (i) is a special case $A = S^k$ of the discussion above. To deduce the claim for negative dimensional homotopy groups we use the isomorphism of orthogonal G-spectra

$$\tau_{k,V} \ : \ \mathrm{sh}^k(\mathrm{sh}^V X) \ \cong \ \mathrm{sh}^V(\mathrm{sh}^k X)$$

whose value at an inner product space U is the map

$$X(U \oplus \tau_{\mathbb{R}^k,V}) \ : \ X(U\oplus\mathbb{R}^k\oplus V) \ \cong \ X(U\oplus V\oplus\mathbb{R}^k) \ .$$

Then the following diagram commutes:

$$\begin{tikzcd}
\pi^G_{-k}(X\wedge S^V) \arrow[r, "(\lambda_X^V)_*"] \arrow[dd, equal] & \pi^G_{-k}(\mathrm{sh}^V X) \arrow[r, "\epsilon_V\circ\psi_X^V"] \arrow[d, equal] & \pi^G_{-k}(X\wedge S^V) \arrow[dd, equal] \\
& \pi^G_0(\mathrm{sh}^k(\mathrm{sh}^V X)) \arrow[d, "(\tau_{k,V})_*"', "\cong"] & \\
\pi^G_0(\mathrm{sh}^k X\wedge S^V) \arrow[r, "(\lambda^V_{\mathrm{sh}^k X})_*"'] & \pi^G_0(\mathrm{sh}^V(\mathrm{sh}^k X)) \arrow[r, "\epsilon_V\circ\psi^V_{\mathrm{sh}^k X}"'] & \pi^G_0(\mathrm{sh}^k X\wedge S^V)
\end{tikzcd}$$

So the claim in dimension $-k$ for the orthogonal G-spectrum X is a consequence of the previously established claim in dimension 0 for the orthogonal G-spectrum $\mathrm{sh}^k X$.

(ii) We start with the morphism $\tilde\lambda_X^V$, which can be treated fairly directly. We treat the case $k \geq 0$ and leave the analogous argument for $k < 0$ to the reader. We define a map in the opposite direction

$$\kappa \ : \ \pi^G_k(\Omega^V \mathrm{sh}^V X) \ \longrightarrow \ \pi^G_k(X) \ .$$

We let $g : S^{U\oplus\mathbb{R}^k} \longrightarrow \Omega^V X(U\oplus V) = (\Omega^V \operatorname{sh}^V X)(U)$ represent a class of the left hand side. The map κ sends $[g]$ to the class represented by the composite

$$S^{U\oplus V\oplus\mathbb{R}^k} \xrightarrow{S^U\wedge\tau_{V,\mathbb{R}^k}} S^{U\oplus\mathbb{R}^k\oplus V} \xrightarrow{g^\flat} X(U\oplus V)\ ,$$

where $g^\flat$ is the adjoint of g. This is compatible with stabilization.

We claim that the map κ is injective. Indeed, if $g : S^{U\oplus\mathbb{R}^k} \longrightarrow \Omega^V X(U\oplus V) = (\Omega^V \operatorname{sh}^V X)(U)$ represents an element in the kernel of κ, then after increasing U, if necessary, the composite $g^\flat \circ (S^U \wedge \tau_{V,\mathbb{R}^k})$ is G-equivariantly null-homotopic. But then $g^\flat$, and hence also its adjoint g, are equivariantly null-homotopic. So κ is injective.

The composite $\kappa \circ (\tilde\lambda^V_X)_*$ sends the class of a G-map $f : S^{U\oplus\mathbb{R}^k} \longrightarrow X(U)$ to the composite

$$S^{U\oplus V\oplus\mathbb{R}^k} \xrightarrow{S^U\wedge\tau_{W,\mathbb{R}^k}} S^{U\oplus\mathbb{R}^k\oplus V} \xrightarrow{(\eta^V_X(U)\circ f)^\flat} X(U\oplus V)\ . \tag{1.27}$$

The adjoint $(\eta^V_X(U)\circ f)^\flat$ coincides with the composite

$$S^{U\oplus\mathbb{R}^k\oplus V} \xrightarrow{f\wedge S^V} X(U)\wedge S^V \xrightarrow{\sigma^{\mathrm{op}}_{U,V}} X(U\oplus V)\ ,$$

so the composite (1.27) is the stabilization $f\diamond V$. Since $f\diamond V$ represents the same class as f, this proves that $\kappa\circ(\tilde\lambda^V_X)_*$ is the identity. Since κ is also injective, the map $(\tilde\lambda^V_X)_*$ is bijective.

If H is a closed subgroup of G we apply the previous argument to the underlying H-representation of V to conclude that $\tilde\lambda^V_X$ induces an isomorphism for π^H_*. So $(\tilde\lambda^V_X)_*$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra.

Now we treat the morphism λ^V_X. Again we show a more general statement, namely that for every pair of G-representations V and W the morphism

$$\Omega^W(\lambda^V_X)\ :\ \Omega^W(X\wedge S^V)\ \longrightarrow\ \Omega^W(\operatorname{sh}^V X)$$

is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. We start with the effect on G-equivariant homotopy groups. For $k\ge 0$ this follows by applying part (i) with $A = S^{\mathbb{R}^k\oplus W}$ and exploiting the natural isomorphism $\pi^G_k(\Omega^W Y)\cong\pi^G(S^{\mathbb{R}^k\oplus W};Y)$. To get the same conclusion for negative dimensional homotopy groups we exploit that $\pi^G_{-k}(Y) = \pi^G_0(\operatorname{sh}^k Y)$, by definition. Moreover, the following diagram commutes

$$\begin{tikzcd}
\pi^G_{-k}(\Omega^W(X\wedge S^V)) \arrow[rr, "\pi^G_{-k}(\Omega^W(\lambda^V_X))"] \arrow[d, equal] & & \pi^G_{-k}(\Omega^W(\operatorname{sh}^V X)) \arrow[d, equal] \\
\pi^G_0(\Omega^W(\operatorname{sh}^k X\wedge S^V)) \arrow[r, "\pi^G_0(\Omega^W(\lambda^V_{\operatorname{sh}^k X}))"'] & \pi^G_0(\Omega^W(\operatorname{sh}^V(\operatorname{sh}^k X))) \arrow[r, "\pi^G_0(\Omega^W\tau)"'] & \pi^G_0(\Omega^W(\operatorname{sh}^k(\operatorname{sh}^V X)))
\end{tikzcd}$$

where τ is the isomorphism of orthogonal G-spectra whose value at U is the homeomorphism

$$(\operatorname{sh}^V(\operatorname{sh}^k X))(U)\ =\ X(U\oplus V\oplus\mathbb{R}^k)\ \xrightarrow{X(U\oplus\tau_{V,\mathbb{R}^k})}\ X(U\oplus\mathbb{R}^k\oplus V)\ =\ (\operatorname{sh}^k(\operatorname{sh}^V X))(U)\ .$$

So the previous argument applied to the spectrum $\operatorname{sh}^k X$ shows that the morphism $\Omega^W(\lambda^V_X)$ also induces isomorphisms on G-equivariant homotopy groups in negative dimensions. If H is any closed subgroup of G, then we consider the underlying H-representations of V and W and conclude that the morphism $\Omega^W(\lambda^V_X)$ induces isomorphisms on π^H_*. This proves that $\Omega^W(\lambda^V_X)$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. The special case $W=0$ proves that λ^V_X is a $\underline{\pi}_*$-isomorphism.

The morphism $\tilde\lambda^V_X$ factors as the composite

$$X\ \xrightarrow{\eta^V_X}\ \Omega^V(X\wedge S^V)\ \xrightarrow{\Omega^V(\lambda^V_X)}\ \Omega^V\operatorname{sh}^V X\ .$$

Since both $\tilde\lambda^V_X$ and $\Omega^V(\lambda^V_X)$ are $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra, so is the adjunction unit η^V_X.

Finally, we treat the adjunction counit ϵ_X^V. The two homomorphisms of orthogonal G-spectra

$$\Omega^V(\tilde\lambda_X^V)\ ,\ \tilde\lambda_{\Omega^V X}^V\ :\ \Omega^V X\ \longrightarrow\ \Omega^V(\Omega^V \operatorname{sh}^V X)$$

are *not* the same; they differ by the involution on the target that interchanges to two V-loop coordinates. An equivariant homotopy of linear isometries from $\tau_V : V \oplus V \longrightarrow V \oplus V$ to $(-\operatorname{Id}) \oplus \operatorname{Id}$ thus induces an equivariant homotopy between the morphism $\Omega^V(\tilde\lambda_X^V)$ and the composite

$$\Omega^V X\ \xrightarrow{\tilde\lambda_{\Omega^V X}^V}\ \Omega^V(\Omega^V \operatorname{sh}^V X)\ \xrightarrow{\Omega^V \operatorname{map}(S^{-\operatorname{Id}},\operatorname{sh}^V X)}\ \Omega^V(\Omega^V \operatorname{sh}^V X)\ .$$

Passing to adjoints shows that the square of morphisms of orthogonal G-spectra

$$\begin{tikzcd}
(\Omega^V X)\wedge S^V \arrow[r, "\epsilon_X^V"] \arrow[d, "\lambda_{\Omega^V X}^V"'] & X \arrow[d, "\tilde\lambda_X^V"] \\
\operatorname{sh}^V \Omega^V X \arrow[r, "\operatorname{sh}^V \operatorname{map}(S^{-\operatorname{Id}},X)"'] & \operatorname{sh}^V \Omega^V X
\end{tikzcd}$$

commutes up to G-equivariant homotopy. Since the other three maps are $\underline{\pi}_*$-isomorphisms, so is ϵ_X^V. □

Next we define the *loop isomorphism*

$$\alpha\ :\ \pi_k^G(\Omega X)\ \longrightarrow\ \pi_{k+1}^G(X)\ . \tag{1.28}$$

We represent a given class in $\pi_k^G(\Omega X)$ by a based G-map $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow \Omega X(V \oplus \mathbb{R}^n)$ and let $\tilde f : S^{V\oplus\mathbb{R}^{n+k+1}} \longrightarrow X(V \oplus \mathbb{R}^n)$ denote the adjoint of f, which represents an element of $\pi_{k+1}^G(X)$. Then we set

$$\alpha[f]\ =\ [\tilde f]\ .$$

Next we define the *suspension isomorphism*

$$-\wedge S^1\ :\ \pi_k^G(X)\ \longrightarrow\ \pi_{k+1}^G(X\wedge S^1)\ . \tag{1.29}$$

We represent a given class in $\pi_k^G(X)$ by a based G-map $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow X(V \oplus \mathbb{R}^n)$; then $f \wedge S^1 : S^{V\oplus\mathbb{R}^{n+k+1}} \longrightarrow X(V \oplus \mathbb{R}^n) \wedge S^1$ represents a class in $\pi_{k+1}^G(X \wedge S^1)$, and we set

$$[f]\wedge S^1\ =\ [f\wedge S^1]\ .$$

Proposition 1.30. *Let G be a compact Lie group, X an orthogonal G-spectrum and k an integer. Then the loop isomorphism* (1.28) *and the suspension isomorphism* (1.29) *are isomorphisms of abelian groups.*

Proof. The inverse to the loop map (1.28) is given by sending the class of a G-map $g : S^{V\oplus\mathbb{R}^{n+k+1}} \longrightarrow X(V \oplus \mathbb{R}^n)$ to the class of its adjoint $S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow \Omega X(V \oplus \mathbb{R}^n)$. The suspension homomorphism is the composite of the two isomorphisms

$$\pi_k^G(X)\ \xrightarrow{(\eta_X)_*}\ \pi_k^G(\Omega(X\wedge S^1))\ \xrightarrow{\alpha}\ \pi_{k+1}^G(X\wedge S^1)\ ;$$

since $(\eta_X)_*$ is an isomorphism by Proposition 1.25 (ii), so is the suspension isomorphism. □

A key feature that distinguishes stable from unstable equivariant homotopy theory – and at the same time an important calculational tool – is the fact that mapping cones give rise to long exact sequences of equivariant homotopy groups. Our next aim is to establish this long exact sequence, see Proposition 1.37.

Construction 1.31 (Mapping cone and homotopy fiber). The *(reduced) mapping cone* Cf of a morphism of based spaces $f : A \longrightarrow B$ is defined by

$$Cf\ =\ (A\wedge[0,1])\cup_f B\ .$$

Here the unit interval $[0,1]$ is pointed by 0, so that $A \wedge [0,1]$ is the reduced cone of A. The mapping cone comes with an embedding $i : B \longrightarrow Cf$ and a projection $p : Cf \longrightarrow A \wedge S^1$; the projection sends B to the basepoint and is given on $A \wedge [0,1]$ by $p(a,x) = a \wedge t(x)$, where

$$t \ : \ [0,1] \ \longrightarrow \ S^1 \qquad \text{defined as} \qquad t(x) \ = \ \frac{2x-1}{x(1-x)} \ . \tag{1.32}$$

What is relevant about the map t is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and $S^1 = \mathbb{R} \cup \{\infty\}$, and that it satisfies $t(1-x) = -t(x)$.

The *mapping cone* Cf of a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is now defined levelwise by

$$(Cf)(V) \ = \ C(f(V)) \ = \ (X(V) \wedge [0,1]) \cup_{f(V)} Y(V) \ ,$$

the reduced mapping cone of $f(V) : X(V) \longrightarrow Y(V)$. The groups G and $O(V)$ act on $(Cf)(V)$ through the given action on $X(V)$ and $Y(V)$ and trivially on the interval. The embeddings $i(V) : Y(V) \longrightarrow C(f(V))$ and projections $p(V) : C(f(V)) \longrightarrow X(V) \wedge S^1$ assemble into morphisms of orthogonal G-spectra

$$i \ : \ Y \ \longrightarrow \ Cf \qquad \text{and} \qquad p \ : \ Cf \ \longrightarrow \ X \wedge S^1 \ . \tag{1.33}$$

We define a *connecting homomorphism* $\partial : \pi^G_{k+1}(Cf) \longrightarrow \pi^G_k(X)$ as the composite

$$\pi^G_{k+1}(Cf) \ \xrightarrow{\pi^G_{k+1}(p)} \ \pi^G_{k+1}(X \wedge S^1) \ \xrightarrow{-\wedge S^{-1}} \ \pi^G_k(X) \ , \tag{1.34}$$

where the first map is the effect of the projection $p : Cf \longrightarrow X \wedge S^1$ on homotopy groups, and the second map is the inverse of the suspension isomorphism $- \wedge S^1 : \pi^G_k(X) \longrightarrow \pi^G_{k+1}(X \wedge S^1)$, compare (1.29).

The *homotopy fiber* is the construction 'dual' to the mapping cone. The homotopy fiber of a continuous map $f : A \longrightarrow B$ of based spaces is the fiber product

$$F(f) \ = \ * \times_B B^{[0,1]} \times_B A \ = \ \{(\lambda, a) \in B^{[0,1]} \times A \mid \lambda(0) = *, \, \lambda(1) = f(a)\} \ ,$$

i.e., the space of paths in B starting at the basepoint and equipped with a lift of the endpoint to A. As basepoint of the homotopy fiber we take the pair consisting of the constant path at the basepoint of B and the basepoint of A. The homotopy fiber comes with maps

$$\Omega B \ \xrightarrow{i} \ F(f) \ \xrightarrow{p} \ A \ ;$$

the map p is the projection to the second factor and the value of the map i on a based loop $\omega : S^1 \longrightarrow B$ is

$$i(\omega) = (\omega \circ t, *) \ ,$$

where $t : [0,1] \longrightarrow S^1$ is defined in (1.32).

The homotopy fiber $F(f)$ of a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra is the orthogonal G-spectrum defined by

$$F(f)(V) \ = \ F(f(V)) \ ,$$

the homotopy fiber of $f(V) : X(V) \longrightarrow Y(V)$. The groups G and $O(V)$ act on $F(f)(V)$ through the given action on $X(V)$ and $Y(V)$ and trivially on the interval. Put another way, the homotopy fiber is the pullback in the cartesian square of orthogonal G-spectra:

$$\begin{array}{ccc} F(f) & \xrightarrow{\quad p \quad} & X \\ \downarrow & & \downarrow{\scriptstyle (*,f)} \\ Y^{[0,1]} & \xrightarrow[\lambda \ \mapsto \ (\lambda(0),\lambda(1))]{} & Y \times Y \end{array}$$

The inclusions $i(V) : \Omega Y(V) \longrightarrow F(f(V))$ and projections $p(V) : F(f(V)) \longrightarrow X(V)$ assemble into morphisms of orthogonal G-spectra

$$i \ : \ \Omega Y \ \longrightarrow \ F(f) \qquad \text{and} \qquad p \ : \ F(f) \ \longrightarrow \ X \ .$$

We define a *connecting homomorphism* $\partial : \pi_{k+1}^G(Y) \longrightarrow \pi_k^G(F(f))$ as the composite

$$\pi_{k+1}^G(Y) \xrightarrow{\alpha^{-1}} \pi_k^G(\Omega Y) \xrightarrow{\pi_k^G(i)} \pi_k^G(F(f)) \ , \tag{1.35}$$

where the first map is the inverse of the loop isomorphism (1.28).

Now we recall the long exact sequences relating the equivariant homotopy groups of source and target of a morphism $f : X \longrightarrow Y$ and the mapping cone respectively homotopy fiber of f. The proof of exactness for the mapping cone sequence will need some elementary homotopies that we spell out in the next proposition.

Proposition 1.36. *Let G be a topological group.*

(i) *For every continuous based map $f : A \longrightarrow B$ of based G-spaces the composites*

$$A \xrightarrow{f} B \xrightarrow{i} Cf \qquad \text{and} \qquad F(f) \xrightarrow{p} A \xrightarrow{f} B$$

are naturally based G-null-homotopic. Moreover, the diagram

$$\begin{tikzcd} & CA \cup_f CB \arrow[dl, "p_A \cup *"'] \arrow[dr, "* \cup p_B"] & \\ A \wedge S^1 \arrow[rr, "f \wedge \tau"'] & & B \wedge S^1 \end{tikzcd}$$

commutes up to natural, based G-homotopy, where τ is the sign involution of S^1 given by $x \mapsto -x$.

(ii) *For every based G-space Z the map $p_Z \cup * : CZ \cup_{Z \times 1} CZ \longrightarrow Z \wedge S^1$ which collapses the second cone is a based G-homotopy equivalence.*

(iii) *Let $f : A \longrightarrow B$ and $\beta : Z \longrightarrow B$ be morphisms of based G-spaces such that the composite $i\beta : Z \longrightarrow Cf$ is equivariantly null-homotopic. Then there exists a based G-map $h : Z \wedge S^1 \longrightarrow A \wedge S^1$ such that $(f \wedge S^1) \circ h : Z \wedge S^1 \longrightarrow B \wedge S^1$ is equivariantly homotopic to $\beta \wedge S^1$.*

Proof. (i) We specify natural G-homotopies by explicit formulae. The map $if : A \longrightarrow Cf$ is null-homotopic by $A \times [0,1] \longrightarrow Cf, (a,s) \mapsto (a,s)$, i.e., the composite of the canonical maps $A \times [0,1] \longrightarrow A \wedge [0,1]$ and $A \wedge [0,1] \longrightarrow Cf$. The map $fp : F(f) \longrightarrow B$ is null-homotopic by $F(f) \times [0,1] \longrightarrow B, (\lambda, a, s) \mapsto \lambda(s)$.

The homotopy for the triangle will be glued together from two pieces. We define a based homotopy $H : CA \times [0,1] \longrightarrow B \wedge S^1$ by the formula

$$H(a,s,u) \ = \ f(a) \wedge t(2-s-u)$$

which is to be interpreted as the basepoint if $2 - s - u \geq 1$. Another based homotopy $H' : CB \times [0,1] \longrightarrow B \wedge S^1$ is given by the formula

$$H'(b,s,u) \ = \ b \wedge t(s-u) \ ,$$

which is to be interpreted as the basepoint if $s \leq u$. The two homotopies are compatible in the sense that

$$H(a,1,u) \ = \ f(a) \wedge t(1-u) \ = \ H'(f(a),1,u) \ ,$$

for all $a \in A$ and $u \in [0,1]$. So H and H' glue and yield a homotopy

$$(CA \cup_f CB) \times [0,1] \cong (CA \times [0,1]) \cup_{f \times [0,1]} (CB \times [0,1]) \xrightarrow{H \cup H'} B \wedge S^1 \ .$$

For $u = 0$ this homotopy starts with the map $* \cup p_B$, and it ends for $u = 1$ with the map $(\tau \wedge f) \circ (p_A \cup *)$.

(ii) Since the functor $Z \wedge -$ is a left adjoint and $Z \wedge \{0,1\} \cong Z \times 1$, the space $CZ \cup_{1 \times Z} CZ$ is homeomorphic to the smash product of Z and the pushout $[0,1] \cup_{\{0,1\}} [0,1]$. This identification

$$CZ \cup_{Z \times 1} CZ \ \cong \ Z \wedge ([0,1] \cup_{\{0,1\}} [0,1])$$

turns the map p_Z into the map

$$Z \wedge (t \cup *) \ : \ Z \wedge ([0,1] \cup_{\{0,1\}} [0,1]) \ \longrightarrow \ Z \wedge S^1 \ .$$

So the claim follows from the fact that the map $t \cup * : [0,1] \cup_{\{0,1\}} [0,1] \longrightarrow S^1$ is a based homotopy equivalence.

(iii) Let $H : CZ = Z \wedge [0,1] \longrightarrow Cf$ be a based, equivariant null-homotopy of the composite $i\beta : Z \longrightarrow Cf$, i.e., $H(z,1) = i(\beta(z))$ for all $z \in Z$. The composite $p_A H : CZ \longrightarrow A \wedge S^1$ then factors as $p_A H = hp_Z$ for a unique G-map $h : Z \wedge S^1 \longrightarrow A \wedge S^1$. We claim that h has the required property.

To prove the claim we need the G-homotopy equivalence $p_Z \cup * : CZ \cup_{Z \times 1} CZ \longrightarrow Z \wedge S^1$ which collapses the second cone. We obtain a sequence of equalities and G-homotopies

$$\begin{aligned}
(f \wedge S^1) \circ h \circ (p_Z \cup *) \ &= \ (f \wedge S^1) \circ (p_A \cup *) \circ (H \cup C(\beta)) \\
&= \ (B \wedge \tau) \circ (f \wedge \tau) \circ (p_A \cup *) \circ (H \cup C(\beta)) \\
&\simeq \ (B \wedge \tau) \circ (* \cup p_B) \circ (H \cup C(\beta)) \\
&= \ (B \wedge \tau) \circ (\beta \wedge S^1) \circ (* \cup p_Z) \\
&= \ (\beta \wedge S^1) \circ (Z \wedge \tau) \circ (* \cup p_Z) \ \simeq \ (\beta \wedge S^1) \circ (p_Z \cup *)
\end{aligned}$$

Here $H \cup C(\beta) : CZ \cup_{1 \times Z} CZ \longrightarrow Cf \cup_B CB \cong CA \cup_f CB$ and τ is the sign involution of S^1. The two homotopies result from part (i) applied to f respectively the identity of Z, and we used the naturality of various constructions. Since the map $p_Z \cup *$ is a G-homotopy equivalence by part (ii), this proves that $(f \wedge S^1) \circ h$ is G-homotopic to $\beta \wedge S^1$. □

Now we are ready to prove the long exact homotopy group sequences for mapping cones and homotopy fibers.

Proposition 1.37. *For every compact Lie group G and every morphism $f : X \longrightarrow Y$ of orthogonal G-spectra the long sequences of equivariant homotopy groups*

$$\cdots \ \longrightarrow \ \pi_{k+1}^G(Cf) \ \xrightarrow{\partial} \ \pi_k^G(X) \ \xrightarrow{\pi_k^G(f)} \ \pi_k^G(Y) \ \xrightarrow{\pi_k^G(i)} \ \pi_k^G(Cf) \ \longrightarrow \ \cdots$$

and

$$\cdots \ \longrightarrow \ \pi_{k+1}^G(Y) \ \xrightarrow{\partial} \ \pi_k^G(F(f)) \ \xrightarrow{\pi_k^G(p)} \ \pi_k^G(X) \ \xrightarrow{\pi_k^G(f)} \ \pi_k^G(Y) \ \longrightarrow \ \cdots$$

are exact.

Proof. We start with exactness of the first sequence at $\pi_k^G(Y)$. The composite of $f : X \longrightarrow Y$ and the inclusion $i : Y \longrightarrow Cf$ is equivariantly null-homotopic, so it induces the trivial map on π_k^G. It remains to show that every element in the kernel of $\pi_k^G(i) : \pi_k^G(Y) \longrightarrow \pi_k^G(Cf)$ is in the image of $\pi_k^G(f)$. We show this for $k \geq 0$; for $k < 0$ we can either use a similar argument or exploit that $\pi_k^G(X) = \pi_0^G(\mathrm{sh}^{\mathbb{R}^{-k}} X)$ and shifting commutes with the formation of mapping cones. We let $\beta : S^{V \oplus \mathbb{R}^k} \longrightarrow Y(V)$ represent an element in the kernel of $\pi_k^G(i)$. By increasing V, if necessary, we can assume that the composite of β with the inclusion $i : Y(V) \longrightarrow (Cf)(V) = C(f(V))$ is equivariantly null-homotopic. Proposition 1.36 (iii) provides a G-map $h : S^{V \oplus \mathbb{R}^k} \wedge S^1 \longrightarrow X(V) \wedge S^1$ such that $(f(V) \wedge S^1) \circ h$ is G-homotopic to $\beta \wedge S^1$. The composite

$$S^{V \oplus \mathbb{R}^{k+1}} \ \xrightarrow{h} \ X(V) \wedge S^1 \ \xrightarrow{\sigma^{\mathrm{op}}_{V,\mathbb{R}}} \ X(V \oplus \mathbb{R})$$

represents an equivariant homotopy class in $\pi_k^G(X)$ and we have

$$\begin{aligned}
\pi_k^G(f)\langle \sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ h \rangle \ &= \ \langle f(V \oplus \mathbb{R}) \circ \sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ h \rangle \\
&= \ \langle \sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ (f(V) \wedge S^1) \circ h \rangle \ = \langle \sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ (\beta \wedge S^1) \rangle \ = \ (-1)^k \cdot \langle \beta \rangle \ .
\end{aligned}$$

To justify the last equation we let $\varphi : V \longrightarrow V \oplus \mathbb{R}$ denote the embedding of the first summand. Then the maps

$$\sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ (\beta \wedge S^1) \ : \ S^{V \oplus \mathbb{R}^k \oplus \mathbb{R}} \ \longrightarrow \ Y(V \oplus \mathbb{R}) \qquad \text{and} \qquad \varphi_*(\beta) \ : \ S^{V \oplus \mathbb{R} \oplus \mathbb{R}^k} \ \longrightarrow \ Y(V \oplus \mathbb{R})$$

differ by the permutation of the source that moves one sphere coordinate past k other sphere coordinates; this permutation has degree k, so Proposition 1.14 (ii) establishes the last equation. Altogether this shows that the class represented by β is the image under $\pi_k^G(f)$ of the class $(-1)^k \cdot \langle \sigma^{\mathrm{op}}_{V,\mathbb{R}} \circ h \rangle$.

We now deduce the exactness at $\pi_k^G(Cf)$ and $\pi_{k-1}^G(X)$ by comparing the mapping cone sequence for $f : X \longrightarrow Y$ to the mapping cone sequence for the morphism $i : Y \longrightarrow Cf$ (shifted to the left). We observe that the collapse map

$$* \cup p \ : \ Ci \cong CY \cup_f CX \ \longrightarrow \ X \wedge S^1$$

is an equivariant homotopy equivalence, and thus induces an isomorphism of equivariant homotopy groups. Indeed, a homotopy inverse

$$r \ : \ X \wedge S^1 \ \longrightarrow \ CY \cup_f CX$$

is defined by the formula

$$r(x \wedge s) \ = \ \begin{cases} (x, 2s) \quad \in CX & \text{for } 0 \le s \le 1/2, \text{ and} \\ (f(x), 2-2s) \in CY & \text{for } 1/2 \le s \le 1, \end{cases}$$

which is to be interpreted levelwise. We omit the explicit G-homotopies $r(* \cup p) \simeq \mathrm{Id}$ and $(* \cup p)r \simeq \mathrm{Id}$.

Now we consider the diagram

$$\begin{tikzcd} Cf \arrow[r, "i_i"] \arrow[dr, "p"'] & Ci \arrow[r, "p_i"] \arrow[d, "*\cup p"] & Y \wedge S^1 \\ & X \wedge S^1 \arrow[ur, "f \wedge S^1"'] & \end{tikzcd}$$

whose upper row is part of the mapping cone sequence for the morphism $i : Y \longrightarrow Cf$. The left triangle commutes on the nose and the right triangle commutes up to G-homotopy. We get a commutative diagram

$$\begin{tikzcd} \pi_k^G(Y) \arrow[r, "\pi_k^G(i)"] \arrow[d, equal] & \pi_k^G(Cf) \arrow[r, "\pi_k^G(i_i)"] \arrow[d, equal] & \pi_k^G(Ci) \arrow[r, "\partial"] \arrow[d, "(-\wedge S^{-1})\circ(*\cup p)_*", "\cong"'] & \pi_{k-1}^G(Y) \arrow[d, equal] \\ \pi_k^G(Y) \arrow[r, "\pi_k^G(i)"'] & \pi_k^G(Cf) \arrow[r, "\partial"'] & \pi_{k-1}^G(X) \arrow[r, "\pi_k^G(f)"'] & \pi_{k-1}^G(Y) \end{tikzcd}$$

(using, for the right square, the naturality of the suspension isomorphism). By the previous paragraph, applied to $i : Y \longrightarrow Cf$ instead of f, the upper row is exact at $\pi_k^G(Cf)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_k^G(Cf)$. But the morphism f was arbitrary, so when applied to $i : Y \longrightarrow Cf$ instead of f, we obtain that the upper row is exact at $\pi_k^G(Ci)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\pi_{k-1}^G(X)$. This finishes the proof of exactness of the first sequence.

Now we come to why the second sequence is exact. We show the claim for $k \ge 0$, the other case being similar. For every $V \in s(\mathcal{U}_G)$ the sequence $F(f)(V) = F(f(V)) \longrightarrow X(V) \longrightarrow Y(V)$ is an equivariant homotopy fiber sequence. So for every based G-CW-complex A, the long sequence of based sets

$$\cdots \ \longrightarrow \ [A, \Omega Y(V)]^G \ \xrightarrow{[A,i(V)]} [A, F(f(V))]^G \ \xrightarrow{[A,p(V)]^G} \ [A, X(V)]^G \ \xrightarrow{[A,f(V)]^G} \ [A, Y(V)]^G$$

is exact. We take $A = S^{V \oplus \mathbb{R}^k}$ and form the colimit over the poset $s(\mathcal{U}_G)$. Since filtered colimits are exact the resulting sequence of colimits is again exact, and that proves the second claim. $\square$

COROLLARY 1.38. *Let G be a compact Lie group and k an integer.*

(i) *For every family of orthogonal G-spectra $\{X_i\}_{i\in I}$ the canonical map*

$$\bigoplus_{i\in I} \pi_k^G(X_i) \;\longrightarrow\; \pi_k^G\left(\bigvee_{i\in I} X_i\right)$$

is an isomorphism.

(ii) *For every finite indexing set I and every family $\{X_i\}_{i\in I}$ of orthogonal G-spectra the canonical map*

$$\pi_k^G\left(\prod_{i\in I} X_i\right) \;\longrightarrow\; \prod_{i\in I} \pi_k^G(X_i)$$

is an isomorphism.

(iii) *For every finite family of orthogonal G-spectra the canonical morphism from the wedge to the product induces isomorphisms of G-equivariant homotopy groups.*

PROOF. (i) We first show the special case of two summands. If X and Y are two orthogonal G-spectra, then the wedge inclusion $i_X : X \longrightarrow X \vee Y$ has a retraction. So the associated long exact homotopy group sequence of Proposition 1.37 splits into short exact sequences

$$0 \;\longrightarrow\; \pi_k^G(X) \;\xrightarrow{\pi_k^G(i_X)}\; \pi_k^G(X\vee Y) \;\xrightarrow{\text{incl}}\; \pi_k^G(C(i_X)) \;\longrightarrow\; 0\ .$$

The mapping cone $C(i_X)$ is isomorphic to $(CX)\vee Y$ and thus G-homotopy equivalent to Y. So we can replace $\pi_k^G(C(i_X))$ by $\pi_k^G(Y)$ and conclude that $\pi_k^G(X\vee Y)$ splits as the sum of $\pi_k^G(X)$ and $\pi_k^G(Y)$, via the canonical map. The case of a finite indexing set I now follows by induction.

In the general case we consider the composite

$$\bigoplus_{i\in I} \pi_k^G(X_i) \;\longrightarrow\; \pi_k^G\left(\bigvee_{i\in I} X_i\right) \;\longrightarrow\; \prod_{i\in I} \pi_k^G(X_i)\ ,$$

where the second map is induced by the projections to the wedge summands. This composite is the canonical map from a direct sum to a product of abelian groups, hence injective. So the first map is injective as well. For surjectivity we consider a G-map $f : S^{V\oplus\mathbb{R}^{n+k}} \longrightarrow \bigvee_{i\in I} X_i(V\oplus\mathbb{R}^n)$ that represents an element in the k-th G-homotopy group of $\bigvee_{i\in I} X_i$. Since the source of f is compact, there is a finite subset J of I such that f has image in $\bigvee_{i\in J} X_i(V\oplus\mathbb{R}^n)$, compare Proposition A.1.12. Then the given class is in the image of $\pi_k^G\left(\bigvee_{i\in J} X_i\right)$; since J is finite, the class is in the image of the canonical map, by the previous paragraph.

(ii) The functor $X \mapsto [S^{V\oplus\mathbb{R}^{n+k}}, X(V\oplus\mathbb{R}^n)]$ commutes with products. For finite indexing sets products of abelian groups are also sums, which commute with filtered colimits, such as the one defining π_k^G.

(iii) This is a direct consequence of (i) and (ii). More precisely, for a finite indexing set I and every integer k the composite map

$$\bigoplus_{i\in I} \pi_k^G(X^i) \longrightarrow \pi_k^G(\bigvee_{i\in I} X^i) \;\longrightarrow\; \pi_k^G(\prod_{i\in I} X^i) \longrightarrow \prod_{i\in I} \pi_k^G(X^i)$$

is an isomorphism, where the first and last maps are the canonical ones. These canonical maps are isomorphisms by parts (i) respectively (ii), hence so is the middle map. □

The equivariant homotopy group functor π_k^G does not in general commute with infinite products. The issue is that π_k^G involves a filtered colimit, and these do not always commute with infinite products. However, this defect is cured when we pass to the equivariant or global stable homotopy category, i.e., π_k^G takes 'derived' infinite products to products. We refer to Remark IV.4.6 below for more details.

We recall that a morphism $f : A \longrightarrow B$ of orthogonal G-spectra is an *h-cofibration* if it has the homotopy extension property, i.e., given a morphism of orthogonal G-spectra $\varphi : B \longrightarrow X$ and a homotopy $H : A \wedge [0,1]_+ \longrightarrow X$ starting with φf, there is a homotopy $\bar{H} : B \wedge [0,1]_+ \longrightarrow X$ starting with φ such that $\bar{H} \circ (f \wedge [0,1]_+) = H$. For every h-cofibration $f : A \longrightarrow B$ of orthogonal G-spectra the collapse map $c : Cf \longrightarrow B/A$ from the mapping cone to the cokernel of f is a homotopy equivalence. Indeed, since the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "-\wedge 1"'] & B \arrow[d] \\ A \wedge [0,1] \arrow[r, "g"'] & Cf \end{tikzcd}$$

is a pushout, the cobase change $g : CA = A \wedge [0,1] \longrightarrow Cf$ also has the homotopy extension property. The cone CA is contractible, so the claim follows from the following more general statement: for every h-cofibration $i : C \longrightarrow D$ such that C is contractible, the quotient map $D \longrightarrow D/C$ is a based homotopy equivalence. The standard proof of this fact in the category of topological spaces (see for example [**68**, Prop. 0.17]) only uses formal properties of homotopies, and carries over to the category of orthogonal G-spectra.

So for every h-cofibration $f : A \longrightarrow B$ of orthogonal G-spectra, the collapse map $c : Cf \longrightarrow B/A$ is a homotopy equivalence, hence it induces isomorphisms of all equivariant homotopy groups. We can thus define a modified connecting homomorphism $\partial : \pi_{k+1}^G(B/A) \longrightarrow \pi_k^G(A)$ as the composite

$$\pi_{k+1}^G(B/A) \xrightarrow{(\pi_{k+1}^G(c))^{-1}} \pi_{k+1}^G(Cf) \xrightarrow{\partial} \pi_k^G(A) \ .$$

We call a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra a *strong level fibration* if for every closed subgroup H of G and every H-representation V the map $f(V)^H : X(V)^H \longrightarrow Y(V)^H$ is a Serre fibration. For every such strong level fibration, the embedding $j : F \longrightarrow F(f)$ of the strict fiber into the homotopy fiber then induces isomorphisms on π_*^G. We can thus define a modified connecting homomorphism $\partial : \pi_{k+1}^G(Y) \longrightarrow \pi_k^G(F)$ as the composite

$$\pi_{k+1}^G(Y) \xrightarrow{\partial} \pi_k^G(F(f)) \xrightarrow{(\pi_k^G(j))^{-1}} \pi_k^G(F) \ .$$

So we deduce:

Corollary 1.39. *Let G be a compact Lie group.*

(i) *Let $f : A \longrightarrow B$ be an h-cofibration of orthogonal G-spectra. Then the long sequence of equivariant homotopy groups*

$$\cdots \longrightarrow \pi_{k+1}^G(B/A) \xrightarrow{\partial} \pi_k^G(A) \xrightarrow{\pi_k^G(f)} \pi_k^G(B) \xrightarrow{\pi_k^G(q)} \pi_k^G(B/A) \longrightarrow \cdots$$

is exact.

(ii) *Let $f : X \longrightarrow Y$ be a strong level fibration of orthogonal G-spectra and $j : F \longrightarrow X$ the inclusion of the pointset level fiber of f. Then the long sequence of equivariant homotopy groups*

$$\cdots \longrightarrow \pi_{k+1}^G(Y) \xrightarrow{\partial} \pi_k^G(F) \xrightarrow{\pi_{k-1}^G(j)} \pi_k^G(X) \xrightarrow{\pi_k^G(f)} \pi_k^G(Y) \longrightarrow \cdots$$

is exact.

Corollary 1.40. *Let G be a compact Lie group and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

a commutative square of orthogonal G-spectra.

(i) *Suppose that the square is a pushout and the induced map $\pi_*^G(f) : \pi_*^G(A) \longrightarrow \pi_*^G(B)$ of G-equivariant homotopy groups is an isomorphism. If in addition f or g is an h-cofibration, then the induced map $\pi_*^G(k) : \pi_*^G(C) \longrightarrow \pi_*^G(D)$ is also an isomorphism.*

(ii) *Suppose that the square is a pullback and the induced map $\pi_*^G(k) : \pi_*^G(C) \longrightarrow \pi_*^G(D)$ of G-equivariant homotopy groups is an isomorphism. If in addition k or h is a strong level fibration, then the induced map $\pi_*^G(f) : \pi_*^G(A) \longrightarrow \pi_*^G(B)$ is also an isomorphism.*

Proof. (i) If f is an h-cofibration, then its long exact homotopy group sequence (Corollary 1.39) shows that all G-equivariant homotopy groups of the cokernel B/A are trivial. Since the square is a pushout, the induced morphism from B/A to any cokernel D/C of k is an isomorphism, so the groups $\pi_*^G(D/C)$ are all trivial. As a cobase change of the h-cofibration f, the morphism k is again an h-cofibration, so its long exact homotopy group sequence shows that $\pi_*^G(k)$ is an isomorphism.

If g is an h-cofibration, then so is its cobase change h. Moreover, any cokernel C/A of g maps by an isomorphism to any cokernel D/B of h, since the square is a pushout. The square induces compatible maps between the two long exact homotopy group sequences of g and h, and the five lemma then shows that $\pi_*^G(k)$ is an isomorphism.

The argument for part (ii) is dual to (i). □

Since the group $\pi_k^G(\Omega^m X)$ is naturally isomorphic to $\pi_{k+m}^G(X)$, looping preserves equivariant stable equivalences. The next proposition generalizes this.

Proposition 1.41. *Let G be a compact Lie group and A a finite based G-CW-complex.*

(i) *Let $f : X \longrightarrow Y$ be a morphism of orthogonal G-spectra with the following property: for every closed subgroup H of G that fixes some non-basepoint of A, the map $\pi_*^H(f) : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ is an isomorphism. Then the morphism $\mathrm{map}(A, f) : \mathrm{map}(A, X) \longrightarrow \mathrm{map}(A, Y)$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra.*

(ii) *The functor $\mathrm{map}(A, -)$ preserves $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra.*

Proof. (i) We start with a special case and let X be an orthogonal G-spectrum whose equivariant homotopy groups $\pi_*^H(X)$ vanish for all closed subgroups H of G that fix some non-basepoint of A. We show first that then the G-equivariant homotopy groups of the orthogonal G-spectrum $\mathrm{map}(A, X)$ vanish.

We argue by induction over the number of equivariant cells in a CW-structure of A. The induction starts when A consists only of the basepoint, in which case $\mathrm{map}(A, X)$ is a trivial spectrum and there is nothing to show. For the inductive step we assume that the groups $\pi_*^G(\mathrm{map}(B, X))$ vanish and A is obtained from B by attaching an equivariant n-cell $G/H \times D^n$ along its boundary $G/H \times \partial D^n$, for some closed subgroup H of G. Then the restriction map $\mathrm{map}(A, X) \longrightarrow \mathrm{map}(B, X)$ is a strong level fibration of orthogonal G-spectra whose fiber is isomorphic to

$$\mathrm{map}(A/B, X) \;\cong\; \mathrm{map}(G/H_+ \wedge S^n, X) \;\cong\; \mathrm{map}(G/H_+, \Omega^n X)\ .$$

The G-equivariant stable homotopy groups of this spectrum are isomorphic to the H-equivariant homotopy groups of $\Omega^n X$, and hence to the shifted H-homotopy groups of X. Since H occurs as a stabilizer of a cell of B, the latter groups vanish by assumption. The long exact sequence of Corollary 1.39 (ii) and the inductive hypothesis for B then show that the groups $\pi_*^G(\mathrm{map}(A, X))$ vanish.

When H is a proper closed subgroup of G we exploit that the underlying H-space of A is H-equivariantly homotopy equivalent to a finite H-CW-complex, see [**83**, Cor. B]. We can thus apply the previous paragraph to the underlying H-spectrum of X and the underlying H-space of A and conclude that the H-equivariant homotopy groups of $\mathrm{map}(A, X)$ vanish. Altogether this proves the special case of the proposition.

The functor $\mathrm{map}(A, -)$ commutes with homotopy fibers; so two applications of the long exact homotopy group sequence of a homotopy fiber (Proposition 1.37) reduce the general case of the first claim to the special case.

Part (ii) is a special case of (i). □

PROPOSITION 1.42. *Let G be a compact Lie group.*

(i) *Let $e_n : X_n \longrightarrow X_{n+1}$ be morphisms of orthogonal G-spectra that are levelwise closed embeddings, for $n \geq 0$. Let X_∞ be a colimit of the sequence $\{e_n\}_{n\geq 0}$. Then for every integer k the canonical map*

$$\mathrm{colim}_{n\geq 0}\ \pi_k^G(X_n) \ \longrightarrow\ \pi_k^G(X_\infty)$$

is an isomorphism.

(ii) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of orthogonal G-spectra that are levelwise closed embeddings, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is a $\underline{\pi}_*$-isomorphism.*

(iii) *Let $f_n : Y_n \longrightarrow Y_{n+1}$ be $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra that are levelwise a closed embeddings, for $n \geq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to a colimit of the sequence $\{f_n\}_{n\geq 0}$ is a $\underline{\pi}_*$-isomorphism.*

PROOF. (i) We let V be a G-representation, and $m \geq 0$ such that $k+m \geq 0$. Since the sphere $S^{V\oplus\mathbb{R}^{m+k}}$ is compact and $X_\infty(V \oplus \mathbb{R}^m)$ is a colimit of the sequence of closed embeddings $X_n(V \oplus \mathbb{R}^m) \longrightarrow X_{n+1}(V \oplus \mathbb{R}^m)$, any based G-map $f : S^{V\oplus\mathbb{R}^{m+k}} \longrightarrow X_\infty(V \oplus \mathbb{R}^m)$ factors through a continuous map

$$\bar{f}\ :\ S^{V\oplus\mathbb{R}^{m+k}} \ \longrightarrow\ X_n(V \oplus \mathbb{R}^m)$$

for some $n \geq 0$, see for example [**79**, Prop. 2.4.2]. Since the canonical map $X_n(V \oplus \mathbb{R}^m) \longrightarrow X_\infty(V \oplus \mathbb{R}^m)$ is injective, $\bar{f}$ is again based and G-equivariant. So the canonical map

$$\mathrm{colim}_{n\geq 0}\,[S^{V\oplus\mathbb{R}^{m+k}}, X_n(V \oplus \mathbb{R}^m)]^G \ \longrightarrow\ [S^{V\oplus\mathbb{R}^{m+k}}, X_\infty(V \oplus \mathbb{R}^m)]^G$$

is bijective. Passing to colimit over m and over the poset $s(\mathcal{U}_G)$ proves the claim.

Part (ii) is a direct consequence of (i), applied to G and all its closed subgroups. Part (iii) is a special case of part (ii) where we set $X_n = Y_0$, $e_n = \mathrm{Id}_{Y_0}$ and $\psi_n = f_{n-1} \circ \cdots \circ f_0 : Y_0 \longrightarrow Y_n$. The morphism ψ_n is then a $\underline{\pi}_*$-isomorphism and Y_0 is a colimit of the constant first sequence. Since the morphism ψ_∞ induced on the colimits of the two sequences is the canonical map $Y_0 \longrightarrow Y_\infty$, part (ii) specializes to claim (iii). □

EXAMPLE 1.43 (Equivariant suspension spectrum). Every based G-space A gives rise to a *suspension spectrum* $\Sigma^\infty A$. This is the orthogonal G-spectrum with V-term

$$(\Sigma^\infty A)(V) \ =\ S^V \wedge A\ ,$$

with $O(V)$-action only on S^V, with G-action only on A, and with structure map $\sigma_{U,V}$ given by the canonical homeomorphism $S^U \wedge S^V \wedge A \cong S^{U\oplus V} \wedge A$. For an unbased G-space Y, we obtain a *unreduced suspension spectrum* $\Sigma^\infty_+ Y$ by first adding a disjoint basepoint and the forming the suspension spectrum as above.

The suspension spectrum functor is homotopical on a large class of G-spaces; however, since the reduced suspension $S^1 \wedge -$ does not preserve weak equivalences in complete generality, a non-degeneracy condition on the basepoint is often needed.

DEFINITION 1.44. A based G-space is *well-pointed* if the basepoint inclusion has the homotopy extension property in the category of unbased G-spaces.

Equivalently, a based G-space (A, a_0) is well-pointed if and only if the inclusion of $A \times \{0\} \cup \{a_0\} \times [0,1]$ into $A \times [0,1]$ has a continuous G-equivariant retraction. Cofibrant based G-spaces are always well-pointed. Also, if we add a disjoint basepoint to any unbased G-space, the result is always well-pointed.

We recall that a continuous map $f : X \longrightarrow Y$ is *m-connected*, for some $m \geq 0$, if the following condition holds: for $0 \leq k \leq m$ and all continuous maps $\alpha : \partial D^k \longrightarrow X$ and $\beta : D^k \longrightarrow Y$ such that $\beta|_{\partial D^k} = f \circ \alpha$, there is a continuous map $\lambda : D^k \longrightarrow X$ such that $\lambda|_{\partial D^k} = \alpha$ and such that $f \circ \lambda$ is homotopic, relative to ∂D^k, to β. An equivalent condition is that the map $\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$ is surjective, and for all $x \in X$ the map $\pi_k(f) : \pi_k(X, x) \longrightarrow \pi_k(Y, f(x))$ is bijective for all $1 \leq k < m$ and surjective for $k = m$. The map f is a weak homotopy equivalence if and only if it is m-connected for every $m \geq 0$.

Proposition 1.45. *Let G be a compact Lie group.*

(i) *Let $f : X \longrightarrow Y$ be a continuous based G-map between well-pointed based G-spaces, and $m \geq 0$. Suppose that for every closed subgroup H of G the fixed point map $f^H : X^H \longrightarrow Y^H$ is $(m - \dim(W_G H))$-connected. Then the induced map of G-equivariant stable homotopy groups*

$$\pi_k^G(\Sigma^\infty f) \ : \ \pi_k^G(\Sigma^\infty X) \ \longrightarrow \ \pi_k^G(\Sigma^\infty Y)$$

is bijective for every integer k with $k < m$ and surjective for $k = m$.

(ii) *For every well-pointed based G-space X and every negative integer k, the G-equivariant homotopy group $\pi_k^G(\Sigma^\infty X)$ is trivial.*

In particular, the suspension spectrum functor Σ^∞ takes G-weak equivalences between well-pointed G-spaces to $\underline{\pi}_*$-isomorphism, and the unreduced suspension spectrum functor Σ^∞_+ takes G-weak equivalences between arbitrary unbased G-spaces to $\underline{\pi}_*$-isomorphism.

Proof. (i) We let V be a G-representation and we let $n \geq 0$ be such that $n+k \geq 0$. Reduced suspension increases the connectivity of continuous maps between well-pointed spaces, compare Proposition A.1.20. So the natural homeomorphism

$$(S^{V \oplus \mathbb{R}^n} \wedge X)^H \ \cong \ S^{V^H \oplus \mathbb{R}^n} \wedge X^H$$

shows that the map

$$(S^{V \oplus \mathbb{R}^n} \wedge f)^H \ : \ (S^{V \oplus \mathbb{R}^n} \wedge X)^H \ \longrightarrow \ (S^{V \oplus \mathbb{R}^n} \wedge Y)^H$$

is $(\dim(V^H) + m + n - \dim(W_G H))$-connected. On the other hand, we claim that the cellular dimension of $S^{V \oplus \mathbb{R}^{n+k}}$ at H, in the sense of [**166**, II.2, p. 106], is at most $\dim(V^H) + n + k - \dim(W_G H)$. Indeed, if any G-CW-structure for $S^{V \oplus \mathbb{R}^{n+k}}$ contains an equivariant cell of the form $G/H \times D^j$, then $(G/H)^H \times \mathring{D}^j$ embeds into $S^{V^H \oplus \mathbb{R}^{n+k}}$, and hence

$$\dim(W_G H) + j \ = \ \dim((G/H)^H) + j \ \leq \ \dim(V^H) + n + k \ .$$

The cellular dimension at H is the maximal j that occurs in this way, so this cellular dimension is at most $\dim(V^H) + n + k - \dim(W_G H)$. We conclude that for $k < m$ the cellular dimension of $S^{V \oplus \mathbb{R}^{n+k}}$ at H is smaller that the connectivity of f^H, and for $k = m$ the former is less than or equal to the latter. So the induced map

$$[S^{V \oplus \mathbb{R}^{n+k}}, S^{V \oplus \mathbb{R}^n} \wedge f]^G \ : \ [S^{V \oplus \mathbb{R}^{n+k}}, S^{V \oplus \mathbb{R}^n} \wedge X]^G \ \longrightarrow \ [S^{V \oplus \mathbb{R}^{n+k}}, S^{V \oplus \mathbb{R}^n} \wedge X]^G$$

is bijective for $k < m$ and surjective for $k = m$, by [**166**, II Prop. 2.7]. Passage to the colimit over $V \in s(\mathcal{U}_G)$ and n then proves the claim.

(ii) This is a special case of (i): the unique map $f : X \longrightarrow *$ to a one-point G-space has the property that $f^H : X^H \longrightarrow *^H$ is 0-connected for every closed subgroup H of G. So for every negative integer k the induced map from $\pi_k^G(\Sigma^\infty X)$ to $\pi_k^G(\Sigma^\infty *)$ is an isomorphism by part (i). Since the latter group is trivial, this proves the claim. $\square$

We end this section with a useful representability result for the functor π_0^H on the category of orthogonal G-spectra, where H is any closed subgroup of a compact Lie group G. We define a *tautological class*

$$e_H \ \in \ \pi_0^H(\Sigma_+^\infty G/H)$$

as the class represented by the H-fixed point eH of G/H.

PROPOSITION 1.46. *Let H be a closed subgroup of a compact Lie group G and $\Phi : G\mathcal{S}p \longrightarrow$ (sets) a set valued functor on the category of orthogonal G-spectra that takes $\underline{\pi}_*$-isomorphisms to bijections. Then evaluation at the tautological class is a bijection*

$$\mathrm{Nat}^{G\mathcal{S}p}(\pi_0^H, \Phi) \ \longrightarrow \ \Phi(\Sigma_+^\infty G/H) \ , \quad \tau \ \longmapsto \ \tau(e_H) \ .$$

PROOF. To show that the evaluation map is injective we show that any natural transformation $\tau : \pi_0^H \longrightarrow \Phi$ is determined by the element $\tau(e_H)$. We let X be any orthogonal G-spectrum and $x \in \pi_0^H(X)$ an H-equivariant homotopy class. The class x is represented by a continuous based H-map

$$f \ : \ S^U \ \longrightarrow \ X(U)$$

for some H-representation U. By increasing U, if necessary, we can assume that U is underlying a G-representation. We can then view f as an H-fixed point in $(\Omega^U \mathrm{sh}^U X)(0) = \Omega^U X(U)$. There is thus a unique morphism of orthogonal G-spectra

$$\hat{f} \ : \ \Sigma_+^\infty G/H \ \longrightarrow \ \Omega^U \mathrm{sh}^U X$$

such that $\hat{f}(0) : G/H_+ \longrightarrow \Omega^U X(U)$ sends the distinguished coset eH to f. This morphism then satisfies

$$\pi_0^H(\hat{f})(e_H) \ = \ \pi_0^H(\tilde{\lambda}_X^U)(x) \quad \text{in} \quad \pi_0^H(\Omega^U \mathrm{sh}^U X) \ ,$$

where $\tilde{\lambda}_X^U : X \longrightarrow \Omega^U \mathrm{sh}^U X$ is the $\underline{\pi}_*$-isomorphism discussed in Proposition 1.25 (ii). The diagram

$$\begin{tikzcd}
\pi_0^H(\Sigma_+^\infty G/H) \arrow[r, "\pi_0^H(\hat{f})"] \arrow[d, "\tau"'] & \pi_0^H(\Omega^U \mathrm{sh}^U X) \arrow[d, "\tau"'] & \pi_0^H(X) \arrow[l, "\pi_0^H(\tilde{\lambda}_X^U)"', "\cong"] \arrow[d, "\tau"] \\
\Phi(\Sigma_+^\infty G/H) \arrow[r, "\Phi(\hat{f})"'] & \Phi(\Omega^U \mathrm{sh}^U X) & \Phi(X) \arrow[l, "\Phi(\tilde{\lambda}_X^U)"', "\cong"]
\end{tikzcd}$$

commutes and the two right horizontal maps are bijective. So

$$\Phi(\hat{f})(\tau(e_H)) \ = \ \tau(\pi_0^H(\hat{f})(e_H)) \ = \ \tau(\pi_0^H(\tilde{\lambda}_X^U)(x)) \ = \ \Phi(\tilde{\lambda}_X^U)(\tau(x)) \ .$$

Since $\Phi(\tilde{\lambda}_X^U)$ is bijective, this proves that τ is determined by its value on the tautological class e_H.

It remains to construct, for every element $y \in \Phi(\Sigma_+^\infty G/H)$, a natural transformation $\tau : \pi_0^H \longrightarrow \Phi$ with $\tau(e_H) = y$. The previous paragraph dictates what to do: we represent a given class $x \in \pi_0^H(X)$ by a continuous based H-map $f : S^U \longrightarrow X(U)$, where U is underlying a G-representation. Then we set

$$\tau(x) \ = \ \Phi(\tilde{\lambda}_X^U)^{-1}(\Phi(\hat{f})(y)) \ .$$

We verify that the element $\tau(x)$ only depends on the class x. To this end we need to show that $\tau(x)$ does not change if we either replace f by a homotopic H-map, or increase it by stabilization along a G-equivariant linear isometric embedding. If $f' : S^U \longrightarrow X(U)$ is H-equivariantly homotopic to f, then the morphism $\hat{f}'$ is homotopic to $\hat{f}$ via a homotopy of morphisms of orthogonal G-spectra

$$K \ : \ (\Sigma_+^\infty G/H) \wedge [0,1]_+ \ \longrightarrow \Omega^U \mathrm{sh}^U X \ .$$

The morphism $q : (\Sigma_+^\infty G/H) \wedge [0,1]_+ \longrightarrow \Sigma_+^\infty G/H$ that maps $[0,1]$ to a single point is a homotopy equivalence, hence a $\underline{\pi}_*$-isomorphism, of orthogonal G-spectra. So $\Phi(q)$ is a bijection. The two embeddings

$i_0, i_1 : (\Sigma^\infty_+ G/H) \longrightarrow \Sigma^\infty_+ G/H \wedge [0,1]_+$ as the endpoints of the interval are right inverse to q, so $\Phi(q) \circ \Phi(i_0) = \Phi(q) \circ \Phi(i_1) = \mathrm{Id}$. Since $\Phi(q)$ is bijective, $\Phi(i_0) = \Phi(i_1)$. Hence

$$\Phi(\hat{f}') \;=\; \Phi(K \circ i_0) \;=\; \Phi(K) \circ \Phi(i_0) \;=\; \Phi(K) \circ \Phi(i_1) \;=\; \Phi(K \circ i_1) \;=\; \Phi(\hat{f}) \;.$$

This shows that $\tau(x)$ does not change if we modify f by an H-homotopy.

Now we let V be another G-representation and $\varphi : U \longrightarrow V$ a G-equivariant linear isometric embedding. Then $\varphi_* f : S^V \longrightarrow X(V)$ is another representative of the class x. A morphism of orthogonal G-spectra

$$\varphi_\sharp \;:\; \Omega^U \operatorname{sh}^U X \;\longrightarrow\; \Omega^V \operatorname{sh}^V X$$

is defined at an inner product space W as the stabilization map

$$\varphi_* \;:\; (\Omega^U \operatorname{sh}^U X)(W) \;=\; \operatorname{map}(S^U, X(W \oplus U)) \;\longrightarrow\; \operatorname{map}(S^V, X(W \oplus V)) \;=\; (\Omega^V \operatorname{sh}^V X)(W) \;.$$

This morphism makes the following diagram of orthogonal G-spectra commute:

$$\begin{tikzcd}
\Sigma^\infty_+ G/H \arrow[r, "\hat{f}"] \arrow[dr, "\widehat{\varphi_* f}"'] & \Omega^U \operatorname{sh}^U X \arrow[d, "\varphi_\sharp"] & X \arrow[l, "\tilde{\lambda}^U_X"'] \arrow[dl, "\tilde{\lambda}^V_X"] \\
& \Omega^V \operatorname{sh}^V X &
\end{tikzcd}$$

So

$$\Phi(\tilde{\lambda}^U_X)^{-1} \circ \Phi(\hat{f}) \;=\; \Phi(\tilde{\lambda}^V_X)^{-1} \circ \Phi(\varphi_\sharp) \circ \Phi(\hat{f}) \;=\; \Phi(\tilde{\lambda}^V_X)^{-1} \circ \Phi(\widehat{\varphi_* f}) \;,$$

and hence the class $\tau(x)$ remains unchanged upon stabilization of f along φ.

Now we know that $\tau(x)$ is independent of the choice of representative for the class x, and it remains to show that τ is natural. But this is straightforward: if $\psi : X \longrightarrow Y$ is a morphism of orthogonal G-spectra and $f : S^U \longrightarrow X(U)$ a representative for $x \in \pi^H_0(X)$ as above, then $\psi(U) \circ f : S^U \longrightarrow Y(U)$ represents the class $\pi^H_0(\psi)(x)$. Moreover, the following diagram of orthogonal G-spectra commutes:

$$\begin{tikzcd}
\Sigma^\infty_+ G/H \arrow[r, "\hat{f}"] \arrow[dr, "\widehat{\psi(U) \circ f}"'] & \Omega^U \operatorname{sh}^U X \arrow[d, "\Omega^U \operatorname{sh}^U \psi"] & X \arrow[l, "\tilde{\lambda}^U_X"'] \arrow[d, "\psi"] \\
& \Omega^U \operatorname{sh}^U Y & Y \arrow[l, "\tilde{\lambda}^U_Y"]
\end{tikzcd}$$

So naturality follows:

$$\begin{aligned}
\tau(\pi^H_0(\psi)(x)) \;&=\; (\Phi(\tilde{\lambda}^U_Y)^{-1} \circ \Phi(\widehat{\psi(U) \circ f}))(y) \\
&=\; (\Phi(\tilde{\lambda}^U_Y)^{-1} \circ \Phi(\Omega^U \operatorname{sh}^U \psi) \circ \Phi(\hat{f}))(y) \\
&=\; (\Phi(\psi) \circ \Phi(\tilde{\lambda}^U_X)^{-1} \circ \Phi(\hat{f}))(y) \;=\; \Phi(\psi)(\tau(x)) \;.
\end{aligned}$$

Finally, the class e_H is represented by the H-map $- \wedge eH : S^0 \longrightarrow S^0 \wedge G/H_+$, which is adjoint to the identity of $\Sigma^\infty_+ G/H$. Hence $\tau(e_H) = \Phi(\mathrm{Id})(y) = y$. □

2. The Wirthmüller isomorphism and transfers

This section establishes the Wirthmüller isomorphism that relates the equivariant homotopy groups of a spectrum over a subgroup with the equivariant homotopy groups of the induced spectrum, see Theorem 2.14; intimately related to the Wirthmüller isomorphism are various transfers that we discuss in Constructions 2.6 and 2.20. We show that transfers are transitive (Proposition 2.27) and commute with inflations (Proposition 2.30). Theorem 4.10 below establishes the 'double coset formula' that expresses the composite of a transfer followed by a restriction as a linear combination of restrictions followed by transfers.

We let H be a closed subgroup of a compact Lie group G. Then the restriction functor from orthogonal G-spectra to orthogonal H-spectra has a left and a right adjoint, and both are given by applying the space level adjoints $G \ltimes_H -$ respectively $\mathrm{map}^H(G,-)$ levelwise. The Wirthmüller isomorphism identifies the G-equivariant homotopy groups of an induced spectrum $G \ltimes_H Y$ with the H-equivariant homotopy groups of $Y \wedge S^L$, where L is a certain H-representation, the tangent representation of the preferred coset eH in the homogeneous space G/H.

For a based H-space A and a based G-space B, the *shearing isomorphism* is the G-equivariant homeomorphism

$$B \wedge (G \ltimes_H A) \;\cong\; G \ltimes_H (i^*B \wedge A)\ , \quad b \wedge [g,a] \;\longmapsto\; [g, (g^{-1}b) \wedge a]\ .$$

Construction 2.1 (Induced spectrum). We let H be a closed subgroup of a compact Lie group G and Y an orthogonal H-spectrum. We denote by $G \ltimes_H Y$ the *induced G-spectrum* whose value at an inner product space V is $(G \ltimes_H Y)(V) = G \ltimes_H Y(V)$, with induced action by the orthogonal group and induced structure maps. When G-acts on V by linear isometries, then by definition the value $(G \ltimes_H Y)(V)$ has the diagonal G-action, through the action on the external G and by functoriality. With this diagonal G-action, $(G \ltimes_H Y)(V)$ is equivariantly isomorphic to $G \ltimes_H Y(i^*V)$ (where H acts diagonally on $Y(i^*H)$), via

$$G \ltimes_H Y(i^*V) \;\cong\; (G \ltimes_H Y)(V)\ , \quad [g,y] \;\longmapsto\; [g, Y(l_g)(y)]\ .$$

Under this isomorphism the structure map of the spectrum $G \ltimes_H Y$ becomes the combination of the shearing isomorphism and the structure map of the H-spectrum Y, i.e., for every G-representation W the following square commutes:

$$\begin{tikzcd}
S^V \wedge (G \ltimes_H Y(i^*W)) \arrow[r, "\cong"] \arrow[d, "\text{shear}"', "\cong"] & S^V \wedge (G \ltimes_H Y)(W) \arrow[dd, "\sigma^{G \ltimes_H Y}_{V,W}"] \\
G \ltimes_H (S^{i^*V} \wedge Y(i^*W)) \arrow[d, "G \ltimes_H (\sigma^Y_{V,W})"'] & \\
G \ltimes_H (Y(i^*V \oplus i^*W)) \arrow[r, "\cong"'] & (G \ltimes_H Y)(V \oplus W)
\end{tikzcd}$$

Construction 2.2. As before we consider a closed subgroup H of a compact Lie group G. To define the Wirthmüller morphism, we need a specific H-equivariant map

$$l_A \;:\; G \ltimes_H A \;\longrightarrow\; A \wedge S^L$$

that is natural for continuous based H-maps in A, see (2.3) below. Here $L = T_H(G/H)$ is the tangent H-representation and S^L its one-point compactification. When H has finite index in G, then L is trivial and then the map $l_A : G \ltimes_H A \longrightarrow A$ is simply the projection onto the wedge summand indexed by the preferred H-coset eH. If the dimension of G is strictly bigger than the dimension of H, then L is non-zero and the definition of the map l_A is substantially more involved.

We consider the left H^2-action on G given by

$$H^2 \times G \;\longrightarrow\; G\ , \quad (h',h) \cdot g \;=\; h'gh^{-1}\ .$$

There is another action of the group Σ_2 on G by inversion, i.e., the non-identity element of Σ_2 acts by $\tau \cdot g = g^{-1}$. These two actions combine into an action of the group $\Sigma_2 \ltimes H^2$ on G, where the semi-direct product is formed with respect to the action of Σ_2 on H^2 that interchanges the two factors (i.e., $\Sigma_2 \ltimes H^2 = \Sigma_2 \wr H$ is the wreath product).

The subgroup H is the $\Sigma_2 \ltimes H^2$-orbit of $1 \in G$, whose stabilizer is the subgroup $\Sigma_2 \times \Delta$, where $\Delta = \{(h,h) \mid h \in H\}$ is the diagonal subgroup of H^2. The differential of the projection $G \longrightarrow G/H$ identifies

$$\nu \;=\; (T_1G)/(T_1H)\ ,$$

(the normal space at $1 \in H$ of the inclusion $H \longrightarrow G$) with the tangent representation $L = T_H(G/H)$. The representation ν is a representation of the stabilizer group $\Sigma_2 \times \Delta$ of 1, and if we identify H with Δ via $h \mapsto (h,h)$, then this identification takes the Δ-action on ν to the tangent H-action on L. Moreover, the differential at 1 of the inversion map $g \mapsto g^{-1}$ is multiplication by -1 of the tangent space; so the above identification of ν with L takes the Σ_2-action to the sign action on L.

We choose an H-invariant inner product on the vector space L. Since H is the H^2-orbit of 1 inside G, there is a slice, i.e., a $\Sigma_2 \times \Delta$-equivariant smooth embedding

$$s \ : \ D(L) \ \longrightarrow \ G$$

of the unit disc of L with $s(0) = 1$ and such that the differential at $0 \in D(L)$ of the composite

$$D(L) \ \xrightarrow{s} \ G \ \xrightarrow{\text{proj}} \ G/H$$

is the identity of L. The property that s is Δ-equivariant means in more concrete terms that the relation

$$s(h \cdot l) \ = \ h \cdot s(l) \cdot h^{-1}$$

holds in G for all $(h,l) \in H \times D(L)$. The Σ_2-equivariance comes down to the relation

$$s(-l) \ = \ s(l)^{-1}$$

for all $l \in D(L)$. After scaling the slice, if necessary, the map

$$D(L) \times H \ \longrightarrow \ G \ , \quad (l,h) \ \longmapsto \ s(l) \cdot h$$

is an embedding whose image is a tubular neighborhood of H in G. For a proof, see for example [**27**, II Thm. 5.4]. This embedding is equivariant for the action of H^2 on the source by

$$(h_1, h_2) \cdot (l,h) \ = \ (h_1 l, \, h_1 h h_2^{-1})$$

and for the action of Σ_2 on the source by

$$\tau \cdot (l,h) \ = \ (-h^{-1} l, \, h^{-1}) \ .$$

The map

$$l_H^G \ : \ G \ \longrightarrow \ S^L \wedge H_+$$

is then defined as the H^2-equivariant collapse map with respect to the above tubular neighborhood. So explicitly,

$$l_H^G(g) \ = \ \begin{cases} (l/(1-|l|)) \wedge h & \text{if } g = s(l) \cdot h \text{ with } (l,h) \in D(L) \times H, \text{ and} \\ \ * & \text{if } g \text{ is not of this form.} \end{cases}$$

Given any based H-space A, we can now form $l \wedge_H A$, where $- \wedge_H -$ refers to the action of the first H-factor in H^2. We obtain an H-equivariant based map

$$l_A \ = \ l_H^G \wedge_H A \ : \ G \ltimes_H A \ \longrightarrow \ (S^L \wedge H_+) \wedge_H A \ \cong \ A \wedge S^L \tag{2.3}$$

that is natural in A. Here H acts by left translation on the source, and diagonally on the target. This map is thus given by

$$l_A[g,a] \ = \ \begin{cases} ha \wedge (l/(1-|l|)) & \text{if } g = s(l) \cdot h \text{ with } (l,h) \in D(L) \times H, \text{ and} \\ \ * & \text{if } g \text{ is not of this form.} \end{cases}$$

If H has finite index in G, then $L = 0$ and the triangle of the following proposition commutes on the nose, by direct inspection. So the essential content of the next result is when the dimension of G exceeds the dimension of H.

PROPOSITION 2.4. *Let H be a closed subgroup of a compact Lie group G, A any based H-space and B any based G-space. Then the following triangle commutes up to H-equivariant based homotopy:*

$$\begin{tikzcd} B \wedge (G \ltimes_H A) \arrow[d, "\text{shear}"', "\cong"] \arrow[dr, "B \wedge l_A"] & \\ G \ltimes_H (i^* B \wedge A) \arrow[r, "l_{i^* B \wedge A}"'] & B \wedge A \wedge S^L \end{tikzcd}$$

PROOF. We can write down an explicit homotopy: we define the map

$$K \ : \ (B \wedge (G \ltimes_H A)) \times [0,1] \ \longrightarrow \ B \wedge A \wedge S^L$$

by

$$K(b \wedge [g,a], t) \ = \ \begin{cases} s(tl)^{-1} b \wedge ha \wedge (l/(1-|l|) & \text{if } g = s(l) \cdot h \text{ for } (l,h) \in D(L) \times H, \text{ and} \\ \ * & \text{if } g \text{ is not in the image of } \bar{s}. \end{cases}$$

Then for all $(l,h) \in D(L) \times H$ with $g = s(l) \cdot h$ we have

$$K(b \wedge [g,a], 0) \ = \ b \wedge ha \wedge (l/(1-|l|)) \ = \ (B \wedge l_A)(b \wedge [g,a])$$

(because $s(0) = 1$), and

$$K(b \wedge [g,a], 1) \ = \ hg^{-1} b \wedge ha \wedge (l/(1-|l|) \ = \ l_{i^* B \wedge A}[g, g^{-1} b \wedge a])$$

(because $s(l)^{-1} = hg^{-1}$). So K is the desired H-equivariant homotopy. $\square$

Now we let Y be an orthogonal H-spectrum. If the group H has finite index in G, then $L = 0$ and l_A is the projection onto the wedge summand indexed by eH; the maps $l_{Y(V)} : G \ltimes_H Y(V) \longrightarrow Y(V)$ then form a morphism of orthogonal H-spectra $l_Y : G \ltimes_H Y \longrightarrow Y$ as V varies over all inner product spaces. This morphism then induces a map of H-equivariant homotopy groups. In general, however, the diagram

$$\begin{tikzcd} S^V \wedge (G \ltimes_H Y(W)) \arrow[r, "S^V \wedge l_{Y(W)}"] \arrow[d, "\cong"', "\text{shear}"] \arrow[dd, bend right=60, "\sigma_{V,W}^{G \ltimes_H Y}"'] & S^V \wedge Y(W) \wedge S^L \arrow[dd, "\sigma_{V,W}^{Y \wedge S^L}"] \\ G \ltimes_H (S^V \wedge Y(W)) \arrow[ur, "l_{S^V \wedge Y(W)}"'] \arrow[d, "G \ltimes_H \sigma_{V,W}^Y"] & \\ G \ltimes_H Y(V \oplus W) \arrow[r, "l_{Y(V \oplus W)}"'] & Y(V \oplus W) \wedge S^L \end{tikzcd}$$

does *not* commute on the nose (because the upper triangle does not commute); so if the dimension of G is bigger than the dimension of H we do *not* obtain a morphism of orthogonal H-spectra in the strict sense. Still, Proposition 2.4 shows that the above diagram does commute up to based H-equivariant homotopy, and this is good enough to yield a well-defined homomorphism

$$(l_Y)_* \ : \ \pi_k^H(G \ltimes_H Y) \ \longrightarrow \ \pi_k^H(Y \wedge S^L) \ .$$

As we just explained, this is an abuse of notation, since $(l_Y)_*$ is in general *not* the induced map of any morphism of orthogonal H-spectra.

We can now consider the composite

$$\text{Wirth}_H^G \ : \ \pi_k^G(G \ltimes_H Y) \ \xrightarrow{\text{res}_H^G} \ \pi_k^H(G \ltimes_H Y) \ \xrightarrow{(l_Y)_*} \ \pi_k^H(Y \wedge S^L) \ , \tag{2.5}$$

which we call the *Wirthmüller map*. We show in Theorem 2.14 below that the Wirthmüller map is an isomorphism.

Construction 2.6 (Transfers). As before we let H be a closed subgroup of a compact Lie group G, and we let Y be an orthogonal H-spectrum. The *external transfer*

$$G \ltimes_H - \ : \ \pi_0^H(Y \wedge S^L) \ \longrightarrow \ \pi_0^G(G \ltimes_H Y) \ , \tag{2.7}$$

is a map in the direction opposite to the Wirthmüller map. The construction involves another equivariant Thom-Pontryagin construction. We choose a G-representation V and a vector $v_0 \in V$ whose stabilizer group is H; this is possible, for example by [**27**, Ch. 0, Thm. 5.2], [**29**, III.5] or [**123**, Prop. 1.4.1]. This data determines a G-equivariant smooth embedding

$$i \ : \ G/H \ \longrightarrow \ V \ , \quad gH \ \longmapsto \ gv_0$$

that identifies G/H with the orbit Gv_0. We let

$$W \ = \ V - (Di)_H(L) \ = \ V - T_{v_0}(G \cdot v_0)$$

denote the orthogonal complement of the image of the tangent space at eH; this is an H-subrepresentation of V, and

$$L \oplus W \ \cong \ V \ , \quad (x, w) \ \longmapsto \ (Di)_H(x) \ + \ w \tag{2.8}$$

is an isomorphism of H-representations.

By multiplying the original vector v_0 with a sufficiently large scalar, if necessary, we can assume that the embedding i is 'wide', in the sense that the exponential map

$$j \ : \ G \times_H D(W) \ \longrightarrow \ V \ , \quad [g, w] \ \longmapsto \ g \cdot (v_0 \ + \ w)$$

is an embedding, where $D(W)$ is the unit disc of the normal H-representation, compare [**27**, Ch. II, Cor. 5.2]. So j defines an equivariant tubular neighborhood of the orbit Gv_0 inside V. The associated Thom-Pontryagin collapse then becomes the G-map

$$c \ : \ S^V \ \longrightarrow \ G \ltimes_H S^W \tag{2.9}$$

defined by

$$c(v) \ = \ \begin{cases} \left[g, \frac{w}{1-|w|}\right] & \text{if } v = g \cdot (v_0 + w) \text{ for some } (g, w) \in G \times D(W), \text{ and} \\ \infty & \text{else.} \end{cases}$$

With the collapse map in place, we can now define the external transfer (2.7). We let U be an H-representation and $f : S^U \longrightarrow Y(U) \wedge S^L$ an H-equivariant based map that represents a class in $\pi_0^H(Y \wedge S^L)$. By enlarging U, if necessary, we can assume that it is the underlying H-representation of a G-representation, see for example [**123**, Prop. 1.4.2]. We stabilize f by W from the right to obtain the H-map

$$f \diamond W \ : \ S^U \wedge S^W \ \xrightarrow{f \wedge S^W} \ Y(U) \wedge S^L \wedge S^W \ \cong_{(2.8)} \ Y(U) \wedge S^V \ \xrightarrow{\sigma^{\mathrm{op}}_{U,V}} \ Y(U \oplus V) \ . \tag{2.10}$$

The composite G-map

$$\begin{aligned} S^{U \oplus V} \ \xrightarrow{S^U \wedge c} \ S^U \wedge (G \ltimes_H S^W) \ &\xrightarrow[\cong]{\text{shear}} G \ltimes_H (S^U \wedge S^W) \\ &\xrightarrow{G \ltimes_H (f \diamond W)} G \ltimes_H (Y(U \oplus V)) \ \cong \ (G \ltimes_H Y)(U \oplus V) \end{aligned}$$

then represents the external transfer

$$G \ltimes_H \langle f \rangle \qquad \text{in} \qquad \pi_0^G(G \ltimes_H Y) \ .$$

The next proposition provides the main geometric input for the Wirthmüller isomorphism.

Proposition 2.11. *Let H be a closed subgroup of a compact Lie group G.*

(i) *The composite*

$$S^V \xrightarrow{c} G \ltimes_H S^W \xrightarrow{l_H^G \wedge_H S^W} S^L \wedge S^W$$

is H-equivariantly homotopic to the map induced by the inverse of the isometry (2.8).

(ii) *Let A be a based H-space and $f, f' : B \longrightarrow G \ltimes_H A$ two continuous based G-maps. If the composites $l_A \circ f, l_A \circ f' : B \longrightarrow A \wedge S^L$ are H-equivariantly homotopic, then the maps $f \wedge S^V, f' \wedge S^V : B \wedge S^V \longrightarrow (G \ltimes_H A) \wedge S^V$ are G-equivariantly homotopic.*

Proof. (i) The composite $(l_H^G \wedge_H S^W) \circ c$ is the collapse map based on the composite smooth embedding

$$j \ : \ D(L) \times D(W) \xrightarrow{(l,w) \mapsto [s(l),w]} G \times_H D(W) \xrightarrow{[g,w] \mapsto g \cdot (v_0 + w)} V \ .$$

Then $j(0,0) = v_0$ and we let

$$D \ = \ (Dj)_{(0,0)} \ : \ L \times W \ \longrightarrow \ V$$

denote the differential of j at $(0,0)$. We observe that

$$D(0,w) \ = \ w$$

because $j(0,w) = v_0 + w$; on the other hand, $j(l,0) = s(l) \cdot v_0$, so the restriction of D to L is the differential at 0 of the composite

$$D(L) \xrightarrow{s} G \xrightarrow{\text{proj}} G/H \xrightarrow{i} V \ .$$

Since the differential of the composite of the first two maps is the identity, we obtain

$$D(l,0) \ = \ (Di)_H(l) \ .$$

Since the differential is additive we conclude that $D(l,w) = (Di)_H(l) + w$, i.e., D equals the isomorphism (2.8).

We consider the H-equivariant homotopy

$$K \ : \ D(L) \times D(W) \times [0,1] \ \longrightarrow \ V \ , \quad K(l,w,t) \ = \ \begin{cases} \frac{j(t \cdot (l,w)) - v_0}{t} \ + \ t \cdot v_0 & \text{for } t > 0, \text{ and} \\ D(l,w) & \text{for } t = 0. \end{cases}$$

That this assignment is continuous (in fact smooth) when t approaches 0 is the defining property of the differential. Moreover, for every $t \in [0,1]$ the map $K(-,-,t) : D(L) \times D(W) \longrightarrow V$ is a smooth equivariant embedding, so it gives rise to a collapse map $c_t : S^V \longrightarrow S^L \wedge S^W$ defined by

$$c_t(v) \ = \ \begin{cases} \left(\frac{l}{1-|l|}, \frac{w}{1-|w|} \right) & \text{if } v = K(l,w,t) \text{ for some } (l,w) \in D(L) \times D(W), \text{ and} \\ \infty & \text{else.} \end{cases}$$

Since K is an isotopy of embeddings, the passage from the embedding to the collapse map is continuous, so the 1-parameter family c_t provides an H-equivariant based homotopy from the collapse map c_0 to the collapse map c_1 associated to the embedding $j = K(-,-,1) : D(L) \times D(W) \longrightarrow V$, i.e., to the map $(l_H^G \wedge_H S^W) \circ c$. Since $D = (Dj)_{(0,0)}$ is the isometry (2.8), another scaling homotopy compares the collapse map c_0 to the one-point compactification of the inverse of (2.8).

(ii) We define a based continuous G-map

$$r \ : \ \text{map}^H(G, A \wedge S^L) \wedge S^V \ \longrightarrow \ G \ltimes_H (A \wedge S^L \wedge S^W)$$

as the composite

$$\begin{aligned} \text{map}^H(G, A \wedge S^L) \wedge S^V \ &\xrightarrow{\text{Id} \wedge c} \ \text{map}^H(G, A \wedge S^L) \wedge (G \ltimes_H S^W) \\ &\xrightarrow{\text{shear}} \ G \ltimes_H \left(i^*(\text{map}^H(G, A \wedge S^L)) \wedge S^W \right) \\ &\xrightarrow{G \ltimes_H (\epsilon \wedge S^W)} \ G \ltimes_H (A \wedge S^L \wedge S^W) \ . \end{aligned}$$

Here $\epsilon : i^*(\mathrm{map}^H(G, A \wedge S^L)) \longrightarrow A \wedge S^L$ is the adjunction counit, i.e., evaluation at $1 \in G$. We expand this definition. We let $\psi : G \longrightarrow A \wedge S^L$ be an H-equivariant based map and $v \in S^V$. If v is not in the image of the tubular neighborhood $j : G \times_H D(W) \longrightarrow V$, then $r(\psi \wedge v)$ is the basepoint. Otherwise, $v = j[g, w] = g \cdot (v_0 + w)$ for some $(g, w) \in G \times D(W)$, and then

$$\begin{aligned} r(\psi \wedge v) &= ((G \ltimes_H (\epsilon \wedge S^W)) \circ \mathrm{shear})(\psi \wedge [g, w/(1-|w|)]) \\ &= [g, \epsilon(g^{-1} \cdot \psi) \wedge w/(1-|w|)] \;=\; \left[g, \psi(g^{-1}) \wedge \frac{w}{1-|w|}\right] \end{aligned} \tag{2.12}$$

We denote by $\tilde{l}_A : A \longrightarrow \mathrm{map}^H(G, A \wedge S^L)$ the H-map defined by $\tilde{l}_A(a)(g) = l_A[g, a]$, and we let $\varphi : V \longrightarrow L \oplus W$ be the inverse of the isometry (2.8). Now we argue that the following square commutes up to H-equivariant based homotopy:

$$\begin{tikzcd} A \wedge S^V \arrow[r, "{[1,-]}"] \arrow[dd, "\tilde{l}_A \wedge S^V"'] & G \ltimes_H (A \wedge S^V) \arrow[d, "\cong"', "G \ltimes_H S^{\varphi}"] \\ & G \ltimes_H (A \wedge S^L \wedge S^W) \arrow[d, "\cong"', "G \ltimes_H (S^{-\mathrm{Id}_L} \wedge S^W)"] \\ \mathrm{map}^H(G, A \wedge S^L) \wedge S^V \arrow[r, "r"'] & G \ltimes_H (A \wedge S^L \wedge S^W) \end{tikzcd} \tag{2.13}$$

To see this we define an H-homotopy

$$K \;:\; (A \wedge S^V) \times [0,1] \;\longrightarrow\; G \ltimes_H (A \wedge S^L \wedge S^W)$$

as follows. We exploit that the map

$$j \;:\; D(L) \times D(W) \;\longrightarrow\; V \;, \quad j(l, w) \;=\; s(l) \cdot (v_0 + w)$$

is a smooth embedding. This map already featured in the proof of part (i), because the collapse map based on j is the composite $(l_H^G \wedge_H S^W) \circ c$. If a vector is of the form $v = j(l, w) = s(l) \cdot (v_0 + w)$ for some $(l, w) \in D(L) \times D(W)$ (necessarily unique), then we set

$$K(a \wedge v, t) \;=\; K(a \wedge (s(l) \cdot (v_0 + w)),\; t) \;=\; \left[s(t \cdot l),\, a \wedge \frac{-l}{1-|l|} \wedge \frac{w}{1-|w|}\right] \;.$$

For $|l| = 1$ or $|w| = 1$ this formula yields the basepoint, so we can extend this definition by sending all elements $a \wedge v$ to the basepoint whenever v is not in the image of the embedding j. We claim that for $t = 0$ we obtain $K(-, 1) = r \circ (\tilde{l}_A \wedge S^V)$. Indeed, if $v = s(l) \cdot (v_0 + w)$ for $(l, w) \in D(L) \times D(W)$ (necessarily unique), then

$$\begin{aligned} (r \circ (\tilde{l}_A \wedge S^V))(a \wedge v) \quad {\scriptstyle (2.12)} &= \left[s(l), \tilde{l}_A(a)(s(l)^{-1}) \wedge \frac{w}{1-|w|}\right] \\ &= \left[s(l), l_A[s(-l), a] \wedge \frac{w}{1-|w|}\right] \\ &= \left[s(l), a \wedge \frac{-l}{1-|l|} \wedge \frac{w}{1-|w|}\right] \;=\; K(a \wedge v, 1) \;. \end{aligned}$$

On the other hand, the map $K(-, 0)$ agrees with the composite

$$\begin{aligned} A \wedge S^V \;&\xrightarrow{A \wedge ((l_H^G \wedge_H S^W) \circ c)}\; A \wedge S^L \wedge S^W \\ &\xrightarrow{[1,-]}\; G \ltimes_H (A \wedge S^L \wedge S^W) \;\xrightarrow{G \ltimes_H (S^{-\mathrm{Id}_L} \wedge S^W)}\; G \ltimes_H (A \wedge S^L \wedge S^W) \;; \end{aligned}$$

part (i) provides an H-homotopy between $((l_H^G \wedge_H S^W) \circ c)$ and the homeomorphism S^φ; so this proves the claim.

We let $L : G \ltimes_H (A \wedge S^V) \longrightarrow \mathrm{map}^H(G, A \wedge S^L) \wedge S^V$ be the G-equivariant extension of the H-map $\tilde{l}_A \wedge S^V$. Since the square (2.13) commutes up to H-equivariant homotopy, the composite

$$G \ltimes_H (A \wedge S^V) \xrightarrow{L} \mathrm{map}^H(G, A \wedge S^L) \wedge S^V \xrightarrow{r} G \ltimes_H (A \wedge S^L \wedge S^W)$$

is G-equivariantly homotopic to the G-homeomorphism $G \ltimes_H ((S^{-\mathrm{Id}_L} \wedge S^W) \circ S^\varphi)$, by adjointness. So the composite of rL with the shearing homeomorphism $(G \ltimes_H A) \wedge S^V \cong G \ltimes_H (A \wedge S^V)$ is also G-homotopic to a homeomorphism. This composite equals the composite

$$(G \ltimes_H A) \wedge S^V \xrightarrow{\Psi_A \wedge S^V} \mathrm{map}^H(G, A \wedge S^L) \wedge S^V \xrightarrow{r} G \ltimes_H (A \wedge S^L \wedge S^W) \ ,$$

where $\Psi_A : G \ltimes_H A \longrightarrow \mathrm{map}^H(G, A \wedge S^L)$ the adjoint of the H-map $l_A : G \ltimes_H A \longrightarrow A \wedge S^L$.

Now we consider based G-maps $f, f' : B \longrightarrow G \ltimes_H A$ such that $l_A \circ f$ and $l_A \circ f' : B \longrightarrow A \wedge S^L$ are H-equivariantly homotopic. Then the two composites

$$B \xrightarrow{f, f'} G \ltimes_H A \xrightarrow{\Psi_A} \mathrm{map}^H(G, A \wedge S^L)$$

are G-equivariantly homotopic, by adjointness. The composite $(\Psi_A \circ f) \wedge S^V = (\Psi_A \wedge S^V) \circ (f \wedge S^V)$ is then G-equivariantly homotopic to $(\Psi_A \circ f') \wedge S^V = (\Psi_A \wedge S^V) \circ (f' \wedge S^V)$. The map $\Psi_A \wedge S^V$ has a G-equivariant retraction, up to G-homotopy, by the previous paragraph. So already the maps $f \wedge S^V$ and $f' \wedge S^V$ are G-homotopic. □

Now we can establish the Wirthmüller isomorphism, which first appeared in [**176**, Thm. 2.1]. Wirthmüller attributes parts of the ideas to tom Dieck and his statement that G-spectra define a 'complete G-homology theory', amounts to Theorem 2.14. Our proof is essentially Wirthmüller's original argument, adapted to the context of equivariant orthogonal spectra.

We recall that

$$\epsilon_L \ : \ \pi_0^H(Y \wedge S^L) \longrightarrow \pi_0^H(Y \wedge S^L)$$

denotes the effect of the involution of $Y \wedge S^L$ induced by the linear isometry $-\mathrm{Id}_L : L \longrightarrow L$ given by multiplication by -1.

Theorem 2.14 (Wirthmüller isomorphism [**176**]). *Let H be a closed subgroup of a compact Lie group G and Y an orthogonal H-spectrum. Let $L = T_H(G/H)$ denote the tangent H-representation. Then the maps*

$$\mathrm{Wirth}_H^G \ : \ \pi_0^G(G \ltimes_H Y) \longrightarrow \pi_0^H(Y \wedge S^L)$$

and

$$(G \ltimes_H -) \circ \epsilon_L \ : \ \pi_0^H(Y \wedge S^L) \longrightarrow \pi_0^G(G \ltimes_H Y)$$

are independent of the choices made in their definitions, and they are mutually inverse isomorphisms.

Proof. We prove the various claims in a specific order. In a first step we show that the Wirthmüller map is left inverse to the map $(G \ltimes_H -) \circ \epsilon_L$, independently of all the choices made in the definitions. We let U be an H-representation and $f : S^U \longrightarrow Y(U) \wedge S^L$ an H-equivariant based map that represents a class in $\pi_0^H(Y \wedge S^L)$. By enlarging U, if necessary, we can assume that it is the underlying H-representation of a G-representation. We also choose a wide G-equivariant embedding $i : G/H \longrightarrow V$ as in Construction 2.6. This provides a decomposition $L \oplus W \cong V$ of H-representations as in (2.8) and a G-equivariant Thom-Pontryagin collapse map $c : S^V \longrightarrow G \ltimes_H S^W$. We let $\varphi : V \cong L \oplus W$ denote the inverse of the linear

isomorphism (2.8). We contemplate the diagram of based H-maps:

$$\begin{tikzcd}
S^U \wedge S^V \arrow[d, "S^U \wedge c"'] \arrow[rr, bend left, "S^U \wedge S^\varphi"] \arrow[dd, bend right=60, "S^U \wedge (l_H^G \wedge_H S^W)"] & & \\
S^U \wedge (G \ltimes_H S^W) \arrow[rr, "S^U \wedge (l_H^G \wedge_H S^W)"'] \arrow[d, "\text{shear}"', "\cong"] & & S^U \wedge S^L \wedge S^W \arrow[d, "S^U \wedge \tau_{L,W}"] \\
G \ltimes_H (S^U \wedge S^W) \arrow[rr, "l_{S^U \wedge S^W}"'] \arrow[d, "G \ltimes_H (f \diamond W)"] & & S^U \wedge S^W \wedge S^L \arrow[d, "(f \diamond W) \wedge S^L"] \\
G \ltimes_H Y(U \oplus V) \arrow[rr, "l_{Y(U \oplus V)}"'] & & Y(U \oplus V) \wedge S^L
\end{tikzcd}$$

The left vertical composite represents the external transfer $G \ltimes_H \langle f \rangle$, so the composite around the lower left corner represents the class $(l_Y)_*(\mathrm{res}_H^G(G \ltimes_H \langle f \rangle))$. The upper triangle commutes up to H-equivariant homotopy by Proposition 2.11 (i). The middle square commutes up to H-homotopy by Proposition 2.4 and the fact that $l_A : G \ltimes_H A \longrightarrow A \wedge S^L$ is defined as the composite of $(l_H^G \wedge_H A)$ and the twist isomorphism $S^L \wedge A \cong A \wedge S^L$. The lower square commutes by naturality of the maps (2.3). Upon expanding the definition (2.10) of $f \diamond W$, the diagram shows that the class $(l_Y)_*(\mathrm{res}_H^G(G \ltimes_H \langle f \rangle))$ is also represented by the composite

$$S^{U \oplus V} \xrightarrow{f \wedge S^\varphi} Y(U) \wedge S^L \wedge S^L \wedge S^W \xrightarrow{Y(U) \wedge S^L \wedge \tau_{L,W}} Y(U) \wedge S^L \wedge S^W \wedge S^L$$
$$\xrightarrow{Y(U) \wedge S^{\varphi^{-1}} \wedge S^L} Y(U) \wedge S^V \wedge S^L \xrightarrow{\sigma_{U,V}^{\mathrm{op}} \wedge S^L} Y(U \oplus V) \wedge S^L \ .$$

The isometry

$$(\varphi^{-1} \oplus L) \circ (L \oplus \tau_{L,W}) \circ (L \oplus \varphi) \ : \ L \oplus V \ \longrightarrow \ V \oplus L$$

is *not* the twist isometry $\tau_{L,V}$; however $(\varphi^{-1} \oplus L) \circ (L \oplus \tau_{L,W}) \circ (L \oplus \varphi)$ is equivariantly homotopic to the composite

$$L \oplus V \xrightarrow{\tau_{L,V}} V \oplus L \xrightarrow{V \oplus - \mathrm{Id}_L} V \oplus L \ .$$

Hence the class $(l_Y)_*(\mathrm{res}_H^G(G \ltimes_H \langle f \rangle))$ is also represented by the composite

$$S^{U \oplus V} \xrightarrow{f \wedge S^V} Y(U) \wedge S^L \wedge S^V \xrightarrow{Y(U) \wedge \tau_{L,V}} Y(U) \wedge S^V \wedge S^L \xrightarrow{\sigma_{U,V}^{\mathrm{op}} \wedge S^{-\mathrm{Id}}} Y(U \oplus V) \wedge S^L \ .$$

So altogether this shows the desired relation

$$\mathrm{Wirth}_H^G(G \ltimes_H \langle f \rangle) \ = \ (l_Y)_*(\mathrm{res}_H^G(G \ltimes_H \langle f \rangle)) \ = \ \epsilon_L \langle f \rangle \ .$$

In particular, the Wirthmüller map is surjective.

Now we show that the Wirthmüller map is injective. We let $f, f' : S^U \longrightarrow G \ltimes_H Y(U)$ be two G-maps that represent classes with the same image under the Wirthmüller map $\mathrm{Wirth}_H^G : \pi_0^G(G \ltimes_H Y) \longrightarrow \pi_0^H(Y \wedge S^L)$. By increasing the G-representation U, if necessary, we can assume that the composites

$$S^U \xrightarrow{f, f'} G \ltimes_H Y(i^*U) \xrightarrow{l_{Y(i^*U)}} Y(i^*U) \wedge S^L$$

are H-equivariantly homotopic. Then by Proposition 2.11 (ii) the two maps

$$S^{U \oplus V} \xrightarrow{f \wedge S^V, f' \wedge S^V} (G \ltimes_H Y(i^*U)) \wedge S^V \ \cong \ (G \ltimes_H Y)(U) \wedge S^V$$

are G-equivariantly homotopic. The maps remain G-homotopic if we furthermore postcompose with the opposite structure map $\sigma_{U,V}^{\mathrm{op}} : (G \ltimes_H Y)(U) \wedge S^V \longrightarrow (G \ltimes_H Y)(U \oplus V)$. This shows that the stabilizations

from the right of f and f' by V become G-homotopic. Since such a stabilization represents the same class in $\pi_0^G(G \ltimes_H Y)$ as the original map, this shows that $\langle f \rangle = \langle f' \rangle$, i.e., the Wirthmüller map is injective.

Now we know that the Wirthmüller map and the map $(G \ltimes_H -) \circ \epsilon_L$ are inverse to each other, no matter which choices of slice $s : D(L) \longrightarrow G$, representation V and G-embedding $G/H \longrightarrow V$ we made. Since the choices for the Wirthmüller map and the choices for the external transfer are independent of each other, the two resulting maps are independent of all choices.

The last thing to show is that the Wirthmüller map and the external transfer are additive maps. If H has finite index in G, the Wirthmüller map is the composite of a restriction homomorphism and the effect of a morphism of orthogonal H-spectra, both of which are additive. In general, however, we need an additional argument, namely naturality. Indeed, for a fixed choice of slice $s : D(L) \longrightarrow G$, the Wirthmüller map $\pi_0^G(G \ltimes_H Y) \longrightarrow \pi_0^H(Y \wedge S^L)$ is natural in the orthogonal H-spectrum Y. Since source and target are reduced additive functors from orthogonal H-spectra to abelian groups, any natural transformation is automatically additive, compare Proposition II.2.13. Since the Wirthmüller map is additive, so is its inverse, and hence also the external transfer. $\square$

The Wirthmüller map was already defined for homotopy groups of all dimensions, but Theorem 2.14 is stated only for 0-dimensional homotopy groups. We now extend this to homotopy groups in all integer dimensions. This extension is a rather formal consequence of the fact that the Wirthmüller maps commute with the loop and suspension isomorphisms

$$\alpha \ : \ \pi_k^G(\Omega X) \ \longrightarrow \ \pi_{k+1}^G(X) \qquad \text{and} \qquad -\wedge S^1 \ : \ \pi_k^G(X) \ \longrightarrow \ \pi_{k+1}^G(X \wedge S^1)$$

defined in (1.28) respectively (1.29).

Proposition 2.15. *Let H be a closed subgroup of a compact Lie group G and Y an orthogonal H-spectrum.*

(i) *The following diagrams commute for all integers k:*

$$\begin{tikzcd}
\pi_k^G(G \ltimes_H (\Omega Y)) \arrow[r, "\mathrm{Wirth}_H^G"] \arrow[d, "\text{assembly}"', "\cong"] & \pi_k^H((\Omega Y) \wedge S^L) \arrow[d, "\cong"', "\text{assembly}"] \\
\pi_k^G(\Omega(G \ltimes_H Y)) \arrow[d, "\alpha"', "\cong"] & \pi_k^H(\Omega(Y \wedge S^L)) \arrow[d, "\cong"', "\alpha"] \\
\pi_{k+1}^G(G \ltimes_H Y) \arrow[r, "\mathrm{Wirth}_H^G"'] & \pi_{k+1}^H(Y \wedge S^L)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_k^G(G \ltimes_H Y) \arrow[r, "\mathrm{Wirth}_H^G"] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^H(Y \wedge S^L) \arrow[d, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^G((G \ltimes_H Y) \wedge S^1) \arrow[d, "b_*"', "\cong"] & \pi_{k+1}^H(Y \wedge S^L \wedge S^1) \arrow[d, "\cong"', "(Y \wedge \tau_{S^L,S^1})_*"] \\
\pi_{k+1}^G(G \ltimes_H (Y \wedge S^1)) \arrow[r, "\mathrm{Wirth}_H^G"'] & \pi_{k+1}^H(Y \wedge S^1 \wedge S^L)
\end{tikzcd}$$

Here $\tau_{S^L,S^1} : S^L \wedge S^1 \longrightarrow S^1 \wedge S^L$ is the twist isomorphism, and the G-equivariant isomorphism

$$b \ : \ (G \ltimes_H Y) \wedge S^1 \ \cong \ G \ltimes_H (Y \wedge S^1)$$

is given by $b([g, y] \wedge t) = [g, y \wedge t]$.

(ii) *The Wirthmüller map*

$$\mathrm{Wirth}_H^G \ : \ \pi_k^G(G \ltimes_H Y) \ \longrightarrow \ \pi_k^H(Y \wedge S^L)$$

is an isomorphism for all $k \in \mathbb{Z}$.

Proof. (i) The loop and suspension isomorphisms commutes with the restriction homomorphisms from G to H, by direct inspection. So the proof comes down to checking that the following diagrams

commute:

$$\begin{tikzcd}
\pi_k^H(G \ltimes_H (\Omega Y)) \arrow[r, "(l_{\Omega Y})_*"] \arrow[d, "\text{assembly}"', "\cong"] & \pi_k^H((\Omega Y) \wedge S^L) \arrow[d, "\cong"', "\text{assembly}"] \\
\pi_k^H(\Omega(G \ltimes_H Y)) \arrow[d, "\alpha"', "\cong"] & \pi_k^H(\Omega(Y \wedge S^L)) \arrow[d, "\cong"', "\alpha"] \\
\pi_{k+1}^H(G \ltimes_H Y) \arrow[r, "(l_Y)_*"'] & \pi_{k+1}^H(Y \wedge S^L)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_k^H(G \ltimes_H Y) \arrow[r, "(l_Y)_*"] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^H(Y \wedge S^L) \arrow[d, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^H((G \ltimes_H Y) \wedge S^1) \arrow[d, "b_*"', "\cong"] & \pi_{k+1}^H(Y \wedge S^L \wedge S^1) \arrow[d, "\cong"', "(Y \wedge \tau_{S^L,S^1})_*"] \\
\pi_{k+1}^H(G \ltimes_H (Y \wedge S^1)) \arrow[r, "(l_{Y \wedge S^1})_*"'] & \pi_{k+1}^H(Y \wedge S^1 \wedge S^L)
\end{tikzcd}$$

This in turn follows from the fact – again verified by direct inspection – that for every based H-space A the following two square commutes:

$$\begin{tikzcd}
G \ltimes_H (\Omega A) \arrow[r, "l_{\Omega A}"] \arrow[d, "\text{assembly}"'] & (\Omega A) \wedge S^L \arrow[d, "\text{assembly}"] \\
\Omega(G \ltimes_H A) \arrow[r, "\Omega l_A"'] & \Omega(A \wedge S^L)
\end{tikzcd}
\qquad
\begin{tikzcd}
(G \ltimes_H A) \wedge S^1 \arrow[r, "l_A \wedge S^1"] \arrow[d, "b"'] & A \wedge S^L \wedge S^1 \arrow[d, "A \wedge \tau_{S^L,S^1}"] \\
G \ltimes_H (A \wedge S^1) \arrow[r, "l_{A \wedge S^1}"'] & A \wedge S^1 \wedge S^L
\end{tikzcd}$$

(ii) We argue by induction over $|k|$, the absolute value of the integer k. The induction starts with $k = 0$, where Theorem 2.14 provides the desired conclusion. If k is positive, the compatibility of the Wirthmüller map with the loop isomorphism, established in part (i), provides the inductive step. If k is negative, the compatibility of the Wirthmüller map with the suspension isomorphism, also established in part (i), provides the inductive step. □

Construction 2.16 (External transfer in integer degrees). So far we only defined the external transfer for 0-dimensional homotopy groups. Theorem 2.14 shows that in dimension 0 the external transfer is inverse to the map $\epsilon_L \circ \mathrm{Wirth}_H^G : \pi_0^G(G \ltimes_H Y) \longrightarrow \pi_0^H(Y \wedge S^L)$. We want the same property in all dimensions, and since the Wirthmüller maps is an isomorphism by Proposition 2.15 we simply define the external transfer isomorphism

$$G \ltimes_H - \ : \ \pi_k^H(Y \wedge S^L) \ \longrightarrow \ \pi_k^G(G \ltimes_H Y) \tag{2.17}$$

as the composite

$$\pi_k^H(Y \wedge S^L) \ \xrightarrow{\epsilon_L} \ \pi_k^H(Y \wedge S^L) \ \xrightarrow{(\mathrm{Wirth}_H^G)^{-1}} \ \pi_k^G(G \ltimes_H Y) \ .$$

The compatibility of the Wirthmüller isomorphism with the loop and suspension isomorphisms then directly implies the analogous compatibility for the external transfer $G \ltimes_H -$, by reading the diagrams of Proposition 2.15 (i) backwards. We record this compatibility for easier reference.

Corollary 2.18. *Let H be a closed subgroup of a compact Lie group G and Y an orthogonal H-spectrum. Then the following diagrams commute for all integers k:*

$$\begin{tikzcd}
\pi_k^H((\Omega Y) \wedge S^L) \arrow[r, "G \ltimes_H -", "\cong"'] \arrow[d, "\text{assembly}"', "\cong"] & \pi_k^G(G \ltimes_H (\Omega Y)) \arrow[d, "\cong"', "\text{assembly}"] \\
\pi_k^H(\Omega(Y \wedge S^L)) \arrow[d, "\alpha"', "\cong"] & \pi_k^G(\Omega(G \ltimes_H Y)) \arrow[d, "\cong"', "\alpha"] \\
\pi_{k+1}^H(Y \wedge S^L) \arrow[r, "\cong", "G \ltimes_H -"'] & \pi_{k+1}^G(G \ltimes_H Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_k^H(Y \wedge S^L) \arrow[r, "G \ltimes_H -", "\cong"'] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^G(G \ltimes_H Y) \arrow[d, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^H(Y \wedge S^L \wedge S^1) \arrow[d, "(Y \wedge \tau_{S^L,S^1})_*"', "\cong"] & \pi_{k+1}^G((G \ltimes_H Y) \wedge S^1) \arrow[d, "\cong"', "b_*"] \\
\pi_{k+1}^H(Y \wedge S^1 \wedge S^L) \arrow[r, "\cong", "G \ltimes_H -"'] & \pi_{k+1}^G(G \ltimes_H (Y \wedge S^1))
\end{tikzcd}$$

Here $\tau_{S^L,S^1} : S^L \wedge S^1 \longrightarrow S^1 \wedge S^L$ *is the twist isomorphism, and the* G*-equivariant isomorphism*

$$b \ : \ (G \ltimes_H Y) \wedge S^1 \ \cong \ G \ltimes_H (Y \wedge S^1)$$

is given by $b([g,y] \wedge t) = [g, y \wedge t]$.

In the next proposition we use the Wirthmüller isomorphism to show that smashing with a cofibrant G-space is homotopical. In [**105**, III Thm. 3.11], Mandell and May give a different proof of this fact which does not use the Wirthmüller isomorphism.

PROPOSITION 2.19. *Let* G *be a compact Lie group and* A *a cofibrant based* G*-space.*

(i) *Let* $f : X \longrightarrow Y$ *be a morphism of orthogonal* G*-spectra with the following property: for every closed subgroup* H *of* G *that fixes some non-basepoint of* A*, the map* $\pi_*^H(f) : \pi_*^H(X) \longrightarrow \pi_*^H(Y)$ *is an isomorphism. Then the morphism* $f \wedge A : X \wedge A \longrightarrow Y \wedge A$ *is a* $\underline{\pi}_*$*-isomorphisms of orthogonal* G*-spectra.*

(ii) *The functor* $- \wedge A$ *preserves* $\underline{\pi}_*$*-isomorphisms of orthogonal* G*-spectra.*

PROOF. (i) Smashing with A commutes with mapping cones, so by the long exact homotopy group sequence of Proposition 1.37 it suffices to show the following special case. Let X be an orthogonal G-spectrum with the following property: for every closed subgroup H of G that occurs as the stabilizer group of a non-basepoint of A, the groups $\pi_*^H(X)$ vanish. Then for all closed subgroups K of G the equivariant homotopy groups $\pi_*^K(X \wedge A)$ vanish.

In a first step we show this when A is a finite dimensional G-CW-complex. We argue by contradiction. If the claim were false, we could find a compact Lie group G of minimal dimension for which it fails. We let A be a G-CW-complex whose dimension n is minimal among all counterexamples. Then A can be obtained from an $(n-1)$-dimensional subcomplex B by attaching equivariant cells $G/H_i \times D^n$, for i in some indexing set I, where H_i is a closed subgroup of G. Then $X \wedge (B/A)$ is isomorphic to a wedge, over the set I, of orthogonal G-spectra $X \wedge (G/H_i)_+ \wedge S^n$. Since equivariant homotopy groups take wedges to sums, the suspension isomorphism and the Wirthmüller isomorphism allow us to rewrite the equivariant homotopy groups of $X \wedge B/A$ as

$$\begin{aligned} \pi_*^G(X \wedge B/A) \ &\cong \ \bigoplus_I \pi_*^G(X \wedge (G/H)_+ \wedge S^n) \\ &\cong \bigoplus_I \pi_{*-n}^G(X \wedge (G/H_i)_+) \ \cong \bigoplus_I \pi_{*-n}^{H_i}(X \wedge S^{L_i}) \ , \end{aligned}$$

where L_i is the tangent representation of H_i in G. Since H_i is the stabilizer of a non-basepoint of A, the groups $\pi_*^K(X)$ vanish for all closed subgroups K of H_i, by hypothesis. If H_i has finite index in G, then $L_i = 0$ and the respective summand thus vanishes. The representation sphere S^{L_i} admits a finite H_i-CW-structure, so if H_i has strictly smaller dimension than G, then the respective summand vanishes by the minimality of G. So altogether we conclude that the groups $\pi_*^G(X \wedge B/A)$ vanish. The groups $\pi_*^G(X \wedge B)$ vanish by the minimality of A. The inclusion of B into A is an h-cofibration of based G-spaces, so the induced morphism $X \wedge B \longrightarrow X \wedge A$ is an h-cofibration of orthogonal G-spectra. So the groups $\pi_*^G(X \wedge A)$ vanish by the long exact sequence of Corollary 1.39 (i). Since A was supposed to be a counterexample to the proposition, we have reached the desired contradiction. Altogether this proves the claim when A admits the structure of a finite dimensional G-CW-complex.

If A admits the structure of a G-CW-complex, possibly infinite dimensional, we choose a skeleton filtration by G-subspaces A_n. Then the G-homotopy groups of $X \wedge A_n$ vanish for all $n \geq 0$, and all the morphisms $X \wedge A_n \longrightarrow X \wedge A_{n+1}$ are h-cofibrations of orthogonal G-spectra. Since equivariant homotopy groups commute with such sequential colimits, the groups $\pi_*^G(X \wedge A)$ vanish as well. An arbitrary cofibrant based G-space is G-homotopy equivalent to a based G-CW-complex, so the groups $\pi_*^G(X \wedge A)$ vanish for all cofibrant A.

Now we let H be an arbitrary subgroup of G. The underlying H-spectrum of $X \wedge A$ is the smash product of the underlying H-spectrum of X and the underlying H-space A. Moreover, the restriction of a cofibrant based G-space is cofibrant as a based H-space by Proposition A.2.14 (i). So by applying the previous argument to these underlying H-objects, we conclude that the groups $\pi_*^H(X \wedge A)$ vanish for all closed subgroups H of G.

Part (ii) is just a special case of (i). □

Now we discuss the transfer maps of equivariant homotopy groups. For a closed subgroup H of G we construct two kinds of transfer maps, the *dimension shifting transfer*

$$\mathrm{Tr}_H^G \ : \ \pi_k^H(X \wedge S^L) \ \longrightarrow \ \pi_k^G(X)$$

and the *degree zero transfer*

$$\mathrm{tr}_H^G \ : \ \pi_k^H(X) \ \longrightarrow \ \pi_k^G(X)$$

that are defined and natural for orthogonal G-spectra X. The key properties of these transfer maps are

- transfers are transitive;
- transfers commute with inflation maps;
- restriction of a degree zero transfer to a closed subgroup satisfies a double coset formula (compare Theorem 4.10).

Construction 2.20 (Transfers). We let H be a closed subgroup of a compact Lie group G. As before we let $L = T_H(G/H)$ denote the tangent space of G/H at the coset eH, which inherits an H-action from the H-action on G/H. Now we let X be an orthogonal G-spectrum. We form the composite

$$\pi_k^H(X \wedge S^L) \ \xrightarrow[\cong]{G \ltimes_H -} \ \pi_k^G(G \ltimes_H X) \ \xrightarrow{\mathrm{act}_*} \ \pi_k^G(X)$$

of the external transfer (2.17) and the effect of the action map (i.e., the adjunction counit) $G \ltimes_H X \longrightarrow X$. We call this composite the *dimension shifting transfer* and also denote it

$$\mathrm{Tr}_H^G \ : \ \pi_k^H(X \wedge S^L) \ \longrightarrow \ \pi_k^G(X) \ . \tag{2.21}$$

The (degree zero) transfer is then defined as the composite,

$$\mathrm{tr}_H^G \ : \ \pi_k^H(X) \ \xrightarrow{(X \wedge i)_*} \ \pi_k^H(X \wedge S^L) \ \xrightarrow{\mathrm{Tr}_H^G} \ \pi_k^G(X) \ , \tag{2.22}$$

where $i : S^0 \longrightarrow S^L$ is the 'inclusion of the origin', the based map sending 0 to 0. Both kinds of transfer are natural for morphisms of orthogonal G-spectra. For finite index inclusions, $L = 0$ and there is no difference between the dimension shifting transfer and the degree zero transfer. The Wirthmüller map is additive and an isomorphism, hence its inverse, the external transfer, is also additive. Since the dimension shifting and degree zero transfers are obtained from there by applying continuous maps, they are also additive.

Example 2.23 (Infinite Weyl group transfers). If H has infinite index in its normalizer, then the degree zero transfer tr_H^G is trivial. Indeed, the inclusion of the normalizer $N_G H$ of H into G induces a smooth embedding

$$W_G H \ = \ (N_G H)/H \ \longrightarrow \ G/H$$

and thus a monomorphism of tangent H-representations

$$T_H(W_G H) \ \longrightarrow \ T_H(G/H) = L \ .$$

If $n \in N$ is an element of the normalizer and $h \in H$, then

$$h \cdot nH \ = \ n \cdot (n^{-1}hn)H \ = \ nH \ ,$$

so $W_G H$ is H-fixed inside G/H. Consequently, the tangent space $T_H(W_G H)$ is contained in the H-fixed space L^H.

If H has infinite index in its normalizer, then the Weyl group $W_G H$ and the tangent space $T_H(W_G H)$ have positive dimension. In particular, L has non-zero H-fixed points. The point 0 in S^L can thus be moved through H-fixed points to the basepoint at infinity. The first map in the composite (2.22) is thus the zero map, hence so is the transfer tr_H^G.

Remark 2.24. As the previous example indicates, the passage from the dimension shifting transfer Tr_H^G to the degree zero transfer tr_H^G loses information. An extreme case is when the subgroup H is normal in G. Then the action of the group H on G/H is trivial; hence also the tangent H-representation L is trivial. Upon choosing an isomorphism $L \cong \mathbb{R}^d$, the Wirthmüller isomorphism identifies $\pi_0^G(G \ltimes_H Y)$ with $\pi_0^H(Y \wedge S^d)$, where $d = \dim(G/H) = \dim(G) - \dim(H)$. The dimension shifting transfer then becomes a natural map

$$\pi_0^H(Y) \;\cong\; \pi_d^H(Y \wedge S^L) \;\xrightarrow{\mathrm{Tr}_H^G}\; \pi_d^G(Y)\ .$$

This transformation is generically non-trivial.

Now we recall some important properties of the transfer maps. We start with the compatibility with the loop and suspension isomorphisms, which is a straightforward consequence of the analogous property for the external transfer, stated in Corollary 2.18.

Proposition 2.25. *Let H be a closed subgroup of a compact Lie group G and X an orthogonal G-spectrum. Then the following diagrams commute for all integers k:*

$$\begin{tikzcd}
\pi_k^H((\Omega X)\wedge S^L) \arrow[r, "\mathrm{Tr}_H^G"] \arrow[d, "\text{assembly}"', "\cong"] & \pi_k^G(\Omega X) \arrow[dd, "\cong"', "\alpha"] \\
\pi_k^H(\Omega(X\wedge S^L)) \arrow[d, "\alpha"', "\cong"] & \\
\pi_{k+1}^H(X\wedge S^L) \arrow[r, "\mathrm{Tr}_H^G"'] & \pi_{k+1}^G(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_k^H(X\wedge S^L) \arrow[r, "\mathrm{Tr}_H^G"] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^G(X) \arrow[dd, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^H(X\wedge S^L\wedge S^1) \arrow[d, "(X\wedge\tau_{S^L,S^1})_*"', "\cong"] & \\
\pi_{k+1}^H(X\wedge S^1\wedge S^L) \arrow[r, "\mathrm{Tr}_H^G"'] & \pi_{k+1}^G(X\wedge S^1)
\end{tikzcd}$$

Here $\tau_{S^L,S^1} : S^L\wedge S^1 \longrightarrow S^1\wedge S^L$ is the twist isomorphism. Moreover, the following diagrams commute:

$$\begin{tikzcd}
\pi_k^H(\Omega X) \arrow[r, "\mathrm{tr}_H^G"] \arrow[d, "\alpha"', "\cong"] & \pi_k^G(\Omega X) \arrow[d, "\cong"', "\alpha"] \\
\pi_{k+1}^H(X) \arrow[r, "\mathrm{tr}_H^G"'] & \pi_{k+1}^G(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_k^H(X) \arrow[r, "\mathrm{tr}_H^G"] \arrow[d, "-\wedge S^1"', "\cong"] & \pi_k^G(X) \arrow[d, "\cong"', "-\wedge S^1"] \\
\pi_{k+1}^H(X\wedge S^1) \arrow[r, "\mathrm{tr}_H^G"'] & \pi_{k+1}^G(X\wedge S^1)
\end{tikzcd}$$

Now we establish the transitivity property for a nested triple of compact Lie groups $K \le H \le G$. We continue to denote by $L = T_H(G/H)$ the tangent H-representation in G, and we write $\bar{L} = T_K(H/K)$ for the tangent K-representation in H. We choose a slice

$$s \;:\; D(L) \;\longrightarrow\; G$$

as in the construction of the map $l_H^G : G \longrightarrow S^L \wedge H_+$ in Construction 2.2; so s is a wide smooth embedding of the unit disc of L satisfying

$$s(0) \;=\; 1 \qquad \text{and} \qquad s(h\cdot l) \;=\; h\cdot s(l)\cdot h^{-1}$$

for all $(h,l) \in H \times D(L)$, and the differential at $0 \in D(L)$ of the composite

$$D(L) \;\xrightarrow{s}\; G \;\xrightarrow{\mathrm{proj}_H}\; G/H$$

is the identity of L. The differential at $0 \in D(L)$ of the composite

$$D(L) \xrightarrow{s} G \xrightarrow{\mathrm{proj}_K} G/K$$

is then a K-equivariant linear monomorphism

$$D(\mathrm{proj}_K \circ s)_0 \ : \ L \ \longrightarrow \ T_K(G/K) \ = \ L(K,G)$$

that provides a splitting to the differential at eK of the projection $q : G/K \longrightarrow G/H$. So the combined map

$$(D(\mathrm{proj}_K \circ s)_0, (Dq)_{eH}) \ : \ L \oplus \bar{L} \ = \ T_H(G/H) \oplus T_K(H/K) \ \longrightarrow \ T_K(G/K) \ = \ L(K,G) \tag{2.26}$$

is an isomorphism of K-representations. Upon one-point compactification this isomorphism induces a homeomorphism of K-spaces

$$S^L \wedge S^{\bar{L}} \ \cong \ S^{L(K,G)} \ .$$

Any two slices are H-equivariantly isotopic (compare [**27**, VI Thm. 2.6]), so the K-equivariant homotopy class of the latter isomorphism is independent of the choice of slice.

Proposition 2.27 (Transitivity of transfers). *Let G be a compact Lie group, $K \leq H \leq G$ nested closed subgroups and X an orthogonal G-spectrum. Then the composite*

$$\pi_k^K(X \wedge S^{L(K,G)}) \ \cong_{(2.26)} \ \pi_k^K(X \wedge S^L \wedge S^{\bar{L}}) \ \xrightarrow{\mathrm{Tr}_K^H} \ \pi_k^H(X \wedge S^L) \ \xrightarrow{\mathrm{Tr}_H^G} \ \pi_k^G(X)$$

agrees with the transfer Tr_K^G. Moreover, the degree zero transfers satisfy

$$\mathrm{tr}_H^G \circ \mathrm{tr}_K^H \ = \ \mathrm{tr}_K^G \ : \ \pi_k^K(X) \ \longrightarrow \ \pi_k^G(X) \ .$$

Proof. We start by establishing transitivity of the Wirthmüller maps. We choose a slice for the inclusion of K into H, i.e., a wide smooth embedding

$$\bar{s} \ : \ D(\bar{L}) \ \longrightarrow \ H$$

satisfying

$$\bar{s}(0) \ = \ 1 \qquad \text{and} \qquad \bar{s}(k \cdot \bar{l}) \ = \ k \cdot \bar{s}(\bar{l}) \cdot k^{-1}$$

for all $(k, \bar{l}) \in K \times D(\bar{L})$, and such that the differential at $0 \in D(\bar{L})$ lifts the identity of $\bar{L}$. We combine the two slices to obtain a slice for K inside G: the K-equivariant map

$$D(L \oplus \bar{L}) \ \longrightarrow \ G \ , \quad (l, \bar{l}) \ \longmapsto \ s(l) \cdot \bar{s}(\bar{l})$$

sends $(0,0)$ to 1, and its differential at $(0,0)$ exactly the identification (2.26). So we can – and will – define the map $l_K^G : G \longrightarrow S^{L(K,G)} \wedge K_+$ from the slice

$$s' \ : \ D(L(K,G)) \ \xrightarrow[(2.26)^{-1}]{\cong} \ D(L \oplus \bar{L}) \ \xrightarrow{(l,\bar{l}) \mapsto s(l) \cdot \bar{s}(\bar{l})} \ G \ .$$

The maps $l_H^G : G \longrightarrow S^L \wedge H_+$ respectively maps $l_K^H : H \longrightarrow S^{\bar{L}} \wedge K_+$ are the Thom-Pontryagin collapses based on the H^2-equivariant smooth embedding

$$\tilde{s} \ : \ D(L) \times H \ \longrightarrow \ G \ , \quad (v,h) \ \longmapsto \ s(v) \cdot h$$

respectively the K^2-equivariant smooth embedding

$$\hat{s} \ : \ D(\bar{L}) \times K \ \longrightarrow \ H \ , \quad (v,k) \ \longmapsto \ \bar{s}(v) \cdot k \ .$$

The composite $S^L \wedge l_K^H$ is thus K^2-equivariantly homotopic to the Thom-Pontryagin collapse based on the K^2-equivariant smooth embedding

$$D(L) \times D(\bar{L}) \times K \ \longrightarrow \ G \ , \quad (l, \bar{l}, k) \ \longmapsto \ s(l) \cdot \bar{s}(\bar{l}) \cdot k$$

The following diagram of K^2-equivariant smooth embeddings then commutes by construction:

$$\begin{tikzcd}
D(L \oplus \bar{L}) \times K \arrow[r, "(2.26)", "\cong"'] \arrow[d, "\mathrm{incl}"'] & D(L(K,G)) \times K \arrow[dd, "(l,k)\mapsto s'(l)\cdot k"] \\
D(L) \times D(\bar{L}) \times K \arrow[d, "(l,\bar{l},k)\mapsto(l,\bar{s}(\bar{l})\cdot k)"'] & \\
D(L) \times H \arrow[r, "(l,h)\mapsto s(l)\cdot h"'] & G
\end{tikzcd}$$

So the associated diagram of K^2-equivariant collapse maps also commutes:

$$\begin{tikzcd}
G_+ \arrow[r, "l_H^G"] \arrow[dd, "l_K^G"'] & S^L \wedge H_+ \arrow[d, "S^L \wedge l_K^H"] \\
 & S^L \wedge S^{\bar{L}} \wedge K_+ \arrow[d, "\Psi"] \\
S^{L(K,G)} \wedge K_+ \arrow[r, "(2.26)^{-1}", "\cong"'] & S^{L\oplus\bar{L}} \wedge K_+
\end{tikzcd}$$

Here $\Psi : S^L \wedge S^{\bar{L}} \longrightarrow S^{L\oplus\bar{L}}$ is the collapse map for the inclusion $D(L \oplus \bar{L}) \longrightarrow D(L) \times D(\bar{L})$. A rescaling homotopy connects Ψ to the canonical homeomorphism $S^L \wedge S^{\bar{L}} \cong S^{L\oplus\bar{L}}$, so the following square commutes up to K^2-equivariant based homotopy:

$$\begin{tikzcd}
G_+ \arrow[r, "l_H^G"] \arrow[d, "l_K^G"'] & S^L \wedge H_+ \arrow[d, "S^L \wedge l_K^H"] \\
S^{L(K,G)} \wedge K_+ \arrow[r, "(2.26)^{-1}", "\cong"'] & S^{L\oplus\bar{L}} \wedge K_+
\end{tikzcd}$$

We can thus conclude that the map l_K^G/K is K-equivariantly homotopic to the composite

$$\begin{aligned}
G/K_+ \;&\cong\; G \ltimes_H (H/K_+) \xrightarrow{l_H^G \wedge_H (H/K_+)} (H/K_+) \wedge S^L \\
&\cong\; H \ltimes_K S^L \xrightarrow{l_K^H \wedge_K S^L} S^L \wedge S^{\bar{L}} \;\cong_{(2.26)}\; S^{L(K,G)} \;.
\end{aligned}$$

Naturality and transitivity of restriction maps then show that the following diagram commutes:

$$\begin{tikzcd}
\pi_k^G(X \wedge G/K_+) \arrow[rrr, "\mathrm{res}_K^G"] \arrow[d, "\cong"'] & & & \pi_k^K(X \wedge G/K_+) \arrow[dddd, "l_K^G/K"] \\
\pi_k^G(X \wedge G \ltimes_H (H/K_+)) \arrow[d, "\mathrm{res}_H^G"] \arrow[ddd, bend right, "\mathrm{Wirth}_H^G"'] & & & \\
\pi_k^H(X \wedge G \ltimes_H (H/K_+)) \arrow[d, "l_H^G \wedge_H (H/K_+)"] & & & \\
\pi_k^H(X \wedge (H/K_+) \wedge S^L) \arrow[d, "\cong"] & & & \pi_k^K(X \wedge S^{L(K,G)}) \arrow[d, "\cong"] \\
\pi_k^H(X \wedge H \ltimes_K S^L)) \arrow[r, "\mathrm{res}_K^H"] \arrow[rrr, bend right, "\mathrm{Wirth}_K^H"'] & \pi_k^K(X \wedge H \ltimes_K S^L) \arrow[rr, "l_K^H \wedge_K S^L"] & & \pi_k^K(X \wedge S^L \wedge S^{\bar{L}})
\end{tikzcd}$$

By Theorem 2.14 the Wirthmüller map is inverse to the composite

$$\pi_k^K(X \wedge S^{L(K,G)}) \xrightarrow{\epsilon_{L(K,G)}} \pi_k^K(X \wedge S^{L(K,G)}) \xrightarrow{G \ltimes_K -} \pi_k^G(G \ltimes_K X) \ .$$

The map $\epsilon_{L(K,G)}$ is induced by the negative of the identity of $L(K,G)$. Under the homeomorphism between $S^{L(K,G)}$ and $S^L \wedge S^{\bar{L}}$, this becomes the smash product of the involutions induced by the negative of the identities of L and $\bar{L}$. So reading the diagram backwards gives a commutative diagram of external transfers

$$\begin{tikzcd}
\pi_k^K(X \wedge S^L \wedge S^{\bar{L}}) \arrow[r, "H \ltimes_K -"] \arrow[ddd, "\cong"'] & \pi_k^H(X \wedge H \ltimes_K S^L) \arrow[r, "\cong"] & \pi_k^H(X \wedge H/K_+ \wedge S^L) \arrow[d, "G \ltimes_H -"] \\
& & \pi_k^G(X \wedge G \ltimes_H H/K_+) \arrow[d, "\cong"] \\
\pi_k^K(X \wedge S^{L(K,G)}) \arrow[rr, "G \ltimes_K -"'] & & \pi_k^G(X \wedge G/K_+)
\end{tikzcd} \tag{2.28}$$

Postcomposing with the effect of the projection $G/K \longrightarrow *$ and exploiting naturality give the claim about the dimension shifting transfer. The second claim follows by precomposing with the inclusion of the origin of $L(K,G)$. $\square$

Example 2.29. We let $K \leq H \leq G$ be nested closed subgroups and X an orthogonal G-spectrum. We let $p : G/K \longrightarrow G/H$ denote the projection. For later reference we show that under the external transfer isomorphisms the effect of morphism $X \wedge p_+ : X \wedge G/K_+ \longrightarrow X \wedge G/H_+$ corresponds to the transfer from K to H. For simplicity we restrict to the case where $\dim(K) = \dim(H)$, i.e., when K has finite index in H; the general case only differs by more complicated notation. If K has finite index in H, then the the differential of the projection $G/K \longrightarrow G/H$ is an isomorphism from the K-representation $T_{eK}(G/K)$ to the underlying K-representation of $L = T_{eH}(G/H)$. We identify these two representations via this isomorphism. We claim

that then the following square commutes:

$$\begin{tikzcd}
\pi_*^K(X \wedge S^L) \arrow[r, "\mathrm{tr}_K^H"] \arrow[d, "G \ltimes_K -"', "\cong"] & \pi_*^H(X \wedge S^L) \arrow[d, "\cong"', "G \ltimes_H -"] \\
\pi_*^G(X \wedge G/K_+) \arrow[r, "(X \wedge p)_*"'] & \pi_*^G(X \wedge G/H_+)
\end{tikzcd}$$

To see this we compose the commutative diagram (2.28) in the proof of Proposition 2.27 that encodes the transitivity of external transfers with the map $\pi_*^G(X \wedge p_+)$ and arrive at another commutative diagram

$$\begin{tikzcd}
\pi_k^K(X \wedge S^L) \arrow[r, "H \ltimes_K -"'] \arrow[rrr, bend left, "\mathrm{tr}_K^H"] \arrow[dd, equal] & \pi_k^H(X \wedge H \ltimes_K S^L) \arrow[r, "\cong"'] & \pi_k^H(X \wedge H/K_+ \wedge S^L) \arrow[r] \arrow[d, "G \ltimes_H -"] & \pi_k^H(X \wedge S^L) \arrow[dd, "G \ltimes_H -"] \\
& & \pi_k^G(X \wedge G \ltimes_H H/K_+) \arrow[d, "\cong"] & \\
\pi_k^K(X \wedge S^L) \arrow[rr, "G \ltimes_K -"'] & & \pi_k^G(X \wedge G/K_+) \arrow[r, "(X \wedge p_+)_*"'] & \pi_k^G(X \wedge G/H_+)
\end{tikzcd}$$

The upper right diagonal map is induced by the projection $H/K \longrightarrow *$ and the right part of the diagram commutes by naturality of the external transfer.

Now we prove a compatibility of transfers with inflations, i.e., restriction along continuous epimorphisms $\alpha : K \longrightarrow G$. For every closed subgroup H of G, the map

$$\bar{\alpha} \ : \ K/J \ \longrightarrow \ G/H \ , \quad kJ \ \longmapsto \ \alpha(k)H$$

is a diffeomorphism, where $J = \alpha^{-1}(H)$. The differential at the coset eJ is an isomorphism

$$(D\bar{\alpha})_{eJ} \ : \ \bar{L} = T_{eJ}(K/J) \ \longrightarrow \ (\alpha|_J)^*(T_{eH}(G/H)) \ = \ (\alpha|_J)^*(L)$$

of J-representations. In the statement and proof of the following proposition a couple of unnamed isomorphisms occur. One of them is the natural isomorphism of K-spaces

$$K \ltimes_J (\alpha|_J)^*(A) \ \cong \ \alpha^*(G \ltimes_H A) \ , \quad [k, a] \ \longmapsto \ [\alpha(k), a] \ .$$

Proposition 2.30. *Let K and G be compact Lie groups and $\alpha : K \longrightarrow G$ a continuous epimorphism. Let H be a closed subgroup of G, set $J = \alpha^{-1}(H)$, and let $\alpha|_J : J \longrightarrow H$ denote the restriction of α.*

(i) *For every orthogonal H-spectrum Y the following diagram commutes:*

$$\begin{tikzcd}
\pi_k^H(Y \wedge S^L) \arrow[r, "(\alpha|_J)^*"] \arrow[d, "G \ltimes_H -"'] & \pi_k^J((\alpha|_J)^*(Y \wedge S^L)) \arrow[r, "\cong"] & \pi_k^J((\alpha|_J)^*(Y) \wedge S^{\bar{L}}) \arrow[d, "K \ltimes_J -"] \\
\pi_k^G(G \ltimes_H Y) \arrow[r, "\alpha^*"'] & \pi_k^K(\alpha^*(G \ltimes_H Y)) \arrow[r, "\cong"'] & \pi_k^K(K \ltimes_J (\alpha|_J)^*(Y))
\end{tikzcd}$$

(ii) *For every orthogonal G-spectrum X the following diagram commutes:*

$$\begin{tikzcd}
\pi_k^H(X \wedge S^L) \arrow[r, "(\alpha|_J)^*"] \arrow[d, "\mathrm{Tr}_H^G"'] & \pi_k^J((\alpha|_J)^*(X \wedge S^L)) \arrow[r, "\cong"] & \pi_k^J((\alpha|_J)^*(X) \wedge S^{\bar{L}}) \arrow[d, "\mathrm{Tr}_J^K"] \\
\pi_k^G(X) \arrow[rr, "\alpha^*"'] & & \pi_k^K(\alpha^*(X))
\end{tikzcd}$$

Moreover, the degree zero transfers satisfy the relation

$$\alpha^* \circ \mathrm{tr}_H^G \;=\; \mathrm{tr}_J^K \circ (\alpha|_J)^*$$

as maps $\pi_k^H(X) \longrightarrow \pi_k^K(\alpha^*(X))$.

(iii) *For every orthogonal G-spectrum X, every $g \in G$ and all closed subgroups $K \leq H$ of G the following diagram commutes:*

$$\begin{tikzcd}
\pi_k^K(X) \arrow[r, "g_\star"] \arrow[d, "\mathrm{tr}_K^H"'] & \pi_k^{{}^gK}(X) \arrow[d, "\mathrm{tr}_{{}^gK}^{{}^gH}"] \\
\pi_k^H(X) \arrow[r, "g_\star"'] & \pi_k^{{}^gH}(X)
\end{tikzcd}$$

Proof. (i) The restriction maps commute with the loop and suspension isomorphisms, and so do the external transfer maps (by Corollary 2.18). So it suffices to prove the claim in dimension $k = 0$. We choose a wide G-equivariant embedding $i : G/H \longrightarrow V$ into some G-representation; from this input data we form the collapse map

$$c_{G/H} \;:\; S^V \;\longrightarrow\; G \ltimes_H S^W$$

and the external transfer $G \ltimes_H -$. Then the composite

$$\bar{i} \;=\; \alpha^*(i) \circ \bar{\alpha} \;:\; K/J \;\longrightarrow\; \alpha^*(G/H) \;\longrightarrow\; \alpha^*(V) \tag{2.31}$$

is a wide K-equivariant embedding, and we can (and will) base the external transfer $K \ltimes_J -$ on this embedding. Since $i : G/H \longrightarrow V$ and $\alpha^*(i) \circ \bar{\alpha}$ have the same image, they define the same decomposition of V into tangent and normal subspaces, i.e.,

$$\bar{W} \;=\; \alpha^*(V) - (D\bar{i})_{eJ}(\bar{L}) \;=\; (\alpha|_J)^*(W)\ .$$

Moreover, the composite

$$S^{\alpha^*(V)} \;=\; \alpha^*(S^V) \;\xrightarrow{\alpha^*(c_{G/H})}\; \alpha^*(G \ltimes_H S^W) \;\cong\; K \ltimes_J (\alpha|_J)^*(S^W) \;=\; K \ltimes_J S^{\bar{W}}$$

is precisely the collapse map based on the wide embedding (2.31). From here the commutativity of the square is straightforward from the definitions.

(ii) The dimension shifting transfer Tr_H^G is defined as the composite of the external transfer $G \ltimes_H -$ and the effect of the action map $G \ltimes_H X \longrightarrow X$, and similarly for Tr_J^K. The action map for the orthogonal K-spectrum $K \ltimes_J \alpha^*(X)$ coincides with the composite

$$K \ltimes_J \alpha^*(X) \;\xrightarrow{\cong}\; \alpha^*(G \ltimes_H X) \;\xrightarrow{\alpha^*(\mathrm{act})}\; \alpha^*(X)\ .$$

So the following diagram commutes by naturality of the restriction map:

$$\begin{tikzcd}
\pi_k^G(G \ltimes_H X) \arrow[r, "\alpha^*"] \arrow[d, "\pi_k^G(\mathrm{act})"'] & \pi_k^K(\alpha^*(G \ltimes_H X)) \arrow[r, "\cong"] \arrow[dr, "\pi_k^K(\alpha^*(\mathrm{act}))"'] & \pi_k^K(K \ltimes_J \alpha^*(X)) \arrow[d, "\pi_k^K(\mathrm{act})"] \\
\pi_k^G(X) \arrow[rr, "\alpha^*"'] & & \pi_k^K(\alpha^*(X))
\end{tikzcd}$$

So part (ii) follows by stacking this commutative diagram to the one of part (i). The second claim follows by precomposing with the inclusion of the origin of L.

Part (iii) follows from part (ii) for the epimorphism $c_g : {}^gH \longrightarrow H$ and the closed subgroup K of H, and naturality of the transfer:

$$\begin{aligned} g_\star \circ \mathrm{tr}_H^K &= (l_g^X)_* \circ c_g^* \circ \mathrm{tr}_H^K \\ &\overset{\text{(ii)}}{=} (l_g^X)_* \circ \mathrm{tr}_{{}^gH}^{{}^gK} \circ c_g^* \;=\; \mathrm{tr}_{{}^gH}^{{}^gK} \circ (l_g^X)_* \circ c_g^* \;=\; \mathrm{tr}_{{}^gH}^{{}^gK} \circ g_\star \end{aligned}$$

□

Now we know how transfers compose and interact with inflations. The remaining compatibility issue is to rewrite the composite

$$\pi_k^H(X) \xrightarrow{\mathrm{tr}_H^G} \pi_k^G(X) \xrightarrow{\mathrm{res}_K^G} \pi_k^K(X)$$

of a transfer map and a restriction map, where H and K are two closed subgroups of a compact Lie group G. The answer is given by the *double coset formula* that we will prove in Theorem 4.10 below.

3. Geometric fixed points

In this section we study the *geometric fixed point homotopy groups* $\Phi_*^G(X)$ of an orthogonal G-spectrum X, an alternative invariant to characterize equivariant stable equivalences. We establish the isotropy separation sequence (3.11) that is often useful for inductive arguments, and we show that equivariant equivalences can also be detected by geometric fixed points (Proposition 3.12). In Proposition 3.13 we show that transfers from proper subgroups are annihilated by geometric fixed points. Theorem 3.18 provides a functorial description of the 0-th equivariant stable homotopy groups of a G-space Y in terms of the path components of the H-fixed points spaces Y^H for closed subgroups H of G with finite Weyl group.

We define the geometric fixed point homotopy groups of an orthogonal G-spectrum X. As before we let $s(\mathcal{U}_G)$ denote the set of finite dimensional G-subrepresentations of the complete G-universe $\mathcal{U}_G$, considered as a poset under inclusion. We obtain a functor from $s(\mathcal{U}_G)$ to sets by

$$V \;\longmapsto\; [S^{V^G}, X(V)^G]\ ,$$

the set of (non-equivariant) homotopy classes of based maps from the fixed point sphere S^{V^G} to the fixed point space $X(V)^G$. An inclusion $V \subseteq W$ in $s(\mathcal{U}_G)$ is sent to the map

$$[S^{V^G}, X(V)^G] \;\longrightarrow\; [S^{W^G}, X(W)^G]$$

that takes the homotopy class of $f : S^{V^G} \longrightarrow X(V)^G$ to the homotopy class of the composite

$$S^{W^G} \cong S^{(V^\perp)^G} \wedge S^{V^G} \xrightarrow{\mathrm{Id}\wedge f} S^{(V^\perp)^G} \wedge X(V)^G \;=\; (S^{V^\perp} \wedge X(V))^G \xrightarrow{(\sigma_{V^\perp,V})^G} X(V^\perp \oplus V)^G \;=\; X(W)^G$$

Definition 3.1. Let G be a compact Lie group and X an orthogonal G-spectrum. The 0-th *geometric fixed point homotopy group* is defined as

$$\Phi_0^G(X) \;=\; \mathrm{colim}_{V\in s(\mathcal{U}_G)}\, [S^{V^G}, X(V)^G]\ . \tag{3.2}$$

If k is an arbitrary integer, we define the k-th geometric fixed point homotopy group $\Phi_k^G(X)$ as the 0-th homotopy group of a suitably looped or shifted spectrum, analogous to the definition of $\pi_k^G(X)$ in (1.11).

The construction comes with a *geometric fixed point map*

$$\Phi^G \;:\; \pi_0^G(X) \;\longrightarrow\; \Phi_0^G(X)\ , \quad [f : S^V \longrightarrow X(V)] \;\longmapsto\; [f^G : S^{V^G} \longrightarrow X(V)^G] \tag{3.3}$$

from the G-equivariant homotopy group to the geometric fixed point homotopy group. For a trivial group, equivariant and geometric fixed point groups coincide and the geometric fixed point map $\Phi^e : \pi_0^e(X) \longrightarrow \Phi_0^e(X)$ is the identity.

EXAMPLE 3.4 (Geometric fixed points of suspension spectra). If A is any based G-space, then the geometric fixed points $\Phi_*^G(\Sigma^\infty A)$ of the suspension spectrum are given by

$$\Phi_k^G(\Sigma^\infty A) \;=\; \mathrm{colim}_{V\in s(\mathcal{U}_G)}\,[S^{V^G\oplus\mathbb{R}^k}, S^{V^G}\wedge A^G]\ .$$

As V ranges over $s(\mathcal{U}_G)$ the dimension of the fixed points grows to infinity. So the composite

$$\pi_k^e(\Sigma^\infty A^G) \;\xrightarrow{p_G^*}\; \pi_k^G(\Sigma^\infty A^G) \;\xrightarrow{\mathrm{incl}_*}\; \pi_k^G(\Sigma^\infty A) \;\xrightarrow{\Phi^G}\; \Phi_k^G(\Sigma^\infty A)$$

is an isomorphism, where p_G^* is restriction along the unique group homomorphism $p_G : G \longrightarrow e$. We will sometimes refer to this isomorphism by saying that 'geometric fixed points commute with suspension spectra'.

CONSTRUCTION 3.5. We let X be an G-orthogonal spectrum and $\alpha : K \longrightarrow G$ a continuous epimorphism. We define *inflation maps*

$$\alpha^* \;:\; \Phi_0^G(X) \;\longrightarrow\; \Phi_0^K(\alpha^* X)$$

on geometric fixed point homotopy groups. We choose a K-equivariant linear isometric embedding $\psi : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ of the restriction along α of the complete G-universe into the complete K-universe. We let $f : S^{V^G} \longrightarrow X(V)^G$ be a based map representing an element in $\Phi_0^G(X)$, for some $V \in s(\mathcal{U}_G)$. Since α is surjective, $V^G = (\alpha^* V)^K$ and $X(V)^G = (\alpha^*(X(V)))^K = ((\alpha^* X)(\alpha^* V))^K$. We use ψ to identify $\alpha^* V$ with $\psi(V)$ as K-representations, and hence also $(\alpha^* V)^K$ with $\psi(V)^K$. This turns f into a based map

$$S^{\psi(V)^K} \;\cong\; S^{(\alpha^* V)^K} \;=\; S^{V^G} \;\xrightarrow{f}\; X(V)^G \;=\; ((\alpha^* X)(\alpha^* V))^K \;\cong\; ((\alpha^* X)(\psi(V)))^K\ .$$

Any two equivariant embeddings of $\alpha^*(\mathcal{U}_G)$ into $\mathcal{U}_K$ are homotopic through K-equivariant linear isometric embeddings, so the restriction map is independent of the choice of ψ. This latter map represents the element $\alpha^*[f]$ in $\Phi_0^K(\alpha^* X)$. The element $\alpha^*[f]$ depends only on the class of f in $\Phi_0^G(X)$, so the inflation map α^* is a well defined homomorphism.

The surjectivity of α is essential to obtain an inflation map α^* on geometric fixed point homotopy groups, and geometric fixed points do not have natural restriction maps to subgroups. These inflation maps between the geometric fixed point homotopy groups are clearly natural in the orthogonal G-spectrum. The next proposition lists the other naturality properties.

PROPOSITION 3.6. *Let G be a compact Lie group and X an orthogonal G-spectrum.*

(i) *For every pair of composable continuous epimorphisms $\alpha : K \longrightarrow G$ and $\beta : L \longrightarrow K$ we have*

$$\beta^* \circ \alpha^* \;=\; (\alpha\beta)^* \;:\; \Phi_0^G(X) \;\longrightarrow\; \Phi_0^L((\alpha\beta)^* X)\ .$$

(ii) *For every element $g \in G$ the composite*

$$\Phi_0^G(X) \;\xrightarrow{c_g^*}\; \Phi_0^G(c_g^* X) \;\xrightarrow{(l_g^X)_*}\; \Phi_0^G(X)$$

is the identity.

(iii) *For every continuous epimorphism $\alpha : K \longrightarrow G$ of compact Lie groups the following square commutes:*

$$\begin{array}{ccc} \pi_0^G(X) & \xrightarrow{\Phi^G} & \Phi_0^G(X) \\ {\scriptstyle\alpha^*}\downarrow & & \downarrow{\scriptstyle\alpha^*} \\ \pi_0^K(\alpha^* X) & \xrightarrow[\Phi^K]{} & \Phi_0^K(\alpha^* X) \end{array}$$

Proof. (i) We choose a K-equivariant linear isometric embedding $\psi : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ and an L-equivariant linear isometric embedding $\varphi : \beta^*(\mathcal{U}_K) \longrightarrow \mathcal{U}_L$. If we then use the L-equivariant linear isometric embedding

$$\varphi \circ \beta^*(\psi) \ : \ (\alpha \circ \beta)^*(\mathcal{U}_G) \ \longrightarrow \ \mathcal{U}_L$$

for the calculation of $(\alpha \circ \beta)^*$, the equality even holds on the level of representatives.

(ii) We let V be a finite dimensional G-subrepresentation of $\mathcal{U}_G$ and $f : S^{V^G} \longrightarrow X(V)^G$ a based map representing an element in $\Phi_0^G(X)$. We use the G-equivariant linear isometry $l_g : c_g^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_G$ given by left multiplication by g. Then $c_g^*(V)$ and $l_g(V)$ have the same underlying sets and the restriction of l_g to $(c_g^*(V))^G$ is the identity onto $(l_g(V))^G$.

The class $c_g^*[f]$ is represented by the composite

$$S^{(l_g(V))^G} \ = \ S^{(c_g^*V)^G} \ = \ S^{V^G} \ \xrightarrow{f} \ X(V)^G \ = \ ((c_g^*X)(c_g^*V))^G \ \xrightarrow{((c_g^*X)(l_g^V))^G} \ ((c_g^*X)(l_g(V)))^G \ .$$

Consequently, $(l_g^X)_*(c_g^*[f])$ is represented by the composite

$$\begin{aligned} S^{(l_g(V))^G} \ &= \ S^{V^G} \ \xrightarrow{f} \ X(V)^G \ = \ ((c_g^*X)(c_g^*V))^G \\ &\xrightarrow{((c_g^*X)(l_g^V))^G} \ ((c_g^*X)(l_g(V)))^G \ \xrightarrow{((l_g^X)(l_g(V)))^G} \ (X(l_g(V)))^G \ . \end{aligned}$$

The G-action on $X(V)$ is defined diagonally, from the external G-action on X and the internal G-action on V. Hence the map $l_g^{X(V)} : c_g^*(X(V)) \longrightarrow X(V)$ is the composite of $(c_g^*X)(l_g^V) : (c_g^*X)(c_g^*V) \longrightarrow (c_g^*X)(V)$ and $(l_g^X)(V) : (c_g^*X)(V) \longrightarrow X(V)$. Since the restriction of $l_g^{X(V)}$ to the G-fixed points is the identity, the composite of $((c_g^*X)(l_g^V))^G$ and $((l_g^X)(V))^G$ is the identity. This shows that $(l_g^X)_*(c_g^*[f])$ is again represented by f, and hence $(l_g^X)_* \circ c_g^* = \mathrm{Id}$.

(iii) We consider a based continuous G-map $f : S^V \longrightarrow X(V)$; then $\alpha^*[f]$ is represented by the K-map

$$\alpha^*(f) \ : \ S^{\alpha^*(V)} \ = \ \alpha^*(S^V) \ \longrightarrow \ \alpha^*(X(V)) \ = \ (\alpha^*X)(\alpha^*(V)) \ ,$$

and so $\Phi(\alpha^*[f])$ is represented by

$$(\alpha^*(f))^K \ : \ (S^{\alpha^*(V)})^K \ = \ \alpha^*(S^V) \ \longrightarrow \ (\alpha^*(X(V)))^K \ = \ ((\alpha^*X)(\alpha^*(V)))^K \ .$$

Since α is surjective, this is the same map as

$$f^G \ : \ (S^V)^G \ \longrightarrow \ (X(V))^G \ .$$

To calculate $\alpha^*(\Phi^G([f]))$ we choose a K-equivariant linear isometric embedding $\psi : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ and conjugate f^G by the isometry

$$(\psi|_{\alpha^*(V)})^K \ : \ (\alpha^*(V))^K \ \cong \ (\psi(V))^K \ .$$

But conjugation by an isometry does not change the stable homotopy class, by Proposition 1.14 (ii) for the trivial group. So

$$\Phi^K(\alpha^*[f]) \ = \ [\alpha(f)^K] \ = \ [f^G] \ = \ \alpha^*(\Phi^G[f]) \ . \qquad \square$$

Remark 3.7 (Weyl group action on geometric fixed points). We let H be a closed subgroup of a compact Lie group G, and X an orthogonal G-spectrum. Every $g \in G$ gives rise to a conjugation homomorphism $c_g : {}^gH \longrightarrow H$ by $c_g(h) = g^{-1}hg$. Moreover, left translation by g is a homomorphism of orthogonal $({}^gH)$-spectra $l_g : c_g^*(X) \longrightarrow X$. So combining the inflation map along c_g with the effect of l_g gives a homomorphism

$$g_\star \ : \ \Phi_0^H(X) \ \xrightarrow{c_g^*} \ \Phi_0^{{}^gH}(c_g^*(X)) \ \xrightarrow{(l_g)_*} \ \Phi_0^{{}^gH}(X) \ .$$

In the special case when g normalizes H, this is a self map of the geometric fixed point group $\Phi_0^H(X)$. If moreover g belongs to H, then the above map $g_\star$ is the identity by Proposition 3.6 (ii). So the maps $g_\star$

define an action of the Weyl group $W_GH = N_GH/H$ on the geometric fixed point homotopy group $\Phi_0^H(X)$. By the same arguments as for equivariant homotopy groups in Remark 1.17, the identity path component of the Weyl group acts trivially, so the action factors over an action of the discrete group $\pi_0(W_GH) = W_GH/(W_GH)^\circ$. Since the geometric fixed point map $\Phi : \pi_0^H(X) \longrightarrow \Phi_0^H(X)$ is compatible with inflation and natural in X, this map is $\pi_0(W_GH)$-equivariant.

Now we recall the interpretation of the geometric fixed point homotopy groups as the equivariant homotopy groups of the smash product of X with a certain universal G-space. This makes the link to other definitions of geometric fixed point spectra. We denote by $\mathcal{P}_G$ the family of proper subgroups of G. We denote by $E\mathcal{P}_G$ a universal space for the family $\mathcal{P}_G$; so $E\mathcal{P}_G$ is a cofibrant G-space without G-fixed points and such that the fixed point space $(E\mathcal{P}_G)^H$ is contractible for every closed proper subgroup H of G. These properties determine $E\mathcal{P}_G$ uniquely up to G-homotopy equivalence.

We denote by $\tilde{E}\mathcal{P}_G$ the reduced mapping cone of the based G-map $(E\mathcal{P}_G)_+ \longrightarrow S^0$ that sends $E\mathcal{P}_G$ to the non-basepoint of S^0. So $\tilde{E}\mathcal{P}_G$ is the unreduced suspension of the universal space $E\mathcal{P}_G$. The G-fixed points of $E\mathcal{P}_G$ are empty and fixed points commute with mapping cones, so the map $S^0 \longrightarrow (\tilde{E}\mathcal{P}_G)^G$ is an isomorphism. For all proper subgroups H of G the map $(E\mathcal{P}_G)^H \longrightarrow (S^0)^H = S^0$ is a weak equivalence, so the mapping cone $(\tilde{E}\mathcal{P}_G)^H$ is contractible.

Example 3.8. We let $\mathcal{U}_G^\perp = \mathcal{U}_G - (\mathcal{U}_G)^G$ be the orthogonal complement of the G-fixed points in the complete G-universe $\mathcal{U}_G$. We claim that the unit sphere $S(\mathcal{U}_G^\perp)$ of this complement is a universal space $E\mathcal{P}_G$ Indeed, the unit sphere $S(\mathcal{U}_G^\perp)$ is G-equivariantly homeomorphic to the space $\mathbf{L}(\mathbb{R},\mathcal{U}_G^\perp)$, so it is cofibrant as a G-space by Proposition I.2.2 (ii). Since $S(\mathcal{U}_G^\perp)$ has no G-fixed points, any stabilizer group is a proper subgroup of G, i.e., in the family $\mathcal{P}_G$. On the other hand, for every proper subgroup H of G there is a G-representation V with $V^G = 0$ but $V^H \neq 0$. Since $\mathcal{U}_G^\perp$ contains infinitely many isomorphic copies of V, the H-fixed points

$$(S(\mathcal{U}_G^\perp))^H \;=\; S((\mathcal{U}_G^\perp)^H)$$

form an infinite dimensional sphere, and hence are contractible. So $S(\mathcal{U}_G^\perp)$ is a universal G-space for the family of proper subgroups.

Since $\tilde{E}\mathcal{P}_G$ is an unreduced suspension of $E\mathcal{P}_G$, it is equivariantly homeomorphic to

$$S(\mathcal{U}_G^\perp \oplus \mathbb{R})\ ,$$

the unit sphere in $\mathcal{U}_G^\perp \oplus \mathbb{R}$. So the unit sphere $S(\mathcal{U}_G^\perp \oplus \mathbb{R})$ is a model for $\tilde{E}\mathcal{P}_G$.

The inclusion $i : S^0 \longrightarrow \tilde{E}\mathcal{P}_G$ induces an isomorphism of G-fixed points $S^0 \cong (\tilde{E}\mathcal{P}_G)^G$. So for every based G-space A the map $- \wedge i : A \longrightarrow \tilde{A} \wedge E\mathcal{P}_G$ induces an isomorphism of G-fixed points. Hence also for every orthogonal G-spectrum the induced map of geometric fixed point homotopy groups

$$\Phi_k^G(X) \;\cong\; \Phi_k^G(X \wedge \tilde{E}\mathcal{P}_G)$$

is an isomorphism. If we compose the inverse with the geometric fixed point homomorphism (3.3), we arrive at a homomorphism $\Phi : \pi_k^G(X \wedge \tilde{E}\mathcal{P}_G) \longrightarrow \Phi_k^G(X)$.

Proposition 3.9. *For every orthogonal G-spectrum X and every integer k, the geometric fixed point map*

$$\Phi \;:\; \pi_k^G(X \wedge \tilde{E}\mathcal{P}_G) \;\longrightarrow\; \Phi_k^G(X)$$

is an isomorphism.

Proof. We claim that for every finite based G-CW-complex A and every based G-space Y the map

$$(-)^G \;:\; \mathrm{map}^G(A, Y \wedge \tilde{E}\mathcal{P}_G) \;\longrightarrow\; \mathrm{map}(A^G, Y^G)$$

that takes a G-map $f : A \longrightarrow Y \wedge \tilde{E}\mathcal{P}_G$ to the induced map on G-fixed points

$$f^G \;:\; A^G \;\longrightarrow\; (Y \wedge \tilde{E}\mathcal{P}_G)^G = Y^G \wedge (\tilde{E}\mathcal{P}_G)^G \cong Y^G$$

is a weak equivalence and Serre fibration.

Indeed, since A is a G-CW-complex, the inclusion of fixed points $A^G \longrightarrow A$ is a G-cofibration and induces a Serre fibration of equivariant mapping spaces

$$\mathrm{map}^G(A, Y \wedge \tilde{E}\mathcal{P}_G) \;\longrightarrow\; \mathrm{map}^G(A^G, Y \wedge \tilde{E}\mathcal{P}_G)\ .$$

Since every G-map from A^G lands in the G-fixed points of $Y \wedge \tilde{E}\mathcal{P}_G$ and because $(Y \wedge \tilde{E}\mathcal{P}_G)^G = Y^G$, the target space is the non-equivariant mapping space $\mathrm{map}(A^G, Y^G)$. The G-space A is built from its fixed points by attaching G-cells $G/H \times D^n$ whose isotropy H is a proper subgroup. Since the H-fixed points of $Y \wedge \tilde{E}\mathcal{P}_G$ are contractible for all proper subgroups H of G, the fibration is also a weak equivalence.

Now we consider a finite dimensional G-representation V. When applied to $A = S^V$ and $Y = X(V)$, the claim implies that the fixed point map

$$[S^V, X(V) \wedge \tilde{E}\mathcal{P}_G]^G \;\longrightarrow\; [S^{V^G}, X(V)^G]$$

is bijective for every G-representation V. Passing to colimits over the poset $s(\mathcal{U}_G)$ proves the result for $k = 0$. The argument in the other dimensions is similar, and we leave it to the reader. □

A consequence of the previous proposition is the following *isotropy separation sequence*. The mapping cone sequence of based G-space

$$(E\mathcal{P}_G)_+ \;\longrightarrow\; S^0 \;\longrightarrow\; \tilde{E}\mathcal{P}_G$$

becomes a mapping cone sequence of G-spectra

$$X \wedge (E\mathcal{P}_G)_+ \;\longrightarrow\; X \;\longrightarrow\; X \wedge \tilde{E}\mathcal{P}_G \tag{3.10}$$

after smashing with any given orthogonal G-spectrum X. So taking equivariant homotopy groups gives a long exact sequence

$$\cdots \;\longrightarrow \pi_k^G(X \wedge (E\mathcal{P}_G)_+) \;\longrightarrow\; \pi_k^G(X) \;\xrightarrow{\Phi}\; \Phi_k^G(X) \;\longrightarrow\; \pi_{k-1}^G(X \wedge (E\mathcal{P}_G)_+) \;\longrightarrow\; \cdots \tag{3.11}$$

where we exploited the identification of Proposition 3.9.

Proposition 3.12. *Let G be a compact Lie group. For a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra the following are equivalent:*

(i) *The morphism $f : X \longrightarrow Y$ is a $\underline{\pi}_*$-isomorphism.*
(ii) *For every closed subgroup H of G and every integer k the map*

$$\Phi_k^H(f) \;:\; \Phi_k^H(X) \;\longrightarrow\; \Phi_k^H(Y)$$

of geometric fixed point homotopy groups is an isomorphism.

Proof. (i)$\Longrightarrow$(ii) If f is is an equivalence of orthogonal G-spectra, then so is $f \wedge \tilde{E}\mathcal{P}_G$ by Proposition 2.19 (ii). Proposition 3.9 then implies that $\Phi_k^H(f) : \Phi_k^H(X) \longrightarrow \Phi_k^H(Y)$ is an isomorphism for all k.

(ii)$\Longrightarrow$(i) We argue by induction on the size of the group G (i.e., of the dimension of G and order of $\pi_0 G$). If G is the trivial group, then the geometric fixed point map $\Phi : \pi_k^e(X) \longrightarrow \Phi_k^e(X)$ does not do anything, and is an isomorphism. Since $\Phi_*^e(f)$ is an isomorphism, so is $\pi_*^e(f)$.

If G is a non-trivial group we know by induction hypothesis that f is an equivalence of orthogonal H-spectra for every closed proper subgroup H of G. Since $E\mathcal{P}_G$ is a cofibrant G-space without G-fixed points, Proposition 2.19 (i) lets us conclude that $f \wedge (E\mathcal{P}_G)_+$ is an equivalence of orthogonal G-spectra. Since $\Phi_*^G(f) : \Phi_*^G(X) \longrightarrow \Phi_*^G(Y)$ is also an isomorphism, the isotropy separation sequence and the five lemma let us conclude that $\pi_*^G(f) : \pi_*^G(X) \longrightarrow \pi_*^G(Y)$ is an isomorphism. □

The next proposition shows that 'geometric fixed points vanish on transfers'. In fact, it is often a helpful slogan to think of geometric fixed points as 'dividing out transfers from proper subgroups' – despite the fact that the kernel of the geometric fixed point map $\Phi : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ is in general larger than the subgroup generated by proper transfers. For finite groups G, the slogan is in fact true *up to torsion*,

i.e., the geometric fixed point map $\Phi : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ is rationally surjective and its kernel is rationally generated by transfers from proper subgroups, compare Proposition 4.30 below. A more general version of part (iii) below will appear in Proposition 4.2 (ii).

PROPOSITION 3.13. *Let K be a closed subgroup of a compact Lie group G and X an orthogonal G-spectrum.*

(i) *Let H be a closed subgroup of G such that K is not subconjugate to H. Then the composite*

$$\pi_0^G(X) \xrightarrow{\mathrm{res}_K^G} \pi_0^K(X) \xrightarrow{\Phi^K} \Phi_0^K(X)$$

annihilates the image of the dimension shifting transfer $\mathrm{Tr}_H^G : \pi_0^H(X \wedge S^L) \longrightarrow \pi_0^G(X)$ and the image of the degree zero transfer $\mathrm{tr}_H^G : \pi_0^H(X) \longrightarrow \pi_0^G(X)$.

(ii) *In particular, the geometric fixed point map $\Phi^G : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ annihilates the image of the dimension shifting transfer and the image of the degree zero transfer from all proper closed subgroups of G.*

(iii) *If the Weyl group $W_G K$ is finite, then the relation*

$$\Phi^K \circ \mathrm{res}_K^G \circ \mathrm{tr}_K^G \;=\; \sum_{gK \in W_G K} \Phi^K \circ g_\star$$

holds as natural transformations from $\pi_0^K(X) \longrightarrow \Phi_0^K(X)$.

PROOF. (i) Let V be any G-representation. The G-space $(G \ltimes_H X)(V)$ is isomorphic to $G \ltimes_H X(i^*V)$. If K is not subconjugate to H, then both G-spaces have only one G-fixed point, the base point. So the geometric fixed point homotopy group $\Phi_0^K(G \ltimes_H X)$ vanishes. The dimension shifting transfer is defined as the composite

$$\pi_0^H(X \wedge S^L) \xrightarrow{G \ltimes_H -} \pi_0^G(G \ltimes_H X) \xrightarrow{\mathrm{act}_*} \pi_0^G(X)\ .$$

The geometric fixed point map is natural for G-maps, so the composite $\Phi^K \circ \mathrm{res}_K^G \circ \mathrm{act}_* : \pi_0^G(G \ltimes_H X) \longrightarrow \Phi_0^K(X)$ factors through the trivial group $\Phi_0^K(G \ltimes_H X)$. Thus the dimension shifting transfer vanishes. The degree 0 transfer factors through the dimension shifting transfer, so it vanishes as well. Part (ii) is the special case of (i) for $K = G$.

(iii) Since the functor π_0^K is represented by the suspension spectrum of G/K (in the sense of Proposition 1.46), is suffices to check the relation for the orthogonal G-spectrum $\Sigma_+^\infty G/K$ and the tautological class e_K.

For every $g \in N_G K$ and every class $x \in \pi_0^K(X)$ we have

$$g_\star(\Phi^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(x)))) \;=\; \Phi^K(g_\star(\mathrm{res}_K^G(\mathrm{tr}_K^G(x)))) \;=\; \Phi^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(x)))$$

because $g_\star \circ \mathrm{res}_K^G = \mathrm{res}_K^G \circ g_\star = \mathrm{res}_K^G$. In other words, classes of the form $\Phi^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(x)))$ are invariant under the action of the Weyl group $W_G K$ specified in Remark 3.7. In the universal case this class lives in the group $\Phi_0^K(\Sigma_+^\infty G/K)$ which is $W_G K$-equivariantly isomorphic to $\pi_0^e(\Sigma_+^\infty W_G K)$ and hence a free module of rank 1 over the integral group ring of the Weyl group $W_G K$, generated by the class $\Phi^K(e_K)$. So the class $\Phi^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(e_K)))$ is an integer multiple of the norm element, i.e.,

$$\Phi^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(e_K))) \;=\; \lambda \cdot \sum_{gK \in W_G K} g_\star(\Phi^K(e_K))$$

for some $\lambda \in \mathbb{Z}$.

It remains to show that $\lambda = 1$. We let $1 \in \pi_0^K(\mathbb{S})$ be the class represented by the identity of S^0. Inspection of the definition in Construction 2.20 reveals that the transfer of the $\mathrm{tr}_K^G(1)$ in $\pi_0^G(\mathbb{S})$ is represented by the G-map

$$S^V \xrightarrow{c} G \ltimes_K S^W \xrightarrow{a} S^V \tag{3.14}$$

where c is the collapse map based on any wide embedding of $i : G/K \longrightarrow V$ into a G-representation, W is the orthogonal complement of the image of the $T_{eK}(G/K)$ under the differential of i, and $a[g,w] = gw$. So the class $\Phi_0^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(1)))$ is represented by the restriction to K-fixed points of the above composite, i.e., by the map

$$S^{V^K} \xrightarrow{c^K} (G \ltimes_K S^W)^K \xrightarrow{a^K} S^{V^K} .$$

Every K-fixed point of $G \ltimes_K S^W$ is of the form $[g,w]$ with $g \in N_G K$ and $w \in S^{W^K}$, i.e., the map

$$(W_G K)_+ \wedge S^{W^K} \longrightarrow (G \ltimes_K S^W) , \quad gK \wedge w \longmapsto [g,w]$$

is a homeomorphism. Since the Weyl group $W_G K$ is finite we have $(T_{eK}(G/K))^K = 0$, and hence $W^K = V^K$. Under these identification, the map c^K becomes a pinch map

$$S^{V^K} \longrightarrow (W_G K)_+ \wedge S^{V^K} \cong \bigvee_{gK \in W_G K} S^{V^K} ,$$

i.e., the projection to each wedge summand has degree 1. On the other hand, the map a^K becomes the fold map

$$(W_G K)_+ \wedge S^{V^K} \longrightarrow S^{V^K} .$$

So the degree of the composite $a^K \circ c^K$ is the order of the Weyl group $W_G K$. We have thus shown that

$$\Phi^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(1))) = |W_G K| \cdot \Phi^K(1)$$

in the group $\Phi_0^K(\mathbb{S})$. On the other hand, the class 1 is invariant under the action of the Weyl group, and hence

$$\Phi^K(\mathrm{res}_K^G(\mathrm{tr}_K^G(1))) = \lambda \cdot \sum_{gK \in W_G K} g_\star(\Phi^K(1)) = \lambda \cdot |W_G K| \cdot \Phi^K(1) .$$

Since the abelian group $\Phi_0^K(\mathbb{S})$ is freely generated by $\Phi^K(1)$, we can compare coefficients in the last two expressions and deduce that $\lambda = 1$. □

The 0-th equivariant homotopy groups of equivariant *spectra* have two extra pieces of structure, compared to equivariant *spaces*: an abelian group structure and transfers. Theorem 3.18 and Proposition IV.1.11 make precise, first for suspension spectra of G-spaces and then for suspension spectra of orthogonal spaces, that at the level of 0-th equivariant homotopy sets, the suspension spectrum 'freely builds in' the extra structure that is available stably.

We introduce specific stabilization maps that relate unstable homotopy sets to stable homotopy groups. We let H be a compact Lie group and Y an H-space. We define a map

$$\sigma^H : \pi_0(Y^H) \longrightarrow \pi_0^H(\Sigma_+^\infty Y) \tag{3.15}$$

by sending the path component of an H-fixed point $y \in Y^H$ to the equivariant stable homotopy class $\sigma^H[y]$ represented by the H-map

$$S^0 \xrightarrow{-\wedge y} S^0 \wedge Y_+ = (\Sigma_+^\infty Y)(0) .$$

By direct inspection, the map σ^H can be factored as the composition

$$\pi_0(Y^H) \xrightarrow{\sigma^e} \pi_0^e(\Sigma_+^\infty Y^H) \xrightarrow{p_H^*} \pi_0^H(\Sigma_+^\infty Y^H) \xrightarrow{\mathrm{incl}_*} \pi_0^H(\Sigma_+^\infty Y) ,$$

where $p_H : H \longrightarrow e$ is the unique group homomorphism.

We recall that for every space Z the non-equivariant stable homotopy group $\pi_0^e(\Sigma_+^\infty Z)$ is a free abelian group generated by the classes $\sigma^e(y)$ for all $y \in \pi_0(Z)$, i.e,

$$\pi_0^e(\Sigma_+^\infty Z) \cong \mathbb{Z}\{\pi_0(Z)\} . \tag{3.16}$$

Indeed, for all $n \geq 2$ the group $\pi_n(S^n \wedge Z_+, *)$ is free abelian, with basis the classes of the maps $- \wedge y : S^n \longrightarrow S^n \wedge Z_+$ as y runs over the path components of Z, see for example [**167**, Prop. 7.1.7]. Passing to the colimit over n proves the claim.

If H is a closed subgroup of a compact Lie group G, and Y is the underlying H-space of a G-space, then the normalizer $N_G H$ leaves Y^H invariant, and the action of $N_G H$ factors over an action of the Weyl group $W_G H = N_G H/H$ on Y^H. This, in turn, induces an action of the component group $\pi_0(W_G H)$ on the set $\pi_0(Y^H)$. For all $g \in G$, then the following square commutes, again by direct inspection:

$$\begin{tikzcd} \pi_0(Y^H) \arrow[r, "\sigma^H"] \arrow[d, "\pi_0(l_g)"'] & \pi_0^H(\Sigma_+^\infty Y) \arrow[d, "g_\star"] \\ \pi_0(Y^{{}^gH}) \arrow[r, "\sigma^{{}^gH}"'] & \pi_0^{{}^gH}(\Sigma_+^\infty Y) \end{tikzcd}$$

Here $l_g : Y^H \longrightarrow Y^{{}^gH}$ is left multiplication by g. In particular, the map σ^H is equivariant for the action of the group $\pi_0(W_G H)$.

After stabilizing along the map $\sigma^H : \pi_0(Y^H) \longrightarrow \pi_0^H(\Sigma_+^\infty Y)$, we can then transfer from H to G. For an element $g \in N_G H$ and a class $x \in \pi_0(Y^H)$ we have

$$\mathrm{tr}_H^G(\sigma^H(\pi_0(l_g)(x))) \;=\; \mathrm{tr}_H^G(g_\star(\sigma^H(x))) \;=\; g_\star(\mathrm{tr}_H^G(\sigma^H(x))) \;=\; \mathrm{tr}_H^G(\sigma^H(x)) \;, \tag{3.17}$$

because transfer commutes with conjugation and inner automorphisms act as the identity. So the composite $\mathrm{tr}_H^G \circ \sigma^H$ coequalizes the $\pi_0(W_G H)$-action on $\pi_0(Y^H)$.

Our proof of the following theorem is based on an inductive argument with the isotropy separation sequence. A different proof, based on the tom Dieck splitting, can be found in [**97**, V Cor. 9.3].

THEOREM 3.18. *Let G be a compact Lie group and Y a G-space.*

(i) *The equivariant homotopy group $\pi_0^G(\Sigma_+^\infty Y)$ is a free abelian group with a basis given by the elements*

$$\mathrm{tr}_H^G(\sigma^H(x)) \;,$$

where H runs through all conjugacy classes of closed subgroups of G with finite Weyl group and x runs through a set of representatives of the $W_G H$-orbits of the elements of the set $\pi_0(Y^H)$.

(ii) *Let $z \in \pi_0^G(\Sigma_+^\infty Y)$ be an equivariant homotopy class such that for every closed subgroup K of G with finite Weyl group the geometric fixed point class*

$$\Phi^K(\mathrm{res}_K^G(z)) \;\in\; \Phi_0^K(\Sigma_+^\infty Y)$$

is trivial. Then $z = 0$.

PROOF. (i) In (3.16) we recalled property (i) when G is a trivial group. For the trivial group the geometric fixed point map $\Phi^e : \pi_0^e(\Sigma_+^\infty Y) \longrightarrow \Phi_0^e(\Sigma_+^\infty Y)$ is the identity, so part (ii) is tautologically true.

Now we let G be any compact Lie group. We let H be a closed subgroup of G with finite Weyl group. By (3.16) we know that the group $\pi_0^e(\Sigma_+^\infty Y^H)$ is free abelian on the set of path components of Y^H; moreover, the Weyl group $W_H K$ permutes the basis elements, i.e., $\pi_0^e(\Sigma_+^\infty Y^H)$ is an integral permutation representation of the group $W_G H$. For any representation M of a finite group W the norm map

$$N \;:\; M \;\longrightarrow\; M \;, \quad x \;\longmapsto\; \sum_{w \in W} w\, x$$

factors over the group of coinvariants

$$M_W \;=\; M/\langle x - wx \,|\, x \in M, w \in W\rangle \;.$$

For the integral permutation representation $M = \mathbb{Z}[S]$ of a W-set S, a special feature is that the induced map $\bar{N} : M_W \longrightarrow M$ is injective.

So part (i) is equivalent to the claim that the map

$$T \ : \ \bigoplus_H (\pi_0^e(\Sigma_+^\infty Y^H))_{W_G H} \ \longrightarrow \ \pi_0^G(\Sigma_+^\infty Y)$$

is an isomorphism, where the sum is indexed by representatives of the conjugacy classes of closed subgroup with finite Weyl groups, and the restriction of T to the H-summand is induced by the composite

$$\pi_0^e(\Sigma_+^\infty Y^H) \ \xrightarrow{p_H^*} \ \pi_0^H(\Sigma_+^\infty Y^H) \ \xrightarrow{\text{incl}} \ \pi_0^H(\Sigma_+^\infty Y) \ \xrightarrow{\text{tr}_H^G} \ \pi_0^G(\Sigma_+^\infty Y) \ .$$

We consider the total geometric fixed point homomorphism

$$\Phi^{\text{total}} \ : \ \pi_0^G(\Sigma_+^\infty Y) \ \longrightarrow \ \prod_{K:\, W_G K \text{ finite}} \Phi_0^K(\Sigma_+^\infty Y) \ , \quad y \ \longmapsto \ (\Phi^K(\text{res}_K^G(y)))_K \ .$$

Property (ii) is equivalent to the claim that Φ^{total} is injective.

We show now that the composite $\Phi^{\text{total}} \circ T$ is injective. We argue by contradiction and suppose that

$$z \ = \ (z_H)_H \ \in \ \bigoplus_H (\pi_0^e(\Sigma_+^\infty Y^H))_{W_G H}$$

is a non-zero element in the source of T such that $\Phi^{\text{total}}(T(z))$ vanishes. We let K be of maximal dimension and with maximal number of path components among all indexing subgroups such that $z_K \neq 0$. Then

$$T(z) \ = \ \text{tr}_K^G(\text{incl}_*(p_K^*(y))) \ + \ \sum_{i=1}^m \text{tr}_{H_i}^G(y_i)$$

with y an element of $\pi_0^e(\Sigma_+^\infty Y^K)$ with non-zero image in the coinvariants $\pi_0^e(\Sigma_+^\infty Y^K))_{W_G K}$, and with certain closed subgroups H_i of G that are not conjugate to K and 'no larger' in the sense that either $\dim(H_i) < \dim(K)$, or $\dim(H_i) = \dim(K)$ and $|\pi_0(H_i)| \leq |\pi_0(K)|$. This means in particular that K is not subconjugate to any of the groups $H_1, \dots, H_m$. Thus

$$\Phi^K(\text{res}_K^G(\text{tr}_{H_i}^G(y_i))) \ = \ 0$$

for all $i = 1, \dots, m$, by Proposition 3.13 (i). Hence

$$\begin{aligned} 0 \ = \ \Phi^K(T(z)) \ &= \ \Phi^K(\text{res}_K^G(\text{tr}_K^G(\text{incl}_*(p_K^*(y))))) \\ &= \ \sum_{gK \in W_G K} \Phi^K(g_\star(\text{incl}_*(p_K^*(y)))) \\ &= \ \sum_{gK \in W_G K} \Phi^K(\text{incl}_*(g_\star(p_K^*(y)))) \\ &= \ \Phi^K(\text{incl}_*(p_K^*(\textstyle\sum_{gK \in W_G K} (l_g)_*(y)))) \ . \end{aligned}$$

The third equation is Proposition 3.13 (iii). Since the composite

$$\pi_0^e(\Sigma_+^\infty Y^K) \ \xrightarrow{p_K^*} \ \pi_0^K(\Sigma_+^\infty Y^K) \ \xrightarrow{\text{incl}_*} \ \pi_0^K(\Sigma_+^\infty Y) \ \xrightarrow{\Phi^K} \ \Phi_0^K(\Sigma_+^\infty Y)$$

is an isomorphism, we conclude that the norm of the element $y \in \pi_0^e(\Sigma_+^\infty Y^K)$ is zero. But since $\pi_0^e(\Sigma_+^\infty Y^K)$ is an integral permutation representation of the Weyl group, this only happens if y maps to 0 in the coinvariants, which contradicts our assumption.

Now we show that the classes $\text{tr}_H^G(\sigma^H(x))$ in the statement of (i) generate the group $\pi_0^G(\Sigma_+^\infty Y)$ (i.e., the homomorphism T is surjective). We argue by induction on the size of G, i.e., by a double induction over the dimension and number of path components of G. The induction starts when G is the trivial group,

which we dealt with above. Now we let G be a non-trivial compact Lie group. We start with the special case $Y = G/K_+$ for a proper closed subgroup K of G. The composite

$$G \ltimes_K \mathbb{S} \xrightarrow{G \ltimes_K (eK)_*} G \ltimes_K (\Sigma^\infty_+ G/K) \xrightarrow{\text{act}} \Sigma^\infty_+ G/K$$

is an isomorphism of orthogonal G-spectra. Hence the composite

$$\pi_0^K(\Sigma^\infty S^V) \xrightarrow[\cong]{G \ltimes_K -} \pi_0^G(G \ltimes \mathbb{S}) \xrightarrow{\pi_0^G(G \ltimes_K (eK)_*)} \pi_0^G(G \ltimes_K (\Sigma^\infty_+ G/K)) \xrightarrow{\pi_0^G(\text{act})} \pi_0^G(\Sigma^\infty_+ G/K)$$

is an isomorphism of abelian groups, where $V = T_K(G/K)$ is the tangent representation and the first map is the external transfer (an isomorphism by Theorem 2.14).

The inclusion $S^0 \longrightarrow S^V$ is an equivariant h-cofibration and its quotient S^V/S^0 is G-homeomorphic to the unreduced suspension of the unit sphere $S(V)$ (with respect to any K-invariant scalar product on V). So the group

$$\pi_0^G(\Sigma^\infty(S^V/S^0)) \;\cong\; \pi_0^G(\Sigma^\infty(S(V)_+ \wedge S^1)) \;\cong\; \pi_{-1}^G(\Sigma^\infty_+ S(V))$$

vanishes by the suspension isomorphism and Proposition 1.45 (ii). The long exact homotopy group sequence of Corollary 1.39 (i) then shows that the map

$$\text{incl}_* \;:\; \pi_0^K(\Sigma^\infty_+ \{0\}) \;\longrightarrow\; \pi_0^K(\Sigma^\infty S^V)$$

is surjective. Since K is a proper closed subgroup of G, it either has smaller dimension or fewer path components, so we know by the inductive hypothesis that the group $\pi_0^K(\Sigma^\infty_+ \{0\})$ is generated by the elements $\text{tr}_L^K(\sigma^L[0])$ where L runs through all conjugacy classes of closed subgroups of K with finite Weyl group.

Putting this all together lets us conclude that the group $\pi_0^G(\Sigma^\infty_+ G/K)$ is generated by the classes

$$\left(\pi_0^G(\text{act} \circ (G \ltimes_K (eK)_*)) \circ (G \ltimes_K -) \circ \text{tr}_L^K \circ \sigma^L\right)[0]$$

for all K-conjugacy classes of closed subgroups $L \leq K$ that have finite Weyl group in K. However, this long expression in fact defines a familiar class, as we shall now see. Indeed, the following diagram commutes by the various naturality properties:

$$\begin{tikzcd}
\pi_0(\{0\}) \arrow[r, "\pi_0((eK)_*)"] \arrow[d, "p_L^* \circ \sigma^e"] \arrow[ddd, bend right=60, "\sigma^L"'] & \pi_0((G/K)^L) \arrow[d, "p_L^* \circ \sigma^e"'] \arrow[ddd, bend left=60, "\sigma^L"] & & \\
\pi_0^L(\Sigma^\infty S^0) \arrow[r, "\pi_0^L((eK)_*)"] \arrow[dd, "\text{incl}_*"] & \pi_0^L(\Sigma^\infty_+ (G/K)^L) \arrow[d, "\text{incl}_*"] & & \\
 & \pi_0^L(\Sigma^\infty_+ G/K) \arrow[d, "\text{incl}_*"] \arrow[dr, "\text{tr}_L^K"] & & \\
\pi_0^L(\Sigma^\infty_+ S^V) \arrow[r, "\pi_0^L((eK)_* \wedge S^V)"] \arrow[d, "\text{tr}_L^K"] & \pi_0^L(\Sigma^\infty_+ G/K \wedge S^V) \arrow[dr, "\text{tr}_L^K"] & \pi_0^K(\Sigma^\infty_+ G/K) \arrow[d, "\text{incl}_*"] \arrow[ddr, bend left, "\text{tr}_K^G"] & \\
\pi_0^K(\Sigma^\infty_+ S^V) \arrow[rr, "\pi_0^K((eK)_* \wedge S^V)"] \arrow[d, "G \ltimes_K -"', "\cong"] & & \pi_0^K(\Sigma^\infty_+ G/K \wedge S^V) \arrow[d, "G \ltimes_K -"'] & \\
\pi_0^G(G \ltimes_K \mathbb{S}) \arrow[rr, "\pi_0^G(G \ltimes_K (eK)_*))"'] & & \pi_0^G(G \ltimes \Sigma^\infty_+ G/K) \arrow[r, "\pi_0^G(\text{act})"'] & \pi_0^G(\Sigma^\infty_+ G/K)
\end{tikzcd}$$

So

$$\big(\pi_0^G(\mathrm{act}\circ(G\ltimes_K(eK)_*))\circ(G\ltimes_K -)\circ \mathrm{tr}_L^K\circ\sigma^L\big)\,[0] \;=\; \mathrm{tr}_K^G(\mathrm{tr}_L^K(\sigma^L((eK)_*[0]))) \;=\; \mathrm{tr}_L^G(\sigma^L[eK])\ .$$

So the group $\pi_0^G(\Sigma_+^\infty G/K)$ is generated by the classes $\mathrm{tr}_L^G(\sigma^L[eK])$ for all K-conjugacy classes of closed subgroups $L\le K$ that have finite Weyl group in K. If the Weyl group of L in the ambient group G happens to have infinite, then $\mathrm{tr}_L^G=0$ and the generator is redundant. Otherwise eK is an L-fixed point of G/K, so the generator is one of the classes mentioned in the statement of (i). This shows the generating property for the G-space G/K.

Next we observe that whenever the claim is true for a family of G-spaces, then it is also true for the disjoint union; this follows from the fact that both fixed points and π_0 commute with disjoint unions, and that equivariant homotopy groups take wedges to direct sums. In particular, the claim holds when Y is a disjoint union of homogeneous space G/H for varying proper closed subgroups H of G.

Now we prove the claim when Y is any G-space without G-fixed points. For every proper closed subgroup H of G we choose representatives for the path components of the H-fixed point space Y^H. These choices determine a continuous G-map

$$f\ :\ Z\ =\ \coprod_H\ \coprod_{[x]\in\pi_0(Y^H)} G/H\ \longrightarrow\ Y$$

by sending gH in the summand indexed by $x\in Y^H$ to gx. Because Y has no G-fixed points, the induced map $\pi_0(f^H):\pi_0(Z^H)\longrightarrow\pi_0(Y^H)$ is then surjective for every closed subgroup H of G, by construction. In the commutative square

$$\begin{tikzcd}
\bigoplus_H(\pi_0^e(\Sigma_+^\infty Z^H))_{W_GH} \arrow[r, "T"] \arrow[d, "\bigoplus(\pi_0^e(\Sigma_+^\infty f^H))_{W_GH}"'] & \pi_0^G(\Sigma_+^\infty Z) \arrow[d, "\pi_0^G(\Sigma_+^\infty f)"] \\
\bigoplus_H(\pi_0^e(\Sigma_+^\infty Y^H))_{W_GH} \arrow[r, "T"'] & \pi_0^G(\Sigma_+^\infty Y)
\end{tikzcd}$$

the right vertical map is then surjective by Proposition 1.45 (i) for $m=0$, and the upper horizontal map is surjective by the above. So the lower horizontal map is surjective as well.

Now we let Y be any G-space, possibly with G-fixed points. The composite

$$\pi_*^e(\Sigma_+^\infty Y^G)\ \xrightarrow{p_G^*}\ \pi_*^G(\Sigma_+^\infty Y^G)\ \xrightarrow{\mathrm{incl}_*}\ \pi_*^G(\Sigma_+^\infty Y)\ \xrightarrow{\Phi^G}\ \Phi_*^G(\Sigma_+^\infty Y)$$

is an isomorphism, i.e., the map $\mathrm{incl}_*\circ p_G^*$ splits the geometric fixed point homomorphism. The long exact isotropy separation sequence (3.11) thus decomposes into short exact sequences and the map

$$(q_*,\,\mathrm{incl}_*\circ p_G^*)\ :\ \pi_*^G(\Sigma_+^\infty(Y\times E\mathcal{P}_G))\oplus\pi_*^e(\Sigma_+^\infty Y^G)\ \longrightarrow\ \pi_*^G(\Sigma_+^\infty Y)$$

is an isomorphism. The G-space $Y\times E\mathcal{P}_G$ has no G-fixed points. So by the previous part, the group $\pi_0^G(\Sigma_+^\infty(Y\times E\mathcal{P}_G))$ is generated by the classes $\mathrm{tr}_H^G(\sigma^H(x))$ for H with finite Weyl group as above and $x\in\pi_0((Y\times E\mathcal{P}_G)^H)$. If H is a proper subgroup, then $(E\mathcal{P}_G)^H$ is contractible and so the projection $q:Y\times E\mathcal{P}_G\longrightarrow Y$ induces a bijection on $\pi_0((-)^H)$. So the generators coming from $Y\times E\mathcal{P}_G$ map to the desired basis elements for Y that are indexed by proper subgroups of G. On the other hand, the group $\pi_*^e(\Sigma_+^\infty Y^G)$ is free with basis the classes $\sigma^e(x)$ for all components $x\in\pi_0(Y^G)$, by (3.16). Because

$$\mathrm{incl}_*(p_G^*(\sigma^e(x)))\ =\ \sigma^G(x)\ ,$$

the basis of the second summand precisely maps to the desired basis elements for Y that are indexed by the group G itself. This completes the argument that the map T is surjective.

Now we can complete the proofs of part (i) and (ii). Since the composite $\Phi^{\mathrm{total}}\circ T$ is injective, the homomorphism T is injective; since T is also surjective, it is bijective, which shows (i) for the group G.

Since T is bijective and $\Phi^{\text{total}} \circ T$ is injective, the homomorphism Φ^{total} is injective, which shows (ii) for the group G. $\square$

EXAMPLE 3.19 (Equivariant 0-stems and Burnside rings). We recall that for a finite group G, the *Burnside ring* $A(G)$ is the Grothendieck group of the abelian monoid, under disjoint, of isomorphism classes of finite G-sets. Every finite G-set is the disjoint union of transitive G-sets, so $A(G)$ is a free abelian group generated by the classes of the G-sets G/H, as H runs over all conjugacy classes of subgroups of G.

The sphere spectrum $\mathbb{S}$ is also the unreduced suspension spectrum of the one-point G-space, $\mathbb{S} \cong \Sigma^\infty_+ *$. So Theorem 3.18 (i) says that the map

$$\psi_G \ : \ A(G) \ \longrightarrow \ \pi_0^G(\mathbb{S}) \ , \quad [G/H] \ \longmapsto \mathrm{tr}_H^G(1)$$

is an isomorphism, a result that is originally due to Segal [**139**]. The Burnside rings for different groups are relation by restriction homomorphisms $\alpha^* : A(G) \longrightarrow A(K)$ along homomorphisms $\alpha : K \longrightarrow G$, induced by restriction of the action along α. The Burnside rings for different groups are relation by restriction maps along homomorphisms $\alpha : K \longrightarrow G$, induced by restriction of the action along α. The Burnside rings also enjoy transfer maps

$$\mathrm{tr}_H^G \ : \ A(H) \ \longrightarrow \ A(G)$$

induced by sending an H-set S to the induced G-set $G \times_H S$.

The compatibility of transfers with inflations (Proposition 2.30 (iii)) implies that for every surjective homomorphism α the relation

$$\alpha^*(\psi_G[G/H]) \ = \ \alpha^*(\mathrm{tr}_H^G(1)) \ = \ \mathrm{tr}_L^K((\alpha|_L)^*(1)) \ = \ \mathrm{tr}_L^K(1) \ = \ \psi_K([K/L]) \ = \ \psi_K(\alpha^*[G/H])$$

holds. In other words, the isomorphisms ψ_G are compatible with inflation. The double coset formula (see Example 4.12 below), and the double coset formula in the Burnside rings imply that the isomorphisms ψ_G are also compatible with transfers.

If G is a compact Lie group of positive dimension, there is no direct interpretation of $\pi_0^G(\mathbb{S})$ in terms of finite G-sets; in that situation, some authors *define* the Burnside ring of G as the group $\pi_0^G(\mathbb{S})$. Theorem 3.18 (i) identifies $\pi_0^G(\mathbb{S})$. as a free abelian group with basis the classes $\mathrm{tr}_H^G(1)$ for all conjugacy classes of closed subgroups H with finite Weyl group. In [**165**, Prop. 5.5.1], tom Dieck gives an interpretation of this Burnside ring in terms of certain equivalence classes of compact G-ENRs.

Theorem 3.18 (ii) provides a convenient detection criterion for elements in the equivariant 0-stems; as we explain now, this can be rephrased as a degree function. For every compact Lie group K the geometric fixed point group $\Phi_0^K(\mathbb{S})$ is the 0-th non-equivariant stable stem, and hence free abelian of rank 1, generated by the class $\Phi^K(1)$. Moreover, if $z \in \pi_0^K(\mathbb{S})$ is represented by the K-map $f : S^V \longrightarrow S^V$ for some K-representation V, then

$$\Phi^K(z) \ = \ [f^K \ : \ S^{V^K} \longrightarrow S^{V^K}] \ = \ \deg(f^K) \cdot \Phi^K(1) \ .$$

So in terms of the basis $\Phi^K(1)$, the geometric fixed point homomorphism $\Phi^K : \pi_0^K(\mathbb{S}) \longrightarrow \Phi_0^K(\mathbb{S})$ is extracting the degree of the restriction of any representative to K-fixed points. We let $C(G)$ denote the set of integer valued, conjugation-invariant functions from the set of closed subgroup of G with finite Weyl group. We define a homomorphism

$$\deg \ : \ \pi_0^G(\mathbb{S}) \ \longrightarrow \ C(G) \tag{3.20}$$

by

$$\deg[f : S^V \longrightarrow S^V](K) \ = \ \deg(f^K \ : \ S^{V^K} \longrightarrow S^{V^K}) \ .$$

Theorem 3.18 (ii) then says that this degree homomorphism is injective. Whenever the group G is non-trivial, the degree homomorphism is *not* surjective. [...continuity condition...] In [**165**, Prop. 5.8.5], tom Dieck exhibits an explicit set of congruences that together with the continuity condition characterize the image of the degree homomorphism (3.20). When G is finite, these congruences – combined with the

isomorphism ψ_G to the Burnside ring $A(G)$ – specialize to the congruences between the cardinalities of the various fixed point sets of a finite G-set, specified for example in [**165**, Prop. 1.3.5].

We will revisit the equivariant 0-stems from a global perspective in Example IV.2.7 and with a view towards products and multiplicative power operations in Example V.4.1.

Theorem 3.18 gives a functorial description of the 0-th equivariant stable homotopy group of an unbased G-space. We will now deduce a similar result for reduced suspension spectra of *based* G-spaces. This version, however, needs a non-degeneracy hypothesis, i.e., we must restrict to well-pointed G-spaces.

THEOREM 3.21. *Let G be a compact Lie group and Y a well-pointed G-space. Then the group $\pi_0^G(\Sigma^\infty Y)$ is a free abelian group with a basis given by the elements*

$$\mathrm{tr}_H^G(\sigma^H(x)) \ ,$$

where H runs through all conjugacy classes of closed subgroups of G with finite Weyl group and x runs through a set of representatives of the $W_G H$-orbits of the non-basepoint components of the set $\pi_0(Y^H)$.

PROOF. If the inclusion of the basepoint $i : \{y_0\} \longrightarrow Y$ is an unbased h-cofibration, then the based map $i_+ : \{y_0\}_+ \longrightarrow Y_+$ is a based h-cofibration. The induced map of suspension spectra $\Sigma^\infty_+ i : \Sigma^\infty_+\{y_0\} \longrightarrow \Sigma^\infty_+ Y$ is an h-cofibration of orthogonal G-spectra, so it gives rise to a long exact sequence of homotopy groups as in Corollary 1.39 (i). The cokernel of $i_+ : \{y_0\}_+ \longrightarrow Y_+$ is G-homeomorphic to Y (with the original basepoint). Also, the map i has a section, so the long exact sequence degenerates into a short exact sequence:

$$0 \ \longrightarrow \ \pi_0^G(\Sigma^\infty_+\{y_0\}) \ \xrightarrow{(\Sigma^\infty_+ i)_*} \ \pi_0^G(\Sigma^\infty_+ Y) \ \xrightarrow{\mathrm{proj}_*} \ \pi_0^G(\Sigma^\infty Y) \ \longrightarrow \ 0$$

Theorem 3.18 (i), applied to the unbased G-spaces $\{y_0\}$ and Y, provides bases of the left and middle group; by naturality the map $(\Sigma^\infty_+ i)_*$ hits precisely the subgroup generated by the basis elements coming from basepoint components of the space Y^H. So the cokernel of $(\Sigma^\infty_+ i)_*$, and hence the group $\pi_0^G(\Sigma^\infty Y)$, has a basis of the desired form. □

4. The double coset formula

The main aim of this section is to establish the the double coset formula for the composite of a transfer followed by a restriction to a closed subgroup, see Theorem 4.10 below. We also discuss various examples and special cases in Example 4.11 through 4.15. We end the section with the a discussion of Mackey functors for finite groups, and prove that rationally and for finite groups, geometric fixed point homotopy groups can be obtained from equivariant homotopy groups by dividing out transfers from proper subgroups (Proposition 4.30).

For use in the proof of the double coset formula, and as a interesting result on its own, we calculate the geometric fixed points of the restriction of any transfer. In fact, when $K = e$ is the trivial subgroup of G, then $\Phi_0^e(X) = \pi_0^e(X)$, then map Φ^e is the identity and the following proposition reduces to a special case of the double coset formula. For the statement we use that the action of a closed subgroup K on a homogeneous space G/H by left translation is smooth, and hence the fixed points $(G/H)^K$ form a disjoint union of closed smooth submanifolds, possibly of varying dimensions. Proposition A.2.22 shows that in fact each component of $(G/H)^K$ is itself a homogeneous space, but that is not important for the following proposition.

We let K be a compact Lie group and B a closed smooth K-manifold. The Mostow-Palais embedding theorem [**115, 122**] provides a smooth K-equivariant embedding $i : B \longrightarrow V$, for some K-representation V. We can assume without loss of generality that V is a subrepresentation of the chosen complete K-universe $\mathcal{U}_K$. We use the inner product on V to the define the normal bundle ν of the embedding at $b \in B$ by

$$\nu_b \ = \ V - (di)(T_b B) \ ,$$

the orthogonal complement of the image of the tangent space T_bB in V. By multiplying with a suitably large scalar, if necessary, we can assume that the embedding is *wide* in the sense that the exponential map

$$D(\nu) \longrightarrow V \ , \quad (b,v) \longmapsto i(b) + v$$

is injective on the unit disc bundle of the normal bundle, and hence a closed G-equivariant embedding. The image of this map is a tubular neighborhood of radius 1 around $i(B)$, and it determines a K-equivariant Thom-Pontryagin collapse map

$$c_B \ : \ S^V \longrightarrow S^V \wedge B_+ \tag{4.1}$$

as follows: every point outside of the tubular neighborhood is sent to the basepoint, and a point $i(b)+v$, for $(b,v) \in D(\nu)$, is sent to

$$c_B(i(b)+v) \ = \ \left(\frac{v}{1-|v|}\right) \wedge b \ .$$

The homotopy class of the map c_B is then an element in the equivariant homotopy group $\pi_0^K(\Sigma_+^\infty B)$. The next result determines the image of the class $[c_B]$ under the geometric fixed point map.

In (3.15) we defined the map $\sigma^K : \pi_0(Y^K) \longrightarrow \pi_0^K(\Sigma_+^\infty Y)$ that produces equivariant stable homotopy classes from fixed point information of a K-space.

Proposition 4.2. *Let K be a compact Lie group.*

(i) *Let B be a closed smooth K-manifold and $i : B \longrightarrow V$ a wide smooth K-equivariant embedding into a K-representation. Then the relation*

$$\Phi^K[c_B] \ = \ \sum_{M \in \pi_0(B^K)} \chi(M) \cdot \Phi^K(\sigma^K[M])$$

holds in the group $\Phi_0^K(\Sigma_+^\infty B)$, where the sum runs over the connected components of the fixed point space B^K.

(ii) *Let G be a compact Lie group containing K and let H be another closed subgroup of G. Then*

$$\Phi^K \circ \mathrm{res}_K^G \circ \mathrm{tr}_H^G \ = \ \sum_{M \in \pi_0((G/H)^K)} \chi(M) \cdot \Phi^K \circ g_\star \circ \mathrm{res}_{K^g}^H \ ,$$

where the sum runs over the connected components of the fixed point space $(G/H)^K$ and $g \in G$ is such that $gH \in M$.

Proof. (i) Since the composite

$$\pi_0^e(\Sigma_+^\infty B^K) \xrightarrow{p_K^*} \pi_0^K(\Sigma_+^\infty B^K) \xrightarrow{\mathrm{incl}_*} \pi_0^K(\Sigma_+^\infty B) \xrightarrow{\Phi^K} \Phi_0^K(\Sigma_+^\infty B)$$

is an isomorphism and the source is free abelian on the path components of the space B^K, the group $\Phi_0^K(\Sigma_+^\infty B)$ is free abelian with basis given by the classes

$$\Phi^K(\sigma^K[M]) \ = \ \Phi^K(\mathrm{incl}_*(p_K^*(\sigma^e[M])))$$

for $M \in \pi_0(B^K)$. The class $\Phi^K[c_B]$ is represented by the non-equivariant map

$$S^{V^K} \xrightarrow{(c_B)^K} S^{V^K} \wedge B_+^K \ .$$

Since K acts smoothly on B, the fixed points B^K are a disjoint union of finitely many closed smooth submanifolds of varying dimensions (possibly non at all).

For each component M of B^K we let $p_M : B_+^K \longrightarrow M_+$ denote the projection, i.e., p_M is the identity on M and sends all other path components of B^K to the basepoint. Then the composite

$$S^{V^K} \xrightarrow{(c_B)^K} S^{V^K} \wedge B_+^K \xrightarrow{S^{V^K} \wedge p_M} S^{V^K} \wedge M_+$$

coincides with

$$S^{V^K} \xrightarrow{c_M} S^{V^K} \wedge M_+ \ ,$$

the collapse map (4.1) based on the non-equivariant wide smooth embedding $(i^K)|_M : M \longrightarrow V^K$. Since M is path connected, the group of based homotopy classes of maps $[S^{V^K}, S^{V^K} \wedge M_+]$ is isomorphic to $\mathbb{Z}$, by (3.16), and an element is determined by the degree of the composite with the projection $S^{V^K} \wedge M_+ \longrightarrow S^{V^K}$. It is a classical fact that the degree of the composite

$$S^{V^K} \xrightarrow{c_M} S^{V^K} \wedge M_+ \xrightarrow{\text{proj}} S^{V^K}$$

is the Euler characteristic of the manifold M, see for example [**11**, Thm. 2.4]. So the summand indexed by M precisely contributes the term $\chi(M) \cdot \Phi^K(\sigma^K[M])$, which proves the desired relation.

(ii) Both sides of the equation are natural transformations on the category of orthogonal G-spectra from the functor π_0^H to the functor Φ_0^K. Since the functor π_0^H is represented by the suspension spectrum of G/H (in the sense of Proposition 1.46), is suffices to check the relation for the orthogonal G-spectrum $\Sigma_+^\infty G/H$ and the tautological class e_H. Inspection of the definition in Construction 2.20 reveals that the transfer of the tautological class $\text{tr}_H^G(e_H)$ in $\pi_0^G(\Sigma_+^\infty G/H)$ is represented by the G-map

$$S^V \xrightarrow{c} G \ltimes_H S^W \xrightarrow{a} S^V \wedge G/H_+$$

where c is the collapse map based on any wide embedding of $i : G/H \longrightarrow V$ into a G-representation, and $a[g, w] = (gw, gH)$. So the class $\text{res}_K^G(\text{tr}_H^G(e_H))$ is represented by the underlying K-map of the above composite, which is precisely the map $c_{G/H}$ for the underlying K-manifold of G/H. Part (i) thus provides the relation

$$\Phi^K(\text{res}_K^G(\text{tr}_H^G(e_H))) \ = \ \Phi^K[c_{G/H}] \ = \sum_{M \in \pi_0((G/H)^K)} \chi(M) \cdot \Phi^K(\sigma^K[M])$$

where the sum runs over the connected components of $(G/H)^K$. On the other hand, if $g \in G$ is such that $gH \in M \subset (G/H)^K$, then $K^g \leq H$ and

$$\sigma^K[M] \ = \ g_\star(\sigma^{K^g}\langle eH \rangle) \ = \ g_\star(\text{res}_{K^g}^H(e_H)) \ ;$$

this proves the desired relation for the universal class e_H. □

Now we can proceed towards the double coset formula for a transfer followed by a restriction. The double coset formula was first proved by Feshbach for Borel cohomology theories [**51**, Thm. II.11] and later generalized to equivariant cohomology theories by Lewis and May [**97**, IV §6]. I am not aware of a written account of the double coset formula in the context of orthogonal G-spectra. Our method of proof for the double coset formula is different from the approach of Feshbach and Lewis-May: we verify the double coset formula after passage to geometric fixed points for all closed subgroups. The detection criterion of Theorem 3.18 (ii) then lets us deduce the double coset formula for the universal example, the tautological class e_H in $\pi_0^H(\Sigma_+^\infty G/H)$.

Before we can state the double coset formula we have to recall some additional concepts, such as orbit type submanifolds and the internal Euler characteristic.

Definition 4.3. Let L be a closed subgroup of a compact Lie group K and B a K-space. The *orbit type space* associated to the conjugacy class of L is the K-invariant subspace

$$B_{(L)} \ = \ \{x \in B \mid \text{the stabilizer group of } x \text{ is conjugate to } L\} \ .$$

The points in the space $B_{(L)}$ are said to have 'orbit type' the conjugacy class (L). We mostly care about orbit type subspaces for equivariant smooth manifolds. Two sources that collect some general information about of orbit type manifolds are Chapters IV and VI of Bredon's [**27**] and Section I.5 of tom Dieck's [**166**]. If B is a smooth K-manifold, then the subspace $B_{(L)}$ is a smooth submanifold of B, and locally closed in B, compare [**166**, I Prop. 5.12] or [**27**, VI Cor. 2.5]. Thus $B_{(L)}$ is called the *orbit type manifold*. If the smooth

K-manifold is compact, then it only has finitely many orbit types – this was first shown by Yang [**177**]; other references are [**27**, IV Prop. 1.2], [**123**, Thm. 1.7.25], or [**166**, I Thm. 5.11]. The orbit type manifolds need not be connected, but each $B_{(L)}$ has only finitely many path components. Also, $B_{(L)}$ need not be closed inside B; but if one orbit type manifold $B_{(L)}$ lies in the closure of another one $B_{(L')}$, then L' is subconjugate to L in K, compare [**27**, IV Thm. 3.3]. For every conjugacy class (L), the quotient map $B_{(L)} \longrightarrow K\backslash B_{(L)}$ is a locally trivial smooth fiber bundle with fiber K/L and structure group the normalizer $N_K L$, compare [**27**, VI Cor. 2.5]. Thus every path component of every orbit space $K\backslash B_{(L)}$ of an orbit type manifold is again a smooth manifold (of varying dimensions), in such a way that the quotient maps are smooth submersions. In particular, if the actions happens to have only one orbit type (i.e., all stabilizer groups are conjugate), then the quotient space $K\backslash B$ is a manifold and inherits a smooth structure from B.

Construction 4.4 (Internal Euler characteristic). We continue to consider a compact Lie group K acting smoothly on a closed smooth manifold B. We let L be a closed subgroup of K and $M \subset K\backslash B_{(L)}$ a connected component of the orbit type manifold for the conjugacy class (L). We let $\bar{M}$ denote the closure of M inside $K\backslash B$ and $\delta M = \bar{M} - M$ the complement of M inside its closure. Since $K\backslash B$ is compact, so is $\bar{M}$. Since $K\backslash B_{(L)}$ is locally closed in $K\backslash B$, the set M is open inside its closure $\bar{M}$. So the complement δM is closed in $\bar{M}$, hence compact. One should beware that while M is a topological manifold (without boundary, and typically not compact), $\bar{M}$ need not be a topological manifold with boundary.

By [**169**, Cor. 3.7 and 3.8] the orbit space $K\backslash B$ admits a triangulation (necessarily finite) such that the orbit type is constant on every open simplex. So if the orbit type component M intersects a particular open simplex, then that simplex is entirely contained in M, i.e., M is a union of open simplices in such a triangulation. This means that $\bar{M}$ is obtained from M by adding some simplices of smaller dimension. Hence M is a simplicial subcomplex of any such triangulation, and δM is a subcomplex of M. In particular, $\bar{M}$ and δM admit the structure of finite simplicial complexes of dimension less than or equal to the dimension of B. Thus the integral singular homology groups of $\bar{M}$ and δM are finitely generated in every degree, and they vanish above the dimension of B. So the Euler characteristics of are well-defined integers. We can thus define the *internal Euler characteristic* of M as

$$\chi^\sharp(M) \;=\; \chi(\bar{M}) \;-\; \chi(\delta M)\ .$$

The internal Euler characteristic $\chi^\sharp(M)$ also has an intrinsic interpretation that does not refer to the ambient space $K\backslash B$. Indeed since the integral homology groups of $\bar{M}$ and δM are finitely generated and vanish for almost all degrees, the same is true for the relative singular homology groups $H_*(\bar{M}, \delta M; \mathbb{Z})$, and the internal Euler characteristic satisfies

$$\chi^\sharp(M) \;=\; \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank}(H_n(\bar{M}, \delta M; \mathbb{Z})) \;=\; \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank}(H^n(\bar{M}, \delta M; \mathbb{Z}))\ .$$

The second equality uses that by the universal coefficient theorem, we can use *co*-homology instead of homology to calculate the Euler characteristic. Since δM is a simplicial subcomplex of $\bar{M}$, it is a neighborhood deformation retract inside $\bar{M}$. So the relative cohomology group $H^n(\bar{M}, \delta M; \mathbb{Z})$ is isomorphic to the reduced cohomology group of the quotient space $\bar{M}/\delta M$, by excision. This quotient space is homeomorphic to $\hat{M}$, the one-point compactification of $M = \bar{M} - \delta M$. Since M is a topological manifold, it is locally compact, and so the cohomology groups of $\hat{M}$ are isomorphic to the compactly supported cohomology groups of M [...ref...]. Altogether, this provides isomorphisms

$$H^n(\bar{M}, \delta M; \mathbb{Z}) \;\cong\; H^n_c(M; \mathbb{Z})\ .$$

So we conclude that the internal Euler characteristic $\chi^\sharp(M)$ can also be defined via the compactly supported Euler characteristic of M, i.e.,

$$\chi^\sharp(M) \;=\; \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank}(H^n_c(M; \mathbb{Z}))\ .$$

Now we let K and H be two closed subgroups of a compact Lie group G. Then the homogeneous space G/H is a smooth manifold and the K-action on G/H by left translations is smooth. The *double coset space* $K\backslash G/H$ is the quotient space of G/H by this K-action, i.e., the quotient of G by the left K- and right H-action by translation. In contrast to a homogeneous space G/H, the double coset space is in general not a smooth manifold. However, the orbit type decomposition expresses the double coset space as the union of certain subspaces that are manifolds of varying dimensions. Indeed, G/H decomposes into orbit type manifolds with respect to K:

$$K\backslash G/H \;=\; \bigcup_{(L)} K\backslash(G/H)_{(L)} \;,$$

where the set-theoretic union is indexed by conjugacy classes of closed subgroups of L; all except finitely many of the subspace $K\backslash(G/H)_{(L)}$ are empty. The double coset formula expresses the composite $\mathrm{res}^G_K \circ \mathrm{tr}^G_H$ as a sum of terms, indexed by all connected components M of orbit type orbit manifolds $K\backslash(G/H)_{(L)}$. The coefficient of the contribution of M is the internal Euler characteristic $\chi^\sharp(M)$.

The operation associated to M is the composite

$$\pi_0^H(X) \;\xrightarrow{\mathrm{res}^H_{K^g\cap H}}\; \pi_0^{K^g\cap H}(X) \;\xrightarrow{g_\star}\; \pi_0^{K\cap {}^gH}(X) \;\xrightarrow{\mathrm{tr}_{K\cap {}^gH}}\; \pi_0^K(X) \;,$$

where $g \in G$ is any element such that $KgH \in M$; since $K \cap {}^gH$ is the stabilizer of the coset gH, the group $K \cap {}^gH$ lies in the conjugacy class (L). The group $K \cap {}^gH$ may have infinite index in its normalizer in K, in which case $\mathrm{tr}^K_{K\cap {}^gH} = 0$, and then the corresponding summand in the double coset formula vanishes.

Proposition 4.5. *Let H and K be closed subgroups of a compact Lie group G and $g \in G$. Then the operation $\mathrm{tr}^K_{K\cap {}^gH} \circ g_\star \circ \mathrm{res}^H_{K^g\cap H}$ only depends on the path component of the double coset KgH in the space $K\backslash(G/H)_{(K\cap {}^gH)}$.*

Proof. We abbreviate $L = K \cap {}^gH$, the stabilizer group of the coset gH. We let $\bar g \in G$ be another group element such that the two double cosets KgH and $K\bar gH$ lie in the same path component of the space $K\backslash(G/H)_{(L)}$. In particular, the group $K \cap {}^{\bar g}H$ is then K-conjugate to L. We denote by

$$[G/H]_L \;=\; \{\gamma H \in G/H \mid K \cap {}^\gamma H \;=\; L\}$$

that space of those cosets whose K-stabilizer group is precisely L (and not just K-conjugate to it). The projection $[G/H]_L \longrightarrow K\backslash(G/H)_{(L)}$ is a principal W_KL-bundle, compare [**123**, Prop. 1.7.35]; so any path from KgH to $K\bar gH$ inside $K\backslash(G/H)_{(L)}$ admits a continuous lift $\omega : [0,1] \longrightarrow (G/H)_L$ with $\omega(0) = gH$. The space $(G/H)_L$ is subspace of $(G/H)^L$, so gH and $\omega(1)$ belong to the same path component of $(G/H)^L$. Proposition A.2.22 then provides an element $z \in (C_GL)^\circ$ in the identity component of the centralizer of L such that $\omega(1) = zgH$. Hence we also have

$$K\bar gH \;=\; K\omega(1) \;=\; KzgH \;,$$

so there are $(k,h) \in K \times H$ with $k\bar gh = zg$. Since $zgH = \omega(1)$ even belongs to $(G/H)_L$, we must have $K \cap {}^{zg}H = L = K \cap {}^gH$. Because the conjugation homomorphism $c_z : L \longrightarrow L$ is the identity and $l^X_z : X \longrightarrow X$ is L-equivariantly homotopic to the identity, the homomorphism

$$z_\star \;=\; (l^X_z)_* \circ c_z^* \;:\; \pi_0^L(X) \;\longrightarrow\; \pi_0^L(X)$$

is the identity. Thus

$$\begin{aligned}\mathrm{tr}^K_{K\cap {}^{\bar g}H} \circ \bar g_\star \circ \mathrm{res}^H_{K^{\bar g}\cap H} \;&=\; \mathrm{tr}^K_{K\cap {}^{k\bar gh}H} \circ k_\star \circ \bar g_\star \circ h_\star \circ \mathrm{res}^H_{K^{k\bar gh}\cap H}\\ &=\; \mathrm{tr}^K_{K\cap {}^{zg}H} \circ z_\star \circ g_\star \circ \mathrm{res}^H_{K^{zg}\cap H} \;=\; \mathrm{tr}^K_{K\cap {}^gH} \circ g_\star \circ \mathrm{res}^H_{K^g\cap H} \;.\end{aligned}$$

□

Our next result is a technical proposition that contains the essential input to the double coset formula from equivariant differential topology.

PROPOSITION 4.6. *Let K be a compact Lie group, B a closed smooth K-manifold and $L \leq J \leq K$ nested closed subgroups. Let N be a connected component of the fixed point space B^L, and M a connected component of the orbit space $K\backslash B_{(J)}$. Let $b \in B$ be any point with stabilizer group J and with $Kb \in M$. Let $W[M,N]$ be the preimage of $N \cap B_{(J)}$ under the map*

$$(K/J)^L \;\longrightarrow\; (B_{(J)})^L \;, \quad kJ \;\longmapsto\; kb \;.$$

Then

$$\chi^\sharp(N \cap B_{(J)}) \;=\; \sum_{M \in \pi_0(K\backslash B_{(J)})} \chi^\sharp(M) \cdot \chi(W[M,N]) \;.$$

PROOF. The projection

$$B_{(J)} \;\xrightarrow{p}\; K\backslash B_{(J)}$$

is a smooth fiber bundle with fiber K/J and structure group $N_K J$, compare [**123**, Thm. 1.7.35] or [**27**, VI Cor. 2.5]. Since L is contained in J the composite

$$(B_{(J)})^L \;\xrightarrow{\text{incl}}\; B_{(J)} \;\xrightarrow{p}\; K\backslash B_{(J)}$$

is surjective and the restriction of p to $(B_{(J)})^L$ is another smooth fiber bundle

$$(B_{(J)})^L \;\longrightarrow\; K\backslash B_{(J)} \;,$$

now with fiber $(K/J)^L$, and the map

$$\Psi_M \;:\; (K/J)^L \;\longrightarrow\; (B_{(J)})^L \;, \quad kJ \;\longmapsto\; kb$$

is the inclusion of the fiber over $Kb \in M$.

For every connected component M of the base $K\backslash B_{(J)}$ the inverse image $p^{-1}(M)$ is open and closed in $(B_{(J)})^L$. For every connected component N of B^L the intersection $N \cap B_{(J)}$ is open and closed in $(B_{(J)})^L$. So the subset $p^{-1}(M) \cap N$ is open and closed in $(B_{(J)})^L$. Moreover, as M varies, the subsets $p^{-1}(M) \cap N$ cover all of $N \cap B_{(J)}$. Moreover, the restriction of the projection to a map

$$p \;:\; p^{-1}(M) \cap N \;\longrightarrow\; M$$

is a smooth fiber bundle with connected base and with fiber $W[M,N]$, by definition of the latter. The internal Euler characteristic is multiplicative on smooth fiber bundles with closed fiber, so

$$\chi^\sharp(p^{-1}(M) \cap N) \;=\; \chi^\sharp(M) \cdot \chi(W[M,N]) \;.$$

The internal Euler characteristic is additive on disjoint unions, so

$$\chi^\sharp(N \cap B_{(J)}) \;=\; \sum_{M \in \pi_0(K\backslash B_{(J)})} \chi^\sharp(N \cap p^{-1}(M)) \;=\; \sum_{M \in \pi_0(K\backslash B_{(J)})} \chi^\sharp(M) \cdot \chi(W[M,N]) \;. \qquad \square$$

EXAMPLE 4.7. For later reference we point out a direct consequence of Proposition 4.6. We apply Proposition 4.6 for $L = e$, the trivial subgroup of K. We let J be any closed subgroup of K, and M a connected component of $K\backslash B_{(J)}$. The space K/J is the disjoint union of its subspaces $W[M,N]$, as N varies over the connected components of B. So summing the formula of Proposition 4.6 over all components of B yields

$$\chi^\sharp(B_{(J)}) \;=\; \sum_{M \in \pi_0(K\backslash B_{(J)})} \chi^\sharp(M) \cdot \chi(K/J) \;,$$

by additivity of Euler characteristics for disjoint unions. Because $\chi(B)$ is the sum of the internal Euler characteristics $\chi^\sharp(B_{(J)})$ over all conjugacy classes of closed subgroups of K, summing up over all conjugacy classes gives the formula

$$\chi(B) \;=\; \sum_{(J) \leq K} \; \sum_{M \in \pi_0(K\backslash B_{(J)})} \chi^\sharp(M) \cdot \chi(K/J) \;. \tag{4.8}$$

We let K be a compact Lie group and B a closed smooth K-manifold. The Mostow-Palais embedding theorem [**115, 122**] provides a wide smooth K-equivariant embedding $i : B \longrightarrow V$, for some K-representation V. The associated collapse map

$$c_B \ : \ S^V \ \longrightarrow \ S^V \wedge B_+$$

was defined (4.1). The homotopy class of the map c_B is then an element in the equivariant homotopy group $\pi_0^K(\Sigma_+^\infty B)$. Theorem 3.18 (i) exhibits a basis of the group $\pi_0^K(\Sigma_+^\infty B)$, and the next theorem expands the class of c_B in that basis. As we shall explain in the proof of Theorem 4.10 below, the double coset formula for $\mathrm{res}_K^G \circ \mathrm{tr}_H^G$ is essentially the special case $B = G/H$ of the following theorem. A compact smooth K-manifold only has finitely many orbit types, so the sum occurring in the following theorem is finite.

Theorem 4.9. *Let K be a compact Lie group and B a closed smooth K-manifold. Then the relation*

$$[c_B] \ = \ \sum_{(J)\leq K} \ \sum_{M\in\pi_0(K\backslash B_{(J)})} \chi^\sharp(M)\cdot \mathrm{tr}_J^K(\sigma^J\langle b_M\rangle)$$

holds in the group $\pi_0^K(\Sigma_+^\infty B)$. Here the sum runs over all connected components M of all orbit type orbit manifolds $K\backslash B_{(J)}$, and the element $b_M \in B$ that occurs is such that $Kb_M \in M$ and with stabilizer group J, and $\langle b_M\rangle$ is the path component of b_M in the space B^J.

Proof. Because the desired relation lies in the 0-th equivariant homotopy group of a suspension spectrum, Theorem 3.18 (ii) applies and shows that we only need to verify the formula after taking geometric fixed points to any closed subgroup L of K. We let J be another closed subgroup of K containing L. We calculate the contribution of the conjugacy class (J) to effect of Φ^L on the right hand side of the formula. For every connected component M of $K\backslash B_{(J)}$ we choose an element $b_M \in B$ with stabilizer group J and $Kb_M \in M$. Then we have

$$\begin{aligned}
&\sum_{M\in\pi_0(K\backslash B_{(J)})} \chi^\sharp(M)\cdot\Phi^L(\mathrm{res}_L^K(\mathrm{tr}_J^K(\sigma^J\langle b_M\rangle)))\\
&\qquad = \sum_{M\in\pi_0(K\backslash B_{(J)})} \ \sum_{W\in\pi_0((K/J)^L)} \chi^\sharp(M)\cdot\chi(W)\cdot\Phi^L(k_\star(\mathrm{res}_{L^k}^J(\sigma^J\langle b_M\rangle)))\\
&\qquad = \sum_{M\in\pi_0(K\backslash B_{(J)})} \ \sum_{W\in\pi_0((K/J)^L)} \chi^\sharp(M)\cdot\chi(W)\cdot\Phi^L(\sigma^L(k_\star(\mathrm{res}_{L^k}^J\langle b_M\rangle)))\\
&\qquad = \sum_{M\in\pi_0(K\backslash B_{(J)})} \ \sum_{N\in\pi_0(B^L)} \chi^\sharp(M)\cdot\chi(W[M,N])\cdot\Phi^L(\sigma^L[N])\\
&\qquad = \sum_{N\in\pi_0(B^L)} \chi^\sharp(N\cap B_{(J)})\cdot\Phi^L(\sigma^L[N]) \ .
\end{aligned}$$

The first equation is Proposition 4.2 (ii), applied to the closed subgroups L and J of K; the element $k \in K$ is chosen so that $kJ \in W \subset (K/J)^L$; in particular this forces the relation $L \leq {}^kJ \leq {}^k\mathrm{stab}(b_M)$, and thus $kb_M \in B^L$. The third equation uses that for every M,

$$(K/J)^L \ = \ \coprod_{N\in\pi_0(B^L)} W[M,N] \ ,$$

that the Euler characteristic is additive on disjoint unions, and that for all $kJ \in W[M,N]$ the element kb_M lies in the path component N of B^L, by the very definition of $W[M,N]$. The fourth equation is Proposition 4.6.

Now we sum up the contributions from the different conjugacy classes of subgroups of K. The smooth K-manifold B is stratified by the relatively closed smooth submanifolds $B_{(J)}$, indexed over the poset of conjugacy classes of subgroup of K, where only finitely many orbit types occur. Intersecting the strata with

a connected component N of the fixed point submanifold B^L gives a stratification of N by relatively closed submanifolds, still indexed over the conjugacy classes of subgroup of K. The internal Euler characteristic is additive for such stratifications [...ref...], so we obtain the relation

$$\chi(N) \;=\; \sum_{(J)\leq K} \chi^\sharp(N\cap B_{(J)})\ .$$

An application of Proposition 4.2 (i) gives

$$\begin{aligned}
\Phi^L(\mathrm{res}^K_L[c_B]) \;&=\; \sum_{N\in\pi_0(B^L)} \chi(N)\cdot\Phi^L(\sigma^L[N])\\
&=\; \sum_{(J)\leq K}\ \sum_{N\in\pi_0(B^L)} \chi^\sharp(N\cap B_{(J)})\cdot\Phi^L(\sigma^L[N])\\
&=\; \sum_{(J)\leq K}\ \sum_{M\in\pi_0(K\backslash B_{(J)})} \chi^\sharp(M)\cdot\Phi^L(\mathrm{res}^K_L(\mathrm{tr}^K_J(\sigma^J\langle b_M\rangle)))\\
&=\; \Phi^L\left(\mathrm{res}^K_L\left(\sum_{(J)\leq K}\ \sum_{M\in\pi_0(K\backslash B_{(J)})} \chi^\sharp(M)\cdot\mathrm{tr}^K_J(\sigma^J\langle b_M\rangle)\right)\right)\ .
\end{aligned}$$

This proves the desired formula after composition with the geometric fixed point map to any closed subgroup of K. □

Now we have all ingredients for the statement and proof of the double coset formula in place.

THEOREM 4.10 (Double coset formula). *Let H and K be closed subgroups of a compact Lie group G. Then for every orthogonal G-spectrum X the relation*

$$\mathrm{res}^G_K\circ\mathrm{tr}^G_H \;=\; \sum_M \chi^\sharp(M)\cdot\mathrm{tr}^K_{K\cap{}^gH}\circ g_\star\circ\mathrm{res}^H_{K^g\cap H}$$

holds as homomorphisms $\pi_0^H(X)\longrightarrow\pi_0^K(X)$. Here the sum runs over all connected components M of all orbit type orbit manifolds $K\backslash(G/H)_{(L)}$, and the element $g\in G$ that occurs is such that $KgH\in M$.

PROOF. Both sides of the double coset formula are natural transformations on the category of orthogonal G-spectra from the functor π_0^H to the functor π_0^K. Since the functor π_0^H is represented by the suspension spectrum of G/H (in the sense of Proposition 1.46), is suffices to check the relation for the orthogonal G-spectrum Σ^∞_+G/H and the tautological class e_H.

Inspection of the definition in Construction 2.20 reveals that the transfer of the tautological class $\mathrm{tr}^G_H(e_H)$ in $\pi_0^G(\Sigma^\infty_+G/H)$ is represented by the G-map

$$S^V\ \xrightarrow{c}\ G\ltimes_H S^W\ \xrightarrow{a}\ S^V\wedge G/H_+$$

where c is the collapse map based on any wide embedding of $i:G/H\longrightarrow V$ into a G-representation, and $a[g,w]=(gw,gH)$. So the class $\mathrm{res}^G_K(\mathrm{tr}^G_H(e_H))$ is represented by the underlying K-map of the above composite, which is precisely the map $c_{G/H}$ for the underlying K-manifold of G/H.

Theorem 4.9 thus yields the formula

$$\mathrm{res}^G_K(\mathrm{tr}^G_H(e_H)) \;=\; [c_{G/H}] \;=\; \sum_{(J)\leq K}\ \sum_{M\in\pi_0(K\backslash(G/H)_{(J)})} \chi^\sharp(M)\cdot\mathrm{tr}^K_J(\sigma^J\langle g_MH\rangle)$$

in the group $\pi_0^K(\Sigma^\infty_+G/H)$. Here $g_M\in G$ is such that $Kg_MH\in M$ and $K\cap{}^{g_M}H=J$. On the other hand,

$$\sigma^J\langle g_MH\rangle \;=\; g_\star(\sigma^{J^{g_M}}\langle eH\rangle) \;=\; g_\star(\mathrm{res}^H_{J^{g_M}}(\sigma^H\langle eH\rangle)) \;=\; g_\star(\mathrm{res}^H_{K^{g_M}\cap H}(e_H))\ ,$$

so this proves the double coset formula for the universal class e_H. □

EXAMPLE 4.11. A special case of the double coset formula is when $K = e$, i.e., when we restrict a transfer all the way to the trivial subgroup of G. In this case there is orbit type component manifolds are the path components of the coset space G/H and the sum in the double coset formula is indexed over the path components of G/H. Since all path components of G/H are homeomorphic, they have the same Euler characteristic $\chi(G/H)/|\pi_0(G/H)|$, so the double coset formula specializes to

$$\mathrm{res}_e^G \circ \mathrm{tr}_H^G \;=\; \sum_{M\in\pi_0(G/H)} \chi(M)\cdot g_\star \circ \mathrm{res}_e^H \;=\; \chi(G/H)/|\pi_0(G/H)|\cdot \sum_{[gH]\in\pi_0(G/H)} g_\star \circ \mathrm{res}_e^H \;.$$

In the special case where the conjugation action of $\pi_0(G)$ on $\pi_0^e(X)$ happens to be trivial (for example for all global stable homotopy types), all the maps $g_\star$ are the identity and the formula simplifies to

$$\mathrm{res}_e^G \circ \mathrm{tr}_H^G \;=\; \chi(G/H)\cdot \mathrm{res}_e^H \;.$$

EXAMPLE 4.12 (Double coset formula for finite index transfers). If H has finite index in G, then the double coset formula simplifies. In this situation the homogeneous space G/H, and hence also the double coset space $K\backslash G/H$, are discrete, all orbit type manifold components are points, and all internal Euler characteristics that occur in the double coset formula are 1. For any other closed subgroup K of G the intersection $K\cap {}^gH$ also has finite index in K, so only finite index transfers show up in the double coset formula, which specializes to the relation

$$\mathrm{res}_K^G \circ \mathrm{tr}_H^G \;=\; \sum_{[g]\in K\backslash G/H} \mathrm{tr}_{K\cap {}^gH}^K \circ g_\star \circ \mathrm{res}_{K^g\cap H}^H \;.$$

EXAMPLE 4.13. We calculate the double coset formula for the maximal torus

$$H \;=\; \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \;:\; \lambda,\mu\in U(1) \right\}$$

of $G = U(2)$. We take $K = N_{U(2)}H$, the normalizer of the maximal torus. Then K is isomorphic to $\Sigma_2 \wr H$, the semidirect product with Σ_2 permuting the two diagonal entries of matrices in H; the group K generated by H and the involution $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; this is Example VI.2 in [**51**]. We calculate the double coset space $K\backslash U(2)/H$ by identifying the space $H\backslash U(2)/H$ and the residual action of the symmetric group Σ_2 on this space. A homeomorphism from $H\backslash U(2)/H$ to the unit interval $[0,1]$ is induced by

$$h \;:\; U(2) \;\longrightarrow\; [0,1]\;, \quad A \;\longmapsto\; |a_{11}|^2\;,$$

the square of the length of the upper left entry a_{11} of A. Lengths are non-negative and every column of a unitary matrix is a unit vector, so the map really lands in the interval $[0,1]$. The number $h(A)$ only depends on the double coset of the matrix A, so the map h factors over the double coset space $H\backslash U(2)/H$. For every $x\in[0,1]$ the vector $(\sqrt{x},\sqrt{1-x})\in\mathbb{C}^2$ has length 1, so it can be complemented to an orthonormal basis, and so it occurs as the first column of a unitary matrix; the map h is thus surjective. On the other hand, if $h(A) = h(B)$ for two unitary matrices A and B, then left multiplication by an element in H makes the first row of B equal to the first row of A. Right multiplication by an element of $1\times U(1)$ then makes the matrices equal. So A and B represent the same element in the double coset space. The induced map

$$\bar{h} \;:\; H\backslash U(2)/H \;\longrightarrow\; [0,1]$$

is thus bijective, hence a homeomorphism. The action of Σ_2 on the orbit space $H\backslash U(2)/H$ permutes the two rows of a matrix; under the homeomorphism $\bar{h}$, this action thus corresponds to the involution of $[0,1]$ sending x to $1-x$. Altogether this specifies a homeomorphism from the double coset space to the interval $[0,1/2]$ that sends a double coset $K\cdot AH$ to the minimum of $|a_{11}|^2$ and $|a_{21}|^2$. The inverse homeomorphism is

$$g \;:\; [0,1/2] \;\cong\; K\backslash U(2)/H\;, \quad g(t) \;=\; K\cdot \begin{pmatrix} \sqrt{1-t} & \sqrt{t} \\ -\sqrt{t} & \sqrt{1-t} \end{pmatrix}\cdot H\;.$$

The orbit type decomposition is as

$$\{0\} \cup (0,1/2) \cup \{1/2\} \ .$$

So as representatives of the orbit types we can choose

$$g(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad g(1/5) = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \qquad \text{and} \qquad g(1/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \ .$$

For the intersections (which are representatives of the conjugacy classes of those subgroups of K with non-empty orbit type manifolds) we get

$$\begin{aligned} K \cap {}^{g(0)}H &= H \\ K \cap {}^{g(1/5)}H &= \Delta \\ K \cap {}^{g(1/2)}H &= \Sigma_2 \times \Delta \ . \end{aligned}$$

Here $\Delta = \left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \ : \ \lambda \in U(1) \right\}$ is the diagonal copy of $U(1)$. The group Δ is normal in K, so it has infinite index in its normalizer, and the corresponding transfer map does not contribute to the double coset formula. The group $\Sigma_2 \times \Delta$ is its own normalizer in K. The internal Euler characteristics of a point is 1, so the double coset formula has two non-trivial summands, and it specializes to

$$\text{res}_K^{U(2)} \circ \text{tr}_H^{U(2)} \ = \ \text{tr}_H^K \ + \ \text{tr}_{\Sigma_2 \times \Delta}^K \circ \gamma_\star \circ \text{res}_{(\Sigma_2 \times \Delta)^\gamma}^H \ : \ \pi_0^H(X) \ \longrightarrow \ \pi_0^K(X) \ , \tag{4.14}$$

where $\gamma = g(1/2) = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$.

Example 4.15. As another example, and for later use in the 'explicit Brauer induction' (see Remark V.4.12 below), we work out the double coset formula for the unitary group $G = U(n)$ with respect to the subgroups

$$H \ = \ U(1) \times U(n-1) \qquad \text{and} \qquad K \ = \ U(k) \times U(n-k) \ ,$$

where $1 \le k \le n-1$. We simplify the notation by writing

$$U(a,b) \ = \ U(a) \times U(b)$$

for the subgroup of those elements of $U(a+b)$ that leave the subspaces $\mathbb{C}^a \oplus 0$ and $0 \oplus \mathbb{C}^b$ invariant. We also use the analogous notation for more than two factors.

Again the relevant double coset space 'is' an interval. Indeed, the continuous map

$$U(n) \ \longrightarrow \ [0,1] \ , \quad A \ \longmapsto \ |a_{11}|^2 + \cdots + |a_{k1}|^2 \ ,$$

the partial length of the entries in the first column of A, is invariant under right multiplication by elements of $U(1,n-1)$, and under left multiplication by elements of $U(k,n-k)$. Every column of a unitary matrix is a unit vector, so the map really lands in the interval $[0,1]$. The map factors over a continuous surjective map

$$h \ : \ U(k,n-k))\backslash U(n)/U(1,n-1) \ \longrightarrow \ [0,1] \ .$$

On the other hand, if $h[A] = h[B]$ for two unitary matrices A and B, then left multiplication by an element in $U(k,n-k)$ makes the first columns of A and B equal. Right multiplication by an element of $1 \times U(n-1)$ then makes the matrices equal. So A and B represent the same element in the double coset space. This shows that the map h is bijective, hence a homeomorphism. The inverse homeomorphism is

$$[0,1] \ \cong \ U(k,n-k)\backslash U(n)/U(1,n-1) \ , \quad t \ \mapsto \ U(k,n-k) \cdot g(t) \cdot U(1,n-1)$$

with

$$g(t) \;=\; \begin{pmatrix} \sqrt{t} & 0 & \cdots & 0 & \sqrt{1-t} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\sqrt{1-t} & 0 & \cdots & 0 & \sqrt{t} \end{pmatrix} .$$

The orbit type decomposition is as

$$\{0\} \cup (0,1) \cup \{1\} \ .$$

As representatives of the orbit types we can choose

$$g(0) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix} \ , \quad g(1/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and the identity $g(1)$. For the intersections (which are representatives of the conjugacy classes of those subgroups of $U(k,n-k)$ with non-empty orbit type manifolds) we get

$$\begin{aligned} U(k,n-k) \cap \ {}^{g(0)}U(1,n-1) \ &= \ U(k,n-k-1,1) \\ U(k,n-k) \cap {}^{g(1/2)}U(1,n-1) \ &= \ \Delta \\ U(k,n-k) \cap \ {}^{g(1)}U(1,n-1) \ &= \ U(1,k-1,n-k) \ . \end{aligned}$$

Here Δ is the subgroup of $U(n)$ consisting of matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

for $(\lambda, A, B) \in U(1) \times U(k-1) \times U(n-k-1)$. The subgroup Δ is normalized by

$$U(1,\, k-1,\, n-k-1,\, 1) \ ,$$

so it has infinite index in its normalizer, and the corresponding transfer map does not contribute to the double coset formula. The double coset formula thus has two non-trivial summands, and comes out as

$$\begin{aligned} \mathrm{res}^{U(n)}_{U(k,n-k)} \circ \mathrm{tr}^{U(n)}_{U(1,n-1)} \ &= \\ \mathrm{tr}^{U(k,n-k)}_{U(1,k-1,n-k)} \circ \mathrm{res}^{U(1,n-1)}_{U(1,k-1,n-k)} \ &+ \ \mathrm{tr}^{U(k,n-k)}_{U(k,n-k-1,1)} \circ g(0)_\star \circ \mathrm{res}^{U(1,n-1)}_{U(1,k,n-k-1)} \ . \end{aligned} \tag{4.16}$$

For easier reference we record that the equivariant homotopy groups of an orthogonal G-spectrum form a Mackey functor. We first recall one of several equivalent algebraic definitions of this concept, where we restrict to the case of finite groups.

Definition 4.17. Let G be a finite group. A *G-Mackey functor* consists of the following data:

- an abelian group $M(H)$ for every subgroup H of G,
- conjugation homomorphisms $g_\star : M(H) \longrightarrow M({}^gH)$ for all $H \le G$ and $g \in G$,
- restriction homomorphisms $\mathrm{res}^H_K : M(H) \longrightarrow M(K)$ for all $K \le H \le G$,
- transfer homomorphisms $\mathrm{tr}^H_K : M(K) \longrightarrow M(H)$ for all $K \le H \le G$.

This data has to satisfy the following conditions:

(i) (Unit conditions)
$$\mathrm{res}^H_H \;=\; \mathrm{tr}^H_H \;=\; \mathrm{Id}_{M(H)}$$
for all subgroups H, and $h_\star = \mathrm{Id}_{M(H)}$ for all $h \in H$.

(ii) (Transitivity conditions)
$$\mathrm{res}^K_L \circ \mathrm{res}^H_K \;=\; \mathrm{res}^H_L \qquad \text{and} \qquad \mathrm{tr}^H_K \circ \mathrm{tr}^K_L \;=\; \mathrm{tr}^H_L$$
for all $L \le K \le H \le G$.

(iii) (Interaction conditions)
$$g_\star \circ g'_\star \;=\; (gg')_\star\ , \qquad \mathrm{tr}^{{}^gH}_{{}^gK} \circ g_\star \;=\; g_\star \circ \mathrm{tr}^H_K \qquad \text{and} \qquad \mathrm{res}^{{}^gH}_{{}^gK} \circ g_\star \;=\; g_\star \circ \mathrm{res}^H_K$$
for all $g, g' \in G$ and $K \le H \le G$,

(iv) (Double coset formula) for every pair of subgroups K, L of H the relation
$$\mathrm{res}^H_L \circ \mathrm{tr}^H_K \;=\; \sum_{[h]\in L\backslash H/K} \mathrm{tr}^L_{L\cap {}^hK} \circ h_\star \circ \mathrm{res}^K_{L^h\cap K} \tag{4.18}$$
holds as maps $M(K) \longrightarrow M(L)$; here $[h]$ runs over a set of representatives for the L-K-double cosets.

To my knowledge, the concept of a Mackey functor goes back to Dress [**43**] and Green [**62**]. Green in fact defined what is nowadays called a *Green functor* (which he called *G-functor* in [**62**, Def. 1.3], which amounts to G-Mackey functor whose values underlie commutative rings, where restriction and conjugation maps are ring homomorphisms, and where the transfer maps satisfy Frobenius reciprocity. Definition 4.17 is the 'down to earth' definition of Mackey functor, and the one that is most useful for concrete calculations. There are two alternative (and equivalent) definitions that are often used:

- as a pair (M_*, M^*) of additive functors on the category of finite G-sets, where M_* is covariant and M^* is contravariant; the two functors must agree on objects and for every every pullback diagram of finite G-sets
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$
the relation
$$M_*(f) \circ M^*(g) \;=\; M^*(h) \circ M_*(k) \;:\; M(C) \;\longrightarrow\; M(B)$$
holds. This is (a special case of) the definition introduced by Dress in [**43**, §6]
- as an additive functor on the category of spans of finite G-sets; the equivalence with Dress' definition is due to Lindner [**99**, Thm. 4].

Our main source of examples of G-Mackey functors comes from orthogonal G-spectra: as we verified in Sections 1, Section 2 and Theorem 4.10, that the restriction, conjugation and transfer maps make the homotopy groups $\pi^H_k(X)$ for varying subgroups H into a G-Mackey functor.

Remark 4.19 (Natural homotopy operations for orthogonal G-spectra). We consider two closed subgroups H and K of a compact Lie group G. We can now describe all natural operations, on the category of orthogonal G-spectra, from the functor π^H_0 to the functor π^K_0. We let L be a closed subgroup of K, and we let $g \in G$ be such that $gH \in (G/H)^L$ (i.e., such that $L^g \le H$). This data gives rise to a natural homomorphism from π^K_0 to π^H_0 as the composite
$$\pi^H_0(X) \;\xrightarrow{\mathrm{res}^H_{L^g}}\; \pi^{L^g}_0(X) \;\xrightarrow{g_\star}\; \pi^L_0(X) \;\xrightarrow{\mathrm{tr}^K_L}\; \pi^K_0(X)\ . \tag{4.20}$$
This composite in fact only depends on the path component of $(W_KL)gH$ in the space $(W_KL)\backslash(G/H)^L$. We claim that the group of natural transformations $\mathrm{Nat}^{G\mathcal{S}p}(\pi^H_0, \pi^K_0)$ is free abelian with basis given by the

operations (4.20), as the L runs over a set of representatives of the conjugacy classes of closed subgroups of K with finite Weyl group, and g runs over a set of representatives of the connected components of the space $(W_K L)\backslash(G/H)^L$. We only sketch the argument, and leave the details to the interested reader: the representability property of Proposition 1.46 reduces the calculation of the natural operations to the calculation of the group $\pi_0^K(\Sigma_+^\infty G/H)$, and our claim comes down to checking that the operations (4.20) correspond to the basis of $\pi_0^K(\Sigma_+^\infty G/H)$, specified in Theorem 3.18 (i).

When the ambient group G is finite, this calculation implies that the natural algebraic structure on the collection of equivariant homotopy groups $\pi_0^H(X)$, for $H \leq G$ and X an orthogonal G-spectrum is precisely that of a G-Mackey functor, i.e., we have not overlooked any natural operations. Even more is true: the vanishing of all equivariant homotopy groups in negative respectively positive dimensions defines a t-structure given on the stable homotopy category of orthogonal G-spectra, and the heart of this t-structure is equivalence to the abelian category of G-Mackey functors. Readers who wish to prove this for themselves can find inspiration in Section IV.4, and specifically Corollary IV.4.11, where we prove the analogous global version.

Proposition 3.13 above shows that the geometric fixed point map (3.3) factors over the quotient of $\pi_0^G(X)$ by the subgroup generated by proper transfers. This should serve as motivation for the following algebraic interlude about Mackey functors for finite groups, where we study the process of 'dividing out transfers' systematically.

Construction 4.21. We let G be a finite group and F a G-Mackey functor. For a subgroup H of G we let $t_H F$ be the subgroup of $F(H)$ generated by transfers from proper subgroup of H, and we define

$$\tau_H F \;=\; F(H)/t_H F\ .$$

For $g \in G$, the conjugation isomorphism $g_\star : F(H) \longrightarrow F({}^gH)$ descends to a homomorphism

$$g_\star \;:\; \tau_H F \;\longrightarrow\; \tau_{{}^gH} F\ . \tag{4.22}$$

Moreover, the transitivity relation $g_\star \circ g'_\star = (gg')_\star$ still holds. In particular, the action of the Weyl group $W_G H$ on $F(H)$ descends to a $W_G H$-action on the quotient group $\tau_H F$.

Now we recall how a G-Mackey functor F can rationally be recovered from the $W_G H$-modules $\tau_H F$ for all subgroups H of G. We let

$$\bar\psi_G^F \;:\; F(G) \;\longrightarrow\; \prod_{H\leq G} \tau_H F$$

be the homomorphism whose H-component is the composite

$$F(G) \;\xrightarrow{\mathrm{res}_H^G}\; F(H) \;\xrightarrow{\mathrm{proj}}\; \tau_H F\ ;$$

here the product is indexed over all subgroups H of G. The group G acts on the product via the maps (4.22), permuting the factors within conjugacy classes. Since inner automorphisms induce the identity and $g_\star \circ \mathrm{res}_H^G = g_\star$, so the map $\bar\psi_G^F$ actually lands in the subgroup of G-invariant tuples. We let

$$\psi_G^F \;:\; F(G) \;\longrightarrow\; \Big(\textstyle\prod_{H\leq G} \tau_H F\Big)^G \tag{4.23}$$

denote the same map as $\bar\psi_G^F$, but now with image the subgroups of G-invariants. While the above description of the target of ψ_G^F is the most natural one, it is somewhat redundant. Indeed, for every $H \leq G$, the image of the restriction map $\mathrm{res}_H^G : F(G) \longrightarrow F(H)$ lands in the subgroup $F(H)^{W_G H}$ of invariants under the Weyl group $W_G H$. Moreover, if we choose representatives of the conjugacy classes of subgroups, then projection from the full product (over all subgroups of G) to the product indexed by the representatives restricts to an isomorphism

$$\Big(\textstyle\prod_{H\leq G} \tau_H F\Big)^G \;\xrightarrow{\cong}\; \prod_{(H)} (\tau_H F)^{W_G H}\ .$$

For explicit calculations, the second description of the target of ψ_G^F is often more convenient.

For a finite group G we set

$$d_G \;=\; \prod\nolimits_{(H)} |W_G H| \;,$$

the product, over all conjugacy classes of subgroups of G, of the orders of the respective Weyl groups. Since the order of $W_G H$ divides the order of G, inverting $|G|$ also inverts the number d_G. The following result is well known, and I do not know who deserves credit for it; the earliest reference that I am aware of is [**159**, Cor. 4.4].

PROPOSITION 4.24. *For every finite group G and every G-Mackey functor F, the kernel and cokernel of morphism ψ_G^F are annihilated by the number d_G. In particular, the morphism ψ_G^F becomes an isomorphism after inverting the order of G.*

PROOF. We reproduce the proof given in [**159**]. We choose representatives for the conjugacy classes of subgroups and number them

$$e = H_1 \;, H_2, \dots, H_n = G$$

in such a way that

- for all $i < j$ and all $g \in G$ the group $H_j \cap {}^g H_i$ is a proper subgroup of H_j, and
- every proper subgroup of H_i is conjugate to one of the groups occurring before H_i.

For example, ordering the groups by non-decreasing order will to the job.

We set

$$K_j \;=\; \ker(\psi_G^F) \cap \sum_{i=1}^{j} \mathrm{tr}_{H_i}^G(F(H_i)) \;\subseteq\; F(G) \;;$$

this defines a nested sequence

$$0 \;=\; K_0 \;\subseteq\; K_1 \;\subseteq\; K_2 \;\subseteq\; \dots\; K_{n-1} \;\subseteq\; K_n \;=\; \ker(\psi_G^F) \;.$$

We show that

$$|W_G H_j| \cdot K_j \;\subseteq\; K_{j-1} \tag{4.25}$$

for all $j = 1, \dots, n$. Altogether this means that

$$d_G \cdot K_n \;=\; \prod\nolimits_{i=1}^{n} |W_G H_i| \cdot K_n \;=\; 0 \;.$$

In the course of proving (4.25) we call an element of $F(H)$ *degenerate* if it maps to zero in $\tau_H F$, i.e., if it is a sum of transfers from proper subgroups of H. So the kernel of ψ_G^F is precisely the subgroup of those elements of $F(G)$ that restrict to degenerate elements on all subgroups.

We write any given element $x \in K_j$ as

$$x \;=\; \mathrm{tr}_{H_j}^G(y) \;+\; \bar{x}$$

for suitable $y \in F(H_j)$ and with $\bar{x}$ a sum of transfers from the groups $H_1, \dots, H_{j-1}$. For $i = 1, \dots, j-1$ the double coset formula for $\mathrm{res}_{H_j}^G \circ \mathrm{tr}_{H_i}^G$ let us write $\mathrm{res}_{H_j}^G(\bar{x})$ as a sum of transfers from proper subgroups of H_j; this uses the hypothesis on the enumeration of the subgroups. So the class $\mathrm{res}_{H_j}^G(\mathrm{tr}_{H_i}^G(\bar{x}))$ is degenerate. Since x is in the kernel of ψ_G^F, the class $\mathrm{res}_{H_j}^G(x)$ is degenerate as well. So the class

$$\mathrm{res}_{H_j}^G(\mathrm{tr}_{H_j}^G(y)) \;=\; \mathrm{res}_{H_j}^G(x) \;-\mathrm{res}_{H_j}^G(\bar{x})$$

is degenerate. The double coset formula expresses $\mathrm{res}_{H_j}^G(\mathrm{tr}_{H_j}^G(y))$ as the sum of the $W_G H_j$-conjugates of y, plus transfers from proper subgroups of H_j. So the element

$$\sum_{gH_j \in N_G H_j} g_\star(y_j) \;\in\; F(H_j)$$

is degenerate and hence the element

$$|W_G H_j| \cdot \mathrm{tr}_{H_j}^G(y_j) \;=\; \mathrm{tr}_{H_j}^G\left(\sum g_\star(y)\right) \;\in\; F(G)$$

is a sum of transfers of proper subgroups of H_j. Every proper subgroup of H_j is conjugate to one of the group $H_1, \dots, H_{j-1}$, so

$$|W_G H_j| \cdot x \;=\; |W_G H_j| \cdot \mathrm{tr}_{H_j}^G(y_j) \;+\; |W_G H_j| \cdot \bar{x}$$

is both in the kernel of ψ_G^F and a sum of transfer from the groups $H_1, \dots, H_{j-1}$. This proves the claim that $|W_G H_j| \cdot x$ belongs to K_{j-1}. Altogether this finishes the proof that the kernel of ψ_G^F is annihilated by the number d_G.

Now we show that the cokernel of ψ_G^F is annihilated by d_G. We let I_j denote the subgroup of $\prod_{i=1}^n (\tau_{H_i} F)^{W_G H_i}$ consisting of those tuples

$$x \;=\; (x_i)_{1 \le i \le n}$$

such that $x_{j+1} = x_{j+2} = \dots = x_n = 0$. This defines a nested sequence

$$0 \;=\; I_0 \;\subseteq\; I_1 \;\subseteq\; I_2 \;\subseteq\; \dots\; I_{n-1} \;\subseteq\; I_n \;=\; \prod_{i=1,\dots,n} F(H_i)^{W_G H_i} \;.$$

We show that

$$|W_G H_j| \cdot I_j \;\subseteq\; \mathrm{Im}(\psi_G^F) \;+\; I_{j-1} \tag{4.26}$$

for all $j = 1, \dots, n$. Altogether this means that

$$d_G \cdot I_n \;\subseteq\; \mathrm{Im}(\psi_G^F) \;,$$

i.e., the cokernel of ψ_G^F is annihilated by d_G.

To prove (4.26) we consider a tuple $x = (x_n)$ in I_j and choose a representative $y \in F(H_j)$ for the 'last' non-zero component, i.e., such that y maps to x_j in $\tau_{H_j} F$. Since x_j is invariant under the action of the Weyl group $W_G H_j$, the element y is at least $W_G H_j$-invariant modulo transfers from proper subgroups of H_j. The double coset formula for $\mathrm{res}_{H_j}^G \circ \mathrm{tr}_{H_j}^G$ thus gives that

$$\mathrm{res}_{H_j}^G(\mathrm{tr}_{H_j}^G(y)) \;\equiv\; \sum_{gH_j \in W_G H_j} g_\star(y) \;\equiv\; |W_G H_j| \cdot y \;,$$

with both congruences modulo transfers from proper subgroups of H_j. So the composite

$$F(G) \;\xrightarrow{\mathrm{res}_{H_j}^G}\; F(H_j) \;\xrightarrow{\mathrm{proj}}\; \tau_{H_j} F$$

takes $\mathrm{tr}_{H_j}^G(y)$ to $|W_G H_j| \cdot x_j$. For $i > j$ the double coset formula for $\mathrm{res}_{H_i}^G \circ \mathrm{tr}_{H_j}^G$ shows that $\mathrm{res}_{H_i}^G(\mathrm{tr}_{H_j}^G(y))$ is a sum of transfers from proper subgroups of H_i. So the element $\mathrm{res}_{H_i}^G(\mathrm{tr}_{H_j}^G(y))$ maps to 0 in $\tau_{H_i} F$ for all $i = j+1, \dots, n$. In other words, the tuple $\psi_G^F(y)$ belongs to I_j. Since $\psi_G^F(y)$ and $|W_G H_j| \cdot x$ both belong to I_j and agree at the component of H_j; we can thus conclude that

$$\psi_G^F(y) - |W_G H_j| \cdot x \;\in\; I_{j-1} \;.$$

This proves (4.26) and finishes the proof that the cokernel of ψ_G^F is annihilated by the number d_G. □

The previous Proposition 4.24 shows that for every G-Mackey functor F the group $\mathbb{Z}[1/|G|] \otimes (\tau_G F)$ is a direct summand of the group $\mathbb{Z}[1/|G|] \otimes F(G)$, and the splitting is natural for morphisms of G-Mackey functors. So the functor

$$G\text{-Mackey} \;\longrightarrow\; \mathcal{A}b \;, \quad F \longmapsto\; \mathbb{Z}[1/|G|] \otimes (\tau_G F)$$

is a natural direct summand of an exact functor. So we can conclude:

COROLLARY 4.27. *Let G be a finite group. The functor that assigns to a G-Mackey functor F the abelian group $\mathbb{Z}[1/|G|] \otimes (\tau_G F)$ is exact.*

REMARK 4.28. With just a little bit more algebraic work, Proposition 4.24 can be refined into the following statement: Let G be a finite groups and R any ring in which the order of G is invertible. Then the functor that assigns to a G-Mackey functor F in R-modules the collection of $R[W_G H]$-modules $\tau_H F$ is an equivalence of categories

$$G\text{-Mackey}_R \;\cong\; \prod_{(H)} R[W_G H]\text{-mod}$$

to the product, over conjugacy classes of subgroups, of the categories of $R[W_G H]$-modules. This applies in particular to $R = \mathbb{Q}$ and shows that rational G-Mackey functors form a semisimple abelian category, i.e., every object is both projective and injective.

Proposition 3.13 above shows that the geometric fixed point map (3.3) factors over the quotient of $\pi_0^G(X)$ by the subgroup generated by proper transfers. We denote by

$$\bar{\Phi} \;:\; \pi_k^G(X)/t(\underline{\pi}_k(X)) \;=\; \tau_G(\underline{\pi}_k(X)) \;\longrightarrow\; \Phi_k^G(X) \tag{4.29}$$

the induced map on the factor group, and call it the *reduced geometric fixed point map*. Our next result shows that for finite groups G, the 'corrected' (i.e., reduced) geometric fixed point map becomes an isomorphism after inverting the order of G. I am convinced that the following proposition is well known among experts, and closely related statements appear in Appendix A of [**65**]; however, I am not aware of a reference for the following statement in this form.

PROPOSITION 4.30. *For every finite group G, every orthogonal G-spectrum X and every integer k the reduced geometric fixed point map*

$$\bar{\Phi} \;:\; \tau_G(\underline{\pi}_k(X)) \;\longrightarrow\; \Phi_k^G(X)$$

becomes an isomorphism after inverting the order of G.

PROOF. We start by showing that for every orthogonal G-spectrum X and every subgroup H of G the transfer map

$$\mathrm{tr}_H^G \;:\; \pi_k^H(X \wedge G/H_+) \;\longrightarrow\; \pi_k^G(X \wedge G/H_+)$$

for the spectrum $X \wedge G/H_+$ is surjective. Indeed, the transfer is defined as the composite

$$\pi_k^H(X \wedge G/H_+) \;\xrightarrow[\cong]{G \ltimes_H -}\; \pi_k^G(G \ltimes_H (X \wedge G/H_+)) \;\xrightarrow{\mathrm{act}_*}\; \pi_k^G(X \wedge G/H_+)\;.$$

The first map – the external transfer – is an isomorphism by Theorem 2.14. The second map is surjective because the action map $G \ltimes_H (X \wedge G/H_+) \longrightarrow X \wedge G/H_+$ has a G-equivariant section

$$X \wedge G/H_+ \;\longrightarrow\; G \ltimes_H (X \wedge G/H_+)\;, \quad x \wedge gH \;\longmapsto\; [g;\, g^{-1}x \wedge eH]\;.$$

Now we let A be a G-CW-complex without G-fixed points. We claim that for every orthogonal G-spectrum X, the entire group $\pi_k^G(X \wedge A_+)$ is generated by transfers from proper subgroups after inverting $|G|$. In a first step we show this when A is a finite dimensional G-CW-complex. We argue by induction over the dimension of A. The induction starts when A is empty, in which case $X \wedge A_+$ is the trivial spectrum and there is nothing to show. Now we consider $n \geq 0$ and assume the claim for all $(n-1)$-dimensional G-CW-complexes without G-fixed points. We suppose that B is obtained from such an $(n-1)$-dimensional G-CW-complex A by attaching equivariant n-cells. After choosing characteristic maps for the n-cells we can identify the quotient B/A as a wedge

$$B/A \;\cong\; \bigvee_{i \in I} S^n \wedge (G/H_i)_+$$

for some indexing set I and certain subgroups H_i of G. Equivariant homotopy groups commute with wedges, so we can identify the homotopy group Mackey functor of $X \wedge (B/A)$ as

$$\underline{\pi}_k(X \wedge B/A) \;\cong\; \bigoplus_{i \in I} \underline{\pi}_k(X \wedge S^n \wedge (G/H_i)_+) \;.$$

Since B has no G-fixed points, the groups H_i are all proper subgroups of G, so the group $\pi_k^G(X \wedge (G/H_i)_+)$ is generated by transfers from the proper subgroup H_i, by the previous paragraph. So altogether the group $\pi_k^G(X \wedge B/A)$ is generated by transfers from proper subgroups (even before any localization).

The inclusion $A \longrightarrow B$ is a G-equivariant h-cofibration, so it induces an h-cofibration of orthogonal G-spectra $X \wedge A_+ \longrightarrow X \wedge B_+$ that results in a long exact sequence of homotopy group Mackey functors as in Corollary 1.39 (i). The functor sending a G-Mackey functor F to $\mathbb{Z}[1/|G|] \otimes (\tau_G F)$ is exact (Corollary 4.27), the groups $\mathbb{Z}[1/|G|] \otimes \tau_G(\underline{\pi}_*(X \wedge A_+))$ vanish by induction, and the groups $\tau_G(\underline{\pi}_*(X \wedge B/A))$ vanish by the previous paragraph. So the groups $\mathbb{Z}[1/|G|] \otimes \tau_G(\underline{\pi}_*(X \wedge B_+))$ vanish by exactness, and this finishes the inductive step.

If A admits the structure of a G-CW-complex, possibly infinite dimensional, we choose a skeleton filtration by G-subspaces A^n. Then the groups $\mathbb{Z}[1/|G|] \otimes \tau_G(\underline{\pi}_*(X \wedge A^n_+))$ vanish for all $n \geq 0$, by the previous paragraph. All the morphisms $X \wedge A_n \longrightarrow X \wedge A_{n+1}$ are h-cofibrations of orthogonal G-spectra. Since equivariant homotopy groups and the functor $\mathbb{Z}[1/|G|] \otimes \tau_G(-)$ both commute with sequential colimits, this shows that the groups $\tau_G(\underline{\pi}_k^G(X \wedge A))$ vanish after inverting $|G|$.

Now we can prove the proposition. The inclusion $S^0 \longrightarrow \tilde{E}\mathcal{P}_G$ gives rise to a commutative square:

$$\begin{tikzcd} \tau_G(\underline{\pi}_k(X)) \arrow[r, "\tau_G(\underline{\pi}_k(X \wedge i))"] \arrow[d, "\bar{\Phi}"'] & \tau_G(\underline{\pi}_k(X \wedge \tilde{E}\mathcal{P}_G)) \arrow[d, "\bar{\Phi}"] \\ \Phi_k^G(X) \arrow[r, "\Phi_k^G(X \wedge i)"'] & \Phi_k^G(X \wedge \tilde{E}\mathcal{P}_G) \end{tikzcd} \tag{4.31}$$

The lower horizontal map is an isomorphism because i identifies S^0 with the fixed point $(\tilde{E}\mathcal{P}_G)^G$. Proposition 3.9 shows that the geometric fixed point map $\Phi : \pi_k^G(X \wedge \tilde{E}\mathcal{P}_G) \longrightarrow \Phi_k^G(X \wedge \tilde{E}\mathcal{P}_G)$ is an isomorphism *before dividing out transfers*. So all transfers from proper subgroups are in fact trivial in $\pi_k^G(X \wedge \tilde{E}\mathcal{P}_G)$, and the right vertical map in the square (4.31) is an isomorphism (even before any localization). By the previous paragraph, the group $\pi_k^G(X \wedge (E\mathcal{P}_G)_+)$ is generated by transfers from proper subgroups after inverting $|G|$, so

$$\mathbb{Z}[1/|G|] \otimes \tau_G(\underline{\pi}_k(X \wedge (E\mathcal{P}_G)_+) \;=\; 0 \;.$$

The functor sending a G-Mackey functor F to the group $\mathbb{Z}[1/|G|] \otimes (\tau_G F)$ is exact (Corollary 4.27), so the isotropy separation sequence (3.11) shows that the upper horizontal map in the square (4.31) becomes an isomorphism after inverting $|G|$. So the left vertical map in the square (4.31) becomes an isomorphism after inverting $|G|$. □

When X is an orthogonal G-spectrum, we can apply the earlier algebraic Proposition 4.24 to the G-Mackey functor $\underline{\pi}_k(X)$; this allows us – after inverting $|G|$ – to reconstruct $\pi_k^G(X)$ from the groups $\tau_H(\underline{\pi}_k(X))$ with their Weyl group action. Moreover, after inverting the group order we can use the reduced geometric fixed point map (4.29), for the underlying H-spectrum of X, to identify the group $\tau_H(\underline{\pi}_k(X))$ with the group $\Phi_k^H(X)$. Under this identification, the map $\bar{\psi}_G^F$ becomes the product of the maps

$$\pi_k^G(X) \;\xrightarrow{\mathrm{res}_H^G}\; \pi_k^H(X) \;\xrightarrow{\Phi^H}\; \Phi_k^H(X) \;.$$

So Propositions 4.24 and 4.30 together prove:

Corollary 4.32. *For every finite group G, every orthogonal G-spectrum X and every integer k the map*

$$(\Phi^H \circ \mathrm{res}^G_H)_H \ : \ \pi^G_k(X) \ \longrightarrow \ \prod_{(H)} \big(\Phi^H_k(X)\big)^{W_G H}$$

becomes an isomorphism after inverting the order of G.

5. Products

In this section we recall the smash product of orthogonal spectra and orthogonal G-spectra and study its formal and homotopical properties. Like the box product of orthogonal spaces, the smash product of orthogonal spectra is another special case of Day's convolution product on categories of enriched functors, compare Appendix A.3. The smash product is intimately related to pairings of equivariant stable homotopy groups that we recall in Construction 5.3; the main properties of these pairings are summarized in Theorem 5.5. When specialized to equivariant ring spectra, these pairings turn the equivariant stable homotopy groups into a graded ring, compare Corollary 5.8. The *geometric* fixed point homotopy groups also support pairings, and Theorem 5.11 and Corollary 5.12 summarize their main properties.

The smash product of orthogonal spectra is characterized by a universal property that we recall now. The indexing category $\mathbf{O}$ for orthogonal spectra has a symmetric monoidal product by direct sum as follows. We denote by $\mathbf{O} \wedge \mathbf{O}$ the category enriched in based spaces whose objects are pairs of inner product spaces and whose morphisms spaces are smash products of morphisms spaces in $\mathbf{O}$. A based continuous functor

$$\oplus \ : \ \mathbf{O} \wedge \mathbf{O} \ \longrightarrow \ \mathbf{O}$$

is defined on objects by orthogonal direct sum, and on morphism spaces by

$$\mathbf{O}(V,W) \wedge \mathbf{O}(V',W') \ \longrightarrow \ \mathbf{O}(V \oplus W, V' \oplus W') \ , \quad (w,\varphi) \wedge (w',\varphi') \ \longmapsto \ ((w,w'), \varphi \oplus \varphi') \ .$$

A *bimorphism* $b : (X,Y) \longrightarrow Z$ from a pair of orthogonal spectra (X,Y) to an orthogonal spectrum Z is a natural transformation

$$b \ : \ X \bar{\wedge} Y \ \longrightarrow \ Z \circ \oplus$$

of continuous functors $\mathbf{O} \wedge \mathbf{O} \longrightarrow \mathbf{T}_*$; here $X \bar{\wedge} Y$ is the 'external smash product' defined by $(X \bar{\wedge} Y)(V,W) = X(V) \wedge Y(W)$. A bimorphism thus consists of based continuous maps

$$b_{V,W} \ : \ X(V) \wedge Y(W) \ \longrightarrow \ Z(V \oplus W)$$

for all inner product spaces V and W that form morphisms of orthogonal spectra in each variable separately. A smash product of two orthogonal spectra is now a universal example of a bimorphism from (X,Y).

Definition 5.1. A *smash product* of two orthogonal spectra X and Y is a pair $(X \wedge Y, i)$ consisting of an orthogonal spectrum $X \wedge Y$ and a universal bimorphism $i : (X,Y) \longrightarrow X \wedge Y$, i.e., a bimorphism such that for every orthogonal spectrum Z the map

$$\mathcal{S}p(X \wedge Y, Z) \ \longrightarrow \ \mathrm{Bimor}((X,Y),Z) \ , \quad f \longmapsto fi = \{f(V \oplus W) \circ i_{V,W}\}_{V,W} \tag{5.2}$$

is bijective.

Since the index category $\mathbf{O}$ is skeletally small and the base category $\mathbf{T}_*$ is cocomplete, every pair of orthogonal spectra has a smash product by Proposition A.3.5. Often only the object $X \wedge Y$ will be referred to as the smash product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (5.2) as the *universal property* of the smash product of orthogonal spectra.

While the smash products for pairs of orthogonal spectra are choices, any collection of choices automatically extends to a symmetric monoidal structure on the category $\mathcal{S}p$: there is a preferred way to extend the chosen smash products to a functor in two variables (Construction A.3.8); this functor has a

preferred symmetric monoidal structure, i.e., there are distinguished natural associativity and symmetry isomorphisms

$$\alpha_{X,Y,Z} \ : \ (X \wedge Y) \wedge Z \ \longrightarrow \ X \wedge (Y \wedge Z) \qquad \text{respectively} \qquad \tau_{X,Y} \ : \ X \wedge Y \ \longrightarrow \ Y \wedge X$$

(see Construction A.3.10). Together with strict unit isomorphisms $\mathbb{S} \wedge X = X = X \wedge \mathbb{S}$, these satisfy the coherence conditions of a symmetric monoidal category, compare Day's Theorem A.3.12. The smash product of orthogonal spectra is *closed* symmetric monoidal in the sense that the smash product is adjoint to an internal Hom spectrum, i.e., there is an adjunction isomorphism

$$\mathrm{Hom}(X \wedge Y, Z) \ \cong \ \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \ .$$

The construction is a special case of Construction A.3.14 for enriched functor categories.

When a compact Lie group G acts on the orthogonal spectra X and Y, then $X \wedge Y$ becomes an orthogonal G-spectrum via the diagonal action. So the smash product lifts to a symmetric monoidal closed structure

$$\wedge \ : \ G\mathcal{S}p \times G\mathcal{S}p \ \longrightarrow \ G\mathcal{S}p$$

on the category of orthogonal G-spectra.

Construction 5.3. Given a compact Lie group G and two orthogonal G-spectra X and Y, we endow the equivariant homotopy groups with an external pairing

$$\times \ : \ \pi_k^G(X) \ \times \ \pi_l^G(Y) \ \longrightarrow \ \pi_{k+l}^G(X \wedge Y) \ , \tag{5.4}$$

where k and l are integers. We let

$$f \ : \ S^{U \oplus \mathbb{R}^{m+k}} \ \longrightarrow \ X(U \oplus \mathbb{R}^m) \qquad \text{and} \qquad g \ : \ S^{V \oplus \mathbb{R}^{n+l}} \ \longrightarrow \ Y(V \oplus \mathbb{R}^n)$$

represent classes in $\pi_k^G(X)$ respectively $\pi_l^G(Y)$, for suitable G-representations U and V. The class $[f] \times [g]$ in $\pi_{k+l}^G(X \wedge Y)$ is then represented by the composite

$$\begin{aligned} S^{U \oplus V \oplus \mathbb{R}^{m+n+k+l}} \ &\cong S^{U \oplus \mathbb{R}^{m+k}} \wedge S^{V \oplus \mathbb{R}^{n+l}} \ \xrightarrow{f \wedge g} \ X(U \oplus \mathbb{R}^m) \wedge Y(V \oplus \mathbb{R}^n) \\ &\xrightarrow{i_{U \oplus \mathbb{R}^m, V \oplus \mathbb{R}^n}} (X \wedge Y)(U \oplus \mathbb{R}^m \oplus V \oplus \mathbb{R}^n) \ \xrightarrow{(X \wedge Y)(U \oplus \tau_{\mathbb{R}^m, V} \oplus \mathbb{R}^n)} \ (X \wedge Y)(U \oplus V \oplus \mathbb{R}^{m+n}) \ . \end{aligned}$$

The first homeomorphism shuffles the sphere coordinates. We omit the verification that the class of the composite only depends on the classes of f and g.

The pairing of equivariant homotopy groups has several expected properties that we summarize in the next theorem.

Theorem 5.5. *Let G be a compact Lie group and X, Y and Z orthogonal G-spectra.*

(i) (Biadditivity) *The product $\times : \pi_k^G(X) \times \pi_l^G(Y) \longrightarrow \pi_{k+l}^G(X \wedge Y)$ is biadditive.*

(ii) (Unitality) *The class $1 \in \pi_0^G(\mathbb{S})$ represented by the identity of S^0 is a two-sided unit for the pairing $\times$.*

(iii) (Associativity) *For all classes $x \in \pi_k^G(X)$, $y \in \pi_l^G(Y)$ and $z \in \pi_m^G(Z)$ the relation*

$$x \times (y \times z) \ = \ (x \times y) \times z$$

holds in $\pi_{k+l+m}^G(X \wedge Y \wedge Z)$.

(iv) (Commutativity) *For all classes $x \in \pi_k^G(X)$ and $y \in \pi_l^G(Y)$ the relation*

$$\tau_*^{X,Y}(x \times y) \ = \ (-1)^{kl} \cdot (y \times x)$$

holds in $\pi_{k+l}^G(Y \wedge X)$, where $\tau^{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ is the symmetry isomorphism of the smash product.

(v) (Restriction) *For all classes* $x \in \pi_k^G(X)$ *and* $y \in \pi_l^G(Y)$ *and all continuous homomorphisms* $\alpha : K \longrightarrow G$ *the relation*

$$\alpha^*(x) \times \alpha^*(y) \;=\; \alpha^*(x \times y)$$

holds in $\pi_{k+l}^K(\alpha^*(X \wedge Y))$.

(vi) (Transfer) *Let* H *be a closed subgroup of* G. *For all* $x \in \pi_k^G(X)$ *and* $z \in \pi_l^H(Y \wedge S^L)$, *the relation*

$$x \times \mathrm{Tr}_H^G(z) \;=\; \mathrm{Tr}_H^G(\mathrm{res}_H^G(x) \times z)$$

holds in $\pi_{k+l}^G(X \wedge Y)$, *where* $L = T_{eH}(G/H)$ *is the tangent* H*-representation. For all* $y \in \pi_l^H(Y)$, *the relation*

$$x \times \mathrm{tr}_H^G(y) \;=\; \mathrm{tr}_H^G(\mathrm{res}_H^G(x) \times y)$$

holds in $\pi_{k+l}^G(X \wedge Y)$.

Proof. (i) We deduce the additivity in the first variable from the general additivity statement in Proposition II.2.13. We consider the two reduced additive functors

$$X \;\longmapsto\; \pi_k^G(X) \qquad \text{and} \qquad X \longmapsto\; \pi_{k+l}^G(X \wedge Y)$$

from the category of orthogonal G-spectra to the category of abelian groups. Proposition II.2.13 shows that for every class $y \in \pi_l^G(Y)$ the natural transformation

$$- \times y \;:\; \pi_k^G(-) \;\longrightarrow\; \pi_{k+l}^G(- \wedge Y)$$

is additive. Additivity in the second variable is proved in the same way. Properties (ii), (iii) and (v) are straightforward consequences of the definitions. The sign $(-1)^{kl}$ in the commutativity relation (iv) is the degree of the map that interchanges a k-sphere with an l-sphere. Indeed, if

$$f \;:\; S^{U \oplus \mathbb{R}^{n+k}} \;\longrightarrow\; X(U \oplus \mathbb{R}^n) \qquad \text{and} \qquad g \;:\; S^{V \oplus \mathbb{R}^{m+l}} \;\longrightarrow\; Y(V \oplus \mathbb{R}^m)$$

represent classes in $\pi_k^G(X)$ respectively $\pi_l^G(Y)$, then the upper horizontal map in the diagram

$$\begin{tikzcd}
S^{U \oplus V \oplus \mathbb{R}^{m+n+k+l}} \arrow[r, "f \times g"] \arrow[d, "\tau_{U,V} \wedge \tau_{m,n} \wedge S^{k+l}"'] & (X \wedge Y)(U \oplus V \oplus \mathbb{R}^{m+n}) \arrow[r, "\tau^{X,Y}(U \oplus V \oplus \mathbb{R}^{m+n})"] & (Y \wedge X)(U \oplus V \oplus \mathbb{R}^{m+n}) \arrow[d, "(Y \wedge X)(\tau_{U,V} \oplus \tau_{m,n})"] \\
S^{V \oplus U \oplus \mathbb{R}^{n+m+k+l}} \arrow[r, "S^{V \oplus U \oplus \mathbb{R}^{n+m}} \wedge \tau_{k,l}"'] & S^{V \oplus U \oplus \mathbb{R}^{n+m+l+k}} \arrow[r, "g \times f"'] & (Y \wedge X)(V \oplus U \oplus \mathbb{R}^{n+m})
\end{tikzcd}$$

represents $\tau_*^{X,Y}([f] \times [g])$, whereas the lower horizontal composites represents $(-1)^{kl} \cdot ([g] \times [f])$. Since the two composite differ by conjugation be a G-equivariant linear isometry, they represent the same class by Proposition 1.14 (ii).

(vi) The following diagram of abelian groups commutes by naturality of the exterior product, and because restriction from G to H is multiplicative:

$$\begin{tikzcd}
\pi_k^G(X) \times \pi_l^G(Y \wedge G/H_+) \arrow[r, "\times"] \arrow[d, "\mathrm{Id} \times \mathrm{res}_H^G"] \arrow[dd, bend right=60, "\mathrm{Id} \times \mathrm{Wirth}_H^G"'] & \pi_{k+l}^G(X \wedge Y \wedge G/H_+) \arrow[d, "\mathrm{res}_H^G"'] \arrow[ddr, bend left=60, "\mathrm{Wirth}_H^G"] \\
\pi_k^G(X) \times \pi_l^H(Y \wedge G/H_+) \arrow[d, "\mathrm{Id} \times (Y \wedge l)_*"] \arrow[dr, "\mathrm{res}_H^G \times \mathrm{Id}"] & \pi_{k+l}^H(X \wedge Y \wedge G/H_+) \arrow[ddr, bend left=60, "(X \wedge Y \wedge l)_*"] \\
\pi_k^G(X) \times \pi_l^H(Y \wedge S^L) \arrow[dr, "\mathrm{res}_H^G \times \mathrm{Id}"'] & \pi_k^H(X) \times \pi_l^H(Y \wedge G/H_+) \arrow[u, "\times"'] \arrow[d, "\mathrm{Id} \times (Y \wedge l)_*"'] \\
& \pi_k^H(X) \times \pi_l^H(Y \wedge S^L) \arrow[r, "\times"'] & \pi_{k+l}^H(X \wedge Y \wedge S^L)
\end{tikzcd}$$

Here $l : G/H \longrightarrow S^L$ is the H-equivariant collapse map defined in (2.3). The two vertical composites are the respective Wirthmüller isomorphisms (see Theorem 2.14). Since the external transfer is inverse to the Wirthmüller isomorphism (up to the effect of the involution $S^{-\mathrm{Id}} : S^L \longrightarrow S^L$), we can read the diagram backwards and conclude that the upper part of the following diagram commutes:

$$\begin{tikzcd}
\pi_k^G(X) \times \pi_l^H(Y \wedge S^L) \arrow[r, "\mathrm{res}_H^G \times \mathrm{Id}"] \arrow[d, "\cong"', "\mathrm{Id} \times (G \ltimes_H -)"] \arrow[ddd, bend right=60, "\mathrm{Id} \times \mathrm{Tr}_H^G"'] & \pi_k^H(X) \times \pi_l^H(Y \wedge S^L) \arrow[r, "\times"] & \pi_{k+l}^H(X \wedge Y \wedge S^L) \arrow[d, "G \ltimes_H -"', "\cong"] \arrow[ddd, bend left=60, "\mathrm{Tr}_H^G"] \\
\pi_k^G(X) \times \pi_l^G(Y \wedge G/H_+) \arrow[rr, "\times"'] \arrow[d, "\mathrm{Id} \times (Y \wedge p)_*"] & & \pi_{k+k}^G(X \wedge Y \wedge G/H_+) \arrow[d, "(X \wedge Y \wedge p)_*"'] \\
\pi_k^G(X) \times \pi_l^G(Y) \arrow[rr, "\times"'] & & \pi_{k+l}^G(X \wedge Y)
\end{tikzcd}$$

Here $p : G/H_+ \longrightarrow S^0$ is the projection. The lower part of the diagram commutes by naturality of the pairings. This proves the first claim involving the dimension shifting transfers. The formula for the degree zero transfers follows by naturality by precomposing with the effect of the inclusion of the origin $S^0 \longrightarrow S^L$. $\square$

Definition 5.6. An *orthogonal ring spectrum* is a monoid in the category of orthogonal spectra with respect to the smash product. For a compact Lie group G, an *orthogonal G-ring spectrum* is a monoid in the category of orthogonal G-spectra with respect to the smash product.

An orthogonal ring spectrum is thus an orthogonal spectrum R equipped with a multiplication morphism $\mu : R \wedge R \longrightarrow R$ and a unit morphism $\eta : \mathbb{S} \longrightarrow R$ such that the associativity and unit diagrams commute (compare (3.15) of Appendix A). A *morphism* of orthogonal ring spectra is a morphism $f : R \longrightarrow S$ of orthogonal spectra that satisfies $f \circ \mu^R = \mu^S \circ (f \wedge f)$ and $f \circ \eta^R = \eta^S$. Via the universal property of the smash product the data contained in the multiplication morphism can be made more explicit: $\mu : R \wedge R \longrightarrow R$ corresponds to a collection of based continuous maps $\mu_{V,W} : R(V) \wedge R(W) \longrightarrow R(V \oplus W)$ that together form a bimorphism. The associativity and unit conditions can also be rephrased in more explicit forms, and then we are requiring that the multiplication and unit maps make $R : \mathbf{O} \longrightarrow \mathbf{T}_*$ into a lax monoidal functor. Most of the time we will specify the data of an orthogonal ring spectrum in the explicit bimorphism form.

An orthogonal ring spectrum R (respective orthogonal G-ring spectrum) is *commutative* if the multiplication morphism satisfies $\mu \circ \tau_{R,R} = \mu$. In the explicit form this is equivalent to the commutativity of

the square

$$\begin{tikzcd} R(V) \wedge R(W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\tau_{R(V),R(W)}"'] & R(V \oplus W) \arrow[d, "R(\tau_{V,W})"] \\ R(W) \wedge R(V) \arrow[r, "\mu_{W,V}"'] & R(W \oplus V) \end{tikzcd}$$

for all inner product spaces V and W. Equivalently, the multiplication and unit maps make $R : \mathbf{O} \longrightarrow \mathbf{T}_*$ into a lax *symmetric* monoidal functor. Commutative orthogonal ring spectra already appear, with an extra pointset topological hypothesis and under the name $\mathscr{I}_*$-prefunctor, in [**107**, IV Def. 2.1].

Since the smash product of orthogonal G-spectra is just the smash product of the underlying orthogonal spectra, endowed with the diagonal G-action, an orthogonal G-ring spectrum is nothing but an orthogonal ring spectrum equipped with a continuous G-action through homomorphisms of orthogonal ring spectra. Via the universal property of the smash product, yet another way to package the data in an orthogonal G-ring spectrum is as a continuous lax monoidal functor from the category $\mathbf{O}$ to the category of based G-spaces.

Given a compact Lie group G and an orthogonal G-ring spectrum R, we define an internal pairing

$$\cdot \ : \ \pi_k^G(R) \times \pi_l^G(R) \ \longrightarrow \ \pi_{k+l}^G(R)$$

on the equivariant homotopy groups of R as the composite

$$\pi_k^G(R) \times \pi_l^G(R) \ \xrightarrow{\times} \ \pi_{k+l}^G(R \wedge R) \ \xrightarrow{\mu_*} \ \pi_{k+l}^G(R) \ . \tag{5.7}$$

Theorem 5.5 then immediately implies:

Corollary 5.8. *Let G be a compact Lie group, R an orthogonal G-ring spectrum and H a closed subgroup of G.*

(i) *The products $\cdot : \pi_k^G(R) \times \pi_l^G(R) \longrightarrow \pi_{k+l}^G(R)$ make the abelian groups $\{\pi_k^G(R)\}_{k\in\mathbb{Z}}$ into a graded ring. The multiplicative unit is the class of the unit map $S^0 \longrightarrow R(0)$.*

(ii) *If the multiplication of R is commutative, then the relation*

$$x \cdot y \ = \ (-1)^{kl} \cdot (y \cdot x)$$

holds for all classes $x \in \pi_k^G(R)$ and $y \in \pi_l^G(R)$.

(iii) (Restriction) *The restriction maps $\mathrm{res}_H^G : \pi_*^G(R) \longrightarrow \pi_*^H(R)$ form a homomorphism of graded rings.*

(iv) (Conjugation) *For every $g \in G$ the conjugation maps $g_\star : \pi_*^H(R) \longrightarrow \pi_*^{{}^gH}(R)$ form a homomorphism of graded rings.*

(v) (Transfer) *For all $x \in \pi_k^G(R)$ and $z \in \pi_l^H(R \wedge S^L)$, the relation*

$$x \cdot \mathrm{Tr}_H^G(z) \ = \ \mathrm{Tr}_H^G(\mathrm{res}_H^G(x) \cdot z)$$

holds in $\pi_{k+l}^G(R)$, where $L = T_{eH}(G/H)$ is the tangent H-representation. For all $y \in \pi_l^H(R)$, the relation

$$x \cdot \mathrm{tr}_H^G(y) \ = \ \mathrm{tr}_H^G(\mathrm{res}_H^G(x) \cdot y)$$

holds in $\pi_{k+l}^G(R)$.

Construction 5.9 (Products on geometric fixed point homotopy groups). Like the equivariant homotopy groups, the geometric fixed point homotopy groups also come with pairings. Given a compact Lie group G and two orthogonal G-spectra X and Y, we define an external pairing

$$\times \ : \ \Phi_k^G(X) \ \times \ \Phi_l^G(Y) \ \longrightarrow \ \Phi_{k+l}^G(X \wedge Y) \ . \tag{5.10}$$

Suppose that V and W are G-representations and

$$f \ : \ S^{V^G \oplus \mathbb{R}^{m+k}} \ \longrightarrow \ X(V \oplus \mathbb{R}^m)^G \qquad \text{and} \qquad g \ : \ S^{W^G \oplus \mathbb{R}^{n+l}} \ \longrightarrow \ Y(W \oplus \mathbb{R}^n)^G$$

are based maps representing classes in $\Phi_k^G(X)$ respectively $\Phi_l^G(Y)$. We denote by $f \times g$ the composite

$$\begin{aligned} S^{(V\oplus W)^G \oplus \mathbb{R}^{m+n+k+l}} &\cong S^{V^G\oplus\mathbb{R}^{m+k}} \wedge S^{W^G\oplus\mathbb{R}^{n+l}} \xrightarrow{f\wedge g} X(V\oplus\mathbb{R}^m)^G \wedge Y(W\oplus\mathbb{R}^n)^G \\ &\cong (X(V\oplus\mathbb{R}^m)\wedge Y(W\oplus\mathbb{R}^n))^G \\ &\xrightarrow{(i_{V\oplus\mathbb{R}^m,W\oplus\mathbb{R}^n})^G} ((X\wedge Y)(V\oplus\mathbb{R}^m\oplus W\oplus\mathbb{R}^n))^G \\ &\xrightarrow{((X\wedge Y)(V\oplus\tau_{\mathbb{R}^m,W}\oplus\mathbb{R}^n))^G} ((X\wedge Y)(V\oplus W\oplus\mathbb{R}^{m+n}))^G \ . \end{aligned}$$

Here the first homeomorphism is induced by the shuffle isometry

$$\begin{aligned} V^G\oplus W^G\oplus\mathbb{R}^{m+n+k+l} &\xrightarrow{V^G\oplus W^G\oplus\mathbb{R}^m\oplus\tau_{\mathbb{R}^n,\mathbb{R}^k}\oplus\mathbb{R}^l} V^G\oplus W^G\oplus\mathbb{R}^{m+k}\oplus\mathbb{R}^{n+l} \\ &\xrightarrow{V^G\oplus\tau_{W^G,\mathbb{R}^{m+k}}\oplus\mathbb{R}^{n+l}} V^G\oplus\mathbb{R}^{m+k}\oplus W^G\oplus\mathbb{R}^{n+l} \ . \end{aligned}$$

We omit the verification that the definition

$$[f]\times[g] \ = \ \langle f\times g\rangle \ \in \ \Phi_{k+l}^G(X\wedge Y)$$

is well-defined.

The pairing of geometric fixed point homotopy groups has properties analogous to that of equivariant homotopy groups (compare Theorem 5.5).

THEOREM 5.11. *Let G be a compact Lie group and X, Y and Z orthogonal G-spectra.*

(i) (Biadditivity) *The product $\times : \Phi_k^G(X)\times\Phi_l^G(Y) \longrightarrow \Phi_{k+l}^G(X\wedge Y)$ is biadditive.*

(ii) (Unitality) *Let $1\in\Phi_0^G(\mathbb{S})$ denote the class represented by the identity of S^0. Then $1\times x = x = x\times 1$.*

(iii) (Associativity) *For all classes $x\in\Phi_k^G(X)$, $y\in\Phi_l^G(Y)$ and $z\in\Phi_j^G(Z)$ the relation*

$$x\times(y\times z) \ = \ (x\times y)\times z$$

holds in $\Phi_{k+l+j}^G(X\wedge Y\wedge Z)$.

(iv) (Commutativity) *For all classes $x\in\Phi_k^G(X)$ and $y\in\Phi_l^G(Y)$ the relation*

$$y\times x \ = \ (-1)^{kl}\cdot(\tau_*^{X,Y}(x\times y))$$

holds in $\Phi_{l+k}^G(Y\wedge X)$, where $\tau^{X,Y} : X\wedge Y \longrightarrow Y\wedge X$ is the symmetry isomorphism of the smash product.

(v) (Inflation) *For all classes $x\in\Phi_k^G(X)$ and $y\in\Phi_l^G(Y)$ and all continuous epimorphisms $\alpha : K\longrightarrow G$ the relation*

$$\alpha^*(x)\times\alpha^*(y) \ = \ \alpha^*(x\times y)$$

holds in $\Phi_{k+l}^K(\alpha^*(X\wedge Y))$.

(vi) (Geometric fixed point map) *For all classes $x\in\pi_k^G(X)$ and $y\in\pi_l^G(Y)$ the relation*

$$\Phi(x)\times\Phi(y) \ = \ \Phi(x\times y)$$

holds in $\Phi_{k+l}^G(X\wedge Y)$.

PROOF. Biadditivity can be shown by the same kind of naturality argument as in the proof of Theorem 5.5 (i). The other conditions are straightforward from the definitions. □

Given an orthogonal G-ring spectrum R, we define an internal pairing '$\cdot$' on the geometric fixed point homotopy groups of R as the composite

$$\Phi_k^G(R)\times\Phi_l^G(R) \ \xrightarrow{\times} \ \Phi_{k+l}^G(R\wedge R) \ \xrightarrow{\mu_*} \ \Phi_{k+l}^G(R) \ .$$

The following properties follow from Theorem 5.11 by applying the effect of the multiplication morphism $\mu : R\wedge R\longrightarrow R$.

Corollary 5.12. *Let G be a compact Lie group and R be an orthogonal G-ring spectrum.*

(i) (Biadditivity) *The products $\cdot : \Phi_k^G(R) \times \Phi_l^G(R) \longrightarrow \Phi_{k+l}^G(R)$ make the abelian groups $\{\Phi_k^G(R)\}_{k\in\mathbb{Z}}$ into a graded ring. The multiplicative unit is the class of the unit map $S^0 \longrightarrow R(0)$.*

(ii) (Commutativity) *If the multiplication of R is commutative, then for all classes $x \in \Phi_k^G(R)$ and $y \in \Phi_l^G(R)$ the relation*

$$x \cdot y \;=\; (-1)^{kl} \cdot y \cdot x$$

holds in $\Phi_{k+l}^G(R)$.

(iii) (Conjugation) *For all closed subgroups H of G and every $g \in G$ the conjugation maps $g_\star : \Phi_*^H(R) \longrightarrow \Phi_*^{{}^gH}(R)$ form a homomorphism of graded rings.*

(iv) (Geometric fixed point map) *The geometric fixed point map $\Phi : \pi_*^G(R) \longrightarrow \Phi_*^G(R)$ is a homomorphism of graded rings.*

CHAPTER IV

Global stable homotopy theory

In this chapter we embark on the investigation of global stable homotopy theory. In Section 1 we specialize the equivariant stable homotopy theory of the previous chapter to global stable homotopy types, which we model by orthogonal spectra (with no additional action of any groups). Section 2 introduces the category of *global functors*, the natural home of the collection of equivariant homotopy groups of a global stable homotopy type (i.e., an orthogonal spectrum). Global functors play the same role for global homotopy theory that is played by the category of abelian groups in ordinary homotopy theory, or by the category of G-Mackey functors for G-equivariant homotopy theory. Global functors are defined as additive functors on the *global Burnside category*; an explicit calculation of the global Burnside category provides the link to other notions of global Mackey functors. In the global context, the pairings on equivariant homotopy groups also give rise to a symmetric monoidal structure on the global Burnside category, and to a symmetric monoidal 'box product' of global functors.

Section 3 establishes the global model structure on the category of orthogonal spectra; more generally, we consider a global family $\mathcal{F}$ and define the $\mathcal{F}$-global model structure, where weak equivalences are tested on equivariant homotopy groups for all Lie groups in $\mathcal{F}$. The $\mathcal{F}$-global model structure is monoidal with respect to the smash product of orthogonal spectra, provided that $\mathcal{F}$ is closed under products. Section 4 collects aspects of global stable homotopy theory that refer to the triangulated structure of the global stable homotopy category. Specific topics are compact generators, Brown representability, a t-structure whose heart is the category of global functors, global Postnikov sections and Eilenberg-Mac Lane spectra of global functors.

Section 5 is a systematic study of the effects of changing the global family. We show that the 'forgetful' functor between the global stable homotopy categories of two nested global families has fully faithful left and right adjoints, which are part of a recollement. We provide characterizations of the global homotopy types in the image of the two adjoints; for example, the right adjoint all the way from the non-equivariant to the global stable homotopy category models Borel cohomology theories. We also relate the global homotopy category to the G-equivariant stable homotopy category for a fixed compact Lie group G; here the forgetful functor also has both adjoints, but these are no more fully faithful as soon as the group G is non-trivial.

The final Section 6 of this chapter establishes an algebraic model for *rational $\mathcal{F}in$-global* stable homotopy theory, i.e., for rational global stable homotopy theory based on the global family of finite groups. Indeed, spectral Morita theory provides a chain of Quillen equivalences to the category of chain complexes of rational global functors on finite groups. On the algebraic side, the abelian category of rational $\mathcal{F}in$-global functors is Morita equivalent to an even simpler category, namely functors from finite groups and conjugacy classes of epimorphisms to $\mathbb{Q}$-vector spaces. Under the two equivalences, homology groups of chain complexes correspond to equivariant stable homotopy groups, respectively geometric fixed point homotopy groups, of spectra.

1. Orthogonal spectra as global homotopy types

In this section we specialize the equivariant stable homotopy theory of Chapter III to global stable homotopy types, which we model by orthogonal spectra (with no additional action of any group).

Given an orthogonal spectrum X and a compact Lie group G, we obtain an orthogonal G-spectrum by letting G act trivially on the values of X. We call this the *underlying orthogonal G-spectrum* of X and use the notation X_G. To simplify notation we omit the subscript '$_G$' when we refer to equivariant homotopy groups, i.e., we simply write $\pi_k^G(X)$ instead of $\pi_k^G(X_G)$.

For global homotopy types (i.e., orthogonal spectra), the notation related to restriction maps simplifies a little, and some special features happen. Indeed, if X is an orthogonal spectrum, then for every continuous homomorphism $\alpha : K \longrightarrow G$ we have $\alpha^*(X_G) = X_K$, because both K and G act trivially. So for global homotopy types the restriction maps become homomorphisms

$$\alpha^* \ : \ \pi_*^G(X) \ \longrightarrow \ \pi_*^K(X) \ .$$

In particular, we have these restriction maps when α is an epimorphism; in that case we refer to α^* as an *inflation map*.

If H is a closed subgroup of G and $g \in G$, the conjugation homomorphism $c_g : {}^gH \longrightarrow H$ is given by $c_g(h) = g^{-1}hg$. We recall that for an orthogonal G-spectrum Y the conjugation homomorphism $g_\star : \pi_0^H(Y) \longrightarrow \pi_0^{{}^gH}(Y)$ is defined as $g_\star = (l_g^Y)_* \circ c_g^*$, where $l_g^Y : Y \longrightarrow Y$ is left multiplication by g, compare (1.18) of Chapter III. If $Y = X_G$ for an orthogonal spectrum X, then l_g^X is the identity; so for global homotopy types we have

$$g_\star \ = \ c_g^* \ : \ \pi_0^H(X) \ \longrightarrow \ \pi_0^{gH}(X) \ . \tag{1.1}$$

A different way to express the significance of the relation (1.1) is that whenever $Y = X_G$ underlies a global homotopy type, then the action of the group

$$\pi_0(W_G H) \ = \ \pi_0((N_G H)/H) \ = \ (N_G H)/(H \cdot (C_G H)^\circ)$$

on $\pi_0^H(X)$ discussed in Construction III.1.17 factors through an action of the quotient group

$$(N_G H)/(H \cdot C_G H) \ .$$

The difference is illustrated most drastically for $H = e$, the trivial subgroup of G. For a general orthogonal G-spectrum the action of $G = W_G e$ in $\pi_0^G(Y)$ is typically non-trivial. If $Y = X_G$ arises from an orthogonal spectrum, then this G-action is trivial.

One should beware that for a general orthogonal G-spectrum Y, a closed subgroup H of G and $g \in G$, the homomorphisms $g_\star : \pi_0^H(Y) \longrightarrow \pi_0^{gH}(Y)$ and $c_g^* : \pi_0^H(Y) \longrightarrow \pi_0^{gH}(c_g^*(Y))$ have different targets, and so cannot even be compared. In the special case when g centralizes the subgroup H, the conjugation homomorphism c_g is the identity of H, c_g^* is the identity of $\pi_0^H(Y)$, and $g_\star$ and c_g^* do have the same target; however, the H-morphism $l_g^Y : Y \longrightarrow Y$ need not induce the identity on π_0^H, so $g_\star = (l_g^Y)_*$ need not be the identity. If $g \in (C_G H)^\circ$ belongs to the identity path component of the centralizer of H in G, then $l_g^Y : Y \longrightarrow Y$ is H-equivariantly homotopic to the identity of Y, and then $g_\star = c_g^* = \mathrm{Id}$.

Remark 1.2 (Global homotopy types are split G-spectra). Obviously, only very special orthogonal G-spectra Y are part of a 'global family', i.e., arise as X_G for an orthogonal spectrum X. However, it is not a priori clear what the homotopical significance of the pointset level condition that G must act trivially on the values of Y (at trivial representations) is. Now we formulate obstructions to 'being global' in terms of the Mackey functor homotopy groups of an orthogonal G-spectrum.

The equivariant homotopy groups $\pi_0^G(X) = \pi_0^G(X_G)$ of an orthogonal spectrum X come equipped with restriction maps along arbitrary continuous group homomorphisms, not necessarily injective. This is in contrast to the situation for a fixed compact Lie group, where one can only restrict to subgroups, or along conjugation maps by elements of the ambient group, but where there are no inflation maps. For every orthogonal spectrum X and compact Lie groups $H \leq G$ the H-homotopy groups of the G-spectrum X_G are, by definition, the H-homotopy groups $\pi_0^H(X)$. So these groups depend only on H, and not on the ambient group G. One obstruction to Y being part of a 'global family' is that the G-Mackey functor structure can

be extended to a 'global functor' (in the sense of Definition 2.2 below). In particular, the G-Mackey functor homotopy groups can be complemented by restriction maps along arbitrary group homomorphisms between the subgroups of G. As the extreme case this includes a restriction map $p^* : \pi_0^e(X) \longrightarrow \pi_0^G(X)$ associated to the unique homomorphism $p : G \longrightarrow e$, splitting the restriction map $\pi_0^G(X) \longrightarrow \pi_0^e(X)$. So one obstruction to being global is that this restriction map from $\pi_0^G(X)$ to $\pi_0^e(X)$ needs to be a split epimorphism. This is the algebraic shadow of the fact that the G-equivariant homotopy types that underlie global homotopy types (i.e., are represented by orthogonal spectra) are 'split' in the sense that there is a morphism from the underlying non-equivariant spectrum to the genuine G-fixed point spectrum that splits the restriction map. Another consequence is that every global functor takes isomorphic values on a pair of isomorphic subgroups of G; for G-Mackey functors this is true when the subgroups are conjugate in G, but not in general when they are merely abstractly isomorphic.

Definition 1.3. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *global equivalence* if the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all compact Lie groups G and all integers k.

We define the *global stable homotopy category* $\mathcal{GH}$ by localizing the category of orthogonal spectra at the class of global equivalences. The global equivalences are the weak equivalences of the *global model structure* on the category of orthogonal spectra, see Theorem 3.26 below. So the methods of homotopical algebra are available for studying global equivalences and the associated global homotopy category. In later sections we will also consider a relative notion of global equivalence, the '$\mathcal{F}$-equivalence', based on a global family $\mathcal{F}$ of compact Lie groups. There we require that the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all integers k and all compact Lie groups G that belong to the global family $\mathcal{F}$.

When we specialize Proposition III.1.25 (ii), Corollary III.1.38, Proposition III.1.41, Proposition III.2.19 (ii) and Proposition III.1.42 to the underlying orthogonal G-spectra of orthogonal spectra, we obtain the following consequences:

Proposition 1.4. (i) *For every orthogonal spectrum X the morphism*

$$\lambda_X \ : \ X \wedge S^1 \ \longrightarrow \ \mathrm{sh}\, X \ , \qquad \textit{its adjoint} \qquad \tilde{\lambda}_X \ : \ X \ \longrightarrow \ \Omega\, \mathrm{sh}\, X$$

and the morphism $\eta_X : X \longrightarrow \Omega(X \wedge S^1)$ are global equivalences.

(ii) *For every finite family of orthogonal spectra the canonical morphism from the wedge to the product is a global equivalence.*

(iii) *For every finite based CW-complex A, the functor* $\mathrm{map}(A, -)$ *preserves global equivalences of orthogonal spectra.*

(iv) *For every cofibrant based space A, the functor $- \wedge A$ preserves global equivalences of orthogonal spectra.*

(v) *Let $e_n : X_n \longrightarrow X_{n+1}$ and $f_n : Y_n \longrightarrow Y_{n+1}$ be morphisms of orthogonal spectra that are levelwise closed embeddings, for $n \geq 0$. Let $\psi_n : X_n \longrightarrow Y_n$ be global equivalences that satisfy $\psi_{n+1} \circ e_n = f_n \circ \psi_n$ for all $n \geq 0$. Then the induced morphism $\psi_\infty : X_\infty \longrightarrow Y_\infty$ between the colimits of the sequences is a global equivalence.*

(vi) *Let $f_n : Y_n \longrightarrow Y_{n+1}$ be global equivalences of orthogonal G-spectra that are levelwise a closed embeddings, for $n \geq 0$. Then the canonical morphism $f_\infty : Y_0 \longrightarrow Y_\infty$ to a colimit of the sequence $\{f_n\}_{n \geq 0}$ is a global equivalence.*

For later use we record another closure property of global equivalences.

Corollary 1.5. *The class of h-cofibrations of orthogonal spectra that are simultaneously global equivalences is closed under cobase change, coproducts and sequential compositions.*

Proof. The class of h-cofibrations is closed under coproducts, cobase change and composition (finite or sequential), compare Corollary A.1.18 (i). The class of global equivalences is closed under coproducts because equivariant homotopy groups take wedges to direct sums (Corollary III.1.38 (i)). The cobase change of an h-cofibration that is also a global equivalence is another global equivalence by Corollary III.1.40 (i).

Every h-cofibration of orthogonal spectra in in particular levelwise a closed embedding. So the class of h-cofibrations that are also global equivalences is closed under sequential composition by Proposition 1.4 (vi). □

For an orthogonal spectrum X the 0-th equivariant homotopy groups $\pi_0^G(X)$ and the restriction maps between them coincide with the homotopy Rep-functor, in the sense of (5.6) of Chapter I, of a certain orthogonal space $\Omega^\bullet X$ that we now recall.

Construction 1.6. We introduce the functor

$$\Omega^\bullet \ : \ \mathcal{S}p \ \longrightarrow \ spc$$

from orthogonal spectra to orthogonal spaces. Given an orthogonal spectrum X, the value of $\Omega^\bullet X$ at an inner product space V is

$$(\Omega^\bullet X)(V) \ = \ \mathrm{map}(S^V, X(V)) \ .$$

If $\varphi : V \longrightarrow W$ is a linear isometric embedding, the induced map

$$\varphi_* \ : \ (\Omega^\bullet X)(V) \ = \ \mathrm{map}(S^V, X(V)) \ \longrightarrow \ \mathrm{map}(S^W, X(W)) \ = \ (\Omega^\bullet X)(W)$$

was defined in (1.9). In particular, the orthogonal group $O(V)$ acts on $(\Omega^\bullet X)(V) = \mathrm{map}(S^V, X(V))$ by conjugation. If $\psi : W \longrightarrow U$ is another isometric embedding, then we have

$$\psi_*(\varphi_* f) \ = \ (\psi\varphi)_* f \ .$$

The assignment $(\varphi, f) \mapsto \varphi_* f$ is continuous in both variables, so we have really defined an orthogonal space $\Omega^\bullet X$. The construction is clearly functorial in the orthogonal spectrum X; moreover, $\Omega^\bullet$ has a left adjoint 'unreduced suspension spectrum' functor Σ^∞_+ that we discuss in Construction 1.7 below.

If G acts on V by linear isometries, then the G-fixed subspace of $(\Omega^\bullet X)(V)$ is the space of G-equivariant based maps from S^V to $X(V)$:

$$((\Omega^\bullet X)(V))^G \ = \ \mathrm{map}^G(S^V, X(V)) \ .$$

The path components of this space are precisely the equivariant homotopy classes of based G-maps, i.e.,

$$\pi_0\left(((\Omega^\bullet X)(V))^G\right) \ = \ \pi_0\left(\mathrm{map}^G(S^V, X(V))\right) \ = \ [S^V, X(V)]^G \ .$$

Passing to colimits over the poset $s(\mathcal{U}_G)$ gives

$$\pi_0^G(\Omega^\bullet X) \ = \ \pi_0^G(X) \ ,$$

i.e., the G-equivariant homotopy group of the orthogonal spectrum X equals the G-equivariant homotopy set of the orthogonal space $\Omega^\bullet X$. A direct inspection shows that the spectrum level restriction maps defined in Construction III.1.15 coincide with the restriction maps for orthogonal spaces introduced in (5.10) of Chapter I.

Construction 1.7 (Suspension spectra of orthogonal spaces). To every orthogonal space Y we can associate an unreduced *suspension spectrum* $\Sigma^\infty_+ Y$ whose value on an inner product space is given by

$$(\Sigma^\infty_+ Y)(V) \ = \ S^V \wedge Y(V)_+ \ ;$$

here the orthogonal group $O(V)$ acts diagonally and the structure map

$$\sigma_{V,W} \ : \ S^V \wedge (S^W \wedge Y(W)_+) \ \longrightarrow \ S^{V\oplus W} \wedge Y(V \oplus W)_+$$

is the combination of the canonical homeomorphism $S^V \wedge S^W \cong S^{V\oplus W}$ and the map $Y(i_W) : Y(W) \longrightarrow Y(V \oplus W)$. If Y is the constant orthogonal space with value a topological space A, then $\Sigma^\infty_+ Y = \Sigma^\infty_+ A$ specializes to the suspension spectrum of A with a disjoint basepoint added. The functor

$$\Sigma^\infty_+ \ : \ spc \ \longrightarrow \ \mathcal{S}p$$

is left adjoint to the functor $\Omega^\bullet$ of Construction 1.6.

Let G be a compact Lie group and $\{V_i\}_{i\geq 1}$ an exhaustive sequence of G-subrepresentations of the complete universe $\mathcal{U}_G$. Given an orthogonal space Y, we denote by $\operatorname{tel}_i Y(V_i)$ the mapping telescope of the sequence of G-spaces

$$Y(V_1) \longrightarrow Y(V_2) \longrightarrow \cdots \longrightarrow Y(V_i) \longrightarrow \cdots ;$$

the maps in the sequence are induced by the inclusions $V_{i-1} \longrightarrow V_i$, so they are G-equivariant, and the telescope inherits a natural G-action. The canonical maps $Y(V_j) \longrightarrow \operatorname{tel}_i Y(V_i)$ induce maps of equivariant homotopy classes

$$\begin{aligned}[S^{V_j\oplus\mathbb{R}^k}, (\Sigma^\infty_+ Y)(V_j)]^G \;=\; [S^{V_j\oplus\mathbb{R}^k}, S^{V_j}\wedge Y(V_j)_+]^G \;&\longrightarrow\; [S^{V_j\oplus\mathbb{R}^k}, S^{V_j}\wedge \operatorname{tel}_i Y(V_i)_+]^G \\ &\longrightarrow\; \pi^G_k(\Sigma^\infty_+ \operatorname{tel}_i Y(V_i)) \;,\end{aligned}$$

where the second map is the canonical one to the colimit. These maps are compatible with stabilization in the source when we increase j. Since the exhaustive sequence is cofinal in the poset $s(\mathcal{U}_G)$, the colimit over j calculates the k-th equivariant homotopy group of $\Sigma^\infty_+ Y$. So altogether, the maps assemble into a natural group homomorphism

$$\pi^G_k(\Sigma^\infty_+ Y) \longrightarrow \pi^G_k(\Sigma^\infty_+ \operatorname{tel}_i Y(V_i)) \;,$$

for $k \geq 0$. For negative values of k, we obtain a similar map by inserting $\mathbb{R}^{-k}$ in second variable of the above sets of equivariant homotopy classes.

Proposition 1.8. *Let G be a compact Lie group and $\{V_i\}_{i\geq 1}$ an exhaustive sequence of G-representations. Then for every orthogonal space Y and every integer k the map*

$$\pi^G_k(\Sigma^\infty_+ Y) \longrightarrow \pi^G_k(\Sigma^\infty_+ \operatorname{tel}_i Y(V_i))$$

is an isomorphism.

Proof. The mapping telescope is the colimit, along h-cofibrations, of the truncated mapping telescopes $\operatorname{tel}_{[0,n]} Y(V_i)$. So the space $S^{V_j}\wedge \operatorname{tel}_i Y(V_i)_+$ is the colimit, along a sequence of closed embeddings, of the spaces $S^{V_j}\wedge \operatorname{tel}_{[0,n]} Y(V_i)_+$. So for every compact based G-space A the canonical map

$$\operatorname{colim}_{n\geq 1} [A, S^{V_j}\wedge \operatorname{tel}_{[0,n]} Y(V_i)_+]^G \longrightarrow [A, S^{V_j}\wedge \operatorname{tel}_i Y(V_i)_+]^G$$

is bijective, compare A.1.10. The canonical map $Y(V_n) \longrightarrow \operatorname{tel}_i Y(V_i)$ factors through $Y(V_n) \longrightarrow \operatorname{tel}_{[0,n]} Y(V_i)$, and this factorization is an equivariant homotopy equivalence. So we can replace $\operatorname{tel}_{[0,n]} Y(V_i)$ by $Y(V_n)$ and conclude that the map

$$\operatorname{colim}_{n\geq 1} [A, S^{V_j}\wedge Y(V_n)_+]^G \longrightarrow [A, S^{V_j}\wedge \operatorname{tel}_i Y(V_i)_+]^G$$

is bijective. We specialize to $A = S^{V_j\oplus\mathbb{R}^k}$ and pass to the colimit over j by stabilization in source and target. The result is a bijection

$$\operatorname{colim}_{j\geq 1}\operatorname{colim}_{n\geq 1} [S^{V_j\oplus\mathbb{R}^k}, S^{V_j}\wedge Y(V_n)_+]^G \longrightarrow \operatorname{colim}_{j\geq 1}[S^{V_j\oplus\mathbb{R}^k}, S^{V_j}\wedge \operatorname{tel}_i Y(V_i)_+]^G \;.$$

The source is isomorphic to the diagonal colimit

$$\operatorname{colim}_{j\geq 1}[S^{V_j\oplus\mathbb{R}^k}, S^{V_j}\wedge Y(V_j)_+]^G$$

by cofinality. Since the exhaustive sequence is cofinal in the poset $s(\mathcal{U}_G)$, the two colimits over j also calculate the colimits over $s(\mathcal{U}_G)$, and hence the k-th equivariant homotopy group of $\Sigma^\infty_+ Y$ respectively $\Sigma^\infty_+ \operatorname{tel}_i Y(V_i)$. This shows the claim for $k \geq 0$. For negative values of k, the argument is similar, by inserting $\mathbb{R}^{-k}$ in second variable of the equivariant homotopy sets. □

Corollary 1.9. *Let $\mathcal{F}$ be a global family. Then the unreduced suspension spectrum functor Σ^∞_+ takes $\mathcal{F}$-equivalences of orthogonal spaces to $\mathcal{F}$-equivalences of orthogonal spectra.*

Proof. Let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence of orthogonal spaces and G a compact Lie group in $\mathcal{F}$. We choose an exhaustive sequence of G-representations $\{V_i\}_{i\geq 1}$. Then the map

$$\mathrm{tel}_i\, f(V_i) \ : \ \mathrm{tel}_i\, X(V_i) \ \longrightarrow \ \mathrm{tel}_i\, Y(V_i)$$

is a G-weak equivalence of G-spaces by Proposition I.4.5 (iii). So the map of suspension spectra $\Sigma^\infty_+ \mathrm{tel}_i\, f(V_i)$ induces isomorphisms on all G-equivariant stable homotopy groups, by Proposition III.1.45. Since the group $\pi^G_k(\Sigma^\infty_+ \mathrm{tel}_i\, X(V_i))$ is naturally isomorphic to the group $\pi^G_k(\Sigma^\infty_+ X)$ (by Proposition 1.8), the morphism of orthogonal suspension spectra $\Sigma^\infty_+ f : \Sigma^\infty_+ X \longrightarrow \Sigma^\infty_+ Y$ induces an isomorphism on π^G_*. □

Now we let Y be an orthogonal space and K a compact Lie group. We define a map

$$\sigma^K \ : \ \pi^K_0(Y) \ \longrightarrow \ \pi^K_0(\Sigma^\infty_+ Y) \tag{1.10}$$

as the effect of the adjunction unit $Y \longrightarrow \Omega^\bullet(\Sigma^\infty_+ Y)$ on the K-equivariant homotopy set π^K_0, using that $\pi^K_0(\Omega^\bullet(\Sigma^\infty_+ Y)) = \pi^K_0(\Sigma^\infty_+ Y)$. If we unravel the definitions, this comes out as follows: if V is a finite dimensional K-subrepresentation of the complete K-universe $\mathcal{U}_K$ and $y \in Y(V)^K$ a K-fixed point, then $\sigma^K[y]$ is represented by the K-map

$$S^V \ \xrightarrow{-\wedge y} \ S^V \wedge Y(V)_+ \ = \ (\Sigma^\infty_+ Y)(V) \ .$$

As K varies, the maps σ^K are compatible with restriction along every continuous homomorphism, since they arise from a morphism of orthogonal spaces. By the same argument as for orthogonal K-spectra given in (3.17), transferring from a closed subgroup L to K annihilates the action of the Weyl group on $\pi^L_0(Y)$.

The next proposition is both a global analog, and a straightforward consequence, of Theorem III.3.18 (i). We call an orthogonal spectrum *globally connective* if its G-equivariant homotopy groups vanish in negative dimensions, for all compact Lie groups G.

Proposition 1.11. *Let Y be an orthogonal space. Then the suspension spectrum $\Sigma^\infty_+ Y$ is globally connective. Moreover, for every compact Lie group K the equivariant homotopy group $\pi^K_0(\Sigma^\infty_+ Y)$ of the suspension spectrum of Y is a free abelian group with a basis given by the elements*

$$\mathrm{tr}^K_L(\sigma^L(x))$$

where L runs through all conjugacy classes of closed subgroups of K with finite Weyl group and x runs through a set of representatives of the $W_K L$-orbits of the set $\pi^L_0(Y)$.

Proof. We consider the functor on the product poset $s(\mathcal{U}_K)^2$ sending (U,V) to the set $[S^V, Y(U)_+ \wedge S^V]^K$. The diagonal is cofinal in $s(\mathcal{U}_K)^2$, thus the induced map

$$\pi^K_0(\Sigma^\infty_+ Y) \ = \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^V, S^V \wedge Y(V)_+]^K \ \longrightarrow \ \mathrm{colim}_{(V,U)\in s(\mathcal{U}_K)^2}[S^V, S^V \wedge Y(U)_+]^K$$

is an isomorphism. The target can be calculated in two steps, so the group we are after is isomorphic to

$$\mathrm{colim}_{U\in s(\mathcal{U}_K)}\left(\mathrm{colim}_{V\in s(\mathcal{U}_K)}[S^V, S^V \wedge Y(U)_+]^K\right) \ = \ \mathrm{colim}_{U\in s(\mathcal{U}_K)}\, \pi^K_0\left(\Sigma^\infty_+ Y(U)\right) \ .$$

We may thus show that the latter group is free abelian with the specified basis.

Theorem III.3.18 (i) identifies the equivariant homotopy group $\pi^K_0\left(\Sigma^\infty_+ Y(U)\right)$ as the free abelian group with basis the classes $\mathrm{tr}^K_L(\sigma^L(x))$, where L runs through all conjugacy classes of closed subgroups of K with finite Weyl group and x runs through a set of representatives of the $W_K L$-orbits of the set $\pi_0(Y(U)^L)$. Passage to the colimit over $U \in s(\mathcal{U}_K)$ yields the proposition. □

We let G be a compact Lie group and V a G-representation. We recall from (5.13) of Chapter I the tautological class

$$u_{G,V} \ \in \ \pi^G_0(\mathbf{L}_{G,V})$$

in the G-equivariant homotopy set of the free orthogonal space $\mathbf{L}_{G,V}$. The *stable tautological class* is

$$e_{G,V} \ = \ \sigma^G(u_{G,V}) \ \in \ \pi^G_0(\Sigma^\infty_+ \mathbf{L}_{G,V}) \ . \tag{1.12}$$

Explicitly, $e_{G,V}$ is the homotopy class of the G-map

$$S^V \longrightarrow S^V \wedge (\mathbf{L}(V,V)/G)_+ = (\Sigma^\infty_+ \mathbf{L}_{G,V})(V) \ , \quad v \longmapsto v \wedge (\mathrm{Id}_V \cdot G) \ .$$

Corollary 1.13. *Let G and K be compact Lie groups and V a faithful G-representation. Then the homotopy group $\pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,V})$ is a free abelian group with basis given by the classes*

$$\mathrm{tr}_L^K(\alpha^*(e_{G,V}))$$

as (L,α) runs over a set of representatives of all $(K\times G)$-conjugacy classes of pairs consisting of a subgroup L of K with finite Weyl group and a continuous homomorphism $\alpha : L \longrightarrow G$.

Proof. The map

$$\mathrm{Rep}(K,G) \longrightarrow \pi_0^K(\mathbf{L}_{G,V}) \ , \quad [\alpha : K \longrightarrow G] \longmapsto \alpha^*(u_{G,V})$$

is bijective according to Proposition I.5.16 (ii). Proposition 1.11 thus says that $\pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,V})$ is a free abelian group with a basis given by the elements

$$\mathrm{tr}_L^K(\sigma^L(\alpha^*(u_{G,V}))) = \mathrm{tr}_L^K(\alpha^*(\sigma^G(u_{G,V}))) = \mathrm{tr}_L^K(\alpha^*(e_{G,V}))$$

where L runs through all conjugacy classes of subgroups of K with finite Weyl group and α runs through a set of representatives of the $W_K L$-orbits of the set $\mathrm{Rep}(L,G)$. The claim follows because $(K \times G)$-conjugacy classes of such pairs (L,α) biject with pairs consisting of a conjugacy class of subgroups (L) and a $W_K L$-equivalence class in $\mathrm{Rep}(L,G)$. □

Example 1.14. We discuss a specific example of Corollary 1.13, with $G = A_3$ the alternating group on three letters and $K = \Sigma_3$ the symmetric group on 3 letters. The group Σ_3 has four conjugacy classes of subgroups, with representatives $\Sigma_3, A_3, (12)$ and e. The groups $\Sigma_3, (12)$ and e admit only trivial homomorphisms to A_3, whereas the alternating group A_3 also has two automorphisms. None of the three endomorphisms of A_3 are conjugate, so the set $\mathrm{Rep}(A_3, A_3)$ has three elements. However, the Weyl group $W_{\Sigma_3} A_3$ has two elements, and its action realizes the two automorphisms of A_3. So while $\pi_0^{A_3}(B_{\mathrm{gl}} A_3) \cong \mathrm{Rep}(A_3, A_3)$ has three elements, it only contributes two generators to the stable group $\pi_0^{\Sigma_3}(\Sigma^\infty_+ B_{\mathrm{gl}} A_3)$. A basis for the free abelian group $\pi_0^{\Sigma_3}(\Sigma^\infty_+ B_{\mathrm{gl}} A_3)$ is thus given by the classes

$$p^*_{\Sigma_3}(1) \ , \qquad \mathrm{tr}_{A_3}^{\Sigma_3}(e_{A_3}) \ , \qquad \mathrm{tr}_{A_3}^{\Sigma_3}(p^*_{A_3}(1)) \ , \qquad \mathrm{tr}_{(12)}^{\Sigma_3}(p^*_{(12)}(1)) \qquad \text{and} \qquad \mathrm{tr}_e^{\Sigma_3}(p^*_e(1)) \ .$$

Here the faithful A_3-representation V is unspecified and we write $e_{A_3} = e_{A_3,V} \in \pi_0^{A_3}(\Sigma^\infty_+ B_{\mathrm{gl}} A_3)$ for the stable tautological class. Moreover $p_H : H \longrightarrow e$ denotes the unique homomorphism to the trivial group and $1 = \mathrm{res}_e^{A_3}(e_{A_3})$ is the restriction of the class e_{A_3} to the trivial group.

Now we discuss how multiplicative features related to the smash product and the homotopy group pairings work out for global homotopy types. In Construction 1.6 we associated an orthogonal space $\Omega^\bullet X$ to every orthogonal spectrum X. This functor is compatible with the smash product of orthogonal spectra and the box product of orthogonal spaces, in the sense of a lax symmetric monoidal transformation

$$(\Omega^\bullet X) \boxtimes (\Omega^\bullet Y) \longrightarrow \Omega^\bullet(X \wedge Y) \tag{1.15}$$

that we review now. This morphism of orthogonal spaces is associated to a bimorphism from $(\Omega^\bullet X, \Omega^\bullet Y)$ to $\Omega^\bullet(X \wedge Y)$ with (V,W)-component the composite

$$\begin{aligned} \mathrm{map}(S^V, X(V)) \times \mathrm{map}(S^W, Y(W)) &\xrightarrow{\wedge} \mathrm{map}(S^{V\oplus W}, X(V) \wedge Y(W)) \\ &\xrightarrow{\mathrm{map}(S^{V\oplus W}, i_{V,W})} \mathrm{map}(S^{V\oplus W}, (X \wedge Y)(V \oplus W)) \ . \end{aligned}$$

The morphism is unital, associative and symmetric. Finally, the homotopy group pairing (see (5.4) of Chapter III) for $k = l = 0$ coincides with the composite

$$\begin{aligned}\pi_0^G(X) \times \pi_0^G(Y) \;=\; \pi_0^G(\Omega^\bullet X) \times \pi_0^G(\Omega^\bullet Y) \;&\xrightarrow{\times}\; \pi_0^G(\Omega^\bullet X \boxtimes \Omega^\bullet Y) \\ &\xrightarrow{(1.15)}\; \pi_0^G(\Omega^\bullet(X \wedge Y)) \;=\; \pi_0^G(X \wedge Y)\ .\end{aligned}$$

Example 1.16 (Orthogonal ring spectra and orthogonal monoid spaces). The monoidal morphism (1.15) for the functor $\Omega^\bullet$ is unital, associative and symmetric. In particular the orthogonal space $\Omega^\bullet R$ associated to an orthogonal ring spectrum becomes an orthogonal monoid space via the composite

$$(\Omega^\bullet R) \boxtimes (\Omega^\bullet R) \;\longrightarrow\; \Omega^\bullet(R \wedge R) \;\xrightarrow{\Omega^\bullet \mu}\; \Omega^\bullet R\ ,$$

and this passage respects commutativity of multiplications. The bimorphism corresponding to the induced product on $\Omega^\bullet R$ thus has as (V, W)-component the composite

$$\begin{aligned}\mathrm{map}(S^V, R(V)) \times \mathrm{map}(S^W, R(W)) \;&\xrightarrow{-\wedge-}\; \mathrm{map}(S^{V\oplus W}, R(V) \wedge R(W)) \\ &\xrightarrow{\mathrm{map}(S^{V\oplus W}, \mu_{V,W})}\; \mathrm{map}(S^{V\oplus W}, R(V \oplus W))\ .\end{aligned}$$

The suspension spectrum functor (see Construction 1.7) takes the box product of orthogonal spaces to the smash product of orthogonal spectra. In more detail: for all inner product spaces V and W the maps

$$\begin{aligned}(\Sigma^\infty_+ X)(V) \wedge (\Sigma^\infty_+ Y)(W) \;&=\; (S^V \wedge X(V)_+) \wedge (S^W \wedge Y(W)_+) \;\cong\; S^{V\oplus W} \wedge (X(V) \times Y(W))_+ \\ &\xrightarrow{S^{V\oplus W} \wedge i_{X,Y}}\; S^{V\oplus W} \wedge (X \boxtimes Y)(V \oplus W)_+ \;=\; (\Sigma^\infty_+(X \boxtimes Y))(V \oplus W)\end{aligned}$$

form a bimorphism, so they correspond to a morphism of orthogonal spectra

$$(\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y) \;\longrightarrow\; \Sigma^\infty_+(X \boxtimes Y)\ . \tag{1.17}$$

Proposition 1.18. *For all orthogonal spaces X and Y the morphisms* (1.17) *are isomorphisms. Together with the isomorphism $\Sigma^\infty_+ * \cong \mathbb{S}$ adjoint to the unique based isomorphism $*_+ \cong S^0$, this makes Σ^∞_+ into a strong symmetric monoidal functor from the category of orthogonal spaces to the category of orthogonal spectra.*

Proof. We define a morphism in the other direction. We start from the maps

$$\begin{aligned}X(V) \;&\xrightarrow{x \mapsto (-\wedge x)}\; \mathrm{map}(S^V, S^V \wedge X(V)_+) \;=\; \mathrm{map}(S^V, (\Sigma^\infty_+ X)(V)) \qquad \text{and} \\ Y(W) \;&\xrightarrow{y \mapsto (-\wedge y)}\; \mathrm{map}(S^W, S^W \wedge Y(W)_+) \;=\; \mathrm{map}(S^W, (\Sigma^\infty_+ Y)(W))\end{aligned}$$

for inner product spaces V and W; these maps are the values at V respectively W of the adjunction unit for the pair $(\Sigma^\infty_+, \Omega^\bullet)$. From these we produce the composite

$$\begin{aligned}X(V) \times Y(W) \;&\longrightarrow\; \mathrm{map}(S^V, (\Sigma^\infty_+ X)(V)) \times \mathrm{map}(S^W, (\Sigma^\infty_+ Y)(W)) \\ &\xrightarrow{\wedge}\; \mathrm{map}(S^{V\oplus W}, (\Sigma^\infty_+ X)(V) \wedge (\Sigma^\infty_+ Y)(W)) \\ &\xrightarrow{(i_{V,W})_*}\; \mathrm{map}(S^{V\oplus W}, ((\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y))(V \oplus W)) \\ &\qquad =\; \Omega^\bullet((\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y))(V \oplus W)\ .\end{aligned}$$

These maps form a bimorphism of orthogonal spaces as V and W vary through the linear isometries category $\mathbf{L}$. The universal property of the box product produces a morphism of orthogonal spaces

$$X \boxtimes Y \;\longrightarrow\; \Omega^\bullet((\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y))\ .$$

This in turn is adjoint to a morphism of orthogonal spectra $\Sigma^\infty_+(X \boxtimes Y) \longrightarrow (\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y)$. We omit the verification that this morphism is indeed inverse to the morphism (1.17). $\square$

Construction 1.19. We can thus produce examples of orthogonal ring spectra as the suspension spectra of orthogonal monoid spaces: the suspension spectrum $\Sigma^\infty_+ M$ of an orthogonal monoid space M becomes an orthogonal ring spectrum via the multiplication map

$$(\Sigma^\infty_+ M) \wedge (\Sigma^\infty_+ M) \;\cong\; \Sigma^\infty_+(M \boxtimes M) \;\xrightarrow{\Sigma^\infty_+ \mu_M}\; \Sigma^\infty_+ M \;.$$

We have $(\Sigma^\infty_+ M)^{\mathrm{op}} = \Sigma^\infty_+(M^{\mathrm{op}})$. If the multiplication of M is commutative, then so is the resulting multiplication on $\Sigma^\infty_+ M$. The functor pair $(\Sigma^\infty_+, \Omega^\bullet)$ is again an adjoint pair when viewed as functors between the categories of orthogonal monoid spaces and orthogonal ring spectra.

This construction contains spherical monoid ring spectra: if M is a topological monoid, then the constant orthogonal space with value M inherits an associative and unital product from M which is commutative if M is. The suspension spectrum with such a constant multiplicative functor is the monoid ring spectrum $\mathbb{S}[M]$.

Construction 1.20. We consider two orthogonal spectra X and Y. In the global context, the equivariant homotopy groups also support inflation maps, which we can use to define another pairing

$$\boxtimes \;:\; \pi^G_k(X) \;\times\; \pi^K_l(Y) \;\longrightarrow\; \pi^{G\times K}_{k+l}(X \wedge Y) \;. \tag{1.21}$$

Here G and K are compact Lie groups and $k, l \in \mathbb{Z}$. The definition of this pairing is simply as the composite

$$\pi^G_k(X) \times \pi^K_l(Y) \;\xrightarrow{p_G^* \times p_K^*}\; \pi^{G\times K}_k(X) \times \pi^{G\times K}_l(Y) \;\xrightarrow{\times}\; \pi^{G\times K}_{k+l}(X \wedge Y) \;,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the projections to the two factors. Theorem III.5.5 (i) and the additivity of inflation maps imply that this pairing is biadditive, and it satisfies certain associativity, commutativity and restriction properties. We do take the time to spell out the most important properties of these pairings in the next theorem.

As we explain in Example 2.20 below, the $\boxtimes$-pairings form a bimorphism of global functors as G and K vary through all compact Lie groups. Moreover, the passage from the 'diagonal' pairings to the 'external' pairings (1.21) can be reversed by taking $K = G$ and restricting to the diagonal; suitably formalized, diagonal and external pairing contains the same amount of information. We refer the reader to Remark 2.25 below for more details.

Theorem 1.22. *Let G, K and L be compact Lie groups and X, Y and Z orthogonal spectra.*

(i) (Biadditivity) *The product $\boxtimes : \pi^G_k(X) \times \pi^K_l(Y) \longrightarrow \pi^{G\times K}_{k+l}(X \wedge Y)$ is biadditive.*

(ii) (Unitality) *Let $1 \in \pi^e_0(\mathbb{S})$ denote the class represented by the identity of S^0. The product is unital in the sense that $1 \boxtimes x = x = x \boxtimes 1$ under the identifications $\mathbb{S} \wedge X = X = X \wedge \mathbb{S}$ and $e \times G \cong G \cong G \times e$.*

(iii) (Associativity) *For all classes $x \in \pi^G_k(X)$, $y \in \pi^K_l(Y)$ and $z \in \pi^L_m(Z)$ the relation*

$$x \boxtimes (y \boxtimes z) \;=\; (x \boxtimes y) \boxtimes z$$

holds in $\pi^{G\times K\times L}_{k+l+m}(X \wedge Y \wedge Z)$.

(iv) (Commutativity) *For all classes $x \in \pi^G_k(X)$ and $y \in \pi^K_l(Y)$ the relation*

$$\tau^{X,Y}_*(x \boxtimes y) \;=\; (-1)^{kl} \cdot \tau^*_{G,K}(y \boxtimes x)$$

holds in $\pi^{G\times K}_{l+k}(Y \wedge X)$, where $\tau^{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ is the symmetry isomorphism of the smash product and $\tau_{G,K} : G \times K \longrightarrow K \times G$ interchanges the factors.

(v) (Restriction) *For all classes $x \in \pi^G_k(X)$ and $y \in \pi^K_l(Y)$ and all continuous homomorphisms $\alpha : \bar{G} \longrightarrow G$ and $\beta : \bar{K} \longrightarrow K$ the relation*

$$\alpha^*(x) \boxtimes \beta^*(y) \;=\; (\alpha \boxtimes \beta)^*(x \boxtimes y)$$

holds in $\pi^{\bar{G}\times\bar{K}}_{k+l}(X \wedge Y)$.

(vi) (Transfer) *For all closed subgroup inclusions $H \leq G$ and $L \leq K$ the square*

$$\begin{tikzcd} \pi_k^H(X) \times \pi_l^L(Y) \arrow[r, "\boxtimes"] \arrow[d, "\mathrm{tr}_H^G \times \mathrm{tr}_L^K"'] & \pi_{k+l}^{H\times L}(X \wedge Y) \arrow[d, "\mathrm{tr}_{H\times L}^{G\times K}"] \\ \pi_k^G(X) \times \pi_l^K(Y) \arrow[r, "\boxtimes"'] & \pi_{k+l}^{G\times K}(X \wedge Y) \end{tikzcd}$$

commutes.

Proof. Parts (i) through (v) follow from the respective parts of Theorem III.5.5 by naturality arguments.

(vi) We start with two special cases, namely $L = K$ respectively $H = G$. The two proofs are analogous, so we only treat the case $L = K$. Indeed:

$$\begin{aligned} \mathrm{tr}_{H\times K}^{G\times K}(x \boxtimes y) &= \mathrm{tr}_{H\times K}^{G\times K}(p_H^*(x) \times p_K^*(y)) = \mathrm{tr}_{H\times K}^{G\times K}(p_H^*(x) \times \mathrm{res}_{H\times K}^{G\times K}(p_K^*(y))) \\ &= \mathrm{tr}_{H\times K}^{G\times K}(p_H^*(x)) \times p_K^*(y) = p_G^*(\mathrm{tr}_H^G(x)) \times p_K^*(y) = \mathrm{tr}_H^G(x) \boxtimes y \end{aligned}$$

We slightly abuse notation by writing p_K for both the projections of $H \times K$ and of $G \times K$ to K. The third equation is the reciprocity relation of Theorem III.5.5 (vi). The fourth equation is the compatibility of transfers with inflations (Proposition III.2.30 (ii)).

The general case is now obtained by combining the two special cases:

$$\mathrm{tr}_H^G(x) \boxtimes \mathrm{tr}_L^K(y) = \mathrm{tr}_{H\times K}^{G\times K}(x \boxtimes \mathrm{tr}_L^K(y)) = \mathrm{tr}_{H\times K}^{G\times K}(\mathrm{tr}_{H\times L}^{H\times K}(x \boxtimes y)) = \mathrm{tr}_{H\times L}^{G\times K}(x \boxtimes y) \qquad \square$$

For all orthogonal spectra (i.e., all global homotopy types), the collection of equivariant homotopy groups $\{\pi_0^G(X)\}_G$ come with restriction and transfer maps, and this data together forms a 'global functor', compare Definition 2.2 below. The geometric fixed point groups have fewer natural operations, and they do *not* allow restriction to subgroups. However, geometric fixed points still have inflation maps, i.e., restriction maps along epimorphisms. Indeed, in Construction III.3.5 we defined inflation maps on geometric fixed point homotopy groups, associated to a continuous epimorphism $\alpha : K \longrightarrow G$ between compact Lie groups. When X is an orthogonal spectrum, representing a global homotopy type, then $\alpha^*(X_G) = X_K$, and the inflation maps become homomorphisms

$$\alpha^* : \Phi_0^G(X) \longrightarrow \Phi_0^K(X) .$$

These inflation maps between the geometric fixed point homotopy groups are clearly natural in the orthogonal spectrum. The next proposition specializes the naturality properties of Proposition III.3.6 to global homotopy types. They can be summarized by saying that geometric fixed point homotopy groups and inflation maps form a contravariant functor on the category of compact Lie groups and conjugacy classes of continuous epimorphisms, and the geometric fixed point map (3.3) is a natural transformation of functors on this category.

Proposition 1.23. *Let X be an orthogonal spectrum.*

(i) *For every pair of composable continuous epimorphisms $\alpha : K \longrightarrow G$ and $\beta : L \longrightarrow K$ we have*

$$\beta^* \circ \alpha^* = (\alpha\beta)^* : \Phi_0^G(X) \longrightarrow \Phi_0^L(X) .$$

(ii) *For every compact Lie group G and every element $g \in G$ the restriction map $c_g^* : \Phi_0^G(X) \longrightarrow \Phi_0^G(X)$ associated to the conjugation homomorphism $c_g : G \longrightarrow G$ is the identity.*

(iii) *For every continuous epimorphism $\alpha : K \longrightarrow G$ of compact Lie groups the following square commutes:*

$$\begin{array}{ccc} \pi_0^G(X) & \xrightarrow{\Phi} & \Phi_0^G(X) \\ {\scriptstyle \alpha^*}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ \pi_0^K(X) & \xrightarrow[\Phi]{} & \Phi_0^K(X) \end{array}$$

(iv) *For all classes $x \in \Phi_k^G(X)$ and $y \in \Phi_l^G(Y)$ and all continuous epimorphisms $\alpha : K \longrightarrow G$ the relation*

$$\alpha^*(x) \times \alpha^*(y) \;=\; \alpha^*(x \times y)$$

holds in $\Phi_{k+l}^K(X \wedge Y)$.

(v) *Let R be an orthogonal ring spectrum. Then for all continuous epimorphisms $\alpha : K \longrightarrow G$ the inflation map $\alpha^* : \Phi_*^G(R) \longrightarrow \Phi_*^K(R)$ is a homomorphism of graded rings.*

Construction 1.24 (Free orthogonal spectra). Given a compact Lie group G and a G-representation V, the functor

$$\mathrm{ev}_{G,V} \;:\; \mathcal{S}p \;\longrightarrow\; G\mathbf{T}_*$$

that sends an orthogonal spectrum X to the based G-space $X(V)$ has a left adjoint

$$F_{G,V} \;:\; G\mathbf{T}_* \;\longrightarrow\; \mathcal{S}p\ . \tag{1.25}$$

The free orthogonal spectrum $F_{G,V}A$ generated by a based G-space A in level V is

$$F_{G,V}A \;=\; \mathbf{O}(V,-) \wedge_G A\ ;$$

the value at an inner product space W is thus given by

$$(F_{G,V}A)(W) \;=\; \mathbf{O}(V,W) \wedge_G A\ .$$

We note that $F_{G,V}A$ consists of a single point in all levels below the dimension of V. The 'freeness' property of $F_{G,V}A$ means that for every orthogonal spectrum X and every based G-map $f : A \longrightarrow X(V)$ there is a unique morphism $\hat{f} : F_{G,V}A \longrightarrow X$ of orthogonal spectra such that the composite

$$A \xrightarrow{\mathrm{Id}\,\wedge -} \mathbf{O}(V,V) \wedge_G A = (F_{G,V}A)(V) \xrightarrow{\hat{f}(V)} X(V)$$

is f. Indeed, the morphism $\hat{f}(W)$ is the composite

$$\mathbf{O}(V,W) \wedge_G A \xrightarrow{\mathrm{Id}\,\wedge f} \mathbf{O}(V,W) \wedge_G X(V) \xrightarrow{\mathrm{act}} X(W)\ .$$

Remark 1.26 (Free spectra as global Thom spectra). The underlying non-equivariant stable homotopy type of $F_{G,V}$ is the Thom spectrum of the negative of the bundle

$$\mathbf{L}(V,\mathbb{R}^\infty) \times_G V \;\longrightarrow\; \mathbf{L}(V,\mathbb{R}^\infty)/G \;=\; BG\ ,$$

the vector bundle over BG associated to the G-representation V. So one should think of the free orthogonal spectrum $F_{G,V} = F_{G,V}S^0$ as the 'global Thom spectrum' associated to a 'virtual global vector bundle', namely the negative of the vector bundle over $B_{\mathrm{gl}}G$ associated to the G-representation V.

The special case $G = O(m)$ of the orthogonal group with $V = \nu_m$, i.e., $\mathbb{R}^m$ with the tautological $O(m)$-action will feature prominently in the rank filtration of the global Thom spectrum $\mathbf{mO}$ in Section VI.2. Non-equivariantly, $F_{O(m),\nu_m}$ is the Thom spectrum of the negative of the tautological m-plane bundle over the Grassmannian $Gr_m(\mathbb{R}^\infty)$; the traditional notation for this Thom spectrum is $MTO(m)$ or simply $MT(m)$. Indeed,

$$F_{O(m),\nu_m}(\mathbb{R}^{m+n}) \;=\; \mathbf{O}(\mathbb{R}^m,\mathbb{R}^{m+n})/O(m)$$

is the Thom space of the orthogonal complement of the tautological m-plane bundle over $\mathbf{L}(\mathbb{R}^m, \mathbb{R}^{m+n})/O(m) = Gr_m(\mathbb{R}^{m+n})$, and this is precisely the $(m+n)$-th space of $MTO(m)$, see for example [**56**, Sec. 3.1]. Similarly, the non-equivariant homotopy type underlying $F_{SO(m),\nu_m}$ is an oriented version of $MTO(m)$, which is usually denoted $MTSO(m)$ or sometimes $MT(m)^+$.

Example 1.27 (Smash product of free orthogonal spectra). The smash product of two free orthogonal spectra is again a free orthogonal spectrum. In more detail, we consider

- two compact Lie groups G and K,
- a G-representation V and a K-representation W, and
- a based G-space A and a based K-space B.

Then $V \oplus W$ is a $(G \times K)$-representation and $A \wedge B$ is a $(G \times K)$-space via

$$(g,k) \cdot (v,w) \;=\; (gv, kw) \qquad \text{respectively} \qquad (g,k) \cdot (a \wedge b) \;=\; ga \wedge kb \;.$$

We claim that the smash product $(F_{G,V}A) \wedge (F_{K,W}B)$ is canonically isomorphic to the free orthogonal spectrum generated by the $(G \times K)$-space $A \wedge B$ in level $V \oplus W$. Indeed, a morphism

$$(F_{G,V}A) \wedge (F_{K,W}B) \;\longrightarrow\; F_{G\times K, V\oplus W}(A \wedge B) \tag{1.28}$$

is obtained by the universal property (see (5.2) of Chapter III) from the bimorphism with (U,U')-component

$$\begin{aligned}(F_{G,V}A)(U) \wedge (F_{K,W}B)(U') \;&=\; (\mathbf{O}(V,U) \wedge_G A) \wedge (\mathbf{O}(W,U') \wedge_K B) \\ &\xrightarrow{\oplus}\; \mathbf{O}(V \oplus W, U \oplus U') \wedge_{G\times K} (A \wedge B) \;=\; ((F_{G\times K, V\oplus W})(A \wedge B))(U \oplus U') \;.\end{aligned}$$

In the other direction, a morphism $F_{G\times K, V\oplus W}(A \wedge B) \longrightarrow F_{G,V}A \wedge F_{G,W}B$ is freely generated by the $(G \times K)$-map

$$A \wedge B \;\longrightarrow\; (F_{G,V}A)(V) \wedge (F_{K,W}B)(W) \;\xrightarrow{i_{V,W}}\; (F_{G,V}A \wedge F_{K,W}B)(V \oplus W) \;.$$

These two maps are inverse to each other.

In Proposition I.2.10 (ii) we have seen that for every compact Lie group G, every G-space A, every G-representation V and every faithful G-representation W the restriction morphism of orthogonal spaces $\rho_{V,W} \times_G A : \mathbf{L}_{G,V\oplus W}A \longrightarrow \mathbf{L}_{G,W}A$ is a global equivalence. One consequence is that the free orthogonal space $\mathbf{L}_{G.W}$ has a well-defined unstable global homotopy type, independent of which faithful G-representation is used. Another consequence is that the induced morphism

$$\Sigma^\infty_+ \rho_{V,W} \times_G A \;:\; \Sigma^\infty_+ \mathbf{L}_{G,V\oplus W}A \;\longrightarrow\; \Sigma^\infty_+ \mathbf{L}_{G,W}A$$

of suspension spectra is a global equivalence of orthogonal spectra, by Corollary 1.9. For an inner product space W, the untwisting homeomorphisms (see (1.2) of Chapter III) descend to homeomorphisms on G-orbit spaces

$$\mathbf{O}(V,W) \wedge_G S^V \;\cong\; S^W \wedge \mathbf{L}(V,W)/G_+ \;.$$

As W varies, these form another 'untwisting isomorphism', an isomorphism of orthogonal spectra

$$F_{G,V}S^V \;\cong\; \Sigma^\infty_+ \mathbf{L}_{G,V} \;. \tag{1.29}$$

So suspension spectra of free orthogonal spaces are free orthogonal spectra. We will now prove a generalization of the fact that $\Sigma^\infty_+ \rho_{V,W}/G$ is a global equivalence for these global Thom spectra. Given G-representations V and W, we define a restriction morphism of orthogonal spectra

$$\lambda_{G,V,W} \;:\; F_{G,V\oplus W}S^V \;\longrightarrow\; F_{G,W} \tag{1.30}$$

as the adjoint of the based G-map

$$S^V \;\longrightarrow\; \mathbf{O}(W, V \oplus W)/G \;=\; F_{G,W}(V \oplus W) \;, \quad v \;\longmapsto\; ((v,0),i) \cdot G \;,$$

where $i : W \longrightarrow V \oplus W$ is the embedding of the second summand. The value of $\lambda_{G,V,W}$ at an inner product space U is then

$$\begin{array}{rcl} \lambda_{G,V,W}(U) \ : \ \mathbf{O}(V \oplus W, U) \wedge_G S^V & \longrightarrow & \mathbf{O}(W,U)/G \\ [(u,\varphi) \wedge v] & \longmapsto & (u + \varphi(v), \varphi \circ i) \cdot G \ . \end{array}$$

THEOREM 1.31. *Let G be a compact Lie group, V a G-representation and W a faithful G-representation. Then the morphism*

$$\lambda_{G,V,W} \ : \ F_{G,V\oplus W} S^V \ \longrightarrow \ F_{G,W}$$

is a global equivalence of orthogonal spectra.

PROOF. To simplify the notation we abbreviate the restriction morphism of orthogonal spaces to

$$\rho \ = \ \rho_{V,W} \ : \ \mathbf{L}(V \oplus W, -) \ \longrightarrow \ \mathbf{L}(W, -) \ .$$

We let K be another compact Lie group and $U \in s(\mathcal{U}_K)$ a finite dimensional K-subrepresentation of the complete K-universe $\mathcal{U}_K$. In a first step we produce a K-representation $U' \in s(\mathcal{U}_K)$ with $U \subseteq U'$ and a continuous $(K \times G)$-equivariant map

$$h \ : \ \mathbf{L}(W,U) \ \longrightarrow \ \mathbf{L}(V \oplus W, U')$$

such that in the diagram

$$\begin{tikzcd} \mathbf{L}(V \oplus W, U) \arrow[r, "i_*"] \arrow[d, "\rho(U)"'] & \mathbf{L}(V \oplus W, U') \arrow[d, "\rho(U')"] \\ \mathbf{L}(W,U) \arrow[r, "i_*"'] \arrow[ur, dashed, "h"] & \mathbf{L}(W,U') \end{tikzcd} \tag{1.32}$$

the lower right triangle commutes, and the upper left triangle commutes up to $(K \times G)$-equivariant fiberwise homotopy over $\mathbf{L}(W,U')$, where $i : U \longrightarrow U'$ is the inclusion.

Since G acts faithfully on W (and hence on $V \oplus W$), both $\mathbf{L}(W,\mathcal{U}_K)$ and $\mathbf{L}(V \oplus W, \mathcal{U}_K)$ are universal spaces for the same family of subgroups of $K \times G$, namely the family $\mathcal{F}(K;G)$ of graph subgroups, compare Proposition I.2.10 (i). Moreover, if Γ is the graph of a continuous homomorphism $\alpha : L \longrightarrow G$ defined on some closed subgroup L of K, then the Γ-fixed points of $\mathbf{L}(W,\mathcal{U}_K)$ are then given by

$$\mathbf{L}(W,\mathcal{U}_K)^\Gamma \ = \ \mathbf{L}^L(\alpha^* W, \mathrm{res}^K_L(\mathcal{U}_K)) \ ,$$

the space of L-equivariant linear isometric embeddings from $\alpha^* W$. The same is true for $V \oplus W$ instead of W, and so the Γ-fixed point map $\rho(\mathcal{U}_K)^\Gamma : \mathbf{L}(V \oplus W, \mathcal{U}_K)^\Gamma \longrightarrow \mathbf{L}(W,\mathcal{U}_K)^\Gamma$ is the restriction map

$$\mathbf{L}^L(\alpha^* V \oplus \alpha^* W, \mathrm{res}^K_L(\mathcal{U}_K)) \ \longrightarrow \ \mathbf{L}^L(\alpha^* W, \mathrm{res}^K_L(\mathcal{U}_K))$$

to the summand $\alpha^* W$. This map is a locally trivial fiber bundle, hence a Serre fibration. We conclude that the restriction map $\rho(\mathcal{U}_K) : \mathbf{L}(V \oplus W, \mathcal{U}_K) \longrightarrow \mathbf{L}(W,\mathcal{U}_K)$ is both a $(K \times G)$-weak equivalence and a $(K \times G)$-fibration.

Since $\mathbf{L}(W,U)$ is cofibrant as a $(K \times G)$-space (by Proposition I.2.2 (ii)), the $(K \times G)$-map $i_* : \mathbf{L}(W,U) \longrightarrow \mathbf{L}(W,\mathcal{U}_K)$ thus admits a $(K \times G)$-equivariant lift $h : \mathbf{L}(W,U) \longrightarrow \mathbf{L}(V \oplus W, \mathcal{U}_K)$ such that $\rho(\mathcal{U}_K) \circ h = i_*$. Since the space $\mathbf{L}(W,U)$ is compact and $\mathbf{L}(V \oplus W, \mathcal{U}_K)$ is the filtered union of the closed subspaces $\mathbf{L}(V \oplus W, U')$ for $U' \in s(\mathcal{U}_K)$, the lift h lands in the subspace $\mathbf{L}(V \oplus W, U')$ for suitably large $U' \in s(\mathcal{U}_K)$, and we may assume that $U \subseteq U'$.

The two maps

$$h \circ \rho(U) \ , \ i_* \ : \ \mathbf{L}(V \oplus W, U) \ \longrightarrow \ \mathbf{L}(V \oplus W, U')$$

become equal after applying $\rho(U') : \mathbf{L}(V \oplus W, U') \longrightarrow \mathbf{L}(V, U')$, hence the composites with $i_* : \mathbf{L}(V \oplus W, U') \longrightarrow \mathbf{L}(V \oplus W, \mathcal{U}_K)$ become equal after applying the $(K \times G)$-equivariant acyclic fibration $\rho(\mathcal{U}_K)$:

$\mathbf{L}(V \oplus W, \mathcal{U}_K) \longrightarrow \mathbf{L}(W, \mathcal{U}_K)$. Since $\mathbf{L}(V \oplus W, U)$ is also $(K \times G)$-cofibrant, there is a fiberwise $(K \times G)$-equivariant homotopy between $h \circ \rho(U)$ and i_* in $\mathbf{L}(V \oplus W, \mathcal{U}_K)$. Again by compactness, the homotopy has image in $\mathbf{L}(V \oplus W, U'')$ for suitably large $U'' \in s(\mathcal{U}_K)$. So after increasing U', if necessary, we have proved the claim subsumed in the diagram (1.32).

Now we lift the data produced in the first step to the Thom spaces of the orthogonal complement bundles. The diagram (1.32) is covered by morphisms of $(K \times G)$-vector bundles:

$$\begin{tikzcd} (U'-U) \times \xi(V \oplus W, U) \times V \arrow[r, "\bar{i}"] \arrow[d, "(U'-U)\times\bar{\rho}(U)"'] & \xi(V \oplus W, U') \times V \arrow[d, "\bar{\rho}(U')"] \\ (U'-U) \times \xi(W, U) \arrow[r, "\bar{i}"'] \arrow[ur, dashed, "\bar{h}"] & \xi(W, U') \end{tikzcd} \tag{1.33}$$

The maps on the total spaces of the bundles are defined as follows: The right vertical morphism is defined by

$$\bar{\rho}(U') \ : \ \xi(V \oplus W, U') \times V \ \longrightarrow \ \xi(W, U') \ , \quad ((u', \varphi), \, v) \ \longmapsto \ (u' + \varphi(v), \, \varphi|_W) \ .$$

The morphism $\bar{\rho}(U)$ is defined in the same way. The lower horizontal morphism is defined by

$$\bar{i} \ : \ (U'-U) \times \xi(W, U) \ \longrightarrow \ \xi(W, U') \ , \quad (u', (u, \varphi)) \ \longmapsto \ (u' + u, \, \varphi) \ .$$

The upper horizontal morphism is defined in the same way, but with $V \oplus W$ instead of W and multiplied by the identity of V. These four outer morphisms in (1.33) are all fiberwise linear isomorphisms; so each of these four bundle maps expresses the source bundle as a pullback of the target bundle. In particular, the square

$$\begin{tikzcd} \xi(V \oplus W, U') \times V \arrow[r, "\bar{\rho}(U')"] \arrow[d] & \xi(W, U') \arrow[d] \\ \mathbf{L}(V \oplus W, U') \arrow[r, "\rho(U')"'] & \mathbf{L}(W, U') \end{tikzcd} \tag{1.34}$$

is a pullback; so the composite

$$(U'-U) \times \xi(W, U) \ \longrightarrow \ \mathbf{L}(W, U) \ \xrightarrow{h} \ \mathbf{L}(V \oplus W, U')$$

and the map of total spaces $\bar{i} : U)(U'-U) \times \xi(W, U) \longrightarrow \xi(W, U')$ assemble into a map

$$\bar{h} \ : \ (U'-U) \times \xi(W, U) \ \longrightarrow \ \xi(V \oplus W, U')$$

that covers h and is a fiberwise linear isomorphism.

In (1.33) (as in (1.32)) the outer square and the lower right triangle commute, but the upper left triangle does *not* commute. We will now show that the upper left triangle commutes up to homotopy of $(K \times G)$-equivariant bundle maps. For this purpose we let

$$H \ : \ \mathbf{L}(V \oplus W, U) \times [0,1] \ \longrightarrow \ \mathbf{L}(V \oplus W, U')$$

be a $(K \times G)$-equivariant homotopy from the map i_* to $h \circ \rho(U)$, such that $\rho(U') \circ H : \mathbf{L}(V \oplus W, U) \times [0,1] \longrightarrow \mathbf{L}(W, U')$ is the constant homotopy from $\rho(U') \circ i_* = i_* \circ \rho(U)$ to itself. Again because the square (1.34) is a pullback, the composite

$$\xi(V \oplus W, U) \times V \times (U'-U) \times [0,1] \ \longrightarrow \ \mathbf{L}(V \oplus W, U) \times [0,1] \ \xrightarrow{H} \ \mathbf{L}(V \oplus W, U')$$

and the map of total spaces

$$(U'-U) \times \xi(V \oplus W, U) \times V \times [0,1] \ \xrightarrow{\text{proj}} \ (U'-U) \times \xi(V \oplus W, U) \times V \ \xrightarrow{\bar{\rho}(U') \circ \bar{i} = \bar{i} \circ \bar{\rho}(U)} \ \xi(W, U')$$

assemble into a map

$$\bar{H} \ : \ (U'-U)\times\xi(V\oplus W,U)\times V\times[0,1] \ \longrightarrow \ \xi(V\oplus W,U')\times V$$

that covers the homotopy H. This lift $\bar{H}$ is a $(K\times G)$-equivariant homotopy of vector bundle morphisms, and for every $t\in[0,1]$, the relation

$$\bar{\rho}(U')\circ\bar{H}(-,t) \ = \ \bar{\rho}(U')\circ\bar{i} \ = \ \bar{\rho}(U')\circ(\bar{h}\circ\bar{\rho}(U))$$

holds by definition of $\bar{H}$. For $t=0$ this shows that $\bar{H}$ starts with $\bar{i}:(U'-U)\times\xi(V\oplus W,U)\times V\longrightarrow\xi(V\oplus W,U')\times V$; for $t=1$ this shows that $\bar{H}$ ends with $\bar{h}\circ\bar{\rho}(U)$, one more time because (1.34) is a pullback. We conclude that $\bar{H}$ makes the upper left triangle in (1.33) commute up to equivariant homotopy of vector bundle maps.

Passing to Thom spaces in (1.33) gives a diagram of $(K\times G)$-equivariant based maps:

$$\begin{tikzcd}
S^{U'-U}\wedge\mathbf{O}(V\oplus W,U)\wedge S^V \arrow[r, "\sigma_{U,U'-U}\wedge S^V"] \arrow[d, "S^{U'-U}\wedge\lambda_{V,W}(U)"'] & \mathbf{O}(V\oplus W,U')\wedge S^V \arrow[d, "\lambda_{V,W}(U')"] \\
S^{U'-U}\wedge\mathbf{O}(W,U) \arrow[r, "\sigma_{U,U'-U}"'] \arrow[ur, dashed, "\bar{h}"] & \mathbf{O}(W,U')
\end{tikzcd}$$

Again, the lower right triangle commutes, and the upper left triangle commutes up to $(K\times G)$-equivariant based homotopy. We pass to G-orbit spaces and obtain a diagram of based G-spaces

$$\begin{tikzcd}
S^{U'-U}\wedge(F_{G,V\oplus W}S^V)(U) \arrow[r, "\sigma_{U,U'-U}"] \arrow[d, "S^{U'-U}\wedge\lambda_{G,V,W}(U)"'] & (F_{G,V\oplus W}S^V)(U') \arrow[d, "\lambda_{G,V,W}(U')"] \\
S^{U'-U}\wedge F_{G,W}(U) \arrow[r, "\sigma_{U,U'-U}"'] \arrow[ur, "\bar{h}/G"] & F_{G,W}(U')
\end{tikzcd}$$

whose lower right triangle commutes, and whose upper left triangle commutes up to K-equivariant based homotopy. Since we had started with an arbitrary K-subrepresentation $U\in s(\mathcal{U}_K)$, this implies that for every based K-space A and G-representation $\bar{U}$ the map on colimits

$$\begin{aligned}
&\operatorname{colim}_{U\in s(\mathcal{U}_K)}[S^U\wedge A,(F_{G,V\oplus W}S^V)(U\oplus\bar{U})]^K \\
&\qquad\longrightarrow\quad \operatorname{colim}_{U\in s(\mathcal{U}_K)}[S^U\wedge A,F_{G,W}(U\oplus\bar{U})]^K
\end{aligned}$$

induced by the morphism $\lambda_{G,V,W}$ is bijective. For $A=\mathbb{S}^k$ and $U=0$ this shows that $\pi_k^G(\lambda_{G,V,W})$ is an isomorphism. For $A=S^0$ and $U=\mathbb{R}^k$ this shows that $\pi_{-k}^G(\lambda_{G,V,W})$ is an isomorphism. So $\lambda_{G,V,W}$ is a global equivalence. □

2. Global functors

This section is devoted to the category $\mathcal{GF}$ of *global functors*, the natural home of the collection of equivariant homotopy groups $\{\pi_0^G(X)\}$ of a global stable homotopy type, i.e., an orthogonal spectrum. The category $\mathcal{GF}$ of global functors is a symmetric monoidal abelian category with enough injectives and projectives that plays the same role for global homotopy theory that is played by the category of abelian groups in ordinary homotopy theory, or by the category of G-Mackey functors for G-equivariant homotopy theory.

An abstract way to motivate the appearance of global functors is as follows. One can consider the globally connective (respectively globally coconnective) orthogonal spectra, i.e., those where all equivariant homotopy groups vanish in negative dimensions (respectively in positive dimensions). It turns out that the

full subcategories of globally connective respectively globally coconnective spectra define a non-degenerate t-structure on the triangulated global stable homotopy category, and the heart of this t-structure is (equivalent to) the abelian category of global functors; we refer the reader to Corollary 4.11 below for details.

We introduce global functors in Definition 2.2 as additive functors on the *global Burnside category*, the category of natural operations between equivariant stable homotopy groups. The abstract definition ensures that equivariant homotopy groups of orthogonal spectra are tautologically global functors. A key result is Theorem 2.6 that describes explicit bases of the morphism groups of the global Burnside category in terms of transfers and restriction operations. This calculation is the key to comparing our notion of global functors to other kinds of global Mackey functors, as well as for all concrete calculations with global functors. Example 2.8 lists interesting examples of global functors: the Burnside ring global functor, represented global functors, constant global functors, the representation ring global functor, and Borel type global functors. Many more examples of global functors are discussed in the remaining sections of this book.

In the global context, the pairing of equivariant homotopy groups give rise to a symmetric monoidal structure on the global Burnside category, compare Theorem 2.16. Hence the abelian category of global functors can also be endowed with a Day type convolution product, the box product of global functors, see Construction 2.18.

Construction 2.1 (Burnside category). We define the pre-additive *Burnside category* $\mathbf{A}$. The objects of $\mathbf{A}$ are all compact Lie groups; morphisms from a group G to K are defined as

$$\mathbf{A}(G,K) \;=\; \mathrm{Nat}(\pi_0^G, \pi_0^K)\ ,$$

the set of natural transformations of functors, from orthogonal spectra to sets, between the equivariant homotopy group functors π_0^G and π_0^K. Composition in the category $\mathbf{A}$ is composition of natural transformations.

It is not a priori clear that the natural transformations from π_0^G to π_0^K form a set (as opposed to a proper class), but this follows from Proposition 2.5 below. The Burnside category $\mathbf{A}$ is skeletally small: isomorphic compact Lie groups are also isomorphic as objects in the category $\mathbf{A}$, and every compact Lie group is isomorphic to a closed subgroup of an orthogonal group $O(n)$. The functor π_0^K is abelian group valued, so the set $\mathbf{A}(G,K)$ is an abelian group under objectwise addition of transformations. Proposition II.2.13, applied to the category of orthogonal spectra, shows that set valued natural transformations between the two reduced additive functors π_0^G and π_0^K are automatically additive. So composition in the Burnside category is additive in each variable, and $\mathbf{A}$ is indeed a pre-additive category.

Definition 2.2. A *global functor* is an additive functor from the Burnside category $\mathbf{A}$ to the category of abelian groups. A morphism of global functors is a natural transformation.

We discuss various explicit examples of interesting global functors in Example 2.8.

Example 2.3. The definition of the Burnside category $\mathbf{A}$ is made so that the collection of equivariant homotopy groups of an orthogonal spectrum is tautologically a global functor. Explicitly, the global homotopy group functor $\underline{\pi}_0(X)$ of an orthogonal spectrum X is defined on objects by

$$(\underline{\pi}_0(X))(G) \;=\; \pi_0^G(X)$$

and on morphisms by evaluating natural transformations at X. It is less obvious that conversely every global functor is isomorphic to the global functor $\underline{\pi}_0(X)$ of some orthogonal spectrum X; we refer the reader to Remark 4.13 below for the construction of Eilenberg-Mac Lane spectra from global functors.

As a category of additive functors out of a skeletally small pre-additive category, the category $\mathcal{GF}$ of global functors has some immediate properties that we collect in the following proposition.

PROPOSITION 2.4. *The category $\mathcal{GF}$ of global functors is an abelian category with enough injectives and projectives.*

PROOF. Any category of additive functors out of a skeletally small additive category is abelian with notions of exactness detected objectwise. A set of projective generators is given by the represented global functors $\mathbf{A}(G,-)$ where G runs through a set of representatives of the isomorphism classes of compact Lie groups. A set of injective cogenerators is given similarly by the global functors

$$\mathrm{Hom}(\mathbf{A}(-,K),\mathbb{Q}/\mathbb{Z}) \ : \ \mathbf{A} \ \longrightarrow \ \mathcal{A}b \ ,$$

where K runs through a set of representatives of the isomorphism classes of compact Lie groups. □

As we shall explain in Construction 2.18 below, the category $\mathcal{GF}$ has a closed symmetric monoidal product $\square$ that arises as a convolution product for a certain symmetric monoidal structure on the Burnside category $\mathbf{A}$.

Our definition of the Burnside category is made so that every orthogonal spectrum X gives rise to a homotopy group global functor without further ado, but it is not clear from the definition how to describe the morphism groups of $\mathbf{A}$ explicitly. Our next aim is to show that each morphism group $\mathbf{A}(G,K)$ is a free abelian group with an explicit basis given by certain composites of a restriction and a transfer morphism. This calculation has two ingredients: We identify natural transformations from π_0^G to π_0^K with the group $\pi_0^K(\Sigma^\infty_+ B_{\mathrm{gl}}G)$, and then we exploit the explicit calculation of the latter group in Corollary 1.13.

PROPOSITION 2.5. *Let G and K be compact Lie groups and V a faithful G-representation. Then evaluation at the stable tautological class is a bijection*

$$\mathbf{A}(G,K) \ = \ \mathrm{Nat}^{\mathcal{S}p}(\pi_0^G,\pi_0^K) \ \longrightarrow \ \pi_0^K(\Sigma^\infty_+\mathbf{L}_{G,V}) \ , \quad \tau \ \longmapsto \ \tau(e_{G,V})$$

to the 0-th K-equivariant homotopy group of the orthogonal spectrum $\Sigma^\infty_+\mathbf{L}_{G,V}$. In other words, the morphism

$$\mathbf{A}(G,-) \ \longrightarrow \ \underline{\pi}_0(\Sigma^\infty_+\mathbf{L}_{G,V})$$

classified by the stable tautological class $e_{G,V}$ is an isomorphism of global functors.

PROOF. To show that the evaluation map is injective we show that any natural transformation $\tau : \pi_0^G \longrightarrow \pi_0^K$ is determined by the element $\tau(e_{G,V})$. We let X be any orthogonal spectrum and $x \in \pi_0^G(X)$ a G-equivariant homotopy class. The class x is represented by a continuous based G-map

$$f \ : \ S^W \ \longrightarrow \ X(W)$$

for some G-representation W. We can stabilize with the representation V and obtain another representative

$$V \diamond f \ : \ S^{V\oplus W} \ \longrightarrow \ X(V\oplus W)$$

for the class x. This G-map is adjoint to a morphism of orthogonal spectra

$$\hat{f} \ : \ \Sigma^\infty_+\mathbf{L}_{G,V\oplus W} \ \longrightarrow \ X$$

that satisfies

$$\hat{f}_*(e_{G,V\oplus W}) \ = \ x \quad \text{in } \pi_0^G(X) \ .$$

The restriction morphism of orthogonal spaces

$$\rho_{G,V,W} \ : \ \mathbf{L}_{G,V\oplus W} \ \longrightarrow \ \mathbf{L}_{G,V}$$

is a global equivalence (by Proposition I.2.10) and sends $u_{G,V\oplus W}$ to $u_{G,V}$. The induced morphism of suspension spectra

$$\Sigma^\infty_+\rho_{G,V,W} \ : \ \Sigma^\infty_+\mathbf{L}_{G,V\oplus W} \ \longrightarrow \ \Sigma^\infty_+\mathbf{L}_{G,V}$$

is thus a global equivalence of orthogonal spectra (by Corollary 1.9), and it sends $e_{G,W\oplus V}$ to $e_{G,V}$. The diagram

$$\begin{tikzcd}
\pi_0^G(\Sigma_+^\infty \mathbf{L}_{G,V}) \arrow[d, "\tau"'] & \pi_0^G(\Sigma_+^\infty \mathbf{L}_{G,V\oplus W}) \arrow[l, "(\Sigma_+^\infty \rho_{G,V,W})_*"', "\cong"] \arrow[r, "\hat{f}_*"] \arrow[d, "\tau"'] & \pi_0^G(X) \arrow[d, "\tau"] \\
\pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V}) & \pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,W\oplus V}) \arrow[l, "\cong"', "(\Sigma_+^\infty \rho_{G,V,W})_*"] \arrow[r, "\hat{f}_*"'] & \pi_0^K(X)
\end{tikzcd}$$

commutes and the two left horizontal maps are isomorphisms. Since

$$x \;=\; \hat{f}_*((\Sigma_+^\infty \rho_{G,V,W})_*^{-1}(e_{G,V}))\ ,$$

naturality yields that

$$\tau(x) \;=\; \tau(\hat{f}_*((\Sigma_+^\infty \rho_{G,V,W})_*^{-1}(e_{G,V}))) \;=\; \hat{f}_*((\Sigma_+^\infty \rho_{G,V,W})_*^{-1}(\tau(e_{G,V})))\ .$$

This shows that the transformation τ is determined by the value $\tau(e_{G,V})$.

It remains to construct, for every element $y \in \pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})$, a natural transformation $\tau : \pi_0^G \longrightarrow \pi_0^K$ with $\tau(e_{G,V}) = y$. The previous paragraph dictates what to do: we represent a given class $x \in \pi_0^G(X)$ by a continuous based G-map $f : S^{V\oplus W} \longrightarrow X(V \oplus W)$ as in the previous paragraph and set

$$\tau(x) \;=\; \hat{f}_*((\Sigma_+^\infty \rho_{G,V,W})_*^{-1}(y))\ .$$

We omit the verification that the element $\tau(x)$ only depends on the class x, that τ is indeed natural, and that it satisfies $\tau(e_{G,V}) = y$; the arguments are of a similar kind as in the analogous result in Proposition III.1.46. □

Now we name a specific basis of the group $\mathbf{A}(G,K)$. For a pair (L,α) consisting of a closed subgroup L of K and a continuous group homomorphism $\alpha : L \longrightarrow G$ we define

$$[L,\alpha] \;=\; \mathrm{tr}_L^K \circ \alpha^* \;\in\; \mathbf{A}(G,K)\ ,$$

the natural transformation whose value at an orthogonal spectrum X is the composite

$$\pi_0^G(X) \;\xrightarrow{\alpha^*}\; \pi_0^L(X) \;\xrightarrow{\mathrm{tr}_L^K}\; \pi_0^K(X)$$

of restriction along α with transfer from L to K.

If L has infinite index in its normalizer, then the transfer map tr_L^K, and hence also the element $[L,\alpha]$, is zero by Example III.2.23. The *conjugate* of (L,α) by a pair $(k,g) \in K \times G$ of group elements is the pair $({}^kL, c_g \circ \alpha \circ c_k)$ consisting of the conjugate subgroup ${}^kL$ and the composite homomorphism

$${}^kL \;\xrightarrow{c_k}\; L \;\xrightarrow{\alpha}\; G \;\xrightarrow{c_g}\; G\ .$$

Since inner automorphisms induce the identity on equivariant homotopy groups (compare Proposition I.5.12),

$$\mathrm{tr}_{{}^kL}^K \circ (c_g \circ \alpha \circ c_k)^* \;=\; \mathrm{tr}_{{}^kL}^K \circ c_k^* \circ \alpha^* \circ g_\star \;=\; c_k^* \circ \mathrm{tr}_L^K \circ \alpha^* \circ g_\star \;=\; \mathrm{tr}_L^K \circ \alpha^*\ .$$

So the transformation $[L,\alpha]$ only depends on the conjugacy class of (L,α), i.e.,

$$[{}^kL, c_g \circ \alpha \circ c_k] \;=\; [L,\alpha] \quad \text{in } \mathbf{A}(G,K).$$

THEOREM 2.6. *Let K and G be compact Lie groups. The morphism group $\mathbf{A}(G,K)$ in the Burnside category is a free abelian group with basis the transformations $[L,\alpha]$, where (L,α) runs over all $(K\times G)$-conjugacy classes of pairs consisting of*

- *a closed subgroup $L \le K$ whose Weyl group $W_K L$ is finite, and*
- *a continuous group homomorphism $\alpha : L \longrightarrow G$.*

PROOF. We let V be any faithful G-representation. By Proposition 1.13 the composite

$$\mathbb{Z}\{[L,\alpha] \mid \ |W_K L| < \infty,\, \alpha : L \longrightarrow G\} \ \longrightarrow\ \mathrm{Nat}(\pi_0^G, \pi_0^K) \ \xrightarrow{\mathrm{ev}}\ \pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})$$

is an isomorphism, where the first map takes $[L,\alpha]$ to $\mathrm{tr}_L^K \circ \alpha^*$ and the second map is evaluation at the stable tautological class $e_{G,V}$. The evaluation map is an isomorphism by Proposition 2.5, so the first map is an isomorphism, as claimed. □

Theorem 2.6 amounts to a complete calculation of the Burnside category, because we also know how to express the composite of two operations, each given in the basis of Theorem 2.6, as a sum of basis elements. Indeed, restrictions are contravariantly functorial and transfers are transitive, and we also know how to expand a transfer followed by a restriction: every group homomorphism is the composite of an epimorphism and a subgroup inclusion; inflations commute with transfers according to Proposition III.2.30, and the restriction of a transfer can be rewritten via the double coset formula as in Theorem III.4.10.

Theorem 2.6 tell us what data is necessary to specify a global functor $M : \mathbf{A} \longrightarrow \mathcal{A}b$. For this, one needs to give the values $M(G)$ at all compact Lie groups G, restriction maps $\alpha^* : M(G) \longrightarrow M(L)$ for all continuous group homomorphisms $\alpha : L \longrightarrow G$ and transfer maps $\mathrm{tr}_L^K : M(L) \longrightarrow M(K)$ for all closed subgroup inclusions $L \leq K$. This data has to satisfy the same kind of relations that the restriction and transfer maps for equivariant homotopy groups satisfy, namely:

- the restriction maps are contravariantly functorial;
- inner automorphisms induce the identity;
- transfers are transitive and tr_K^K is the identity;
- the transfer tr_L^K is zero if the Weyl group $W_K L$ is infinite;
- transfer along an inclusion $H \leq G$ interacts with inflation along an epimorphism $\alpha : K \longrightarrow G$ according to
$$\alpha^* \circ \mathrm{tr}_H^G \ = \ \mathrm{tr}_L^K \circ (\alpha|_L)^* \ : \ M(H) \ \longrightarrow \ M(K) \ ,$$
where $L = \alpha^{-1}(H)$;
- for all pairs of closed subgroups H and K of G, the double coset formula holds, see Theorem III.4.10.

We note that the hypothesis that inner automorphisms acts as the identity implies that the restriction map α^* only depends on the homotopy class of α. More precisely, suppose that $\alpha, \alpha' : K \longrightarrow G$ are homotopic through continuous group homomorphisms. Then α and α' belong to the same path component of the space $\hom(K,G)$ of continuous homomorphisms, and so they are conjugate by an element of G, compare Proposition A.2.21.

This explicit description allows us to relate our notion of global functor to other 'global' versions of Mackey functors. For example, our category of global functors is equivalent to the category of *functors with regular Mackey structure* in the sense of Symonds [**157**, §3, p. 177]. As we shall explain in Remark 2.10 below, global functors defined on finite groups are equivalent to *inflation functors* in the sense of Webb [**173**] and to the 'global $(\emptyset,\infty)$-Mackey functors' in the sense of Lewis [**95**]. Our global functors are *not* equivalent to the *global Mackey functors* in the sense of tom Dieck [**166**, Ch. VI (8.14), Ex. 5]; indeed, in the indexing category Ω for tom Dieck's global Mackey functors the group $\mathrm{Hom}_\Omega(G,K)$ has a $\mathbb{Z}$-basis indexed by $(G\times K)$-conjugacy classes $[L,\alpha]$ where the Weyl group $W_K L$ is allowed to be infinite.

EXAMPLE 2.7 (Sphere spectrum). The *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}(V) = S^V$, the one-point compactification of the inner product space V. The orthogonal group acts as the one-point compactification of its action on V. The structure map $\sigma_{V,W} : S^V \wedge S^W \longrightarrow S^{V\oplus W}$ is the canonical homeomorphism. The equivariant homotopy groups of the sphere spectrum are the equivariant stable stems. The sphere spectrum is the suspension spectrum of the constant one-point orthogonal space $\mathbf{L}_{e,0}$,

$$\mathbb{S} \ \cong \ \Sigma_+^\infty \mathbf{L}_{e,0} \ .$$

The trivial representation is faithful as a representation of the trivial group, so $\mathbf{L}_{e,0} = B_{\mathrm{gl}}e$ is a global classifying space for the trivial group. By Corollary 1.13 the group $\pi_0^K(\mathbb{S})$ is a free abelian group with basis the classes $\mathrm{tr}_L^K(p_L^*(1))$ where L runs over all conjugacy classes of closed subgroups of K with finite Weyl group, and where $p_L : L \longrightarrow e$ is the unique homomorphism. For finite groups, this is originally due to Segal [**139**], and for general compact Lie groups to tom Dieck, as a corollary to his splitting theorem (see Satz 2 and Satz 3 of [**164**]).

The class $1 \in \pi_0(\mathbb{S})$ represented by the identity of S^0 is the stable tautological class $e_{e,0}$ (compare (1.12)). By Proposition 2.5, the action on the unit $1 \in \pi_0(\mathbb{S})$ is an isomorphism of global functors

$$\mathbb{A} \;=\; \mathbf{A}(e,-) \;\longrightarrow\; \underline{\pi}_0(\mathbb{S})$$

from the Burnside ring global functor $\mathbb{A}$ to the 0-th homotopy global functor of the sphere spectrum.

Example 2.8. (i) The *Burnside ring global functor* is the represented global functor $\mathbb{A} = \mathbf{A}(e,-)$ of morphisms out of the trivial group e. By Theorem 2.6, the value $\mathbb{A}(K) = \mathbf{A}(e,K)$ at a compact Lie group K is a free abelian group with basis the set of conjugacy classes of closed subgroups $L \leq K$ with finite Weyl group. When K is finite, then the Weyl group condition is vacuous and $\mathbb{A}(K)$ this is canonically isomorphic to the Burnside ring of K, by sending the operation $[L, p_L] = \mathrm{tr}_K^L \circ p_L^* \in \mathbb{A}(K)$ to the class of the K-set K/L (where $p_L : L \longrightarrow e$ is the unique homomorphism). As we discussed in Example 2.7, the Burnside ring global functor $\mathbb{A}$ is realized by the sphere spectrum $\mathbb{S}$. More generally, the represented functors $\mathbf{A}(G,-)$ are other examples of global functors, and we have seen in Proposition 2.5 that the represented global functor $\mathbf{A}(G,-)$ is realized by the suspension spectrum of a global classifying space $B_{\mathrm{gl}}G$ of the compact Lie group G.

(ii) In [**165**, Sec. 5.5], tom Dieck gives a very different construction of the Burnside ring global functor. We let G be a compact Lie group. A *G-ENR* is a G-space equivariantly homeomorphic to a G-retract of a G-invariant open subset of a finite dimensional G-representation. The acronym 'ENR' stands for *euclidean neighborhood retract*. Examples of G-ENRs are smooth compact G-manifolds and finite G-CW-complexes.

Tom Dieck calls two compact G-ENRs X and Y *equivalent* if for every closed subgroup H of G the Euler characteristics of the H-fixed point spaces X^H and Y^H coincide; here Euler characteristics are taken with respect to compactly supported Alexander-Spanier cohomology, and there is some work involved in showing that a compact G-ENR has a well defined Euler characteristic. Then $A(G)$ is defined as the set of equivalence classes of compact G-ENRs. The set $A(G)$ is naturally a commutative ring, with addition induced by disjoint union and multiplication induced by cartesian product of G-ENRs.

Tom Dieck shows in [**165**, Prop. 5.5.1], that $A(G)$ is a free abelian group with basis the classes of the homogeneous spaces G/H, where H runs over the conjugacy classes of closed subgroups with finite Weyl group. Moreover, the class of a general compact G-ENR X is expressed in terms of this basis by the formula

$$[X] \;=\; \sum_{(H)} \chi^{\mathrm{AS}}(G\backslash X_{(H)}) \cdot [G/H] \;;$$

a compact G-ENR has only finitely many orbit types, so the sum is in fact finite. Restriction of scalars along a continuous homomorphism $\alpha : K \longrightarrow G$ induces a ring homomorphism $\alpha^* : A(G) \longrightarrow A(K)$. If H is a closed subgroup of G, then extension of scalars – sending an H-ENR Y to $G \times_H Y$ – induces an additive transfer map $\mathrm{tr}_H^G : A(H) \longrightarrow A(G)$ that satisfies reciprocity with respect to restriction from G to H. By explicit comparison of bases, the homomorphisms

$$A(G) \;\longrightarrow\; \mathbf{A}(G)\;, \quad [G/H] \;\longmapsto\; \mathrm{tr}_H^G$$

define an isomorphism of global functors.

(iii) Given an abelian group M, the *constant global functor* $\underline{M}$ is given by $\underline{M}(G) = M$ and all restriction maps are identity maps. The transfer $\mathrm{tr}_H^G : \underline{M}(H) \longrightarrow \underline{M}(G)$ is multiplication by the Euler characteristic of the homogeneous space G/H. In particular, if H is a subgroup of finite index of G, then tr_H^G is multiplication

by the index $[G:H]$. In this example, the double coset formula is a special case of the Euler characteristic formula (4.8) of Example III.4.7, namely for the K-manifold $B = G/H$.

There is a well-known point set level model of an Eilenberg-Mac Lane spectrum $\mathcal{H}M$ that we recall in Construction VI.1.9 below. However, contrary to what one may expect, the 0-th homotopy group global functor $\underline{\pi}_0(\mathcal{H}M)$ is *not* isomorphic to the constant global functor $\underline{M}$. More precisely, the restriction map $p_G^* : \pi_0^e(\mathcal{H}M) \longrightarrow \pi_0^G(\mathcal{H}M)$ is an isomorphism for finite groups G, but not for general compact Lie groups, see Example VI.1.16.

(iv) The *unitary representation ring global functor* $\mathbf{RU}$ assigns to a compact Lie group G the unitary representation ring $\mathbf{RU}(G)$, i.e., the Grothendieck group of finite dimensional complex G-representations, with product induced by tensor product of representations. The restriction maps $\alpha^* : \mathbf{RU}(G) \longrightarrow \mathbf{RU}(K)$ are induced by restriction of representations along a continuous homomorphism $\alpha : K \longrightarrow G$. The transfer maps $\mathrm{tr}_H^G : \mathbf{RU}(H) \longrightarrow \mathbf{RU}(G)$ along a closed subgroup inclusion $H \leq G$ are given by the *smooth induction* of Segal [**144**, § 2]. If H is a subgroup of finite index of G, then this induction sends the class of an H-representation to the induced G-representation $\mathrm{map}^G(H, V)$ (which is then isomorphic to $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$); in general, induction may send actual representations to virtual representations. In the generality of compact Lie groups, the double coset formula for $\mathbf{RU}$ was proved by Snaith [**147**, Thm. 2.4]. We look more closely at the representation global functor in Example V.4.9, and we show in Theorem VI.5.17 that $\mathbf{RU}$ is realized by the global K-theory spectrum $\mathbf{KU}$ (see Construction VI.5.7).

(v) Given any generalized cohomology theory E (in the non-equivariant sense), we can define a global functor $\underline{E}$ by setting

$$\underline{E}(G) \;=\; E^0(BG)\ ,$$

the 0-th E-cohomology of a classifying space of the group G. The contravariant functoriality in group homomorphisms $\alpha : K \longrightarrow G$ comes from the covariant functoriality of the classifying space construction. The transfer map for a subgroup inclusion $H \leq G$ comes from the stable transfer map (i.e., Becker-Gottlieb transfer)

$$\Sigma_+^\infty BG \;\longrightarrow\; \Sigma_+^\infty BH\ .$$

The double coset formula was proved in this context by Feshbach [**51**, Thm. II.11]. We return to global Borel theories in more detail in Example 5.21 below. We will show in Proposition 5.27 that the global functor $\underline{E}$ is realized by an orthogonal spectrum bE, the 'global Borel theory' associated to E.

(v) The Borel cohomology construction of part (iv) has a 'relative' generalization where we start with a compact Lie group K and a generalized K-equivariant cohomology theory, represented by an orthogonal K-spectrum Z; the previous paragraph is the special case where K is a trivial group. Then we can define a global functor $\underline{Z}$ by setting

$$\underline{Z}(G) \;=\; Z^0(B(K,G))\ ,$$

the 0-th K-equivariant Z-cohomology of a classifying K-space $B(K,G)$ for (K,G)-bundles. We will show in Remark 5.33 that this global functor $\underline{Z}$ is realized by an orthogonal spectrum RZ, where R is a right adjoint of the derived forgetful functor $U : \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$ from the global stable homotopy category to the G-equivariant stable homotopy category.

Remark 2.9. If we fix a compact Lie group G and let H run through all subgroups of G, then the collection of H-equivariant homotopy groups $\pi_0^H(X)$ of an orthogonal spectrum X forms a *Mackey functor* for the group G, with respect to the restriction to subgroups, conjugation and transfer maps. As we already discussed in Remark 1.2, the G-Mackey functors that come up this way are special, and not all G-Mackey functors arise this way. To illustrate this we compare Mackey functors for the group $C_3 = \{1, \tau, \tau^2\}$ with three elements to additive functors on the full subcategory of $\mathbf{A}$ spanned by the group e and C_3. Generating

operations can be displayed as follows:

$$\begin{tikzcd} F(C_3) \arrow[loop above, "\alpha^*"] \arrow[d, shift left, "\mathrm{res}"] \\ F(e) \arrow[u, shift left, bend left, "p^*"] \arrow[u, shift left, bend right, "\mathrm{tr}"'] \end{tikzcd} \qquad \begin{tikzcd} F(C_3) \arrow[d, shift left, "\mathrm{res}"] \\ F(e) \arrow[u, shift left, bend left, "\mathrm{tr}"'] \arrow[loop below, "\tau"] \end{tikzcd}$$

global functor on e and C_3 $\qquad$ C_3-Mackey functor

Here $\mathrm{res} = \mathrm{res}_e^{C_3}$ and $\mathrm{tr} = \mathrm{tr}_e^{C_3}$ are the restriction and transfer maps that are present in both cases. A global functor also comes with restriction maps along the epimorphism $p : C_3 \longrightarrow e$ and along the automorphism $\alpha : C_3 \longrightarrow C_3$ with $\alpha(\tau) = \tau^2$, and the relations are

$$\mathrm{res} \circ p^* \;=\; \mathrm{Id}_{F(e)} \qquad \text{and} \qquad \mathrm{res} \circ \mathrm{tr} \;=\; 3 \cdot \mathrm{Id}_{F(e)}$$

as well as $\alpha^* \circ \alpha^* = \mathrm{Id}_{F(C_3)}$, $\alpha^* \circ p^* = p^*$, $\mathrm{res} \circ \alpha^* = \mathrm{res}$ and $\alpha^* \circ \mathrm{tr} = \mathrm{tr}$. In contrast, C_3-Mackey functors have an additional action of C_3 (the Weyl group of e in C_3) on $F(e)$, and this action satisfies the relation

$$\mathrm{res} \circ \mathrm{tr} \;=\; \mathrm{Id}_{F(e)} + \tau + \tau^2 \ .$$

Remark 2.10. The full subcategory $\mathbf{A}_{\mathcal{F}in}$ of the Burnside category $\mathbf{A}$ spanned by *finite* groups has a different, more algebraic description, as we shall now recall. This alternative description is often taken as the definition in algebraic treatments of global functors. The category of 'global functors on finite groups', i.e., additive functors from $\mathbf{A}_{\mathcal{F}in}$ to abelian groups, is thus equivalent to the category of 'inflation functors' in the sense of Webb [**173**, p.271] and to the 'global $(\emptyset, \infty)$-Mackey functors' in the sense of Lewis [**95**].

We define the additive *combinatorial Burnside category* $\mathbb{A}^{\mathrm{c}}$. The objects of $\mathbb{A}^{\mathrm{c}}$ are all finite groups. The abelian group $\mathbb{A}^{\mathrm{c}}(G, K)$ of morphisms from a group G to K is the Grothendieck group of finite K-G-bisets where the right G-action is free. In the special case $G = e$ of the trivial group as source we obtain $\mathbb{A}^{\mathrm{c}}(e, K)$, the Burnside ring of finite K-sets. Composition

$$\circ \;:\; \mathbb{A}^{\mathrm{c}}(K, L) \times \mathbb{A}^{\mathrm{c}}(G, K) \;\longrightarrow\; \mathbb{A}^{\mathrm{c}}(G, L)$$

is induced by the balanced product over K, i.e., it is the biadditive extension of

$$(S, T) \;\longmapsto\; S \times_K T \ .$$

Here S has a left L-action and a commuting free right K-action, whereas T has a left K-action and a commuting free right G-action. The balanced product $S \times_K T$ than inherits a left L-action from S and a free right G-action from T. Since the balanced product is associative up to isomorphism, this defines a pre-additive category. So the category $\mathbb{A}^{\mathrm{c}}$ is the 'group completion' of the category $\mathbb{A}^{+}_{\mathcal{F}in}$, the restriction of the effective Burnside category $\mathbb{A}^{+}$ (compare Construction II.2.28) to finite groups.

We define additive maps

$$\Psi_{G,K} \;:\; \mathbf{A}(G, K) \;\longrightarrow\; \mathbb{A}^{\mathrm{c}}(G, K)$$

that will turn out to give an additive equivalence of categories (restricted to finite groups). The map $\Psi_{G,K}$ sends a basis element $[L, \alpha]$ to the class of the K-G-biset

$$K \times_{(L,\alpha)} G \;=\; K \times G/(kl, g) \sim (k, \alpha(l)g)$$

whose right G-action is free. Every transitive G-free K-G-biset is isomorphic to one of this form, and $K \times_{(L,\alpha)} G$ is isomorphic, as a K-G-biset, to $K \times_{(L',\alpha')} G$ if and only if (L, α) is conjugate to (L', α'). So the map $\Psi_{K,G}$ sends the basis of $\mathbf{A}(G, K)$ of Theorem 2.6 to a basis of $\mathbb{A}^{\mathrm{c}}(G, K)$, and it is thus an isomorphism.

We claim that the maps $\Psi_{G,K}$ form a functor as G and K vary through all finite groups; this then shows that Ψ is an additive equivalence of categories from the full subcategory of $\mathbf{A}_{\mathcal{F}in}$ to $\mathbb{A}^{\mathrm{c}}$. The functoriality boils down to the fact that in both categories restriction, inflation and transfer interact with each other in exactly the same way. We omit the details.

We will now use the 'exterior' homotopy group pairings of Construction 1.20 to define a biadditive functor

$$\times \ : \ \mathbf{A} \times \mathbf{A} \ \longrightarrow \ \mathbf{A}$$

that is given on objects by the product of Lie groups, and that extends to a symmetric monoidal structure on the Burnside category $\mathbf{A}$.

Proposition 2.11. *Let G, G', K and K' be compact Lie groups. Given operations $\tau \in \mathbf{A}(G, G')$ and $\psi \in \mathbf{A}(K, K')$, there is a unique operation*

$$\tau \times \psi \ \in \ \mathbf{A}(G \times K, \, G' \times K')$$

with the following property: for all orthogonal spectra X and Y and all classes $x \in \pi_0^G(X)$ and $y \in \pi_0^K(Y)$ the relation

$$(\tau \times \psi)(x \boxtimes y) \ = \ \tau(x) \boxtimes \psi(y) \tag{2.12}$$

holds in $\pi_0^{G' \times K'}(X \wedge Y)$.

Proof. We choose a faithful G-representation V and a faithful K-representation W, which have associated stable tautological classes (1.12)

$$e_{G,V} \ \in \ \pi_0^G(\Sigma^\infty_+ \mathbf{L}_{G,V}) \qquad \text{and} \qquad e_{K,W} \ \in \ \pi_0^K(\Sigma^\infty_+ \mathbf{L}_{K,W}) \ .$$

Combining the isomorphism (1.17) with the one from Example I.3.39 shows that the orthogonal spectrum $\Sigma^\infty_+ \mathbf{L}_{G,V} \wedge \Sigma^\infty_+ \mathbf{L}_{K,W}$ is isomorphic to $\Sigma^\infty_+ \mathbf{L}_{G \times K, V \oplus W}$ in a way that matches the class $e_{G,V} \boxtimes e_{K,W}$ with the class $e_{G \times K, V \oplus W}$. Proposition 2.5 then shows that the pair $(\Sigma^\infty_+ \mathbf{L}_{G,V} \wedge \Sigma^\infty_+ \mathbf{L}_{K,W}, e_{G,V} \boxtimes e_{K,W})$ represents the functor $\pi_0^{G \times K}$. There is thus a unique operation $\tau \times \psi \in \mathbf{A}(G \times K, G' \times K')$ that satisfies

$$(\tau \times \psi)(e_{G,V} \boxtimes e_{K,W}) \ = \ \tau(e_{G,V}) \boxtimes \psi(e_{K,W}) \tag{2.13}$$

in $\pi_0^{G' \times K'}(\Sigma^\infty_+ \mathbf{L}_{G,V} \wedge \Sigma^\infty_+ \mathbf{L}_{K,W})$.

The relation (2.13) is a special case of (2.12), and it remains to show that the operation $\tau \times \psi$ satisfies the relation (2.12) in complete generality. As we already argued in the proof of Proposition 2.5, there is a G-representation $\bar{V}$ and a morphism of orthogonal spectra

$$f \ : \ \Sigma^\infty_+ \mathbf{L}_{G, V \oplus \bar{V}} \ \longrightarrow \ X$$

that satisfies $f_*(e_{G, V \oplus \bar{V}}) = x$. Similarly, there is a K-representation $\bar{W}$ and a morphism of orthogonal spectra

$$g \ : \ \Sigma^\infty_+ \mathbf{L}_{K, W \oplus \bar{W}} \ \longrightarrow \ Y$$

that satisfies $g_*(e_{K, W \oplus \bar{W}}) = y$. The morphism

$$\Sigma^\infty_+ \rho_{G,V,\bar{V}} \ : \ \Sigma^\infty_+ \mathbf{L}_{G, V \oplus \bar{V}} \ \longrightarrow \ \Sigma^\infty_+ \mathbf{L}_{G,V}$$

satisfies

$$(\Sigma^\infty_+ \rho_{G,V,\bar{V}})(e_{G, V \oplus \bar{V}}) \ = \ e_{G,V} \ ,$$

and similarly for the triple $(K, W, \bar{W})$. Naturality then yields

$$\begin{aligned}
&(\Sigma^\infty_+ \rho_{G,V,\bar{V}} \wedge \Sigma^\infty_+ \rho_{K,W,\bar{W}})_*((\tau \times \psi)(e_{G,V\oplus\bar{V}} \boxtimes e_{K,W\oplus\bar{W}})) \\
&\qquad = (\tau \times \psi)((\Sigma^\infty_+ \rho_{G,V,\bar{V}})_*(e_{G,V\oplus\bar{V}}) \boxtimes (\Sigma^\infty_+ \rho_{K,W,\bar{W}})_*(e_{K,W\oplus\bar{W}})) \\
&\qquad = (\tau \times \psi)(e_{G,V} \boxtimes e_{K,W}) \\
&\qquad = \tau(e_{G,V}) \boxtimes \psi(e_{K,W}) \qquad (2.13) \\
&\qquad = \tau((\Sigma^\infty_+ \rho_{G,V,\bar{V}})_*(e_{G,V\oplus\bar{V}})) \boxtimes \psi((\Sigma^\infty_+ \rho_{K,W,\bar{W}})_*(e_{K,W\oplus\bar{W}})) \\
&\qquad = (\Sigma^\infty_+ \rho_{G,V,\bar{V}} \wedge \Sigma^\infty_+ \rho_{K,W,\bar{W}})_*(\tau(e_{G,V\oplus\bar{V}}) \boxtimes \psi(e_{K,W\oplus\bar{W}}))\ .
\end{aligned}$$

The morphism $\Sigma^\infty_+ \rho_{G,V,\bar{V}} \wedge \Sigma^\infty_+ \rho_{K,W,\bar{W}}$ is isomorphic to $\Sigma^\infty_+ \rho_{G\times K, V\oplus W, \bar{V}\oplus\bar{W}}$. The morphism $\rho_{G\times K, V\oplus W, \bar{V}\oplus\bar{W}}$ is a global equivalence of orthogonal spaces (by Proposition I.2.10 (ii)), and so the morphism $\Sigma^\infty_+ \rho_{G,V,\bar{V}} \wedge \Sigma^\infty_+ \rho_{K,W,\bar{W}}$ is a global equivalence of orthogonal spectra (by Corollary 1.9). So in particular it induces an isomorphism on $\pi_0^{G'\times K'}$, and we can conclude that

$$(\tau \times \psi)(e_{G,V\oplus\bar{V}} \boxtimes e_{K,W\oplus\bar{W}}) \ = \ \tau(e_{G,V\oplus\bar{V}}) \boxtimes \psi(e_{K,W\oplus\bar{W}})\ .$$

Now the relation (2.12) follows by simple naturality:

$$\begin{aligned}
(\tau \times \psi)(x \boxtimes y) \ &= \ (\tau \times \psi)(f_*(e_{G,V\oplus\bar{V}}) \boxtimes g_*(e_{K,W\oplus\bar{W}})) \\
&= \ (\tau \times \psi)((f \wedge g)_*(e_{G,V\oplus\bar{V}} \boxtimes e_{K,W\oplus\bar{W}})) \\
&= \ (f \wedge g)_*((\tau \times \psi)(e_{G,V\oplus\bar{V}} \boxtimes e_{K,W\oplus\bar{W}})) \\
&= \ (f \wedge g)_*((\tau(e_{G,V\oplus\bar{V}}) \boxtimes \psi(e_{K,W\oplus\bar{W}}))) \\
&= \ f_*(\tau(e_{G,V\oplus\bar{V}})) \boxtimes g_*(\psi(e_{K,W\oplus\bar{W}})) \\
&= \ \tau(f_*(e_{G,V\oplus\bar{V}})) \boxtimes \psi(g_*(e_{K,W\oplus\bar{W}})) \ = \ \tau(x) \boxtimes \psi(y)
\end{aligned}$$

□

Example 2.14. We identify the product of two generating operations in the stable Burnside category. We recall that for a pair (L, α) consisting of a closed subgroup L of K and a continuous group homomorphism $\alpha : L \longrightarrow G$ we defined

$$[L, \alpha] \ = \ \mathrm{tr}_L^K \circ \alpha^* \ \in \ \mathbf{A}(G, K)\ .$$

By Theorem 2.6, a certain subset of these operations forms a basis of the abelian group $\mathbf{A}(G, K)$. Using parts (v) and (vi) of Theorem 1.22 we deduce that

$$\begin{aligned}
[L \times L', \alpha \times \alpha'](x \boxtimes y) \ &= \ \mathrm{tr}_{L\times L'}^{K\times K'}((\alpha \times \alpha')^*(x \boxtimes y)) \\
&= \ \mathrm{tr}_{L\times L'}^{K\times K'}(\alpha^*(x) \boxtimes (\alpha')^*(y)) \\
&= \ \mathrm{tr}_L^K(\alpha^*(x)) \boxtimes \mathrm{tr}_{L'}^{K'}((\alpha')^*(y)) \ = \ [L, \alpha](x) \boxtimes [L', \alpha'](y)\ .
\end{aligned}$$

So the operation $[L \times L', \alpha \times \alpha']$ has the property that characterizes the operation $[L, \alpha] \times [L', \alpha']$. Hence on the generating operations the monoidal product in $\mathbf{A}$ is given by the formula

$$[L, \alpha] \times [L'\alpha'] \ = \ [L \times L', \alpha \times \alpha']\ . \qquad (2.15)$$

Now we are ready to establish the monoidal structure of the global Burnside category.

Theorem 2.16. *Let G, G', K and K' be compact Lie groups.*

(i) *The map*

$$\times \ : \ \mathbf{A}(G, K) \times \mathbf{A}(G', K') \ \longrightarrow \ \mathbf{A}(G \times G', K \times K')$$

is biadditive.

(ii) *As the Lie groups vary, the maps of* (i) *form a functor* $\times : \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$.

(iii) *The restriction operations along the group isomorphisms*

$$a_{G,K,L} \ : \ G \times (K \times L) \ \cong \ (G \times K) \times L \ ,$$
$$G \times K \ \cong \ K \times G \qquad \textit{respectively} \qquad G \times e \ \cong \ G \ \cong \ e \times G$$

make the functor $\times$ into a symmetric monoidal structure on the global Burnside category.

Proof. (i) We show additivity in the first variable, the other case being analogous. The relation

$$\begin{aligned}((\tau \times \psi) + (\tau' \times \psi))(x \boxtimes y) &= (\tau \times \psi)(x \boxtimes y) + (\tau' \times \psi)(x \boxtimes y) \\ &= (\tau(x) \boxtimes \psi(y)) + (\tau'(x) \boxtimes \psi(y)) \\ &= (\tau(x) + \tau'(x)) \boxtimes \psi(y) \ = \ (\tau + \tau')(x) \boxtimes \psi(y)\end{aligned}$$

shows that the operation $(\tau \times \psi') + (\tau' \times \psi)$ has the property that characterizes the operation $(\tau + \tau') \times \psi$. So

$$(\tau + \tau') \times \psi \ = \ (\tau \times \psi) + (\tau' \times \psi) \ .$$

(ii) The relation

$$\begin{aligned}((\tau' \times \psi') \circ (\tau \times \psi))(x \boxtimes y) \ &= \ (\tau' \times \psi')((\tau \times \psi)(x \boxtimes y)) \ = \ (\tau' \times \psi')(\tau(x) \boxtimes \psi(y)) \\ &= \ \tau'(\tau(x)) \boxtimes \psi'(\psi(y)) \ = \ (\tau' \circ \tau)(x) \boxtimes (\psi' \circ \psi)(y)\end{aligned}$$

shows that the operation $(\tau' \times \psi') \circ (\tau \times \psi)$ has the property that characterizes the operation $(\tau' \circ \tau) \times (\psi' \circ \psi)$. So

$$(\tau' \circ \tau) \times (\psi' \circ \psi) \ = \ (\tau' \times \psi') \circ (\tau \times \psi) \ .$$

A similar (but even shorter) argument shows that $\mathrm{Id}_G \times \mathrm{Id}_K = \mathrm{Id}_{G \times K}$.

(iii) We start with naturality of the associativity isomorphism. We consider operations $\tau \in \mathbf{A}(G, G')$, $\psi \in \mathbf{A}(K, K')$ and $\kappa \in \mathbf{A}(M, M')$. The relation

$$\begin{aligned}((\tau \times (\psi \times \kappa)) \circ a^*_{G,K,L})((x \boxtimes y) \boxtimes z) \ &= \ (\tau \times (\psi \times \kappa))(a^*_{G,K,L}((x \boxtimes y) \boxtimes z)) \\ &= \ ((\tau \times (\psi \times \kappa))(x \boxtimes (y \boxtimes z))) \\ &= \ \tau(x) \boxtimes (\psi(y) \boxtimes \kappa(z)) \\ &= \ a^*_{G',K',L'}((\tau(x) \boxtimes \psi(y)) \boxtimes \kappa(z)) \\ &= \ a^*_{G',K',L'}(((\tau \times \psi) \times \kappa)((x \boxtimes y) \boxtimes z))\end{aligned}$$

shows that the operation $(a^*_{G',K',L'})^{-1} \circ (\tau \times (\psi \times \kappa)) \circ a^*_{G,K,L}$ has the property that characterizes the operation $(\tau \times \psi) \times \kappa$. So

$$(\tau \times (\psi \times \kappa)) \circ a^*_{G,K,L} \ = \ a^*_{G',K',L'} \circ ((\tau \times \psi) \times \kappa) \ .$$

The arguments for naturality of the unit and symmetry isomorphisms are similar, and we omit them.

The unit, associativity (pentagon) and symmetry (hexagon) coherence relations in $\mathbf{A}$ follow from the corresponding coherence relations for the product of groups, and the fact that passage from group homomorphisms to restriction operations is functorial. $\square$

Remark 2.17. The restriction of the monoidal structure on the Burnside category to finite groups has an interpretation in terms of the cartesian product of bisets: under the equivalence of categories $\mathbf{A}_{\mathcal{F}in} \cong \mathbb{A}^c$ described in Remark 2.10, it corresponds to the monoidal structure

$$\mathbb{A}^{\mathrm{c}}(G, K) \times \mathbb{A}^{\mathrm{c}}(G', K') \ \longrightarrow \ \mathbb{A}^{\mathrm{c}}(G \times G', K \times K') \ , \quad ([S], [S']) \ \longmapsto [S \times S'] \ .$$

Here S is a right free K-G-biset and S' is a right free K'-G'-biset; the cartesian product $S \times S'$ is then a right free $(K \times K')$-$(G \times G')$-biset. Indeed, the equivalence $\mathbf{A}_{\mathcal{F}in} \cong \mathbb{A}^c$ discussed in Remark 2.10 sends the basis element $[L, \alpha] \in \mathbf{A}(G, K)$ to the class of the biset $K \times_{(L,\alpha)} G$. So the equivalence is monoidal because

$$(K \times_{(L,\alpha)} G) \times (K' \times_{(L',\alpha')} G') \qquad \text{and} \qquad (K \times K') \times_{(L \times L', \alpha \times \alpha')} G \times G'$$

are isomorphic as $(K \times K')$-$(G \times G')$-bisets.

Construction 2.18 (Box product of global functors). Since global functors are additive functors on the Burnside category $\mathbf{A}$, the symmetric monoidal product on $\mathbf{A}$ gives rise to a symmetric monoidal convolution product on the category of global functors. This is a special case of the general construction of Day [**40**] that we review in Appendix A.3. We now make this convolution product more explicit. We denote by $\mathbf{A} \otimes \mathbf{A}$ the pre-additive category whose objects are pairs of compact Lie groups, and with morphism groups

$$(\mathbf{A} \otimes \mathbf{A})((G, G'), (K, K')) \;=\; \mathbf{A}(G, K) \otimes \mathbf{A}(G', K') \;.$$

We let F, F' and F'' be global functors. We denote by $F \otimes F' : \mathbf{A} \otimes \mathbf{A} \longrightarrow \mathcal{A}b$ the objectwise tensor product given on objects by

$$(F \otimes F')(G, G') \;=\; F(G) \otimes F'(G') \;.$$

A *bimorphism* is a natural transformation

$$F \otimes F' \;\longrightarrow\; F'' \circ \times$$

of biadditive functors on the category $\mathbf{A} \otimes \mathbf{A}$. Since the morphism groups in the Burnside category are generated by transfers and restrictions, this means more explicitly, that a bimorphism is a collection of group homomorphisms

$$b_{G,\bar{G}} \;:\; F(G) \otimes F'(\bar{G}) \;\longrightarrow\; F''(G \times \bar{G})$$

for all compact Lie groups G and G', such that for all continuous group homomorphisms $\alpha : K \longrightarrow G$ and $\bar{\alpha} : \bar{K} \longrightarrow \bar{G}$ and for all closed subgroups $H \leq G$ and $\bar{H} \leq \bar{G}$ the diagram

$$\begin{tikzcd}
F(H) \otimes F'(\bar{H}) \arrow[r, "\mathrm{tr}_H^G \otimes \mathrm{tr}_{\bar{H}}^{\bar{G}}"] \arrow[d, "b_{H,\bar{H}}"'] & F(G) \otimes F'(\bar{G}) \arrow[r, "\alpha^* \otimes \bar{\alpha}^*"] \arrow[d, "b_{G,\bar{G}}"'] & F(K) \otimes F'(\bar{K}) \arrow[d, "b_{K,\bar{K}}"] \\
F''(H \times \bar{H}) \arrow[r, "\mathrm{tr}_{H \times \bar{H}}^{G \times \bar{G}}"'] & F''(G \times \bar{G}) \arrow[r, "(\alpha \times \bar{\alpha})^*"'] & F''(K \times \bar{K})
\end{tikzcd}$$

commutes. Here we exploited that the generating operations multiply as described in (2.15). Equivalently: for every compact Lie group G the maps $\{b_{G,\bar{G}}\}_{\bar{G}}$ form a morphism of global functors $F(G) \otimes F'(-) \longrightarrow F''(G \times -)$ and for every compact Lie group $\bar{G}$ the maps $\{b_{G,\bar{G}}\}_G$ form a morphism of global functors $F(-) \otimes F'(\bar{G}) \longrightarrow F''(- \times \bar{G})$.

A *box product* of F and F' is a universal example of a global functor with a bimorphism from F and F'. More precisely, a box product is a pair $(F \Box F', i)$ consisting of a global functor $F \Box F'$ and a universal bimorphism $i : F \otimes F' \longrightarrow (F \Box F') \circ \times$, i.e., a bimorphism such that for every global functor F'' the map

$$\mathcal{GF}(F \Box F', F'') \;\longrightarrow\; \mathrm{Bimor}((F, F'), F'') \;, \quad f \longmapsto fi \tag{2.19}$$

is bijective. Box products exist by the general theory (see Proposition A.3.5), and they are unique up to preferred isomorphism (see Remark A.3.4). Often only the global functor $F \Box F'$ will be referred to as the box product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will refer to the bijection (2.19) as the *universal property* of the box product of global functors.

The universal property guarantees that given any collection of choices of box product $F \Box F'$ for all pairs of global functors, $F \Box F'$ is an additive functor in both variables and there are preferred associativity and commutativity isomorphisms

$$\alpha_{F,F',F''} \;:\; (F \Box F') \Box F'' \;\cong\; F \Box (F' \Box F'') \qquad \text{and} \qquad \tau_{F,F'} \;:\; F \Box F' \;\cong\; F' \Box F \;.$$

Moreover, we can arrange that the Burnside ring global functor $\mathbb{A} = \mathbf{A}(e, -)$ is a strict unit in the sense that

$$F \Box \mathbb{A} \;=\; F \;=\; \mathbb{A} \Box F \;.$$

Altogether this structure makes the category of global functors into a symmetric monoidal category, with the Burnside ring global functor $\mathbb{A}$ as a strict unit object, compare Theorem A.3.12. The box product of representable global functors is again representable, by Proposition A.3.13. In the case at hand the maps

$$\times \ : \ \mathbf{A}(G,K) \otimes \mathbf{A}(\bar{G},\bar{K}) \ \longrightarrow \ \mathbf{A}(G \times \bar{G}, K \times \bar{K})$$

form a bimorphism from $(\mathbf{A}(G,-), \mathbf{A}(\bar{G},-))$ to $\mathbf{A}(G \times \bar{G},-)$; the induced morphism of global functors

$$\mathbf{A}(G,-) \,\square\, \mathbf{A}(\bar{G},-) \ \longrightarrow \ \mathbf{A}(G \times \bar{G},-)$$

is an isomorphism. As explained in Construction A.3.14, the box product of global functors is *closed* symmetric monoidal, but we will not use that.

Example 2.20. Given orthogonal spectra X and Y the external product maps

$$\boxtimes \ : \ \pi_0^G(X) \otimes \pi_0^K(Y) \ \longrightarrow \ \pi_0^{G \times K}(X \wedge Y)$$

form a bimorphism of global functors by Theorem 1.22. So the universal property of the box product produces a morphism of global functors

$$\underline{\pi}_0(X) \,\square\, \underline{\pi}_0(Y) \ \longrightarrow \ \underline{\pi}_0(X \wedge Y) \ . \tag{2.21}$$

We show in Proposition 4.16 below that whenever X and Y are globally connective and at least one of them is flat, then the smash product $X \wedge Y$ is again globally connective and the morphism (2.21) is an isomorphism of global functors.

Remark 2.22. We claim that the box product is right exact in both variables. To see this we recall that the values of any Day type convolution product can be described as an enriched coend (compare Remark A.3.6). In our present situation this says that $F \square M$ is a cokernel of a certain homomorphism of global functors

$$\begin{aligned} d \ : \ \bigoplus_{H,H',G,G'} & \mathbf{A}(H \times H',-) \otimes \mathbf{A}(G,H) \otimes \mathbf{A}(G',H') \otimes F(G) \otimes M(G') \\ & \longrightarrow \bigoplus_{G,G'} \mathbf{A}(G \times G',-) \otimes F(G) \otimes M(G') \ . \end{aligned} \tag{2.23}$$

The left sum is indexed over all quadruples (H,H',G,G') in a set of representatives of isomorphism classes of compact Lie groups, the right sum is indexed over pairs (G,G') of such groups. The map d is the difference of two homomorphism; one of them sums the tensor products of

$$\mathbf{A}(H \times H',-) \otimes \mathbf{A}(G,H) \otimes \mathbf{A}(G',H') \ \longrightarrow \ \mathbf{A}(G \times G',-) \ , \quad \varphi \otimes \tau \otimes \tau' \ \longmapsto \ \varphi \circ (\tau \times \tau')$$

and the identity on $F(G) \otimes M(G')$. The other map sums the tensor product of the identity of the global functor $\mathbf{A}(H \times H',-)$ and the action maps $\mathbf{A}(G,H) \otimes F(G) \longrightarrow F(H)$ respectively $\mathbf{A}(G',H') \otimes M(G') \longrightarrow M(H')$. Cokernels of global functors are calculated objectwise, so the value $(F \square M)(K)$ is a cokernel of the morphism of abelian groups that we obtain by plugging K into the free variable above. Theorem 2.6 describes explicit free generators for the morphism groups in the Burnside category; using this, the value $(F \square M)(K)$ can be expanded into a cokernel of a morphism between two huge sums of tensor products of values of F and M.

Now we consider a short exact sequence

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$$

of global functors. This gives rise to a commutative diagram of global functors

$$\begin{array}{ccc}
\bigoplus \mathbf{A}(H\times H',-)\otimes \mathbf{A}(G,H)\otimes \mathbf{A}(G',H')\otimes F(G)\otimes M(G') & \longrightarrow & \bigoplus \mathbf{A}(G\times G',-)\otimes F(G)\otimes M(G') \\
\downarrow & & \downarrow \\
\bigoplus \mathbf{A}(H\times H',-)\otimes \mathbf{A}(G,H)\otimes \mathbf{A}(G',H')\otimes F(G)\otimes M'(G') & \longrightarrow & \bigoplus \mathbf{A}(G\times G',-)\otimes F(G)\otimes M'(G') \\
\downarrow & & \downarrow \\
\bigoplus \mathbf{A}(H\times H',-)\otimes \mathbf{A}(G,H)\otimes \mathbf{A}(G',H')\otimes F(G)\otimes M''(G') & \longrightarrow & \bigoplus \mathbf{A}(G\times G',-)\otimes F(G)\otimes M''(G') \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

with exact columns. The induced sequence of horizontal cokernels

$$F\Box M \longrightarrow F\Box M' \longrightarrow F\Box M'' \longrightarrow 0$$

is thus also exact.

A closer look reveals that we did not use anything special about the preadditive indexing category $\mathbf{A}$ in the above proof of right exactness; so the same argument shows that any Day convolution product of additive functors from a symmetric monoidal preadditive category to abelian groups is right exact.

While the box product of global functors has many properties familiar from the tensor product of modules over a commutative ring, there is one aspect where these constructions are fundamentally different: *projectives are not generally flat* in the category of global functors. In other words, for most projective global functors P, the functor $-\Box P$ does *not* send monomorphisms to monomorphisms. Because projectives are not generally flat, one has to be careful when deriving the box product. This kind of phenomenon has been analyzed in great detail by Lewis in [**95**]; Lewis' notation for the category $\mathbb{A}^c$ is $\mathcal{B}_*(\emptyset,\infty)$, our $\mathcal{F}in$-global functors are his 'global $(\emptyset,\infty)$-Mackey functors' and the category of $\mathcal{F}in$-global functors is denoted $\mathfrak{M}_*(\emptyset,\infty)$. Theorem 6.10 of [**95**] shows that the representable functor $\mathbf{A}_{C_p}$ is not flat, where C_p is a cyclic group of prime order p.

We now remark that bimorphisms of global functors can be identified with another kind of structure that we call 'diagonal products'.

Definition 2.24. Let X, Y and Z be global functors. A *diagonal product* is a natural transformation $X\otimes Y \longrightarrow Z$ of Rep-functors to abelian groups that satisfies reciprocity, where $X\otimes Y$ is the objectwise tensor product of Rep-functors to abelian groups.

Remark 2.25. More explicitly, a diagonal product consists of additive maps

$$\nu_G \ : \ X(G)\otimes Y(G) \ \longrightarrow \ Z(G)$$

for every compact Lie group G that are natural for restriction along homomorphisms $\alpha : K \longrightarrow G$ and satisfy the reciprocity relation

$$\mathrm{tr}_H^G(\nu_H(x\otimes \mathrm{res}_H^G(y))) \ = \ \nu_G(\mathrm{tr}_H^G(x)\otimes y)$$

for all closed subgroups H of G and all classes $x\in X(H)$ and $y\in Y(G)$, and similarly in the other variable. Any bimorphism $\mu : (X,Y)\longrightarrow Z$ gives rise to a diagonal product as follows. For a group G we define ν_G as the composite

$$X(G)\otimes Y(G) \ \xrightarrow{\mu_{G,G}} \ Z(G\times G) \ \xrightarrow{\Delta_G^*} \ Z(G)$$

where $\Delta_G : G \longrightarrow G \times G$ is the diagonal. For a group homomorphism $\alpha : K \longrightarrow G$ we have $\Delta_G \circ \alpha = (\alpha \times \alpha) \circ \Delta_K$, so the diagram

$$\begin{tikzcd}
X(G) \otimes Y(G) \arrow[r, "\mu_{G,G}"] \arrow[d, "\alpha^* \otimes \alpha^*"'] & Z(G \times G) \arrow[r, "\Delta_G^*"] \arrow[d, "(\alpha \times \alpha)^*"] & Z(G) \arrow[d, "\alpha^*"] \\
X(K) \otimes Y(K) \arrow[r, "\mu_{K,K}"'] & Z(K \times K) \arrow[r, "\Delta_K^*"'] & Z(K)
\end{tikzcd}$$

commutes.

Since there is only one double coset for the left Δ_G-action and the right $H \times G$-action on $G \times G$, the double coset formula becomes

$$\Delta_G^* \circ \mathrm{tr}_{H \times G}^{G \times G} \;=\; \mathrm{tr}_H^G \circ \Delta_H^* \circ \mathrm{res}_{H \times H}^{H \times G} \;.$$

We conclude that

$$\begin{aligned}
\mathrm{tr}_H^G(\nu_H(x \otimes \mathrm{res}_H^G(y))) \;&=\; \mathrm{tr}_H^G(\Delta_H^*(\mu_{H,G}(x \otimes y))) \;=\; \Delta_G^*(\mathrm{tr}_{H \times G}^{G \times G}(\mu_{H,G}(x \otimes y))) \\
&=\; \Delta_G^*(\mu_{G,G}(\mathrm{tr}_H^G(x) \otimes y)) \;=\; \nu_G(\mathrm{tr}_H^G(x) \otimes y) \;,
\end{aligned}$$

the reciprocity relation for the diagonal product ν. The reciprocity in the other variable is similar.

Conversely, given a diagonal product ν, we define a bimorphism as follows. For compact Lie groups G and K we define the component $\mu_{G,K}$ as the composite

$$X(G) \otimes Y(K) \;\xrightarrow{p_G^* \otimes p_K^*}\; X(G \times K) \otimes Y(G \times K) \;\xrightarrow{\nu_{G \times K}}\; Z(G \times K) \;,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the projections. If the diagonal product ν was defined from an external product μ as above, then

$$\nu_{G \times K} \circ (p_G^* \otimes p_K^*) \;=\; \Delta_{G \times K}^* \circ \mu_{G \times K, G \times K} \circ (p_G^* \otimes p_K^*) \;=\; \Delta_{G \times K}^* \circ (p_G \times p_K)^* \circ \mu_{G,K} \;=\; \mu_{G,K}$$

because the composite $(p_G \times p_K) \circ \Delta_{G \times K}$ is the identity. So the external product can be recovered from the diagonal product.

Given homomorphisms $\alpha : G \longrightarrow G'$ and $\beta : K \longrightarrow K'$, we have $p_{G'} \circ (\alpha \times \beta) = \alpha \circ p_G$ and $p_{K'} \circ (\alpha \times \beta) = \beta \circ p_K$, so the left part of the diagram

$$\begin{tikzcd}
X(G) \otimes Y(K) \arrow[r, "p_G^* \otimes p_K^*"] \arrow[d, "\alpha^* \otimes \beta^*"'] & X(G \times K) \otimes Y(G \times K) \arrow[r, "\nu_{G \times K}"] \arrow[d, "(\alpha \times \beta)^* \otimes (\alpha \times \beta)^*"] & Z(G \times K) \arrow[d, "(\alpha \times \beta)^*"] \\
X(G') \otimes Y(K') \arrow[r, "p_G^* \otimes p_K^*"'] & X(G' \times K') \otimes Y(G' \times K') \arrow[r, "\nu_{G' \times K'}"'] & Z(G' \times K')
\end{tikzcd}$$

commutes. The right part commutes by naturality of the diagonal product ν.

For naturality with respect to transfers we let H be a closed subgroup of G, and we consider classes $x \in X(H)$ and $y \in Y(K)$. Then

$$\begin{aligned}
\mathrm{tr}_{H \times K}^{G \times K}(x \times y) \;&=\; \mathrm{tr}_{H \times K}^{G \times K}(p_H^*(x) \cdot p_K^*(y)) \;=\; \mathrm{tr}_{H \times K}^{G \times K}(p_H^*(x) \cdot \mathrm{res}_{H \times K}^{G \times K}(\bar{p}_K^*(y))) \\
&=\; \mathrm{tr}_{H \times K}^{G \times K}(p_H^*(x)) \cdot \bar{p}_K^*(y) \;=\; p_G^*(\mathrm{tr}_H^G(x)) \cdot p_K^*(y) \;=\; \mathrm{tr}_H^G(x) \times y \;.
\end{aligned}$$

Here $p : H \times K \longrightarrow K$ and $\bar{p} : G \times K \longrightarrow K$ are the projections to K, and the third equality is reciprocity. The argument for transfer naturality in the K-variable is similar.

3. Global model structures for orthogonal spectra

In this section we establish the level and global model structures on the category of orthogonal spectra. Many arguments are parallel to the unstable analogs in Section I.3, so there is a certain amount of repetition. The main model structure of interest for us is the *global model structure*, see Theorem 3.26. The weak equivalences in this model structure are the global equivalences and the cofibrations are the flat cofibrations. More generally, we consider a global family $\mathcal{F}$ and define the *$\mathcal{F}$-global model structure*, see Theorem 3.21 below, with weak equivalences the $\mathcal{F}$-equivalences, i.e., those morphisms inducing isomorphisms of G-equivariant homotopy groups for all G in $\mathcal{F}$. Proposition 3.29 shows that the $\mathcal{F}$-global model structure is monoidal with respect to the smash product of orthogonal spectra, provided that $\mathcal{F}$ is closed under products.

There is a functorial way to write an orthogonal spectrum as a sequential colimit of spectra which are made from the information below a fixed level. We refer to this as the *skeleton filtration* of an orthogonal spectrum. Again the word 'filtration' should be used with caution because the maps from the skeleta to the orthogonal spectrum need not be injective. As in the unstable situation in Section I.3, the skeleton filtration is a special case of the general skeleton filtration on certain enriched functor categories that we discuss in Appendix A.3 Indeed, if we specialize the base category to $\mathcal{V} = \mathbf{T}_*$, the category of based spaces under smash product, and the index category to $\mathcal{D} = \mathbf{O}$, then the functor category $\mathcal{D}^*$ becomes the category $\mathcal{S}p$ of orthogonal spectra. The dimension function needed in the construction and analysis of skeleta is the vector space dimension. The general theory provides for every orthogonal spectrum X and every $m \geq 0$ an *m-skeleton*,

$$\mathrm{sk}^m X \;=\; l_m(X^{\leq m}) \;,$$

the extension of the restriction of X to $\mathbf{O}^{\leq m}$, and a natural morphism $i_m : \mathrm{sk}^m X \longrightarrow X$, the counit of the adjunction $(l_m, (-)^{\leq m})$. The *m-th latching space* of X is the based $O(m)$-space

$$L_m X \;=\; (\mathrm{sk}^{m-1} X)(\mathbb{R}^m) \;;$$

it comes with a natural based $O(m)$-equivariant map

$$\nu_m = i_{m-1}(\mathbb{R}^m) \;:\; L_m X \;\longrightarrow\; X(\mathbb{R}^m) \;,$$

the *m-th latching map*. The value

$$i_m(V) \;:\; (\mathrm{sk}^m X)(V) \;\longrightarrow\; X(V)$$

of this morphism is an isomorphism for all inner product spaces V of dimension at most m. We also agree to set $\mathrm{sk}^{-1} X = *$, the trivial orthogonal spectrum, and $L_0 X = *$, a one-point space.

The different skeleta are related by natural morphisms $j_m : \mathrm{sk}^m X \longrightarrow \mathrm{sk}^{m+1} X$, for all $m \geq 0$, such that $i_{m+1} \circ j_m = i_m$. The sequence of skeleta stabilizes to X in a very strong sense. For every inner product space V, the maps $j_m(V)$ and $i_m(V)$ are isomorphisms as soon as $m \geq \dim(V)$. In particular, $X(V)$ is a colimit, with respect to the morphisms $i_m(V)$, of the sequence of maps $j_m(V)$. Since colimits in the category of orthogonal spectra are created objectwise, we deduce that the orthogonal spectrum X is a colimit, with respect to the morphisms i_m, of the sequence of morphisms j_m.

Example 3.1. Let G be a compact Lie group and V a G-representation of dimension n. Then the free orthogonal spectrum (1.25) $F_{G,V} A$ generated by a based G-space A in level V is 'purely n-dimensional' in the following sense. The space $(F_{G,V} A)_m$ is trivial for $m < n$, and hence the latching space $L_m(F_{G,V} A)$ is trivial for $m \leq n$. For $m > n$ the latching map $\nu_m : L_m(F_{G,V} A) \longrightarrow (F_{G,V} A)_m$ is an isomorphism. So the skeleton $\mathrm{sk}^m(F_{G,V} A)$ is trivial for $m < n$ and $\mathrm{sk}^m(F_{G,V} A) = F_{G,V} A$ is the entire spectrum for $m \geq n$.

We denote by

$$G_m \;=\; F_{O(m),\mathbb{R}^m} \;:\; O(m)\mathbf{T}_* \;\longrightarrow\; \mathcal{S}p$$

the left adjoint to the evaluation functor $X \mapsto X(\mathbb{R}^m)$, i.e., the free functor (1.25) indexed by the tautological $O(m)$-representation. As a special case of the previous example, the orthogonal spectrum $G_m A$ is purely m-dimensional for every based $O(m)$-space A. Proposition A.3.19 specializes to:

PROPOSITION 3.2. *For every orthogonal spectrum X and every $m \geq 0$ the commutative square*

$$\begin{tikzcd} G_m L_m X \arrow[r, "G_m \nu_m"] \arrow[d] & G_m X(\mathbb{R}^m) \arrow[d] \\ \mathrm{sk}^{m-1} X \arrow[r, "j_{m-1}"'] & \mathrm{sk}^m X \end{tikzcd} \tag{3.3}$$

is a pushout of orthogonal spectra. The two vertical morphisms are adjoint to the identity of $L_m X$ respectively $X(\mathbb{R}^m)$.

Proposition A.3.27 is a fairly general recipe for constructing level model structures on categories such as orthogonal spectra. We specialize the general construction to the category of orthogonal spectra. For a morphism $f : X \longrightarrow Y$ of orthogonal spectra and $m \geq 0$ we have a commutative square of $O(m)$-spaces:

$$\begin{tikzcd} L_m X \arrow[r, "L_m f"] \arrow[d, "\nu_m^X"'] & L_m Y \arrow[d, "\nu_m^Y"] \\ X(\mathbb{R}^m) \arrow[r, "f(\mathbb{R}^m)"'] & Y(\mathbb{R}^m) \end{tikzcd}$$

We thus get a natural morphism of based $O(m)$-spaces

$$\nu_m f \ = \ f(\mathbb{R}^m) \cup \nu_m^Y \ : \ X(\mathbb{R}^m) \cup_{L_m X} L_m Y \ \longrightarrow \ Y(\mathbb{R}^m) \ .$$

DEFINITION 3.4. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *flat cofibration* if the latching morphism $\nu_m f : X(\mathbb{R}^m) \cup_{L_m X} L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-cofibration for all $m \geq 0$. An orthogonal spectrum Y is *flat* if the morphism from the trivial spectrum to it is a flat cofibration, i.e., for every $m \geq 0$ the latching map $\nu_m : L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-cofibration.

PROPOSITION 3.5. *Let $i : X \longrightarrow Y$ be a flat cofibration of orthogonal spectra. Then for every $m \geq 0$, the morphism $i(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-cofibration.*

PROOF. We use the relative skeleton filtration of the morphism i, compare (3.23) of Appendix A.3. It shows that the morphism i_n is the finite composite

$$X(\mathbb{R}^m) = (\mathrm{sk}^{-1}[i])(\mathbb{R}^m) \xrightarrow{(j_0[i])(\mathbb{R}^m)} (\mathrm{sk}^0[i])(\mathbb{R}^m) \xrightarrow{(j_1[i])(\mathbb{R}^m)} \dots \xrightarrow{(j_n[i])(\mathbb{R}^m)} (\mathrm{sk}^n[i])(\mathbb{R}^m) = Y(\mathbb{R}^m) \ ,$$

so it suffices to show that for all $k \geq 0$, the morphism $j_k[i] : \mathrm{sk}^{k-1}[i] \longrightarrow \mathrm{sk}^k[i]$ is levelwise an equivariant cofibration for the relevant orthogonal group. The pushout square (3.3) in level $k+n$ is a pushout of $O(k+n)$-spaces:

$$\begin{tikzcd} O(k+n) \ltimes_{O(k)\times O(n)} ((X(\mathbb{R}^k) \cup_{L_k X} L_k Y) \wedge S^n) \arrow[r] \arrow[d, "O(k+n)\ltimes(\nu_k i \wedge S^n)"'] & (\mathrm{sk}^{k-1}[i])_{k+n} \arrow[d, "(j_k[i])_{k+n}"] \\ O(k+n) \ltimes_{O(k)\times O(n)} (Y(\mathbb{R}^k) \wedge S^n) \arrow[r] & (\mathrm{sk}^k[i])_{k+n} \end{tikzcd}$$

Since S^n can be given the structure of a based $O(n)$-CW-complex, the functor $O(k+n) \ltimes_{O(k)\times O(n)} (- \wedge S^n)$ takes $O(k)$-cofibrations to $O(k+n)$-cofibrations; so the left vertical morphism, and hence also $(j_k[i])_{k+n}$, is an $O(k+n)$-cofibration. $\square$

We let $\mathcal{F}$ be a global family in the sense of Definition I.4.1, i.e., a non-empty class of compact Lie groups that is closed under isomorphisms, closed subgroups and quotient groups. As in the unstable situation in Section I.4, we now develop the *$\mathcal{F}$-level model structure* on the category of orthogonal spectra, in which the $\mathcal{F}$-level equivalences are the weak equivalences. This model structure has a 'global' (or 'stable') version, see Theorem 3.21 below.

We recall that $\mathcal{F}(m)$ denotes the family of those closed subgroups of the orthogonal group $O(m)$ that belong to the global family $\mathcal{F}$.

Definition 3.6. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is

- an *$\mathcal{F}$-level equivalence* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-equivalence for all $m \geq 0$;
- an *$\mathcal{F}$-level fibration* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-fibration for all $m \geq 0$;
- an $\mathcal{F}$-cofibration if the latching morphism $\nu_m f : X(\mathbb{R}^m) \cup_{L_m X} L_m Y \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-cofibration or all $m \geq 0$.

In other words, $f : X \longrightarrow Y$ is an $\mathcal{F}$-level equivalence (respectively $\mathcal{F}$-level fibration) if for every $m \geq 0$ and every subgroup H of $O(m)$ that belongs to the family $\mathcal{F}$ the map $f(\mathbb{R}^m)^H : X(\mathbb{R}^m)^H \longrightarrow Y(\mathbb{R}^m)^H$ is a weak equivalence (respectively Serre fibration).

Let G be any group from the family $\mathcal{F}$ and V a faithful G-representation of dimension m. We let $\alpha : \mathbb{R}^m \longrightarrow V$ be a linear isometry and define a homomorphism $c_\alpha : G \longrightarrow O(m)$ by 'conjugation by α', i.e., we set

$$(c_\alpha(g))(x) \;=\; \alpha^{-1}(g \cdot \alpha(x))$$

for $g \in G$ and $x \in \mathbb{R}^m$. We restrict the $O(m)$-action on $X(\mathbb{R}^m)$ to a G-action along the homomorphism c_α. Then the map

$$c_\alpha^*(X(\mathbb{R}^m)) \;\longrightarrow\; X(V)\;, \quad x \longmapsto [\alpha, x]$$

is a G-equivariant homeomorphism, natural in X; it restricts to a natural homeomorphism

$$X(\mathbb{R}^m)^{\bar{G}} \;\longrightarrow\; X(V)^G\;, \quad x \longmapsto [\alpha, x]$$

where $\bar{G} = c_\alpha(G)$ is the image of c_α. This implies:

Proposition 3.7. *Let $\mathcal{F}$ be a global family and $f : X \longrightarrow Y$ a morphism of orthogonal spectra.*

(i) *The morphism f is an $\mathcal{F}$-level equivalence if and only if for every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is an $(\mathcal{F} \cap G)$-equivalence.*

(ii) *The morphism f is an $\mathcal{F}$-level fibration if and only if for every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is an $(\mathcal{F} \cap G)$-fibration.*

Now we are ready to establish the $\mathcal{F}$-level model structure.

Proposition 3.8. *Let $\mathcal{F}$ be a global family. The $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations form a model structure, the $\mathcal{F}$-*level model structure*, on the category of orthogonal spectra. The $\mathcal{F}$-level model structure is proper, topological and cofibrantly generated.*

Proof. The first part is a special cases of Proposition A.3.27, in the following way. We let $\mathcal{C}(m)$ be the $\mathcal{F}(m)$-projective model structure on the category of based $O(m)$-spaces, i.e., the based version of the model structure of Proposition A.2.10. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition A.3.27 become the $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations.

The consistency condition (see Definition A.3.26) becomes the following condition: for all $m, n \geq 0$ and every acyclic cofibration $i : A \longrightarrow B$ in the $\mathcal{F}(m)$-projective model structure on based $O(m)$-spaces, every cobase change, in the category of based $O(m+n)$-spaces, of the map

$$\mathbf{O}(\mathbb{R}^m, \mathbb{R}^{m+n}) \wedge_{O(m)} i \;:\; \mathbf{O}(\mathbb{R}^m, \mathbb{R}^{m+n}) \wedge_{O(m)} A \;\longrightarrow\; \mathbf{O}(\mathbb{R}^m, \mathbb{R}^{m+n}) \wedge_{O(m)} B$$

is an $\mathcal{F}(m+n)$-weak equivalence. We show a stronger statement, namely that the functor

$$\mathbf{O}(\mathbb{R}^m, \mathbb{R}^{m+n}) \wedge_{O(m)} - \ : \ O(m)\mathbf{T}_* \ \longrightarrow \ O(m+n)\mathbf{T}_*$$

takes acyclic cofibrations in the $\mathcal{F}(m)$-projective model structure to acyclic cofibrations in the projective model structure on the category of based $O(m+n)$-spaces (i.e., the $\mathcal{A}ll$-projective model structure, where $\mathcal{A}ll$ is the family of all closed subgroups of $O(m+n)$). Since the functor under consideration is a left adjoint, it suffices to prove the claim for the generating acyclic cofibrations, i.e., the maps

$$(O(m)/H \times j^k)_+$$

for all $k \geq 0$ and all $H \in \mathcal{F}(m)$, where $j^k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion. Up to isomorphism, the functor under consideration takes this map to the map

$$O(m+n) \ltimes_{H \times O(n)} S^n \wedge j^k_+$$

where H acts trivially on S^n. Since the projective model structure on the category of based $O(m+n)$-spaces is topological, it suffices to show that $O(m+n) \ltimes_{H \times O(n)} S^n$ is cofibrant in this model structure. Since S^n is $O(n)$-equivariantly homeomorphic to the reduced mapping cone of the map $O(n)/O(n-1)_+ \longrightarrow S^0$, it suffices to show that the two $O(m+n)$-spaces

$$O(m+n) \times_{H \times O(n)} (O(n)/O(n-1)) \qquad \text{and} \qquad O(m+n)/(H \times O(n))$$

are cofibrant in the unbased sense. Both of these $O(m+n)$-spaces are homogeneous spaces, hence they admit smooth $O(m+n)$-actions, and hence an $O(m+n)$-equivariant CW-structure by Illman's theorem [**82**, Cor. 7.2]. In particular, both spaces are $O(m+n)$-cofibrant.

We describe explicit sets of generating cofibrations and generating acyclic cofibrations for the $\mathcal{F}$-level model structure. We let $I_{\mathcal{F}}$ be the set of all morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (2.11) of Section A.2. Then the set $I_{\mathcal{F}}$ detects the acyclic fibrations in the $\mathcal{F}$-level model structure by Proposition A.3.27 (iii). Similarly, we let $J_{\mathcal{F}}$ be the set of all morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (2.12) of Section A.2. Again by Proposition A.3.27 (iii), $J_{\mathcal{F}}$ detects the fibrations in the $\mathcal{F}$-level model structure. The $\mathcal{F}$-level model structure is topological by Proposition A.2.8, where we take $\mathcal{G}$ as the set of free orthogonal spectra $F_{H,\mathbb{R}^m}$ for all $m \geq 0$ and all $H \in \mathcal{F}(m)$.

The $\mathcal{F}(m)$-projective model structure on the category of based $O(m)$-spaces is right proper for all $m \geq 0$. Limits in the category of orthogonal spectra are constructed levelwise. Since weak equivalences and fibrations are also defined levelwise, right properness is inherited levelwise. The argument for left properness is similar, but not completely analogous because cofibrations are *not* defined levelwise. Since $\mathcal{F}$-cofibrations are in particular flat cofibrations, they are levelwise $O(m)$-cofibrations (Proposition 3.5). Colimits in the category of orthogonal spectra are also constructed levelwise, so left properness for the $\mathcal{F}$-level model structure is a consequence of left properness of the $\mathcal{F}(m)$-projective model structure on $O(m)$-spaces for all m. □

When $\mathcal{F} = \langle e \rangle$ is the minimal global family consisting of all trivial groups, then the $\langle e \rangle$-level equivalences (respectively $\langle e \rangle$-level fibrations) are the level equivalences (respectively level fibrations) of orthogonal spectra in the sense of Definition 6.1 of [**104**]. Hence the $\langle e \rangle$-cofibrations are the 'q-cofibrations' in the sense of [**104**, Def. 6.1]. For the minimal global family, the $\langle e \rangle$-level model structure thus specializes to the level model structure of [**104**, Thm. 6.5].

For easier reference we spell out the case $\mathcal{F} = \mathcal{A}ll$ of the maximal global family of all compact Lie groups. In this case $\mathcal{A}ll(m)$ is the family of all closed subgroups of $O(m)$, and the $\mathcal{A}ll$-cofibrations specialize to the flat cofibrations. We introduce special names for the $\mathcal{A}ll$-level equivalences and the $\mathcal{A}ll$-level fibrations (which coincide with the injective $\mathcal{A}ll$-fibrations), analogous to the unstable situation in Section I.3.

Definition 3.9. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *strong level equivalence* (respectively *strong level fibration*) if for every $m \geq 0$ the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-weak equivalence. (respectively an $O(m)$-fibration).

For the global family $\mathcal{F} = \mathcal{A}ll$, Proposition 3.8 specializes to the following strong level model structure:

Proposition 3.10. *The strong level equivalences, strong level fibrations and flat cofibrations form a model structure, the* strong level model structure*, on the category of orthogonal spectra. The strong level model structure is proper, topological and cofibrantly generated.*

Now we introduce and discuss an important class of orthogonal spectra.

Definition 3.11. An orthogonal spectrum X is a *global Ω-spectrum* if for every compact Lie group G, every faithful G-representation W and an arbitrary G-representation V the adjoint structure map

$$\tilde{\sigma}_{V,W} \ : \ X(W) \ \longrightarrow \ \mathrm{map}(S^V, X(V \oplus W))$$

is a G-weak equivalence.

The global Ω-spectra will turn out to be the fibrant objects in the global model structure on orthogonal spectra, see Theorem 3.26 below. This means that global Ω-spectra abound, because every orthogonal spectrum admits a global equivalence to a global Ω-spectrum.

Remark 3.12. Global Ω-spectra are a very rich kind of structure, because they encode compatible equivariant infinite loop spaces for all compact Lie groups at once. For a global Ω-spectrum X and a compact Lie group G the associated orthogonal G-spectrum X_G is 'eventually an Ω-G-spectrum' in the sense that the Ω-G-spectrum condition of [**105**, III Def. 3.1] holds for all 'sufficiently large' (i.e., faithful) G-representations. However, if G is a non-trivial group, then the associated orthogonal G-spectrum X_G is in general *not* an Ω-G-spectrum since there is no control over the G-homotopy type of the values at non-faithful representations.

For every compact Lie group G and every faithful G-representation V, the G-space

$$X[G] \ = \ \Omega^V X(V) \tag{3.13}$$

is a 'genuine' equivariant infinite loop space, i.e., deloopable in the direction of every representation. Indeed, for every G-representation W, the G-map

$$\Omega^V(\tilde{\sigma}_{V,W}) \ : \ X[G] \ = \ \Omega^V X(V) \ \longrightarrow \ \Omega^W(\Omega^V X(V \oplus W))$$

is a G-weak equivalence, so the global Ω-spectrum X provides a W-deloop $\Omega^V X(V \oplus W)$ of $X[G]$. The G-space $X[G]$ is also independent, up to G-weak equivalence, of the choice of faithful G-representation. Indeed, if $\bar{V}$ is another faithful G-representation, then the G-maps

$$\Omega^V X(V) \ \xrightarrow{\Omega^V(\tilde{\sigma}_{V,\bar{V}})} \ \Omega^V(\Omega^{\bar{V}} X(V \oplus \bar{V})) \ \cong \ \Omega^{\bar{V}}(\Omega^V, X(\bar{V} \oplus V)) \ \xleftarrow{\Omega^{\bar{V}}(\tilde{\sigma}_{\bar{V},V})} \ \Omega^{\bar{V}} X(\bar{V})$$

are G-weak equivalences.

As G varies, the equivariant infinite loop spaces X_G are closely related to each other. For example, if H is a subgroup of G, then any faithful G-representation is also faithful as an H-representation. So $X[H]$ is H-weakly equivalent to the restriction of the G-equivariant infinite loop space $X[G]$.

Remark 3.14. Let X be a global Ω-spectrum. Specialized to the trivial group, the condition in Definition 3.11 says that X is in particular a non-equivariant Ω-spectrum in the sense that the adjoint structure map $\tilde{\sigma}_{V,\mathbb{R}} : X(V) \longrightarrow \Omega X(V \oplus \mathbb{R})$ is a weak equivalence of (non-equivariant) spaces for every inner product space V.

If X is a global Ω-spectrum, then so is the shifted spectrum $\operatorname{sh} X$ and the function spectrum $\operatorname{map}(K, X)$ for every cofibrant based space K. Indeed, if W is a faithful G-representation, then $W \oplus \mathbb{R}$, the sum with a trivial 1-dimensional representation, is also faithful. So the adjoint structure map

$$\tilde{\sigma}^{X}_{V,W\oplus\mathbb{R}} \ : \ X(W \oplus \mathbb{R}) \ \longrightarrow \ \Omega^V(X(V \oplus W \oplus \mathbb{R}))$$

is a G-weak equivalence for every G-representation V. But this map is also the adjoint structure map

$$\tilde{\sigma}^{\operatorname{sh} X}_{V,W} \ : \ (\operatorname{sh} X)(W) \ \longrightarrow \ \Omega^V((\operatorname{sh} X)(V \oplus W))$$

of the shifted spectrum.

The argument for mapping spectra is similar. The mapping space functor $\operatorname{map}(K, -)$ takes the G-weak equivalence $\tilde{\sigma}_{V,W} : X(W) \longrightarrow \Omega^V X(V \oplus W)$ to a G-weak equivalence

$$\operatorname{map}(K, X(W)) \ \xrightarrow{\operatorname{map}(K,\tilde{\sigma}_{V,W})} \ \operatorname{map}(K, \Omega^V X(V \oplus W)) \ ,$$

and this map is G-homeomorphic to the adjoint structure map

$$\operatorname{map}(K, X)(W) \ \longrightarrow \ \Omega^V(\operatorname{map}(K, X)(V \oplus W))$$

of the mapping spectrum $\operatorname{map}(K, X)$.

Definition 3.15. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *global fibration* if it is a strong level fibration and for every compact Lie group G, every G-representation V and every faithful G-representation W the square

$$\begin{tikzcd} X(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"] \arrow[d, "f(W)^G"'] & \operatorname{map}^G(S^V, X(V \oplus W)) \arrow[d, "\operatorname{map}^G(S^V, f(V\oplus W))"] \\ Y(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"'] & \operatorname{map}^G(S^V, Y(V \oplus W)) \end{tikzcd} \tag{3.16}$$

is homotopy cartesian.

We state a useful criterion for checking when a morphism is a global fibration. We recall from Construction III.1.31 the homotopy fiber $F(f)$ of a morphism $f : X \longrightarrow Y$ of orthogonal spectra along with the natural map $i : \Omega Y \longrightarrow F(f)$. We apply this for the shifted morphism $\operatorname{sh} f : \operatorname{sh} X \longrightarrow \operatorname{sh} Y$ and precompose with the morphism $\tilde{\lambda}_Y$ (see (1.23) of Chapter III) and denote by $\xi(f)$ the resulting composite

$$Y \ \xrightarrow{\tilde{\lambda}_Y} \ \Omega(\operatorname{sh} Y) \ \xrightarrow{i} \ F(\operatorname{sh} f) \ .$$

We note that the orthogonal spectrum $F(\mathrm{Id}_{\operatorname{sh} X})$ has a preferred contraction, so part (ii) of the next proposition is a way to make precise that the sequence

$$X \ \xrightarrow{f} \ Y \ \xrightarrow{\xi(f)} \ F(\operatorname{sh} f)$$

is a 'strong level homotopy fiber sequence'.

Proposition 3.17. *Let $f : X \longrightarrow Y$ be a strong level fibration of orthogonal spectra. Then the following two conditions are equivalent.*

(i) *The morphism f is a global fibration.*
(ii) *The strict fiber of f is a global Ω-spectrum and the commutative square*

$$\begin{tikzcd} X \arrow[r, "\xi(\mathrm{Id}_X)"] \arrow[d, "f"'] & F(\mathrm{Id}_{\operatorname{sh} X}) \arrow[d] \\ Y \arrow[r, "\xi(f)"'] & F(\operatorname{sh} f) \end{tikzcd}$$

is homotopy cartesian in the strong level model structure.

PROOF. (i)$\Longrightarrow$(ii) Since the square (3.16) is homotopy cartesian and the vertical maps are Serre fibrations, the induced map on strict fibers $(f^{-1}(*)(W))^G \longrightarrow \mathrm{map}^G(S^V, f^{-1}(*)(V \oplus W))$ is a weak equivalence. So the strict fiber is a global Ω-spectrum.

The square of condition (ii) factors as the composite of two commutative squares:

$$\begin{tikzcd} X \arrow[r, "\tilde{\lambda}_X"] \arrow[d, "f"'] & \Omega\,\mathrm{sh}\,X \arrow[r, "i"] \arrow[d, "\Omega\,\mathrm{sh}\,f"] & F(\mathrm{Id}_{\mathrm{sh}\,X}) \arrow[d] \\ Y \arrow[r, "\tilde{\lambda}_Y"'] & \Omega\,\mathrm{sh}\,Y \arrow[r, "i"'] & F(\mathrm{sh}\,f) \end{tikzcd}$$

Specializing the global fibration property (3.16) for $W = \mathbb{R}$ with trivial G-action shows that the left square is homotopy cartesian in the strong level model structure. For every continuous based map $g : A \longrightarrow B$ the square

$$\begin{tikzcd} \Omega A \arrow[r, "i"] \arrow[d, "\Omega g"'] & F(\mathrm{Id}_A) \arrow[d] \\ \Omega B \arrow[r, "i"'] & F(g) \end{tikzcd}$$

is homotopy cartesian; applying this to $g = ((\mathrm{sh}\,f)(V))^G$ for all G-representations V shows that the right square is homotopy cartesian in the strong level model structure. So the composite square is homotopy cartesian in the strong level model structure.

(ii)$\Longrightarrow$(i) We let G be a compact Lie group, V a G-representation, and W a faithful G-representation. We consider the commutative diagram

$$\begin{tikzcd} X(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"] \arrow[d, "f(W)^G"'] & \mathrm{map}^G(S^V, X(V \oplus W)) \arrow[d, "\mathrm{map}^G(S^V, f(V \oplus W))"] \\ Y(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"'] \arrow[d, "(\xi(f)(W))^G"'] & \mathrm{map}^G(S^V, Y(V \oplus W)) \arrow[d, "\mathrm{map}^G(S^V, (\xi(f)(V \oplus W)))"] \\ F(f)(W \oplus \mathbb{R})^G \arrow[r, "\sim", "(\tilde{\sigma}_{V,W})^G"'] & \mathrm{map}^G(S^V, F(f)(V \oplus W \oplus \mathbb{R})) \end{tikzcd}$$

Since f is a strong level fibration the natural morphism from the strict fiber to the homotopy fiber is a strong level equivalence. So the lower horizontal map is a weak equivalence because the strict fiber, and hence the homotopy fiber, is a global Ω-spectrum. Moreover, the two vertical columns are homotopy fiber sequences by hypothesis, so the upper square is homotopy cartesian, both with respect to the strong level model structure. $\square$

DEFINITION 3.18. Let $\mathcal{F}$ be a global family.

- A morphism $f : X \longrightarrow Y$ of orthogonal spectra is an *$\mathcal{F}$-equivalence* if the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all G in $\mathcal{F}$ and all integers k.
- An orthogonal spectrum X is an *$\mathcal{F}$-Ω-spectrum* if for every compact Lie group G in $\mathcal{F}$, every G-representation V and every faithful G-representation W the adjoint structure map

$$\tilde{\sigma}_{V,W} \ : \ X(W) \ \longrightarrow \ \mathrm{map}(S^V, X(V \oplus W))$$

  is a G-weak equivalence.

When $\mathcal{F} = \mathcal{A}ll$ is the maximal global family of all compact Lie groups, then an $\mathcal{A}ll$-equivalence is just a global equivalence in the sense of Definition 1.3. Also, an $\mathcal{A}ll$-Ω-spectrum is the same as a global Ω-spectrum in the sense of Definition 3.11. When $\mathcal{F} = \langle e \rangle$ is the minimal global family of all trivial groups, then the $\langle e \rangle$-equivalences are just the traditional, non-equivariant, stable equivalences of orthogonal spectra, also known as 'π_*-isomorphisms'. The $\langle e \rangle$-Ω-spectra are just the traditional, non-equivariant, Ω-spectra.

The following diagram collects various notions of equivalences and their implications that play a role in this book:

$$\begin{tikzcd}
\text{homotopy equivalence} \arrow[r, Rightarrow] & \text{strong level equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{global equivalence} \arrow[d, Rightarrow] \\
& \mathcal{F}\text{-level equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathcal{F}\text{-equivalence} \arrow[d, Rightarrow] \\
& \text{level equivalence} \arrow[r, Rightarrow] & \text{stable equivalence}
\end{tikzcd}$$

The following proposition collects various useful relations between $\mathcal{F}$-equivalence, $\mathcal{F}$-level equivalences and $\mathcal{F}$-Ω-spectra.

Proposition 3.19. *Let $\mathcal{F}$ be a global family.*

(i) *Every $\mathcal{F}$-level equivalence of orthogonal spectra is an $\mathcal{F}$-equivalence.*
(ii) *Let X be an $\mathcal{F}$-Ω-spectrum. Then for every G in $\mathcal{F}$, every faithful G-representation V and every $k \geq 0$ the stabilization map*

$$[S^{V \oplus \mathbb{R}^k}, X(V)]^G \ \longrightarrow\ \pi_k^G(X)\ , \quad [f] \ \longmapsto\ \langle f \rangle$$

is bijective.
(iii) *Let X be an $\mathcal{F}$-Ω-spectrum such that $\pi_k^G(X) = 0$ for every integer k and all G in $\mathcal{F}$. Then for every group G in $\mathcal{F}$ and every faithful G-representation V the space $X(V)$ is G-weakly contractible.*
(iv) *Every $\mathcal{F}$-equivalence between $\mathcal{F}$-Ω-spectra is an $\mathcal{F}$-level equivalence.*
(v) *Every $\mathcal{F}$-equivalence that is also a global fibration is an $\mathcal{F}$-level equivalence.*

Proof. (i) We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-level equivalence, and we need to show that the map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all integers k and all G in the global family $\mathcal{F}$. We start with the case $k = 0$. We let G be a group from the family $\mathcal{F}$ and V a finite dimensional G-subspace of the complete G-universe $\mathcal{U}_G$ such that G acts faithfully on V. By Proposition 3.7 (i) the map $f(V) : X(V) \longrightarrow Y(V)$ is a G-weak equivalence. Since the representation sphere S^V can be given a G-CW-structure, the induced map on G-homotopy classes

$$[S^V, f(V)]^G \ :\ [S^V, X(V)]^G \ \longrightarrow\ [S^V, Y(V)]^G$$

is bijective. The faithful G-representations are cofinal in the poset $s(\mathcal{U}_G)$, so taking the colimit over $V \in s(\mathcal{U}_G)$ shows that $\pi_0^G(f) : \pi_0^G(X) \longrightarrow \pi_0^G(Y)$ is an isomorphism. For $k > 0$ we exploit that $\pi_k^G(X)$ is naturally isomorphic to $\pi_0^G(\Omega^k X)$ and the k-fold loop of an $\mathcal{F}$-level equivalence is again an $\mathcal{F}$-level equivalence. For $k < 0$ we exploit that $\pi_k^G(X)$ is naturally isomorphic to $\pi_0^G(\mathrm{sh}^{-k} X)$ and that every shift of an $\mathcal{F}$-level equivalence is again an $\mathcal{F}$-level equivalence.

(ii) We start with the special case where V is a G-invariant subspace of the complete G-universe $\mathcal{U}_G$ used to define $\pi_k^G(X)$. Since V is faithful, so is every G-representation that contains V. So the directed system whose colimit is $\pi_k^G(X)$ consists of isomorphisms 'above V'. So the canonical map

$$[S^{V \oplus \mathbb{R}^k}, X(V)]^G \ \longrightarrow\ \pi_k^G(X)$$

is bijective.

If V is an arbitrary G-representation we can choose a G-equivariant linear isometry $\alpha : V \longrightarrow \bar{V}$ to a finite dimensional G-subrepresentation of the complete universe $\mathcal{U}_G$. The map in question is the composite

$$[S^{V\oplus\mathbb{R}^k}, X(V)]^G \;\longrightarrow\; [S^{\bar{V}\oplus\mathbb{R}^k}, X(\bar{V})]^G \;\longrightarrow\; \pi_k^G(X)$$

of 'conjugation by α' and the canonical map that was shown to be bijective is the previous paragraph.

(iii) We adapt an argument of Lewis-May-Steinberger [**97**, I 7.12] to our context. Every $\mathcal{F}$-Ω-spectrum is in particular a non-equivariant Ω-spectrum; every non-equivariant Ω-spectrum with trivial homotopy groups is levelwise weakly contractible, so this takes care of trivial groups.

Now we let G be a non-trivial group in $\mathcal{F}$. We argue by a nested induction over the 'size' of G: we induct over the dimension of G and, for fixed dimension, over the cardinality of the finite set $\pi_0 G$ of path components of G. Every proper subgroup H of G either has strictly smaller dimension than G, or the same dimension but fewer path components. So we know by induction that the fixed point space $X(V)^H$ is weakly contractible for every proper subgroup H of G. So it remains to analyze the G-fixed points of $X(V)$.

We let $W = V - V^G$ be the orthogonal complement of the fixed subspace V^G of V. Then W is a faithful G-representation with trivial fixed points. The cofiber sequence of G-CW-complexes

$$S(W)_+ \;\longrightarrow\; D(W)_+ \;\longrightarrow\; S^W$$

induces a fiber sequence of equivariant mapping spaces

$$\mathrm{map}^G(S^W, X(V)) \;\longrightarrow\; \mathrm{map}^G(D(W)_+, X(V)) \;\longrightarrow\; \mathrm{map}^G(S(W)_+, X(V))\ .$$

Since $W^G = 0$, the G-fixed points $S(W)^G$ are empty, so any G-CW-structure on $S(W)$ uses only equivariant cells of the form $G/H \times D^n$ for proper subgroups H of G. But for proper subgroups H, all the fixed point spaces $X(V)^H$ are weakly contractible by induction. Hence the space $\mathrm{map}^G(S(W)_+, X(V))$ is weakly contractible. Since the unit disc $D(W)$ is equivariantly contractible, the space $\mathrm{map}^G(D(W)_+, X(V))$ is homotopy equivalent to $X(V)^G$, and we conclude that evaluation at the fixed point $0 \in W$ is a weak equivalence

$$\mathrm{map}^G(S^W, X(V)) \;\simeq\; X(V)^G\ .$$

We have $X(V) = X(W \oplus V^G) = (\mathrm{sh}^{V^G} X)(W)$. The stabilization map

$$\pi_k\left(\mathrm{map}^G(S^W, X(V))\right) \;=\; [S^{W\oplus\mathbb{R}^k}, (\mathrm{sh}^{V^G} X)(W)]^G \;\longrightarrow\; \pi_k^G(\mathrm{sh}^{V^G} X) \cong \pi_{k-\dim(V^G)}^G(X)$$

is bijective for all $k \geq 0$, by part (ii). Since we assumed that the G-equivariant homotopy groups of X vanish, the space $\mathrm{map}^G(S^W, X(V))$ is path connected and has vanishing homotopy groups, i.e., it is weakly contractible. So $X(V)^G$ is weakly contractible, and this completes the proof of (iii).

(iv) Let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence between $\mathcal{F}$-Ω-spectra. We let F denote the homotopy fiber of f, and we let G be a group from the family $\mathcal{F}$. For every G-representation V the G-space $F(V)$ is then G-homeomorphic to the homotopy fiber of $f(V) : X(V) \longrightarrow Y(V)$. So F is again an $\mathcal{F}$-Ω-spectrum. The long exact sequence of homotopy groups (see Proposition III.1.37) implies that $\pi_*^H(F) = 0$ for all H in $\mathcal{F}$.

If G acts faithfully on V, then by the $\mathcal{F}$-Ω-spectrum property, the space $X(V)$ is G-weakly equivalent to $\Omega X(V \oplus \mathbb{R})$ and similarly for Y. So the map $f(V)$ is G-weakly equivalent to

$$\Omega f(V \oplus \mathbb{R}) \;:\; \Omega X(V \oplus \mathbb{R}) \;\longrightarrow\; \Omega X(V \oplus \mathbb{R})\ .$$

Hence we have a homotopy fiber sequence of G-spaces

$$X(V) \;\xrightarrow{f(V)}\; Y(V) \;\longrightarrow\; F(V \oplus \mathbb{R})\ .$$

Since F is an $\mathcal{F}$-Ω-spectrum with vanishing equivariant homotopy groups for groups in $\mathcal{F}$, the space $F(V\oplus\mathbb{R})$ is G-weakly contractible by part (iii). So $f(V)$ is a G-weak equivalence. The morphism f is then an $\mathcal{F}$-level equivalence by the criterion of Proposition 3.7.

(v) We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence and a global fibration. Then the strict fiber $f^{-1}(*)$ of f is a global Ω-spectrum with trivial G-equivariant homotopy groups for all $G \in \mathcal{F}$. So $f^{-1}(*)$ is $\mathcal{F}$-level equivalent

to the trivial spectrum by part (iii). Since f is a strong level fibration, the embedding $f^{-1}(*) \longrightarrow F(f)$ of the strict fiber into the homotopy fiber $F(f)$ is a strong level equivalence, so the homotopy fiber $F(f)$ is $\mathcal{F}$-level equivalent to the trivial spectrum. The homotopy cartesian square of Proposition 3.17 (ii) then shows that f is an $\mathcal{F}$-level equivalence. □

Our next result, in fact the main result of this section, is the $\mathcal{F}$-global model structure. This model structure is compactly generated, and we spell out an explicit set of generating cofibrations and generating acyclic cofibrations. The set $I_{\mathcal{F}}$ was defined in the proof of Proposition 3.8 as the set of morphisms $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (2.12) of Section A.2. Similarly, the set $J_{\mathcal{F}}$ is the set of morphisms $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (2.12) of Section A.2.

Given any compact Lie group G and G-representations V and W we recall from (1.30) the morphism

$$\lambda_{G,V,W} = \lambda_{V,W}/G \ : \ F_{G,V\oplus W}S^V \ \longrightarrow \ F_{G,W} \ .$$

If the representation W is faithful, then this morphism is a global equivalence by Theorem 1.31. We set

$$K_{\mathcal{F}} \ = \ \bigcup_{G,V,W} \mathcal{Z}(\lambda_{G,V,W}) \ , \tag{3.20}$$

the set of all pushout products of sphere inclusions $i_m : \partial D^m \longrightarrow D^m$ with the mapping cylinder inclusions of the morphisms $\lambda_{G,V,W}$ (compare Construction I.3.16); here the union is over a set of representatives of the isomorphism classes of triples (G, V, W) consisting of a compact Lie group G in the family $\mathcal{F}$, a G-representation V and a faithful G-representation W.

Theorem 3.21 ($\mathcal{F}$-global model structure). *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-equivalences and $\mathcal{F}$-cofibrations are part of a model structure on the category of orthogonal spectra, the* $\mathcal{F}$-global model structure.

(ii) *A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a fibration in the $\mathcal{F}$-global model structure precisely when it is an $\mathcal{F}$-level fibration and for every compact Lie group G in $\mathcal{F}$, every G-representation V and every faithful G-representation W the square*

$$\begin{array}{ccc} X(W)^G & \xrightarrow{(\tilde{\sigma}_{V,W})^G} & \mathrm{map}^G(S^V, X(V\oplus W)) \\ {\scriptstyle f(W)^G}\downarrow & & \downarrow{\scriptstyle \mathrm{map}^G(S^V, f(V\oplus W))} \\ Y(W)^G & \xrightarrow[(\tilde{\sigma}_{V,W})^G]{} & \mathrm{map}^G(S^V, Y(V\oplus W)) \end{array} \tag{3.22}$$

is homotopy cartesian.

(iii) *The fibrant objects in the $\mathcal{F}$-global model structure are the $\mathcal{F}$-Ω-spectra.*

(iv) *A morphism of orthogonal spectra is:*
- *an acyclic fibration in the $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $I_{\mathcal{F}}$;*
- *a fibration in the $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$.*

(v) *The $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

(vi) *The adjoint functor pair*

$$spc \ \underset{\Omega^\bullet}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \ \mathcal{S}p$$

is a Quillen pair for the two $\mathcal{F}$-global model structures on orthogonal spaces and orthogonal spectra.

Proof. The category of orthogonal spectra is complete and cocomplete (MC1), the $\mathcal{F}$-equivalences satisfy the 2-out-of-3 property (MC2) and the classes of $\mathcal{F}$-equivalences, $\mathcal{F}$-global fibrations and $\mathcal{F}$-cofibrations are closed under retracts (MC3). The $\mathcal{F}$-level model structure (Proposition 3.8) shows that every morphism of orthogonal spectra can be factored as an $\mathcal{F}$-cofibration followed by an $\mathcal{F}$-level equivalence. Since $\mathcal{F}$-level equivalences are in particular $\mathcal{F}$-equivalences, this provides one of the factorizations as required by MC5.

The morphism $\lambda_{G,V,W}$ represents the map $(\tilde{\sigma}_{V,W})^G : X(W)^G \longrightarrow \mathrm{map}^G(S^V, X(V \oplus W))^G$; by Proposition I.3.17, the right lifting property with respect to the union $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ thus characterizes the morphisms in part (ii). We apply the small object argument (see for example [**47**, 7.12] or [**79**, Thm. 2.1.14]) to the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$. All morphisms in $J_{\mathcal{F}}$ are flat cofibrations and $\mathcal{F}$-level equivalences; $\mathcal{F}$-level equivalences are $\mathcal{F}$-equivalences by Proposition 3.19 (i). Since $F_{G,V\oplus W}S^V$ and $F_{G,W}$ are flat, the morphisms in $K_{\mathcal{F}}$ are also flat cofibrations, and they are $\mathcal{F}$-equivalences because the morphisms $\lambda_{G,V,W}$ are. The small object argument provides a functorial factorization of every morphism $\varphi : X \longrightarrow Y$ of orthogonal spectra as a composite

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ and q has the right lifting property with respect to $J_{\mathcal{F}} \cup K_{\mathcal{F}}$; in particular, the morphism q satisfies the conditions of part (ii). All morphisms in $J_{\mathcal{F}} \cup K_{\mathcal{F}}$ are $\mathcal{F}$-equivalences and $\mathcal{F}$-cofibrations, hence also h-cofibrations (by Proposition A.1.18 applied to the strong level model structure). By Corollary IV.1.5 (or rather its modification for $\mathcal{F}$-equivalences), the class of h-cofibrations that are simultaneously $\mathcal{F}$-equivalences is closed under coproducts, cobase change and sequential composition. So the morphism i is an $\mathcal{F}$-cofibration and an $\mathcal{F}$-equivalence.

Now we show the lifting properties MC4. By Proposition 3.19 (v) a morphism that is both a global fibration and an $\mathcal{F}$-equivalence is an $\mathcal{F}$-level equivalence, and hence an acyclic fibration in the $\mathcal{F}$-level model structure. So every morphism that is simultaneously a global fibration and an $\mathcal{F}$-equivalence has the right lifting property with respect to $\mathcal{F}$-cofibrations. Now we let $j : A \longrightarrow B$ be an $\mathcal{F}$-cofibration that is also an $\mathcal{F}$-equivalence and we show that it has the left lifting property with respect to all morphisms mentioned in (ii). We factor $j = q \circ i$, via the small object argument for $J_{\mathcal{F}} \cup K_{\mathcal{F}}$, where $i : A \longrightarrow W$ is a $(J_{\mathcal{F}} \cup K_{\mathcal{F}})$-cell complex and $q : W \longrightarrow B$ has the properties of part (ii). Then q is an $\mathcal{F}$-equivalence since j and i are, so q is an acyclic fibration in the $\mathcal{F}$-level model structure, again by Proposition 3.19 (v). Since j is an $\mathcal{F}$-cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting property with respect to all morphisms of part (ii). But then j itself has this lifting property. This finishes the verification of the model category axioms. Alongside we have also specified sets of generating flat cofibrations $I_{\mathcal{F}}$ and generating acyclic cofibrations $J_{\mathcal{F}} \cup K_{\mathcal{F}}$. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of flat cofibrations. So the $\mathcal{F}$-model structure is cofibrantly generated.

Left properness of the $\mathcal{F}$-global model structure follows from Corollary III.1.40 (i) and the fact that $\mathcal{F}$-cofibrations are h-cofibrations. Right properness follows from Corollary III.1.40 (ii) and the fact that the morphisms of (ii) are in particular $\mathcal{F}$-level fibrations.

It remains to show that the $\mathcal{F}$-global model structure is topological. The cofibrations in the $\mathcal{F}$.global model structure coincide with the cofibrations in the $\mathcal{F}$-level model structure, so the pushout product of a cofibration of spaces with an $\mathcal{F}$-cofibration is an $\mathcal{F}$-cofibration by Proposition 3.8. Similarly, the pushout product of an acyclic cofibration of spaces with an $\mathcal{F}$-cofibration is an $\mathcal{F}$-level equivalence by Proposition 3.8, hence an $\mathcal{F}$-equivalence. Finally, we have to show that pushout products of cofibrations of

spaces with $\mathcal{F}$-cofibrations that are also $\mathcal{F}$-equivalences are again $\mathcal{F}$-equivalences. It suffices to consider a generating cofibration $i_k : \partial D^k \longrightarrow D^k$ of spaces and a generating acyclic cofibration in the set $J_{\mathcal{F}} \cup K_{\mathcal{F}}$. The morphisms in $J_{\mathcal{F}}$ are $\mathcal{F}$-level equivalences, hence taken care of by Proposition 3.8 again. The pushout product of i_k and a morphism $i_m \Box c(j)$ in $\mathcal{Z}(\lambda_{G,V,W})$ is isomorphic to $i_{k+m} \Box c(j)$, hence again an $\mathcal{F}$-cofibration and $\mathcal{F}$-equivalence.

Part (vi) is straightforward from the definitions. □

In the case $\mathcal{F} = \langle e \rangle$ of the minimal global family of trivial groups, the $\langle e \rangle$-equivalences are the (non-equivariant) stable equivalences of orthogonal spectra, and the $\langle e \rangle$-global model structure coincides with the stable model structure established by Mandell, May, Shipley and the author in [**104**, Thm. 9.2].

The same proof as in the unstable situation in Corollary I.4.10 applies to prove the following characterization of $\mathcal{F}$-equivalences.

Corollary 3.23. *Let $f : A \longrightarrow B$ be a morphism of orthogonal spectra and $\mathcal{F}$ a global family. Then the following conditions are equivalent.*

(i) *The morphism f is an $\mathcal{F}$-equivalence.*
(ii) *The morphism can be written as $f = w_2 \circ w_1$ for an $\mathcal{F}$-level equivalence w_2 and a global equivalence w_1.*
(iii) *For some (hence any) $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ in the $\mathcal{F}$-level model structure and every $\mathcal{F}$-Ω-spectrum Y the induced map*

$$[f^c, Y] \ : \ [B^c, Y] \ \longrightarrow \ [A^c, Y]$$

on homotopy classes of morphisms is bijective.

Remark 3.24 (Mixed global model structures). Cole's 'mixing theorem' for model structures [**37**, Thm. 2.1] allows to construct many more $\mathcal{F}$-model structures on the category of orthogonal spectra. We consider two global families such that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is an $\mathcal{F}$-equivalence and every fibration in the $\mathcal{E}$-global model structure is a fibration in the $\mathcal{F}$-global model structure. By Cole's theorem [**37**, Thm. 2.1] the $\mathcal{F}$-equivalences and the fibrations of the $\mathcal{E}$-global model structure are part of a model structure, the *$\mathcal{E}$-mixed $\mathcal{F}$-global model structure* on the category of orthogonal spectra. By [**37**, Prop. 3.2] the cofibrations in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure are precisely the retracts of all composites $h \circ g$ in which g is an $\mathcal{F}$-cofibration and h is simultaneously an $\mathcal{E}$-equivalence and an $\mathcal{E}$-cofibration. In particular, an orthogonal spectrum is cofibrant in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure if it is $\mathcal{E}$-cofibrant and $\mathcal{E}$-equivalent to an $\mathcal{F}$-cofibrant orthogonal spectrum [**37**, Cor. 3.7]. The $\mathcal{E}$-mixed $\mathcal{F}$-global model structure is again proper (Propositions 4.1 and 4.2 of [**37**]).

When $\mathcal{F} = \langle e \rangle$ is the minimal family of trivial groups, this provides infinitely many $\mathcal{E}$-mixed model structures on the category of orthogonal spectra that are all Quillen equivalent. In the extreme case, the $\mathcal{A}ll$-mixed $\langle e \rangle$-model structure is the $\mathbb{S}$-model structure of Stolz [**150**, Prop. 1.3.10].

Remark 3.25 ($\mathcal{F}in$-global homotopy theory via symmetric spectra). We denote by $\mathcal{F}in$ the global family of finite groups. The $\mathcal{F}in$-global stable homotopy theory has another very natural model, namely the category of symmetric spectra in the sense of Hovey, Shipley and Smith [**80**]. In [**71, 72**] Hausmann has established a global model structure on the category of symmetric spectra, and he showed that the forgetful functor is a right Quillen equivalence from the category of orthogonal spectra with the $\mathcal{F}in$-global model structure to the category of symmetric spectra with the global model structure. Symmetric spectra cannot model global homotopy types for all compact Lie groups, basically because compact Lie groups of positive dimensions do not have any faithful permutation representations.

For easier reference we spell out the special case $\mathcal{F} = \mathcal{A}ll$ for the maximal family of all compact Lie groups, resulting in the *global model structure* on the category of orthogonal spectra.

Theorem 3.26 (Global model structure). *The global equivalences, global fibrations and flat cofibrations form a model structure, the* global model structure *on the category of orthogonal spectra. The fibrant objects*

in the global model structure are the global Ω-spectra. The global model structure is proper, topological and compactly generated.

One of the points of model category structures is that they facilitate the analysis of and constructions in the homotopy category (which only depends on the class of weak equivalences). An example of this is the existence and constructions of products and coproducts in a homotopy category. We take the time to make this explicit for the $\mathcal{F}$-global stable homotopy category, for which we write $\mathcal{GH}_{\mathcal{F}}$.

PROPOSITION 3.27. *Let $\mathcal{F}$ be a global family.*

(i) *For every family of orthogonal spectra, the coproduct (i.e., wedge) of orthogonal spectra is also a coproduct in the homotopy category $\mathcal{GH}_{\mathcal{F}}$.*

(ii) *Let $\{X_i\}_{i\in I}$ be a family of orthogonal spectra such that the canonical map*

$$\pi_k^G\big(\prod_{i\in I} X_i\big) \;\longrightarrow\; \prod_{i\in I}\pi_k^G(X_i)$$

is an isomorphism for every compact Lie group G in $\mathcal{F}$ and every integer k. Then the product $\prod_{i\in I} X_i$ of orthogonal spectra is also a product of $\{X_i\}_{i\in I}$ in the homotopy category $\mathcal{GH}_{\mathcal{F}}$.

In particular, the $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$ has all set indexed coproducts and products.

PROOF. (i) Since the $\mathcal{F}$-equivalences are part of the $\mathcal{F}$-global model structure, general model category theory guarantees that coproducts in $\mathcal{GH}_{\mathcal{F}}$ can be constructed by taking the pointset level coproduct of $\mathcal{F}$-cofibrant approximations, see for example [**79**, Ex. 1.3.11]. Since equivariant homotopy groups take wedges of orthogonal spectra to direct sums (Corollary III.1.38 (i)), any wedge of $\mathcal{F}$-equivalences is again an $\mathcal{F}$-equivalence. So the pointset level wedge maps by an $\mathcal{F}$-equivalence to the wedge of the $\mathcal{F}$-cofibrant approximations, and these two are isomorphic in $\mathcal{GH}_{\mathcal{F}}$.

(ii) This part is essentially dual to (i), with the caveat that infinite products of $\mathcal{F}$-equivalences need not be $\mathcal{F}$-equivalences in general. To construct a product in $\mathcal{GH}_{\mathcal{F}}$ of the given family, the abstract recipe is to choose $\mathcal{F}$-equivalences $f_i : X_i \longrightarrow X_i^{\mathcal{F}}$ to $\mathcal{F}$-Ω-spectra, and then form the pointset level product of the replacements. For all groups G in $\mathcal{F}$ we consider the commutative square

$$\begin{tikzcd}
\pi_k^G\left(\prod_{i\in I} X_i\right) \arrow[r] \arrow[d, "(\prod f_i)_*"'] & \prod_{i\in I}\pi_k^G(X_i) \arrow[d, "\prod (f_i)_*"] \\
\pi_k^G\left(\prod_{i\in I} X_i^{\mathcal{F}}\right) \arrow[r] & \prod_{i\in I}\pi_k^G(X_i^{\mathcal{F}})
\end{tikzcd}$$

the upper map is an isomorphism by hypothesis, and the right map is an isomorphism since each f_i is an $\mathcal{F}$-equivalence. The lower map is also an isomorphism: since all $X_i^{\mathcal{F}}$ are $\mathcal{F}$-Ω-spectra, the colimit system

$$\begin{aligned}
\pi_k^G\big(\prod\nolimits_{i\in I} X_i^{\mathcal{F}}\big) \;&=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)}\left[S^{V\oplus\mathbb{R}^k}, \prod\nolimits_{i\in I} X_i^{\mathcal{F}}(V)\right]^G \\
&=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)}\left(\prod\nolimits_{i\in I}[S^{V\oplus\mathbb{R}^k}, X_i^{\mathcal{F}}(V)]^G\right)
\end{aligned}$$

consists of isomorphisms starting at all *faithful* G-representations V. Since faithful representations are cofinal in the poset $s(\mathcal{U}_G)$, in this particular situation the colimit commutes with the product. Altogether we can conclude that in our situation the morphism

$$\prod f_i \;:\; \prod\nolimits_{i\in I} X_i \;\longrightarrow\; \prod\nolimits_{i\in I} X_i^{\mathcal{F}}$$

is an $\mathcal{F}$-equivalence. Since the right hand side is a product in $\mathcal{GH}_{\mathcal{F}}$ of the family $\{X_i\}_{i\in I}$, so is the left hand side. $\square$

We close this section by studying the interaction of the smash product of orthogonal spectra with the level and global model structures. Given two morphisms $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ of orthogonal spectra we denote by $f\Box g$ the *pushout product* morphism defined as

$$f\Box g = (f \wedge Y) \cup (B \wedge g) \ : \ A \wedge Y \cup_{A\wedge X} B \wedge X \ \longrightarrow \ B \wedge Y \ .$$

We introduce another piece of notation that is convenient for the discussion of the monoidal properties: when I and I' are two sets of morphisms of orthogonal spectra, then we define

$$I \ \sqsubset \ I'$$

to mean that every morphism in I is isomorphic to a morphism in I'. Moreover, we write $I\Box I'$ for the set of pushout product morphisms $f\Box g$ for all $f \in I$ and all $g \in I'$. If $\mathcal{E}$ and $\mathcal{F}$ are global families, then we denote by $\mathcal{E} \times \mathcal{F}$ the smallest global family that contains all groups of the form $G \times K$ for $G \in \mathcal{E}$ and $K \in \mathcal{F}$.

The sets $I_{\mathcal{F}}$ and $J_{\mathcal{F}}$ of generating cofibrations respectively generating acyclic cofibrations for the $\mathcal{F}$-level model structure were defined in the proof of Proposition 3.8. They consist of the morphisms

$$F_{H,\mathbb{R}^m} \wedge (i_k)_+ \qquad \text{respectively} \qquad F_{H,\mathbb{R}^m} \wedge (j_k)_+$$

for all $k \geq 0$, all $m \geq 0$ and all compact Lie groups H in $\mathcal{F}(m)$. Here we used that the orthogonal spectrum $F_{G,V}(G/H_+)$ is isomorphic to $F_{H,V}$. The set $K_{\mathcal{F}}$ of acyclic cofibrations for the $\mathcal{F}$-global model structure was defined in (3.20).

Proposition 3.28. *Let $\mathcal{E}$ and $\mathcal{F}$ be two global families. Then the following relations hold for the sets of generating cofibrations and acyclic cofibrations:*

$$I_{\mathcal{E}} \,\Box\, I_{\mathcal{F}} \ \sqsubset \ I_{\mathcal{E}\times\mathcal{F}} \ , \qquad I_{\mathcal{E}} \,\Box\, J_{\mathcal{F}} \ \sqsubset \ J_{\mathcal{E}\times\mathcal{F}} \qquad \text{and} \qquad I_{\mathcal{E}} \,\Box\, K_{\mathcal{F}} \ \sqsubset \ K_{\mathcal{E}\times\mathcal{F}} \ .$$

Proof. We start with two key observations concerning the generating cofibrations $i_k : \partial D^k \longrightarrow D^k$ and the generating acyclic cofibrations $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ for the model structure of spaces: the pushout product $i_k\Box i_m$ of two sphere inclusions is homeomorphic to the map i_{k+m}; similarly, the pushout product $i_k\Box j_m$ is homeomorphic to the map j_{k+m}.

The first relation $I_{\mathcal{E}}\Box I_{\mathcal{F}} \sqsubset I_{\mathcal{E}\times\mathcal{F}}$ is then a consequence of the compatibilities between smash products and the isomorphism (1.28) for the smash product of two free orthogonal spectra:

$$(F_{G,V} \wedge (i_k)_+)\Box(F_{K,W} \wedge (i_m)_+) \ \cong \ (F_{G,V} \wedge F_{K,W}) \wedge (i_k\Box i_m)_+ \ \cong \ F_{G\times K, V\oplus W} \wedge (i_{k+m})_+$$

The second relation $I_{\mathcal{E}}\Box J_{\mathcal{F}} \sqsubset J_{\mathcal{E}\times\mathcal{F}}$ is proved in the same way:

$$(F_{G,V} \wedge (i_k)_+)\Box(F_{K,W} \wedge (j_m)_+) \ \cong \ (F_{G,V} \wedge F_{K,W}) \wedge (i_k\Box j_m)_+ \ \cong \ F_{G\times K, V\oplus W} \wedge (j_{k+m})_+$$

For the third relation we recall that $c_{K,U,W}$ is the mapping cylinder inclusion of the global equivalence

$$\lambda_{K,U,W} \ : \ F_{K,U\oplus W}S^U \ \longrightarrow \ F_{K,W} \ .$$

The claim $I_{\mathcal{E}}\Box K_{\mathcal{F}} \sqsubset K_{\mathcal{E}\times\mathcal{F}}$ then follows from

$$(F_{G,V} \wedge (i_k)_+)\Box(c_{K,U,W}\Box(i_m)_+) \ \cong \ (F_{G,V} \wedge c_{K,U,W})\Box(i_k\Box i_m)_+ \ \cong \ c_{G\times K, V\oplus U, W} \wedge (i_{k+m})_+ \ . \qquad \Box$$

Certain pushout product properties are now formal consequences.

Proposition 3.29. *Let $\mathcal{E}$ and $\mathcal{F}$ be two global families.*

(i) *The pushout product of an $\mathcal{E}$-cofibration with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-cofibration.*
(ii) *The pushout product of a flat cofibration with an $\mathcal{F}$-cofibration that is also an $\mathcal{F}$-equivalence is a flat cofibration and global equivalence.*
(iii) *Let $\mathcal{F}$ be a multiplicative global family, i.e., $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. Then the $\mathcal{F}$-global model structure satisfies the pushout product property with respect to the smash product of orthogonal spectra.*

Proof. (i) It suffices to show the claim for a set of generating cofibrations, where it follows from the relation $I_{\mathcal{E}}\Box I_{\mathcal{F}} \sqsubset I_{\mathcal{E}\times\mathcal{F}}$ established in Proposition 3.28.

(ii) Again it suffices to check the pushout product of any generating flat cofibration with a generating acyclic cofibration for the $\mathcal{F}$-global model structure. For generators, the claim follows from the relation

$$I_{\mathcal{A}ll}\Box(J_{\mathcal{F}} \cup K_{\mathcal{F}}) \;=\; (I_{\mathcal{A}ll}\Box J_{\mathcal{F}}) \cup (I_{\mathcal{A}ll}\Box K_{\mathcal{F}}) \;\sqsubset\; J_{\mathcal{A}ll\times\mathcal{F}} \cup K_{\mathcal{A}ll\times\mathcal{F}} \;=\; J_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$$

established in Proposition 3.28.

(iii) The pushout product of two $\mathcal{F}$-cofibrations is an $\mathcal{F}$-cofibration by part (i) and the hypothesis that $\mathcal{F}$ is multiplicative. Since $\mathcal{F}$-cofibrations are in particular flat cofibrations, the pushout product of two $\mathcal{F}$-cofibrations one of which is also an $\mathcal{F}$-equivalence is another $\mathcal{F}$-equivalence by part (ii). $\Box$

In the special case where $\mathcal{E} = \mathcal{F} = \langle e\rangle = \mathcal{E}\times\mathcal{F}$ are the trivial global families, part (iii) of the previous proposition specializes to Proposition 12.6 of [**104**].

The sphere spectrum $\mathbb{S}$ is the unit object for the smash product of orthogonal spectra, and it is 'free', i.e., $\langle e\rangle$-cofibrant. Thus $\mathbb{S}$ is cofibrant in the $\mathcal{F}$-global model structure for every global family $\mathcal{F}$. So if $\mathcal{F}$ is multiplicative, then with respect to the smash product, the $\mathcal{F}$-global model structure is a symmetric monoidal model category in the sense of [**79**, Def. 4.2.6]. A corollary is that the homotopy category $\mathcal{GH}_{\mathcal{F}}$, i.e., the localization of the category of orthogonal spectra at the class of $\mathcal{F}$-equivalences, inherits a closed symmetric monoidal structure, compare [**79**, Thm. 4.3.3]. The *derived smash product*

$$\wedge^{\mathbb{L}}_{\mathcal{F}} \;:\; \mathcal{GH}_{\mathcal{F}} \times \mathcal{GH}_{\mathcal{F}} \;\longrightarrow\; \mathcal{GH}_{\mathcal{F}}\;, \tag{3.30}$$

i.e., the induced symmetric monoidal product on $\mathcal{GH}_{\mathcal{F}}$, is any total left derived functor of the smash product.

Corollary 3.31. *For every multiplicative global family $\mathcal{F}$, the $\mathcal{F}$-global homotopy category $\mathcal{GH}_{\mathcal{F}}$ is closed symmetric monoidal under the derived smash product* (3.30).

The value of the derived smash product at a pair (X, Y) of orthogonal spectra can be calculated as

$$X \wedge^{\mathbb{L}}_{\mathcal{F}} Y \;=\; X^c \wedge Y^c\;,$$

where $X^c \longrightarrow X$ and $Y^c \longrightarrow Y$ are cofibrant replacements in the $\mathcal{F}$-global model structure, i.e., $\mathcal{F}$-equivalences with $\mathcal{F}$-cofibrant sources. By the following 'flatness theorem', it actually suffices to 'resolve' only one of the factors, and it is enough to require the source of the 'resolution' to be flat (as opposed to being $\mathcal{F}$-cofibrant).

Theorem 3.32. *Let $\mathcal{F}$ be a global family.*

(i) *Smashing with a flat orthogonal spectrum preserves $\mathcal{F}$-equivalences.*
(ii) *Smashing with any orthogonal spectrum preserves $\mathcal{F}$-equivalences between flat orthogonal spectra.*

Proof. (i) [... restore the argument...]

(ii) We start with a special case where C is a flat orthogonal spectrum whose G-equivariant homotopy groups vanish for all $G \in \mathcal{F}$; we show that then for every orthogonal spectrum A the smash product $A \wedge C$ has vanishing G-equivariant homotopy groups for all $G \in \mathcal{F}$. To see this we choose a global equivalence $\varphi : A^\flat \longrightarrow A$ with flat source and an $\mathcal{F}$-equivalence $\psi : \bar{C} \longrightarrow C$ with $\mathcal{F}$-cofibrant source. Then the unique morphism $* \longrightarrow \bar{C}$ is an $\mathcal{F}$-cofibration and $\mathcal{F}$-equivalence, so $A^\flat \wedge \bar{C}$ is $\mathcal{F}$-equivalent to a trivial orthogonal spectrum by Proposition 3.29 (ii). Since $\mathcal{F}$-cofibrant spectra are in particular flat, the morphism

$$A^\flat \wedge \psi \;:\; A^\flat \wedge \bar{C} \;\longrightarrow\; A^\flat \wedge C$$

is an $\mathcal{F}$-equivalence by part (i). Moreover, the morphism $\varphi \wedge C : A^\flat \wedge C \longrightarrow A \wedge C$ is even a global equivalence by part (i). Altogether this shows that $A \wedge C$ is $\mathcal{F}$-equivalent to a trivial spectrum.

Now we treat the general case. We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence between flat orthogonal spectra. The mapping cone Cf is then flat and $\mathcal{F}$-equivalent to a trivial spectrum. So for every orthogonal spectrum A the smash product $A \wedge Cf$ is $\mathcal{F}$-equivalent to a trivial spectrum, by the previous paragraph.

Since $A \wedge Cf$ is isomorphic to the mapping cone of $A \wedge f : A \wedge X \longrightarrow A \wedge Y$, the long exact homotopy groups sequences show that $A \wedge f$ is an $\mathcal{F}$-equivalence. □

Now we can prove another important relationship between the global model structures and the smash product, namely the *monoid axiom* [**133**, Def. 3.3]. As in the unstable situation in Proposition I.4.14 we only discuss the weaker form of the monoid axiom with sequential (as opposed to more general transfinite) compositions.

PROPOSITION 3.33 (Monoid axiom). *We let $\mathcal{F}$ be a global family. For every flat cofibration $i : A \longrightarrow B$ that is also an $\mathcal{F}$-equivalence and every orthogonal spectrum Y the morphism*

$$i \wedge Y \ : \ A \wedge Y \ \longrightarrow \ B \wedge Y$$

is an h-cofibration and an $\mathcal{F}$-equivalence. Moreover, the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential compositions.

PROOF. Given Theorem 3.32, this is a standard argument, similar to the proofs of the monoid axiom in the non-equivariant context ([**104**, Prop. 12.5], [**150**, Prop. 1.3.10]) and the G-equivariant context ([**105**, III Prop. 7.4], [**150**, Prop. 2.3.27]). Every flat cofibration is an h-cofibration (Corollary A.1.18 (iii)), and h-cofibrations are closed under smashing with any orthogonal spectrum, so $i \wedge Y$ is an h-cofibration. Since i is a h-cofibration and $\mathcal{F}$-equivalence, its cokernel B/A is $\mathcal{F}$-stably contractible by the long exact homotopy group sequence (Corollary III.1.39). But B/A is also flat as a cokernel of a flat cofibration, so the spectrum $(B/A) \wedge Y$ is $\mathcal{F}$-stably contractible by Theorem 3.32 (ii). Since $i \wedge Y$ is an h-cofibration with cokernel isomorphic to $(B/A) \wedge Y$, the long exact homotopy group sequence then shows that $i \wedge Y$ is an $\mathcal{F}$-equivalence.

The proof that the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential compositions is the same as for the special case $\mathcal{F} = \mathcal{A}ll$ (i.e., for global equivalences) in Corollary 1.5. □

Every $\mathcal{F}$-cofibration is in particular a flat cofibration. So the monoid axiom implies the monoid axiom in the $\mathcal{F}$-global model structure. If the global family $\mathcal{F}$ is closed under products, Theorem [**133**, Thm. 4.1] applies to the $\mathcal{F}$-global model structure and shows the following lifting results. The additional claims in part (i) about the behavior of the forgetful functor on the cofibrations are proved as in the unstable analog in Corollary I.4.16.

COROLLARY 3.34. *Let R be an orthogonal ring spectrum and $\mathcal{F}$ a multiplicative global family.*

(i) *The category of R-modules admits the* $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spectra is an $\mathcal{F}$-equivalence (respectively fibration in the $\mathcal{F}$-global model structure). This model structure is cofibrantly generated. Every cofibration in this $\mathcal{F}$-global model structure is an h-cofibration of underlying orthogonal spectra. If the underlying orthogonal spectrum of R is $\mathcal{F}$-cofibrant, then every cofibration of R-modules is a $\mathcal{F}$-cofibration of underlying orthogonal spectra.*

(ii) *If R is commutative, then with respect to $\wedge_R$ the $\mathcal{F}$-global model structure of R-modules is a monoidal model category that satisfies the monoid axiom.*

(iii) *If R is commutative, then the category of R-algebras admits the* $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spectra is an $\mathcal{F}$-equivalence (respectively fibration in the $\mathcal{F}$-global model structure). Every cofibrant R-algebra is also cofibrant as an R-module.*

Strictly speaking, Theorem 4.1 of [**133**] does not apply verbatim to the $\mathcal{F}$-global model structures because the hypothesis that every object is small (with respect to some regular cardinal) is not satisfied. However, in our situation the sources of the generating cofibrations and generating acyclic cofibrations are small with respect to sequential compositions of h-cofibrations, and this suffices to run the countable small object argument (compare also Remark 2.4 of [**133**]).

PROPOSITION 3.35. *Let R be an orthogonal ring spectrum and N a right R-module that is cofibrant in the $\mathcal{A}ll$-global model structure of Corollary* 3.34 (i). *Then for every global family $\mathcal{F}$, the functor $N \wedge_R -$ takes $\mathcal{F}$-equivalences of left R-modules to $\mathcal{F}$-equivalences of orthogonal spaces.*

PROOF. The argument is completely parallel to the unstable precursor in Proposition I.4.17. We call a right R-module N *homotopical* if the functor $N \wedge_R -$ takes $\mathcal{F}$-equivalences of left R-modules to $\mathcal{F}$-equivalences of orthogonal spectra. Since the $\mathcal{A}ll$-global model structure on the category of right R-modules is obtained by lifting the global model structure of orthogonal spectra along the free and forgetful adjoint functor pair, every cofibrant right R-module is a retract of an R-module that arises as the colimit of a sequence

$$* = M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_k \longrightarrow \dots \tag{3.36}$$

in which each M_k is obtained from M_{k-1} as a pushout

$$\begin{tikzcd} A_k \wedge R \arrow[r, "f_k \wedge R"] \arrow[d] & B_k \wedge R \arrow[d] \\ M_{k-1} \arrow[r] & M_k \end{tikzcd}$$

for some flat cofibration $f_k : A_k \longrightarrow B_k$ between flat orthogonal spectra. For example, f_k can be chosen as a disjoint union of morphisms in the set I^{str} of generating flat cofibrations. We show by induction on k that each module M_k is homotopical. The induction starts with the trivial R-module, where there is nothing to show. Now we suppose that M_{k-1} is homotopical, and we claim that then M_k is homotopical as well. To see this we consider an $\mathcal{F}$-equivalence of left R-modules $\varphi : X \longrightarrow Y$. Then $M_k \wedge_R \varphi$ is obtained by passing to horizontal pushouts in the following commutative diagram of orthogonal spectra:

$$\begin{tikzcd} M_{k-1} \wedge_R X \arrow[d, "M_{k-1} \wedge_R \varphi"'] & A_k \wedge X \arrow[l] \arrow[r, "f_k \wedge X"] \arrow[d, "A_k \wedge \varphi"] & B_k \wedge X \arrow[d, "B_k \wedge \varphi"] \\ M_{k-1} \wedge_R Y & A_k \wedge Y \arrow[l] \arrow[r, "f_k \wedge Y"'] & B_k \wedge Y \end{tikzcd}$$

Here we have exploited that $(A_k \wedge R) \wedge_R X$ is naturally isomorphic to $A_k \wedge X$. In the diagram, the left vertical morphism is an $\mathcal{F}$-equivalence by hypothesis. The middle and right vertical morphisms are $\mathcal{F}$-equivalences because smash product with a flat orthogonal spectrum preserves $\mathcal{F}$-equivalences (Theorem 3.32 (ii)). Moreover, since the morphism f_k is a flat cofibration, it is an h-cofibration (by Corollary A.1.18 (iii)), and so the morphisms $f_k \wedge X$ and $f_k \wedge Y$ are h-cofibrations. Corollary III.1.40 (i) then implies that the induced morphism on horizontal pushouts $M_k \wedge_R \varphi$ is again an $\mathcal{F}$-equivalence.

Now we let M be a colimit of the sequence (3.36). Then $M \wedge_R X$ is a colimit of the sequence $M_k \wedge_R X$. Moreover, since $f_k : A_k \longrightarrow B_k$ is an h-cofibration, so is the morphism $f_k \wedge R$, and hence also its cobase change $M_{k-1} \longrightarrow M_k$. So the sequence whose colimit is $M \wedge_R X$ consists of h-cofibrations, which are in particular levelwise closed embeddings. The same is true for $M \wedge_R Y$. Since each M_k is homotopical and colimits of orthogonal spectra along closed embeddings are homotopical (see Proposition III.1.42 (ii)), we conclude that the morphism $M \wedge_R \varphi : M \wedge_R X \longrightarrow M \wedge_R Y$ is an $\mathcal{F}$-equivalence, so that M is homotopical. Since $\mathcal{F}$-equivalences are closed under retracts, the class of homotopical R-modules is closed under retracts, and so every cofibrant right R-module is homotopical. □

4. Triangulated global stable homotopy categories

As the homotopy category of a stable model structure, the global stable homotopy category $\mathcal{GH}$ comes with a natural structure of a triangulated category. The shift functor is the suspension of orthogonal

spectra, and the distinguished triangles arise from mapping cone sequences. In this section we collect the aspects of global stable homotopy theory that are best expressed in terms of the triangulated structure.

More generally, we work in the triangulated $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$, where $\mathcal{F}$ is any global family. Theorem 4.3 provides a set of compact generators of $\mathcal{GH}_{\mathcal{F}}$, namely the suspension spectra of global classifying spaces of the groups in $\mathcal{F}$. A consequence is Brown representability for cohomological and homological functors out of $\mathcal{GH}_{\mathcal{F}}$, compare Corollary 4.5. Corollary 4.11 shows that the classes of $\mathcal{F}$-connective and $\mathcal{F}$-coconnective spectra form a non-degenerate t-structure on $\mathcal{GH}_{\mathcal{F}}$; moreover, taking 0-th equivariant homotopy groups is an equivalence from the heart of this t-structure to the category of $\mathcal{F}$-global functors. Immediate consequences are global Postnikov sections and the existence of Eilenberg-Mac Lane spectra that realize global functors. Proposition 4.16 establishes another connection between the smash product and the algebra of global functors: for connective orthogonal spectra the box product of global functors calculates the 0-th homotopy groups of a derived smash product.

The last topic of this section are certain distinguished triangles in the global stable homotopy category that arise from special representations of compact Lie groups G, namely when G acts faithfully and transitively on the unit sphere. The stabilizer of a unit vector is then a closed subgroup H such that G/H is diffeomorphic to a sphere, and Theorem 4.22 establishes a distinguished triangle that relates the global classifying space of G to certain free orthogonal spectra of G and H. The main examples of this situation are the tautological representations of the groups $O(m)$, $SO(m)$, $U(m)$, $SU(m)$ and $Sp(m)$; these will show up again in Section VI.2 in the rank filtrations of the global Thom spectra.

The $\mathcal{F}$-global homotopy category $\mathcal{GH}$ is the homotopy category of a stable model structure, so it is naturally a triangulated category, for example by [**79**, Sec. 7.1] or [**136**, Thm. A.12]. The shift functor is modeled by the pointset level suspension of orthogonal spectra. More precisely, the suspension functor $-\wedge S^1 : \mathcal{S}p \longrightarrow \mathcal{S}p$ of Construction III.1.21 preserves $\mathcal{F}$-equivalences, so it descends to a functor on the $\mathcal{F}$-global stable homotopy category

$$-\wedge S^1 \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{GH}_{\mathcal{F}}$$

for which we use the same name. The distinguished triangles are defined from mapping cone sequences, i.e., a triangle is distinguished if and only if it is isomorphic, in $\mathcal{GH}_{\mathcal{F}}$, to a sequence of the form

$$X \ \xrightarrow{f} \ Y \ \xrightarrow{i} \ Cf \ \xrightarrow{p} \ X\wedge S^1$$

for some morphism of orthogonal spectra $f : X \longrightarrow Y$; here the morphisms i and p were defined in (1.33).

Example 4.1 (Shift preserves distinguished triangles). The shift functor $\mathrm{sh} : \mathcal{S}p \longrightarrow \mathcal{S}p$ of Construction III.1.21 preserves $\mathcal{F}$-equivalences, so it descends to a functor on the $\mathcal{F}$-global stable homotopy category

$$\mathrm{sh} \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{GH}_{\mathcal{F}}$$

for which we use the same name. Moreover, shifting commutes with smashing with a based space *on the nose*, i.e., $(\mathrm{sh}\, X)\wedge A = \mathrm{sh}(X\wedge A)$ – so we can (and will) leave out parentheses in such expressions. Since the suspension functor on $\mathcal{GH}_{\mathcal{F}}$ is induced by smashing with S^1, the shift functor commutes with the suspension functor, again on the nose, both on the pointset level and also on the level of the $\mathcal{F}$-global stable homotopy category. We will now argue that shifting also preserves distinguished triangles on the nose; equivalently, the derived shift is an exact functor of triangulated categories if we equip it with the identity isomorphism $\mathrm{sh}\circ(-\wedge S^1) = (-\wedge S^1)\circ \mathrm{sh}$.

To prove our claim we consider a distinguished triangle in $\mathcal{GH}_{\mathcal{F}}$:

$$A \ \xrightarrow{f} \ B \ \xrightarrow{g} \ C \ \xrightarrow{h} \ A\wedge S^1$$

The morphism $\lambda_A : A \wedge S^1 \longrightarrow \operatorname{sh} A$ is a natural global equivalence by Proposition III.1.25 (iii), so all vertical morphisms in the following diagram are isomorphisms in $\mathcal{GH}_{\mathcal{F}}$:

$$\begin{tikzcd}
A \wedge S^1 \arrow[r, "f\wedge S^1"] \arrow[d, "\lambda_A"'] & B \wedge S^1 \arrow[r, "g\wedge S^1"] \arrow[d, "\lambda_B"'] & C \wedge S^1 \arrow[r, "-h\wedge S^1"] \arrow[d, "\lambda_C"] & A \wedge S^1 \wedge S^1 \arrow[d, "\lambda_A\wedge S^1"] \\
\operatorname{sh} A \arrow[r, "\operatorname{sh} f"'] & \operatorname{sh} B \arrow[r, "\operatorname{sh} g"'] & \operatorname{sh} C \arrow[r, "\operatorname{sh} h"'] & \operatorname{sh} A \wedge S^1
\end{tikzcd}$$

The claim now follows from the following three observations:

- the suspension functor in any triangulated category preserves distinguished triangles *up to a sign*; so the upper triangle above is distinguished;
- the left and middle square commutes by naturality of the λ-morphism;
- the right square commutes because the two morphisms $\lambda_{A\wedge S^1}, \lambda_A \wedge S^1 : A \wedge S^1 \wedge S^1 \longrightarrow \operatorname{sh} A \wedge S^1$ differ by the twist involution of $S^1 \wedge S^1$; since this involution has degree -1, we obtain
$$(\operatorname{sh} h) \circ \lambda_C \;=\; \lambda_{A\wedge S^1} \circ (h \wedge S^1) \;=\; -(\lambda_A \wedge S^1) \circ (h \wedge S^1)\ .$$

Definition 4.2. Let $\mathcal{T}$ be a triangulated category which has infinite sums. An object C of $\mathcal{T}$ is *compact* if for every family $\{X_i\}_{i\in I}$ of objects the canonical map
$$\bigoplus_{i\in I} \mathcal{T}(C,\, X_i) \;\longrightarrow\; \mathcal{T}(C,\, \bigoplus_{i\in I} X_i)$$
is an isomorphism. A set $\mathcal{G}$ of objects of $\mathcal{T}$ is called a set of *weak generators* if the following condition holds: if X is an object such that the groups $\mathcal{T}(G[k], X)$ are trivial for all $k \in \mathbb{Z}$ and all $G \in \mathcal{G}$, then X is a zero object. The triangulated category $\mathcal{T}$ is *compactly generated* if it has sums and a set of compact weak generators.

Being compactly generated has strong formal consequences for a triangulated category, see Theorem 4.4 below.

If G is from a global family $\mathcal{F}$, then the functor $\pi_0^G : \mathcal{S}p \longrightarrow \mathcal{A}b$ takes $\mathcal{F}$-equivalences to isomorphisms. So the universal property of a localization provides a unique factorization
$$\pi_0^G \;:\; \mathcal{GH}_{\mathcal{F}} \;\longrightarrow\; \mathcal{A}b$$
through the $\mathcal{F}$-global stable homotopy category. We will abuse notation and use the same symbol for the equivariant homotopy group functor on the category of orthogonal spectra and for its 'derived' functor defined on $\mathcal{GH}_{\mathcal{F}}$. This abuse of notation is mostly harmless, but there is one point where it can create confusion, namely in the context of infinite products; we refer the reader to Remark 4.6 for this issue.

We recall from Definition I.2.11 that the global classifying space $B_{\mathrm{gl}}G$ of a compact Lie group G is the free orthogonal space $\mathbf{L}_{G,V} = \mathbf{L}(V,-)/G$, for some faithful G-representation V. The choice of faithful representation is omitted from the notation because the global homotopy type of $B_{\mathrm{gl}}G$ does not depend on it. The suspension spectrum of $B_{\mathrm{gl}}G$ comes with a stable tautological class
$$e_G \;=\; e_{G,V} \in\; \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}}G)$$
defined in (1.12).

In the proof of the next proposition we will start using the shorthand notation
$$[\![X, Y]\!]_{\mathcal{F}} \;=\; \mathcal{GH}_{\mathcal{F}}(X, Y)$$
for the abelian group of morphisms from X to Y in the triangulated $\mathcal{F}$-global stable homotopy category.

Theorem 4.3. *Let $\mathcal{F}$ be a global family and G a compact Lie group in $\mathcal{F}$.*

(i) *The pair $(\Sigma^\infty_+ B_{\mathrm{gl}}G, e_G)$ represents the functor $\pi_0^G : \mathcal{GH}_{\mathcal{F}} \longrightarrow$ (sets).*

(ii) *The orthogonal spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is a compact object of the $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$.*

(iii) *As G varies through a set of representatives of isomorphism classes of groups in $\mathcal{F}$, the spectra $\Sigma^\infty_+ B_{\mathrm{gl}}G$ form a set of weak generators for the $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$.*

In particular, the $\mathcal{F}$-global stable homotopy category $\mathcal{GH}_{\mathcal{F}}$ is compactly generated.

Proof. (i) We need to show that for every orthogonal spectrum X the map

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}} \longrightarrow \pi_0^G(X)\ , \quad f \longmapsto f_*(e_G)$$

is bijective. Since both sides take $\mathcal{F}$-equivalences in X to bijections, we can assume that X is an $\mathcal{F}$-Ω-spectrum, and hence fibrant in the $\mathcal{F}$-global model structure. For G in the family $\mathcal{F}$, the orthogonal spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is $\mathcal{F}$-cofibrant. So the localization functor induces a bijection

$$\mathcal{S}p(\Sigma^\infty_+ B_{\mathrm{gl}}G, X)/\text{homotopy} \longrightarrow [\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}}$$

from the set of homotopy classes of morphisms of orthogonal spectra to the set of morphisms in $\mathcal{GH}_{\mathcal{F}}$.

We let V be the faithful G-representation that is implicit in the definition of the global classifying space $B_{\mathrm{gl}}G$. By the freeness property of $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$, morphisms from $\Sigma^\infty_+ B_{\mathrm{gl}}G$ to X biject with based G-maps $S^V \longrightarrow X(V)$, and similarly for homotopies. The composite

$$[S^V, X(V)]^G \xrightarrow{\cong} [\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}} \xrightarrow{f \mapsto f_*(e_G)} \pi_0^G(X)$$

is the stabilization map, and hence bijective by Proposition 3.19 (ii). Since the left map and the composite are bijective, so is the evaluation map at the stable tautological class.

(ii) By Proposition 3.27 the wedge of any family of orthogonal spectra is a coproduct in $\mathcal{GH}_{\mathcal{F}}$. We have a commutative square

$$\begin{array}{ccc} \bigoplus_{i\in I} [\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\, X_i]\!]_{\mathcal{F}} & \longrightarrow & [\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\, \bigoplus_{i\in I} X_i]\!]_{\mathcal{F}} \\ \downarrow & & \downarrow \\ \bigoplus_{i\in I} \pi_0^G(X_i) & \longrightarrow & \pi_0^G\left(\bigvee_{i\in I} X_i\right) \end{array}$$

in which the vertical maps are evaluation at the stable tautological class, which are isomorphisms by part (i). The lower horizontal map is an isomorphism by Corollary III.1.38 (i), hence so is the upper horizontal map. This shows that $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is compact as an object of the triangulated category $\mathcal{GH}_{\mathcal{F}}$.

(iii) If X is an orthogonal spectrum such that the graded abelian group $[\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\, X]\!]_*$ is trivial for every group G in $\mathcal{F}$, then X is $\mathcal{F}$-equivalent to the trivial orthogonal spectrum by part (i); so X is a zero object in $\mathcal{GH}_{\mathcal{F}}$. This proves that the spectra $\Sigma^\infty_+ B_{\mathrm{gl}}G$ form a set of weak generators $\mathcal{GH}_{\mathcal{F}}$ as G varies over $\mathcal{F}$. □

We let $\mathcal{T}$ be a triangulated category with sums. A *localizing subcategory* of $\mathcal{T}$ is a full subcategory $\mathcal{X}$ which is closed under sums and under extensions in the following sense: if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle in $\mathcal{T}$ such that two of the objects A, B or C belong to $\mathcal{X}$, then so does the third. A set of *compact* objects is a set of weak generators in the sense of Definition 4.2 if and only if the smallest localizing subcategory containing the set is all of $\mathcal{T}$, see for example [**134**, Lemma 2.2.1].

A covariant functor E from $\mathcal{T}$ to the category of abelian groups is called *homological* if for every distinguished triangle (f, g, h) in $\mathcal{T}$ the sequence of abelian groups

$$E(A) \xrightarrow{E(f)} E(B) \xrightarrow{E(g)} E(C) \xrightarrow{E(h)} E(A[1])$$

is exact. A contravariant functor E from $\mathcal{T}$ to the category of abelian groups is called *cohomological* if for every distinguished triangle (f, g, h) in $\mathcal{T}$ the sequence of abelian groups

$$E(A[1]) \xrightarrow{E(h)} E(C) \xrightarrow{E(g)} E(B) \xrightarrow{E(f)} E(A)$$

is exact.

THEOREM 4.4 (Brown representability). *Let $\mathcal{T}$ be a compactly generated triangulated category.*

(i) *Every cohomological functor $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ that takes sums in $\mathcal{T}$ to products of abelian groups is representable, i.e., isomorphic to the functor $\mathcal{T}(-,X)$ for some object X of $\mathcal{T}$.*
(ii) *Every homological functor $\mathcal{T} \longrightarrow \mathcal{A}b$ that takes products in $\mathcal{T}$ to products of abelian groups is representable, i.e., isomorphic to the functor $\mathcal{T}(Y,-)$ for some object Y of $\mathcal{T}$.*
(iii) *An exact functor $F : \mathcal{T} \longrightarrow \mathcal{S}$ to another triangulated category has a right adjoint if and only if it takes sums in $\mathcal{T}$ to sums in $\mathcal{S}$.*
(iv) *An exact functor $F : \mathcal{T} \longrightarrow \mathcal{S}$ to another triangulated category has a left adjoint if and only if it takes products in $\mathcal{T}$ to products in $\mathcal{S}$.*

A proof of part (i) of this form of Brown representability can be found in [**117**, Thm. 3.1] or [**90**, Thm. A]. A proof of part (ii) of this form of Brown representability can be found in [**118**, Thm. 8.6.1] or [**90**, Thm. B]. Part (iii) is a formal consequences of part (i): if F preserves sums, then for every object X of $\mathcal{T}$ the functor

$$\mathcal{S}(F(-),X) \ : \ \mathcal{T}^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b$$

is cohomological and takes sums to products. Hence the functor is representable by an object RX in $\mathcal{T}$ and an isomorphism

$$\mathcal{T}(A,RX) \ \cong \ \mathcal{S}(FA,X) \ ,$$

natural in A. Once this representing data is chosen, the assignment $X \mapsto RX$ extends canonically to a functor $R : \mathcal{S} \longrightarrow \mathcal{T}$ that is right adjoint to U. In much the same way, part (iv) is a formal consequences of part (ii).

Hence Theorem 4.3 entitles us to the following corollary.

COROLLARY 4.5. *Let $\mathcal{F}$ be a global family and $\mathcal{S}$ a triangulated category.*

(i) *Every localizing subcategory of the $\mathcal{F}$-global stable homotopy category which contains the spectrum $\Sigma^\infty_+ B_{\mathrm{gl}} G$ for every group G of $\mathcal{F}$ is all of $\mathcal{GH}_{\mathcal{F}}$.*
(ii) *Every cohomological functor on $\mathcal{GH}_{\mathcal{F}}$ that takes sums to products is representable.*
(iii) *Every homological functor on $\mathcal{GH}_{\mathcal{F}}$ that takes products to products is representable.*
(iv) *An exact functor $F : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{S}$ has a right adjoint if and only if it preserves sums.*
(v) *An exact functor $F : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{S}$ has a left adjoint if and only if it preserves products.*

REMARK 4.6 (Equivariant homotopy groups of infinite products). In Corollary III.1.38 (ii) we showed that for every compact Lie group G the functor $\pi_0^G : \mathcal{S}p \longrightarrow \mathcal{A}b$ preserves *finite* products. However, it is *not* true that π_0^G, as a functor on the category of orthogonal spectra, preserves infinite products in general.

On the other hand, the 'derived' functor $\pi_0^G : \mathcal{GH} \longrightarrow \mathcal{A}b$ is representable, by the spectrum $\Sigma^\infty_+ B_{\mathrm{gl}} G$, so it preserves infinite products. This is no contradiction because an infinite product of orthogonal spectra is *not* in general a product in the global homotopy category. To calculate a product in $\mathcal{GH}$ of a family $\{X_i\}_{i\in I}$ of orthogonal spectra, one has to choose stable equivalences $f_i : X_i \longrightarrow X_i^{\mathrm{f}}$ to global Ω-spectra. For an infinite indexing set, the morphism

$$\prod_{i\in I} f_i \ : \ \prod_{i\in I} X_i \ \longrightarrow \ \prod_{i\in I} X_i^{\mathrm{f}}$$

may fail to be a global equivalence, and then the target, but not the source, of this map is a product in $\mathcal{GH}$ of the family $\{X_i\}_{i\in I}$. So when considering infinite products it is important to be aware of our abuse of notation and to remember that the symbol π_0^G has two different meanings.

The preferred set of generators $\{\Sigma^\infty_+ B_{\mathrm{gl}} G\}$ of the global stable homotopy category has another special property, it is 'positive' in the following sense: for all compact Lie groups G and K and all $n < 0$ the group

$$[\![\Sigma^n \Sigma^\infty_+ B_{\mathrm{gl}} G, \ \Sigma^\infty_+ B_{\mathrm{gl}} K]\!] \ \cong \ \pi_n^G(\Sigma^\infty_+ B_{\mathrm{gl}} K)$$

is trivial, because the underlying orthogonal G-spectrum of $\Sigma^\infty_+ B_{\mathrm{gl}}K$ is the suspension spectrum of a G-space. A set of positive compact generators in this sense has strong implications, as we shall now explain. Even though our main interest is in the global stable homotopy category, we will work in general triangulated categories for some time, because we feel that this makes the arguments more transparent.

A 't-structure' as introduced by Beilinson, Bernstein and Deligne in [**16**, Def. 1.3.1] axiomatizes the situation in the derived category of an abelian category given by cochain complexes whose cohomology vanishes in positive respectively negative dimensions.

Definition 4.7. A *t-structure* on a triangulated category $\mathcal{T}$ is a pair $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ of full subcategories satisfying the following three conditions, where $\mathcal{T}^{\le n} = \mathcal{T}^{\le 0}[-n]$ and $\mathcal{T}^{\ge n} = \mathcal{T}^{\ge 0}[-n]$:

(a) For all $X \in \mathcal{T}^{\le 0}$ and all $Y \in \mathcal{T}^{\ge 1}$ we have $\mathcal{T}(X, Y) = 0$.
(b) $\mathcal{T}^{\le 0} \subset \mathcal{T}^{\le 1}$ and $\mathcal{T}^{\ge 0} \supset \mathcal{T}^{\ge 1}$.
(c) For every object X of $\mathcal{T}$ there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

such that $A \in \mathcal{T}^{\le 0}$ and $B \in \mathcal{T}^{\ge 1}$.

A t-structure is *non-degenerate* if $\bigcap_{n \le 0} \mathcal{T}^{\le n} = \{0\}$ and $\bigcap_{n \ge 0} \mathcal{T}^{\ge n} = \{0\}$.

The original definition of t-structures is motivated by derived categories of cochain complexes as the main examples, and the subcategory $\mathcal{T}^{\le 0}$ behaves like complexes with trivial cohomology in positive degrees. We are mainly interested in spectra, where a homological (as opposed to *co*homological) grading is more common. So we turn a t-structure into homological notation by setting

$$\mathcal{T}_{\ge n} = \mathcal{T}^{\le -n} \qquad \text{and} \qquad \mathcal{T}_{\le n} = \mathcal{T}^{\ge -n} \ .$$

In this homological notation $\mathcal{T}_{\ge n} = \mathcal{T}_{\ge 0}[n]$ and $\mathcal{T}_{\le n} = \mathcal{T}_{\le 0}[n]$, and the conditions for a pair $(\mathcal{T}_{\ge 0}, \mathcal{T}_{\le 0}) = (\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ to be a t-structure become:

(a') For all $X \in \mathcal{T}_{\ge 0}$ and all $Y \in \mathcal{T}_{\le -1}$ we have $\mathcal{T}(X, Y) = 0$.
(b') $\mathcal{T}_{\ge 0} \subset \mathcal{T}_{\ge -1}$ and $\mathcal{T}_{\le 0} \supset \mathcal{T}_{\le -1}$.
(c') For every object X of $\mathcal{T}$ there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

such that $A \in \mathcal{T}_{\ge 0}$ and $B \in \mathcal{T}_{\le -1}$.

Some of the basic results of Beilinson, Bernstein and Deligne about t-structures are (in our homological notation):

- for every $n \in \mathbb{Z}$, the inclusion $\mathcal{T}_{\ge n} \longrightarrow \mathcal{T}$ has a right adjoint $\tau_{\ge n} : \mathcal{T} \longrightarrow \mathcal{T}_{\ge n}$, and the inclusion $\mathcal{T}_{\le n} \longrightarrow \mathcal{T}$ has a left adjoint $\tau_{\le n} : \mathcal{T} \longrightarrow \mathcal{T}_{\le n}$ [**16**, Prop. 1.3.3].
- given choices of adjoints as in the previous item, then for all $a \le b$ there is a preferred natural isomorphism of functors between $\tau_{\ge a} \circ \tau_{\le b}$ and $\tau_{\le b} \circ \tau_{\ge a}$ [**16**, Prop. 1.3.5].
- The *heart*

$$\mathscr{H} \ = \ \mathcal{T}_{\ge 0} \cap \mathcal{T}_{\le 0} \ = \ \mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0} \ ,$$

  viewed as a full subcategory of $\mathcal{T}$, is an abelian category and $\tau_{\le 0} \circ \tau_{\ge 0} : \mathcal{T} \longrightarrow \mathscr{H}$ is a homological functor [**16**, Thm. 1.3.6]. Two composable morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ form a short exact sequence in $\mathscr{H}$ if and only if there is a morphism $\delta : C \longrightarrow A[1]$ (necessarily unique) such that the triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} A[1]$$

  is distinguished.

In general the heart of a t-structure need not have enough injectives, nor enough projectives, nor infinite sums. The following proposition gives a sufficient condition for when the heart of the t-structure has all these properties, and can even be identified with a module category. For a set of objects $\mathcal{C}$, we denote by $\mathrm{End}(\mathcal{C})$ the 'endomorphism category', i.e., the full pre-additive subcategory of $\mathcal{T}$ with object set $\mathcal{C}$. By an $\mathrm{End}(\mathcal{C})$-module we mean an additive functor

$$M \ : \ \mathrm{End}(\mathcal{C})^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b$$

from the opposite category of $\mathrm{End}(\mathcal{C})$. So when $\mathcal{C} = \{C\}$ consists of a single object, then the $\mathrm{End}(\mathcal{C})$-modules are just the right modules over the endomorphism ring $\mathrm{End}(C) = \mathcal{T}(C, C)$. The tautological functor

$$\mathcal{T} \ \longrightarrow \ \text{mod-}\,\mathrm{End}(\mathcal{C}) \tag{4.8}$$

takes an object X to $\mathcal{T}(-, X)|_{\mathrm{End}(\mathcal{C})}$, the restriction of the contravariant Hom-functor to the full subcategory $\mathrm{End}(\mathcal{C}) \subset \mathcal{T}$.

We let $\mathcal{T}$ be a triangulated category with infinite sums and we let $\mathcal{C}$ be a set of compact objects of $\mathcal{T}$. We denote by $\langle\mathcal{C}\rangle_+$ the smallest class of objects of $\mathcal{T}$ that contains $\mathcal{C}$, is closed under sums (possibly infinite) and is *closed under cones* in the following sense: if

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

is a distinguished triangle such that A and B belong to the class, then so does C. Any non-empty class of objects that is closed under cones contains all zero objects (because a zero object is a cone of any identity morphism) and is closed under suspension (because $A[1]$ is a cone of the morphism from A to a zero object).

The following results are well known in the triangulated category community, but I was unable to find a reference in the generality that we need. For a single compact object (as opposed to a set of compact objects), part (i) of the following proposition can be found in Lemma 6.1 of [**5**]; part (ii) is essentially Theorem A.1 of [**5**].

Proposition 4.9. *Let $\mathcal{T}$ be a triangulated category with infinite sums and let $\mathcal{C}$ be a set of compact weak generators of $\mathcal{T}$ such that the group $\mathcal{T}(C[n], C')$ is trivial for all $C, C' \in \mathcal{C}$ and all $n < 0$.*

(i) *The class $\langle\mathcal{C}\rangle_+$ coincides with the class of those objects X of $\mathcal{T}$ such that $\mathcal{T}(C[n], X) = 0$ for all $n < 0$.*
(ii) *We denote by $\langle\mathcal{C}\rangle_{\leq 0}$ the class of objects X of $\mathcal{T}$ such that $\mathcal{T}(C[n], X) = 0$ for all $C \in \mathcal{C}$ and all $n > 0$. Then the pair $(\langle\mathcal{C}\rangle_+, \langle\mathcal{C}\rangle_{\leq 0})$ is a non-degenerate t-structure on the category $\mathcal{T}$.*
(iii) *The heart $\mathscr{H} = \langle\mathcal{C}\rangle_+ \cap \langle\mathcal{C}\rangle_{\leq 0}$ of the t-structure consists of those objects X of $\mathcal{T}$ such that $\mathcal{T}(C[n], X) = 0$ for all $C \in \mathcal{C}$ and all $n \neq 0$.*
(iv) *Every sum in $\mathcal{T}$ of objects in the heart $\mathscr{H} = \langle\mathcal{C}\rangle_+ \cap \langle\mathcal{C}\rangle_{\leq 0}$ belongs to $\mathscr{H}$. In particular, the heart has infinite sums and the inclusion $\mathscr{H} \longrightarrow \mathcal{T}$ preserves them.*
(v) *The set $\mathcal{P} = \{\tau_{\leq 0}C \mid C \in \mathcal{C}\}$ belongs to the heart $\mathscr{H}$ and is a set of small projective generators of the heart.*
(vi) *The restriction of the tautological functor* (4.8) *to the heart is an equivalence of categories*

$$\mathscr{H} \ \xrightarrow{\cong} \ \text{mod-}\,\mathrm{End}(\mathcal{C}) \ .$$

Proof. We start by proving one half of property (i). We let $\mathcal{X}$ denote the class of $\mathcal{T}$-objects X such that $\mathcal{T}(C[n], X) = 0$ for all $n < 0$. Then $\mathcal{C} \subset \mathcal{X}$ by the positivity hypothesis. Since the objects in $\mathcal{C}$ are compact, the class $\mathcal{X}$ is closed under arbitrary sums. Finally, the class $\mathcal{X}$ is closed under cones: Given a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

with $X, Y \in \mathcal{X}$, the exact sequence

$$\mathcal{T}(C[n], Y) \ \longrightarrow \ \mathcal{T}(C[n], Z) \ \longrightarrow \ \mathcal{T}(C[n-1], X)$$

shows that Z again belongs to $\mathcal{X}$. Altogether this shows that $\langle\mathcal{C}\rangle_+ \subseteq \mathcal{X}$.

Now we prove (ii), i.e., we verify the axioms of a t-structure. For axiom (a') we consider the class $\mathcal{A}$ of $\mathcal{T}$-objects A such that $\mathcal{T}(A,X)=0$ for all $X\in\mathcal{T}_{\leq 0}$. Then $\mathcal{C}[1]\subset\mathcal{A}$ by definition of $\langle\mathcal{C}\rangle_{\leq 0}$. Since $\mathcal{T}(-,X)$ takes sums to products, the class $\mathcal{A}$ is closed under arbitrary sums. Moreover, the class $\mathcal{A}$ is closed under cones: given a distinguished triangle

$$A\longrightarrow B\longrightarrow C\longrightarrow A[1]$$

with $A,B\in\mathcal{A}$, then for every $X\in\langle\mathcal{C}\rangle_{\leq 0}$ the exact sequence

$$\mathcal{T}(A[1],X)\ \longrightarrow\ \mathcal{T}(C,X)\ \longrightarrow\ \mathcal{T}(B,X)$$

and the fact that $\mathcal{A}[1]\subset\mathcal{A}$ shows that C again belongs to $\mathcal{A}$. Altogether this shows that $\langle\mathcal{C}\rangle_+[1]\subseteq\mathcal{A}$, i.e., property (a') holds. Property (b') is a direct consequence of the definitions. It remains to construct the distinguished triangle required by axiom (c'). By induction on n we construct objects A_n in $\langle\mathcal{C}\rangle_+$ and morphisms $i_n:A_n\longrightarrow A_{n+1}$ and $u_n:A_n\longrightarrow X$ such that $u_{n+1}i_n=u_n$. We start with

$$A_0\ =\ \bigoplus_{C\in\mathcal{C},k\geq 0,x\in\mathcal{T}(C[k],X)} C[k]\ .$$

Then A_0 belongs to $\langle\mathcal{C}\rangle_+$ and the canonical map $u_0:A_0\longrightarrow X$ (i.e., the morphism x on the summand indexed by x) induces a surjection $\mathcal{T}(C[k],u_0):\mathcal{T}(C[k],A_0)\longrightarrow\mathcal{T}(C[k],X)$ for all $C\in\mathcal{C}$ and $k\geq 0$.

In the inductive step we suppose that A_n and $u_n:A_n\longrightarrow X$ have already been constructed. We define

$$D_n\ =\ \bigoplus_{C\in\mathcal{C},k\geq 0,x\in\ker(\mathcal{T}(C[k],u_n))} C[k]\ ,$$

which comes with a tautological morphism $\tau:D_n\longrightarrow A_n$, again given by x on the summand indexed by x. We choose a distinguished triangle

$$D_n\xrightarrow{\tau}A_n\xrightarrow{i_n}A_{n+1}\longrightarrow D_n[1]\ .$$

Since D_n and A_n belong to the class $\langle\mathcal{C}\rangle_+$, we also have $A_{n+1}\in\langle\mathcal{C}\rangle_+$. Since $u_n\tau=0$ (by definition), we can choose a morphism $u_{n+1}:A_{n+1}\longrightarrow X$ such that $u_{n+1}i_n=u_n$. This completes the inductive construction.

Now we choose a homotopy colimit $(A,\{\varphi_n:A_n\longrightarrow A\}_n)$, of the sequence of morphisms $i_n:A_n\longrightarrow A_{n+1}$. Since all the objects A_n are in $\langle\mathcal{C}\rangle_+$, so is A. Since a homotopy colimit in $\mathcal{T}$ is a weak colimit, we can choose a morphism $u:A\longrightarrow X$ such that $u\varphi_n=u_n$ for all $n\geq 0$. We choose a distinguished triangle

$$A\ \xrightarrow{u}\ X\ \xrightarrow{v}\ B\ \longrightarrow\ A[1]\ .$$

We claim that $B\in\langle\mathcal{C}\rangle_{\leq -1}$, i.e., that $\mathcal{T}(C[k],B)=0$ for all $k\geq 0$ and $C\in\mathcal{C}$. The map

$$\mathcal{T}(C[k],u_0)\ =\ \mathcal{T}(C[k],u)\circ\mathcal{T}(C[k],\varphi_0)\ :\ \mathcal{T}(C[k],A_0)\ \longrightarrow\ \mathcal{T}(C[k],X)$$

is surjective, hence $\mathcal{T}(C[k],u):\mathcal{T}(C[k],A)\longrightarrow\mathcal{T}(C[k],X)$ is also surjective. To show that $\mathcal{T}(C[k],u)$ is injective we let $\alpha:C[k]\longrightarrow A$ be an element such that $u\alpha=0$. Since C is compact, there is an $n\geq 0$ and a morphism $\alpha':C[k]\longrightarrow A_n$ such that $\alpha=\varphi_n\alpha'$. Then $u_n\alpha'=u\varphi_n\alpha'=u\alpha=0$. So α' indexes one of the summands of D_n. Thus α' factors through the tautological morphism $\tau:D_n\longrightarrow A_n$ as $\alpha'=\tau\alpha''$, and hence

$$\alpha\ =\ \varphi_n\alpha'\ =\ \varphi_{n+1}i_n\tau\alpha''\ =\ 0$$

since i_n and τ are consecutive morphisms in a distinguished triangle. Hence $\mathcal{T}(C[k],u)$ is also injective, hence bijective. The exact sequence

$$\mathcal{T}(C[k],A)\ \xrightarrow{u_*}\ \mathcal{T}(C[k],X)\ \xrightarrow{v_*}\ \mathcal{T}(C[k],B)\ \xrightarrow{w_*}\ \mathcal{T}(C[k-1],A)\ \xrightarrow{u_*}\ \mathcal{T}(C[k-1],X)$$

then shows that $\mathcal{T}(C[k],B)=0$ for all $k\geq 1$. Because $A\in\langle\mathcal{C}\rangle_+$ we also have $\mathcal{T}(C[-1],A)=0$ by the already established part of (i). So for $k=0$ the exact sequence above shows that also $\mathcal{T}(C,B)=0$, and hence $B\in\langle\mathcal{C}\rangle_{\leq -1}$. So $(\langle\mathcal{C}\rangle_+,\langle\mathcal{C}\rangle_{\leq 0})$ is a t-structure on $\mathcal{T}$.

Now we show that the t-structure is non-degenerate. If an object X of $\mathcal{T}$ belongs to $\langle\mathcal{C}\rangle_+[m]$, then in particular $\mathcal{T}(\mathcal{C}[n],X)=0$ for all $n\leq m$ by the already established part of (i). So if X lies in the intersection

$\bigcap_{m\geq 0}\langle\mathcal{C}\rangle_+[m]$, then X is a zero object because $\mathcal{C}$ is a set of weak generators. If X belongs to $\langle\mathcal{C}\rangle_{\leq m}$ for all $m \leq 0$, then also $\mathcal{T}(\mathcal{C}[n], X) = 0$ for all $n \in \mathbb{Z}$ by the definition of $\langle\mathcal{C}\rangle_{\leq 0}$. So again X is a zero object because $\mathcal{C}$ is a set of weak generators. This shows that the t-structure is non-degenerate and finishes the proof of (ii).

Now we prove the other half of part (i). We suppose that $\mathcal{T}(C[n], X) = 0$ for all $C \in \mathcal{C}$ and all $n < 0$. Part (ii) provides a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \langle\mathcal{C}\rangle_+$ and $B \in \langle\mathcal{C}\rangle_{\leq -1}$. For $n \geq 0$ the group $\mathcal{T}(C[n], B)$ is trivial because $B \in \langle\mathcal{C}\rangle_{\leq -1}$. For $n < 0$ the group $\mathcal{T}(C[n], X)$ is trivial by hypothesis and the group $\mathcal{T}(C[n-1], A)$ is trivial because $A \in \langle\mathcal{C}\rangle_+$ and the already established part of (i). The exact sequence

$$\mathcal{T}(C[n], X) \longrightarrow \mathcal{T}(C[n], B) \longrightarrow \mathcal{T}(C[n-1], A)$$

shows that the group $\mathcal{T}(C[n], B)$ is also trivial. So no shift of any object of $\mathcal{C}$ has a non-trivial morphism to B, and thus B is a zero object because $\mathcal{C}$ is a set of weak generators. But then the morphism $A \longrightarrow X$ must be an isomorphism, so $X \in \langle\mathcal{C}\rangle_+$.

Part (iii) is the combination of part (i) and the definition of $\langle\mathcal{C}\rangle_{\leq 0}$.

(iv) We let $\{X_i\}_{i\in I}$ be a family of objects in $\mathscr{H}$. Then for every $C \in \mathcal{C}$ and every integer $n \neq 0$ we have

$$\mathcal{T}(C[n], \bigoplus X_i) \cong \bigoplus \mathcal{T}(C[n], X_i) = 0$$

because C is compact; so by (iii) the sum of the X_i also belongs to the heart. Since the heart is a full subcategory of $\mathcal{T}$, this sum is also a coproduct in the abelian category $\mathscr{H}$.

(v) For every $C \in \mathcal{C}$ the object $\tau_{\leq 0}C$ belongs to $\langle\mathcal{C}\rangle_{\leq 0}$ by definition. Since

$$\tau_{\leq 0}C = \tau_{\leq 0}(\tau_{\geq 0}C) = \tau_{\geq 0}(\tau_{\leq 0}C) ,$$

the object $\tau_{\leq 0}C$ also belongs to $\langle\mathcal{C}\rangle_+$, hence to the heart $\mathscr{H}$.

Now we let $\{X_i\}_{i\in I}$ be a family of objects in the heart. Then their sum in $\mathcal{T}$ is also a sum in the heart by (iv). The vertical maps in the commutative square

$$\begin{array}{ccc} \bigoplus \mathscr{H}(\tau_{\leq 0}C, X_i) & \longrightarrow & \mathscr{H}(\tau_{\leq 0}C, \bigoplus X_i) \\ \cong \downarrow & & \downarrow \cong \\ \bigoplus \mathcal{T}(C, X_i) & \longrightarrow & \mathcal{T}(C, \bigoplus X_i) \end{array}$$

are isomorphisms by the universal property of the truncation morphism $C \longrightarrow \tau_{\leq 0}C$, and the lower horizontal map is an isomorphism because C is compact. So the upper horizontal map is an isomorphism, and that shows that the object $\tau_{\leq 0}C$ is small in the abelian category $\mathscr{H}$.

We let $X \in \mathscr{H}$ be any object of the heart. Then $\mathcal{T}(C[n], X) = 0$ for all $C \in \mathcal{C}$ and all $n \neq 0$ by (iii). So if in addition also $\mathscr{H}(\tau_{\leq 0}C, X) = 0$, then

$$\mathcal{T}(C, X) \cong \mathscr{H}(\tau_{\leq 0}C, X) = 0 ,$$

and hence $X = 0$ because $\mathcal{C}$ is a set of weak generators for the triangulated category $\mathcal{T}$. Altogether this shows that $\mathcal{P}$ is a set of generators for the abelian category $\mathscr{H}$.

It remains to show that for all $C \in \mathcal{C}$ the object $\tau_{\leq 0}C$ is projective in the abelian category $\mathscr{H}$. For this we consider any epimorphism $f : X \longrightarrow Y$ in the heart $\mathscr{H}$, so that there is a distinguished triangle in $\mathcal{T}$

$$F \longrightarrow X \xrightarrow{f} Y \longrightarrow F[1] .$$

such that the object F again belongs to $\mathscr{H}$. In the commutative diagram

$$\begin{tikzcd}
\mathscr{H}(\tau_{\leq 0}C, X) \arrow[r, "\mathscr{H}(\tau_{\leq 0}C, f)"] \arrow[d, "\cong"'] & \mathscr{H}(\tau_{\leq 0}C, Y) \arrow[d, "\cong"] & \\
\mathcal{T}(C, X) \arrow[r, "\mathcal{T}(C,f)"'] & \mathcal{T}(C, Y) \arrow[r] & \mathcal{T}(C[-1], F) = 0
\end{tikzcd}$$

the lower row is then exact, where the vertical maps are induced by the adjunction unit $C \longrightarrow \tau_{\leq 0}C$. Moreover, the group $\mathcal{T}(C[-1], F)$ is trivial by (iii) because $F \in \mathscr{H}$. So the morphism $\mathcal{T}(C, f)$ is surjective, hence so if $\mathscr{H}(\tau_{\leq 0}C, f)$, and this shows that the object $\tau_{\leq 0}C$ is projective.

(vi) We let $\mathcal{A}$ be any abelian category with infinite sums and a set $\mathcal{P}$ of small projective generators. It is well-known that then the functor

$$\mathcal{A}(\mathcal{P}, -) \ : \ \mathcal{A} \ \longrightarrow \ \text{mod-}\operatorname{End}(\mathcal{P})$$

is an equivalence of categories; a proof in the case of a single generator can for example be found in [**10**, II Thm. 1.3] or [**132**, Thm. 2.5]. This general fact applies to the heart $\mathscr{H}$ of the t-structure by (v). The claim follows because the pre-additive category $\operatorname{End}(\mathcal{P})$ generated by the set $\mathcal{P}$ in the heart $\mathscr{H}$ is isomorphic to the pre-additive category generated by the original set $\mathcal{C}$ in the triangulated category $\mathcal{T}$. A preferred isomorphism from $\operatorname{End}(\mathcal{C})$ to $\operatorname{End}(\mathcal{P})$ is given on objects by $C \mapsto \tau_{\leq 0}C$, and on morphisms by the isomorphisms

$$\mathcal{T}(C, C') \ \xrightarrow{\mathcal{T}(C, \eta_{C'})} \ \mathcal{T}(C, \tau_{\leq 0}C') \ \xleftarrow{\mathcal{T}(\eta_C, \tau_{\leq 0}C')} \ \mathscr{H}(\tau_{\leq 0}C, \tau_{\leq 0}C')$$

induced by the adjunction units $\eta_C : C \longrightarrow \tau_{\leq 0}C$ and $\eta_{C'} : C' \longrightarrow \tau_{\leq 0}C'$. □

We specialize Proposition 4.9 to the $\mathcal{F}$-global stable homotopy category for a global family $\mathcal{F}$. By Theorem 4.3 the set

$$\mathcal{C}_{\mathcal{F}} \ = \ \{\Sigma^\infty_+ B_{\mathrm{gl}} G\}_{[G] \in \mathcal{F}}$$

is a set of compact weak generators for the triangulated category $\mathcal{GH}_{\mathcal{F}}$, where $[G]$ indicates that we choose representatives from the isomorphism classes of compact Lie groups in $\mathcal{F}$. This generating set is 'positive' in the sense of Proposition 4.9.

Definition 4.10. Let $\mathcal{F}$ be a global family. An orthogonal spectrum X is *$\mathcal{F}$-connective* if the homotopy group $\pi_n^G(X)$ is trivial for every group G in $\mathcal{F}$ and every $n < 0$. An orthogonal spectrum X is *$\mathcal{F}$-coconnective* if the homotopy group $\pi_n^G(X)$ is trivial for every group G in $\mathcal{F}$ and every $n > 0$.

Since the spectrum $(\Sigma^\infty_+ B_{\mathrm{gl}} G)[n]$ represents the functor π_n^G (by Theorem 4.3 (i) and the suspension isomorphism), the class $\langle \mathcal{C}_{\mathcal{F}} \rangle_{\leq 0}$ is precisely the class of $\mathcal{F}$-coconnective spectra. So Proposition 4.9 specializes to:

Corollary 4.11. *Let $\mathcal{F}$ be a global family.*

(i) *The class $\langle \Sigma^\infty_+ B_{\mathrm{gl}} G : [G] \in \mathcal{F} \rangle_+$ coincides with the class of $\mathcal{F}$-connective spectra.*
(ii) *The classes of $\mathcal{F}$-connective spectra and $\mathcal{F}$-coconnective spectra form a non-degenerate t-structure on $\mathcal{GH}_{\mathcal{F}}$ whose heart consists of those orthogonal spectra X such that $\pi_n^G(X) = 0$ for all $G \in \mathcal{F}$ and all $n \neq 0$.*
(iii) *The functor*

$$\underline{\pi}_0 \ : \ \mathscr{H} \ \longrightarrow \ \mathcal{GF}_{\mathcal{F}}$$

is an equivalence of categories from the heart of the t-structure to the category of $\mathcal{F}$-global functors.

Proof. Parts (i) and (ii) are special cases of Proposition 4.9. For part (iii) it suffices to show that the full pre-additive subcategory $\mathrm{End}(\mathcal{C}_{\mathcal{F}}) \subset \mathcal{GH}_{\mathcal{F}}$ with object set $\{\Sigma^\infty_+ B_{\mathrm{gl}} G\}_{[G]\in\mathcal{F}}$ is anti-equivalent to the full subcategory of the Burnside category $\mathbf{A}$ with object class $\mathcal{F}$, in such a way that the tautological functor corresponds to the functor $\underline{\pi}_0$. Then Proposition 4.9 (vi) provides the claim. The equivalence $\mathrm{End}(\mathcal{C}_{\mathcal{F}})^{\mathrm{op}} \longrightarrow \mathbf{A}_{\mathcal{F}}$ is given by the inclusion on objects, and on morphisms by the isomorphisms

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}} G, \Sigma^\infty_+ B_{\mathrm{gl}} K]\!]_{\mathcal{F}} \;\cong\; \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} K) \;\cong\; \mathbf{A}(K,G)$$

specified in Theorem 4.3 (i) respectively Proposition 2.5. □

Remark 4.12 (Postnikov sections). For the standard t-structure on the global homotopy category (i.e., Corollary 4.11 for $\mathcal{F} = \mathcal{A}ll$) the truncation functor

$$\tau_{\leq n} \;:\; \mathcal{GH} \;\longrightarrow\; \mathcal{GH}_{\leq n}\;,$$

left adjoint to the inclusion, provides a 'global Postnikov section': For every orthogonal spectrum X the spectrum $\tau_{\leq n} X$ satisfies $\underline{\pi}_k(\tau_{\leq n} X) = 0$ for $k > n$ and the adjunction unit $X \longrightarrow X_{\leq n}$ induces an isomorphism on the global functor $\underline{\pi}_k$ for every $k \leq n$.

Remark 4.13 (Eilenberg-Mac Lane spectra). In the case $\mathcal{F} = \mathcal{A}ll$ of the maximal global family, part (iii) of Corollary 4.11 in particular provides an Eilenberg-Mac Lane spectrum for every global functor M, i.e., an orthogonal spectrum HM such that $\underline{\pi}_k(HM) = 0$ for all $k \neq 0$ and such that the global functor $\underline{\pi}_0(HM)$ is isomorphic to M, and these properties characterize HM up to preferred isomorphism in $\mathcal{GH}$. Moreover, a choice of inverse to the equivalence $\underline{\pi}_0$ of Corollary 4.11 (iii), composed with the inclusion of the heart, provides an Eilenberg-Mac Lane functor

$$H \;:\; \mathcal{GF} \;\longrightarrow\; \mathcal{GH}$$

to the global homotopy category.

A general fact about t-structures, proved in [**16**, Rem. 3.1.17 (i)], is that for all objects A and B of the heart not only do morphisms in the heart $\mathscr{H}$ coincide with morphisms in the ambient triangulated category $\mathcal{T}$ (simply because the heart is defined as a full subcategory), but also the Yoneda extension group $\mathrm{Ext}^1_{\mathscr{H}}(A,B)$ is isomorphic to the group $\mathcal{T}(A, B[1])$, naturally in both variables. In the case of the global homotopy category, this specializes to the following property: for every short exact sequence of global functors

$$0 \;\longrightarrow\; A \;\xrightarrow{i}\; B \;\xrightarrow{j}\; C \;\longrightarrow\; 0$$

there is a unique morphism $\delta : HC \longrightarrow HA \wedge S^1$ in the global stable homotopy category such that the diagram

$$HA \;\xrightarrow{Hi}\; HB \;\xrightarrow{Hj}\; HC \;\xrightarrow{\delta}\; HA \wedge S^1$$

is a distinguished triangle. Moreover, the assignment

$$\mathrm{Ext}^1_{\mathcal{GF}}(C,A) \;\longrightarrow\; [\![HC, HA \wedge S^1]\!]\;, \quad [i,j] \longmapsto \delta$$

is a natural group isomorphism. One should beware, though, that for $n \geq 2$ there is no such simple relationship between the group $\mathrm{Ext}^n_{\mathcal{GF}}(C,A)$ and the morphism group $[\![HC, HA \wedge S^n]\!]$.

Example 4.14 (Non-standard t-structures). In addition to the 'standard' t-structure specified in Corollary 4.11 we can use Proposition 4.9 to exhibit non-degenerate t-structures on $\mathcal{GH}_{\mathcal{F}}$ with different (i.e., inequivalent) hearts. We illustrate this in the simplest non-trivial case for the global family $\mathcal{F} = \langle C_2 \rangle$ generated by a cyclic group of order 2. The 'standard' set of compact generators for $\mathcal{GH}_{\langle C_2\rangle}$ is then $\mathcal{C}_2 = \{\mathbb{S}, \Sigma^\infty_+ B_{\mathrm{gl}} C_2\}$. Corollary 4.11 identifies the heart of the standard t-structure with the category of $\langle C_2\rangle$-global functors or, equivalently, with the category of $\mathrm{End}(\mathcal{C}_2)$-modules. The information contained in an

$\mathrm{End}(\mathcal{C}_2)$-module M consists of two abelian groups $M(C_2)$ and $M(e)$ and morphisms $p^*, \mathrm{tr} : M(e) \longrightarrow M(C_2)$ and $\mathrm{res} : M(C_2) \longrightarrow M(e)$ that satisfy the relations

$$\mathrm{res} \circ p^* \;=\; \mathrm{Id}_{M(e)} \qquad \text{and} \qquad \mathrm{res} \circ \mathrm{tr} \;=\; 2 \cdot \mathrm{Id}_{M(e)} \;,$$

compare Remark 2.9.

There is some redundancy in this 'standard' presentation of the heart of the standard t-structure, because

$$M(C_2) \;=\; p^*(M(e)) \;\oplus\; \ker(\mathrm{res}) \tag{4.15}$$

naturally splits off a copy of $M(e)$. We get a more economical presentation from a modified set of generators. We denote by $\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2$ the *reduced* suspension spectrum of the global classifying space, i.e., of the mapping cone of the morphism

$$\mathbf{L}_{e,\sigma} \;\longrightarrow\; \mathbf{L}_{C_2,\sigma} \;=\; B_{\mathrm{gl}} C_2 \;,$$

where σ is the 1-dimensional sign representation of C_2. Since the previous morphism has a retraction in $\mathcal{GH}$, we obtain a splitting

$$\Sigma^\infty_+ B_{\mathrm{gl}} C_2 \;\cong\; \mathbb{S} \oplus \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2$$

in the triangulated category $\mathcal{GH}$, reflecting the splitting (4.15). The set $\mathcal{C}_2' = \{\mathbb{S}, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2\}$ is thus another set of compact generators for $\mathcal{GH}_{\langle C_2 \rangle}$. However, $\langle \mathcal{C}_2 \rangle_+ = \langle \mathcal{C}_2' \rangle_+$ and so the t-structure generated by $\mathcal{C}_2'$ is again the standard t-structure, and the heart has not changed. However, the endomorphism category $\mathrm{End}(\mathcal{C}_2')$ is not equivalent to $\mathrm{End}(\mathcal{C}_2)$: the group $[\![\mathbb{S}, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2]\!]$ is trivial, and the groups $[\![\mathbb{S}, \mathbb{S}]\!]$, and $[\![\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2, \mathbb{S}]\!]$ and $[\![\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2]\!]$ are free abelian of rank 1. So an $\mathrm{End}(\mathcal{C}_2')$-module N consists of two abelian groups $N(C_2)$ and $N(e)$ and a morphism $\mathrm{tr} : N(e) \longrightarrow N(C_2)$, not subject to any relations. Even though $\mathrm{End}(\mathcal{C}_2')$ is not equivalent to $\mathrm{End}(\mathcal{C}_2)$, they are Morita equivalent, i.e., the category of $\mathrm{End}(\mathcal{C}_2')$-modules is equivalent to the category of $\mathrm{End}(\mathcal{C}_2)$-modules (as it must be, since both are equivalent to $\mathcal{GF}_{\langle C_2 \rangle}$).

A small modification of $\mathcal{C}_2'$, however, returns a different non-degenerate t-structure on $\mathcal{GH}_{\langle C_2 \rangle}$ with a heart that is not equivalent to the standard heart. We set $\mathcal{C}_2'' = \{\mathbb{S} \wedge S^1, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2\}$, i.e., the sphere spectrum is suspended once; this is another set of compact generators for $\mathcal{GH}_{\langle C_2 \rangle}$. The positivity condition is still satisfied because

$$[\![\mathbb{S} \wedge S^1, (\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2) \wedge S^1]\!] \;\cong\; [\![\mathbb{S}, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2]\!] \;\cong\; \pi_0^e(\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2) \;=0 \;.$$

So Proposition 4.9 applies and shows that $(\langle \mathcal{C}_2'' \rangle_+, \langle \mathcal{C}_2'' \rangle_{\leq 0})$ is a non-degenerate t-structure on $\mathcal{GH}_{\langle C_2 \rangle}$ whose heart $\mathscr{H}'$ consists of all orthogonal spectra X such that $\pi_k^e(X) = 0$ for $k \neq 1$, and $[\![\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2, X]\!]_k = 0$ for $k \neq 0$. Since $\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2$ represents the kernel of the restriction map from C_2 to the trivial group, the second condition is equivalent to the requirement that the restriction map $\mathrm{res}_e^{C_2} : \pi_k^{C_2}(X) \longrightarrow \pi_k^e(X)$ is an isomorphism for all $k \neq 0$. The group $[\![\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2, \mathbb{S} \wedge S^1]\!]$ is trivial and the group

$$[\![\mathbb{S} \wedge S^1, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2]\!] \;\cong\; \pi_1^e(\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2)$$

is cyclic of order 2. So an $\mathrm{End}(\mathcal{C}_2'')$-module P consists of two abelian groups $P(C_2)$ and $P(e)$ and a morphism $g : N(C_2) \longrightarrow N(e)$ such that $2g = 0$. This module category, and hence the heart of this non-standard t-structure, is not equivalent to the standard heart. The following diagram displays the three module

categories schematically:

$$\begin{tikzcd}[column sep=small] \text{mod-}\operatorname{End}(\mathcal{C}_2) & & \text{mod-}\operatorname{End}(\mathcal{C}_2') & & \text{mod-}\operatorname{End}(\mathcal{C}_2'') \\ M(C_2) \arrow[d, bend left=60, "\mathrm{res}"] & & N(C_2) & & P(C_2) \arrow[d, "g"'] \\ M(e) \arrow[u, bend left=60, "p^*"] \arrow[u, bend right=60, "\mathrm{tr}"'] & & N(e) \arrow[u, bend right=60, "\mathrm{tr}"'] & & N(e) \end{tikzcd}$$

$$\mathrm{res}\circ p^* = \mathrm{Id}\ , \quad \mathrm{res}\circ \mathrm{tr} \ = \ 2\cdot \mathrm{Id} \qquad\qquad 2g = 0$$

We recall from (3.30) that the symmetric monoidal derived smash product $\wedge^{\mathbb{L}}$ on the global stable homotopy category is obtained as the total left derived functor of the smash product of orthogonal spectra. The box product of global functors was introduced in Construction 2.18. A canonical morphism of global functors

$$(\underline{\pi}_0 X)\,\square\,(\underline{\pi}_0 Y) \ \longrightarrow\ \underline{\pi}_0(X\wedge Y)$$

was defined in Example 2.20; when applied to flat replacements of X and Y this becomes the morphism of the following proposition. We recall that an orthogonal spectrum is *globally connective* if its G-equivariant homotopy groups vanish in negative dimensions, for all compact Lie groups G. So globally connective is the same as $\mathcal{A}ll$-connective in the sense of Definition 4.10.

Proposition 4.16. *For all globally connective spectra X and Y the orthogonal spectrum $X\wedge^{\mathbb{L}} Y$ is globally connective and the natural morphism*

$$(\underline{\pi}_0 X)\,\square\,(\underline{\pi}_0 Y) \ \longrightarrow\ \underline{\pi}_0(X\wedge^{\mathbb{L}} Y)$$

is an isomorphism of global functors.

Proof. We fix a compact Lie group K and let $\mathcal{Y}$ be the class of globally connective orthogonal spectra Y such that $\Sigma^\infty_+ B_{\mathrm{gl}}K\wedge^{\mathbb{L}} Y$ is globally connective and the natural morphism

$$\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\,\square\,\underline{\pi}_0 Y \ \longrightarrow\ \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K\wedge^{\mathbb{L}} Y)$$

is an isomorphism of global functors. The class $\mathcal{Y}$ is closed under sums and we claim that it is also closed under cones. We let

$$A \ \longrightarrow\ B \ \longrightarrow\ C \ \longrightarrow\ \Sigma A$$

be a distinguished triangle in $\mathcal{GH}$ such that A and B belong to $\mathcal{Y}$. Since A is globally connective the global functor $\underline{\pi}_{-1}(A)$ vanishes; since A belongs to $\mathcal{Y}$ the global functor $\underline{\pi}_{-1}(\Sigma^\infty_+ B_{\mathrm{gl}}K\wedge^{\mathbb{L}} A)$ vanishes. Since $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\square-$ is right exact (by Remark 2.22), the upper row in the commutative diagram

$$\begin{tikzcd} \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\square(\underline{\pi}_0 A) \arrow[r] \arrow[d] & \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\square(\underline{\pi}_0 B) \arrow[r] \arrow[d] & \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\square(\underline{\pi}_0 C) \arrow[r] \arrow[d] & 0 \\ \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K\wedge^{\mathbb{L}} A) \arrow[r] & \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K\wedge^{\mathbb{L}} B) \arrow[r] & \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K\wedge^{\mathbb{L}} C) \arrow[r] & 0 \end{tikzcd}$$

is exact. The lower row is exact because smashing with $\Sigma^\infty_+ B_{\mathrm{gl}}K$ preserves distinguished triangles. The two left vertical maps are isomorphisms because A and B belong to the class $\mathcal{Y}$. So the right vertical map is an isomorphism and C belongs to $\mathcal{Y}$ as well. Moreover, we have

$$\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\,\square\,\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}G) \ \cong\ \mathbf{A}(K,-)\,\square\,\mathbf{A}(G,-) \ \cong\ \mathbf{A}(K\times G,-) \ \cong \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}(K\times G))$$

by Proposition 2.5 and Proposition A.3.13, and

$$(\Sigma^\infty_+ B_{\mathrm{gl}} K) \wedge (\Sigma^\infty_+ B_{\mathrm{gl}} G) \;\cong\; \Sigma^\infty_+ (B_{\mathrm{gl}} K \boxtimes B_{\mathrm{gl}} G) \;\cong\; \Sigma^\infty_+ B_{\mathrm{gl}}(K \times G)$$

by (1.17), respectively by (3.41) of Chapter I. This shows that the class $\mathcal{Y}$ is closed under sums and cones and contains the suspension spectra $\Sigma^\infty_+ B_{\mathrm{gl}} G$ for all compact Lie groups G. Corollary 4.11 (i) (for the global family $\mathcal{F} = \mathcal{A}ll$) then shows that $\mathcal{Y}$ is the class of all globally connective orthogonal spectra. This proves the proposition in the special case $X = \Sigma^\infty_+ B_{\mathrm{gl}} K$

Now we perform the same argument in the other variable. We fix a globally connective spectrum Y and let $\mathcal{X}$ denote the class of globally connective orthogonal spectra X such that $X \wedge^{\mathbb{L}} Y$ is globally connective and the natural morphism of the proposition is an isomorphism of global functors. The class $\mathcal{X}$ is again closed under sums and cones, by the same arguments as above. Moreover, for every compact Lie group K the suspension spectrum $\Sigma^\infty_+ B_{\mathrm{gl}} K$ belongs to the class $\mathcal{X}$ by the previous paragraph. Again Corollary 4.11 (i) shows that $\mathcal{X}$ is the class of all globally connective orthogonal spectra. □

The free orthogonal spectrum $F_{G,V} = \mathbf{O}(V,-)/G$ generated by a G-representation V was defined in Construction 1.24. Now we identify the functor represented by $F_{G,V}$ in the global stable homotopy category. The element

$$(0, \mathrm{Id}_V) \cdot G \;\in\; \mathbf{O}(V,V)/G = \; F_{G,V}(V)$$

is a G-fixed point, so the G-map

$$S^V \;\longrightarrow\; \mathbf{O}(V,V)/G \wedge S^V \;=\; (F_{G,V} \wedge S^V)(V)\;, \quad v \;\longmapsto\; (0, \mathrm{Id}_V) \cdot G \wedge (-v)$$

represents an equivariant homotopy class

$$a_{G,V} \;\in\; \pi_0^G(F_{G,V} \wedge S^V)\;. \tag{4.17}$$

The morphism of orthogonal G-spectra $\lambda^V_E : E \wedge S^V \longrightarrow \mathrm{sh}^V E$ was defined in (1.23) of Chapter III; its value at an inner product space U is the opposite structure map $\sigma^{\mathrm{op}}_{U,V} : E(U) \wedge S^V \longrightarrow E(U \oplus V) = (\mathrm{sh}^V E)(U)$. The morphism λ^V_E is a G-equivariant stable equivalence by Proposition III.1.25 (ii).

The following lemma is a direct consequence of Proposition III.1.25 (i).

Lemma 4.18. *The morphism $\lambda^V_{F_{G,V}} : F_{G,V} \wedge S^V \longrightarrow \mathrm{sh}^V F_{G,V}$ takes the element $a_{G,V}$ to the class in $\pi_0^G(\mathrm{sh}^V F_{G,V})$ that is represented by the G-fixed point $(0, \mathrm{Id}_V) \cdot G$ of $(\mathrm{sh}^V F_{G,V})(0)$.*

We emphasize that the representative of $a_{G,V}$ involves the involution $S^{-\mathrm{Id}} : S^V \longrightarrow S^V$, which represents a certain unit in the ring $\pi_0^G(\mathbb{S})$ that squares to the identity. Lemma 4.18 is our reason for choosing this particular normalization. We could remove the involution from the definition of $a_{G,V}$, thereby changing the class $a_{G,V}$ into $\epsilon_V(a_{G,V})$, but then the unit would instead appear in other formulas later.

Theorem 4.19. *Let G be a compact Lie group and V a faithful G-representation. Then the pair $(F_{G,V}, a_{G,V})$ represents the functor*

$$\mathcal{GH} \longrightarrow (\mathrm{sets})\;, \quad E \;\longmapsto\; E_0^G(S^V)\;.$$

The free orthogonal spectrum $F_{G,V}$ is a compact object of the global stable homotopy category $\mathcal{GH}$.

Proof. We define

$$b_{G,V} \;=\; (\lambda^V_{F_{G,V}})_*(a_{G,V}) \;\in\; \pi_0^G(\mathrm{sh}^V F_{G,V})\;;$$

Lemma 4.18 shows that the class $b_{G,V}$ is represented by the G-fixed point

$$(0, \mathrm{Id}_V) \cdot G \;\in\; \mathbf{O}(V,V)/G \;=\; (\mathrm{sh}^V F_{G,V})(0)\;.$$

Since the morphism $\lambda^V_{F_{G,V}}$ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra, we may thus show that the pair $(F_{G,V}, b_{G,V})$ represents the functor

$$\mathcal{GH} \longrightarrow (\mathrm{sets})\;, \quad E \;\longmapsto\; \pi_0^G(\mathrm{sh}^V E)\;,$$

i.e., for every orthogonal spectrum E the map

$$[\![F_{G,V}, E]\!] \longrightarrow \pi_0^G(\mathrm{sh}^V E)\ , \quad f \longmapsto (\mathrm{sh}^V f)_*(b_{G,V})$$

is bijective. Since both sides take global equivalences in E to bijections, we can assume that E is a global Ω-spectrum. The orthogonal spectrum $F_{G,V}$ is flat, and hence cofibrant in the global model structure. So the localization functor induces a bijection

$$\mathcal{S}p(F_{G,V}, E)/\text{homotopy} \longrightarrow [\![F_{G,V}, E]\!]$$

from the set of homotopy classes of morphisms of orthogonal spectra to the set of morphisms in $\mathcal{GH}$. By the freeness property, morphisms from $F_{G,V}$ to E biject with G-fixed points of $E(V)$, and homotopies correspond to paths of G-fixed points. The composite

$$\pi_0(E(V)^G) \xrightarrow{\cong} [\![F_{G,V}, E]\!] \xrightarrow{f \mapsto (\mathrm{sh}^V f)_*(b_{G,V})} \pi_0^G(\mathrm{sh}^V E)$$

is the stabilization map of the G-Ω-spectrum $\mathrm{sh}^V E$, and hence bijective. Since the left map and the composite are bijective, so is the evaluation map at the class $b_{G,V}$.

Now we prove that $F_{G,V}$ is a compact object in $\mathcal{GH}$. In the commutative square

$$\begin{array}{ccc} \bigoplus_{i\in I}[\![F_{G,V}, X_i]\!] & \longrightarrow & [\![F_{G,V}, \bigoplus_{i\in I} X_i]\!] \\ \downarrow & & \downarrow \\ \bigoplus_{i\in I}\pi_0^G(X_i \wedge S^V) & \longrightarrow & \pi_0^G\left((\bigvee_{i\in I} X_i) \wedge S^V\right) \end{array}$$

both vertical maps are evaluation at the class $a_{G,V}$, which are isomorphisms by part (i). The lower horizontal map is an isomorphism by Corollary III.1.38 (i), hence so is the upper horizontal map. This shows that $F_{G,V}$ is compact as an object of the triangulated category $\mathcal{GH}$. □

Now we construct certain distinguished triangles in the global stable homotopy category that relate free orthogonal spectra for a compact Lie group G and a free orthogonal spectrum for a closed subgroup H that occurs as the stabilizer of a unit vector in a transitive faithful G-representation. Equivalently, we are looking for ways to present spheres as homogeneous spaces. The main case we care about is $G = O(m)$ acting tautologically on $\mathbb{R}^m$ with stabilizer group $H = O(m-1)$. This special case will become relevant in Section VI.2 when we analyze the rank filtration of the global Thom spectrum **mO**. Similarly, we can let $G = SO(m)$ act tautologically on $\mathbb{R}^m$ with stabilizer group $H = SO(m-1)$, and this shows up in the rank filtration of the global Thom spectrum **mSO**; here we need $m \geq 2$, since $SO(1)$ does not act transitively on $S(\mathbb{R})$. Other examples are $U(m)$ or $SU(m)$ (the latter for $m \geq 2$) acting on the underlying $\mathbb{R}$-vector space of $\mathbb{C}^m$, with stabilizer groups $U(m-1)$ respectively $SU(m-1)$. Similarly, we can consider the tautological representation of $Sp(m)$ on the underlying $\mathbb{R}$-vector space of $\mathbb{H}^m$ with stabilizer group $Sp(m-1)$. These examples show up in rank filtrations of the global Thom spectra **mU**, **mSU** and **mSp**. There are also some 'exceptional' examples, such as the exceptional Lie group G_2, the group of $\mathbb{R}$-algebra automorphism of the octonions $\mathbb{O}$. Here we take V as the tautological 7-dimensional representation on the imaginary octonions, with stabilizer group isomorphic to $SU(3)$. For a complete list of examples we refer the reader to [**17**, Ch. 7.B, Ex. 7.13] and the references given therein.

Construction 4.20. We consider a compact Lie group G and a G-representation V such that G acts faithfully and transitively on the unit sphere $S(V)$. We choose a unit vector $v \in S(V)$ and let H be the stabilizer group of v. Then the underlying H-representation of V decomposes as

$$V = (\mathbb{R} \cdot v) \oplus L$$

where L is the orthogonal complement of v. We use the letter L for this complement because it essentially 'is' the tangent representation $T_H(G/H)$. More precisely, the differential at eH of the smooth G-equivariant

embedding

$$G/H \longrightarrow V \ , \quad gH \longmapsto gv$$

is an H-equivariant isomorphism $(Di)_H : T_H(G/H) \cong L$.

We define a based G-map

$$\begin{aligned} r \ : \ S^V \ &\longrightarrow \ \mathbf{O}(L,V)/H \ = \ F_{H,L}(V) \qquad \text{by} \\ r(g \cdot t \cdot v) \ &= \ g \cdot ((t^2-1)/t \cdot v, \mathrm{incl}) \cdot H \ , \end{aligned} \tag{4.21}$$

where $g \in G$ and $t \in [0,\infty]$. We let

$$T \ : \ \Sigma^\infty_+ B_{\mathrm{gl}} G \ = \ \Sigma^\infty_+ \mathbf{L}_{G,V} \ \longrightarrow \ F_{H,L}$$

denote the adjoint of r.

We define a morphism of orthogonal spectra

$$i \ : \ F_{H,L} \ \longrightarrow \ \mathrm{sh}\, F_{G,V}$$

as the adjoint of the H-fixed point

$$\psi \cdot G \ \in \ \mathbf{O}(V, L \oplus \mathbb{R})/G \ = \ (\mathrm{sh}\, F_{G,V})(L) \ ,$$

where $\psi : V \longrightarrow L \oplus \mathbb{R}$ is the H-equivariant isometry given by $\psi(x) = (x - \langle x,v\rangle \cdot v, \ \langle x,v\rangle)$. The value at an inner product space W is then the map

$$\begin{aligned} i(W) \ : \ F_{H,L}(W) = \mathbf{O}(L,W)/H \ &\longrightarrow \ \mathbf{O}(V, W \oplus \mathbb{R})/G \ = \ (\mathrm{sh}\, F_{G,V})(W) \\ (w,\varphi) \cdot H \qquad &\longmapsto \qquad ((w,0), (\varphi \oplus \mathbb{R}) \circ \psi) \cdot G \ . \end{aligned}$$

We define a morphism of orthogonal spectra $a : F_{G,V} \longrightarrow \Sigma^\infty_+ \mathbf{L}_{G,V} = \Sigma^\infty_+ B_{\mathrm{gl}} G$ as the adjoint of the G-fixed point

$$0 \wedge \mathrm{Id} \cdot G \ \in \ S^V \wedge \mathbf{L}(V,V)/G_+ \ = \ (\Sigma^\infty_+ \mathbf{L}_{G,V})(V) \ .$$

We recall that the morphism $\lambda_X : X \wedge S^1 \longrightarrow \mathrm{sh}\, X$ was defined in (1.23) of Chapter III, and that it is a global equivalence by Proposition 1.4 (i). The stable tautological class

$$e_G \ = \ e_{G,V} \ \in \ \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} G)$$

was defined in (1.12). In the next theorem we abuse notation and use the same symbols for morphisms of orthogonal spectra and their images in the global stable homotopy category.

THEOREM 4.22. *Let G be a compact Lie group and V a G-representation such that G acts faithfully and transitively on the unit sphere $S(V)$; let H the stabilizer group of a unit vector of V.*

(i) *The sequence*

$$F_{G,V} \ \xrightarrow{a} \ \Sigma^\infty_+ B_{\mathrm{gl}} G \ \xrightarrow{T} \ F_{H,L} \ \xrightarrow{-\lambda^{-1}_{F_{G,V}} \circ i} \ F_{G,V} \wedge S^1$$

is a distinguished triangle in the global stable homotopy category.

(ii) *The morphism T satisfies the relation*

$$T_*(e_G) \ = \ \mathrm{Tr}^G_H(a_{H,L})$$

in the group $\pi_0^G(F_{H,L})$.

PROOF. We consider the based map $q : G/H_+ \longrightarrow S^0$ that sends G/H to the non-basepoint. We identify the mapping cone of q with the sphere S^V via the G-equivariant homeomorphism

$$h \ : \ Cq \ \cong \ S^V$$

that is induced by the map

$$G/H \times [0,1] \ \longrightarrow \ S^V \ , \quad (gH, x) \ \longmapsto \ g \cdot (1-x)/x \cdot v \ .$$

Under this identification the mapping cone inclusion $i : S^0 \longrightarrow Cq$ becomes the inclusion $\iota : \{0,\infty\} = S^0 \longrightarrow S^V$, and the projection $p : Cq \longrightarrow G/H_+ \wedge S^1$ becomes the map

$$p \circ h^{-1} \ : \ S^V \ \longrightarrow \ G/H_+ \wedge S^1 \ , \quad g \cdot t \cdot v \ \longmapsto \ gH \wedge \frac{1-t^2}{t} \ , \tag{4.23}$$

where $g \in G$ and $t \in [0,\infty]$. The free functor $F_{G,V}$ takes mapping cone sequences of based G-spaces to mapping cone sequences of orthogonal spectra, so the sequence

$$F_{G,V}(G/H_+) \ \xrightarrow{F_{G,V}q} \ F_{G,V} \ \xrightarrow{F_{G,V}\iota} \ F_{G,V}S^V \ \xrightarrow{F_{G,V}(p\circ h^{-1})} \ F_{G,V}(G/H_+ \wedge S^1)$$

is a distinguished triangle in the global stable homotopy category.

We define a morphism $\Lambda : F_{G,V}(G/H_+ \wedge S^1) \longrightarrow F_{H,L}$ as the adjoint of the G-map

$$\begin{aligned} G/H_+ \wedge S^1 \ &\longrightarrow \ \mathbf{O}(L,V)/H \ = \ F_{H,L}(V) \\ gH \ \wedge t \ &\longmapsto \ g \cdot (-t \cdot v, \text{ incl}) \cdot H \ . \end{aligned}$$

The morphism Λ is the composite of an isomorphism $F_{G,V}(G/H_+ \wedge S^1) \cong F_{H,\mathbb{R}\oplus L}S^1$ and the morphism $\lambda_{\mathbb{R},L}/H : F_{H,\mathbb{R}\oplus L}S^1 \longrightarrow F_{H,L}$ defined in (1.30). The morphism $\lambda_{\mathbb{R},L}/H$ is a global equivalence by Theorem 1.31, hence Λ is a global equivalence as well. The composite

$$\Sigma^\infty_+ B_{\mathrm{gl}}G \ \xrightarrow{\text{untwist}^{-1}} \ F_{G,V}S^V \ \xrightarrow{F_{H,L}(p\circ h^{-1})} \ F_{G,V}(G/H_+ \wedge S^1) \ \xrightarrow{\Lambda} \ F_{H,L}$$

coincides with the morphism T, by direct inspection of the effects at the inner product space V.

Our next claim is that the following diagram of orthogonal spectra commutes up to homotopy:

$$\begin{tikzcd} F_{G,V}(G/H_+) \wedge S^1 \arrow[r, "(F_{G,V}q)\wedge S^1"] \arrow[d, "\Lambda"'] & F_{G,V} \wedge S^1 \arrow[d, "\lambda_{F_{G,V}}"] \\ F_{H,L} \arrow[r, "i"'] & \operatorname{sh} F_{G,V} \end{tikzcd} \tag{4.24}$$

The representing property of $F_{G,V}$ reduces this to evaluating the square at V and showing that the two resulting G-maps from $G/H_+ \wedge S^1$ to $(\operatorname{sh} F_{G,V})(V)$ are equivariantly homotopic. Since G-maps out of G/H correspond to H-fixed points, this in turn reduces to the claim that the two maps

$$S^1 \ \longrightarrow \ (\mathbf{O}(V, V \oplus \mathbb{R})/G)^H \ = \ ((\operatorname{sh} F_{G,V})(V))^H$$

that send $t \in S^1$ to

$$((-t \cdot v, 0), i_0) \cdot G \qquad \text{respectively} \qquad ((0,t), i_1) \cdot G$$

are based homotopic through H-fixed points, where $i_0, i_1 : V \longrightarrow V \oplus \mathbb{R}$ are given by

$$i_0(x) \ = \ (x - \langle x, v\rangle \cdot v, \ \langle x, v\rangle) \qquad \text{respectively} \qquad i_1(x) \ = \ (x, 0) \ .$$

A homotopy that witnesses this is

$$S^1 \times [0,1] \ \longrightarrow \ (\mathbf{O}(V, V \oplus \mathbb{R})/G)^H \ , \quad (t,s) \ \longmapsto \ ((-t \cdot \cos(\pi s/2) \cdot v, t \cdot \sin(\pi s/2)), \ i_s) \cdot G \ ,$$

with

$$i_s \ : \ V \ \longrightarrow \ V \oplus \mathbb{R} \ , \quad i_s(x) \ = \ (x + (\sin(\pi s/2) - 1) \cdot \langle x, v\rangle \cdot v, \cos(\pi s/2) \cdot \langle x, v\rangle) \ .$$

This shows the claim that the square (4.24) is homotopy commutative.

So altogether we have produced a commutative diagram in the global stable homotopy category:

$$\begin{tikzcd}
F_{G,V} \arrow[r, "F_{G,V}\iota"] \arrow[d, equal] & F_{G,V}S^V \arrow[r, "F_{G,V}(p\circ h^{-1})"] \arrow[d, "\text{untwist}"', "\cong"] & F_{G,V}(G/H_+\wedge S^1) \arrow[r, "-(F_{G,V}q\wedge S^1)"] \arrow[d, "\sim"', "\Lambda"] & F_{G,V}\wedge S^1 \arrow[d, "\sim"', "\lambda_{F_{G,V}}"] \\
F_{G,V} \arrow[r, "a"'] & \Sigma^\infty_+ B_{\mathrm{gl}}G \arrow[r, "T"'] & F_{H,L} \arrow[r, "-i"'] & \mathrm{sh}\, F_{G,V}
\end{tikzcd}$$

The upper sequence is obtained by rotating a distinguished triangle, so it is distinguished. Since all vertical morphisms are isomorphisms in $\mathcal{GH}$, we conclude that the sequence $(a, T, -\lambda^{-1}_{F_{G,V}}\circ i)$ is indeed a distinguished triangle in $\mathcal{GH}$.

It remains to analyze the effect of the morphism T on the stable tautological class e_G. We consider the wide G-equivariant embedding

$$i \;:\; G/H \;\longrightarrow\; V\;, \quad gH \;\longmapsto\; gv\;.$$

This embedding was already used to identify the tangent space $T_H(G/H)$ at the preferred coset with the subspace L inside V; the inclusion $L \longrightarrow V$ corresponds to the differential $(Di)_H$. The orthogonal complement of L is precisely the line spanned by the vector v, and the subgroup H acts trivially on this complement.

As described more generally in Construction III.2.6, the wide embedding gives rise to a G-equivariant Thom-Pontryagin collapse map

$$c \;:\; S^V \;\longrightarrow\; G/H_+\wedge S^1\;,$$

see (2.9) of Chapter III. In the more general context of an arbitrary closed subgroup, the target of the collapse map is $G\ltimes_H S^{V-L}$; but in our situation H acts trivially on $\mathbb{R}\cdot v = V-L$, so we can (and will) identify $V-L\cong\mathbb{R}$ via $x\mapsto\langle x,v\rangle$. The collapse map c sends the origin and all vectors of length at least 2 to the basepoint; if $x\in V$ satisfies $0<|x|<2$, then $x = g\cdot t\cdot v$ for some $g\in G$ and $t\in(0,2)$. The collapse map c sends such a vector x to

$$gH\wedge\frac{t-1}{1-|t-1|}\;\in\; G/H_+\wedge S^1\;.$$

Now we calculate the class $\mathrm{Tr}^G_H(a_{H,L})$. We factor the external transfer $G\ltimes_H - \,:\, \pi^G_0(E\wedge S^L)\longrightarrow \pi^G_0(G\ltimes_H E)$ as the composite

$$\pi^H_0(E\wedge S^L)\;\xrightarrow{(\lambda^L_E)_*}\;\pi^H_0(\mathrm{sh}^L E)\;\xrightarrow{G\boxtimes_H -}\;\pi^G_0(G\ltimes_H E)\;.$$

Here $\lambda^L_E : E\wedge S^L\longrightarrow \mathrm{sh}^L E$ is the morphism of orthogonal G-spectra defined in (1.23) of Chapter III, and the second homomorphism is defined as follows. We let U be an H-representation and $f : S^U\longrightarrow E(U\oplus L) = (\mathrm{sh}^L E)(U)$ an H-equivariant based map that represents a class in $\pi^H_0(\mathrm{sh}^L E)$. By enlarging U, if necessary, we can assume that it is the underlying H-representation of a G-representation. The composite G-map

$$\begin{aligned}
S^{U\oplus V}\;&\xrightarrow{S^U\wedge c}\;S^U\wedge(G/H_+\wedge S^1)\;\xrightarrow[\cong]{\text{shear}}\;G\ltimes_H(S^{i^*U}\wedge S^1)\\
&\xrightarrow{G\ltimes_H(f\wedge S^1)}\;G\ltimes_H(E(i^*U\oplus L)\wedge S^1)\\
&\xrightarrow{G\ltimes_H\sigma^{\mathrm{op}}_{U\oplus L,\mathbb{R}}}\;G\ltimes_H(E(i^*U\oplus L\oplus\mathbb{R}))\\
&\cong\quad G\ltimes_H(E(i^*U\oplus i^*V))\;\cong\;(G\ltimes_H E)(U\oplus V)
\end{aligned}$$

then represents the class

$$G\boxtimes_H\langle f\rangle\qquad\text{in}\qquad\pi^G_0(G\ltimes_H E)\;.$$

The identification $L\oplus\mathbb{R}\cong V$ above is via $(x,\lambda)\mapsto x+\lambda\cdot v$. The relation $G\boxtimes_H((\lambda^L_E)_*(x)) = G\ltimes_H x$ is straightforward from the definitions.

By Lemma 4.18, the class $(\lambda^L_{F_{H,L}})_*(a_{H,L})$ is represented by the H-fixed point

$$(0, \mathrm{Id}_L) \cdot H \ \in \ \mathbf{O}(L,L)/H \ = \ (\mathrm{sh}^L F_{H,L})(0) \ .$$

So the class $\mathrm{Tr}^G_H(a_{H,L}) = \mathrm{act}_*(G \boxtimes_H ((\lambda^L_{F_{H,L}})_*(a_{H,L})))$ is represented by the composite G-map

$$\begin{aligned} S^V \ &\xrightarrow{c} \ G/H_+ \wedge S^1 \ \xrightarrow{G \ltimes_H ((0,\mathrm{Id}_L)H \wedge S^1)} \ G \ltimes_H (\mathbf{O}(L,L)/H \wedge S^1) \\ &\xrightarrow{G \ltimes_H \sigma^{\mathrm{op}}_{L,\mathbb{R}}} \ G \ltimes_H (\mathbf{O}(L, L \oplus \mathbb{R})/H) \ \xrightarrow[\cong]{\psi_*^{-1}} \ G \ltimes_H (\mathbf{O}(L,V)/H) \ \xrightarrow{\mathrm{act}} \ \mathbf{O}(L,V)/H \ . \end{aligned}$$

Expanding all the definitions identifies this composite as the map

$$S^V \ \longrightarrow \ \mathbf{O}(L,V)/H \ , \quad g \cdot t \cdot v \ \longmapsto \ g \cdot (\psi(t) \cdot v, \mathrm{incl}) \cdot H \ ,$$

for $g \in G$ and $t \in [0, \infty]$, with

$$\psi \ : \ [0,\infty] \ \longrightarrow \ S^1 \ , \quad \psi(t) \ = \ \begin{cases} \frac{t-1}{1-|t-1|} & \text{if } t \in (0,2), \text{ and} \\ * & \text{else.} \end{cases}$$

The function ψ is homotopic, relative to $\{0, \infty\}$, to the function sending t to $(t^2-1)/t$. Any relative homotopy between these two functions induces a based G-equivariant homotopy between the representative of the class $\mathrm{Tr}^G_H(a_{H,L})$ and the map r define in (4.21). So r itself is a representative of the class $\mathrm{Tr}^G_H(a_{H,L})$. Since $T : \Sigma^\infty_+ B_{\mathrm{gl}} G \longrightarrow F_{H,L}$ was defined as the adjoint to the G-map r, we deduce the desired relation $\mathrm{Tr}^G_H(a_{H,L}) = [r] = T_*(e_G)$. □

Remark 4.25. Part (ii) of Theorem 4.22 can be interpreted as saying that for global homotopy types, the morphism of orthogonal spectra $T : \Sigma^\infty_+ B_{\mathrm{gl}} G \longrightarrow F_{H,L}$ represents the dimension shifting transfer from H to G. Indeed, the pair $(\Sigma^\infty_+ B_{\mathrm{gl}} G, e_G)$ represents the functor π^G_0 (by Theorem 4.3 (i)), and the pair $(F_{H,L}, a_{H,L})$ represents the functor $E \mapsto \pi^H_0(E \wedge S^L)$ (by Theorem 4.19). Naturality of transfers then promotes the relation $T_*(e_G) = \mathrm{Tr}^G_H(a_{H,L})$ to a commutative diagram:

$$\begin{tikzcd} \llbracket F_{H,L}, E \rrbracket \arrow[r, "\llbracket T, E \rrbracket"] \arrow[d, "f \mapsto f_*(a_{H,L})"', "\cong"] & \llbracket \Sigma^\infty_+ B_{\mathrm{gl}} G, E \rrbracket \arrow[d, "\cong"', "f \mapsto f_*(e_G)"] \\ \pi^H_0(E \wedge S^L) \arrow[r, "\mathrm{Tr}^G_H"'] & \pi^G_0(E) \end{tikzcd}$$

5. Change of families

In this section we compare the global stable homotopy categories for two different global families $\mathcal{F}$ and $\mathcal{E}$, where we suppose that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is also an $\mathcal{F}$-equivalence, so we get a 'forgetful' functor on the homotopy categories

$$U = U^{\mathcal{E}}_{\mathcal{F}} \ : \ \mathcal{GH}_{\mathcal{E}} \ \longrightarrow \ \mathcal{GH}_{\mathcal{F}}$$

from the universal property of localizations. The global model structures are stable, so the two global homotopy categories $\mathcal{GH}_{\mathcal{E}}$ and $\mathcal{GH}_{\mathcal{F}}$ have preferred triangulated structures, and the forgetful functor is canonically an exact functor of triangulated categories. We show in Theorem 5.1 that this forgetful functor has a left and a right adjoint, both fully faithful, and that this data is part of a recollement of triangulated categories. If the global families $\mathcal{E}$ and $\mathcal{F}$ are multiplicative, then the smash product of orthogonal spectra can be derived to symmetric monoidal products on $\mathcal{GH}_{\mathcal{E}}$ and on $\mathcal{GH}_{\mathcal{F}}$ (see Corollary 3.31). The forgetful functor is strongly monoidal with respect to these derived smash products. Indeed, the derived smash product in $\mathcal{GH}_{\mathcal{E}}$ can be calculated by flat approximation up to $\mathcal{E}$-equivalence; every $\mathcal{E}$-equivalence is also an $\mathcal{F}$-equivalence, so these flat approximations can also be used to calculate the derived smash product

in $\mathcal{GH}_{\mathcal{F}}$. Theorem 5.1 below also exhibits symmetric monoidal structures on the two adjoints of the forgetful functor.

The special case with $\mathcal{F}$ the trivial global family allows a quick calculation of the Picard group of $\mathcal{GH}_{\mathcal{E}}$, with the possibly disappointing answer that it is free abelian of rank 1, generated by the global sphere spectrum (see Theorem 5.6). Propositions 5.9 and 5.18 provide characterizations of the global homotopy types in the image of the left and right adjoints. As we explain in Example 5.21, the 'absolute' right adjoint from the non-equivariant to the full global stable homotopy category models Borel cohomology theories. Construction 5.24 exhibits a particularly nice lift of the global Borel functor to the pointset level, i.e., a lax symmetric monoidal endofunctor of the category of orthogonal spectra.

At the end of this section we also relate the global stable homotopy category to the G-equivariant stable homotopy category G-$\mathcal{SH}$ for a fixed compact Lie group G. There is another forgetful functor

$$U \;=\; U_G \;:\; \mathcal{GH} \;\longrightarrow\; G\text{-}\mathcal{SH}$$

which is an exact functor of triangulated categories and has both a left adjoint and a right adjoint, compare Theorem 5.31 below. Here, however, the adjoints are *not* fully faithful as soon as the group G is non-trivial.

THEOREM 5.1. *Let $\mathcal{F}$ and $\mathcal{E}$ be two global families such that $\mathcal{F} \subseteq \mathcal{E}$.*

(i) *The forgetful functor*

$$U \;:\; \mathcal{GH}_{\mathcal{E}} \;\longrightarrow\; \mathcal{GH}_{\mathcal{F}}$$

has a left adjoint L and a right adjoint R, and both adjoints are fully faithful.

(ii) *If the global families $\mathcal{E}$ and $\mathcal{F}$ are multiplicative, then the right adjoint has a preferred lax symmetric monoidal structure $(RA) \wedge^{\mathbb{L}}_{\mathcal{E}} (RB) \longrightarrow R(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$.*

(iii) *If the global families $\mathcal{E}$ and $\mathcal{F}$ are multiplicative, then the left adjoint has a preferred strong symmetric monoidal structure, i.e., a natural isomorphism between $L(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$ and $(LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB)$.*

PROOF. By Proposition 3.27 the categories $\mathcal{GH}_{\mathcal{E}}$ and $\mathcal{GH}_{\mathcal{F}}$ have infinite sums and infinite products, and the forgetful functor preserves both.

(i) As spelled out in Corollary 4.5 (v), the existence of the left adjoint is an abstract consequence of the fact that $\mathcal{GH}_{\mathcal{E}}$ is compactly generated and that the functor U preserves products. But instead of arguing by hand that U preserves products, we give an alternative construction of the left adjoint by model category theory. Indeed, it is immediate from the characterization of fibrations in the global model structures in Theorem 3.21 (ii) that the identity functor is a right Quillen functor from the $\mathcal{E}$-global to the $\mathcal{F}$-global model structure. So its derived functor has a (derived) left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$, see for example [**79**, Lemma 1.3.10]. Since the right Quillen functor (i.e., the identity) preserves all weak equivalences, the adjunction unit $A \longrightarrow U(LA)$ is an isomorphism in the $\mathcal{F}$-global homotopy category. So the left adjoint is fully faithful.

By Proposition 3.27 sums in both $\mathcal{GH}_{\mathcal{E}}$ and $\mathcal{GH}_{\mathcal{F}}$ are represented by the pointset level wedge of orthogonal spectra. So the forgetful functor U preserves sums. As spelled out in Corollary 4.5 (iv), the existence of the right adjoint is then an abstract consequence of the fact that $\mathcal{GH}_{\mathcal{E}}$ is compactly generated.

The fact that R is fully faithful is also a formal consequence of properties of the adjoint pair (L, U). As we saw above, the adjunction unit $\eta_A : A \longrightarrow U(LA)$ is an isomorphism for every object A of $\mathcal{GH}_{\mathcal{F}}$. So the map

$$(\eta_A)^* \;:\; [\![U(LA), X]\!]_{\mathcal{F}} \;\longrightarrow\; [\![A, X]\!]_{\mathcal{F}}$$

is bijective for every object X of $\mathcal{GH}_{\mathcal{F}}$. The adjunction (U, R) lets us rewrite the left hand side as $[\![LA, RX]\!]_{\mathcal{E}}$, and the adjunction (L, U) lets us rewrite this further to $[\![A, U(RX)]\!]_{\mathcal{F}}$. Under these substitutions, the map $(\eta_A)^*$ becomes the map induced by the adjunction counit $\epsilon_X : U(RX) \longrightarrow X$. This adjunction counit is thus a natural isomorphism, and so R is also fully faithful.

(ii) The lax monoidal structure of the right adjoint R is a formal consequence of the strong monoidal structure of the forgetful functor U. Indeed, for every pair of orthogonal spectra A and B this strong

monoidal structure and the adjunction counits provide a morphism

$$U\big((RA)\wedge^{\mathbb{L}}_{\mathcal{E}}(RB)\big)\ \cong\ (URA)\wedge^{\mathbb{L}}_{\mathcal{F}}(URB)\ \xrightarrow{\epsilon_A\wedge^{\mathbb{L}}\epsilon_B}\ A\wedge^{\mathbb{L}}_{\mathcal{F}}B$$

whose adjoint $(RA)\wedge^{\mathbb{L}}_{\mathcal{E}}(RB)\longrightarrow R(A\wedge^{\mathbb{L}}_{\mathcal{F}}B)$ is associative, commutative and unital.

(iii) The strong symmetric monoidal structure on U and the adjunction units provide a morphism

$$A\wedge^{\mathbb{L}}_{\mathcal{F}}B\ \xrightarrow{\eta_A\wedge^{\mathbb{L}}\eta_B}\ U(LA)\wedge^{\mathbb{L}}_{\mathcal{F}}U(LB)\ \cong\ U\big((LA)\wedge^{\mathbb{L}}_{\mathcal{E}}(LB)\big)$$

whose adjoint $\lambda_{A,B}: L(A\wedge^{\mathbb{L}}_{\mathcal{F}}B)\longrightarrow (LA)\wedge^{\mathbb{L}}_{\mathcal{E}}(LB)$ is associative, commutative and unital. We claim that the morphism $\lambda_{A,B}$ is in fact an isomorphism in $\mathcal{GH}_{\mathcal{E}}$ (and hence it can be turned around). We can assume that A and B are $\mathcal{F}$-cofibrant so that $LA=A$ and $LB=B$. Since $\mathcal{F}$ is multiplicative, the pointset level smash product $A\wedge B$ is again $\mathcal{F}$-cofibrant by Proposition 3.29 (i). So the value of the left adjoint L on $A\wedge B$ is also given by $A\wedge B$. □

Remark 5.2 (Recollements). Theorem 5.1 implies that for all pairs of nested global families $\mathcal{F}\subseteq\mathcal{E}$ the diagram of triangulated categories and exact functors

$$\begin{tikzcd}
\mathcal{GH}(\mathcal{E};\mathcal{F}) \arrow[r, "i_*"] & \mathcal{GH}_{\mathcal{E}} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "U"] & \mathcal{GH}_{\mathcal{F}} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"]
\end{tikzcd}$$

is a *recollement* in the sense of [**16**, Sec. 1.4]. Here $\mathcal{GH}(\mathcal{E};\mathcal{F})$ denotes the '$\mathcal{E}$-global homotopy category with support outside $\mathcal{F}$', i.e., the full subcategory of $\mathcal{GH}_{\mathcal{E}}$ of spectra all of whose $\mathcal{F}$-equivariant homotopy groups vanish. The functor $i_*:\mathcal{GH}(\mathcal{E};\mathcal{F})\longrightarrow\mathcal{GH}_{\mathcal{E}}$ is the inclusion, and i^* (respectively $i^!$) is a left adjoint (respectively right adjoint) of i_*.

Remark 5.3. In Theorem 5.1 the left adjoint $L:\mathcal{GH}_{\mathcal{F}}\longrightarrow\mathcal{GH}_{\mathcal{E}}$ of the forgetful functor $U:\mathcal{GH}_{\mathcal{E}}\longrightarrow\mathcal{GH}_{\mathcal{F}}$ is obtained as the total left derived functor of the identity functor on orthogonal spectra with respect to change of model structure from the $\mathcal{F}$-global to the $\mathcal{E}$-global model structure. So one can calculate the value of the left adjoint L on an orthogonal spectrum X by choosing any cofibrant replacement in the $\mathcal{F}$-global model structure, i.e., an $\mathcal{F}$-equivalence $X_{\mathcal{F}}\longrightarrow X$ with $\mathcal{F}$-cofibrant source. (One can do this in the $\mathcal{F}$-level model structure and thereby arrange that the map is even an $\mathcal{F}$-level equivalence.) The global homotopy type (so in particular the $\mathcal{E}$-global homotopy type) of $X_{\mathcal{F}}$ is then well-defined. Indeed, since the identity is a left Quillen functor from the $\mathcal{F}$-global to the global model structure (the special case $\mathcal{E}=\mathcal{A}ll$), every acyclic cofibration in the $\mathcal{F}$-global model structure is a global equivalence. Ken Brown's lemma (see the proof of [**30**, I.4 Lemma 1] or [**79**, Lemma 1.1.12]) then implies that every $\mathcal{F}$-equivalence between $\mathcal{F}$-cofibrant orthogonal spectra is a global equivalence.

It seems worth spelling out the extreme case when $\mathcal{F}=\langle e\rangle$ is the minimal global family of trivial groups and when $\mathcal{E}=\mathcal{A}ll$ is the maximal global family of all compact Lie groups: An orthogonal spectrum X is $\langle e\rangle$-cofibrant if for every $m\geq 0$ the latching morphism $W: L_mX\longrightarrow X(\mathbb{R}^m)$ is an $O(m)$-cofibration and $O(m)$ acts freely on the complement of the image. These are precisely the orthogonal spectra called 'q-cofibrant' in [**104**]. Then every non-equivariant stable equivalence between such $\langle e\rangle$-cofibrant orthogonal spectra is a global equivalence.

Remark 5.4. We let $\mathcal{F}\subseteq\mathcal{E}$ be a pair of nested multiplicative global families. By Theorem 5.1 (ii) the right adjoint $R:\mathcal{GH}_{\mathcal{F}}\longrightarrow\mathcal{GH}_{\mathcal{E}}$ to the forgetful functor comes with a lax symmetric monoidal structure, so it takes '$\mathcal{F}$-global homotopy ring spectra' (i.e., monoid objects in $\mathcal{GH}_{\mathcal{F}}$ under $\wedge^{\mathbb{L}}_{\mathcal{F}}$) to '$\mathcal{E}$-global homotopy ring spectra', preserving commutativity of multiplications. Similarly, the symmetric monoidal structure on the left adjoint can be used to upgrade L to a functor on homotopy ring spectra.

The formal properties of the change-of-family functors established in Theorem 5.1 facilitate an easy and rather formal argument to identify the Picard group of the global stable homotopy category. The result is that there are no 'exotic' invertible objects, i.e., the only smash invertible objects of $\mathcal{GH}$ are the suspensions and desuspensions of the global sphere spectrum. The same is true more generally for the $\mathcal{F}$-global stable homotopy category relative to any multiplicative global family $\mathcal{F}$, see Theorem 5.6 below.

We recall that an object X of a monoidal category $(\mathcal{C}, \Box, I)$ is *invertible* if there is another object Y such that both $X\Box Y$ and $Y\Box X$ are isomorphic to the unit object I. If the isomorphism classes of invertible objects form a set, then the *Picard group* $\mathrm{Pic}(\mathcal{C})$ is this set, with the group structure induced by the monoidal product. Every strong monoidal functor between monoidal categories takes invertible objects to invertible objects, and thus induces a group homomorphism of Picard groups.

Part of the calculation of $\mathrm{Pic}(\mathcal{GH})$ involves a very general argument that we spell out explicitly.

Proposition 5.5. *Let $(\mathcal{C}, \Box, I)$ be a monoidal category, $P : \mathcal{C} \longrightarrow \mathcal{C}$ a strong monoidal functor and $\epsilon : P \longrightarrow \mathrm{Id}_{\mathcal{C}}$ a monoidal transformation such that $\epsilon_I : PI \longrightarrow I$ is an isomorphism. Then the induced endomorphism of the Picard group*

$$\mathrm{Pic}(P) \ : \ \mathrm{Pic}(\mathcal{C}) \ \longrightarrow \ \mathrm{Pic}(\mathcal{C})$$

is the identity.

Proof. We let X be any invertible object, and Y an inverse of X. Since ϵ is a monoidal transformation, the composite

$$(PX)\Box(PY) \ \xrightarrow{\cong} \ P(X\Box Y) \ \xrightarrow{\epsilon_{X\Box Y}} \ X\Box Y$$

agrees with the morphism $\epsilon_X \Box \epsilon_Y$, where the first isomorphism is the strong monoidal structure on P. Since $X\Box Y$ is isomorphic to the unit object and ϵ_I is an isomorphism, the morphism $\epsilon_{X\Box Y}$ is also an isomorphism. So the composite $\epsilon_X \Box \epsilon_Y$ is an isomorphism. Since

$$\epsilon_X \Box \epsilon_Y \ = \ (X\Box\epsilon_Y) \circ (\epsilon_X \Box PY) \ = \ (\epsilon_X \Box Y) \circ (PX \Box \epsilon_Y)$$

we conclude that $\epsilon_X \Box PY$ has a left inverse and $\epsilon_X \Box Y$ has a right inverse. Since Y and PY are invertible objects, the functors $-\Box Y$ and $-\Box PY$ are equivalences of categories, so already $\epsilon_X : PX \longrightarrow X$ has both a left and a right inverse. So ϵ_X is an isomorphism, and hence $\mathrm{Pic}(P)[X] = [PX] = [X]$. $\Box$

Now we have all necessary ingredients to determine the Picard group of the $\mathcal{F}$-global stable homotopy category.

Theorem 5.6. *For every multiplicative global family $\mathcal{F}$, the Picard group of the $\mathcal{F}$-global stable homotopy category is free abelian of rank 1, generated by the suspension of the global sphere spectrum.*

Proof. The forgetful functor

$$U \ = \ U^{\mathcal{F}}_{\langle e \rangle} \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{GH}_{\langle e \rangle} \ = \ \mathcal{SH}$$

to the non-equivariant stable homotopy category and its left adjoint

$$L \ : \ \mathcal{SH} \ \longrightarrow \ \mathcal{GH}_{\mathcal{F}}$$

both have strong monoidal structures, the latter by Theorem 5.1 (iii). Moreover, the adjunction counit $\epsilon : LU \longrightarrow \mathrm{Id}$ is a monoidal transformation, and the morphism $\epsilon_{\mathbb{S}} : L(U\mathbb{S}) \longrightarrow \mathbb{S}$ is an isomorphism. We apply Proposition 5.5 to the composite endofunctor LU of $\mathcal{GH}_{\mathcal{F}}$ and conclude that the composite homomorphism

$$\mathrm{Pic}(\mathcal{GH}_{\mathcal{F}}) \ \xrightarrow{\mathrm{Pic}(U)} \ \mathrm{Pic}(\mathcal{SH}) \ \xrightarrow{\mathrm{Pic}(L)} \ \mathrm{Pic}(\mathcal{GH}_{\mathcal{F}})$$

is the identity. In particular, the homomorphism $\mathrm{Pic}(U)$ induced by the forgetful functor is injective. The Picard group of the non-equivariant stable homotopy category is free abelian of rank 1, generated by the suspension of the non-equivariant sphere spectrum. This generator is the image of the suspension of the

global sphere spectrum. So the homomorphism $\mathrm{Pic}(U)$ is surjective, hence an isomorphism. So $\mathrm{Pic}(\mathcal{GH}_{\mathcal{F}})$ is also free abelian of rank 1, generated by the suspension of the global sphere spectrum. $\square$

Now we develop criteria that characterize global homotopy types in the essential image of one of the adjoints to a forgetful change-of-family functor. The following terminology is convenient here.

Definition 5.7. Let $\mathcal{F}$ be a global family. An orthogonal spectrum is *left induced* from $\mathcal{F}$ if it is in the essential image of the left adjoint $L_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$. Similarly, an orthogonal spectrum is *right induced* from $\mathcal{F}$ if it is in the essential image of the right adjoint $R_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$.

We start with a criterion, for certain 'reflexive' global families, that characterizes the left induced homotopy types in terms of geometric fixed points.

Definition 5.8. A global family $\mathcal{F}$ is *reflexive* if for every compact Lie group K there is a compact Lie group uK, belonging to $\mathcal{F}$, and a continuous homomorphism $p : K \longrightarrow uK$ that is initial among continuous homomorphisms from K to groups in $\mathcal{F}$.

In other words, $\mathcal{F}$ is reflexive if and only if the inclusion into the category of all compact Lie groups has a left adjoint. As always with adjoints, the universal pair (uK, p) is then unique up to unique isomorphism under K. Moreover, the universal homomorphism $p : K \longrightarrow uK$ is necessarily surjective. Indeed, the image of p is a closed subgroup of uK, hence also in the global family $\mathcal{F}$. So if the image of p were strictly smaller than K, then p would not be initial among morphisms into groups from $\mathcal{F}$. Some examples of reflexive global families are the minimal global family $\langle e\rangle$ of trivial groups, the global family $\mathcal{F}in$ of finite groups and the global family $\mathcal{A}b$ of abelian compact Lie groups. The maximal family of all compact Lie groups is also reflexive, but in this case the following proposition has no content.

A reflexive global family $\mathcal{F}$ is in particular multiplicative. Indeed, for $G, K \in \mathcal{F}$ the projections $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ factor through continuous homomorphisms $q_G : u(G \times K) \longrightarrow G$ respectively $q_K : u(G \times K) \longrightarrow K$. The composite

$$G \times K \ \xrightarrow{p}\ u(G \times K) \ \xrightarrow{(q_G, q_K)}\ G \times K$$

is then the identity, so the universal homomorphism $p : G \times K \longrightarrow u(G \times K)$ is injective. Since $u(G \times K)$ belongs to $\mathcal{F}$, so does $G \times K$.

Proposition 5.9. *Let $\mathcal{F}$ be a reflexive global family. Then an orthogonal spectrum X is left induced from $\mathcal{F}$ if and only if for every compact Lie group K the inflation map*

$$p^* \ : \ \Phi_*^{uK}(X) \ \longrightarrow \ \Phi_*^K(X)$$

associated to the universal morphism $p : K \longrightarrow uK$ is an isomorphism between the geometric fixed point homotopy groups for uK and K.

Proof. We let $\mathcal{X}$ be the full subcategory of $\mathcal{GH}$ consisting of the orthogonal spectra X such that for every compact Lie group K the inflation map $p^* : \Phi_*^{uK}(X) \longrightarrow \Phi_*^K(X)$ is an isomorphism. We need to show that $\mathcal{X}$ coincides with the class of spectra left induced from $\mathcal{F}$.

Geometric fixed point homotopy groups commute with sums and take exact triangles to long exact sequences. So $\mathcal{X}$ is closed under sums and triangles, i.e., it is a localizing subcategory of the global homotopy category. Now we claim that for every group G in $\mathcal{F}$ the suspension spectrum of the global classifying space $B_{\mathrm{gl}}G$ belongs to $\mathcal{X}$. Since $p : K \longrightarrow uK$ is initial among morphisms into groups from $\mathcal{F}$, precomposition with p is a bijection between the sets of conjugacy classes of homomorphisms into G; moreover, the image of a homomorphism $\alpha : uK \longrightarrow G$ agrees with the image of $\alpha \circ p : K \longrightarrow G$, because p is surjective. Proposition I.5.16 (i) identifies the fixed points of the orthogonal space $B_{\mathrm{gl}}G$ as a disjoint union, over conjugacy classes of homomorphisms, of centralizers of images. So the restriction map along p is a weak equivalence of fixed points spaces

$$p^* \ : \ ((B_{\mathrm{gl}}G)(\mathcal{U}_{uK}))^{uK} \ = \ ((B_{\mathrm{gl}}G)(\mathcal{U}_{uK}))^{K} \ \simeq \ ((B_{\mathrm{gl}}G)(\mathcal{U}_{K}))^{K} \ .$$

Geometric fixed points commute with suspension spectra (see Example III.3.4), in the sense of an isomorphism

$$\Phi_*^K(\Sigma_+^\infty B_{\mathrm{gl}}G) \;\cong\; \pi_*^e(\Sigma_+^\infty\left((B_{\mathrm{gl}}G)(\mathcal{U}_K)\right)^K)\ ,$$

natural for inflation maps. So together this implies the claim for the suspension spectrum of $B_{\mathrm{gl}}G$.

Now we have shown that $\mathcal{X}$ is a localizing subcategory of the global stable homotopy category that contains the suspension spectra of global classifying spaces of all groups in $\mathcal{F}$. The left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$ is fully faithful and $\mathcal{GH}_{\mathcal{F}}$ is generated by the suspension spectra of the global classifying spaces in $\mathcal{F}$ (by Theorem 4.3). So $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$ is an equivalence onto the full triangulated subcategory generated by the suspension spectra $\Sigma_+^\infty B_{\mathrm{gl}}G$ for all $G \in \mathcal{F}$. So the image of L is contained in $\mathcal{X}$.

Now suppose that conversely X is an orthogonal spectrum in $\mathcal{X}$. The adjunction counit $\epsilon_X : L(UX) \longrightarrow X$ is an $\mathcal{F}$-equivalence, so it induces isomorphisms of geometric fixed point groups for all groups in $\mathcal{F}$. By the hypothesis on X and naturality of the inflation maps p^*, the morphism ϵ_X induces isomorphism of geometric fixed point groups for all compact Lie groups. So ϵ_X is a global equivalence, and in particular X is left induced from $\mathcal{F}$. □

Remark 5.10. The same proof as in Proposition 5.9 yields the following relative version of the proposition. We let $\mathcal{F} \subset \mathcal{E}$ be global families and assume that $\mathcal{F}$ is *reflexive relative to* $\mathcal{E}$, i.e., for every compact Lie group K from the family $\mathcal{E}$ there is a compact Lie group uK, belonging to $\mathcal{F}$, and a continuous homomorphism $p : K \longrightarrow uK$ that is initial among homomorphisms to groups in $\mathcal{F}$. Then an orthogonal spectrum X is in the essential image of the relative left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ if and only if for every compact Lie group K in $\mathcal{E}$ the universal morphism $p : K \longrightarrow uK$ induces isomorphisms

$$p^* \;:\; \Phi_*^{uK}(X) \;\longrightarrow\; \Phi_*^K(X)$$

between the geometric fixed point homotopy groups of uK and K.

Example 5.11. The minimal global family $\mathcal{F} = \langle e\rangle$ of trivial groups is reflexive, and the unique morphism $K \longrightarrow e$ to any trivial group is universal. So Proposition 5.9 characterizes the global homotopy types in the essential of the left adjoint $L : \mathcal{SH} = \mathcal{GH}_{\langle e\rangle} \longrightarrow \mathcal{GH}$ from the non-equivariant stable homotopy category to the global stable homotopy category: an orthogonal spectrum X is left induced from the trivial family if and only if for every compact Lie group K the unique homomorphism $p : K \longrightarrow e$ induces an isomorphism

$$p^* \;:\; \Phi_*^e(X) \;\longrightarrow\; \Phi_*^K(X)\ .$$

The geometric fixed point homotopy groups $\Phi_*^e(X)$ with respect to the trivial group are isomorphic to $\pi_*^e(X)$, the stable homotopy groups of the underlying non-equivariant spectrum. So the global homotopy types in the essential of the left adjoint $L : \mathcal{SH} \longrightarrow \mathcal{GH}$ are precisely the orthogonal spectra with 'constant geometric fixed points'.

Here are some specific examples of left induced global homotopy types.

Example 5.12 (Suspension spectra). The orthogonal sphere spectrum $\mathbb{S}$ and the suspension spectrum of every based space are left induced from the trivial global family $\langle e\rangle$. Indeed, geometric fixed points commute with suspension spectra in the following sense: the G-geometric fixed point spectrum of the suspension spectrum of any based G-space A is stably equivalent to the suspension spectrum of the G-fixed point space A^G, compare Example III.3.4. So when A has trivial G-action,

$$\Phi_*^G(\Sigma^\infty A) \;\cong\; \pi_*(\Sigma^\infty A)\ .$$

So the suspension spectrum $\Sigma^\infty A$ has 'constant geometric fixed points', and it is left induced from the trivial family by the criterion of Example 5.11.

Example 5.13 (Global classifying spaces and free orthogonal spectra). If G is a compact Lie group from a global family $\mathcal{F}$, then the suspension spectrum of the global classifying space $B_{\mathrm{gl}}G$ is left induced from $\mathcal{F}$. To see this, we can refer to the proof of Proposition 5.9; alternatively, we may show that $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is $\mathcal{F}$-cofibrant, i.e., has the left lifting property with respect to morphisms that are both $\mathcal{F}$-level equivalences and $\mathcal{F}$-level fibrations. We recall that $B_{\mathrm{gl}}G = \mathbf{L}_{G,V} = \mathbf{L}(V,-)/G$ is a free orthogonal space, where V is any faithful G-representation. So morphisms $\Sigma^\infty_+ B_{\mathrm{gl}}G \longrightarrow X$ of orthogonal spectra biject with continuous based G-maps $S^V \longrightarrow X(V)$; since S^V can be given the structure of a based G-CW-complex, it has the left lifting property with respect to G-weak equivalences that are also G-fibrations, and the claim follows by adjointness.

The same kind of reasoning shows that the free orthogonal spectra $F_{G,V}$ introduced in Construction 1.24 are left induced from $\mathcal{F}$ whenever G belongs to $\mathcal{F}$ and V is a faithful G-representation.

Example 5.14 ($\mathbf{\Gamma}$-spaces). We let $\mathbf{\Gamma}$ denote the category whose objects are the based sets $n^+ = \{0,1,\dots,n\}$, with basepoint 0, and with morphisms all based maps. A $\mathbf{\Gamma}$-*space* is a functor from $\mathbf{\Gamma}$ to the category of spaces which is reduced (i.e., the value at any one-point set is a one-point space).

A $\mathbf{\Gamma}$-space $F : \mathbf{\Gamma} \longrightarrow \mathbf{T}$ can be extended to a continuous functor on the category of based spaces by a coend construction. The value of the extended functor on a based space K is given by

$$F(K) \;=\; \int^{n^+\in\mathbf{\Gamma}} F(n^+)\times K^n \;=\; \left(\coprod_{n\ge 0} F(n^+)\times K^n\right)/\sim\,, \tag{5.15}$$

where we use that $K^n = \mathrm{map}(n^+, K)$ is contravariantly functorial in n^+. In more detail $F(K)$ is obtained from the disjoint union of the spaces $F(n^+)\times K^n$ by modding out the equivalence relation generated by

$$(F(\alpha)(x);\, k_1,\dots,k_n) \;\sim\; (x;\, k_{\alpha(1)},\dots,k_{\alpha(m)})$$

for all $x\in F(m^+)$, all $(k_1,\dots,k_n)$ in K^n, and all morphisms $\alpha : m^+ \longrightarrow n^+$ in $\mathbf{\Gamma}$. Here $k_{\alpha(i)}$ is to be interpreted as the basepoint of K whenever $\alpha(i)=0$. We write $[x;\, k_1,\dots,k_n]$ for the equivalence class in $F(K)$ of a tuple $(x;\, k_1,\dots,k_n)\in F(n^+)\times K^n$. The assignment $(F,K)\mapsto F(K)$ is functorial in the $\mathbf{\Gamma}$-space F and in the based space K. We will not distinguish notationally between the original $\mathbf{\Gamma}$-space and its extension. The extended functor is continuous and comes with a continuous, based *assembly map*

$$\alpha \;:\; K\wedge F(L) \;\longrightarrow\; F(K\wedge L)\;,\quad \alpha(k\wedge[x;\, l_1,\dots,l_n]) \;=\; [x;\, k\wedge l_1,\dots,k\wedge l_n]\;. \tag{5.16}$$

The assembly map is natural in all three variables and associative and unital.

We can now define an orthogonal spectrum $F(\mathbb{S})$ by evaluating the $\mathbf{\Gamma}$-space F on spheres. In other words, the value of $F(\mathbb{S})$ at an inner product space V is given by

$$F(\mathbb{S})(V) \;=\; F(S^V)$$

and the structure map $\sigma_{V,W} : S^V\wedge F(\mathbb{S})(W) \longrightarrow F(\mathbb{S})(V\oplus W)$ is the assembly map (5.16) for $K=S^V$ and $L=S^W$, followed by the effect of F on the canonical homeomorphism $S^V\wedge S^W \cong S^{V\oplus W}$. The $O(V)$-action on $F(\mathbb{S})(V)$ is via the action on S^V and the continuous functoriality of F.

Proposition 5.17. *Let F be a $\mathbf{\Gamma}$-space and G a compact Lie group.*

(i) *If G is connected, then for every based G-space K the map*

$$F(K^G) \;\longrightarrow\; (F(K))^G$$

induced by the fixed point inclusion $K^G \longrightarrow K$ is a homeomorphism.

(ii) *The projection $p : G \longrightarrow \pi_0 G = \bar{G}$ to the group of path components induces an isomorphism*

$$p^* \;:\; \Phi^{\bar{G}}_*(F(\mathbb{S})) \;\longrightarrow\; \Phi^G_*(F(\mathbb{S}))$$

of geometric fixed point homotopy groups of the orthogonal spectrum $F(\mathbb{S})$.

(iii) *The orthogonal spectrum $F(\mathbb{S})$ obtained by evaluation of F on spheres is left induced from the global family $\mathcal{F}in$ of finite groups.*

PROOF. (i) Since K^G is a closed subspace of K, the induced map $F(K^G) \longrightarrow F(K)$ is a closed embedding by [...this should need a condition on F...]. Since the image of this maps is contained in the closed subspace $F(K)^G$, also the map $F(K^G) \longrightarrow (F(K))^G$ is a closed embedding. So it only remains to show that the map is surjective. We consider a point of $F(K)$ represented by a tuple $(x;\, k_1, \dots, k_n) \in F(n^+) \times K^n$. We assume that the number n has been chosen minimally, so that the entries k_i are pairwise distinct and different from the basepoint of K. If the point $[x;\, k_1, \dots, k_n]$ of $F(K)$ is G-fixed, then for every group element g the tuple $(x;\, gk_1, \dots, gk_n)$ is equivalent to the original tuple, so there is a unique permutation $\sigma(g) \in \Sigma_n$ such that

$$(x;\, gk_1, \dots, gk_n) \;=\; (F(\sigma)(x);\, k_{\sigma(g)(1)}, \dots, k_{\sigma(g)(n)})\ .$$

Since G acts continuously on K, the map $\sigma : G \longrightarrow \Sigma_n$ is a continuous homomorphism. Since G is connected, the homomorphism σ must be trivial, i.e., $\sigma(g) = 1$ for all $g \in G$. Thus the points $k_1, \dots, k_n$ are all G-fixed.

(ii) We can calculate G-fixed points by first taking fixed points with respect to the normal subgroup G° (the path component of the identity) and then fixed points with respect to the quotient $\bar{G} = G/G^\circ = \pi_0 G$. So for $k \geq 0$, the G-geometric fixed points of the orthogonal spectrum $F(\mathbb{S})$ can be rewritten as

$$\begin{aligned}
\Phi_k^G(F(\mathbb{S})) \;&=\; \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, [S^{V^G \oplus \mathbb{R}^k},\, F(S^V)^G] \\
&\cong\; \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, [(S^{V^{G^\circ} \oplus \mathbb{R}^k})^{\bar{G}},\, F(S^{V^{G^\circ}})^{\bar{G}}] \\
&\cong\; \mathrm{colim}_{W \in s(\mathcal{U}_{\bar{G}})}\, [S^{W^{\bar{G}} \oplus \mathbb{R}^k},\, F(S^W)^{\bar{G}}] \;=\; \Phi_k^{\bar{G}}(F(\mathbb{S}))\ .
\end{aligned}$$

The second step uses the homeomorphism $(F(S^V))^{G^\circ} \cong F(S^{V^{G^\circ}})$ of part (i). The third step uses that $(\mathcal{U}_G)^{G^\circ}$ is a complete universe for the finite group $\bar{G}$ and as V runs through $s(\mathcal{U}_G)$, the G°-fixed points V^{G° exhaust $(\mathcal{U}_G)^{G^\circ}$. The composite isomorphism is given by the inflation map p^*. The argument for $k < 0$ is similar.

(iii) The global family $\mathcal{F}in$ of finite groups is reflexive, and for every compact Lie group K the projection $K \longrightarrow \pi_0 K$ to the finite group of path components is universal (with respect to $\mathcal{F}in$). Part (ii) verifies the geometric fixed point criterion, so by Proposition 5.9 the orthogonal spectrum $F(\mathbb{S})$ is left induced from the global family of finite groups. □

Now we look more closely at right induced global homotopy types. For a global family $\mathcal{F}$ and a compact Lie group G we denote by $\mathcal{F} \cap G$ the family of those closed subgroups of G that belong to $\mathcal{F}$, and $E(\mathcal{F} \cap G)$ is a universal G-space for the family $\mathcal{F} \cap G$. We also need the equivariant cohomology theory represented by an orthogonal spectrum X. If A is a G-space, we define the G-cohomology $X_G^k(A)$ as

$$X_G^k(A) \;=\; [\![\Sigma_+^\infty \mathbf{L}_{G,V} A, X]\!]^k\ ,$$

the group of degree k maps in $\mathcal{GH}$ from the suspension spectrum of the free orthogonal space $\mathbf{L}_{G,V} A$ to X. Here V is an implicitly chosen faithful G-representation. By the adjunction between the global stable homotopy category on the G-equivariant stable homotopy category that we discuss in Theorem 5.31 below, the group $X_G^k(A)$ is isomorphic to the value at A of the G-cohomology theory represented by the underlying G-spectrum X_G.

PROPOSITION 5.18. *An orthogonal spectrum X is right induced from a global family $\mathcal{F}$ if and only if for every compact Lie group group G and every cofibrant G-space A the map*

$$X_G^*(A) \;\longrightarrow\; X_G^*(A \times E(\mathcal{F} \cap G))$$

induced by the projection $A \times E(\mathcal{F} \cap G) \longrightarrow A$ is an isomorphism.

Proof. For every G-space A the projection

$$A \times E(\mathcal{F} \cap G) \;\longrightarrow\; A$$

is an $\mathcal{F}$-equivalence; moreover, the source is $(\mathcal{F} \cap G)$-projective, so the free orthogonal space

$$\mathbf{L}_{G,V} \times_G (A \times E(\mathcal{F} \cap G))$$

is left induced from the global family $\mathcal{F}$. This implies that

$$L_{\mathcal{F}}(U_{\mathcal{F}}(\mathbf{L}_{G,V} \times_G A)) \;\cong\; \mathbf{L}_{G,V} \times_G (A \times E(\mathcal{F} \cap G))$$

in the unstable global homotopy category. Hence

$$\begin{aligned} X^*_G(A \times E(\mathcal{F} \cap G)) \;&=\; [\![\Sigma^\infty_+ \mathbf{L}_{G,V} \times_G (A \times E(\mathcal{F} \cap G)), X]\!] \\ &\cong\; [\![\Sigma^\infty_+ L_{\mathcal{F}}(U_{\mathcal{F}}(\mathbf{L}_{G,V} \times_G A)), X]\!] \\ &\cong\; [\![L_{\mathcal{F}}(U_{\mathcal{F}}(\Sigma^\infty_+ \mathbf{L}_{G,V} \times_G A)), X]\!] \;\cong\; [\![\Sigma^\infty_+ \mathbf{L}_{G,V} \times_G A, R_{\mathcal{F}}(U_{\mathcal{F}}(X))]\!]\;. \end{aligned}$$

for every orthogonal spectrum X. Under this composite isomorphism, the map of the proposition becomes the map

$$X^*_G(A) \;=\; [\![\Sigma^\infty_+ \mathbf{L}_{G,V} \times_G A, X]\!] \;\longrightarrow\; [\![\Sigma^\infty_+ \mathbf{L}_{G,V} \times_G A, R_{\mathcal{F}}(U_{\mathcal{F}}(X))]\!]$$

induced by the adjunction unit $X \longrightarrow R_{\mathcal{F}}(U_{\mathcal{F}}(X))$.

If X is right induced from $\mathcal{F}$, then this adjunction unit is an isomorphism, hence so is the map $X^*_G(A) \longrightarrow X^*_G(A \times E(\mathcal{F} \cap G))$. If, on the other hand, this map is an isomorphism for all G-spaces A, then for $A = *$ we deduce that the map

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]^* \;\longrightarrow\; [\![\Sigma^\infty_+ B_{\mathrm{gl}}G, R_{\mathcal{F}}(U_{\mathcal{F}}(X))]\!]^*$$

is an isomorphism. Since the suspension spectrum of $B_{\mathrm{gl}}G$ represents π^G_0 (by Theorem 4.3 for the global family of all compact Lie groups), this shows that the adjunction unit $X \longrightarrow R_{\mathcal{F}}(U_{\mathcal{F}}(X))$ is a global equivalence. So X is right induced from $\mathcal{F}$. □

Remark 5.19. Essentially the same proof also shows the following relative version of Proposition 5.18. We let $\mathcal{F} \subseteq \mathcal{E}$ be two nested global families. Then an orthogonal spectrum X is in the essential image of the relative right adjoint $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ if and only if for every group G in $\mathcal{E}$ and every cofibrant G-space A the map

$$X^*_G(A) \;\longrightarrow\; X^*_G(A \times E(\mathcal{F} \cap G))$$

induced by the projection $A \times E(\mathcal{F} \cap G) \longrightarrow A$ is an isomorphism.

Example 5.20. We let X be a global Ω-spectrum with the property that for every inner product space V, the $O(V)$-space $X(V)$ is cofree, i.e., for some (hence any) universal free $O(V)$-space E the map

$$\mathrm{const} \;:\; X(V) \;\longrightarrow\; \mathrm{map}(E, X(V))$$

that sends a point to the corresponding constant map is an $O(V)$-weak equivalence. We claim that then the orthogonal spectrum X is right induced from the trivial global family $\langle e \rangle$. We use the criterion of Proposition 5.18 and show that for every compact Lie group group G, every cofibrant G-space A and every integer k the map

$$\Pi^* \;:\; X^k_G(A) \;\longrightarrow\; X^k_G(A \times EG)$$

induced by the projection $\Pi : A \times EG \longrightarrow A$ is an isomorphism.

We start with the case $k = 0$. We let V be any faithful G-representation. The projection Π is a weak equivalence of underlying non-equivariant spaces, and source and target are cofibrant as G-spaces. So the G-map

$$S^V \wedge \Pi_+ \;:\; S^V \wedge (A \times EG)_+ \;\longrightarrow\; S^V \wedge A_+$$

is also a weak equivalence of underlying non-equivariant spaces, and source and target are G-cofibrant in the based sense. Since $X(V)$ is cofree as an $O(V)$-space, it is also cofree as a G-space, where G acts via the representation homomorphism $G \longrightarrow O(V)$. So the induced map

$$[S^V \wedge \Pi_+, X(V)]^G \ : \ [S^V \wedge A_+, X(V)]^G \ \longrightarrow \ [S^V \wedge (A \times EG)_+, X(V)]^G$$

is a weak equivalence.

We recall that

$$X^0_G(A) \ = \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\, X]\!] \ ,$$

the group of morphisms in the global stable homotopy category. Since X is a global Ω-spectrum and the orthogonal suspension spectrum $\Sigma^\infty_+ \mathbf{L}_{G,V} A$ is flat, the localization map

$$\mathcal{S}p(\Sigma^\infty_+ \mathbf{L}_{G,V} A,\, X)/\text{homotopy} \ \longrightarrow \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\, X]\!]$$

is bijective. By the freeness property, the left hand side bijects with the set $[S^V \wedge A_+, X(V)]^G$. So by the previous paragraph the map $\Pi^* : X^0_G(A) \longrightarrow X^0_G(A \times EG)$ is bijective.

For $k > 0$ we apply the same argument to the global Ω-spectrum $\mathrm{sh}^k X$ (which also has cofree levels) and exploit the natural isomorphism

$$(\mathrm{sh}^k X)^0_G(A) \ = \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\, \mathrm{sh}^k X]\!] \ \cong \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\, X]\!]^k \ = \ X^k_G(A) \ .$$

For $k < 0$ we apply the same argument to the global Ω-spectrum $\Omega^{-k} X$ (which also has cofree levels) and exploit the natural isomorphism

$$(\Omega^{-k} X)^0_G(A) \ = \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\, \Omega^{-k} X]\!] \ \cong \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\, X]\!]^k \ = \ X^k_G(A) \ .$$

As we explain in more detail in Example VI.5.33 below, the global K-theory spectrum $\mathbf{KU}$ is right induced from the global family of finite cyclic groups. This fact is a reformulation in our present language of the generalization, due to Adams, Haeberly, Jackowski and May [**1**], of the Atiyah-Segal completion theorem.

Example 5.21 (Global Borel theories). We let E be a non-equivariant generalized cohomology theory. Then we obtain a global functor $\underline{E}$ by setting

$$\underline{E}(G) \ = \ E^0(BG) \ ,$$

the 0-th E-cohomology of the classifying space of the group G. The contravariant functoriality in group homomorphisms $\alpha : K \longrightarrow G$ comes from the covariant functoriality of the classifying space construction. The transfer map for a subgroup inclusion $H \subset G$ comes from the Becker-Gottlieb transfer [**11**]

$$\Sigma^\infty_+ BG \ \xrightarrow{\mathrm{tr}} \ \Sigma^\infty_+ BH$$

associated to the fiber bundle $EG/H \longrightarrow EG/G = BG$ with fiber G/H, using that EG/H is a classifying space for H. Strictly speaking, in [**11**] Becker and Gottlieb only define a stable transfer map for locally trivial fiber bundles with smooth compact manifold fiber whenever the base is a finite CW-complex. To get the transfer above one approximates EG (and hence BG) by its finite dimensional skeleta. The verification of the double coset formula for this global functor is due to Feshbach [**51**, Thm. II.11].

More generally, we can consider the Borel equivariant cohomology theory represented by E. For a compact Lie group G and a cofibrant G-space A, its value is

$$E^*(EG \times_G A) \ ,$$

the E-cohomology of the Borel construction (also known as homotopy orbit construction). Here EG is a universal free G-space, which is unique up to equivariant homotopy equivalence. We claim that these Borel cohomology theories associated to E are represented by a specific global homotopy type, namely the result of applying the right adjoint

$$R \ : \ \mathcal{SH} \ \longrightarrow \ \mathcal{GH}$$

to the forget functor $U : \mathcal{GH} \longrightarrow \mathcal{SH}$ to any non-equivariant spectrum that represents E. To verify this claim we choose a faithful G-representation V and recall from Proposition I.2.10 (i) that the G-space $\mathbf{L}(V, \mathbb{R}^\infty)$ is a universal free G-space. So for every G-space A the underlying non-equivariant homotopy type of the closed orthogonal space $\mathbf{L}_{G,V} A$ is

$$(\mathbf{L}_{G,V} A)(\mathbb{R}^\infty) \ = \ \mathbf{L}(V, \mathbb{R}^\infty) \times_G A \ = \ EG \times_G A \ ,$$

the homotopy orbit space $EG \times_G A$. The adjunction isomorphism for (U, R) thus provides an isomorphism

$$\begin{aligned} (RE)^0_G(A) \ &= \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A, RE]\!] \\ &\cong \ [U(\Sigma^\infty_+ \mathbf{L}_{G,V} A), E] \ = \ [\Sigma^\infty_+ (EG \times_G A), E] \ = \ E^0(EG \times_G A) \ . \end{aligned}$$

When G acts trivially on A, then $EG \times_G A = (EG/G) \times A = BG \times A$, and the above specializes to an isomorphism

$$(RE)^0_G(A) \ \cong \ E^0(BG \times A) \ . \tag{5.22}$$

When A is a one-point G-space, this bijection gives rise to a composite isomorphism

$$\pi^G_0(RE) \ \cong \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V}, RE]\!] \ \cong \ E^0(BG) \ , \tag{5.23}$$

where the first one is inverse to evaluation at the stable tautological class $e_{G,V} \in \pi^G_0(\Sigma^\infty_+ \mathbf{L}_{G,V})$. We claim that the isomorphisms (5.23) are compatible with restriction maps arising from continuous group homomorphisms $\alpha : K \longrightarrow G$. For this purpose we also choose a faithful K-representation W. This data gives rise to a composite morphism of global classifying spaces

$$B_{\mathrm{gl}}\alpha \ : \ B_{\mathrm{gl}}K \ = \ \mathbf{L}_{K,\alpha^*(V)\oplus W} \ \xrightarrow{\rho_{\alpha^*(V),W}/K} \ \mathbf{L}_{K,\alpha^*(V)} \ \xrightarrow{\mathrm{proj}} \ \mathbf{L}_{G,V} \ = \ B_{\mathrm{gl}}G \ .$$

The first morphism restricts a linear isometric embedding from $\alpha^*(V) \oplus W$ to $\alpha^*(V)$, and the second morphism is the quotient map from K-orbits to G-orbits. On the underlying non-equivariant homotopy types (i.e., after evaluating at $\mathbb{R}^\infty$), the morphism $B_{\mathrm{gl}}\alpha$ classifies the homomorphism α. Moreover, the morphism has the 'correct' behavior on the unstable tautological classes in that sense of the relation

$$(B_{\mathrm{gl}}\alpha)_*(u_{K,W}) \ = \ \alpha^*(u_{G,V}) \quad \text{in } \pi^K_0(B_{\mathrm{gl}}G) \ ,$$

by direct verification from the definitions. The analogous relation for the stable tautological classes

$$\begin{aligned} (\Sigma^\infty_+ B_{\mathrm{gl}}\alpha)_*(e_{K,W}) \ &= \ (\Sigma^\infty_+ B_{\mathrm{gl}}\alpha)_*(\sigma^K(u_{K,W})) \ = \ \sigma^K((B_{\mathrm{gl}}\alpha)_*(u_{K,W})) \\ &= \ \sigma^K(\alpha^*(u_{G,V})) \ = \ \alpha^*(\sigma^G(u_{G,V})) \ = \ \alpha^*(e_{G,V}) \end{aligned}$$

in the stable group $\pi^K_0(\Sigma^\infty_+ B_{\mathrm{gl}}G)$ then follows by naturality of the suspension maps $\sigma^K : \pi^K_0(Y) \longrightarrow \pi^K_0(\Sigma^\infty_+ Y)$ and its compatibility with restriction. This proves the compatibility of the isomorphisms (5.23) with restriction maps. Compatibility with transfers is essentially built in, as both the transfers in Construction III.2.6 and the Becker-Gottlieb transfer in [**11**] are defined as the Thom-Pontryagin construction based on smooth equivariant embedding of G/H into a G-representation; we omit the formal proof. In any case, the group isomorphisms (5.23) together form an isomorphism of global functors between $\underline{\pi}_0(RE)$ and $\underline{E}$. This proves the claim that the 'global Borel theories' are precisely the ones right induced from non-equivariant stable homotopy theory (i.e., from the global family of trivial Lie groups).

Construction 5.24. We introduce a specific pointset level lift

$$b \ : \ \mathcal{S}p \ \longrightarrow \ \mathcal{S}p$$

of the right adjoint $R : \mathcal{SH} \longrightarrow \mathcal{GH}$ to the category of orthogonal spectra. Given an orthogonal spectrum E we define a new orthogonal spectrum bE as follows. For an inner product space V we set

$$(bE)(V) \ = \ \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty)_+, E(V)) \ , \tag{5.25}$$

the space of all continuous maps from $\mathbf{L}(V,\mathbb{R}^\infty)$ to $E(V)$. The orthogonal group $O(V)$ acts on this mapping space by conjugation, through its actions on $\mathbf{L}(V,\mathbb{R}^\infty)$ and on $E(V)$. We define structure maps $\sigma_{V,W}$: $S^V \wedge (bE)(W) \longrightarrow (bE)(V\oplus W)$ as the composite

$$\begin{aligned} S^V \wedge \operatorname{map}(\mathbf{L}(W,\mathbb{R}^\infty),E(W)) \quad &\xrightarrow{\text{assembly}} \quad \operatorname{map}(\mathbf{L}(W,\mathbb{R}^\infty)_+, S^V\wedge E(W)) \\ &\xrightarrow{\operatorname{map}(\operatorname{res}_W,\sigma^E_{V,W})} \operatorname{map}(\mathbf{L}(V\oplus W,\mathbb{R}^\infty)_+, E(V\oplus W)) \end{aligned}$$

where $\operatorname{res}_W : \mathbf{L}(V\oplus W,\mathbb{R}^\infty) \longrightarrow \mathbf{L}(W,\mathbb{R}^\infty)$ is the map that restrict an isometric embedding from $V\oplus W$ to W. In the functorial description of orthogonal spectra, the structure maps are given by

$$\begin{aligned} \mathbf{O}(V,W)\wedge \operatorname{map}(\mathbf{L}(V,\mathbb{R}^\infty)_+,E(V)) \ &\longrightarrow\ \operatorname{map}(\mathbf{L}(W,\mathbb{R}^\infty)_+,E(W)) \\ (w,\varphi)\wedge \quad f \qquad\qquad &\longmapsto\ \{\ \psi\mapsto X(w,\varphi)(f(\psi\circ\varphi))\ \}\ . \end{aligned}$$

The endofunctor b on the category of orthogonal spectra comes with a natural transformation

$$i_E\ :\ E\ \longrightarrow\ bE \tag{5.26}$$

whose value at an inner product space V sends a point $x\in E(V)$ to the constant map $\mathbf{L}(V,\mathbb{R}^\infty)\longrightarrow E(V)$ with value x. Said another way, the map $E(V)\longrightarrow \operatorname{map}(\mathbf{L}(V,\mathbb{R}^\infty),E(V)) = (bE)(V)$ is induced by the unique map $\mathbf{L}(V,\mathbb{R}^\infty)\longrightarrow *$. Since $\mathbf{L}(V,\mathbb{R}^\infty)$ is contractible, the morphism $i_E : E\longrightarrow bE$ is a non-equivariant level equivalence, hence a non-equivariant stable equivalence.

The next result shows that the global Borel construction b takes Ω-spectra to global Ω-spectra, and that the functor b realizes, in a certain precise way, the right adjoint to the forgetful functor from the global homotopy category to the traditional stable homotopy category. Since the morphism $i_E : E\longrightarrow bE$ is a non-equivariant stable equivalence, it becomes invertible in the non-equivariant stable homotopy category $\mathcal{SH}$. Part (ii) of the following proposition shows that the morphism $i_E^{-1} : bE\longrightarrow E$ is the counit of the adjunction (U,R).

Proposition 5.27. *Let E be an orthogonal Ω-spectrum.*

(i) *The orthogonal spectrum bE is a global Ω-spectrum and right induced from the trivial global family.*
(ii) *For every orthogonal spectrum A both of the two group homomorphisms*

$$[\![A,bE]\!]\ \xrightarrow{U}\ \mathcal{SH}(A,bE)\ \xrightarrow{(i_E)_*^{-1}}\ \mathcal{SH}(A,E)\ .$$

are bijective.

Proof. (i) We let G be a compact Lie group and V and W two G-representations such that W is faithful. Since E is an Ω-spectrum, the adjoint structure map

$$\tilde\sigma^E_{V,W}\ :\ E(W)\ \longrightarrow\ \Omega^V E(V\oplus W)$$

is a non-equivariant weak equivalence. The spaces $\mathbf{L}(W,\mathbb{R}^\infty)$ and $\mathbf{L}(V\oplus W,\mathbb{R}^\infty)$ are cofibrant as G-spaces by Proposition I.2.2 (ii). Because W, and hence also $V\oplus W$, is a faithful G-representation, the induced G-action on $\mathbf{L}(W,\mathbb{R}^\infty)$ and $\mathbf{L}(V\oplus W,\mathbb{R}^\infty)$ is free. So the induced map

$$\operatorname{map}(\mathbf{L}(W,\mathbb{R}^\infty),\tilde\sigma_{V,W})\ :\ (bE)(W)=\operatorname{map}(\mathbf{L}(W,\mathbb{R}^\infty),E(W))\ \longrightarrow\ \operatorname{map}(\mathbf{L}(W,\mathbb{R}^\infty),\Omega^V E(V\oplus W))$$

is a G-weak equivalence. Moreover, the restriction map $\operatorname{res}_W : \mathbf{L}(V\oplus W,\mathbb{R}^\infty)\longrightarrow \mathbf{L}(W,\mathbb{R}^\infty)$ is a G-homotopy equivalence (by Proposition I.2.10 (ii)), hence it induces another G-homotopy equivalence

$$\begin{aligned} \operatorname{map}(\operatorname{res}_W,\Omega^V E(V\oplus W))\ :\ &\operatorname{map}(\mathbf{L}(W,\mathbb{R}^\infty),\Omega^V E(V\oplus W)) \\ &\longrightarrow\ \operatorname{map}(\mathbf{L}(V\oplus W,\mathbb{R}^\infty),\Omega^V E(V\oplus W)) \end{aligned}$$

on mapping spaces. The target of this last map is G-homeomorphic to

$$\operatorname{map}(S^V,\operatorname{map}(\mathbf{L}(V\oplus W,\mathbb{R}^\infty),E(V\oplus W)))\ =\ \Omega^V((bE)(V\oplus W))\ ;$$

under this homeomorphism, the composite of the two G-weak equivalences becomes the adjoint structure map

$$\tilde{\sigma}^{bE}_{V,W} \ : \ (bE)(W) \ \longrightarrow \ \Omega^V((bE)(V \oplus W)) \ .$$

So we have shown that $\tilde{\sigma}^{bE}_{V,W}$ is a G-weak equivalence, and that means that bE is a global Ω-spectrum.

The $O(V)$-space $\mathbf{L}(\mathbb{R}^\infty, V)$ is a universal free $O(V)$-space by Proposition I.2.10 (i). So the $O(V)$-space $(bE)(V) = \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty)_+, E(V))$ is cofree. Since bE is also a global Ω-spectrum, the criterion of Example 5.20 shows that it is right induced from the trivial global family.

Part (ii) is a formal consequence of (i): since bE is right induced from the trivial global family, the forgetful functor induces a bijection $U : [\![A, bE]\!] \cong \mathcal{SH}(A, bE)$. Since the morphism $i_E : E \longrightarrow bE$ becomes an isomorphism in $\mathcal{SH}$, it induces another bijection on $\mathcal{SH}(A, -)$. $\square$

We can also endow the functor b with a lax symmetric monoidal transformation

$$\mu_{E,F} \ : \ bE \wedge bF \ \longrightarrow \ b(E \wedge F) \ .$$

To construct $\mu_{E,F}$ we start from the $(O(V) \times O(W))$-equivariant maps

$$\begin{aligned} \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty), E(V)) \wedge \mathrm{map}(\mathbf{L}(W, \mathbb{R}^\infty), F(W)) \ &\xrightarrow{\wedge} \ \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty) \times \mathbf{L}(V, \mathbb{R}^\infty), E(V) \wedge F(W)) \\ &\xrightarrow{\mathrm{map}(\mathrm{res}_{V,W}, i_{V,W})} \ \mathrm{map}(\mathbf{L}(V \oplus W, \mathbb{R}^\infty), (E \wedge F)(V \oplus W)) \end{aligned}$$

that constitute a bimorphism from (bE, bF) to $b(E \wedge F)$. Here

$$\mathrm{res}_{V,W} \ : \ \mathbf{L}(V \oplus W, \mathbb{R}^\infty) \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty) \times \mathbf{L}(W, \mathbb{R}^\infty)$$

maps an embedding of $V \oplus W$ to its restrictions to V and W. The morphism $\mu_{E,F}$ is associated to this bimorphism via the universal property of the smash product.

Remark 5.28. Various 'completion' maps (also called 'bundling maps') fit in here as follows. For this we suppose that E is a commutative orthogonal ring spectrum and a positive Ω-spectrum. Then the morphism $i_E : E \longrightarrow bE$ is a kind of 'global completion map'. For every compact Lie group G it induces a ring homomorphism of G-equivariant homotopy groups

$$\pi_0^G(E) \ \longrightarrow \ \pi_0^G(bE) \ \cong \ E^0(BG) \ .$$

When $E = \mathbb{S}$ is the sphere spectrum and G is finite, Carlsson's theorem [**35**] (proving the Segal conjecture) shows that the map

$$\mathbb{A}(G) \cong \pi_0^G(\mathbb{S}) \ \longrightarrow \ \pi^0(BG)$$

is completion of the Burnside ring at the augmentation ideal. The sphere spectrum is the suspension spectrum of a global classifying space of the trivial group; more generally, for the global classifying space $B_{\mathrm{gl}}K$ of a finite group K the 'forgetful' map

$$\mathbf{A}(K, G) \cong \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}}K) \ \longrightarrow \ [\Sigma^\infty_+ BG, \Sigma^\infty_+ BK]$$

is again completion at the augmentation ideal of the Burnside ring $A(G)$, see [**96**, Thm. A].

Since the 'global Borel cohomology' functor $b : \mathcal{S}p \longrightarrow \mathcal{S}p$ is lax symmetric monoidal, it takes orthogonal ring spectra to orthogonal ring spectra, in a way preserving commutativity and module structures. Since the transformation i_E is monoidal, it becomes a homomorphism of orthogonal ring spectra when E is an orthogonal ring spectrum. We let $\mathbb{S} \longrightarrow \mathbb{S}^{\mathrm{f}}$ be a 'positively fibrant replacement', i.e., a morphism of commutative orthogonal ring spectra that is a non-equivariant stable equivalence and whose target is a positive Ω-spectrum; such a replacement exists and is homotopically unique by the positive model structure for commutative orthogonal ring spectra of [**104**, Thm. 15.1]. The ultra-commutative ring spectrum $\hat{\mathbb{S}} = b(\mathbb{S}^{\mathrm{f}})$ thus comes with a commutative ring spectrum structure, and we call it the *completed sphere spectrum*. Moreover, for every $\hat{\mathbb{S}}$-module spectrum E the map

$$\hat{\mathbb{S}} \wedge bE \ = \ b(\mathbb{S}^{\mathrm{f}}) \wedge bE \ \longrightarrow \ b(\mathbb{S}^{\mathrm{f}} \wedge E) \ \xrightarrow{b(\mathrm{act})} \ bE$$

makes the orthogonal spectrum bE into a module spectrum over the completed sphere spectrum. Since $\hat{\mathbb{S}}$ is non-equivariantly stably equivalent to $\mathbb{S}$, this shows that for every group G the equivariant homotopy group

$$\pi_k^G(bE) \;\cong\; E^{-k}(BG)$$

is naturally a module over the commutative ring $\pi_0^G(\hat{\mathbb{S}})$.

For the global K-theory spectrum (compare Construction VI.5.7 below) and any compact Lie group G, the map

$$\mathbf{RU}(G) \cong \pi_0^G(\mathbf{KU}) \;\longrightarrow\; K^0(BG)$$

is the map of the Atiyah-Segal completion theorem [**9**], and hence a completion at the augmentation ideal of the representation ring. For the Eilenberg-Mac Lane spectrum $\mathcal{H}\mathbb{Z}$ (see Construction VI.1.9), the global functor $G \mapsto H^0(BG;\mathbb{Z})$ is constant with value $\mathbb{Z}$; the map

$$\pi_0^G(H\mathbb{Z}) \;\longrightarrow\; H^0(BG;\mathbb{Z})$$

is surjective and an isomorphism modulo torsion for all compact Lie groups whose identity path component is commutative (compare Example VI.1.16).

Remark 5.29. We close this section by remarking that the right adjoint $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ to the forgetful functor $U : \mathcal{GH}_{\mathcal{E}} \longrightarrow \mathcal{GH}_{\mathcal{F}}$ does not in general preserve infinite sums, and the left adjoint L does not preserve infinite products. So the class of left induced spectra is *not* closed under products in the ambient category and the class of right induced spectra is *not* closed under coproducts in the ambient category. The reader may want to recall from Remark 4.6 the subtleties involved with infinite products in $\mathcal{GH}$.

We illustrate this by specific examples in the extreme case $\mathcal{F} = \langle e\rangle$ and $\mathcal{E} = \mathcal{A}ll$. In the non-equivariant stable homotopy category the canonical map

$$\bigoplus_{i\geq 0} \Sigma^i H\mathbb{F}_2 \;\longrightarrow\; \prod_{i\geq 0} \Sigma^i H\mathbb{F}_2$$

from the coproduct to the product of infinitely many suspended copies of the mod-2 Eilenberg-Mac Lane spectrum is an isomorphism. Since the right adjoint and equivariant homotopy groups preserve products, the canonical map

$$\pi_0^G\left(R\left(\bigoplus\nolimits_{i\geq 0} \Sigma^i H\mathbb{F}_2\right)\right) \;\xrightarrow{\cong}\; \prod\nolimits_{i\geq 0} \pi_0^G(R(\Sigma^i H\mathbb{F}_2)) \;\cong\; \prod\nolimits_{i\geq 0} H^i(BG,\mathbb{F}_2)$$

is an isomorphism of abelian groups. In the special case $G = C_2$, the cyclic group of order 2, the group above then becomes an infinite product of copies of $\mathbb{F}_2$. On the other hand,

$$\pi_0^{C_2}\left(\bigoplus\nolimits_{i\geq 0} R(\Sigma^i H\mathbb{F}_2)\right) \;\cong\; \bigoplus\nolimits_{i\geq 0} \pi_0^{C_2}(R(\Sigma^i H\mathbb{F}_2)) \;\cong\; \bigoplus\nolimits_{i\geq 0} H^i(BC_2,\mathbb{F}_2)$$

is a countable direct sum of copies of $\mathbb{F}_2$. So the canonical map

$$\bigoplus_{i\geq 0} R\left(\Sigma^i H\mathbb{F}_2\right) \;\longrightarrow\; R\left(\bigoplus_{i\geq 0} \Sigma^i H\mathbb{F}_2\right)$$

is not a global equivalence.

A similar, but slightly more involved, argument shows that the left adjoint does not preserve products. As before the canonical map

$$\bigoplus_{i<0} \Sigma^i H\mathbb{F}_2 \;\longrightarrow\; \prod_{i<0} \Sigma^i H\mathbb{F}_2$$

is an isomorphism in $\mathcal{SH}$, where now sum and product are taken over all $i < 0$ (as opposed to $i \geq 0$). Since the left adjoint and equivariant homotopy groups preserves coproducts, the canonical map

$$\bigoplus_{i<0} \pi_0^G\left(L(\Sigma^i H\mathbb{F}_2)\right) \;\longrightarrow\; \pi_0^G\left(L\left(\prod_{i<0} \Sigma^i H\mathbb{F}_2\right)\right) \tag{5.30}$$

is an isomorphism of abelian groups. Again we specialize to $G = C_2$, the cyclic group of order 2. If X is a non-equivariant homotopy type, then LX has 'constant geometric fixed points' in the sense of Remark 5.11. So the geometric fixed point map $\Phi : \pi_0^{C_2}(LX) \longrightarrow \Phi_0^{C_2}(LX) \cong \pi_0^e(LX)$ has a section and the isotropy separation sequence (see (3.11) of Chapter III) splits. So the C_2-equivariant homotopy groups decompose as

$$\pi_0^{C_2}(LX) \;\cong\; \pi_0^{C_2}(LX \wedge (EC_2)_+) \oplus \Phi_0^{C_2}(LX) \;\cong\; \pi_0^e(X \wedge (BC_2)_+) \oplus\; \pi_0^e(X)\ .$$

The second step uses the Adams isomorphism and the fact that any global homotopy type has trivial G-action upon restriction to a trivial G-universe. When $X = \Sigma^i H\mathbb{F}_2$ for negative i, then the second summand is trivial and hence

$$\pi_0^{C_2}(L(\Sigma^i H\mathbb{F}_2)) \;\cong\; \pi_0^e(\Sigma^i H\mathbb{F}_2 \wedge (BC_2)_+) \;\cong\; H_{-i}(BC_2, \mathbb{F}_2)\ .$$

So the group (5.30) is a countably infinite sum of copies of $\mathbb{F}_2$. On the other hand,

$$\pi_0^{C_2}\left(\prod_{i<0} L(\Sigma^i H\mathbb{F}_2)\right) \;\cong\; \prod_{i<0} \pi_0^{C_2}(L(\Sigma^i H\mathbb{F}_2)) \;\cong\; \prod_{i<0} \pi_0^e\left(\Sigma^i H\mathbb{F}_2 \wedge (BC_2)_+\right) \;\cong\; \prod_{i<0} H_{-i}(BC_2, H\mathbb{F}_2)\ ,$$

again by the split isotropy separation sequence. This is an infinite product of copies of $\mathbb{F}_2$, so the canonical map

$$L\left(\prod_{i<0} \Sigma^i H\mathbb{F}_2\right) \;\longrightarrow\; \prod_{i<0} L(\Sigma^i H\mathbb{F}_2)$$

is not a global equivalence.

In the rest of this section we fix a compact Lie group G and relate the global homotopy category to the G-equivariant stable homotopy category (based on a complete universe). Not surprisingly, the model for the G-equivariant stable homotopy category most convenient for our purposes is the category of orthogonal G-spectra. We denote by G-$\mathcal{SH}$ the *G-equivariant stable homotopy category*, i.e., any localization of the category $G\mathcal{S}p$ of orthogonal G-spectra at the class of $\underline{\pi}_*$-isomorphisms. Various stable model structures have been established with $\underline{\pi}_*$-isomorphisms as weak equivalences, for example by Mandell-May [**105**, III Thm. 4.2], Stolz [**150**, Thm. 2.3.27] and Hill-Hopkins-Ravenel [**76**, Prop. B.63]. In particular, as for every stable model category, the homotopy category G-$\mathcal{SH}$ comes with a preferred structure of a triangulated category.

A functor

$$(-)_G \;:\; \mathcal{S}p \;\longrightarrow\; G\mathcal{S}p\ , \quad X \;\longmapsto\; X_G$$

from orthogonal spectra to orthogonal G-spectra is given by endowing an orthogonal spectrum with the trivial G-action. Clearly, every global equivalence of orthogonal spectra becomes a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. Since the trivial action functor takes global equivalences to $\underline{\pi}_*$-isomorphisms, we get a 'forgetful' functor on the homotopy categories

$$U = U_G \;:\; \mathcal{GH} \;\longrightarrow\; G\text{-}\mathcal{SH}$$

form the universal property of localizations.

We will show now that the forgetful functor has both a left and a right adjoint. Moreover, U is canonically an exact functor of triangulated categories. The 'equivariant' smash product of orthogonal G-spectra is simply the smash product of the underlying non-equivariant orthogonal spectra with diagonal G-action. So the trivial action functor $(-)_G : \mathcal{S}p \longrightarrow G\mathcal{S}p$ is strong symmetric monoidal. The smash

product of orthogonal spectra and of orthogonal G-spectra can be derived to symmetric monoidal products on $\mathcal{GH}$ and on G-$\mathcal{SH}$ (see Corollary 3.31). The forgetful functor is strongly monoidal with respect to these derived smash products.

When $G = e$ is a trivial group, the next theorem reduces to the change of family functor of Theorem 5.1, with $\mathcal{E} = \mathcal{A}ll$ and $\mathcal{F} = \langle e \rangle$.

THEOREM 5.31. *For every compact Lie group G the forgetful functor*

$$U \ : \ \mathcal{GH} \ \longrightarrow \ G\text{-}\mathcal{SH}$$

preserves sums and products, and it has a left adjoint and a right adjoint. The left adjoint has a preferred lax symmetric comonoidal structure. The right adjoint has a preferred lax symmetric monoidal structure.

PROOF. Sums in $\mathcal{GH}$ and G-$\mathcal{SH}$ are represented in both cases by the pointset level wedge; for $\mathcal{GH}$ we state this explicitly in Proposition 3.27 (i); for G-$\mathcal{SH}$ we can run the argument based on the stable model structure for orthogonal G-spectra established in [**105**, III Thm 4.2]. On the pointset level, the forgetful functor preserves wedges, so the derived forgetful functor preserves sums. As spelled out in Corollary 4.5 (iv), the existence of the right adjoint is an abstract consequence of the fact that $\mathcal{GH}$ is compactly generated and that the functor U preserves sums.

The forgetful functor also preserves infinite products, but the argument here is slightly more subtle because products in $\mathcal{GH}$ are not generally represented by the pointset level product, and because equivariant homotopy groups do not in general commute with infinite pointset level products, compare Remark 4.6. We let $\{X_i\}_{i\in I}$ be a family of orthogonal spectra. By replacing each factor by a globally equivalent spectrum, if necessary, we can assume without loss of generality that each X_i is a global Ω-spectrum. Since global Ω-spectra are the fibrant objects in a model structure underlying $\mathcal{GH}$, the pointset level product $\prod_{i\in I} X_i$ then represents the product in $\mathcal{GH}$.

Even though X_i is a global Ω-spectrum, the underlying orthogonal G-spectrum $(X_i)_G$ need *not* be a G-Ω-spectrum. However, as we spell out in the proof of Proposition 3.27 (ii), the natural map

$$\pi_k^K(\prod_{i\in I} X_i) \ \longrightarrow \ \prod_{i\in I} \pi_k^K(X_i)$$

is an isomorphism for all compact Lie groups K and all integers k. Again we can run the argument of Proposition 3.27 (ii) in the stable model structure for orthogonal G-spectra [**105**, III Thm 4.2], and conclude that in this situation, the pointset level product is also a product in G-$\mathcal{SH}$. So the derived forgetful functor preserves products. The existence of the left adjoint is then again an abstract consequence of the fact that $\mathcal{GH}$ is compactly generated, compare Corollary 4.5 (v).

The same formal argument as in part (iii) of Theorem 5.1 shows how to turn the strong monoidal structure of the forgetful functor U into a lax comonoidal structure $L(A \wedge^{\mathbb{L}} B) \longrightarrow (LA) \wedge^{\mathbb{L}} (LB)$ of the left adjoint. In contrast to Theorem 5.1 (iii), however, this morphism is usually *not* an isomorphism, so we cannot turn it around into a monoidal structure on L. The same formal argument as in Theorem 5.1 (ii) constructs the lax monoidal structure on R from the strong monoidal structure of the forget functor U. □

Theorem 5.31 looks similar to the change-of-family Theorem 5.1, but there is one important difference: if the group G is non-trivial, then neither of the two adjoints to the forgetful functor $U : \mathcal{GH} \longrightarrow G$-$\mathcal{SH}$ is fully faithful.

REMARK 5.32. We mention an alternative way to construct the two adjoints to the forgetful functor $U : \mathcal{GH} \longrightarrow G$-$\mathcal{SH}$, by exhibiting the pointset forgetful functor $(-)_G : \mathcal{S}p \longrightarrow G\mathcal{S}p$ as a left and a right Quillen functor for suitable model structures (as opposed to using Brown representability). We sketch this for the left adjoint, where we can use the stable model structure of orthogonal G-spectra established by Mandell and May in [**105**, III Thm 4.2]. However, we cannot argue directly with the functor $(-)_G : \mathcal{S}p \longrightarrow G\mathcal{S}p$, since it is *not* a right Quillen functor. Indeed, if it were a right Quillen functor, then it

would preserve fibrant objects. However, a global Ω-spectrum is typically *not* a G-Ω-spectrum when given the trivial G-action.

What saves us is that a global Ω-spectrum is 'eventually' a G-Ω-spectrum, i.e., starting at faithful representations. This allows us to modify $(-)_G$ into a right Quillen functor as follows. We choose a faithful G-representation V and let

$$\Omega^V \operatorname{sh}^V \ : \ \mathcal{S}p \ \longrightarrow \ G\mathcal{S}p$$

denote the functor that takes an orthogonal spectrum X to the orthogonal G-spectrum with U-th level

$$(\Omega^V \operatorname{sh}^V X)(U) \ = \ \operatorname{map}(S^V, X(U \oplus V)) \ .$$

We emphasize that the G-action on $\Omega^V \operatorname{sh}^V X$ is non-trivial, despite the fact that we started with an orthogonal spectrum without a G-action. A natural morphism of orthogonal G-spectra $\tilde{\lambda}^V_X : X \longrightarrow \Omega^V \operatorname{sh}^V X$ is given by the adjoint of the morphism $\lambda^V_X \ : \ X \wedge S^V \ \longrightarrow \ \operatorname{sh}^V X$ defined in (1.23) of Chapter III; the morphism $\tilde{\lambda}^V_X$ is a $\underline{\pi}_*$-isomorphism by Proposition III.1.25 (ii). In particular, the functor $\Omega^V \operatorname{sh}^V$ also takes global equivalences of orthogonal spectra to $\underline{\pi}_*$-isomorphisms of orthogonal G-spectra, and the derived functor of $\Omega^V \operatorname{sh}^V$ is naturally isomorphic to the forgetful functor $U : \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$. The argument can then be completed by showing that the functor $\Omega^V \operatorname{sh}^V$ is a right Quillen functor from the global model structure on orthogonal spectra to the stable model structure on orthogonal G-spectra from [**105**, III Thm 4.2]. Then by general model category theory, the derived functor of $\Omega^V \operatorname{sh}^V$, and hence also the forgetful functor U, has a left adjoint.

The existence of the right adjoint to U can also be established by model category reasoning. For this one can use the $\mathbb{S}$-model structure on orthogonal spectra constructed by Stolz [**150**, Thm. 2.3.27]. We leave it to the interested reader to show that the forgetful functor $(-)_G : \mathcal{S}p \longrightarrow G\mathcal{S}p$ is a left Quillen functor from the global model structure on orthogonal spectra to the stable $\mathbb{S}$-model structure on orthogonal G-spectra.

The left adjoint $L : G\text{-}\mathcal{SH} \longrightarrow \mathcal{GH}$ to the forgetful functor is an exact functor of triangulated categories that preserves infinite sums. The G-equivariant stable homotopy category is compactly generated by the unreduced suspension spectra of all the coset spaces G/H, for all closed subgroups H of G. So L is essentially determined by its values on these generators. The sequence of bijections

$$\mathcal{GH}(L(\Sigma^\infty_+ G/H), X) \ \cong \ G\text{-}\mathcal{SH}(\Sigma^\infty_+ G/H, UX) \ \cong \ \pi_0^H(X) \ \cong \ \mathcal{GH}(\Sigma^\infty_+ B_{\mathrm{gl}} H, X)$$

shows that the left adjoint L takes the unreduced suspension spectrum of the coset space G/H to the suspension spectrum of the global classifying space of H. In the special case $H = G$ the spectrum $\Sigma^\infty_+ G/H$ is the equivariant sphere spectrum $\mathbb{S}_G$, and we obtain that

$$L(\mathbb{S}_G) \ \cong \ \Sigma^\infty_+ B_{\mathrm{gl}} G \ .$$

Now

$$G\text{-}\mathcal{SH}(\mathbb{S}_G, \mathbb{S}_G) \ \cong \ \pi_0^G(\mathbb{S}) \ \cong \ \mathbb{A}(G) \ = \ \mathbf{A}(e, G)$$

is the Burnside ring, whereas

$$\mathcal{GH}(L(\mathbb{S}_G), L(\mathbb{S}_G)) \ \cong \ \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} G) \ \cong \ \ \mathbf{A}(G, G)$$

is the double Burnside ring. The map $L : G\text{-}\mathcal{SH}(\mathbb{S}_G, \mathbb{S}_G) \longrightarrow \mathcal{GH}(L(\mathbb{S}_G), L(\mathbb{S}_G))$ corresponds to the ring homomorphism

$$\mathbb{A}(G) \ = \ \mathbf{A}(e, G) \ \longrightarrow \ \mathbf{A}(G, G) \ , \quad \operatorname{tr}^G_H \circ p^*_H \ \longmapsto \ \operatorname{tr}^G_H \circ \operatorname{res}^G_H$$

from the Burnside ring to the double Burnside ring; this homomorphism is never surjective unless G is trivial, so the left adjoint is not full.

Remark 5.33. Now we let K be another compact Lie group. Theorem 5.31 constructs a right adjoint $R : K\text{-}\mathcal{SH} \longrightarrow \mathcal{GH}$ to the forgetful functor, and this right adjoint assigns a global homotopy type to every K-homotopy type. An obvious question is how to describe the G-equivariant cohomology theory represented by RZ in terms of the K-cohomology theory represented by Z. When K is the trivial group, the right

adjoint R specializes to the change of family functor $R : \mathcal{SH} \longrightarrow \mathcal{GH}$ and Example 5.21 identifies RZ as the global Borel theory associated to the cohomology theory represented by Z. As well shall now explain, for general K the answer is a 'relative' version of a global Borel theory.

We let G be another compact Lie group and V a faithful G-representation. We recall from Proposition I.2.10 that the $(K \times G)$-space $\mathbf{L}(V, \mathcal{U}_K)$ is a universal space for the family $\mathcal{F}(K;G)$ of graph subgroups. This justifies writing $E(K;G)$ for this universal space $\mathbf{L}(V, \mathcal{U}_K)$. Then for every cofibrant G-space A, the adjunction (U, R) provides an isomorphism

$$\begin{aligned}(RZ)^0_G(A) \;&=\; [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A,\, RZ]\!] \;\cong\; K\text{-}\mathcal{SH}(U(\Sigma^\infty_+ \mathbf{L}_{G,V} A),\, Z)\\ &=\; K\text{-}\mathcal{SH}(\Sigma^\infty_+ \mathbf{L}(V, \mathcal{U}_K) \times_G A, Z) \;=\; Z^0_K(E(K;G) \times_G A)\ .\end{aligned}$$

In the special case where A is a one-point G-space, the group $(RZ)^0_G(A)$ becomes the G-equivariant stable homotopy group of RZ. On the other hand, $E(K;G)/G = B(K;G)$ is a classifying K-space for principal G-bundles. So in this case, the isomorphism specializes to an isomorphism

$$\pi^G_0(RZ) \;\cong\; Z^0_K(B(K;G))\ .$$

Remark 5.34. The discussion in this section could be done relative to a global family $\mathcal{F}$, as long as $\mathcal{F}$ contains the compact Lie group G under consideration (and hence also all its subgroups). Indeed, if $\mathcal{F}$ contains G, then every $\mathcal{F}$-equivalence of orthogonal spectra is a $\underline{\pi}_*$-isomorphism of underlying orthogonal G-spectra. Hence the trivial action functor descends to a 'forgetful' functor on the homotopy categories

$$U^{\mathcal{F}}_G \;:\; \mathcal{GH}_{\mathcal{F}} \;\longrightarrow\; G\text{-}\mathcal{SH}$$

by the universal property of localizations. The same arguments as in Theorem 5.31 show the existence of both adjoints to this forgetful functor, with the same kind of monoidal properties.

Theorem 5.31 discusses the maximal case of the global family $\mathcal{F} = \mathcal{A}ll$ of all compact Lie groups. The minimal case is the global family $\langle G \rangle$ generated by G, i.e., the class of compact Lie groups that are isomorphic to a quotient of a closed subgroup of G. All the forgetful functors $U^{\mathcal{F}}_G$ then factor as composites

$$\mathcal{GH}_{\mathcal{F}} \;\xrightarrow{U^{\mathcal{F}}_{\langle G \rangle}}\; \mathcal{GH}_{\langle G \rangle} \;\xrightarrow{U^{\langle G \rangle}_G}\; G\text{-}\mathcal{SH}$$

of a change-of-family functor and a family-to-group functor. The various adjoints then compose accordingly.

The functor $X \mapsto X_G$ is fully faithful on the pointset level, but its homotopical 'derived analog' U is *not* fully faithful. For example, for every compact Lie group G there is an orthogonal spectrum X that is not globally stably contractible, but such that the groups $\pi^H_*(X)$ are trivial for all closed subgroups H of G, and hence X_G is a zero object in G-$\mathcal{SH}$. Another hint is the fact that the equivariant homotopy groups of a global homotopy type, restricted to G and its subgroups, have more structure than is available for a general G-homotopy type, and satisfy certain relations that do not hold for general orthogonal G-spectra, compare Remark 1.2.

Also, whenever G is non-trivial, then the global homotopy category $\mathcal{GH}_{\langle G \rangle}$ associated to the global family generated by G is different from the G-equivariant stable homotopy category G-$\mathcal{SH}$. In other words, if G is non-trivial, then the forgetful family-to-group functor $U^{\langle G \rangle}_G : \mathcal{GH}_{\langle G \rangle} \longrightarrow G\text{-}\mathcal{SH}$ is *not* an equivalence, and neither of its adjoints is fully faithful.

6. Rational finite global homotopy theory

In this section we establish an algebraic model for *rational $\mathcal{F}in$-global* stable homotopy theory, i.e., for rational global stable homotopy theory based on the global family of finite groups. The principal result is Theorem 6.3, providing a chain of Quillen equivalences between the category of orthogonal spectra with the rational $\mathcal{F}in$-global model structure and the category of chain complexes of rational global functors on finite groups. Under this equivalence, the homotopy group global functor for spectra corresponds to the homology group global functor for complexes.

We then explain how this algebraic model can be simplified further. It is well known that rational G-Mackey functors for a fixed finite group naturally split into contributions indexed by conjugacy classes of subgroups. While the analogous global context is *not* semisimple, the category of rational $\mathcal{F}in$-global functors is Morita equivalent to a simpler one, namely contravariant functors from the category Out of finite groups and conjugacy classes of epimorphisms to $\mathbb{Q}$-vector spaces, by Theorem 6.9 below. So rational $\mathcal{F}in$-global stable homotopy theory is also modeled by the complexes of such functors. Under the composite equivalence, the geometric fixed point homotopy group functors for spectra corresponds to the homology group Out-functors for complexes, see Corollary 6.11.

As before we let $\mathcal{F}in$ denote the global family of finite groups. By the results of the previous sections, the associated global stable homotopy category $\mathcal{GH}_{\mathcal{F}in}$ indexed on finite groups is a compactly generated triangulated category with a symmetric monoidal derived smash product. We call an object X of the category $\mathcal{GH}_{\mathcal{F}in}$ *rational* if the equivariant homotopy groups $\pi_k^G(X)$ are uniquely divisible (i.e., $\mathbb{Q}$-vector spaces) for all finite groups G. In this section we will give an algebraic model of the *rational global stable homotopy category* indexed on finite groups, i.e., the full subcategory $\mathcal{GH}_{\mathcal{F}in}^{\mathbb{Q}}$ of rational spectra in $\mathcal{GH}_{\mathcal{F}in}$. Theorem 6.3 below shows that the homotopy types in $\mathcal{GH}_{\mathcal{F}in}^{\mathbb{Q}}$ are determined by a chain complex of global functors, up to quasi-isomorphism. More precisely, we construct an equivalence of triangulated categories from $\mathcal{GH}_{\mathcal{F}in}^{\mathbb{Q}}$ to the unbounded derived category of rational global functors on finite groups.

We let G and K be compact Lie groups. We recall from Proposition 2.5 that the evaluation map

$$\mathbf{A}(G,K) \ \longrightarrow\ \pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}}G)\ , \quad \tau \longmapsto \tau(e_G)$$

is an isomorphism, where $e_G \in \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is the stable tautological class. More precisely, the definition of the global classifying space $B_{\mathrm{gl}}G$ involves an implicit choice of faithful G-representation V that is omitted from the notation, and e_G is the class denote $e_{G,V}$ in (1.12). Like for every suspension spectrum, the group $\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is trivial for $k<0$.

Proposition 6.1. *Let G and K be finite groups. Then for every $k>0$, the equivariant homotopy group $\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is torsion.*

Proof. We show first that the geometric fixed point homotopy group $\Phi_k^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is torsion for $k>0$. This part of the argument needs G to be finite, but K could be any compact Lie group. Geometric fixed points commute with suspension spectra, i.e., the groups $\Phi_*^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ are isomorphic to the non-equivariant stable homotopy groups of the fixed point space $((B_{\mathrm{gl}}G)(\mathcal{U}_K))^K$. Proposition I.5.16 (i) identifies these fixed points as

$$((B_{\mathrm{gl}}G)(\mathcal{U}_K))^K \ \simeq \coprod_{[\alpha]\in \mathrm{Rep}(K,G)} BC(\alpha)\ ,$$

where the disjoint union is indexed by conjugacy classes of homomorphisms from K to G, and $C(\alpha)$ is the centralizer of the image of $\alpha : K \longrightarrow G$. Since G is finite, so are all the centralizers $C(\alpha)$, hence the classifying space $BC(\alpha)$ has no rational homology, hence no rational stable homotopy, in positive dimensions. So we conclude that the rationalized stable homotopy groups of the space $((B_{\mathrm{gl}}G)(\mathcal{U}_K))^K$ vanish in positive dimensions.

If K is also finite, then the k-th rationalized equivariant stable homotopy group of any orthogonal K-spectrum can be recovered from the k-th rationalized geometric fixed point homotopy groups for all subgroups L of K, as described in Corollary III.4.32. So when both G and K are finite, then also the equivariant homotopy group $\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$ is torsion for all $k>0$. □

The conclusion of Proposition 6.1 is no longer true if we drop the finiteness hypothesis on one of the two groups G or K. For example, for $G = e$ we have $\Sigma^\infty_+ B_{\mathrm{gl}} G = \mathbb{S}$, and the dimension shifting transfer $\mathrm{Tr}_e^{U(1)}(1)$ is an element of infinite order in the group $\pi_1^{U(1)}(\mathbb{S})$. On the other hand, for $K = e$ the group $\pi_k^K(\Sigma^\infty_+ B_{\mathrm{gl}} G)$ is the non-equivariant stable homotopy group of the ordinary classifying space BG. For $G = U(1)$ the group $\pi_k^e(\Sigma^\infty_+ BU(1))$ contains a free summand of rank 1 whenever $k \geq 0$ is even.

Now we can establish an algebraic model for the rational $\mathcal{F}in$-global homotopy category. We let $\mathcal{A}$ be a pre-additive category, such as the $\mathcal{F}in$-Burnside category $\mathbf{A}_{\mathcal{F}in}$. We denote by $\mathcal{A}$-mod the category of additive functors from $\mathcal{A}$ to the category of $\mathbb{Q}$-vector spaces. This is an abelian category, and the represented functors $\mathcal{A}(a, -)$, for all objects a of $\mathcal{A}$, form a set of finitely presented projective generators of $\mathcal{A}$-mod. The category of $\mathbb{Z}$-graded chain complexes in the abelian category $\mathcal{A}$-mod then admits the *projective model structure* with the quasi-isomorphisms as weak equivalences. The fibrations in the projective model structure are those chain morphisms that are surjective in every chain complex degree and at every object of $\mathcal{A}$. This projective model structure for complexes of $\mathcal{A}$-modules is a special case of [**36**, Thm. 5.1]. Indeed, the projective (in the usual sense) $\mathcal{A}$-modules together with the epimorphisms form a projective class (in the sense of [**36**, Def. 1.1]), and this class is determined (in the sense of [**36**, Sec. 5.2]) by the set of represented functors.

We also need the rational version of the $\mathcal{F}$-global model structure, for a global family $\mathcal{F}$. We call a morphism $f : X \longrightarrow Y$ of orthogonal spectra a *rational $\mathcal{F}$-equivalence* if the map

$$\mathbb{Q} \otimes \pi_k(f) \ : \ \mathbb{Q} \otimes \pi_k^G(X) \ \longrightarrow \ \mathbb{Q} \otimes \pi_k^G(Y)$$

is an isomorphism for all integers k and all groups G in the family $\mathcal{F}$:

THEOREM 6.2 (Rational $\mathcal{F}$-global model structure). *Let $\mathcal{F}$ be a global family.*

(i) *The rational $\mathcal{F}$-equivalences and $\mathcal{F}$-cofibrations are part of a model structure on the category of orthogonal spectra, the* rational $\mathcal{F}$-global model structure.
(ii) *The fibrant objects in the rational $\mathcal{F}$-global model structure are the $\mathcal{F}$-Ω-spectra X such that for all $G \in \mathcal{F}$ the equivariant homotopy groups $\pi_*^G(X)$ are uniquely divisible.*
(iii) *The rational $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

Theorem 6.2 is obtained by Bousfield localization of the $\mathcal{F}$-global model structure on orthogonal spectra, and one can use a similar proof as for the rational stable model structure on sequential spectra in [**135**, Lemma 4.1]. We omit the details. Now we can state and prove the main result of this section.

THEOREM 6.3. *There is a chain of Quillen equivalences between the category of orthogonal spectra with the rational $\mathcal{F}in$-global model structure and the category of chain complexes of rational global functors on finite groups. In particular, this induces an equivalence of triangulated categories*

$$\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in} \ \longrightarrow \ \mathcal{D}\left(\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}\right) \ .$$

The equivalence can be chosen so that the homotopy group global functor on the left hand side corresponds to the homology group global functor on the right hand side.

PROOF. We prove this as a special case of the 'generalized tilting theorem' of Brooke Shipley and the author. Indeed, by Theorem 4.3 the suspension spectra of the global classifying spaces $B_{\mathrm{gl}} G$ are compact generators of the global homotopy category $\mathcal{GH}_{\mathcal{F}in}$ as G varies through all finite groups. So the rationalizations $(\Sigma^\infty_+ B_{\mathrm{gl}} G)_{\mathbb{Q}}$ are compact generators of the rational global homotopy category $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$. If k is any integer, then the morphism vector spaces between two such objects are given by

$$\begin{aligned} [\![(\Sigma^\infty_+ B_{\mathrm{gl}} K)_{\mathbb{Q}},\ (\Sigma^\infty_+ B_{\mathrm{gl}} G)_{\mathbb{Q}}]\!]_k \ &\cong \ \pi_k^K((\Sigma^\infty_+ B_{\mathrm{gl}} G)_{\mathbb{Q}}) \ \cong \ \mathbb{Q} \otimes \pi_k^K(\Sigma^\infty_+ B_{\mathrm{gl}} G) \\ &\cong \ \begin{cases} \mathbb{Q} \otimes \mathbf{A}(G, K) & \text{for } k = 0, \text{ and} \\ \quad 0 & \text{for } k \neq 0. \end{cases} \end{aligned}$$

The vanishing for $k > 0$ is Proposition 6.1.

The rational $\mathcal{F}in$-global model structure on orthogonal spectra is topological (hence simplicial), compactly generated, proper and stable; so we can apply the Tilting Theorem [**134**, Thm. 5.1.1]. This theorem yields a chain of Quillen equivalences between orthogonal spectra in the rational $\mathcal{F}in$-global model structure and the category of chain complexes of $\mathbb{Q} \otimes \mathbf{A}_{\mathcal{F}in}$-modules, i.e., additive functors from the rationalized Burnside category $\mathbb{Q} \otimes \mathbf{A}_{\mathcal{F}in}$ to abelian groups. This functor category is equivalent to the category of additive functors from $\mathbf{A}_{\mathcal{F}in}$ to $\mathbb{Q}$-vector spaces, and this proves the theorem. $\square$

Remark 6.4. There is an important homological difference between global functors on finite groups and Mackey functors for one fixed finite group. Indeed, for a finite group G, the rationalized Burnside ring $\mathbb{Q} \otimes A(G)$ has plenty of idempotents that can be used to split a rational Mackey functor for the group G into smaller pieces. The end result is that the category of rational Mackey functors over G is equivalent to a product, indexed over conjugacy classes (H) of subgroups of G, of the module categories over the rational group rings $\mathbb{Q}[W_G H]$ of the Weyl group. In particular, the abelian category of rational G-Mackey functors is semisimple, every object is projective and injective and the derived category is equivalent, by taking homology, to the category of graded rational Mackey functors over G.

There is no analog of this for rational $\mathcal{F}in$-global functors. For example, the rationalized augmentation

$$\mathbb{Q} \otimes \mathbb{A} = \mathbb{Q} \otimes \mathbf{A}(e, -) \ \longrightarrow \ \underline{\mathbb{Q}}$$

from the rationalized Burnside ring global functor to the constant global functor for the group $\mathbb{Q}$ does *not* split on finite groups. The new phenomenon is that any splitting would have to be natural for inflation maps.

Let us be even more specific. In the constant global functor $\underline{\mathbb{Q}}$ we have

$$2 \cdot p^*(1) = \mathrm{tr}_e^{C_2}(1) \qquad \text{in} \qquad \underline{\mathbb{Q}}(C_2) = \mathbb{Q} \ ,$$

where $p : C_2 \longrightarrow e$ is the unique group homomorphism. So for any morphism of global functors $\varphi : \underline{\mathbb{Q}} \longrightarrow N$ the image $\varphi(e)(1)$ of the unit element under the map $\varphi(e) : \mathbb{Q} \longrightarrow N(e)$ must satisfy

$$\mathrm{tr}_e^{C_2}(\varphi(e)(1)) \ = \ 2 \cdot p^*(\varphi(e)(1)) \ .$$

But in the Burnside ring $\mathbb{A}(e)$, and also in its rationalization, 0 is the only element in the kernel of $\mathrm{tr}_e^{C_2} - 2 \cdot p^*$; so every morphism of global functors from $\underline{\mathbb{Q}}$ to $\mathbb{Q} \otimes \mathbb{A}$ is zero.

While the abelian category $\mathcal{GF}_{\mathcal{F}in}^{\mathbb{Q}}$ is not semisimple, we can still 'divide out transfers' and thereby replace $\mathcal{GF}_{\mathcal{F}in}^{\mathbb{Q}}$ by an equivalent, but simpler category. We let Out denote the category of finite groups and conjugacy classes of surjective group homomorphisms. We write

$$\text{mod-}\,\mathrm{Out} \ = \ \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathcal{A}b)$$

for the category of 'right Out-modules', i.e., contravariant functors from Out to the category of abelian groups. To a $\mathcal{F}in$-global functor $F : \mathbf{A}_{\mathcal{F}in} \longrightarrow \mathcal{A}b$ we can associate a right Out-module $\tau F : \mathrm{Out}^{\mathrm{op}} \longrightarrow \mathcal{A}b$, the *reduced functor* as follows. On objects we set

$$(\tau F)(G) \ = \ \tau_F G \ = \ F(G)/tF(G) \ ,$$

the quotient of the group $F(G)$ by the subgroup $tF(G)$ generated by the images of all transfer maps $\mathrm{tr}_H^G : F(H) \longrightarrow F(G)$ for all proper subgroups H of G. If $\alpha : K \longrightarrow G$ is a surjective group homomorphism and $H \leq G$ a proper subgroup, then $L = \alpha^{-1}(H)$ is a proper subgroup of K and the relation

$$\alpha^* \circ \mathrm{tr}_H^G \ = \ \mathrm{tr}_L^K \circ (\alpha|_L)^*$$

as maps $F(H) \longrightarrow F(K)$ shows that the inflation map $\alpha^* : F(G) \longrightarrow F(K)$ passes to a homomorphism $\alpha^* : (\tau F)(G) \longrightarrow (\tau F)(K)$ of quotient groups. We will now argue that the reduction functor

$$\tau \ : \ \mathcal{GF}_{\mathcal{F}in} \ \longrightarrow \ \text{mod-}\,\mathrm{Out}$$

is rationally an equivalence of categories, compare Theorem 6.9 below. By construction, the projection maps $F(G) \longrightarrow \tau F(G)$ form a natural transformation from the restriction of the global functor F to the category Out to τF.

EXAMPLE 6.5. We let M be an abelian group and $\underline{M}$ the constant global functor with value M, compare Example 2.8 (iii). So we have $\tau\underline{M}(G) = M/cM$ where c is the greatest common divisor of the indices of all proper subgroups of G. If G is not a p-group for any prime p, then this greatest common divisor is 1. If G is a non-trivial p-group, then G has a proper subgroup of index p. So we have

$$(\tau\underline{M})(G) \;=\; \begin{cases} M & \text{if } G = e, \\ M/pM & \text{if } G \text{ is a non-trivial } p\text{-group, and} \\ 0 & \text{else.} \end{cases}$$

The inflation maps in $\tau\underline{M}$ are quotient maps.

EXAMPLE 6.6. The Burnside ring $\mathbb{A}(K) = \mathbf{A}(e, K)$ of a finite group K is freely generated, as an abelian group, by the transfers $\mathrm{tr}_L^K(1)$ where L runs through representatives of the conjugacy classes of subgroups of K. So $(\tau\mathbb{A})(K)$ is free abelian of rank 1, generated by the class of the multiplicative unit 1. All restriction maps preserve the unit, so the reduced functor $\tau\mathbb{A}$ of the global Burnside ring functor is isomorphic to the constant functor $\mathrm{Out}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ with value $\mathbb{Z}$,

$$\tau\mathbb{A} \;\cong\; \underline{\mathbb{Z}}\ .$$

As before we denote by $\mathbf{A}_G = \mathbf{A}(G, -)$ the global functor generated by a compact Lie group G. We will now present $\tau(\mathbf{A}_G)$ as an explicit quotient of a sum of representable Out-modules. For every closed subgroup H of G the restriction map res_H^G is a morphism in $\mathbf{A}(G, H)$. If H is finite, the Yoneda lemma provides a unique morphism

$$\mathrm{Out}_H \;\longrightarrow \tau(\mathbf{A}_G)$$

of Out-functors from the representable functor $\mathrm{Out}_H = \mathbb{Z}[\mathrm{Out}(-, H)]$ to $\tau(\mathbf{A}_G)$ that sends the identity of H to the class of res_H^G in $\tau(\mathbf{A}_G)(H)$. For every element $g \in G$ the conjugation isomorphism $c_g : {}^gH \longrightarrow H$ given by $c_g(\gamma) = g^{-1}\gamma g$ induces an isomorphism

$$g_\star \circ - \;:\; \mathbf{A}_G(H) \;\longrightarrow\; \mathbf{A}_G({}^gH)$$

by postcomposition. We have

$$g_\star \circ [\mathrm{res}_H^G] \;=\; [g_\star \circ \mathrm{res}_H^G] \;=\; [\mathrm{res}_{{}^gH}^G \circ g_\star] \;=\; [\mathrm{res}_{{}^gH}^G]$$

in $(\tau\mathbf{A}_G)({}^gH)$. The Yoneda lemma translates this into the fact that the triangle of Out-functors

$$\begin{tikzcd} \mathrm{Out}_H \arrow[d, "c_g \circ -"'] \arrow[dr] & \\ \mathrm{Out}_{{}^gH} \arrow[r] & \tau(\mathbf{A}_G) \end{tikzcd}$$

commutes. So the direct sum of the transformations $\mathrm{Out}_H \longrightarrow \tau\mathbf{A}_G$ factors over a natural transformation

$$\Big(\bigoplus\nolimits_{H \le G} \mathrm{Out}_H\Big)/G \;\longrightarrow\; \tau(\mathbf{A}_G)\ . \tag{6.7}$$

The source of this morphism can be rewritten if we choose representatives of the conjugacy classes of subgroups in H:

$$\bigoplus_{(H)} (\mathrm{Out}_H / W_G H) \;\longrightarrow\; \tau(\mathbf{A}_G)\ .$$

Now the sum is indexed by conjugacy classes of subgroups of G.

PROPOSITION 6.8. *For every finite group G, the morphism* (6.7) *is an isomorphism of* Out*-modules.*

Proof. By Theorem 2.6 the abelian group $\mathbf{A}_G(K) = \mathbf{A}(G,K)$ is freely generated by the elements $\mathrm{tr}_L^K \circ \alpha^*$ where (L,α) runs through representatives of the conjugacy classes of pairs consisting of a subgroup L of K and a continuous homomorphism $\alpha : L \longrightarrow G$. So $(\tau\mathbf{A}_G)(K)$ is a free abelian group with basis the classes of α^* for all conjugacy classes of homomorphisms $\alpha : K \longrightarrow G$.

On the other hand, the group $(\mathrm{Out}_H / W_G H)(K)$ is free abelian with basis given by $W_G H$-orbits of conjugacy classes of epimorphisms $\alpha : K \longrightarrow H$. The map (6.7) sends the basis element represented by α to the basis element represented by the composite of α with the inclusion $H \longrightarrow G$. So the homomorphism (6.7) takes a basis of the source to a basis of the target, and is thus an isomorphism. □

We recalled in Proposition III.4.24 above how the value of a G-Mackey functor, for a finite group G, can rationally be recovered from the groups $(\tau F)(H)$ for all subgroups H of G: the map

$$\psi_G^F \ : \ F(G) \ \longrightarrow \ \Big(\prod\nolimits_{H\le G} (\tau F)(H)\Big)^G$$

whose H-component is the composite

$$F(G) \ \xrightarrow{\mathrm{res}_H^G} \ F(H) \ \xrightarrow{\mathrm{proj}} (\tau F)(H)$$

becomes an isomorphism after tensoring with $\mathbb{Q}$. When applied to a $\mathcal{F}in$-global functor F, we see that F can rationally be recovered from the Out-module τF. This is the key input to the following equivalence of categories. I suspect that the following proposition is also well known, but I have been unable to find an explicit reference.

Theorem 6.9. *The restriction of the functor τ to the full subcategory of rational $\mathcal{F}in$-global functors is an equivalence of categories*

$$\tau \ : \ \mathcal{GF}_{\mathcal{F}in}^{\mathbb{Q}} \ \longrightarrow \ \text{mod-}\,\mathrm{Out}_{\mathbb{Q}} \ = \ \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q})$$

onto the category of rational Out*-modules.*

Proof. Since global functors are an enriched functor category and the functor

$$\tau \ : \ \mathcal{GF}_{\mathcal{F}in} \ \longrightarrow \ \text{mod-}\,\mathrm{Out}$$

commutes with colimits, τ has a right adjoint

$$\rho \ : \ \text{mod-}\,\mathrm{Out} \ \longrightarrow \ \mathcal{GF}_{\mathcal{F}in} \ .$$

If ρX is an Out-module, the value of the $\mathcal{F}in$-global functor ρX at a finite group G is necessarily given by

$$(\rho X)(G) \ = \ \text{mod-}\,\mathrm{Out}(\tau(\mathbf{A}_G), X) \ ,$$

the group of Out-module homomorphisms from $\tau(\mathbf{A}_G)$ to X. The global functoriality in G is via τ, i.e., as the composite

$$\begin{aligned} \mathbf{A}(G,K) \otimes (\rho X)(G) \ &\longrightarrow \ \mathcal{GF}(\mathbf{A}_K, \mathbf{A}_G) \otimes (\rho X)(G) \\ &\xrightarrow{\tau\otimes \mathrm{Id}} \ \text{mod-}\,\mathrm{Out}(\tau(\mathbf{A}_K), \tau(\mathbf{A}_G)) \otimes (\rho X)(G) \ \xrightarrow{\circ} \ (\rho X)(K) \ . \end{aligned}$$

We rewrite the definition of $(\rho X)(G)$ using the description of the Out-module $\tau(\mathbf{A}_G)$ given in (6.7). Indeed, by Proposition 6.8, precomposition with (6.7) induces an isomorphism

$$(\rho X)(G) \ \cong \ \text{mod-}\,\mathrm{Out}\left(\Big(\bigoplus\nolimits_{H\le G} \mathrm{Out}_H\Big)/G, X\right) \ \cong \ \Big(\prod\nolimits_{H\le G} X(H)\Big)^G \ .$$

So for every global functor F, this description shows that $(\rho(\tau(F)))(G)$ is isomorphic to the target of the morphism ψ_G^F. A closer analysis reveals that for $X = \tau F$, the above isomorphism identifies ψ_G^F with the value of the adjunction unit $\eta : F \longrightarrow \rho(\tau F)$ at G. So Proposition III.4.24 shows that for every rational $\mathcal{F}in$-global functor F the adjunction unit $\eta_F : F \longrightarrow \rho(\tau F)$ is an isomorphism. This implies that the restriction of the left adjoint τ to the category of rational $\mathcal{F}in$-global functors is fully faithful.

Now we consider a rational Out-module X. Then $\rho(X)$ is a rational $\mathcal{F}in$-global functor, so $\eta_{\rho(X)} : \rho(X) \longrightarrow \rho(\tau(\rho(X)))$ is an isomorphism by the previous paragraph. Since $\eta_{\rho(X)}$ is right inverse to $\rho(\epsilon_X)$, the morphism $\rho(\epsilon_X) : \rho(\tau(\rho(X))) \longrightarrow \rho(X)$ is an isomorphism of $\mathcal{F}in$-global functors. By Proposition 6.8 the represented Out-module Out_G is a direct summand of $\tau(\mathbf{A}_G)$. So the group

$$\text{mod-}\,\mathrm{Out}(\mathrm{Out}_G, X) \;\cong\; X(G)$$

is a direct summand of the group

$$\text{mod-}\,\mathrm{Out}(\tau(\mathbf{A}_G), X) \;\cong\; \mathcal{GF}(\mathbf{A}_G, \rho X) \;\cong\; (\rho X)(G)\ ,$$

and this splitting is natural for morphisms of Out-modules in X. In particular, the morphism $\epsilon_X(G) : (\tau(\rho X))(G) \longrightarrow X(G)$ is a direct summand of the morphism

$$\rho(\epsilon_X)(G) \;:\; (\rho(\tau(\rho X)))(G) \;\longrightarrow\; (\rho X)(G)\ .$$

The latter is an isomorphism by the previous paragraph, so the morphism $\epsilon_X(G)$ is also an isomorphism. This shows that for every rational Out-module X the adjunction counit $\epsilon_X : \tau(\rho(X)) \longrightarrow X$ is an isomorphism.

Altogether we have now seen that when restricted to rational objects on both sides, the unit and counit of the adjunction (τ, ρ) are isomorphisms. This proves the theorem. □

The rational equivalence τ of abelian categories prolongs to an equivalence of derived categories by applying τ dimensionwise to chain complexes. The combination with the equivalence of triangulated categories of Theorem 6.3 is then a chain of two exact equivalences of triangulated categories

$$\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in} \;\underset{\cong}{\longrightarrow}\; \mathcal{D}\left(\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}\right) \;\underset{\cong}{\xrightarrow{\mathcal{D}(\tau)}}\; \mathcal{D}(\text{mod-}\,\mathrm{Out}_{\mathbb{Q}})\ . \tag{6.10}$$

The next proposition shows that this composite equivalence is an algebraic model for the geometric fixed point homotopy groups.

For every orthogonal spectrum X and every compact Lie group G, the geometric fixed point map $\Phi : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ annihilates all transfers from proper subgroups by Proposition III.3.13. So the geometric fixed point map factors over a homomorphism

$$\bar{\Phi} \;:\; \tau(\underline{\pi}_k(X))(G) \;\longrightarrow\; \Phi_k^G(X)$$

that we called the reduced geometric fixed point map above. The geometric fixed point maps are compatible with inflations (Proposition 1.23 (iii)), so as G varies among finite groups, the reduced geometric fixed point maps form a morphism of Out-modules. When we apply Proposition III.4.30 to the underlying orthogonal G-spectrum of an orthogonal spectrum, it specializes to the following:

Corollary 6.11. *For every orthogonal spectrum X, every finite group G and every integer k the map*

$$\bar{\Phi} \;:\; \tau(\underline{\pi}_k(X))(G) \;\longrightarrow\; \Phi_k^G(X)$$

of becomes an isomorphism after tensoring with $\mathbb{Q}$. So for varying finite groups G, these maps form a rational isomorphism of Out*-functors $\tau(\underline{\pi}_k(X)) \cong \underline{\Phi}_k(X)$.*

As a corollary we obtain that the combined equivalence κ of triangulated categories (6.10) from the rational finite global homotopy category $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$ to the derived category of the abelian category $\text{mod-}\,\mathrm{Out}_{\mathbb{Q}}$ comes with a natural isomorphism

$$\Phi_*^G(X) \;\cong\; H_*(\kappa(X))\ ,$$

for every object X of $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$, between the geometric fixed point homotopy groups and the homology Out-modules of $\kappa(X)$.

Example 6.12. As before we let $\mathbf{RU}$ denote the unitary representation ring global functor, compare Example 2.8 (iv). Artin's theorem says that the representation ring $\mathbf{RU}(G)$ is rationally generated by the representations induced from cyclic subgroups of G. So if G is not itself cyclic, then $\mathbf{RU}_{\mathbb{Q}}(G) = \mathbb{Q} \otimes \mathbf{RU}(G)$ is generated by transfers from proper subgroups, and hence $\tau\mathbf{RU}_{\mathbb{Q}}(G) = 0$.

Now suppose that $G = C_n = \mathbb{Z}/n\mathbb{Z}$ is cyclic of order n with generator γ. We let z_n denote the representation of C_n on $\mathbb{C}$ where γ acts by multiplication by the primitive n-th root of unity $\zeta_n = e^{2\pi i/n}$. Then $\mathbf{RU}(C_n) = \mathbb{Z}[z]/(z^n - 1)$. If $n = p^h$ for a prime p, we have

$$\mathrm{ind}_{p^{h-1}}^{p^h}(z_{p^{h-1}}^i) \;=\; z^i \cdot (1 + z^q + z^{2q} + \cdots + z^{(p-1)q}) \;,$$

where $q = p^{h-1}$. So for $G = C_{p^h}$ the subgroup generated by transfers from proper subgroups is the ideal of $\mathbf{RU}(C_n) = \mathbb{Z}[z]/(z^n - 1)$ generated by $1 + z^q + z^{2q} + \cdots + z^{(p-1)q}$. This factor ring is isomorphic to $\mathbb{Z}(\zeta_{p^h})$, the ring of integers in the cyclotomic number field $\mathbb{Q}(\zeta_{p^h})$. Now suppose that $n = kl$ where k and l are coprime. Then $C_n = C_k \times C_l$ for uniquely determined cyclic subgroups C_k and C_l of order k respectively l. So we have $\mathbf{RU}(C_n) = \mathbf{RU}(C_k \times C_l) \cong \mathbf{RU}(C_k) \otimes \mathbf{RU}(C_l)$, the isomorphism given by tensor product of representations. Moreover, the maximal subgroups of C_n are of the form $H \times C_l$ or $C_k \times H'$ for maximal subgroups H of C_k or H' of C_l. So

$$(\tau\mathbf{RU}_{\mathbb{Q}})(C_n) \;=\; \tau\mathbf{RU}_{\mathbb{Q}}(C_k \times C_l) \;\cong\; \tau\mathbf{RU}_{\mathbb{Q}}(C_k) \otimes \tau\mathbf{RU}_{\mathbb{Q}}(C_l) \;.$$

So altogether we conclude that

$$\tau\mathbf{RU}_{\mathbb{Q}}(G) \;\cong\; \begin{cases} \mathbb{Q}(\zeta_n) & \text{if } G = C_n, \text{ and} \\ 0 & \text{if } G \text{ is not cyclic.} \end{cases}$$

The isomorphism sends the primitive n-th root of unity ζ_n to the residue class of the representation z. The projection map $p : C_{ni} \longrightarrow C_n$ induces

$$p^* \;:\; \mathbb{Q}(\zeta_n) \;\longrightarrow\; \mathbb{Q}(\zeta_{ni}) \;, \quad p^*(\zeta_n) = \zeta_{ni}^i \;.$$

The degree of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ is $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$, which is also the order of $\mathrm{Aut}(C_n) = \mathrm{Out}(C_n)$. In fact, $(\tau\mathbf{RU}_{\mathbb{Q}})(C_n)$ is free of rank 1 over the group ring $\mathbb{Q}[\mathrm{Out}(C_n)]$.

CHAPTER V

Global power functors

This chapter is devoted to an in depth study of 'global power functors', which is our name for the algebraic structure on the equivariant homotopy groups of ultra-commutative ring spectra. While most of the results in this chapter are of algebraic nature, the motivation comes from global equivariant homotopy theory. Indeed:

- Theorem 1.9 shows that the 0-th equivariant homotopy groups of an ultra-commutative ring spectrum form a global power functor;
- Proposition VI.1.6 shows that the power operations are precisely the natural additional structure that the global functor $\underline{\pi}_0(R)$ of an ultra-commutative ring spectrum R;
- Theorem VI.1.7 below shows that every global power functor can be realized by an ultra-commutative ring spectrum.

So we hope that altogether these are enough good reasons to for studying the category of global power functors.

Section 1 introduces the formal setup for encoding the power operations on ultra-commutative ring spectra. We introduce global power functors as global Green functors equipped with additional power operations, satisfying a list of axioms reminiscent of the property of the power maps $x \mapsto x^m$ in a commutative ring. We also show that the global functor $\underline{\pi}_0(R)$ of an ultra-commutative ring spectrum R supports power operations, and is an example of a global power functor.

Section 2 gives both a monadic and a comonadic description of the category of global power functors. We introduce the comonad of 'exponential sequences' on the category of global Green functors, and show that its coalgebras are equivalent to global power functors. A formal consequence is that the category of global power functors has all limits and colimits, and that they are created in the category of global Green functors. We discuss localization of global Green functors and global power functors at a multiplicative subset of the underlying ring, including rationalization of global power functors.

Section 3 discusses the existence, properties and interrelationship of various free functors. It features the algebraic categories of Rep-functors, abelian Rep-monoids, global power monoids, global functors, global Green functors and global power functors. These categories are related by two kinds of 'forgetful' functors; one of them forgets the additive structure (including the transfers) and remembers only the multiplicative structure. The other one forgets power operations respectively multiplications. All these forgetful functors have left adjoint free functors, and all these functor pairs are compatible in various ways that we specify. The study of this algebraic structure is relevant to topology because the six algebraic categories mentioned above are the homes of the 0-th equivariant homotopy sets/groups of orthogonal spaces, E_∞-orthogonal monoid spaces, ultra-commutative monoids, orthogonal spectra E_∞-orthogonal ring spectra, respectively ultra-commutative ring spectra.

In Section 4 we discuss various examples of global power functors, such as the Burnside ring global power functor, the global functor represented by an abelian compact Lie group, free global power functors, constant global power functors, and the complex representation ring global functor.

1. Power operations

Section 1 introduces the formal setup for encoding the power operations on ultra-commutative ring spectra. In Definition 1.5 we introduce global power functors, which are global Green functors equipped with additional power operations, satisfying a list of axioms reminiscent of the property of the power maps $x \mapsto x^m$ in a commutative ring. Theorem 1.9 shows that the global functor $\underline{\pi}_0(R)$ of an ultra-commutative ring spectrum R supports power operations, and is an example of a global power functor.

In Section II.2 we introduced power operations and transfers on the equivariant homotopy sets of ultra-commutative monoids. For every ultra-commutative ring spectrum R, the orthogonal space $\Omega^\bullet R$ inherits a commutative multiplication, making it an ultra-commutative monoid (compare Example IV.1.16). Moreover, $\pi_0^G(\Omega^\bullet R) = \pi_0^G(R)$, so this endows the 0-th stable equivariant homotopy groups $\pi_0^G(R)$ with multiplicative power operations and transfers, natural for homomorphisms of ultra-commutative ring spectra. Since these operations come from the multiplicative (as opposed to the 'additive' structure) of the ring spectrum, we now switch to a multiplicative notation and write

$$P^m \ : \ \pi_0^G(R) \ \longrightarrow \ \pi_0^{\Sigma_m \wr G}(R) \ , \tag{1.1}$$

(instead of $[m]$) for the multiplicative power operations, and we write N_H^G (instead of tr_H^G) for multiplicative transfers. Such multiplicative transfer are often referred to as *norm maps*, and that is also the terminology we will use. Since we will use these power operations a lot, we take the time to expand the definition: the operation P^m takes the class represented by a based G-map $f : S^V \longrightarrow R(V)$, for some G-representation V, to the class of the $(\Sigma_m \wr G)$-map

$$S^{V^m} = (S^V)^{\wedge m} \ \xrightarrow{f^{\wedge m}} \ R(V)^{\wedge m} \ \xrightarrow{\mu_{V,\dots,V}} R(V^m) \ ,$$

where $\mu_{V,\dots,V}$ is the iterated, $(\Sigma_m \wr G)$-equivariant multiplication map of R.

In this section we study the power operations for ultra-commutative ring spectra in some detail. We package the resulting algebraic structure on the global functor $\underline{\pi}_0(R)$ as a *global power functor*, see Definition 1.5. For a different perspective and a detailed algebraic study of global power functors (restricted to finite groups), including the relationship to the concepts of λ-rings, τ-rings and β-rings, we refer the reader to Ganter's paper [**57**].

Some of the main results in this section are the following: for every ultra-commutative ring spectrum R the global functor $\underline{\pi}_0(R)$ is naturally a global power functor (Theorem 1.9) and all the natural operations on $\underline{\pi}_0(R)$ are generated by restrictions, transfers and power operations (Proposition 1.6), so we are not missing any additional structure. Theorem 1.7 below shows that every global power functor is realized by an ultra-commutative Eilenberg-Mac Lane ring spectrum.

Definition 1.2. A *global Green functor* is a commutative monoid in the category $\mathcal{GF}$ of global functors under the monoidal structure given by the box product, compare (2.19) of Chapter IV.

As we explain after Definition IV.2.24, this commutative multiplication on a global Green functor R can be made more explicit in two equivalent ways:

- as a commutative ring structure on the group $R(G)$ for every compact Lie group, subject to the requirement that all restrictions maps are ring homomorphisms and the transfer maps satisfy reciprocity;
- as a unit element $1 \in R(e)$ and biadditive, commutative, associative and unital external pairings $\times : R(G) \times R(H) \longrightarrow R(G \times H)$ that are morphisms of global functors in each variable separately.

For orthogonal spectra (as opposed to orthogonal spaces), the equivariant homotopy set π_0^G has two pieces of additional structure, namely an addition and transfer maps. We clarify next how the power operations of ultra-commutative ring spectra interact with this structure. For $m \geq 2$ the power operation P^m is *not* additive, but it satisfies various properties reminiscent of the map $x \mapsto x^m$ in a commutative

ring. We formalize these properties into the concept of a *global power functor*. Conditions (i) through (vi) in the following definition express the fact that a global power functor has an underlying 'multiplicative' global power monoid, in the sense of Definition II.2.8, if we forget the additive structure. The definition makes use of certain embeddings between products and wreath products:

$$\begin{aligned} \Phi_{i,j} \ : \ (\Sigma_i \wr G) \ \times \ (\Sigma_j \wr G) \quad &\longrightarrow \quad \Sigma_{i+j} \wr G \\ ((\sigma;\, g_1, \dots, g_i),\, (\sigma';\, g_{i+1}, \dots, g_{i+j})) \ &\longmapsto \ (\sigma + \sigma';\, g_1, \dots, g_{i+j}) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \Psi_{k,m} \ : \ \Sigma_k \wr (\Sigma_m \wr G) \quad &\longrightarrow \quad \Sigma_{km} \wr G \\ (\sigma;\, (\tau_1;\, h^1), \dots, (\tau_k;\, h^k)) \ &\longmapsto \ (\sigma \natural (\tau_1, \dots, \tau_k);\, h^1 + \dots + h^k) \ . \end{aligned} \tag{1.4}$$

These monomorphisms were defined (and 'explained') in Construction II.2.3.

Definition 1.5. A *global power functor* is a global Green functor R equipped with maps

$$P^m \ : \ R(G) \ \longrightarrow \ R(\Sigma_m \wr G)$$

for all compact Lie groups G and $m \geq 1$, called *power operations*, that satisfy the following relations.

(i) (Unit) $P^m(1) = 1$ for the unit $1 \in R(e)$.
(ii) (Identity) $P^1 = \mathrm{Id}$ under the identification $G \cong \Sigma_1 \wr G$ sending g to $(1; g)$.
(iii) (Naturality) For every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups and all $m \geq 1$ the relation

$$P^m \circ \alpha^* \ = \ (\Sigma_m \wr \alpha)^* \circ P^m$$

holds as maps $R(G) \longrightarrow R(\Sigma_m \wr K)$.
(iv) (Multiplicativity) For all compact Lie groups G, all $m \geq 1$ and all classes $x, y \in R(G)$ the relation

$$P^m(x \cdot y) \ = \ P^m(x) \cdot P^m(y)$$

holds in the group $R(\Sigma_m \wr G)$.
(v) (Restriction) For all compact Lie groups G, all $m > k > 0$ and all $x \in R(G)$ the relation

$$\Phi^*_{k,m-k}(P^m(x)) \ = \ P^k(x) \times P^{m-k}(x)$$

holds in $R((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G))$.
(vi) (Transitivity) For all compact Lie groups G, all $k, m \geq 1$ and all $x \in R(G)$ the relation

$$\Psi^*_{k,m}(P^{km}(x)) \ = \ P^k(P^m(x))$$

holds in $R(\Sigma_k \wr (\Sigma_m \wr G))$.
(vii) (Additivity) For all compact Lie groups G, all $m \geq 1$, and all $x, y \in R(G)$ the relation

$$P^m(x + y) \ = \ \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}(P^k(x) \times P^{m-k}(y))$$

holds in $R(\Sigma_m \wr G)$, where $\mathrm{tr}_{k,m-k}$ is the transfer associated to the embedding $\Phi_{k,m-k} : (\Sigma_k \wr G) \ \times (\Sigma_{m-k} \wr G) \longrightarrow \Sigma_m \wr G$ defined in (1.3). Here $P^0(x)$ is the multiplicative unit 1.
(viii) (Transfer) For every subgroup H of a compact Lie group G and every $m \geq 1$ the relation

$$P^m \circ \mathrm{tr}^G_H \ = \ \mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H} \circ P^m$$

holds as maps $R(H) \longrightarrow R(\Sigma_m \wr G)$.

A *morphism* of global power functors is a morphism of global Green functors that also commutes with the power operations.

In a global power functor the relation $P^m(0) = 0$ also holds for every $m \geq 1$ and all G. Indeed, the additivity relation gives

$$P^m(0) \;=\; P^m(0+0) \;=\; \sum_{i+j=m} \mathrm{tr}_{i,j}(P^i(0) \cdot P^{m-i}(0)) \;.$$

Since $P^0(0) = 1$, subtracting $2P^m(0)$ gives

$$-P^m(0) \;=\; \sum_{i=1}^{m-1} \mathrm{tr}_{i,j}(P^i(0) \cdot P^{m-i}(0)) \;.$$

Starting from $P^1(0) = 0$ this shows inductively that $P^m(0) = 0$.

Remark 1.6 (Global power functors versus global Tambara functors). A global power functor gives rise to two underlying global power monoids, the *additive* and the *multiplicative* one. As we explained in Construction II.2.32, applied to the multiplicative global power monoid, the power operations P^m lead to 'multiplicative transfers', $N_H^G : R(H) \longrightarrow R(G)$ that are called *norm maps*, for every subgroup H of finite index in G. For the convenience of the reader, we make the definition of the norm maps in a global power functor R explicit. We suppose that H has index m in G, and we choose a 'H-basis' of G, i.e., an ordered m-tuple $\bar{g} = (g_1, \ldots, g_m)$ of elements in disjoint H-orbits such that

$$G \;=\; \bigcup_{i=1}^{m} g_i H \;.$$

The wreath product $\Sigma_m \wr H$ acts freely and transitively from the right on the set of all such H-bases of G, by the formula

$$(g_1, \ldots, g_m) \cdot (\sigma;\, h_1, \ldots, h_m) \;=\; (g_{\sigma(1)} h_1, \ldots, g_{\sigma(m)} h_m) \;.$$

We obtain a continuous homomorphism $\Psi_{\bar{g}} : G \longrightarrow \Sigma_m \wr H$ by requiring that

$$\gamma \cdot \bar{g} \;=\; \bar{g} \cdot \Psi_{\bar{g}}(\gamma) \;.$$

The norm $N_H^G : R(H) \longrightarrow R(G)$ is then the composite

$$R(H) \;\xrightarrow{P^m}\; R(\Sigma_m \wr H) \;\xrightarrow{\Psi_{\bar{g}}^*}\; R(G) \;.$$

Any other H-basis is of the form $\bar{g}\omega$ for a unique $\omega \in \Sigma_m \wr H$. We have $\Psi_{\bar{g}\omega} = c_\omega \circ \Psi_{\bar{g}}$ as maps $G \longrightarrow \Sigma_m \wr H$, where $c_\omega(\gamma) = \omega^{-1}\gamma\omega$. Since inner automorphisms induce the identity in any Rep-functor, we have

$$\Psi_{\bar{g}}^* = \Psi_{\bar{g}\omega}^* \;:\; R(\Sigma_m \wr H) \;\longrightarrow\; R(G) \;.$$

So the norm N_H^G does not depend on the choice of basis $\bar{g}$.

The norms maps satisfy a number of important relations, by Proposition II.2.33 applied to the multiplicative monoid of the global power functor R. There relations – turned into multiplicative notation – are as follows.

(i) (Transitivity) We have $N_G^G = \mathrm{Id}_{R(G)}$ and for nested subgroups $H \subseteq G \subseteq F$ of finite index the relation

$$N_G^F \circ N_H^G \;=\; N_H^F$$

holds as maps $R(H) \longrightarrow R(F)$.

(ii) (Multiplicative double coset formula) For every subgroup K of G (not necessarily of finite index) the relation

$$\mathrm{res}_K^G \circ N_H^G \;=\; \prod_{[g] \in K\backslash G/H} N_{K \cap {}^g H}^K \circ g_\star \circ \mathrm{res}_{K^g \cap H}^H$$

holds as maps $R(H) \longrightarrow R(K)$. Here $[g]$ runs over a set of representatives of the finite set of K-H-double cosets.

(iii) (Epimorphic restriction) For every continuous epimorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation
$$\alpha^* \circ N_H^G \;=\; N_L^K \circ (\alpha|_L)^*$$
holds as maps from $R(H) \longrightarrow R(K)$, where $L = \alpha^{-1}(H)$.

(iv) For every $m \geq 1$ the power m-th power operation can be recovered as
$$P^m \;=\; N_K^{\Sigma_m \wr G} \circ q^* \;,$$
where K is the subgroup of $\Sigma_m \wr G$ consisting of all $(\sigma;\, g_1, \dots, g_m)$ such that $\sigma(m) = m$ and $q : K \longrightarrow G$ is defined by $q(\sigma;\, g_1, \dots, g_m) = g_m$.

In particular, the power operations define the norm maps, but they can also be reconstructed from the norm maps. So the information in a global power functor could be packaged in an equivalent but different way using norm maps instead of power operations. The algebraic structure that arises then is the global analog of a *TNR-functor* in the sense of Tambara [**158**], nowadays also called a *Tambara functor*; here the acronym stands form 'Transfer, Norm and Restriction'. This observation can be stated as an equivalence of categories between global power functors and a certain category of 'global Tambara functors' or 'global TNR functors'; we shall not pursue this further. Our reason for favoring power operations over norm maps is that they satisfy explicit and intuitive formulas with respect to the rest of the structure (restriction, transfer, sum, product,...). The norm maps also satisfy universal formulas when applied to sums and transfers, but the author finds them harder to describe and to remember.

For a fixed finite group G, Brun [**31**, Sec. 7.2] has constructed norm maps on the 0-th equivariant homotopy group Mackey functor of every commutative orthogonal G-ring spectrum, and he showed that this structure is a TNR-functor. So when restricted to finite groups, the global power functor structure on $\underline{\pi}_0(R)$ for an ultra-commutative ring spectrum, obtained in the following Theorem 1.9, could also be deduced by using Brun's TNR-structure for the underlying orthogonal G-ring spectrum R_G for every compact Lie group G, and then turning the norm maps into power operations as explained in Construction II.2.32. However, Brun's construction is rather indirect and this would hide the simple and explicit nature of the power operations.

In order to show that the 0-th equivariant homotopy group functor of an ultra-commutative ring spectrum satisfies the transfer axiom (viii) of a global power functor, we study the interplay between power operations, the Wirthmüller isomorphism and the degree shifting transfer. To state the results we first have to generalize power operations from equivariant homotopy groups to the equivariant homology theories.

We let R be an orthogonal spectrum, G a compact Lie group and A a based G-space. We define the G-equivariant R-homology group of A as the group
$$R_0^G(A) \;=\; \pi_0^G(R \wedge A) \;=\; \operatorname{colim}_{V \in s(\mathcal{U}_G)} [S^V, R(V) \wedge A]^G \;.$$
Every continuous group homomorphism $\alpha : K \longrightarrow G$ induces a restriction homomorphism
$$\alpha^* \;:\; R_0^G(A) \;\longrightarrow\; R_0^K(\alpha^*(A))$$
that generalizes the restriction homomorphism $\alpha^* : \pi_0^G(R) \longrightarrow \pi_0^K(R)$. Again α^* is defined by applying restriction of scalars to any representative of a given equivariant homology class.

Now we let R be an orthogonal ring spectrum (not necessarily commutative). Then the equivariant homology theories represented by R inherit multiplications in the form of bilinear maps
$$\times \;:\; R_0^G(A) \times R_0^G(B) \;\longrightarrow\; R_0^G(A \wedge B) \;.$$
We define this pairing simply as the composite
$$\begin{aligned} R_0^G(A) \times R_0^G(B) \;=\; \pi_0^G(R \wedge A) \times \pi_0^G(R \wedge B) \;&\xrightarrow{\times}\; \pi_0^G((R \wedge A) \wedge (R \wedge B)) \\ &\xrightarrow{\mu_*}\; \pi_0^G(R \wedge A \wedge B) \;=\; R_0^G(A \wedge B) \;, \end{aligned}$$

where the first map is the homotopy group pairing of Construction III.5.3 and $\mu : (R \wedge A) \wedge (R \wedge B) \longrightarrow R \wedge A \wedge B$ stems from the multiplication of R. For $A = B = S^0$ this construction reduces to the pairings of equivariant homotopy groups (defined in (5.7) of Chapter IV).

Now we suppose that the ring spectrum R is ultra-commutative. Given a based G-space A, we write $A^{(m)} = A^{\wedge m}$ for its m-fold smash power, which is naturally a based $(\Sigma_m \wr G)$-space. Then we define the m-th power operation

$$P^m \ : \ R_0^G(A) \ \longrightarrow \ R_0^{\Sigma_m \wr G}(A^{(m)})$$

by the obvious generalization of (1.1): the operation P^m takes the class represented by a based G-map $f : S^V \longrightarrow R(V) \wedge A$, for some G-representation V, to the class of the $(\Sigma_m \wr G)$-map

$$S^{V^m} = (S^V)^{(m)} \ \xrightarrow{f^{(m)}} \ (R(V) \wedge A)^{(m)} \ \xrightarrow{\text{shuffle}} \ R(V)^{(m)} \wedge A^{(m)} \ \xrightarrow{\mu_{V,\dots,V} \wedge A^{(m)}} R(V^m) \wedge A^{(m)} \ ,$$

where $\mu_{V,\dots,V}$ is the iterated multiplication map of R. We omit the straightforward verification that the power operations in equivariant R-homology are compatible with restriction maps: for every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups and every based G-space A, the relation

$$P^m \circ \alpha^* \ = \ (\Sigma_m \wr \alpha)^* \circ P^m$$

holds as maps from $R_0^G(A)$ to $R_0^{\Sigma_m \wr K}(\alpha^*(A)^{(m)})$, exploiting that $(\Sigma_m \wr \alpha)^*(A^{(m)}) = \alpha^*(A)^{(m)}$ as $(\Sigma_m \wr K)$-spaces.

The following proposition makes precise in which way the power operations in equivariant R-homology are compatible with the Wirthmüller isomorphism of Theorem III.2.14. To give the precise statement we have to introduce additional notation. We let H be a closed subgroup of a compact Lie group G. As before we let

$$L \ = \ T_{eH}(G/H)$$

denote the tangent H-representation, the tangent space of G/H at the distinguished coset eH. We write

$$\gamma \ : \ (G/H)^m \ \cong \ (\Sigma_m \wr G)/(\Sigma_m \wr H) \ , \quad (g_1 H, \dots, g_m H) \ \longmapsto \ (1; g_1, \dots, g_m) \cdot (\Sigma_m \wr H)$$

for the distinguished $(\Sigma_m \wr G)$-equivariant diffeomorphism. The differential of γ at $(H, \dots, H)$ is a $(\Sigma_m \wr H)$-equivariant linear isometry

$$(D\gamma)_{(H,\dots,H)} \ : \ L^m \ \cong \ T_{\Sigma_m \wr H}\big((\Sigma_m \wr G)/(\Sigma_m \wr H)\big) \ .$$

In the next proposition and its corollaries, we will use this equivariant isometry to identify L^m with the tangent representation of $\Sigma_m \wr H$ inside $\Sigma_m \wr G$.

Proposition 1.7. *Let R be an ultra-commutative ring spectrum and H a closed subgroup of a compact Lie group G. Then the following diagram commutes*

$$\begin{tikzcd}
\pi_0^G(R \wedge G/H_+) \arrow[r, "\mathrm{Wirth}_H^G", "\cong"'] \arrow[d, "P^m"'] & \pi_0^H(R \wedge S^L) \arrow[d, "P^m"] \\
\pi_0^{\Sigma_m \wr G}(R \wedge (G/H)_+^m) \arrow[d, "(R \wedge \gamma)_*"', "\cong"] & \pi_0^{\Sigma_m \wr H}(R \wedge (S^L)^{\wedge m}) \arrow[d, "\cong"] \\
\pi_0^{\Sigma_m \wr G}(R \wedge (\Sigma_m \wr G)/(\Sigma_m \wr H)_+) \arrow[r, "\cong", "\mathrm{Wirth}_{\Sigma_m \wr H}^{\Sigma_m \wr G}"'] & \pi_0^{\Sigma_m \wr H}(R \wedge S^{L^m})
\end{tikzcd}$$

where the horizontal maps are the respective Wirthmüller isomorphisms.

PROOF. We choose a slice as in the definition of the Wirthmüller map in Construction III.2.2, i.e., a smooth embedding $s : D(L) \longrightarrow G$ that satisfies the relations

$$s(0) = 1 \ , \quad s(h \cdot l) = h \cdot s(l) \cdot h^{-1} \qquad \text{and} \qquad s(-l) = s(l)^{-1}$$

for all $(l, h) \in D(L) \times H$, and such that the differential at 0 of the composite

$$D(L) \ \xrightarrow{s} \ G \ \xrightarrow{\text{proj}} \ G/H$$

is the identity. The collapse map

$$\lambda_H^G \ = \ l_H^G/H \ : \ G/H \ \longrightarrow \ S^L$$

is then given by the formula

$$\lambda_H^G(gH) \ = \ \begin{cases} l/(1-|l|) & \text{if } g = s(l) \cdot h \text{ with } (l,h) \in D(L) \times H, \text{ and} \\ \quad * & \text{if } g \text{ is not of this form.} \end{cases}$$

We define a slice for the pair $(\Sigma_m \wr G, \Sigma_m \wr H)$ from the slice s for the pair (G, H), namely as the smooth embedding

$$\bar{s} \ : \ D(L^m) \ \longrightarrow \ \Sigma_m \wr G \ , \quad \bar{s}(l_1, \dots, l_m) \ = \ (1;\, s(l_1), \dots, s(l_m)) \ .$$

Clearly, $\bar{s}(0, \dots, 0)$ is the multiplicative unit,

$$\bar{s}(-l_1, \dots, -l_m) \ = \ (1;\, s(-l_1), \dots, s(-l_m)) \ = \ (1;\, s(l_1)^{-1}, \dots, s(l_m)^{-1}) \ = \ \bar{s}(-l_1, \dots, -l_m)^{-1} \ ,$$

and

$$\begin{aligned} \bar{s}((\sigma^{-1};\, h_1, \dots, h_m) \cdot (l_1, \dots, l_m)) \ &= \ \bar{s}(h_{\sigma(1)} l_{\sigma(1)}, \dots, h_{\sigma(m)} l_{\sigma(m)}) \\ &= \ (1;\, s(h_{\sigma(1)} l_{\sigma(1)}), \dots, s(h_{\sigma(m)} l_{\sigma(m)})) \\ &= \ (1;\, h_{\sigma(1)} s(l_{\sigma(1)}) h_{\sigma(1)}^{-1}, \dots, h_{\sigma(m)} s(l_{\sigma(m)}) h_{\sigma(m)}^{-1}) \\ &= \ (\sigma^{-1};\, h_1, \dots, h_m) \cdot (1;\, s(l_1), \dots, s(l_m)) \ (\sigma;\, h_{\sigma(1)}^{-1}, \dots, h_{\sigma(m)}^{-1}) \\ &= \ (\sigma^{-1};\, h_1, \dots, h_m) \cdot \bar{s}(l_1, \dots, l_m) \ (\sigma^{-1};\, h_1, \dots, h_m)^{-1} \ , \end{aligned}$$

for all $(l_1, \dots, l_m) \in D(L^m)$ and all $(\sigma^{-1};\, h_1, \dots, h_m) \in \Sigma_m \wr H$. Finally, the differential of the composite

$$D(L^m) \ \xrightarrow{\bar{s}} \ \Sigma_m \wr G \ \xrightarrow{\text{proj}} (\Sigma_m \wr G)/(\Sigma_m \wr H) \ \xrightarrow{\gamma^{-1}} \ (G/H)^m$$

is the identity, so we have indeed defined a slice. We let

$$\lambda_{\Sigma_m \wr H}^{\Sigma_m \wr G} \ : \ (\Sigma_m \wr G)/(\Sigma_m \wr H) \ \longrightarrow \ S^{L^m}$$

denote the collapse map based on the slice $\bar{s}$. The composite

$$(G/H)^m \ \xrightarrow{\gamma} \ (\Sigma_m \wr G)/(\Sigma_m \wr H) \ \xrightarrow{\lambda_{\Sigma_m \wr H}^{\Sigma_m \wr G}} \ S^{L^m}$$

sends a point of the form $(s(l_1)H, \dots, s(l_m)H)$ with $(l_1, \dots, l_m) \in D(L^m)$ to the point

$$\frac{(l_1, \dots, l_m)}{1 - \sqrt{|l_1|^2 + \cdots + |l_m|^2}} \ ;$$

all other points of $(G/H)^m$ are sent to the basepoint at infinity. A scaling homotopy thus witnesses that the following diagram that commutes up to $(\Sigma_m \wr H)$-equivariant homotopy:

$$\begin{tikzcd} (G/H)^m \arrow[r, "(\lambda_H^G)^m"] \arrow[d, "\gamma"', "\cong"] & (S^L)^m \arrow[d, "\cong"] \\ (\Sigma_m \wr G)/(\Sigma_m \wr H) \arrow[r, "\lambda_{\Sigma_m \wr H}^{\Sigma_m \wr G}"'] & S^{L^m} \end{tikzcd}$$

Now we contemplate the diagram:

$$\begin{tikzcd}
\pi_0^G(R\wedge G/H_+) \arrow[r, "\mathrm{res}^G_H"] \arrow[d, "P^m"'] & \pi_0^H(R\wedge G/H_+) \arrow[r, "(R\wedge\lambda^G_H)_*"] \arrow[d, "P^m"'] & \pi_0^H(R\wedge S^L) \arrow[d, "P^m"] \\
\pi_0^{\Sigma_m\wr G}(R\wedge (G/H)^m_+) \arrow[r, "\mathrm{res}^{\Sigma_m\wr G}_{\Sigma_m\wr H}"] \arrow[d, "(R\wedge\gamma)_*"', "\cong"] & \pi_0^{\Sigma_m\wr H}(R\wedge (G/H)^m_+) \arrow[r, "((\lambda^G_H)^m)_*"] \arrow[d, "(R\wedge\gamma)_*"', "\cong"] & \pi_0^{\Sigma_m\wr H}(R\wedge (S^L)^m) \arrow[d, "\cong"] \\
\pi_0^{\Sigma_m\wr G}(R\wedge (\Sigma_m\wr G)/(\Sigma_m\wr H)_+) \arrow[r, "\mathrm{res}^{\Sigma_m\wr G}_{\Sigma_m\wr H}"'] & \pi_0^{\Sigma_m\wr H}(R\wedge (\Sigma_m\wr G)/(\Sigma_m\wr H)_+) \arrow[r, "(R\wedge\lambda^{\Sigma_m\wr G}_{\Sigma_m\wr H})_*"'] & \pi_0^{\Sigma_m\wr H}(R\wedge S^{L^m})
\end{tikzcd}$$

The two squares on the left commute by compatibility of power operations with restriction and by naturality of restriction. The upper right square commutes by naturality of power operations. The lower right square commutes by the previous paragraph. Since the upper and lower horizontal composites are the respective Wirthmüller maps, this proves the proposition. □

A direct consequence of the previous proposition is that power operations are compatible with dimension shifting and degree zero transfer maps.

Corollary 1.8. *Let R be an ultra-commutative ring spectrum and H a closed subgroup of a compact Lie group G. Then the following two diagrams commute:*

$$\begin{tikzcd}
\pi_0^H(R\wedge S^L) \arrow[r, "\mathrm{Tr}^G_H"] \arrow[d, "P^m"'] & \pi_0^G(R) \arrow[dd, "P^m"] \\
\pi_0^{\Sigma_m\wr H}(R\wedge (S^L)^{\wedge m}) \arrow[d, "(R\wedge\gamma)_*"', "\cong"] & \\
\pi_0^{\Sigma_m\wr H}(R\wedge S^{\bar{L}}) \arrow[r, "\mathrm{Tr}^{\Sigma_m\wr G}_{\Sigma_m\wr H}"'] & \pi_0^{\Sigma_m\wr G}(R)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_0^H(R) \arrow[r, "\mathrm{tr}^G_H"] \arrow[d, "P^m"'] & \pi_0^G(R) \arrow[d, "P^m"] \\
\pi_0^{\Sigma_m\wr H}(R) \arrow[r, "\mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_m\wr H}"'] & \pi_0^{\Sigma_m\wr G}(R)
\end{tikzcd}$$

Proof. The dimension shifting transfer

$$\mathrm{Tr}^G_H \ : \ \pi_0^H(R\wedge S^L) \ \longrightarrow \ \pi_0^G(R)$$

is defined as the inverse of the Wirthmüller isomorphism $\pi_0^H(R\wedge S^L) \longrightarrow \pi_0^G(R\wedge G/H_+)$ and the effect of the unique G-map $p : G/H \longrightarrow *$. The first claim thus follows from Proposition 1.7 and the commutativity of the diagram

$$\begin{tikzcd}
\pi_0^G(R\wedge G/H_+) \arrow[r, "(R\wedge p_+)_*"] \arrow[d, "P^m"'] & \pi_0^G(R) \arrow[d, "P^m"] \\
\pi_0^{\Sigma_m\wr G}(R\wedge (G/H)^m_+) \arrow[r, "(R\wedge p^m_+)_*"'] \arrow[d, "(R\wedge\gamma)_*"', "\cong"] & \pi_0^{\Sigma_m\wr G}(R) \arrow[d, equal] \\
\pi_0^{\Sigma_m\wr G}(R\wedge (\Sigma_m\wr G)/(\Sigma_m\wr H)_+) \arrow[r, "(R\wedge\bar{p}_+)_*"'] & \pi_0^{\Sigma_m\wr G}(R)
\end{tikzcd}$$

The degree zero transfer is obtained from the dimension shifting transfer by precomposing with the effect of the map $S^0 \longrightarrow S^L$, the inclusion of the origin into the tangent representation. If we raise the inclusion

of the origin of S^L to the m-th power, the canonical homeomorphism $(S^L)^m \longrightarrow S^{L^m}$ identifies it with the inclusion of the origin of S^{L^m}. So the power operations are also compatible with degree zero transfers. $\square$

Much of the next result is contained, at least implicitly, in Greenlees' and May's construction of norm maps [**64**, Sec. 7-9], simply because an ultra-commutative ring spectrum is an example of a '$\mathcal{GI}_*$-FSP' in the sense of [**64**, Def. 5.5].

THEOREM 1.9. *Let R be an ultra-commutative ring spectrum. The power operations* (1.1) *make the global functor $\underline{\pi}_0(R)$ into a global power functor.*

PROOF. The properties (i) through (vi) only involve the multiplication, power operations and restriction maps, so they are special cases of Proposition II.2.15 for the ultra-commutative monoid $\Omega^\bullet R$. The transfer relation (viii) is proved in Corollary 1.8. The most involved argument remaining is required for the additivity formula identifying the behavior of power operations on sums.

We first show an external version of the additivity relation. We consider two based G-spaces A and B and equivariant homology classes $x \in R_0^G(A)$ and $y \in R_0^G(B)$. We let

$$x \oplus y \ \in \ R_0^G(A \vee B)$$

be the unique class such that

$$(p_A)_*(x \oplus y) \ = \ x \qquad \text{and} \qquad (p_B)_*(x \oplus y) \ = \ y \ ,$$

where $p_A : A \vee B \longrightarrow A$ and $p_B : A \vee B \longrightarrow B$ are the projections. We show the relation

$$P^m(x \oplus y) \ = \ \sum_{i=0}^{m} \psi^i_*((\Sigma_m \wr G) \ltimes_{\Sigma_{i,m-i} \wr G} (P^i(x) \times P^{m-i}(y))) \tag{1.10}$$

in the group $R_0^{\Sigma_m \wr G}((A \vee B)^{(m)})$, where

$$\psi^i \ : \ (\Sigma_m \wr G) \ltimes_{\Sigma_{i,m-i} \wr G} (A^{(i)} \wedge B^{\wedge(m-i)}) \ \longrightarrow \ (A \vee B)^{(m)}$$

is the $\Sigma_m \wr G$-equivariant extension of the map

$$\mathrm{incl}_A^{(i)} \wedge \mathrm{incl}_B^{(m-i)} \ : \ A^{(i)} \wedge B^{(m-i)} \ \longrightarrow \ (A \vee B)^{(m)} \ .$$

The space $(A \vee B)^{(m)}$ decomposes $(\Sigma_m \wr G)$-equivariantly as the wedge of the images of the maps $\psi^0, \dots, \psi^m$; since R_0^G is additive on wedges, it suffices to show the relation after projection to each of the wedge summands. So we let

$$\bar{\psi}^i \ : \ (A \vee B)^{(m)} \ \longrightarrow \ (\Sigma_m \wr G) \ltimes_{\Sigma_{i,m-i} \wr G} (A^{(i)} \wedge B^{\wedge(m-i)})$$

be the right inverse to ψ^i that sends the other wedge summands to the basepoint, i.e., so that $\bar{\psi}^i \circ \psi^j$ is constant for $j \neq i$. The relation (1.10) thus follows if we can show

$$\bar{\psi}^i_*(P^m(x \oplus y)) \ = \ (\Sigma_m \wr G) \ltimes_{\Sigma_{i,m-i} \wr G} (P^i(x) \times P^{m-i}(y)) \tag{1.11}$$

in the group

$$R_0^{\Sigma_m \wr G}((\Sigma_m \wr G) \ltimes_{\Sigma_{i,m-i} \wr G} (A^{(i)} \wedge B^{\wedge(m-i)}))$$

for all $0 \leq i \leq m$. Now it suffices, in turn, to show (1.11) after applying the Wirthmüller isomorphism, i.e., the composite

$$\begin{aligned} R_0^{\Sigma_m \wr G}((\Sigma_m \wr G) \ltimes_{\Sigma_{i,m-i} \wr G} (A^{(i)} \wedge B^{(m-i)})) \ &\xrightarrow{\mathrm{res}^{\Sigma_m \wr G}_{\Sigma_{i,m-i} \wr G}} R_0^{\Sigma_{i,m-i} \wr G}((\Sigma_m \wr G) \ltimes_{\Sigma_{i,m-i} \wr G} (A^{(i)} \wedge B^{(m-i)})) \\ &\xrightarrow{(l_{A^{(i)} \wedge B^{\wedge(m-i)}})_*} R_0^{\Sigma_{i,m-i} \wr G}(A^{(i)} \wedge B^{(m-i)}) \ . \end{aligned}$$

Here

$$l_{A^{(i)} \wedge B^{(m-i)}} \ : \ (\Sigma_m \wr G) \ltimes_{\Sigma_{i,m-i} \wr G} (A^{(i)} \wedge B^{(m-i)}) \ \longrightarrow \ A^{(i)} \wedge B^{(m-i)}$$

is the projection to the summand indexed by the preferred coset. We obtain

$$\begin{aligned}(l_{A^{(i)}\wedge B^{(m-i)}})_*(\mathrm{res}^{\Sigma_m\wr G}_{\Sigma_{i,m-i}\wr G}(\bar\psi^i_*(P^m(x\oplus y)))) &= (l_{A^{(i)}\wedge B^{(m-i)}}\circ\bar\psi^i)_*(\mathrm{res}^{\Sigma_m\wr G}_{\Sigma_{i,m-i}\wr G}(P^m(x\oplus y)))\\ &= (p_A^{(i)}\wedge p_B^{(m-i)})_*(P^i(x\oplus y)\times P^{m-i}(x\oplus y))\\ &= P^i((p_A)_*(x\oplus y))\times P^{m-i}((p_B)_*(x\oplus y))\\ &= P^i(x)\times P^{m-i}(y)\ .\end{aligned}$$

Since the Wirthmüller isomorphism is inverse to the map $(\Sigma_m\wr G)\ltimes_{\Sigma_{i,m-i}\wr G}-$ (compare Theorem III.2.14), this proves (1.11), and hence (1.10).

Now we obtain the additivity relation by specializing (1.10) to $A=B=S^0$ and using naturality for the fold map $\nabla: S^0\vee S^0\longrightarrow S^0$. Then

$$\begin{aligned}P^m(x+y) &= P^m(\nabla_*(x\oplus y)) = \nabla_*^{(m)}(P^m(x\oplus y))\\ &\overset{(1.10)}{=} \sum_{i=0}^m(\nabla^{(m)}\circ\psi^i)_*((\Sigma_m\wr G)\ltimes_{\Sigma_{i,m-i}\wr G}(P^i(x)\times P^{m-i}(y)))\\ &= \sum_{i=0}^m \mathrm{tr}^{\Sigma_m\wr G}_{\Sigma_{i,m-i}\wr G}(P^i(x)\times P^{m-i}(y))\ .\end{aligned}$$

The third equation uses that the following square commutes

$$\begin{tikzcd}(\Sigma_m\wr G)\ltimes_{\Sigma_{i,m-i}\wr G}((S^0)^{(i)}\wedge(S^0)^{(m-i)}) \arrow[r, "\nabla^{(m)}\circ\psi^i"] \arrow[d, "\cong"'] & (S^0)^{(m)} \arrow[d, "\cong"] \\ (\Sigma_m\wr G)/(\Sigma_{i,m-i}\wr G)_+ \arrow[r, "p_+"'] & S^0\end{tikzcd}$$

where the vertical maps are the canonical homeomorphisms. □

Remark 1.12 (Relation to classical power operations). We still owe the explanation how power operations for ultra-commutative ring spectra refine the classical power operations defined in the (non-equivariant) cohomology theory represented by an H_∞-ring spectrum. We recall from [**34**, I.§4] that an H_∞-structure, is an algebra structure over the monad

$$L\mathbb{P}\ :\ \mathcal{SH}\ \longrightarrow\ \mathcal{SH}$$

on the stable homotopy category that can be obtained be suitably deriving the 'symmetric algebra' monad

$$\mathbb{P}\ :\ \mathcal{S}p\ \longrightarrow\ \mathcal{S}p$$

on the category of orthogonal spectra (whose algebras are commutative orthogonal ring spectra). This is not the full truth, because Bruner, May, McClure and Steinberger use a different model for the stable homotopy category, so strictly speaking one would have to translate the relevant parts of [**34**] to the context of orthogonal spectra. If we did that, the derived functor of the m-symmetric power of orthogonal spectra would be modeled by the m-th *extended power*

$$D_mX\ =\ (E\Sigma_m)_+\wedge_{\Sigma_m}X^{\wedge m}\ .$$

Specifying an H_∞-structure on an orthogonal spectrum E thus amounts to specifying morphisms, in the non-equivariant stable homotopy category,

$$\mu_m\ :\ D_mE\ \longrightarrow\ E$$

from the m-th extended power to E; the algebra structure over the monad $L\mathbb{P}$ then translates into a specific collection of relations among the morphisms μ_m that are spelled out in [**34**, Ch. I Def. 3.1].

For every space A, the H_∞-structure gives rise to power operations

$$\mathcal{P}_m \ : \ E^0(X) \ \longrightarrow \ E^0(B\Sigma_m \times X) \tag{1.13}$$

in E-cohomology defined in [**34**, Ch. I Def. 4.1] as the following composite:

$$\begin{aligned} E^0(A) \ = \ [\Sigma^\infty_+ A, E] \ \xrightarrow{D_m} [D_m(\Sigma^\infty_+ A), D_m E] \ &\xrightarrow{[D_m(\Sigma^\infty_+ A), \mu_n]} \ [D_m(\Sigma^\infty_+ A), E] \\ \cong \quad [\Sigma^\infty_+(D_m A), E] \ = \ E^0(D_m A) \ &\xrightarrow{E^0(E\Sigma_m \times_{\Sigma_m} \Delta)} \ E^0(B\Sigma_m \times A) \ . \end{aligned}$$

Here $[-,-]$ denotes the group of morphisms in the stable homotopy category $\mathcal{SH}$, $D_m A = E\Sigma_m \times_{\Sigma_m} A^m$ is the space level extended power, and $\Delta : A \longrightarrow A^m$ is the diagonal. Depending on the context, the power operations (1.13) are often processed further; in favorable cases, $E^0(B\Sigma_m \times A)$ can sometimes be explicitly described as a functor of $E^0(A)$, and the power operations can be translated into a specific kind of algebraic structure.

Now we let R be an ultra-commutative ring spectrum. Then the underlying H_∞-structure is given by the composite morphism

$$D_m R \ = \ (E\Sigma_m)_+ \wedge_{\Sigma_m} R^{\wedge m} \ \longrightarrow \Sigma_m \backslash R^{\wedge m} \ = \ \mathbb{P}^m R \ \xrightarrow{\text{mult}} \ R$$

where the first morphism collapses $E\Sigma_m$ to a point and the second map is induced by the iterated multiplication $R^{\wedge m} \longrightarrow R$. The definition (1.1) of the power operations on equivariant homotopy groups directly extends to power operations

$$P^m \ : \ R^0_G(A) \ \longrightarrow \ R^0_{\Sigma_m \wr G}(A^m) \ ;$$

here A is any G-space and A^m is considered as a $(\Sigma_m \wr G)$-space via the action

$$(\sigma; g_1, \dots, g_i) \cdot (a_1 \dots, a_m) \ = \ (g_{\sigma^{-1}(1)} a_{\sigma^{-1}(1)} \dots, g_{\sigma^{-1}(m)} a_{\sigma^{-1}(m)}) \ .$$

The operation P^m takes the class represented by a based G-map $f : S^V \wedge A_+ \longrightarrow R(V)$, for some G-representation V, to the class of the $(\Sigma_m \wr G)$-map

$$S^{V^m} \wedge A^m_+ \cong (S^V \wedge A_+)^{\wedge m} \ \xrightarrow{f^{\wedge m}} \ R(V)^{\wedge m} \ \xrightarrow{\mu_{V,\dots,V}} R(V^m) \ ,$$

where $\mu_{V,\dots,V}$ is the iterated, $(\Sigma_m \wr G)$-equivariant multiplication map of R. When A is a one-point G-space, then $R^0_G(A)$ is isomorphic to $\pi^G_0(R)$, and the power operation just defined reduces to the operation (1.1). We refrain from spelling out the naturality properties of these extended operations.

A forgetful homomorphism

$$R^0_G(A) \ \longrightarrow \ R^0(EG \times_G A)$$

is defined as the composite

$$R^0_G(A) \ = \ [\![\Sigma^\infty_+ \mathbf{L}_{G,V} A, R]\!] \ \xrightarrow{U} \ [U(\Sigma^\infty_+ \mathbf{L}_{G,V} A), R] \ = \ [\Sigma^\infty_+(EG \times_G A), R] \ = \ R^0(EG \times_G A) \ ,$$

where $U : \mathcal{GH} \longrightarrow \mathcal{SH}$ is the forgetful functor from the global stable homotopy category to the non-equivariant stable homotopy category. Then the following diagram commutes:

$$\begin{tikzcd} R^0(A) \arrow[r, "D_m"] \arrow[d, "P^m"'] & {[D_m(\Sigma^\infty_+ A), D_m R]} \arrow[d, "(\mu_m)_*"'] \arrow[dd, bend left, "\mathcal{P}_m"] \\ R^0_{\Sigma_m}(A^m) \arrow[r, "U"] \arrow[d, "R^0_{\Sigma_m}(\Delta)"'] & R^0(E\Sigma_m \times_{\Sigma_m} A) \arrow[d, "R^0(E\Sigma_m \times_{\Sigma_m} \Delta)"'] \\ R^0_{\Sigma_m}(A) \arrow[r, "U"'] & R^0(B\Sigma_m \times A) \end{tikzcd}$$

This makes precise in which way power operations for ultra-commutative ring spectra refine power operations for H_∞-ring spectra.

Remark 1.14 (G_∞-ring spectra). As we recalled in the previous remark, non-equivariant power operations on a ring valued cohomology theory already arise from a weaker structure than a commutative (or equivalently E_∞-multiplication): all that is needed is an H_∞-structure, compare [**34**, I.§4]. This suggests a global analog of an H_∞-structure that, for lack of better name, we call a G_∞*-structure*. We only briefly sketch the ingredients to the definition. The 'symmetric algebra' monad $\mathbb{P}$ takes global equivalences between flat orthogonal spectra to global equivalences between flat orthogonal spectra, by Theorem VI.1.3 below and Ken Brown's lemma. So the monad $\mathbb{P}$ can also be derived with respect to global equivalences. The result is a monad

$$\mathbb{G} \;=\; L_{\mathrm{gl}}\mathbb{P} \;:\; \mathcal{GH} \;\longrightarrow\; \mathcal{GH}$$

on the global stable homotopy category, whose algebras we call G_∞-ring spectra.

The underlying non-equivariant homotopy type of a G_∞-ring spectrum comes with an H_∞-structure, so we arrive at a square of forgetful functors between categories of structured ring spectra with different degrees of commutativity:

$$\begin{tikzcd}
\mathrm{Ho}(\text{ultra-commutative ring spectra}) \arrow[r] \arrow[d] & (G_\infty\text{-ring spectra}) \arrow[d] \\
\mathrm{Ho}(\text{commutative ring spectra}) \arrow[r] & (H_\infty\text{-ring spectra})
\end{tikzcd}$$

We emphasize that, like H_∞-ring spectra, the category of G_∞-ring spectra is not the homotopy category of any natural model category.

Example 1.15 (Units of an ultra-commutative ring spectrum). In Example II.2.17 we defined the naive units of an orthogonal monoid space. When R is an orthogonal ring spectrum, then the naive units of the multiplicative orthogonal monoid space $\Omega^\bullet R$ satisfy

$$\pi_0^G((\Omega^\bullet R)^{n\times}) \;=\; \{x \in \pi_0^G(R) \mid \mathrm{res}_e^G(x) \text{ is a unit in } \pi_0^e(R)\}\ ,$$

the multiplicative submonoid of $\pi_0^G(R)$ of elements that become invertible when restricted to the trivial group. One should beware that these naive units may contain non-invertible elements, i.e., the orthogonal monoid space $(\Omega^\bullet R)^{n\times}$ may not be group-like. The author does not know of a construction of 'true global units' for general orthogonal ring spectra, i.e., without the ultra-commutativity hypothesis.

When the ring spectrum R is ultra-commutative, then there is a more refined construction

$$GL_1(R) \;=\; (\Omega^\bullet R)^\times\ ,$$

the *global units* of R. Indeed, if R is ultra-commutative, then $\Omega^\bullet R$ is an ultra-commutative monoid, so we can form the 'true' global units, the homotopy fiber of the multiplication morphism, see Construction II.5.19. Then $GL_1(R)$ is a group-like ultra-commutative monoid and for every compact Lie group G,

$$\pi_0^G(GL_1(R)) \;=\; (\pi_0^G(R))^\times\ ,$$

the multiplicative submonoid of units of the commutative ring $\pi_0^G(R)$. Moreover, the power operations in $\underline{\pi}_0(R)$ correspond to the power operations in $\underline{\pi}_0(GL_1(R))$.

Remark 1.16. In the non-equivariant context, $GL_1(R)$ is an infinite loop space, i.e., weakly equivalent to the 0-th space of an Ω-spectrum of units. This fact has a global generalization as follows. As we hope to explain elsewhere, every ultra-commutative monoid M has a *global delooping* $\mathbf{B}M$, an orthogonal spectrum that is a $\mathcal{F}in$-global Ω-spectrum. It also comes with a natural morphism of orthogonal spaces

$M \longrightarrow \Omega^\bullet(\mathbf{B}M)$ that is a $\mathcal{F}in$-global equivalence whenever M is group-like. Since $GL_1(R)$ is an ultra-commutative monoid, it has a global delooping

$$gl_1(R) \;=\; \mathbf{B}(GL_1(R))\;.$$

Since the ultra-commutative monoid $GL_1(R)$ is group-like, the morphism of orthogonal spaces

$$\xi \;:\; GL_1(R) \;\longrightarrow\; \Omega^\bullet(\mathbf{B}(GL_1(R))) \;=\; \Omega^\bullet(gl_1(R))$$

is a $\mathcal{F}in$-global equivalence. For every compact Lie group G, this induces a map

$$(\pi_0^G(R))^\times \;\cong\; \pi_0^G(GL_1(R)) \;\cong\; \pi_0^G((GL_1(R))^c) \;\xrightarrow{\underline{\pi}_0(\xi)}\; \pi_0^G(\Omega^\bullet(gl_1(R))) \;=\; \pi_0^G(gl_1(R))\;.$$

These maps are compatible with restriction along continuous homomorphisms and they are bijective whenever G is finite. Moreover, the maps take the multiplication respectively norm operations in $(\underline{\pi}_0(R))^\times$ to addition respectively finite index transfers in $\underline{\pi}_0(gl_1(R))$.

Remark 1.17 (Picard groups). The global units $GL_1(R)$ of an ultra-commutative ring spectrum R ought to have an interesting delooping $\mathrm{pic}(R)$ that records the information about invertible modules over the equivariant ring spectra underlying R. At present I have no construction of this delooping as an ultra-commutative monoid, but I describe the evidence for expecting its existence.

For every compact Lie group G the underlying orthogonal G-ring spectrum R_G of R has a symmetric monoidal model category of modules, i.e., orthogonal G-spectra with an action by R (where G acts trivially on R). The equivalences we consider here are R-linear morphisms that are $\underline{\pi}_*$-isomorphisms of underlying orthogonal G-spectra; the construction of such a symmetric monoidal model category can be found in [**105**, III Thm. 7.6]. We let

$$\mathrm{Pic}(R)(G) \;=\; \mathrm{Pic}(\mathrm{Ho}(R_G\text{-mod}))$$

be the resulting Picard group, i.e., the set of isomorphism classes, in the homotopy category of R_G-modules, of objects that are invertible under the derived smash product. For a continuous group homomorphism $\alpha : K \longrightarrow G$ the restriction functor $\alpha^* : R_G\text{-mod} \longrightarrow R_K\text{-mod}$ derives to a strong symmetric monoidal functor

$$R\alpha^* \;:\; \mathrm{Ho}(R_G\text{-mod}) \;\longrightarrow\; \mathrm{Ho}(R_K\text{-mod})\;.$$

So $R\alpha^*$ preserves invertibility and induces a group homomorphism

$$\alpha^* \;:\; \mathrm{Pic}(R)(G) \longrightarrow\; \mathrm{Pic}(R)(K)\;.$$

For a second homomorphism $\beta : L \longrightarrow K$ the functors $(R\beta^*)\circ(R\alpha^*)$ and $R(\alpha\circ\beta)^*$ are naturally isomorphic. Moreover, for every element $g \in G$ the restriction functor $Rg_\star$ is naturally isomorphic to the identity functor of $\mathrm{Ho}(R_G\text{-mod})$, via left multiplication by g. So the assignment $G \mapsto \mathrm{Pic}(R)(G)$ becomes a functor

$$\mathrm{Pic}(R) \;:\; \mathrm{Rep}^{\mathrm{op}} \;\longrightarrow\; \mathcal{A}b\;.$$

But the ultra-commutativity gives more. For every finite index subgroup $H \le G$, the norm construction of Hill, Hopkins and Ravenel derives to a strong symmetric monoidal functor

$$N_H^G \;:\; \mathrm{Ho}(R_H\text{-mod}) \;\longrightarrow\; \mathrm{Ho}(R_G\text{-mod})\;,$$

compare [**76**, Prop. B.105]. So also N_H^G preserves invertibility and induces a group homomorphism

$$N_H^G \;:\; \mathrm{Pic}(R)(H) \longrightarrow\; \mathrm{Pic}(R)(G)\;.$$

These norm maps are transitive and they extend the abelian Rep-monoid to a global power monoid $\mathrm{Pic}(R)$.

We expect that there is a 'natural' ultra-commutative monoid $\mathrm{pic}(R)$ such that $\underline{\pi}_0(R) \cong \mathrm{Pic}(R)$ as global power monoids and such that $\Omega(\mathrm{pic}(R))$ is globally equivalent, as an ultra-commutative monoid, to $GL_1(R)$. The G-fixed points $(\mathrm{pic}(R))^G$ ought to have the homotopy type, as an E_∞-space, of the nerve of the category of invertible R_G-modules and $\underline{\pi}_*$-isomorphisms. Despite the strong evidence for its existence, we cannot presently construct $\mathrm{pic}(R)$ as an ultra-commutative monoid in our formalism.

2. Comonadic description of global power functors

In this section we show that the category of global power functors is both monadic and comonadic over the category of global Green functors. The situation is formally similar to the context of λ-rings, which are both monadic and comonadic over the category of commutative rings.

We have already considered exponential sequences in the context of global power monoids in Section II.2; we showed there that the functor of exponential sequences can be made into a comonad on the category of abelian Rep-monoids, and that the coalgebras over this comonad 'are' precisely the global power monoids. Now we revisit this idea in the context of global power functors, which have an 'additive' and a 'multiplicative' structure. Construction 2.1 introduces an addition and transfer maps on the exponential sequences of the multiplicative part of a global Green functor. Theorem 2.8 shows that with the additional structure, the collection exponential sequence of a global Green functor is another global Green functor. Theorem 2.10 extends the functor of exponential sequences to a comonad on the category of global Green functors, and it shows that the category of its coalgebras is equivalent to the category of global power functors.

A formal consequence of this fact is that the category of global power functors has all limits and colimits, and that they are created in the category of global Green functors. Proposition 2.14 discusses localization of global Green functors at a multiplicative subset of the underlying ring; while the structure of global Green functor always 'survives localization', this is not generally true for global power functors. Theorem 2.15 exhibits a necessary and sufficient condition so that a localization of a global power functor inherits power operations. For localization at a set of integers primes this condition is always satisfied (Example 2.16), so in particular global power functors can be rationalized. Proposition 2.18 shows that the category of global power functors is also monadic over the category of global Green functors.

We describe global power functors as the coalgebras over a certain comonad 'exp' on the category of global Green functors. When restricted to finite groups, most of the results about this comonad are contained in the PhD thesis of J. Singer [**146**], a former student of the author. Also for finite groups (as opposed to compact Lie groups), this comonadic description has independently been obtained by Ganter [**57**]. The exp comonad we use now is a 'lift' of the exp monad used for global power monoids in Construction II.2.40, i.e., the underlying multiplicative global power monoid of $\exp(R)$ only depends on the underlying multiplicative global power monoid of R, and the additive structure, including the transfers, go along free for the ride.

Construction 2.1. We let R be a global Green functor and G a compact Lie group. We recall from Construction II.2.40 that

$$\exp(R;G) \ \subset \ \prod\nolimits_{m\geq 0} R(\Sigma_m \wr G)$$

denotes the set of *exponential sequences*, i.e., of those families $(x_m)_m$ that satisfy $x_0 = 1$ in $R(\Sigma_0 \wr G) = R(e)$ and

$$\Phi^*_{k,m-k}(x_m) \ = \ x_k \times x_{m-k}$$

in $R((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G))$ for all $0 < k < m$, where

$$\Phi_{k,m-k} \ : \ (\Sigma_k \wr G) \ \times \ (\Sigma_{m-k} \wr G) \ \longrightarrow \ \Sigma_m \wr G$$

is the monomorphism defined in (1.3). Also in Construction II.2.40 we defined a multiplication on the set $\exp(R;G)$ by coordinatewise multiplication in $R(\Sigma_m \wr G)$, i.e.,

$$(x \cdot y)_m \ = \ x_m \cdot y_m \ ,$$

and restriction maps

$$\alpha^* \ : \ \exp(R;G) \ \longrightarrow \ \exp(R;K) \qquad \text{by} \qquad \big(\alpha^*(x)\big)_m \ = \ (\Sigma_m \wr \alpha)^*(x_m)$$

for every continuous group homomorphism $\alpha : K \longrightarrow G$. The restriction maps are multiplicative, unital and contravariantly functorial.

Now we also add an addition and transfers to the picture. We introduce another binary operation $\oplus$ on $\exp(R;G)$ by

$$(x \oplus y)_m \;=\; \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}(x_k \times y_{m-k}) \;, \tag{2.2}$$

where $x = (x_m)$, $y = (y_m)$ and $\mathrm{tr}_{k,m-k} : R((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G)) \longrightarrow R(\Sigma_m \wr G)$ is the transfer associated to the monomorphism $\Phi_{k,m-k}$. If H is a closed subgroup of G, then we define a transfer map tr_H^G : $\exp(R;H) \longrightarrow \exp(R;G)$ by

$$\big(\mathrm{tr}_H^G(x)\big)_m \;=\; \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}(x_m) \;. \tag{2.3}$$

We have to check that $\oplus$ and tr_H^G preserve exponential sequence, but that will be part of the following proposition.

Proposition 2.4. *Let R be a global Green functor and G a compact Lie group.*

(i) *The set $\exp(R;G)$ of exponential sequences is an abelian group under $\oplus$.*
(ii) *The addition $\oplus$ and the componentwise multiplication make $\exp(R;G)$ into a commutative ring.*
(iii) *For every continuous group homomorphism $\alpha : K \longrightarrow G$, the restriction map $\alpha^* : \exp(R;G) \longrightarrow \exp(R;K)$ is a ring homomorphism*
(iv) *For every closed subgroup H of G, the transfer is an additive map*

$$\mathrm{tr}_H^G \;:\; \exp(R;H) \;\longrightarrow\; \exp(R;G)$$

that satisfies reciprocity with respect to restriction from G to H.
(v) *For varying G, the ring structures, the restriction maps α^* and the transfer maps tr_H^G make $\exp(R) = \exp(R;-)$ into a global Green functor.*

Proof. This proof is quite lengthy, although not difficult and mostly a matter of appropriate book keeping.

(i) We start by showing that the sum, with respect to $\oplus$, of two exponential sequences is again exponential. The key step in this verification is an application of the double coset formula, for which we need to understand the $(\Sigma_i \times \Sigma_{m-i})$-$(\Sigma_k \times \Sigma_{m-k})$-double cosets inside Σ_m. We parametrize these double cosets by pairs (a,b) of natural numbers satisfying

$$0 \le a \le i \;, \quad 0 \le b \le m-i \qquad \text{and} \qquad a+b = k \;. \tag{2.5}$$

For each such pair we define a permutation $\chi(a,b) \in \Sigma_m$ by

$$\chi(a,b)(j) \;=\; \begin{cases} j & \text{for } 1 \le j \le a, \\ j-a+i & \text{for } a+1 \le j \le a+b, \\ j-b & \text{for } a+b+1 \le j \le i+b, \\ j & \text{for } i+b+1 \le j \le m. \end{cases}$$

In other words, $\chi(a,b)$ is the unique $(k,m-k)$-shuffle such that

$$\chi(a,b)(\{1,\dots,k\}) \;=\; \{1,\dots,a\} \;\cup\; \{i+1,\dots,i+b\} \;.$$

Then the permutations $\chi(a,b)$ form a set of double coset representatives for the subgroups $\Sigma_i \times \Sigma_{m-i}$ and $\Sigma_k \times \Sigma_{m-k}$ inside Σ_m, for all pairs (a,b) subject to (2.5).

When applying the double coset formula we will need the formulas

$$(\Sigma_i \times \Sigma_{m-i})^{\chi(a,b)} \;\cap\; (\Sigma_k \times \Sigma_{m-k}) \;=\; \Sigma_a \times \Sigma_b \times \Sigma_{i-a} \times \Sigma_{m-i-b}$$

and

$$(\Sigma_i \times \Sigma_{m-i}) \;\cap\; {}^{\chi(a,b)}(\Sigma_k \times \Sigma_{m-k}) \;=\; \Sigma_a \times \Sigma_{i-a} \times \Sigma_b \times \Sigma_{m-i-b} \;.$$

Thus the double coset formula becomes

$$\begin{aligned}\Phi^*_{i,m-i}(\mathrm{tr}_{k,m-k}(x\times y)) &= \sum_{a,b} \mathrm{tr}^{\Sigma_i\times\Sigma_{m-i}}_{(\Sigma_i\times\Sigma_{m-i})\cap\chi(a,b)(\Sigma_k\times\Sigma_{m-k})}\Big(\chi(a,b)_\star\Big(\mathrm{res}^{\Sigma_k\times\Sigma_{m-k}}_{(\Sigma_i\times\Sigma_{m-i})\chi(a,b)\cap(\Sigma_k\times\Sigma_{m-k})}(x\times y)\Big)\Big)\\ &= \sum_{a,b} \mathrm{tr}^{\Sigma_i\times\Sigma_{m-i}}_{\Sigma_a\times\Sigma_{i-a}\times\Sigma_b\times\Sigma_{m-i-b}}\Big(\chi(a,b)_\star\Big(\mathrm{res}^{\Sigma_k}_{\Sigma_a\times\Sigma_b}(x)\times\mathrm{res}^{\Sigma_{m-k}}_{\Sigma_{i-a,m-i-b}}(y)\Big)\Big) \qquad (2.6)\end{aligned}$$

The two sums run over all pairs (a,b) of natural numbers satisfying $0\le a\le i$, $0\le b\le m-i$ and $a+b=k$. Now we consider exponential sequences $x,y\in\exp(R;G)$ and calculate

$$\begin{aligned}\Phi^*_{i,m-i}((x\oplus y)_m) &= \sum_{k=0}^{m}\Phi^*_{i,m-i}(\mathrm{tr}_{k,m-k}(x_k\times y_{m-k}))\\ &\overset{(2.6)}{=} \sum_{a,b}\mathrm{tr}^{\Sigma_i\times\Sigma_{m-i}}_{\Sigma_i\times\Sigma_{i-a}\times\Sigma_b\times\Sigma_{m-i-b}}(\chi(a,b)_\star(x_a\times x_b\times y_{i-a}\times y_{m-i-b}))\\ &= \sum_{a,b}\mathrm{tr}_{a,i-a}(x_a\times y_{i-a})\times\mathrm{tr}_{b,m-i-b}(x_b\times y_{m-i-b}) \;=\; (x\oplus y)_i\times(x\oplus y)_{m-i}\ .\end{aligned}$$

Here the last two sums run over all pairs (a,b) of natural numbers satisfying $0\le a\le i$ and $0\le b\le m-i$. This shows that the sequence $x\oplus y$ is again exponential.

The following square of group monomorphisms commutes:

$$\begin{array}{ccc}(\Sigma_j\wr G)\times(\Sigma_k\wr G)\times(\Sigma_l\wr G) & \xrightarrow{(\Sigma_j\wr G)\times\Phi_{k,l}} & (\Sigma_j\wr G)\times(\Sigma_{k+l}\wr G)\\ \downarrow{\scriptstyle \Phi_{j,k}\times(\Sigma_l\wr G)} & & \downarrow{\scriptstyle \Phi_{j,k+l}}\\ (\Sigma_{j+k}\wr G)\times(\Sigma_l\wr G) & \xrightarrow[\Phi_{j+k,l}]{} & \Sigma_{j+k+l}\wr G\end{array}$$

So for all $x\in R(\Sigma_j\wr G)$, $y\in R(\Sigma_k\wr G)$, and $z\in R(\Sigma_l\wr G)$, the relation

$$\mathrm{tr}_{j,k+l}(x\times\mathrm{tr}_{k,l}(y\times z)) \;=\; \mathrm{tr}_{j+k,l}(\mathrm{tr}_{j,k}(x\times y)\times z)$$

holds in the group $R(\Sigma_{j+k+l}\wr G)$. By unraveling the definitions, this becomes the associativity of the operation $\oplus$.

Also, the following square of group monomorphisms commutes:

$$\begin{array}{ccc}(\Sigma_k\wr G)\times(\Sigma_l\wr G) & \xrightarrow{\Phi_{k,l}} & \Sigma_{k+l}\wr G\\ \downarrow{\scriptstyle \mathrm{twist}} & & \downarrow{\scriptstyle c_\chi}\\ (\Sigma_l\wr G)\times(\Sigma_k\wr G) & \xrightarrow[\Phi_{l,k}]{} & \Sigma_{l+k}\wr G\end{array}$$

Here $\chi=(\chi_{k,l};1,\dots,1)$, for the shuffle permutation $\chi_{k,l}\in\Sigma_{k+l}$. So for all $x\in R(\Sigma_k\wr G)$ and $y\in R(\Sigma_l\wr G)$, the relation

$$\mathrm{tr}_{l,k}(y\times x) \;=\; \chi_\star(\mathrm{tr}_{k,l}(x\times y)) \;=\; \mathrm{tr}_{k,l}(x\times y)$$

holds in the group $R(\Sigma_{k+l}\wr G)$. By unraveling the definitions, this implies the commutativity of the operation $\oplus$.

The sequence $\underline{0}$ with $\underline{0}_0=1$ and $\underline{0}_m=0$ for $m\ge 1$ is a neutral element for $\oplus$. Given an exponential sequence x we define a sequence y inductively by $y_0=1$ and by

$$y_m \;=\; -\sum_{k=1}^{m}\mathrm{tr}_{k,m-k}(x_k\times y_{m-k})$$

for $m \geq 1$. We claim that the sequence y is again exponential, and we show the relation

$$\Phi^*_{i,m-i}(y_m) \;=\; y_i \times y_{m-i}$$

by induction on m. The induction starts with $m = 1$, where there is nothing to show. Now we assume that $m \geq 2$; then

$$\begin{aligned}
-\Phi^*_{i,m-i}(y_m) \;&=\; \sum_{k=1}^{m} \Phi^*_{i,m-i}(\mathrm{tr}_{k,m-k}(x_k \times y_{m-k})) \\
&=\; \sum_{a+b\geq 1} \mathrm{tr}^{\Sigma_i\times\Sigma_{m-i}}_{\Sigma_a\times\Sigma_{i-a}\times\Sigma_b\times\Sigma_{m-i-b}}\,(\chi(a,b)_\star(x_a \times x_b \times y_{i-a} \times y_{m-i-b})) \qquad (2.6)\\
&=\; \sum_{a,b\geq 1} \mathrm{tr}_{a,i-a}(x_a \times y_{i-a}) \times \mathrm{tr}_{b,m-i-b}(x_b \times y_{m-i-b}) \\
&\quad +\; \sum_{a=1}^{i} \mathrm{tr}_{a,i-a}(x_a \times y_{i-a}) \times y_{m-i} \;+\; \sum_{b=1}^{m-i} y_i \times \mathrm{tr}_{b,m-i-b}(x_b \times y_{m-i-b}) \\
&=\; y_i \times y_{m-i} \;-\; y_i \times y_{m-i} \;-\; y_i \times y_{m-i} \;=\; -y_i \times y_{m-i}\;.
\end{aligned}$$

In the sums, (a, b) runs over all pairs of natural numbers satisfying $a \leq i$ and $b \leq m - i$ plus the conditions attached to the summation symbols. The second equation uses the inductive hypothesis. This shows that the sequence y is again exponential. The relation $x \oplus y = \underline{0}$ holds by construction, so y is an inverse of x with respect to $\oplus$. Altogether this shows that $\exp(R; G)$ is an abelian group under $\oplus$.

(ii) We already know from Proposition II.2.41 that the componentwise multiplication preserves exponential sequence and defines a commutative monoid structure on $\exp(R; G)$. The exponential sequence $(1)_{m\geq 0}$ is the multiplicative unit. Part (i) shows that $\exp(R; G)$ is an abelian group under $\oplus$. It remains to show distributivity of multiplication over $\oplus$. We let $x, y, z \in \exp(R; G)$ be exponential sequences. Then

$$\begin{aligned}
((x \oplus y) \cdot z)_m \;&=\; \left(\sum_{k=0}^{m} \mathrm{tr}_{k,m-k}(x_k \times y_{m-k})\right) \cdot z_m \\
&=\; \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}((x_k \times y_{m-k}) \cdot \Phi^*_{k,m-k}(z_m)) \\
&=\; \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}((x_k \times y_{m-k}) \cdot (z_k \times z_{m-k})) \\
&=\; \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}((x_k \cdot z_k) \times (y_{m-k} \cdot z_{m-k})) \;=\; ((x \cdot z) \oplus (y \cdot z))_m\;.
\end{aligned}$$

The second equation is reciprocity in the global Green functor R.

(iii) We let $\alpha : K \longrightarrow G$ be any continuous group homomorphism. We already know from Proposition II.2.41 that α^* preserves exponential sequences and is multiplicative and unital. The additivity of α^* uses the relation

$$(\Sigma_m \wr \alpha)^* \circ \mathrm{tr}_{k,m-k} \;=\; \mathrm{tr}_{k,m-k} \circ ((\Sigma_k \wr \alpha) \times (\Sigma_{m-k} \wr \alpha))^* \qquad (2.7)$$

as maps from $R((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G))$ to $R(\Sigma_m \wr K)$. To prove (2.7) we distinguish two cases. If α is surjective, then so is $\Sigma_m \wr \alpha$, and

$$(\Sigma_m \wr \alpha)^{-1}((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G)) \;=\; (\Sigma_k \wr K) \times (\Sigma_{m-k} \wr K)\;.$$

So for epimorphisms, the relation (2.7) is a special case of compatibility of transfer with inflation. If H is a closed subgroup of G, then $\Sigma_m \wr G$ consists of a single double coset for the left $(\Sigma_m \wr H)$-action and right

$((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G))$-action, and

$$(\Sigma_m \wr H) \ \cap \ ((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G)) \ = \ (\Sigma_k \wr H) \times (\Sigma_{m-k} \wr H) \ .$$

So the double coset formula specializes to

$$\mathrm{res}^{\Sigma_m \wr G}_{\Sigma_m \wr H} \circ \mathrm{tr}_{k,m-k} \ = \ \mathrm{tr}_{k,m-k} \circ \mathrm{res}^{(\Sigma_k \wr G)\times(\Sigma_{m-k}\wr G)}_{(\Sigma_k \wr H)\times(\Sigma_{m-k}\wr H)} \ ,$$

which is precisely the relation (2.7) for the inclusion $H \longrightarrow G$. Every homomorphism factors as an epimorphism followed by a subgroup inclusion, so relation (2.7) follows in general. Now we can show that α^* is additive:

$$\begin{aligned}
(\alpha^*(x \oplus y))_n \ &= \ \sum_{k=0}^{m} (\Sigma_m \wr \alpha)^*(\mathrm{tr}_{k,m-k}(x_k \times y_{m-k})) \\
&\overset{(2.7)}{=} \ \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}\left(((\Sigma_k \wr \alpha) \times (\Sigma_{m-k} \wr \alpha))^*(x_k \times y_{m-k})\right) \\
&= \ \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}\left((\Sigma_k \wr \alpha)^*(x_k) \times (\Sigma_{m-k} \wr \alpha)^*(y_{m-k})\right) \ = \ (\alpha^*(x) \oplus \alpha^*(y))_m \ .
\end{aligned}$$

(iv) We show first that transfers preserve exponential sequences. We let H be a closed subgroup of G and $x \in \exp(R;H)$. Then

$$\begin{aligned}
\Phi^*_{k,m-k}\left(\mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H}(x_m)\right) \ &= \ \mathrm{tr}^{(\Sigma_k \wr G)\times(\Sigma_{m-k}\wr G)}_{(\Sigma_k \wr H)\times(\Sigma_{m-k}\wr H)}(\Phi^*_{k,m-k}(x_m)) \\
&= \ \mathrm{tr}^{(\Sigma_k \wr G)\times(\Sigma_{m-k}\wr G)}_{(\Sigma_k \wr H)\times(\Sigma_{m-k}\wr H)}(x_k \times x_{m-k}) \ = \ \mathrm{tr}^{\Sigma_k \wr G}_{\Sigma_k \wr H}(x_k) \times \mathrm{tr}^{\Sigma_{m-k}\wr G}_{\Sigma_{m-k}\wr H}(x_{m-k}) \ .
\end{aligned}$$

The first equality is the double coset formula, exploiting that $\Sigma_m \wr G$ consists of a single double coset for the left $((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G))$-action and right $(\Sigma_m \wr H)$-action, and

$$((\Sigma_k \wr G) \times (\Sigma_{m-k} \wr G)) \ \cap \ (\Sigma_m \wr H) \ = \ (\Sigma_k \wr H) \times (\Sigma_{m-k} \wr H) \ .$$

So the sequence $\mathrm{tr}^G_H(x)$ is again exponential.

To see that the transfer is additive we observe that

$$\mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H} \circ \mathrm{tr}_{k,m-k} \ = \ \mathrm{tr}_{k,m-k} \circ \mathrm{tr}^{(\Sigma_k \wr G)\times(\Sigma_{m-k}\wr G)}_{(\Sigma_k \wr H)\times(\Sigma_{m-k}\wr H)}$$

as maps from $R((\Sigma_k \wr H) \times (\Sigma_{m-k} \wr H))$ to $R(\Sigma_m \wr G)$, by transitivity of transfers. Thus

$$\begin{aligned}
(\mathrm{tr}^G_H(x \oplus y))_m \ &= \ \sum_{k=0}^{m} \mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H}(\mathrm{tr}_{k,m-k}(x_k \times y_{m-k})) \\
&= \ \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}\left(\mathrm{tr}^{(\Sigma_k \wr G)\times(\Sigma_{m-k}\wr G)}_{(\Sigma_k \wr H)\times(\Sigma_{m-k}\wr H)}(x_k \times y_{m-k})\right) \\
&= \ \sum_{k=0}^{m} \mathrm{tr}_{k,m-k}\left(\mathrm{tr}^{\Sigma_k \wr G}_{\Sigma_k \wr H}(x_k) \times \mathrm{tr}^{\Sigma_{m-k}\wr G}_{\Sigma_{m-k}\wr H}(y_{m-k})\right) \ = \ (\mathrm{tr}^G_H(x) \oplus \mathrm{tr}^G_H(y))_m \ .
\end{aligned}$$

The reciprocity for restriction and transfer is a direct consequence of the reciprocity for the global Green functor R:

$$(\mathrm{tr}^G_H(x) \cdot y)_m \ = \ \mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H}(x_m) \cdot y_m \ = \ \mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H}(x_m \cdot \mathrm{res}^{\Sigma_m \wr G}_{\Sigma_m \wr H}(y_m)) \ = \ (\mathrm{tr}^G_H(x \cdot \mathrm{res}^G_H(y)))_m \ .$$

(v) Clearly, the map tr^G_G is the identity of $\exp(R;G)$. For nested subgroup $K \leq H \leq G$, we have

$$\mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H} \circ \mathrm{tr}^{\Sigma_m \wr H}_{\Sigma_m \wr K} \ = \ \mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr K}$$

by transitivity of transfers. This implies the relation $\mathrm{tr}^G_H \circ \mathrm{tr}^H_K = \mathrm{tr}^G_K$ as maps from $\exp(R;K)$ to $\exp(R;G)$.

The normalizer of $\Sigma_m \wr H$ in $\Sigma_m \wr G$ is $\Sigma_m \wr N_G(H)$; hence the Weyl group of $\Sigma_m \wr H$ in $\Sigma_m \wr G$ is isomorphic to $(W_G(H))^m$. In particular, if the Weyl group of H in G is infinite, then for every $m \geq 1$, the Weyl group of $\Sigma_m \wr H$ in $\Sigma_m \wr G$ is infinite. So in this situation,

$$(\mathrm{tr}_H^G(x))_m \;=\; \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}(x_m) \;=\; \begin{cases} 1 & \text{for } m = 0, \text{ and} \\ 0 & \text{for } m \geq 1. \end{cases}$$

In other words, $\mathrm{tr}_H^G(x) = \underline{0}$, the neutral element for the operation $\oplus$.

For a surjective continuous group homomorphism $\alpha : K \longrightarrow G$, the homomorphism $\Sigma_n \wr \alpha$ is again surjective. We let H be a closed subgroup of G and set $L = \alpha^{-1}(H)$. Then

$$(\Sigma_m \wr \alpha)^{-1}(\Sigma_m \wr H) \;=\; \Sigma_m \wr L \qquad \text{and} \qquad (\Sigma_m \wr \alpha)|_{\Sigma_m \wr L} \;=\; \Sigma_m \wr (\alpha|_L)\ .$$

So

$$(\Sigma_m \wr \alpha)^* \circ \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G} \;=\; \mathrm{tr}_{\Sigma_m \wr L}^{\Sigma_m \wr K} \circ (\Sigma_m \wr (\alpha|_L))^*\ .$$

Plugging this into the definitions shows that $\alpha^* \circ \mathrm{tr}_H^G = \mathrm{tr}_L^K \circ (\alpha|_L)^*$ as maps from $\exp(R; H)$ to $\exp(R; K)$.

Maybe the most involved verification of this part is the double coset formula for the restriction of a transfer in $\exp(R)$. We let K and H be two closed subgroups of a compact Lie group G. Then the coset space

$$(\Sigma_m \wr G)/(\Sigma_m \wr H)$$

is homeomorphic to $(G/H)^m$. Thus the double coset space

$$(\Sigma_m \wr K)\backslash(\Sigma_m \wr G)/(\Sigma_m \wr H)$$

is homeomorphic to

$$\Sigma_m \backslash (K\backslash G/H)^m\ ,$$

the m-th symmetric power of the double coset space $K\backslash G/H$.

We suppose that H has finite index in G. We choose a set $\{\gamma_1, \ldots, \gamma_k\} \subset G$ of representatives of the double cosets $K\backslash G/H$. Then the elements

$$\gamma^{[i]} \;=\; (1; \gamma_1^{[i_1]}, \ldots, \gamma_k^{[i_k]}) \;\in\; \Sigma_m \wr G$$

form a set of representatives of the double cosets $(\Sigma_m \wr K)\backslash(\Sigma_m \wr G)/(\Sigma_m \wr H)$, where $i = (i_1, \ldots, i_k)$ ranges of all k-tuples of non-negative integers such that $i_1 + \cdots + i_k = m$, and where $\gamma_j^{[i_j]}$ means 'γ_j, repeated i_j times'. Moreover,

$$(\Sigma_m \wr K) \cap {}^{\gamma^{[i]}}(\Sigma_m \wr H) \;=\; (\Sigma_{i_1} \wr (K \cap {}^{\gamma_1}H)) \times \cdots \times (\Sigma_{i_k} \wr (K \cap {}^{\gamma_k}H))\ .$$

For $x \in \exp(R;H)$, the double coset formula for the global functor R then gives:

$$\begin{aligned}
\big(\mathrm{res}^G_K(\mathrm{tr}^G_H(x))\big)_m &= \mathrm{res}^{\Sigma_m \wr G}_{\Sigma_m \wr K}(\mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H}(x_m)) \\
&= \sum_{i_1+\cdots+i_k=m} \mathrm{tr}^{\Sigma_m \wr K}_{(\Sigma_m \wr K)\cap{}^{\gamma^{[i]}}(\Sigma_m \wr H)}((\gamma_i)_\star(\mathrm{res}^{\Sigma_m \wr H}_{(\Sigma_m \wr K)^{\gamma^{[i]}}\cap(\Sigma_m \wr H)}(x_m))) \\
&= \sum_{i_1+\cdots+i_k=m} \mathrm{tr}^{\Sigma_m \wr K}_{\prod_{j=1}^k(\Sigma_{i_j} \wr (K\cap{}^{\gamma_j}H))}((\gamma_i)_\star(\mathsf{X}_{j=1}^k\ \mathrm{res}^{\Sigma_{i_j} \wr H}_{\Sigma_{i_j} \wr (K^{\gamma_j}\cap H)}(x_{i_j}))) \\
&= \sum_{i_1+\cdots+i_k=m} \mathrm{tr}_{i_1,\ldots,i_m}\Big(\mathsf{X}_{j=1}^k\ \mathrm{tr}^{\Sigma_{i_j} \wr K}_{\Sigma_{i_j} \wr (K\cap{}^{\gamma_j}H)}((1;\gamma_j^{[i_j]})_\star(\mathrm{res}^{\Sigma_{i_j} \wr H}_{\Sigma_{i_j} \wr (K^{\gamma_j}\cap H)}(x_{i_j})))\Big) \\
&= \sum_{i_1+\cdots+i_k=m} \mathrm{tr}_{i_1,\ldots,i_m}\big(\mathsf{X}_{j=1}^k\ (\mathrm{tr}^K_{K^{\gamma_j}\cap H}((\gamma_j)_\star(\mathrm{res}^H_{K^{\gamma_j}\cap H}(x))))_{i_j}\big) \\
&= \left(\sum_{j=1}^k \mathrm{tr}^K_{K^{\gamma_j}\cap H}((\gamma_j)_\star(\mathrm{res}^H_{K^{\gamma_j}\cap H}(x)))\right)_m
\end{aligned}$$

The second equation is the exponential property of x, the third one is transitivity of transfers. [... add the general case...] □

The previous proposition establishes the functor exp of exponential sequences as an endofunctor of the category of global Green functors. Now we make this endofunctor into a comonad. The transformations are exactly the same that we already used in the context of global power monoids in Construction II.2.40. A natural transformation of global Green functors

$$\eta_R\ :\ \exp(R)\ \longrightarrow\ R$$

is given by $\eta(x) = x_1$, using the identification $G \cong \Sigma_1 \wr G$ via $g \mapsto (1;g)$. A natural transformation

$$\kappa_R\ :\ \exp(R)\ \longrightarrow\ \exp(\exp(R))$$

is given at a compact Lie group G by

$$(\kappa(x)_m)_k\ =\ \Psi^*_{k,m}(x_{km})\ \in\ R(\Sigma_k \wr (\Sigma_m \wr G))\ ;$$

here the restriction is along the monomorphism (1.4)

$$\begin{aligned}
\Psi_{k,m}\ :\ \Sigma_k \wr (\Sigma_m \wr G) &\longrightarrow \Sigma_{km} \wr G \\
(\sigma;\,(\tau_1;\,h^1),\ldots,(\tau_k;\,h^k)) &\longmapsto (\sigma\natural(\tau_1,\ldots,\tau_k);\,h^1+\cdots+h^k)\ .
\end{aligned}$$

The analog of the following for finite groups is Satz 2.17 in [**146**].

THEOREM 2.8. *Let R be a global Green functor.*

(i) *For every compact Lie group G and for every exponential sequence $x \in \exp(R;G)$, the sequence $\kappa(x)$ is an element of $\exp(\exp(R);G)$.*
(ii) *As the group varies, the maps κ form a morphism of global Green functors $\kappa_R : \exp(R) \longrightarrow \exp(\exp(R))$, natural in R.*
(iii) *The natural transformations*

$$\eta\ :\ \exp\ \longrightarrow\ \mathrm{Id} \qquad \textit{and} \qquad \kappa\ :\ \exp\ \longrightarrow\ \exp\circ\exp$$

make the functor exp *into a comonad on the category of global Green functors.*

PROOF. Parts (i) and (iii) are set theoretic properties that only depend on the underlying multiplicative abelian Rep-monoid of R. So parts (i) and (iii) are special case of the analogous proposition for the exp comonad on global power monoids that we established in Proposition II.2.42.

Most of part (ii) is also taken care of by Proposition II.2.42, but the additive structure is a new piece of data; so we also have to check that $\kappa_R(G) : \exp(R;G) \longrightarrow \exp(\exp(R);G)$ is additive and compatible with transfers as G varies. The most involved part of this argument is the additivity of $\kappa_R(G)$, and for this verification we have to confront the double coset formula for the subgroups

$$\Sigma_k \wr' \Sigma_m \;=\; \Phi_{k,m}(\Sigma_k \wr (\Sigma_m \wr G)) \qquad \text{and} \qquad \Sigma_i \times' \Sigma_{km-i} \;=\; \Phi_{i,km-i}((\Sigma_i \wr G) \times (\Sigma_{km-i} \wr G))$$

of the wreath product $\Sigma_{km} \wr G$. To a large extent, the group G acts like a dummy, which is why we omit it from the notation for $\Sigma_k \wr' \Sigma_m$ and $\Sigma_i \times' \Sigma_{km-i}$. We specify a bijection between the set of double cosets

$$\Sigma_k \wr' \Sigma_m \backslash \Sigma_{km} \wr G / \Sigma_i \times' \Sigma_{km-i}$$

and the set of tuples $(a_0, \dots, a_m)$ of natural numbers that satisfy the two conditions

$$a_0 + \cdots + a_m \;=\; k \qquad \text{and} \qquad \sum_{j=0}^{m} j \cdot a_j \;=\; i \;. \tag{2.9}$$

For $1 \le b \le k$ we let $j_b \in \{1, \dots, m\}$ be the unique number such that

$$a_0 + \cdots + a_{j_b - 1} \;<\; b \;\le\; a_0 + \cdots + a_{j-1} + a_{j_b} \;.$$

We define

$$A(a_0, \dots, a_m) \;=\; \bigcup_{b=1}^{k} \{(b-1)m + 1, \dots, (b-1)m + j_b\} \;\subseteq\; \{1, \dots, km\} \;;$$

because

$$\sum_{b=1}^{k} j_b \;=\; \sum_{j=1}^{m} a_j \cdot j \;=\; i \;,$$

the set $A(a_0, \dots, a_m)$ has exactly i elements. We let $\bar\sigma = \bar\sigma(a_0, \dots, a_m) \in \Sigma_{km}$ be any permutation such that

$$\{\bar\sigma(1), \dots, \bar\sigma(i)\} \;=\; A(a_0, \dots, a_m) \;.$$

Then the elements

$$\sigma(a_0, \dots, a_m) \;=\; (\bar\sigma(a_0, \dots, a_m);\, 1, \dots, 1) \;\in\; \Sigma_{km} \wr G$$

are a complete set of double coset representations as $(a_0, \dots, a_m)$ ranges over those tuples that satisfy (2.9). Moreover,

$$(\Sigma_k \wr' \Sigma_m)^{\sigma(a_0,\dots,a_m)} \cap (\Sigma_i \times' \Sigma_{km-i}) \;=\; \left(\prod_{j=0}^{m}{}' \Sigma_{a_j} \wr' \Sigma_j \right) \times' \left(\prod_{j=0}^{m}{}' \Sigma_{a_j} \wr' \Sigma_{m-j} \right)$$

and

$$(\Sigma_k \wr' \Sigma_m) \cap {}^{\sigma(a_0,\dots,a_m)}(\Sigma_i \times' \Sigma_{km-i}) \;=\; \prod_{j=0}^{m}{}' (\Sigma_{a_j} \wr' \Sigma_j) \times' (\Sigma_{a_j} \wr' \Sigma_{m-j}) \;.$$

[...fix notation...] For $x_i \in R(\Sigma_i \wr G)$ and $y_{km-i} \in R(\Sigma_{km-i} \wr G)$, the double coset formula this becomes the relation

$$\begin{aligned} &\Psi^*_{k,m}(\mathrm{tr}_{i,km-i}(x_i \times y_{km-i})) \\ &\quad = \sum \mathrm{tr}^{\Sigma_k \wr \Sigma_m \wr G}_{\prod'(\Sigma_{a_j} \wr' \Sigma_j) \times' (\Sigma_{a_j} \wr' \Sigma_{m-j})} \left(\sigma(a_0, \dots, a_m)_\star \left(\mathrm{res}^{(\Sigma_i \wr G) \times (\Sigma_{km-i} \wr G)}_{(\prod' \Sigma_{a_j} \wr' \Sigma_j) \times' (\prod' \Sigma_{a_j} \wr' \Sigma_{m-j})} (x_i \times y_{km-i}) \right) \right) \end{aligned}$$

where the sum ranges over all tuples $(a_0,\dots,a_m)$ that satisfy (2.9). If x_i and y_{km-i} are the respective components of two exponential sequences $x,y\in\exp(R;G)$, then

$$\begin{aligned}\sigma(a_0,\dots,a_m)_\star&\left(\mathrm{res}^{(\Sigma_i\wr G)\times(\Sigma_{km-i}\wr G)}_{(\prod'\Sigma_{a_j}\wr'\Sigma_j)\times'(\prod'\Sigma_{a_j}\wr'\Sigma_{m-j})}(x_i\times y_{km-i})\right)\\ &=\ \sigma(a_0,\dots,a_m)_\star\left(\mathrm{res}^{\Sigma_i\wr G}_{\prod'\Sigma_{a_j}\wr'\Sigma_j}(x_i)\times\mathrm{res}^{\Sigma_{km-i}\wr G}_{\prod'\Sigma_{a_j}\wr'\Sigma_{m-j}}(y_{km-i})\right)\\ &=\ \sigma(a_0,\dots,a_m)_\star\left(\prod\nolimits_{j=0}^m\Psi^*_{a_j,j}(x_{a_jj})\times\prod\nolimits_{j=0}^m\Psi^*_{a_j,m-j}(y_{a_j(m-j)})\right)\\ &=\ \prod_{j=0}^m\left(\Psi^*_{a_j,j}(x_{a_jj})\times\Psi^*_{a_j,m-j}(y_{a_j(m-j)})\right)\ .\end{aligned}$$

Given $x,y\in\exp(R;G)$ we have

$$(\kappa(x)\oplus\kappa(y))_m\ =\ \bigoplus_{j=0}^m\ \mathrm{tr}_{j,m-j}(\kappa(x)_j\times\kappa(y)_{m-j})\ ,$$

where the sum on the right is taken in the group $\exp(R;\Sigma_m\wr G)$ under $\oplus$. Expanding this further we arrive at the expression

$$\begin{aligned}((\kappa(x)\oplus\kappa(y))_m)_k\ &=\ \sum_{a_0+\dots+a_m=k}\mathrm{tr}_{a_0,\dots,a_m}\left(\prod_{j=0}^m\mathrm{tr}_{j,m-j}(\kappa(x)_j\times\kappa(y)_{m-j})_{a_j}\right)\\ &=\ \sum_{a_0+\dots+a_m=k}\mathrm{tr}_{a_0,\dots,a_m}\left(\prod_{j=0}^m\mathrm{tr}^{\Sigma_{a_j}\wr\Sigma_m}_{\Sigma_{a_j}\wr\Sigma_{j,m-j}}(\Psi^*_{a_j,j}(x_{a_jj})\times\Psi^*_{a_j,m-j}(y_{a_j(m-j)}))\right)\\ &=\ \sum_{a_0+\dots+a_m=k}\mathrm{tr}^{\Sigma_k\wr\Sigma_m\wr G}_{\prod'(\Sigma_{a_j}\wr'\Sigma_j)\times'(\Sigma_{a_j}\wr'\Sigma_{m-j})}\left(\prod_{j=0}^m\Psi^*_{a_j,j}(x_{a_jj})\times\Psi^*_{a_j,m-j}(y_{a_j(m-j)})\right)\\ &=\ \sum_{a_0+\dots+a_m=k}\mathrm{tr}^{\Sigma_k\wr\Sigma_m\wr G}_{\prod'(\Sigma_{a_j}\wr'\Sigma_j)\times'(\Sigma_{a_j}\wr'\Sigma_{m-j})}\left(\prod_{j=0}^m\Psi^*_{a_j,j}(x_{a_jj})\times\Psi^*_{a_j,m-j}(y_{a_j(m-j)})\right)\\ &=\ \sum_{i=0}^{km}\Psi^*_{k,m}(\mathrm{tr}_{i,km-i}(x_i\times y_{km-i}))\ =\ (\kappa(x\oplus y)_m)_k\end{aligned}$$

in the group $R(\Sigma_k\wr\Sigma_m\wr G)$. Here $\mathrm{tr}_{a_0,\dots,a_m}$ is shorthand notation for the transfer along the monomorphism

$$\Phi^{\Sigma_m\wr G}_{a_0,\dots,a_m}\ :\ (\Sigma_{a_0}\wr(\Sigma_m\wr G))\times\dots\times(\Sigma_{a_j}\wr(\Sigma_m\wr G))\times\dots\times(\Sigma_{a_m}\wr(\Sigma_m\wr G))\ \longrightarrow\ \Sigma_k\wr(\Sigma_m\wr G)\ ,$$

the analog of the monomorphism (1.3) with multiple inputs (and for the group $\Sigma_m\wr G$ instead of G). Hence the map $\kappa_R(G):\exp(R;G)\longrightarrow\exp(\exp(R);G)$ is additive.

The compatibility of κ with transfers needs another application of a double coset formula. Indeed, for every closed subgroup H of G, the group $\Sigma_{km}\wr G$ consists of a single double coset for the left $(\Sigma_k\wr(\Sigma_m\wr G))$-action and right $(\Sigma_{km}\wr H)$-action, and

$$(\Sigma_k\wr\Sigma_m\wr G)\ \cap\ (\Sigma_{km}\wr H)\ =\ \Sigma_k\wr\Sigma_m\wr H\ .$$

So for every $x\in\exp(R;H)$, the relations

$$\begin{aligned}(\kappa(\mathrm{tr}^G_H(x))_m)_k\ =\ \Psi^*_{k,m}\left(\mathrm{tr}^{\Sigma_{km}\wr G}_{\Sigma_{km}\wr H}(x_{km})\right)\ &=\ \mathrm{tr}^{\Sigma_k\wr\Sigma_m\wr G}_{\Sigma_k\wr\Sigma_m\wr H}(\Psi^*_{k,m}(x_{km}))\\ &=\ \mathrm{tr}^{\Sigma_k\wr\Sigma_m\wr G}_{\Sigma_k\wr\Sigma_m\wr H}((\kappa(x)_m)_k)\ =\ \left((\mathrm{tr}^G_H(\kappa(x)))_m\right)_k\end{aligned}$$

holds in $R(\Sigma_k \wr \Sigma_m \wr G)$. So $\kappa \circ \mathrm{tr}_H^G = \mathrm{tr}_H^G \circ \kappa$. Altogether this shows that the maps $\kappa_R(G)$ are ring homomorphisms and compatible with restriction and transfers, so they form a morphism of global Green functors. □

Now we can finally get to the main result of this section, identifying global power functors with coalgebras over the comonad of exponential sequences. We suppose that R is a global Green functor and $P : R \longrightarrow \exp(R)$ a natural transformation of global Green functors. For every compact Lie group G, a sequence of operations $P^m : R(G) \longrightarrow R(\Sigma_m \wr G)$ is then defined by

$$P^m(x) \;=\; (P(x))_m \;,$$

i.e., $P^m(x)$ is the m-th component of the exponential sequence $P(x)$.

Theorem 2.10 (Comonadic description of global power functors).

(i) *Let R be a global Green functor and $P : R \longrightarrow \exp(R)$ a morphism of global Green functors that makes R into a coalgebra over the comonad* $(\exp, \eta, \kappa)$. *Then the operations $P^m : R(G) \longrightarrow R(\Sigma_m \wr G)$ make R into a global power functor.*

(ii) *The functor*

$$(\exp\text{-coalgebras}) \;\longrightarrow\; (\text{global power functors}) \;, \quad (R, P : R \longrightarrow \exp(R)) \;\longmapsto\; (R, \{P^m\}_{m\geq 0})$$

is an isomorphism of categories.

Proof. (i) The fact that $P : R \longrightarrow \exp(R)$ takes values in exponential sequences is equivalent to the restriction condition of the power operations. The fact that $P : R \longrightarrow \exp(R)$ is a transformation of global Green functors encodes simultaneously the unit, contravariant naturality, transfer, multiplicativity and additivity relations of a global power functor. The identity relation $P^1 = \mathrm{Id}$ is equivalent to the counit condition of a coalgebra, i.e., that the composite

$$R \;\xrightarrow{P}\; \exp(R) \;\xrightarrow{\eta_R}\; R$$

is the identity. The transitivity relation is equivalent to

$$\exp(P) \circ P \;=\; \kappa_R \circ P \;,$$

the coassociativity condition of a coalgebra.

Part (ii) is essentially reading part (i) backwards, and we omit the details. □

The interpretation of global power functors as coalgebras over a comonad has some useful consequences that are not so easy to see directly from the original definition in terms of power operations and explicit relations. In general, the forgetful functor from any category of coalgebras to the underlying category has a right adjoint 'cofree' functor. In particular, colimits in a category of coalgebras are created in the underlying category. In our situation that means:

Corollary 2.11. (i) *Colimits in the category of global power functors exist and are created in the underlying category of global Green functors.*

(ii) *For every global Green functor R, the maps*

$$P^m \;:\; \exp(R; G) \;\longrightarrow \exp(R; \Sigma_m \wr G) \;, \quad P^m(x) \;=\; \kappa(x)_m = (\Psi^*_{k,m}(x_{km}))_{k\geq 0} \;.$$

make the global Green functor $\exp(R)$ *into a global power functor.*

(iii) *When viewed as a functor to the category of global power functors as in* (ii), *the functor* exp *is right adjoint to the forgetful functor.*

Example 2.12 (Coproducts). We let R and S be two global Green functors. Global Green functors are the commutative monoids, with respect to the box product, in the category of global functors. So the box product $R\Box S$ is the coproduct in the category of global Green functors, with multiplication defined as the composite

$$R\Box S\Box R\Box S \xrightarrow{R\Box\tau_{S,R}\Box S} R\Box R\Box S\Box S \xrightarrow{\mu_R\Box\mu_S} R\Box S\ .$$

If $P : R \longrightarrow \exp(R)$ and $P' : S \longrightarrow \exp(S)$ are global power structures on R and S, then $R\Box S$ has preferred power operations specified by the morphism of global Green functors

$$R\Box S \xrightarrow{P\Box P'} \exp(R)\Box\exp(S) \longrightarrow \exp(R\Box S)$$

where the second morphism is the canonical one from the coproduct of exp to the values of exp at a coproduct. With these power operations, $R\Box S$ becomes a coproduct of R and S in the category of global power functors, by Corollary 2.11 (i).

This abstract definition of the power operations on $R\Box S$ can be made more explicit. Indeed, the power operations on $R\Box S$ are determined by the formula

$$P^m(x\times y) \;=\; \Delta^*(P^m(x)\times P^m(y))$$

for all compact Lie groups G and K and classes $x\in R(G)$ and $y\in S(K)$, and by the relations of the power operations. Here $\Delta : \Sigma_m \wr (G\times K) \longrightarrow (\Sigma_m\wr G)\times(\Sigma_m\wr K)$ is the diagonal monomorphism (see (2.12) of Chapter I).

The coproduct of global power functors is realized by the coproduct of ultra-commutative ring spectra in the following sense. If E and F are ultra-commutative ring spectra, then the ring spectra morphisms $E\longrightarrow E\wedge F$ and $F\longrightarrow E\wedge F$ induced morphisms of global power functors $\underline{\pi}_0(E)\longrightarrow\underline{\pi}_0(E\wedge F)$ and $\underline{\pi}_0(F)\longrightarrow\underline{\pi}_0(E\wedge F)$, and together they define a morphism from the coproduct of global power functors

$$\underline{\pi}_0(E)\,\Box\,\underline{\pi}_0(F) \longrightarrow \underline{\pi}_0(E\wedge F)\ .$$

If E and F are globally connective and at least one of them is flat as an orthogonal spectrum, then this is an isomorphism of global functors by Proposition IV.4.16, hence an isomorphism of global power functors.

Example 2.13 (Localization of global power functors). Now we discuss localizations of global power functors. We first consider a global Green functor R and a multiplicative subset $S\subseteq R(e)$ in the 'underlying ring', i.e., the value at the trivial group. We define a global Green functor $R[S^{-1}]$ and a morphism of global Green functors $i : R\longrightarrow R[S^{-1}]$. The value at a compact Lie group G is the ring

$$R[S^{-1}](G) \;=\; R(G)[p_G^*(S)^{-1}]\ ,$$

the localization of the ring $R(G)$ at the multiplicative subset obtained as the image of S under the ring homomorphism $p_G^* : R(e)\longrightarrow R(G)$. The value of the morphism i at G is the localization map $R(G)\longrightarrow R(G)[p_G^*(S)^{-1}]$. If $\alpha : K\longrightarrow G$ is a continuous homomorphism, the relation $p_G\circ\alpha = p_K$ implies that the ring homomorphism

$$\alpha^* \;:\; R(G) \longrightarrow R(K)$$

takes the set $p_G^*(S)$ to the set $p_K^*(S)$. So the universal property of localization provides a unique ring homomorphism

$$\alpha[S^{-1}]^* \;:\; R[S^{-1}](G) \;=\; R(G)[p_G^*(S)^{-1}] \longrightarrow R(K)[p_K^*(S)^{-1}] \;=\; R[S^{-1}](K)$$

such that $\alpha[S^{-1}]\circ i(G) = i(K)\circ\alpha^*$. Again by the universal property of localizations, this data produces a contravariant functor from the category Rep to the category of commutative rings.

Now we let H be a closed subgroup of G. We consider $R(H)$ as a module over $R(G)$ via the restriction homomorphism $\mathrm{res}_H^G : R(G)\longrightarrow R(H)$. Because $\mathrm{res}_H^G(p_G^*(S)) = (p_G)|_H^*(S) = p_H^*(S)$, the localization $R(H)[p_H^*(S)^{-1}] = R[S^{-1}](H)$ is also a localization of $R(H)$ at $p_G^*(S)$ as an $R(G)$-module. The reciprocity

formula means that the transfer map $\mathrm{tr}_H^G : R(H) \longrightarrow R(G)$ is a homomorphism of $R(G)$-modules. The composite $R(G)$-linear map

$$R(H) \xrightarrow{\mathrm{tr}_H^G} R(G) \xrightarrow{i(G)} R(G)[p_G^*(S)^{-1}] \;=\; R[S^{-1}](G)$$

thus extends over a unique $R[S^{-1}](G)$-linear map

$$\mathrm{tr}[S^{-1}]_H^G \;:\; R[S^{-1}](H) \;=\; R(H)[p_G^*(S)^{-1}] \;\longrightarrow\; R[S^{-1}](G)$$

such that $\mathrm{tr}[S^{-1}]_H^G \circ i(H) = i(G) \circ \mathrm{tr}_H^G$. Reciprocity for $\mathrm{tr}[S^{-1}]_H^G$ is equivalent to $R[S^{-1}](G)$-linearity. The other necessary properties of transfers, such as transitivity, compatibility with inflations, vanishing for infinite Weyl groups, and the double coset formula all follow from corresponding properties in the global Green functor R and the universal properties of localization. So altogether this shows that the objectwise localizations assemble into a new global Green functor $R[S^{-1}]$; the homomorphisms $i(G)$ altogether form a morphism of global Green functors $i : R \longrightarrow R[S^{-1}]$ by construction. The following universal property is also straightforward from the universal property of localizations of commutative rings and modules.

Proposition 2.14. *Let R be a global Green and S a multiplicative subset of the underlying ring $R(e)$. Let $f : R \longrightarrow R'$ be a morphism of global Green functors such that all elements of the set $f(e)(S)$ are invertible in the ring $R'(e)$. Then there is a unique homomorphism of global Green functors $\bar{f} : R[S^{-1}] \longrightarrow R'$ such that $\bar{f}i = f$.*

Now we let R be a global power functor. If we want the localization $R[S^{-1}]$ to inherit power operations, then we need an extra hypothesis on the multiplicative subset S.

Theorem 2.15. *Let R be a global power functor and S a multiplicative subset of the underlying ring $R(e)$. Suppose that for every $m \geq 1$ the multiplicative subset*

$$P^m(S) \;\subset\; R(\Sigma_m)$$

becomes invertible in the ring $R[S^{-1}](\Sigma_m)$.

(i) *There is a unique extension of the global Green functor $R[S^{-1}]$ to a global power functor such that the morphism $i : R \longrightarrow R[S^{-1}]$ is a morphism of global power functors.*
(ii) *Let $f : R \longrightarrow R'$ be a morphism of global power functors such that all elements of the set $f(e)(S)$ are invertible in the ring $R'(e)$. Then there is a unique homomorphism of global power functors $\bar{f} : R[S^{-1}] \longrightarrow R'$ such that $\bar{f}i = f$.*

Proof. (i) We use the comonadic description of global power functors given in Theorem 2.10. This exhibits the power operations of R as a morphism of global Green functors $P : R \longrightarrow \exp(R)$.

The multiplication in the ring $\exp(R; e)$ is componentwise, and for every $s \in S$ the element $P^m(s)$ has a multiplicative inverse $t_m \in R[S^{-1}](\Sigma_m)$ by hypothesis. We claim that the sequence $t = (t_m)_{m\geq 0}$ is again exponential, and hence an element of the ring $\exp(R[S^{-1}], e)$. Indeed, for $0 < i < m$ we have

$$\begin{aligned}\Phi_{i,m-i}^*(t_m) \cdot (P^i(s) \times P^{m-i}(s)) \;&=\; \Phi_{i,m-i}^*(t_m) \cdot \Phi_{i,m-i}^*(P^m(s)) \\ &=\; \Phi_{i,m-i}^*(t_m \cdot P^m(s)) \;=\; \Phi_{i,m-i}^*(1) \;=\; 1\ .\end{aligned}$$

Since $t_i \times t_{m-i}$ is also inverse to $P^i(s) \times P^{m-i}(s)$ and inverse are unique, we conclude that $\Phi_{i,m-i}^*(t_m) = t_i \times t_{m-i}$. This shows that the inverses form another exponential sequence.

The relation

$$P(s) \cdot t \;=\; (P^m(s) \cdot t_m)_m \;=\; 1$$

holds in the ring $\exp(R[S^{-1}]; e)$, by construction. So the composite morphism

$$R \xrightarrow{P} \exp(R) \xrightarrow{\exp(i)} \exp(R[S^{-1}])$$

takes S to invertible elements. So the universal property of Proposition 2.14 provides a unique morphism of global Green functors

$$P[S^{-1}] \ : \ R[S^{-1}] \ \longrightarrow \ \exp(R[S^{-1}])$$

such that $P[S^{-1}] \circ i = \exp(i) \circ P$. Now we observe that

$$\begin{aligned}
\exp(P[S^{-1}]) \circ P[S^{-1}] \circ i \ &= \ \exp(P[S^{-1}]) \circ \exp(i) \circ P \\
&= \ \exp(P[S^{-1}] \circ i) \circ P \\
&= \ \exp(\exp(i) \circ P) \circ P \\
&= \ \exp(\exp(i)) \circ \exp(P) \circ P \\
&= \ \exp(\exp(i)) \circ \kappa_R \circ P \\
&= \ \kappa_{R[S^{-1}]} \circ \exp(i) \circ P \\
&= \ \kappa_{R[S^{-1}]} \circ P[S^{-1}] \circ i \ .
\end{aligned}$$

The uniqueness clause in the universal property of Proposition 2.14 then implies that

$$\exp(P[S^{-1}]) \circ P[S^{-1}] \ = \ \kappa_{R[S^{-1}]} \circ P[S^{-1}] \ .$$

So the morphism $P[S^{-1}]$ is a coalgebra structure over the exp comonad. The relation $P[S^{-1}] \circ i = \exp(i) \circ P$ then says that i is a morphism of exp-coalgebras, so this completes the proof.

(ii) Since morphisms of global power functors are in particular morphisms of global Green functors, the uniqueness clause follows from Proposition 2.14. If $f : R \longrightarrow R'$ is a morphism of global power functors such that all elements of $f(e)(S)$ are invertible in the ring $R'(e)$, then Proposition 2.14 provides a homomorphism of global Green functors $\bar{f} : R[S^{-1}] \longrightarrow R'$ such that $\bar{f} i = f$. We need to show that $\bar{f}$ is also compatible with the power operations. We let

$$P[S^{-1}] \ : \ R[S^{-1}] \ \longrightarrow \ \exp(R[S^{-1}]) \qquad \text{and} \qquad P' \ : \ R' \ \longrightarrow \ \exp(R')$$

be the morphisms of global Green functors that encode the exp-coalgebra structure. We observe that

$$\begin{aligned}
\exp(\bar{f}) \circ P[S^{-1}] \circ i \ &= \ \exp(\bar{f}) \circ \exp(i) \circ P \\
&= \ \exp(f) \circ P \ = \ P' \circ f \ = \ P' \circ \bar{f} \circ i \ ;
\end{aligned}$$

the third equation is the hypothesis that f is a morphism of global power functors. This is an equality between morphisms of global Green functors, so the uniqueness of Proposition 2.14 shows that $\exp(\bar{f}) \circ P[S^{-1}] = P' \circ \bar{f}$, i.e., f is a morphism of exp-coalgebras. □

Example 2.16 (Localization at a subring of $\mathbb{Q}$). We use Theorem 2.15 to show that global power functors can always be rationalized; more generally, power operations 'survive' localization at any subring of the ring $\mathbb{Q}$ of rational numbers. We consider a global power functor R and a natural number $n \geq 2$. We claim that for every $m \geq 1$ the element $P^m(n)$ of $R(\Sigma_m)$ becomes invertible in the ring $R(\Sigma_m)[1/n]$. We argue by induction over m; we start with $m = 1$, where the relation $P^1(n) = n$ shows the claim. Now we consider $m \geq 2$ and assume that for all $0 \leq i < m$ the element $P^i(n)$ has an inverse t_i in the ring $R(\Sigma_i)[1/n]$. Then for every n-tuple $(i_1, \ldots, i_n)$ that satisfies $0 \leq i_j < m$ and $i_1 + \cdots + i_n = m$ we get the relation

$$\begin{aligned}
\mathrm{res}^{\Sigma_m}_{\Sigma_{i_1} \times \cdots \times \Sigma_{i_n}} (P^m(n)) \cdot (t_{i_1} \times \cdots \times t_{i_n}) \ &= \ (P^{i_1}(n) \times \cdots \times P^{i_n}(n)) \cdot (t_{i_1} \times \cdots \times t_{i_n}) \\
&= \ (P^{i_1}(n) \cdot t_{i_1}) \times \cdots \times (P^{i_n}(n) \cdot t_{i_n}) \ = \ 1 \times \cdots \times 1
\end{aligned}$$

in the ring $R(\Sigma_{i_1} \times \cdots \times \Sigma_{i_n})[1/n]$. Thus we get

$$\begin{aligned} P^m(n) &= P^m(\underbrace{1+\cdots+1}_{n}) = \sum_{i_1+\cdots+i_n=m} \mathrm{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(P^{i_1}(1)\times\cdots\times P^{i_n}(1)) \\ &= n\cdot P^m(1) + \sum_{i_1+\cdots+i_n=m,\ i_j<m} \mathrm{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(1\times\cdots\times 1) \\ &= n + \sum_{i_1+\cdots+i_n=m,\ i_j<m} \mathrm{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(\mathrm{res}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(P^m(n))\cdot(t_{i_1}\times\cdots\times t_{i_n})) \\ &= n + \sum_{i_1+\cdots+i_n=m,\ i_j<m} P^m(n)\cdot \mathrm{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(t_{i_1}\times\cdots\times t_{i_n}) \end{aligned}$$

Rearranging the terms gives

$$P^m(n)\cdot\left(1-\sum_{i_1+\cdots+i_n=m,\ i_j<m} \mathrm{tr}^{\Sigma_m}_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_n}}(t_{i_1}\times\cdots\times t_{i_n})\right) = n\ ;$$

so $P^m(n)$ has an inverse in the ring $R(\Sigma_m)[1/n]$, and this completes the inductive step.

Now we let S be any multiplicative subset of the ring of integers. By the previous paragraph, Theorem 2.15 applies and provides a unique structure of global power ring on the global Green functor $R[S^{-1}]$ such that the morphism $i : R \longrightarrow R[S^{-1}]$ is a morphism of global power functors. In particular, if we let S be the set of all positive integers, we can conclude that the rationalization $\mathbb{Q}\otimes R$ of R has a unique structure of global power functor such that the localization map $R \longrightarrow \mathbb{Q}\otimes R$ is a morphism of global power functors.

Remark 2.17 (Monadic description of global power functors). Now we explain that the category of global power functors in not only comonadic, but also monadic over the category of global Green functors. In fact, both categories are examples of algebras over *multisorted algebraic theories* (also called *colored theories*). The 'sorts' (or 'colors') are the compact Lie groups and the content of this claim is that the structure of global Green functors respectively global power functors can be specified by giving the values $R(G)$ at every compact Lie group, together with n-ary operations for different $n \geq 0$ and varying inputs and output, and relations between composites of those operations.

In the case of global Green functors, the operations to be specified are

- the constants given by the additive and multiplicative units in the rings $R(G)$,
- the unary operations given by the additive inverse map in $R(G)$, the restriction maps and transfers,
- and the binary operations specifying the addition and multiplication in the rings $R(G)$.

The relations include, among others, the neutrality, associativity and commutativity of addition and multiplication; the distributivity in the rings $R(G)$; the additivity of restriction and transfers; the functoriality of restrictions and transitivity of transfers; and the double coset and reciprocity formulas.

Global power functors have additional unary operations, the power operations, and additional relations as listed in Definition 1.5.

Proposition 2.18. *The forgetful functor from the category of global power functors to the category of global Green functors has a left adjoint. The category of global power functors is isomorphic to the category of algebras over the monad of this adjunction.*

Proof. For the existence of the left adjoint we have to show that for every global Green functor R the functor

$$(\text{global power functor}) \longrightarrow (\text{sets})\ , \quad S \longmapsto \mathrm{Green}(R, US)$$

is representable. This is a formal consequence of the existence of free global power functors, colimits of global power functors and the fact that global power functors are a multi-sorted theory. We explain this in more detail, without completely formalizing the argument.

We choose a set of compact Lie groups $\{K_i\}_{i\in I}$ that contains one compact Lie group from every isomorphism class. We form the global power functor

$$L \;=\; \square_{i\in I, x\in R(K_i)} C_{i,x} \;,$$

a coproduct, indexed by all pairs (i,x) consisting of an index $i \in I$ and an element $x \in R(K_i)$, of free global power functors

$$C_{i,x} \;=\; C_{K_i}$$

generated by the compact Lie group K_i. On this free global power functor we impose the minimal amount of relations so that the maps $R(K_i) \longrightarrow L(K_i)$ that send $x \in R(K_i)$ to the generator indexed by (i,x) becomes a morphism of global Green functors. Here 'imposing relations' means that we form another box product L' of free global power functors, with one box factor for each relation between elements in the various sets $R(K_i)$. For example, we include one factor for the sum of each pair of elements in the same set $R(K_i)$, another factor for the product of each pair of elements in the same set $R(K_i)$, and more factors for zero elements, multiplicative units, all restriction relations and all transfers relations. Then we form a coequalizer, in the category of global power functors

$$L' \rightrightarrows L \longrightarrow F$$

where the two morphisms from L' to L restrict, on each box factor, to the morphism that represents the respective relation. The resulting global power functor then represents the functor $\mathrm{Green}(R, U(-))$, so we can take F as the value of the left adjoint on R.

Since the global power functors are equivalent to the coalgebras over the exp-comonad, the forgetful functor creates all colimits, in particular coequalizers. So by Beck's monadicity theorem (see for example [**103**, VI.7 Thm. 1]), the tautological functor from global power functors to algebras over the adjunction monad is an isomorphism of categories. □

Example 2.19 (Limits). The category of global power functors has limits, and they are defined 'groupwise'. A product is a special case of a limit, and the product of global power functors is realized by the product of ultra-commutative ring spectra. If E and F are ultra-commutative ring spectra, then so is the product $E \times F$ of the underlying orthogonal spectra, and the canonical map

$$\underline{\pi}_0(E \times F) \;\longrightarrow\; \underline{\pi}_0(E) \times \underline{\pi}_0(F)$$

is an isomorphism of global power functors (by Corollary III.1.38 (ii)).

3. Free structures

In this book we introduce and study various kinds of 'global' topological structures that can be arranged in the form of a commutative diagram with forgetful functor in the vertical direction, and with adjoint functor pairs in the horizontal direction:

$$\begin{tikzcd}
\text{(ultra-commutative ring spectra)} \arrow[r, shift right, "\Omega^\bullet"'] \arrow[d] & \text{(ultra-commutative monoids)} \arrow[l, shift right, "\Sigma^\infty_+"'] \arrow[d] \\
(E_\infty\text{-ring spectra}) \arrow[r, shift right, "\Omega^\bullet"'] \arrow[d] & (E_\infty\text{-}\boxtimes\text{-monoids}) \arrow[l, shift right, "\Sigma^\infty_+"'] \arrow[d] \\
\text{(orthogonal spectra)} \arrow[r, shift right, "\Omega^\bullet"'] & \text{(orthogonal spaces)} \arrow[l, shift right, "\Sigma^\infty_+"']
\end{tikzcd}$$

There is a corresponding diagram of algebraic categories and forgetful functors that can be arranged in the same form:

$$\begin{tikzcd}
\text{(global power functors)} \arrow[r, "R\longmapsto R^\times"] \arrow[d, "U"'] & \text{(global power monoids)} \arrow[d, "U"] \\
\text{(global Green functors)} \arrow[r, "R\longmapsto R^\times"'] \arrow[d, "U"'] & \text{(abelian Rep-monoids)} \arrow[d, "U"] \\
\text{(global functors)} \arrow[r, "\text{forget}"'] & \text{(Rep-functors)}
\end{tikzcd}$$

In the horizontal direction we forget the additive structure (including the transfers) and remember the multiplicative structure. In other words, to a global Green functor R we assign the underlying multiplicative abelian Rep-monoid $R^\times$. So $R^\times(G) = (R(G), \cdot)$, the multiplicative monoid of the ring $R(G)$; the restriction maps for $R^\times$ are the same as for R. If R is a global power functor R, then the abelian Rep-monoid $R^\times$ also supports power operations, given by the multiplicative power operations P^m of R. In the vertical direction we first forget the multiplicative power operations and then the multiplication. The correspondence between the two diagrams is that at each of the six vertices, the algebraic category encodes the full natural structure on the equivariant homotopy sets $\underline{\pi}_0$ for objects in the topological category. This is proved for orthogonal spaces in Proposition I.5.16, for ultra-commutative monoids in two different incarnations in Theorem II.2.26 (iii) and Proposition II.2.36, and for ultra-commutative ring spectra in Proposition VI.1.6. For E_∞-$\boxtimes$-monoids and for E_∞-ring spectra, the analogous correspondence could be established by the same methods; we refrain from doing this since we don't have any immediate application. For orthogonal spectra, we have taken a different route and *defined* global functors as the category of the natural algebraic structure; Theorem IV.2.6 provides the explicit calculation of the algebraic structure encoded in a global functor. Strictly speaking, the respective theorems only identify the natural *unary* operations, but Remark II.2.27 illustrates in the example of ultra-commutative monoids how one can identify the natural n-ary operations for all $n \geq 0$.

In this section we want construct and discuss left adjoints to the horizontal algebraic functors. Since the horizontal functors remember the multiplicative structure and forget the additive structure, their left adjoints will have to 'freely build in the additive structure', including the additive transfers. A special feature of our situation is that the additive and multiplicative structures are in some sense 'independent'. More formally, the vertical forgetful functors commute with the horizontal left adjoints, i.e., the left adjoints can be chosen so that the following diagram commutes:

$$\begin{tikzcd}
\text{(global power functors)} \arrow[d] & \text{(global power monoids)} \arrow[l, "L"'] \arrow[d] \\
\text{(global Green functors)} \arrow[d] & \text{(abelian Rep-monoids)} \arrow[l, "L"'] \arrow[d] \\
\text{(global functors)} & \text{(Rep-functors)} \arrow[l, "L"]
\end{tikzcd}$$

We start by explicitly describing the left adjoint L : (Rep-functors) $\longrightarrow \mathcal{GF}$. The forgetful functor factors through the intermediate category of abelian Rep-groups, which is isomorphic to the category of additive functors from $\mathbb{Z}\,\mathrm{Rep}^{\mathrm{op}}$ to abelian groups:

$$\mathcal{GF} = \mathrm{Fun}^{\oplus}(\mathbf{A}, \mathcal{A}b) \;\longrightarrow\; \mathrm{Fun}^{\oplus}(\mathbb{Z}\,\mathrm{Rep}^{\mathrm{op}}, \mathcal{A}b) \;\cong\; \mathrm{Fun}(\mathrm{Rep}^{\mathrm{op}}, \mathcal{A}b) \;\longrightarrow\; \mathrm{Fun}(\mathrm{Rep}^{\mathrm{op}}, (\mathrm{sets}))\ .$$

We can thus define a left adjoint as the composite of the two left adjoints. The first forgetful functor has a left adjoint that takes the free abelian group objectwise. The left forgetful functor is restriction along the additive functor

$$j \ : \ \mathbb{Z}\,\mathrm{Rep}^{\mathrm{op}} \ \longrightarrow \ \mathbf{A}$$

that sends the conjugacy class of a continuous homomorphism $\alpha : K \longrightarrow G$ to the restriction operation $\alpha^* \in \mathbf{A}(G,K)$. So a left adjoint is given by enriched Kan extension along the additive functor j. If we spell this out we arrive at the following formula for the composite left adjoint:

$$LX \ = \ \mathrm{coker}\left(\bigoplus_{[K],[H]} \mathbb{Z}[X(H)] \otimes \mathbb{Z}[\mathrm{Rep}(K,H)] \otimes \mathbf{A}(K,-) \ \longrightarrow \ \bigoplus_{[H]} \mathbb{Z}[X(H)] \otimes \mathbf{A}(H,-) \right)$$

Colimits in the category of global functors are pointwise; so to extract the value of LX at a compact Lie group G, one simply inserts G into the variable slot.

We make the free global functor LX generated by a Rep-functor X more explicit. We let H be a closed subgroup of a compact Lie group G. We consider the composite

$$X(H) \ \longrightarrow \ (LX)(H) \ \xrightarrow{\mathrm{tr}_H^G} \ (LX)(G)$$

of the adjunction unit and the transfer map in the global functor LX. For every element $gH \in W_H G$ in the Weyl group we have

$$\mathrm{tr}_H^G \ = \ g_\star \circ \mathrm{tr}_H^G \ = \ \mathrm{tr}_H^G \circ g_\star \ ,$$

so the map is constant on $W_G H$-orbits of $X(H)$. The factorization through $W_G H \backslash X(H)$ is a set-theoretic map to an abelian group, so it extends uniquely to a homomorphism

$$\mathbb{Z}[W_G H \backslash X(H)] \ \longrightarrow \ (LX)(G)$$

on the free abelian group.

Proposition 3.1. *Let X be a* Rep*-functor. Then for every compact Lie group G, the map*

$$\bigoplus_{(H)} \mathbb{Z}[W_G H \backslash X(H)] \ \longrightarrow \ (LX)(G)$$

is an isomorphism, where the sum is indexed by the set of conjugacy classes of closed subgroups H of G with finite Weyl group.

The abelian Rep-monoids are the commutative monoids in the category of Rep-functors with respect to the pointwise cartesian product of functors. The global Green functors are the commutative monoids in the category of global functors with respect to the convolution $\Box$-product. A key property is thus that the left adjoint is strong symmetric monoidal with respect to these two symmetric monoidal structures. We define a monoidal transformation. We consider two Rep-functors X and Y and a compact Lie group G. Then the composite maps

$$\begin{aligned} X(G) \times Y(G) \ &\xrightarrow{\eta_X(G) \otimes \eta_Y(G)} (LX)(G) \otimes (LY)(G) \\ &\xrightarrow{i_{G,G}} ((LX)\Box(LY))(G \times G) \ \xrightarrow{\Delta_G^*} \ ((LX)\Box(LY))(G) \end{aligned}$$

are natural for restriction maps in G; here $\eta_X : X \longrightarrow LX$ is the adjunction unit, i is the universal bimorphism from (LX, LY) to $(LX)\Box(LY)$, and $\Delta_G : G \longrightarrow G \times G$ is the diagonal. So the above maps define a morphism of Rep-functors $X \times Y \longrightarrow (LX)\Box(LY)$. Adjoint to this is a morphism of global functors

$$L(X \times Y) \ \longrightarrow \ (LX)\Box(LY) \ . \tag{3.2}$$

There is a unique morphism of Rep-functors $\mathrm{Rep}(-,e) \longrightarrow \mathbf{A}(e,-) = \mathbf{A}$ that sends the identity in $\mathrm{Rep}(e,e)$ to the identity operation in $\mathbf{A}(e,e)$. Adjoint to this is a morphism of global functors

$$L(\mathrm{Rep}(-,e)) \ \longrightarrow\ \mathbb{A}\ . \tag{3.3}$$

PROPOSITION 3.4. *For all pairs of* Rep*-functors X and Y the morphism* (3.2) *of global functors*

$$L(X \times Y) \ \longrightarrow\ (LX)\Box(LY)$$

is an isomorphism. The morphism of Rep*-functors* (3.3) *is an isomorphism $L(\mathrm{Rep}(e,-)) \cong \mathbb{A}$. Moreover, these morphisms make the free functor*

$$L\ :\ (\text{Rep-functors}) \ \longrightarrow\ (\text{global functors})$$

into a strong symmetric monoidal functor for the product of Rep*-functors and the box product of global functors.*

PROOF. This is largely formal. The morphism (3.3) is an isomorphism because both $(L(\mathrm{Rep}(-,e), \eta_{\mathrm{Rep}(-,e)}(\mathrm{Id}_e))$ and $(\mathbb{A}, \mathrm{Id}_{\pi_0^e})$ represent the functor that sends a global functor F to the set $F(e)$. The monoidal products $\Box$ and $\times$ preserves colimits in both variables, and the free functor preserves colimits. So source and target of the natural transformation preserve colimits in each variable. Every Rep-functor is a coend of representable Rep-functors, so it suffices to check the claim when $X = \mathrm{Rep}(-,G)$ and $Y = \mathrm{Rep}(-,K)$ are represented Rep-functors. But $L\,\mathrm{Rep}(-,G)$ is representable by G, and convolution products of represented functors are represented functors (Proposition A.3.13). So

$$\begin{aligned}(L\,\mathrm{Rep}(-,G))\Box(L\,\mathrm{Rep}(-,K)) \ &\cong\ \mathbf{A}(G,-)\Box\mathbf{A}(K,-) \ \cong\ \mathbf{A}(G\times K,-)\\ &\cong\ L(\mathrm{Rep}(-,G\times K)) \ \cong\ L(\mathrm{Rep}(-,G)\times \mathrm{Rep}(-,K))\ .\end{aligned}$$

We omit the verification that the isomorphisms (3.2) are associative, commutative and unital. $\Box$

Now we discuss the free global Green functor generated by an abelian Rep-monoid, and the free global power functor generated by a global power monoid. We show that both kinds of free objects are given by the free global functor of the underlying Rep-functor, and that, loosely speaking, the multiplications and power operations 'come along free for the ride'. The next theorem makes precise that for every abelian Rep-monoid M, the global functor $L(UM)$ 'is' the free global Green functor generated by M. Similarly, for every global power monoid M, the global functor $L(UM)$ 'is' the free global power functor generated by M. The following terminology will be convenient for talking about the various free objects.

DEFINITION 3.5. (i) A *free global functor* generated by a Rep-functor X is a pair (L,i) consisting of a global functor L and a morphism of Rep-functors $i : X \longrightarrow UL$ with the following property: for every global functor F and every morphism of Rep-functors $j : X \longrightarrow UF$ there is a unique morphism of global functors $\varphi : L \longrightarrow F$ such that $(U\varphi)\circ i = j$.

(ii) A *free global Green functor* generated by an abelian Rep-monoid M is a pair (R,i) consisting of a global Green functor R and a morphism of abelian Rep-monoids $i : M \longrightarrow R^\times$ with the following property: for every global Green functor F and every morphism of abelian Rep-monoids $j : M \longrightarrow F^\times$ there is a unique morphism of global Green functors $\varphi : R \longrightarrow F$ such that $\varphi^\times \circ i = j$.

(iii) A *free global power functor* generated by a global power monoid M is a pair (R,i) consisting of a global power functor R and a morphism of global power monoids $i : M \longrightarrow R^\times$ with the following property: for every global power functor F and every morphism of global power monoids $j : M \longrightarrow F^\times$ there is a unique morphism of global power functors $\varphi : R \longrightarrow F$ such that $\varphi^\times \circ i = j$.

In other words, (L,i) is a free global functor for X if and only if it represents the functor

$$\mathcal{GF} \ \longrightarrow\ (\text{sets})\ , \quad F \ \longmapsto\ \text{Rep-functors}(X, UF)\ .$$

Similarly, free global Green functors and free global power functors are representing data for the functors

$$\text{(global Green functors)} \longrightarrow \text{(sets)}\ , \quad F \longmapsto \text{(abelian Rep-monoids)}(M, F^\times)$$
$$\text{(global power functors)} \longrightarrow \text{(sets)}\ , \quad F \longmapsto \text{(global power monoids)}(M, F^\times)$$

THEOREM 3.6. (i) *Let M be an abelian* Rep-*monoid and (L, i) a free global functor generated by the underlying* Rep-*functor of M. Then the global functor L has a unique structure of global Green functor such that $i : M \longrightarrow L^\times$ is a morphism of abelian* Rep-*monoids to the multiplicative abelian* Rep-*monoid of L. Moreover, with this structure the pair (L, i) is a free global Green functor generated by M.*

(ii) *Let M be a global power monoid and (R, i) a free global Green functor generated by the underlying abelian* Rep-*monoid of M. Then the global Green functor R has a unique structure of global power functor such that $i : M \longrightarrow R^\times$ is a morphism of global power monoids to the multiplicative global power monoids of R. Moreover, with this structure the pair (R, i) is a free global power functor generated by M.*

PROOF. (i) This is a formal consequence of the fact that the free functor L is strong symmetric monoidal, compare Proposition 3.4. In such a situation, it automatically lifts to a functor on commutative monoid objects, and that lift is automatically the structured free functor. Since I do not know of a convenient and complete reference, I make this a little more explicit. We let M be an abelian Rep-monoid and denote by UM the underlying Rep-functor. Then the composites

$$(L(UM))\Box(L(UM)) \ \cong_{(3.2)}\ L(UM \times UM) \ \xrightarrow{L\mu}\ L(UM)$$

and

$$\mathbb{A} \ \cong_{(3.3)}\ L(\mathrm{Rep}(-, e)) \ \xrightarrow{L\eta}\ L(UM)$$

make the global functor $L(UM)$ into a global Green functor, where $\mu : UM \times UM \longrightarrow UM$ and $\eta : \mathrm{Rep}(, e) \longrightarrow UM$ are the multiplication respectively unit of the monoid structure of M. The unit of the adjunction $\mathrm{Id} \longrightarrow LU$ is a monoidal transformation by the very construction of the morphisms (3.2) and (3.3); so the unit morphism for M is a morphism of abelian Rep-functors.

Now we let R be any global Green functor and $f : M \longrightarrow R^\times$ a morphism of abelian Rep-monoids. Since R is underlying a global Green functor, the underlying morphism $Uf : UM \longrightarrow U(R^\times) = (UR)^\times$ of Rep-functors has an adjoint morphism of global functors $f^\sharp : L(UM) \longrightarrow UR$. [...finish...]

(ii) This, too, is a formal argument, similar to the one showing that colimits in categories of coalgebras are created in the underlying category, and based on the following facts:

- global power functors are comonadic over the category of global Green functors (Theorem 2.10),
- global power monoids are comonadic over the category of abelian Rep-monoids (Theorem II.2.44),
- and the forgetful functor commutes with the two comonads, i.e., $\exp(R^\times) = \exp(R)^\times$ are the same abelian Rep-monoid for every global Green functor R.

We let UM denote the underlying abelian Rep-monoid of M. By the comonadic nature of global power monoids, the power operations of M provide a morphism of abelian Rep-monoids

$$P \ : \ UM \ \longrightarrow\ \exp(UM)$$

that is a coalgebra structure for the exp-comonad. The composite

$$UM \ \xrightarrow{P}\ \exp(UM) \ \xrightarrow{\exp(i)}\ \exp(R^\times) \ = \ \exp(R)^\times$$

is a morphism of abelian Rep-monoids to the multiplicative Rep-monoid of the global Green functor $\exp(R)$; the universal property of (R, i) provides a unique morphism of global Green functors $Q : R \longrightarrow \exp(R)$ such that

$$Q^\times \circ i \ = \ \exp(i) \circ P \ : \ UM \ \longrightarrow\ \exp(R^\times) = \exp(R)^\times\ . \tag{3.7}$$

We claim that the morphism Q is an exp-coalgebra structure on the global Green functor R. To see this, we observe the relations

$$\begin{aligned}(\exp(Q)\circ Q)^\times\circ i \;&=\; \exp(Q)^\times\circ\exp(i)\circ P \;=\; \exp(Q^\times\circ i)\circ P\\ &\underset{(3.7)}{=}\; \exp(\exp(i)\circ P)\circ P \;=\; \exp(\exp(i))\circ\exp(P)\circ P\\ &=\; \exp(\exp(i))\circ\kappa_{UM}\circ P \;=\; \kappa_{R^\times}\circ\exp(i)\circ P\\ &\underset{(3.7)}{=}\; (\kappa_R)^\times\circ\exp(i)\circ P \;=\; (\kappa_R\circ Q)^\times\circ i\ .\end{aligned}$$

The universal property of the pair (R,i) then implies the coassociativity relation $\exp(Q)\circ Q = \kappa_R\circ Q$. Similarly, we observe that

$$(\eta_R\circ Q)^\times\circ i \;=\; (\eta_R)^\times\circ Q^\times\circ i \;=_{(3.7)}\; \eta_{R^\times}\circ\exp(i)\circ P \;=\; i\circ\eta_{UM}\circ P \;=\; i\ .$$

The universal property then implies the counitality relation $\eta_R\circ Q = \mathrm{Id}_R$. So R is indeed an exp-coalgebra with respect to the morphism Q, i.e., a global power functor. The relation (3.7) says that $i : M \longrightarrow R^\times$ is a morphism of global power monoids. On the other hand, if $\bar{Q} : R \longrightarrow \exp(R)$ is any global power structure such that such that i is also a morphism of global power monoids for $\bar{Q}$, then the relation (3.7) must also hold with $\bar{Q}$ instead of Q. But then $Q^\times\circ i = \bar{Q}^\times\circ i$, and hence $Q = \bar{Q}$ by the universal property of (R,i). This proves the uniqueness of the global power structure.

Now we prove the final claim, namely that the coalgebra structure Q makes (R,i) a free global power functor generated by M. So we consider a global power functor F with structure morphism $P_F : F \longrightarrow \exp(F)$ and a morphism of global power monoids $j : M \longrightarrow F^\times$. The universal property of (R,i) provides a unique morphism of global Green functors $\varphi : R \longrightarrow F$ such that $j = \varphi^\times\circ i$. In particular, there is at most one morphism of global power functors with this property; it remains to show that φ is is even a morphism of global power functors. We observe the relations

$$(P_F\circ\varphi)^\times\circ i \;=\; P_F^\times\circ j \;=\; \exp(j)\circ P \;=\; \exp(\varphi^\times)\circ\exp(i)\circ P \;=_{(3.7)}\; (\exp(\varphi)\circ Q)^\times\circ i\ .$$

The universal property of (R,i) then shows that $P_F\circ\varphi = \exp(\varphi)\circ Q$, so φ is a morphism of global power functors. □

Now we explain in which sense the suspension spectrum construction realizes various free algebraic structures. We let Y be an orthogonal space, and $\Sigma^\infty_+ Y$ its unreduced suspension spectrum as defined in Construction IV.1.7. In (1.10) of Chapter IV we defined a 'stabilization' morphism of Rep-functors

$$\sigma \;:\; \underline{\pi}_0(Y) \;\longrightarrow\; \underline{\pi}_0(\Sigma^\infty_+ Y)$$

from the homotopy group Rep-functor of Y to the underlying Rep-functor of the global functor $\underline{\pi}_0(\Sigma^\infty_+ Y)$.

Now we consider an ultra-commutative monoid M. Then the suspension spectrum inherits the structure of an ultra-commutative ring spectrum, see Construction IV.1.19. In that situation, the morphism σ is induced by a morphism of orthogonal spaces $\eta : M \longrightarrow \Omega^\bullet(\Sigma^\infty_+ M)$, the unit of the adjunction $(\Sigma^\infty_+, \Omega^\bullet)$; η is a morphism of ultra-commutative monoids to the multiplicative monoid of the ring spectrum $\Sigma^\infty_+ M$, so the morphism $\sigma : \underline{\pi}_0(M) \longrightarrow \underline{\pi}_0(\Sigma^\infty_+ M)$ is a morphism of global power monoids to the multiplicative global power monoid of the global power functor $\underline{\pi}_0(\Sigma^\infty_+ M)$. The following theorem shows that the morphism σ is in fact universal in several ways.

Theorem 3.8. (i) *For every orthogonal space Y the pair $(\underline{\pi}_0(\Sigma^\infty_+ Y), \sigma)$ is a free global functor generated by the* Rep-*functor $\underline{\pi}_0(Y)$.*

(ii) *For every ultra-commutative monoid M the pair $(\underline{\pi}_0(\Sigma^\infty_+ M), \sigma)$ is a free global power functor generated by the global power monoid $\underline{\pi}_0(M)$.*

Proof. (i) This is the combination of the explicit description of a free global functor in Proposition 3.1 and the calculation of $\underline{\pi}_0(\Sigma^\infty_+ Y)$ in Proposition IV.1.11.

(ii) This is the combination of part (i) and Theorem 3.6. Indeed, $\underline{\pi}_0(\Sigma^\infty_+ M)$ is a global power functor and $\sigma : \underline{\pi}_0(M) \longrightarrow \underline{\pi}_0(\Sigma^\infty_+ M)^\times$ is a morphism of global power monoids. Since the underlying global functor of $\underline{\pi}_0(\Sigma^\infty_+ M)$ is freely generated by the underlying Rep-functor of $\underline{\pi}_0(M)$ (by part (i)), the pair $(\underline{\pi}_0(\Sigma^\infty_+ M), \sigma)$ is a free global Green functor generated by the underlying abelian Rep-monoid of $\underline{\pi}_0(M)$ (by Theorem 3.6 (i)), and hence a free global power functor generated by the global power monoid $\underline{\pi}_0(M)$ (by Theorem 3.6 (ii)). □

Remark 3.9. There are more variations of Theorem 3.8 for other kinds of structure. We mention two more without giving complete details. If M is an orthogonal monoid space (not necessarily commutative), then $\Sigma^\infty_+ M$ is an orthogonal ring spectrum. Moreover, $\underline{\pi}_0(M)$ then has the structure of a Rep-monoid (not necessarily abelian), and $\underline{\pi}_0(\Sigma^\infty_+ M)$ has the structure of a 'global ring functor', i.e., a monoid object (not necessarily commutative) in the category of global functors under the convolution box product. Put another way, a global ring functor is a global Green functor without the commutativity requirement for the multiplication.

If M is an E_∞-orthogonal monoid space, the $\Sigma^\infty_+ M$ is an E_∞-orthogonal ring spectrum. Then $\underline{\pi}_0(M)$ then has the structure of an abelian Rep-monoid and $\underline{\pi}_0(\Sigma^\infty_+ M)$ has the structure of a global Green functor'. Two additional versions of Theorem 3.8 are then as follows:

- For every orthogonal monoid space M the pair $(\underline{\pi}_0(\Sigma^\infty_+ M), \sigma)$ is a free global ring functor generated by the Rep-monoid $\underline{\pi}_0(M)$.
- For every E_∞-orthogonal monoid space M the pair $(\underline{\pi}_0(\Sigma^\infty_+ M), \sigma)$ is a free global Green functor generated by the abelian Rep-monoid $\underline{\pi}_0(M)$.

4. Examples

In this section we discuss various examples of and constructions with global power functors, and how they are realized topologically by examples of or constructions with ultra-commutative ring spectra. These examples include the Burnside ring global power functor (Example 4.1), the global functor represented by an abelian compact Lie group (Theorem 4.3), free global power functors (Construction 4.5), constant global power functors (Example 4.6), the complex representation ring global functor (Example 4.9), a global view on 'explicit Brauer induction' (Remark 4.12) and the global Green functor of class functions (Example 4.15).

Example 4.1 (Burnside ring global functor). The Burnside ring global functor $\mathbb{A} = \mathbf{A}(e, -)$ is the unit object for the box product of global functors, and hence an initial object in the category of global Green functors. Initial objects are examples of colimits, so Corollary 2.11 (i) implies that $\mathbb{A}$ has a unique structure of global power functor, and with this structure it is an initial global power functor. Indeed, there is a unique morphism $P : \mathbb{A} \longrightarrow \exp(\mathbb{A})$ of global Green functors (since $\mathbb{A}$ is initial), and the coalgebra diagrams commute (again since $\mathbb{A}$ is initial). With these power operations, $\mathbb{A}$ is also an initial global power functor.

We can make the power operations in the Burnside ring functor more explicit. Indeed, the group $\mathbb{A}(G)$ is free abelian with a basis given by the elements $t_H = \mathrm{tr}^G_H(p^*_H(1))$ for every conjugacy class of subgroups $H \leq G$ with finite Weyl group, where $p_H : H \longrightarrow e$ is the unique homomorphism. On these generators, the naturality properties of a global power functor force the power operations to be given by

$$\begin{aligned} P^m(t_H) \;&=\; P^m(\mathrm{tr}^G_H(p^*_H(1))) \;=\; \mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H}((\Sigma_m \wr p_H)^*(P^m(1))) \\ &=\; \mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H}((\Sigma_m \wr p_H)^*(p^*_{\Sigma_m}(1))) \;=\; \mathrm{tr}^{\Sigma_m \wr G}_{\Sigma_m \wr H}(p^*_{\Sigma_m \wr H}(1)) \;=\; t_{\Sigma_m \wr H}\ . \end{aligned} \tag{4.2}$$

This determines the power operations in general by the additivity property, and also shows the uniqueness.

When restricted to *finite* groups, the ring $\mathbb{A}(G)$ is isomorphic to the Grothendieck group of finite G-sets, and in this description the power operations are given by raising a finite G-set to a power, i.e., the power map

$$P^m \;:\; \mathbb{A}(G) \;\longrightarrow\; \mathbb{A}(\Sigma_m \wr G)$$

takes the class of a finite G-set S to the class of the $(\Sigma_m \wr G)$-set S^m. Indeed, for the additive generator $[G/H] = t_H$ of $\mathbb{A}(G)$ this is the relation (4.2), and for general finite G-sets it follows from the additivity formula for power operations and the fact that for two finite G-sets S and T the power $(S \coprod T)^m$ is $(\Sigma_m \wr G)$-equivariantly isomorphic to the coproduct

$$\coprod_{i=0}^{m} (\Sigma_m \wr G) \times_{(\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G)} S^i \times T^{m-i} \ .$$

The canonical power operations in the Burnside ring global functor correspond to the homotopy theoretic power operations for the global sphere spectrum. Indeed, since $\mathbb{A}$ is initial in both the category of global Green functors and in the category of global power functors, any isomorphism of global Green functors is automatically compatible with power operations. In other words, we can conclude that the square

$$\begin{array}{ccc} \mathbb{A}(G) & \xrightarrow{P^m} & \mathbb{A}(\Sigma_m \wr G) \\ \cong \downarrow & & \downarrow \cong \\ \pi_0^G(\mathbb{S}) & \xrightarrow[P^m]{} & \pi_0^{\Sigma_m \wr G}(\mathbb{S}) \end{array}$$

commutes for all G and m without even having to go back to the definition of the operations in $\underline{\pi}_0(\mathbb{S})$; the vertical maps are the action on the generator $1 \in \pi_0^e(\mathbb{S})$.

The previous example generalizes to the representable global functor $\mathbf{A}(A, -)$ for every *abelian* compact Lie group, which has a preferred structure of a global power functor. Since this representable global functor is freely generated by the identity 1_A in $\mathbf{A}(A, A)$, the power operations are all determined by naturality from the effect on this generator.

THEOREM 4.3. *Let A be an abelian compact Lie group A. The represented global functor $\mathbf{A}(A, -)$ has a unique structure of global power functor in which the multiplication is the composite*

$$\mathbf{A}(A, -) \Box \mathbf{A}(A, -) \ \cong \ \mathbf{A}(A \times A, -) \ \xrightarrow{\mathbf{A}(\mu^*, -)} \ \mathbf{A}(A, -) \ ,$$

where $\mu : A \times A \longrightarrow A$ is the multiplication of A and $\mu^* \in \mathbf{A}(A, A \times A)$ is the associated restriction morphism. The power operations satisfy

$$P^m(1_A) \ = \ p_m^* \ \in \mathbf{A}(A, \Sigma_m \wr A) \ , \tag{4.4}$$

the inflation operation of the continuous epimorphism

$$p_m \ : \ \Sigma_m \wr A \ \longrightarrow \ A \ , \quad (\sigma; a_1, \dots, a_m) \ \longmapsto \ a_1 \cdot \dots \cdot a_m \ .$$

Moreover, for every global power functor R the map

$$(\text{global power functors})(\mathbf{A}(A, -), R) \ \longrightarrow \ \{x \in R(A) \mid P^m(x) = p_m^*(x) \text{ for all } m \geq 1\}$$

sending a morphism $f : \mathbf{A}(A, -) \longrightarrow R$ to the class $f_A(1_A) \in R(A)$ is bijective.

PROOF. We showed in Proposition II.2.45 that the represented Rep-functor $\mathrm{Rep}(-, A)$ has a unique structure of global power monoid. Since $\mathbf{A}(A, -)$ is a free global functor generated by the representable Rep-functor $\mathrm{Rep}(-, A)$, the global power structure on $\mathrm{Rep}(-, A)$ induces a unique global Green functor structure on $\mathbf{A}(A, -)$ such that the morphism $i : \mathrm{Rep}(-, A) \longrightarrow \mathbf{A}(A, -)^\times$ sending the generator to the generator is a morphism of abelian Rep-monoids, by Theorem 3.6 (i). Since the multiplication in $\mathrm{Rep}(-, A)$ is induces by the multiplication in A, the multiplication of $\mathbf{A}(A, -)$ must also be induced by μ^*.

Theorem 3.6 (ii) then shows that the global Green structure on $\mathbf{A}(A, -)$ extends uniquely to the structure of a global power functor such that $i : \mathrm{Rep}(-, A) \longrightarrow \mathbf{A}(A, -)^\times$ is a morphism of global power monoids. Since the relation $P^m(1_A) = p_m^*$ holds in $\mathrm{Rep}(-, A)$ (by Proposition II.2.45), it also holds in

the global power functor $\mathbf{A}(A,-)$. Also by Theorem 3.6 (ii), $(\mathbf{A}(A,-),i)$ is a free global power functor generated by the global power monoid $\mathrm{Rep}(-,A)$. So morphisms of global power functors $\mathbf{A}(A,-)\longrightarrow R$ biject with morphisms of global power monoids $\mathrm{Rep}(-,A)\longrightarrow R^\times$, and Proposition II.2.45 implies the claim. $\square$

The global power functor $\mathbf{A}(A,-)$ described in Theorem 4.3 is realized by an ultra-commutative ring spectrum. Indeed, in Example II.3.22 we provided a global classifying space $B^{\otimes}_{\mathrm{gl}}A$ for the abelian group A with a commutative multiplication. The associated unreduced suspension spectrum $\Sigma^\infty_+ B^{\otimes}_{\mathrm{gl}}A$ is then an ultra-commutative ring spectrum. The Rep-functor $\underline{\pi}_0(B^{\otimes}_{\mathrm{gl}}A)$ is freely generated by the unstable tautological class $u_A\in\pi_0^A(B^{\otimes}_{\mathrm{gl}}A)$; by Proposition II.2.45 there is only one structure of global power monoid on $\mathrm{Rep}(-,A)$, so in particular the power operations on the generator are necessarily given by $P^m(u_A)=p_m^*(u_A)$.

Theorem IV.2.5 shows that $\underline{\pi}_0(\Sigma^\infty_+ B^{\otimes}_{\mathrm{gl}}A)$ is freely generated, as a global functor, by the stable tautological class $e_A\in\pi_0^A(\Sigma^\infty_+ B^{\otimes}_{\mathrm{gl}}A)$, the stabilization of the unstable tautological class u_A. On the other hand, the global power functor $\underline{\pi}_0(\Sigma^\infty_+ B^{\otimes}_{\mathrm{gl}}A)$ is freely generated by the global power monoid $\underline{\pi}_0(B^{\otimes}_{\mathrm{gl}}A)$, by Theorem 3.8 (ii). So the multiplication and power operations in $\pi_0^A(\Sigma^\infty_+ B^{\otimes}_{\mathrm{gl}}A)$ must be the ones that we characterized algebraically in Theorem 4.3. In particular,

$$P^m(e_A)\ =\ p_m^*(e_A)\qquad \text{in } \pi_0^{\Sigma_m\wr A}(\Sigma^\infty_+ B^{\otimes}_{\mathrm{gl}}A).$$

Construction 4.5 (Free global power functors). For a compact Lie group K we construct a *free global power functor* C_K generated by K. The underlying global functor is

$$C_K\ =\ \bigoplus_{m\geq 0}\mathbf{A}(\Sigma_m\wr K,-)\ ,$$

the direct sum of the global functors represented by the wreath products $\Sigma_m\wr K$, including the trivial group $\Sigma_0\wr K=e$. The multiplication $\mu: C_K\Box C_K\longrightarrow C_K$ that makes this into a global Green functor restricted to the (m,n)-summand is the morphism

$$\mathbf{A}(\Sigma_m\wr K,-)\Box\mathbf{A}(\Sigma_n\wr K,-)\ \longrightarrow\ C_K$$

that corresponds, via the universal property of the box product, to the bimorphism with (G,G')-component

$$\begin{aligned}\mathbf{A}(\Sigma_m\wr K,G)\Box\mathbf{A}(\Sigma_n\wr K,G')\ &\xrightarrow{\times}\ \mathbf{A}((\Sigma_m\wr K)\times(\Sigma_n\wr K),G\times G')\\ &\xrightarrow{\mathbf{A}(\Phi^*_{m,n},G\times G')}\ \mathbf{A}(\Sigma_{m+n}\wr K,G\times G')\ \xrightarrow{\mathrm{incl}}\ C_K(G\times G')\ ;\end{aligned}$$

here $\Phi^*_{m,n}$ is the restriction map associated to the embedding (2.5)

$$\Phi_{m,n}\ :\ (\Sigma_m\wr K)\times(\Sigma_n\wr K)\ \longrightarrow\ \Sigma_{m+n}\wr K\ .$$

The multiplication is associative because

$$\Phi_{m+n,k}\circ(\Phi_{m,n}\times(\Sigma_k\wr K))=\Phi_{m,n+k}\circ((\Sigma_m\wr K)\times\Phi_{n,k})\ :\ (\Sigma_m\wr K)\times(\Sigma_n\wr K)\times(\Sigma_k\wr K)\ \longrightarrow\ \Sigma_{m+n+k}\wr K\ .$$

The multiplication is commutative because the group homomorphisms

$$\Phi_{m,n}\ ,\ \Phi_{n,m}\circ\tau_{\Sigma_m\wr K,\Sigma_n\wr K}\ :\ (\Sigma_m\wr K)\times(\Sigma_n\wr K)\longrightarrow\ \Sigma_{m+n}\wr K$$

are conjugate, so they represent the same morphism in $\mathbf{A}((\Sigma_m\wr K)\times(\Sigma_n\wr K),\ \Sigma_{m+n}\wr K)$. The unit is the inclusion $\mathbf{A}(e,-)\longrightarrow C_K$ of the summand indexed by $m=0$.

The global Green functor C_K can be made into a global power functor in a unique way such that the relation

$$P^m(1_K)\ =\ 1_{\Sigma_m\wr K}$$

holds in the m-th summand of $C_K(\Sigma_m \wr K)$, where $1_K \in \mathbf{A}(K,K)$ and $1_{\Sigma_m \wr K} \in \mathbf{A}(\Sigma_m \wr K, \Sigma_m \wr K)$ are the identity operations. Indeed, C_K is generated as a global functor by the classes $1_{\Sigma_m \wr K}$ for all $k \geq 0$, so there is at most one such global power structure, and every morphism of global power functors out of C_K is determined by its values on the class 1_K. The existence of a global power structure on C_K with this property could be justified purely algebraically, but we show it by realizing C_K by an ultra-commutative ring spectrum.

The unreduced suspension spectrum

$$\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}K) \;\cong\; \bigvee_{m\geq 0} \Sigma^\infty_+ B_{\mathrm{gl}}(\Sigma_m \wr K)$$

of the free ultra-commutative monoid (compare Example II.1.7) generated by a global classifying space of K is an ultra-commutative ring spectrum. According to Proposition IV.2.5, its 0-th homotopy group global functor is given additively by

$$\underline{\pi}_0(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}K)) \;\cong\; \bigoplus_{m\geq 0} \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}(\Sigma_m \wr K)) \;\cong\; \bigoplus_{m\geq 0} \mathbf{A}(\Sigma_m \wr K, -)\ .$$

Under this isomorphism, the stable tautological class $e_K \in \pi_0^K(\Sigma^\infty_+ B_{\mathrm{gl}}K)$ maps to the generator $1_K \in \mathbf{A}(K,K)$. The class e_K is the stabilization of the unstable tautological class $u_K \in \pi_0^K(B_{\mathrm{gl}}K)$, whose m-th power is $[m](u_K) = u_{\Sigma_m \wr K}$ in $\pi_0^{\Sigma_m \wr K}(\mathbb{P}(B_{\mathrm{gl}}K))$ (see (2.22) of Chapter II). The stabilization map $\sigma : \underline{\pi}_0(B_{\mathrm{gl}}K) \longrightarrow \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)$ commutes with power operations, so this shows that

$$P^m(e_K) \;=\; P^m(\sigma^K(u_K)) \;=\; \sigma^{\Sigma_m \wr K}([m](u_K)) \;=\; \sigma^{\Sigma_m \wr K}(u_{\Sigma_m \wr K}) \;=\; e_{\Sigma_m \wr K}$$

in $\pi_0^{\Sigma_m \wr K}(\Sigma^\infty_+ \mathbb{P}(B_{\mathrm{gl}}K))$.

We warn the reader that the previous example of the free global power functor C_K is *not* the symmetric algebra, with respect to the box product of global functors, of the represented global functor $\mathbf{A}(K,-)$. The issue is that the global functors

$$\mathbf{A}(K,-)^{\square m}/\Sigma_m \;\cong\; \mathbf{A}(K^m,-)/\Sigma_m \qquad \text{and} \qquad \mathbf{A}(\Sigma_m \wr K, -)$$

are typically not isomorphic. The restriction map $\mathrm{res}^{\Sigma_m \wr K}_{K^m} \in \mathbf{A}(\Sigma_m \wr K, K^m)$ induces a morphism of represented global functors

$$-\circ \mathrm{res}^{\Sigma_m \wr K}_{K^m} \;:\; \mathbf{A}(K^m,-) \;\longrightarrow \mathbf{A}(\Sigma_m \wr K, -)$$

that equalizes the Σ_m-action on the source because every permutation of the factors of K^m becomes an inner automorphism in $\Sigma_m \wr K$. So the morphism factors over a morphism of global functors

$$\mathbf{A}(K^m,-)/\Sigma_m \;\longrightarrow\; \mathbf{A}(\Sigma_m \wr K, -)$$

which, however, is generally *not* an isomorphism (already for $K = e$ and $m = 2$). The box product symmetric algebra

$$\bigoplus_{m\geq 0} \mathbf{A}(K,-)^{\square m}/\Sigma_m \;\cong\; \bigoplus_{m\geq 0} \mathbf{A}(K^m,-)/\Sigma_m$$

also has a universal property: it is the free global Green functor generated by K. However, this box product symmetric algebra does not seem to have natural power operations.

Example 4.6 (Constant global power functors and Eilenberg-Mac Lane spectra). We let B be a commutative ring. Then the constant global functor $\underline{B}$ (Example IV.2.8 (iii)) is naturally a global Green functor, via the ring structure of B. For every compact Lie group G, every exponential sequence $x \in \exp(\underline{B}; G)$ satisfies $x_n = (x_1)^n$; so an exponential sequence is completely determined by the element x_1, and the morphism

$$\eta_{\underline{B}} \;:\; \exp(\underline{B}) \;\longrightarrow\; \underline{B}$$

is an isomorphism of global Green functors. So when restricted to constant global Green functors, the exp comonad is isomorphic to the identity. Thus $\underline{B}$ has a unique structure of coalgebra over the comonad exp, with structure morphism the inverse of $\eta_{\underline{B}}$.

The preferred global power structure on the constant global functor $\underline{B}$ can also be derived more directly. In fact, since the power operation P^m has to be an equivariant refinement of the m-th power map in $R(G)$ (compare Remark II.2.9), the only possibility to define P^m is as

$$P^m \ : \ \underline{B}(G) \ = B \ \longrightarrow \ B = \underline{B}(\Sigma_m \wr G) \ , \quad b \ \longmapsto \ b^m \ ,$$

the mth power in the ring B.

Since $\underline{B}$ is constant, every morphism $R \longrightarrow \underline{B}$ of global functors is determined by the map $R(e) \longrightarrow \underline{B}(e) = B$. Moreover, every ring homomorphism $\psi : R(e) \longrightarrow B$ extends uniquely to a morphism of global power functors $\hat{\psi} : R \longrightarrow \underline{B}$ by defining its value at a compact Lie group as the composite

$$R(G) \ \xrightarrow{p_G^*} \ R(e) \ \xrightarrow{\psi} \ B \ .$$

In other words, the functor

$$(\text{commutative rings}) \ \longrightarrow \ (\text{global power functors}) \ , \quad B \ \longmapsto \ \underline{B}$$

is right adjoint to the functor that takes a global power functor R to the ring $R(e)$.

The author does not know an explicit pointset level model for an ultra-commutative ring spectrum that realizes the constant global power functor $\underline{B}$. The Eilenberg-Mac Lane spectrum $\mathcal{H}B$, discussed in Construction 1.9 below, is an ultra-commutative ring spectrum and tries to realize $\underline{B}$: the global power functor $\underline{\pi}_0(\mathcal{H}B)$ is indeed constant *on finite groups*, but the restriction maps are not generally isomorphisms, compare Example 1.16. The morphism of global power functors $\underline{\pi}_0(\mathcal{H}B) \longrightarrow \underline{B}$ adjoint to the identification $\pi_0^e(\mathcal{H}B) \cong B$ is thus an isomorphism at finite groups (and some other compact Lie groups), but *not* an isomorphism in general.

Example 4.7 (Monoid rings). Let R be a global Green functor and M a commutative monoid. We denote by $R[M]$ the *monoid ring functor*; its value at a compact Lie group G is given by

$$(R[M])(G) \ = \ R(G)[M] \ ,$$

the monoid ring of M over $R(G)$. The structure as global functor is induced from the structure of R and constant in M. The multiplication and unit are induced from the multiplication and units of R and M. The global Green functor $R[M]$ can be characterized as follows by the functor that it represents: for every global Green functor S, morphisms $R[M] \longrightarrow S$ biject with pairs consisting of a morphism of global Green functors $R \longrightarrow S$ and a monoid homomorphism $M \longrightarrow (S(e), \cdot)$ to the multiplicative monoid of the underlying ring $S(e)$.

Now suppose that R is even a global power functor. Then $R[M]$ inherits a natural structure as global power functor; indeed, we define the power operation

$$P^n \ : \ R(G)[M] \ \longrightarrow \ R(\Sigma_n \wr G)[M]$$

by $P^n(r \cdot m) = P^n(r) \cdot m^n$ for $r \in R(G)$ and $m \in M$, and then we extend this by additivity to general elements in $R(G)[M]$. In this upgraded setting, the global power functor $R[M]$ has a similar characterization as in the previous paragraph: for a global power functor S, we let

$$\mathrm{Mon}(S) \ = \ \{x \in S(e) \mid P^m(x) = p_{\Sigma_m}^*(x) \text{ for all } m \geq 1\}$$

be the set of *monoid-like elements* of S; here $p_{\Sigma_m} : \Sigma_m \longrightarrow e$ is the unique homomorphism. The set $\mathrm{Mon}(S)$ is a submonoid of the multiplicative monoid of $S(e)$. Then morphisms $R[M] \longrightarrow S$ biject with pairs consisting of a morphism of global power functors $R \longrightarrow S$ and a monoid homomorphism $M \longrightarrow \mathrm{Mon}(S)$ to the monoid-like elements of S.

We let E be an ultra-commutative ring spectrum and M a commutative monoid. Then the monoid ring spectrum $E[M] = M_+ \wedge E$ is another ultra-commutative ring spectrum. The unit of M induces a morphism of ultra-commutative ring spectra $E \longrightarrow E[M]$ that induces a morphism of global power functors $\underline{\pi}_0(E) \longrightarrow \underline{\pi}_0(E[M])$. Similarly, the unit of E induces a monoid homomorphism $M \longrightarrow M_+ \wedge E(0)$, and this induces a monoid homomorphism $M \longrightarrow \pi_0^e(E[M])$ to the multiplicative monoid of the ring $\pi_0^e(E[M])$. The universal property of the algebraic monoid ring combines these two pieces of data into a morphism of global power functors

$$(\underline{\pi}_0 E)[M] \;\longrightarrow\; \underline{\pi}_0(E[M]) \;. \tag{4.8}$$

Additively, the left hand side is a direct sum of copies of $\underline{\pi}_0(E)$, indexed by the elements of M. Similarly, the orthogonal spectrum $E[M]$ is a wedge of copies of E, indexed by the elements of M. So the right hand side $\underline{\pi}_0(E[M])$ is also a direct sum of copies of $\underline{\pi}_0(E)$ (by Corollary III.1.38 (i)). We conclude that the preferred morphism of global power functors (4.8) is an isomorphism.

Example 4.9 (Representation ring global functor). As G varies over all compact Lie groups, the unitary representation rings $\mathbf{RU}(G)$ form the *unitary representation ring global functor* $\mathbf{RU}$, compare Example IV.2.8 (iv). This is classical in the restricted realm of finite groups, but somewhat less familiar for compact Lie groups in general. The restriction maps $\alpha^* : \mathbf{RU}(G) \longrightarrow \mathbf{RU}(K)$ are induced by restriction of representations along a homomorphism $\alpha : K \longrightarrow G$. The transfer maps $\mathrm{tr}_H^G : \mathbf{RU}(H) \longrightarrow \mathbf{RU}(G)$ along a closed subgroup inclusion $H \leq G$ are given by the *smooth induction* of Segal [**144**, § 2]. If H is a subgroup of finite index of G, then this induction sends the class of an H-representation W to the induced G-representation $\mathrm{map}^G(H, W)$; in general, induction sends actual representations to virtual representations.

The key tool for understanding the smooth induction is Segal's character formula [**144**, p. 119] that we now recall. Given an element $g \in G$, we let $F_1, \ldots, F_k$ be the connected components of the fixed point space $(G/H)^g$, and we choose $a_i, \ldots, a_k \in G$ such that $a_i H \in F_i$. Then for every class $x \in \mathbf{RU}(H)$, the character of the induced representation is given in terms of the character of x by

$$\chi(\mathrm{tr}_H^G(x))(g) \;=\; \sum_{i=1}^{k} \chi(F_i) \cdot \chi(x)(a_i^{-1} g a_i) \;. \tag{4.10}$$

Segal had originally stated this formula only for *regular* elements g, which means in particular that $(G/H)^g$ consists of isolated fixed points. So for regular g, each component F_i is a point, and thus $\chi(F_i) = 1$. The regular elements are dense in G, and since the character is continuous, it is determined by its values on regular elements. Oliver showed in [**120**, Prop. 2.3], that the formula (4.10) is in fact continuous as a function of g. Since the formula agrees with Segal's for regular elements, it describes the character of the induced representation in general. When H has finite index in G, then $(G/H)^g$ is finite and each component F_i is a point. So then $\chi(F_i) = 1$ and the formula (4.16) simplifies to the classical formula for the character of an induced representation.

In the generality of compact Lie groups, the double coset formula for $\mathbf{RU}$ was proven by Snaith [**147**, Thm. 2.4]. The representation rings also have well-known power operations

$$P^m \;:\; \mathbf{RU}(G) \;\longrightarrow\; \mathbf{RU}(\Sigma_m \wr G) \;;$$

on the class of a G-representation V, the power operation is represented by the tensor power,

$$P^m[V] \;=\; [V^{\otimes m}]$$

using the canonical action of $\Sigma_m \wr G$ on $V^{\otimes m}$. Since power operations are not additive, one has to argue why this assignment extends to virtual vector bundles. The standard way is to assemble all power operations on representations into a map

$$P \;:\; \mathbf{RU}^+(G) \;\longrightarrow\; \exp(\mathbf{RU}; G) \;, \quad P([V]) \;=\; \big([V^{\otimes m}]\big)_{m \geq 0}$$

from the monoid of isomorphism classes of G-representations. If W is another G-representation, then

$$(V \oplus W)^{\otimes m} \qquad \text{and} \qquad \bigoplus_{k=0}^{m} \mathrm{tr}^{\Sigma_m \wr G}_{(\Sigma_k \wr G)\times(\Sigma_{m-k}\wr G)}(V^{\otimes k} \otimes W^{\otimes(m-k)})$$

are isomorphic as $(\Sigma_m \wr G)$-representations, because tensor product distributes over direct sum. This means that the total power map P is a monoid homomorphism from $\mathbf{RU}^+(G)$ to the group $\exp(\mathbf{RU}; G)$ under $\oplus$. So the total power operation extends uniquely to a group homomorphism

$$P \ : \ \mathbf{RU}(G) \ \longrightarrow \ \exp(\mathbf{RU}; G)$$

on the representation ring. The operation P^m is then the m-th factor of this extension.

The representation ring global functor $\mathbf{RU}$ is realized by the ultra-commutative ring spectrum $\mathbf{KU}$, the unitary global K-theory spectrum, see Theorem 5.17 below.

Another special property of the representation ring global functor is the following:

Proposition 4.11. *Let G be a connected compact Lie group, T a maximal torus and $N = N_G T$ the maximal torus normalizer. Then*

$$\mathrm{tr}^G_N(1) \ = \ 1$$

in the ring $\mathbf{RU}(G)$. Moreover, the composite

$$\mathbf{RU}(G) \ \xrightarrow{\mathrm{res}^G_N} \ \mathbf{RU}(N) \ \xrightarrow{\mathrm{tr}^G_N} \ \mathbf{RU}(G)$$

is the identity.

Proof. If $g \in G$ is such that $tgN = gN$ for all $t \in T$, then $T^g \subseteq N$. Since T^g is connected, it is contained in the identity component $N^\circ = T$. So we conclude that $T^g = T$, i.e., $g \in N$. Altogether this shows that $(G/N)^T = \{N\}$.

Now we can prove the proposition. We let $g \in T$ be any regular element of the maximal torus. Then T is the closure of the subgroup generated by g, and hence

$$(G/N)^g \ = \ (G/N)^T$$

is a single point. The character formula (4.10) thus shows that

$$\chi(\mathrm{tr}^G_N(1))(g) \ = \ 1 \ .$$

Since the regular elements are dense in T, the restriction of the character $\chi(\mathrm{tr}^G_N(1))$ to T is identically 1. Since every element of G is conjugate to an element in T, the character of $\mathrm{tr}^G_N(1)$ is identically 1. Since elements of $\mathbf{RU}(G)$ are characterized by their characters, this shows the first relation. The reciprocity formula then yields

$$\mathrm{tr}^G_N(\mathrm{res}^G_N(x)) \ = \ \mathrm{tr}^G_N(1) \cdot x \ = \ x$$

for all $x \in \mathbf{RU}(G)$. □

Remark 4.12 (Brauer induction). By Brauer's theorem [**26**, Thm. I] the complex representation ring of a finite group is generated, as an abelian group, by representations that are induced from 1-dimensional representations of subgroups. Segal generalized this result to compact Lie groups in [**144**, Prop. 3.11 (ii)], where 'induction' refers to the smooth induction (that generalizes the classical induction for finite index inclusions). In fact, in the world of compact Lie group, Segal's smooth induction for not necessarily finite index subgroups makes the proof quite transparent, as we shall now recall. In our language the statement can be expressed by saying that the representation ring global functor $\mathbf{RU}$ is 'cyclic' in the sense that it is generated by a single element, the class $x \in \mathbf{RU}(T)$ of the tautological 1-dimensional representation of the circle group $T = U(1)$. Equivalently, the morphism of global functors

$$\mathrm{ev}_x \ : \ \mathbf{A}(T, -) \ \longrightarrow \ \mathbf{RU}$$

classified by the element x is an epimorphism. We recall the argument: we let $i : T \times U(n-1) \longrightarrow U(n)$ be the block sum embedding and $q : T \times U(n-1) \longrightarrow T$ the projection to the first factor. The character formula (4.10) for induced representations shows that the element

$$\mathrm{tr}^{U(n)}_{T\times U(n-1)}(q^*(x)) \;=\; i_!(q^*(x)) \;\in\; \mathbf{RU}(U(n)) \tag{4.13}$$

has the same character as the tautological n-dimensional representation of $U(n)$. Since characters determine unitary representations of compact Lie groups, $i_!(q^*(x))$ equals the class of the tautological representation τ_n of $U(n)$. Any unitary representation of a compact Lie group G of dimension n is isomorphic to $\alpha^*(\tau_n)$ for a continuous homomorphism $\alpha : G \longrightarrow U(n)$; so the class of such a representation equals

$$\alpha^*(i_!(q^*(x))) \;\in\; \mathbf{RU}(G)\ .$$

So the global functor $\mathbf{RU}$ is generated by the single class $x = \tau_1$.

An interesting line of investigation, dubbed *explicit Brauer induction*, started with Snaith's paper [**147**]. Informally speaking, an 'explicit Brauer induction' is a section to the map

$$\mathbf{A}(T,G) \;\longrightarrow\; \mathbf{RU}(G)$$

that is specified by a direct recipe, for example an explicit formula, and has naturality properties as the group G varies. So such a map is an 'explicit and natural' way to write a (virtual) representation as a sum of induced representations of 1-dimensional representations. The first explicit Brauer induction was Snaith's formula [**147**, Thm. (2.16)]; however, Snaith's maps are not additive and not compatible with restriction to subgroups. Later Boltje [**23**] specified a different explicit Brauer induction formula by purely algebraic means; Symonds [**157**] gave a topological interpretation of Boltje's construction. The Boltje-Symonds maps are additive and natural for restriction along group homomorphisms; the maps are not (and in fact cannot be) in general compatible with transfers. In our present language, the Boltje-Symonds maps form a natural transformation of Rep-abelian groups from $\mathbf{RU}$ to $\mathbf{A}(T,-)$. We recall the construction of these maps; we follow Symonds' approach, as his reasoning is very much in the spirit of global homotopy theory.

The starting point is the formula (4.13) that expresses the class of the tautological $U(n)$-representation in $\mathbf{RU}(U(n))$ as a smooth induction of a specific 1-dimensional representation of the subgroup $T\times U(n-1)$, namely the one whose character is the projection

$$q \;:\; T\times U(n-1) \;\longrightarrow\; T$$

to the first factor. This formula suggest a class in $\mathbf{A}(T,U(n))$ as the image of the tautological $U(n)$-representation; if we also want naturality and additivity, then this fixes things completely:

THEOREM 4.14 (Boltje [**23**], Symonds [**157**]). *There is a unique natural transformation of* Rep-*abelian groups*

$$b \;:\; \mathbf{RU} \;\longrightarrow\; \mathbf{A}(T,-)$$

that satisfies

$$b_{U(n)}[\mathbb{C}^n] \;=\; \mathrm{tr}^{U(n)}_{T\times U(n-1)} \circ q^* \;=\; \in\; \mathbf{A}(T,U(n))\ .$$

Moreover:

(i) *The transformation b is a section to the morphism of global power functors* $\mathrm{ev}_x : \mathbf{A}(T,-) \longrightarrow \mathbf{RU}$ *given by evaluation at the class $x \in \mathbf{RU}(T)$.*

(ii) *The value of b_G at the 1-dimensional representation with character $\chi : G \longrightarrow U(1)$ is given by*

$$b_G[\chi^*(\mathbb{C})] \;=\; \chi^* \;\in\; \mathbf{A}(T,G)\ .$$

PROOF. Every class in $\mathbf{RU}(G)$ is a formal difference of classes of actual representations, and every n-dimensional representation is the restriction of $\mathbb{C}^n$ along some continuous homomorphism $G \longrightarrow U(n)$. So uniqueness is a consequence of naturality and additivity.

Conversely, this also suggests how to define the transformation. If V is any n-dimensional unitary G-representation, then V is isomorphic to $\alpha^*(\mathbb{C}^n)$ for some continuous homomorphism $\alpha : G \longrightarrow U(n)$ that is unique up to conjugacy. So we set

$$b_G[V] \;=\; b[\alpha^*(\mathbb{C}^n)] \;=\; \alpha^* \circ \mathrm{tr}^{U(n)}_{T\times U(n-1)} \circ q^* \;\in\; \mathbf{A}(T,G)\ .$$

This defines set theoretic maps

$$b_G \;:\; \mathbf{RU}^+(G) \;\longrightarrow\; \mathbf{A}(T,G)$$

from the monoid of isomorphism classes of unitary G-representations, and these maps are automatically compatible with restriction along group homomorphisms. The double coset formula (4.16) of Chapter IV for $\mathrm{res}^{U(n+m)}_{U(n)\times U(m)} \circ \mathrm{tr}^{U(n+m)}_{T\times_{U(n+m-1)}}$ implies the relation

$$\mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(b_{U(n+m)}[\mathbb{C}^{n+m}]) \;=\; b_{U(n)}[\mathbb{C}^n] \oplus b_{U(m)}[\mathbb{C}^m]$$

in the group $\mathbf{A}(T, U(n)\times U(m))$. Hence the maps b_G are additive, and so they extend uniquely to a group homomorphism

$$b_G \;:\; \mathbf{RU}(G) \;\longrightarrow\; \mathbf{A}(T,G)$$

on the Grothendieck group, for which we use the same name. These homomorphisms are still compatible with restriction along group homomorphisms.

It remains to show the additional properties. The transformation $\mathrm{ev}_x \circ b : \mathbf{RU} \longrightarrow \mathbf{RU}$ is additive and natural for restriction along continuous homomorphisms, so for property (i) it suffices to show the relation $\mathrm{ev}_x \circ b = \mathrm{Id}$ in the universal examples, i.e., for the tautological representations of the unitary groups $U(n)$. This universal example is taken care of by the formula (4.13).

Property (ii) holds because $U(1) = T$ and for $n = 1$ the map q is the identity of T. □

While this definition of the 'explicit Brauer map' $b : \mathbf{RU} \longrightarrow \mathbf{A}(T,-)$ is very slick, it is not yet particularly explicit. To write the class $b_G[V]$ as a $\mathbb{Z}$-linear combination of transfers of 1-dimensional representations of subgroups of G, one would now have to write the classifying homomorphism $\alpha : G \longrightarrow U(n)$ as the composite of an epimorphism and a subgroup inclusion and then expand the term $\alpha^* \circ \mathrm{tr}^{U(n)}_{T\times U(n-1)}$ using the compatibility of transfers with inflation, and the double coset formula for the restriction of a transfer. Readers desperate for a truly explicit formula can find one in [**23**, Thm. (2.1)] or [**24**, Thm. 2.24 (e)].

Example 4.15. We define a global Green functor Cl of *class functions* on objects by

$$\mathrm{Cl}(G) \;=\; \mathrm{map}(G,\mathbb{C})^G\ ,$$

the ring of $\mathbb{C}$-valued class functions on G, i.e., continuous functions $\varphi : G \longrightarrow \mathbb{C}$ such that $\varphi(\gamma g \gamma^{-1}) = \varphi(g)$ for all $g, \gamma \in G$. As far as the author is aware, the global Green functor Cl does *not* extend to a global power functor.

We give the set $\mathrm{Cl}(G)$ the pointwise ring structure. A group homomorphism $\alpha : K \longrightarrow G$ induces a map

$$\alpha^* \;:\; \mathrm{Cl}(G) \;\longrightarrow\; \mathrm{Cl}(K)\ , \quad \alpha^*(\varphi)(k) \;= \varphi(\alpha(k))\ .$$

This defines the contravariant functoriality of Cl. If H is a closed subgroup of G, then a transfer map

$$\mathrm{tr}^G_H \;:\; \mathrm{Cl}(H) \;\longrightarrow\; \mathrm{Cl}(G)$$

was defined by Oliver [**120**], motivated by the character formula for Segal's smooth induction for representations. Given a class function $\varphi \in \mathrm{Cl}(H)$ and an element $g \in G$, we let $F_1, \ldots, F_k$ be the connected

components of the space $(G/H)^g$, and we choose $a_i, \dots, a_k \in G$ such that $a_i H \in F_i$. Then by [**120**, Prop. 2.3], the rule

$$(\mathrm{tr}_H^G(\varphi))(g) \;=\; \sum_{i=1}^{k} \chi(F_i) \cdot \varphi(a_i^{-1} g a_i) \tag{4.16}$$

defines a class function $\mathrm{tr}_H^G(\varphi) \in \mathrm{Cl}(G)$, where $\chi(F_i)$ is the Euler characteristic of F_i. When H has finite index in G, then $(G/H)^g$ is finite and each component F_i is a point. So then $\chi(F_i) = 1$ and the formula (4.16) simplifies to the classical formula

$$(\mathrm{tr}_H^G(\varphi))(g) \;=\; \sum_{aH \in (G/H)^g} \varphi(a^{-1} g a)$$

for the character of an induced representation.

Proposition 4.17. *The functor* Cl *of class functions defines a Green functor.*

Proof. Clearly, the restriction maps are contravariantly functorial, and inner automorphisms induce the identity, by the very definition of class functions. The most difficult parts of the argument have already been verified by Oliver, namely the double coset formula in [**120**, Lemma 2.4] and the transitivity of these transfers in [**120**, Lemma 2.6]. It is straightforward that transfers commute with inflation; indeed, if $\alpha : K \longrightarrow G$ is a continuous epimorphism, $L = \alpha^*(H)$ is the preimage of H and $k \in K$, then

$$(K/L)^k \;\xrightarrow{\cong}\; (\alpha^*(G/H))^k \;=\; (G/H)^{\alpha(k)} \;, \quad \kappa L \;\longmapsto\; \alpha(\kappa) H$$

is a homeomorphism. We let $F_1, \dots, F_k$ be the connected components of $(G/H)^{\alpha(k)}$ and choose $b_1, \dots, b_k \in K$ such that $\alpha(b_i)H \in F_i$. The cosets $b_1 L, \dots b_k L$ then represent the path components of $(K/L)^k$, so for every class function $\varphi \in \mathrm{Cl}(H)$,

$$\begin{aligned}(\alpha^* \circ \mathrm{tr}_H^G)(\varphi)(k) \;&=\; (\mathrm{tr}_H^G(\varphi))(\alpha(k)) \;=\; \sum_{i=1}^{k} \chi(F_i) \cdot \varphi(\alpha(b_i)^{-1} \alpha(k) \alpha(b_i)) \\ &=\; \sum_{i=1}^{k} \chi(F_i) \cdot (\alpha|_L^*(\varphi))(b_i^{-1} k b_i) \;=\; (\mathrm{tr}_L^K \circ \alpha|_L^*)(\varphi)(k) \;.\end{aligned}$$

Now we suppose that the Weyl group of H is infinite, and we argue that then $\mathrm{tr}_H^G(\varphi) = 0$. The Weyl group acts freely and smoothly from the right on G/H by

$$G/H \times W_G H \;\longrightarrow\; G/H \;, \quad (gH, nH) \;\longmapsto\; (gn)H \;.$$

This action is by G-equivariant maps, so it restricts to a smooth action on the fixed point space $(G/H)^g$ for every $g \in G$. Since $W_G H$ has positive dimension, it contains a subgroup $T \leq W_G H$ that is isomorphic to the circle group $U(1)$. This way we obtain a smooth and free T-action on $(G/H)^g$. Each path component F_i of $(G/H)^g$ is T-invariant, and multiplicativity of Euler characteristic for smooth fiber bundles shows that

$$\chi(F_i) \;=\; \chi(T) \times \chi(F_i/T) \;=\; 0 \;.$$

Since these Euler characteristics occur as factors in the defining formula (4.16), we conclude that $\mathrm{tr}_H^G(\varphi)(g) = 0$ for all $g \in G$. This shows that the class functions form a global functor with respect to the restriction and transfer maps.

Clearly, the restriction maps are ring homomorphisms. For $\varphi \in \mathrm{Cl}(H)$, $\psi \in \mathrm{Cl}(G)$ and $g \in G$ we have the relation

$$\begin{aligned}\left(\mathrm{tr}_H^G(\varphi)\cdot\psi\right)(g) &= \left(\sum_{i=1}^k \chi(F_i)\cdot\varphi(a_i^{-1}ga_i)\right)\cdot\psi(g)\\ &= \sum_{i=1}^k \chi(F_i)\cdot\varphi(a_i^{-1}ga_i)\cdot\psi(a_i^{-1}ga_i) \;=\; \mathrm{tr}_H^G\left(\varphi\cdot\mathrm{res}_H^G(\psi)\right)(g)\ ;\end{aligned}$$

the second equation is the hypothesis that ψ is a class function for G. This establishes reciprocity, so Cl is a global Green functor. □

The maps

$$\chi(G)\ :\ \mathbf{RU}(G)\ \longrightarrow\ \mathrm{Cl}(G)$$

that send the class of a virtual representation to its character define the *character morphism* of global Green functors

$$\chi\ :\ \mathbf{RU}\ \longrightarrow\ \mathrm{Cl}\ .$$

Indeed, compatibility of characters with restriction, sum and product is well known. When $[G:H]$ is finite, the compatibility with transfers is the classical formula for the character of an induced representation. When H is a general closed subgroup of G, then it is Segal's formula [**144**, p. 119] for the character of the smooth induction. Representations are determined by their characters, so the character morphism χ is a monomorphism. The character map extends to a morphism of global Green functors

$$\tilde{\chi}\ :\ \mathbb{C}\otimes\mathbf{RU}\ \longrightarrow\ \mathrm{Cl}$$

which is a $\mathcal{F}in$-isomorphism. The global Green functor Cl shares certain properties with the representation ring global functor $\mathbf{RU}$. For example, if G is a connected compact Lie group and $N = N_G T$ the normalizer of a maximal torus, then

$$\mathrm{tr}_N^G(1)\ =\ 1$$

in the ring $\mathrm{Cl}(G)$. Moreover, the composite

$$\mathrm{Cl}(G)\ \xrightarrow{\mathrm{res}_N^G}\ \mathrm{Cl}(N)\ \xrightarrow{\mathrm{tr}_N^G}\ \mathrm{Cl}(G)$$

is the identity. Indeed, the proof of the corresponding property in $\mathbf{RU}$ in Proposition 4.11 was via characters, so it also works for Cl.

Example 4.18. The right adjoint $R : \mathcal{SH} \longrightarrow \mathcal{GH}$ to the forgetful functor from the global to the non-equivariant stable homotopy category is modeled on the pointset level by the functor $b : \mathcal{S}p \longrightarrow \mathcal{S}p$ discussed in Construction IV.5.24. The global homotopy type of bE is that of a Borel cohomology theory, and in particular,

$$\pi_0^G(bE)\ \cong\ E^0(BG)\ ,$$

natural in G for transfers and restriction maps. The functor b is lax symmetric monoidal, so it takes an ultra-commutative ring spectrum R to an ultra-commutative ring spectrum bR; the power operations

$$P^m\ :\ \pi_0^e(bE)\ \longrightarrow\ \pi_0^{\Sigma_m}(bE)$$

correspond to the classical power operations

$$\mathcal{P}_m\ :\ E^0(*)\ \longrightarrow\ E^0(B\Sigma_m)\ ,$$

compare the more general Remark 1.12.

CHAPTER VI

Ultra-commutative ring spectra

This final chapter is devoted to ultra-commutative ring spectra. On the point set level, these objects are simply commutative orthogonal ring spectra; we use the term *'ultra-commutative'* to emphasize that care about their homotopy theory with respect to multiplicative morphisms that are global equivalences. We refer to the introduction of Chapter II for further justification of the adjective 'ultra-commutative'. In short, the slogan "E_∞=commutative" is not true globally and a strictly commutative multiplication encodes a large amount of structure that deserves a special name.

Commutative orthogonal ring spectra are called '$\mathscr{I}_*$-prefunctors' in [**107**, IV Def. 2.1].

1. Global model structure

In this section we construct a model structure on the category of ultra-commutative ring spectra with global equivalences as the weak equivalences, see Theorem 1.5. The strategy is the same is in the unstable situation in Section II.1: we establish a 'positive' version of the global model structure and lift it to commutative monoid objects with the help of the general lifting theorem [**175**, Thm. 3.2].

With the help of this global model structure we calculate the algebra of natural operations on the homotopy groups of ultra-commutative ring spectra: we show that these operations are freely generated by restrictions, transfers and power operations. The precise result can be found in Proposition 1.6.

Definition 1.1. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *positive cofibration* if it is a flat cofibration and the map $f(0) : X(0) \longrightarrow Y(0)$ is a homeomorphism. An orthogonal spectrum is a *positive global Ω-spectrum* if for every compact Lie group G, every G-representation V and ever faithful G-representation W with $W \neq 0$ the adjoint structure map

$$\tilde{\sigma}_{V,W} \ : \ X(W) \ \longrightarrow \ \mathrm{map}(S^V, X(V \oplus W))$$

is a G-weak equivalence.

If G is a non-trivial compact Lie group, then any faithful G-representation is automatically non-trivial. So a positive global Ω-spectrum is a global Ω-spectrum (in the absolute sense) if the adjoint structure map $\tilde{\sigma}_{0,\mathbb{R}} : X(0) \longrightarrow \Omega X(\mathbb{R})$ is a non-equivariant weak equivalence.

Proposition 1.2 (Positive global model structure). *The global equivalences and positive cofibrations are part of a proper topological model structure, the* positive global model structure *on the category of orthogonal spectra. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a fibration in the positive global model structure if and only if for every compact Lie group G, every G-representation V and every faithful G-representation W with $W \neq 0$ the square*

$$\begin{tikzcd} X(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"] \arrow[d, "f(W)^G"'] & \mathrm{map}^G(S^V, X(V \oplus W)) \arrow[d, "\mathrm{map}^G(S^V, f(V \oplus W))"] \\ Y(W)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"'] & \mathrm{map}^G(S^V, Y(V \oplus W)) \end{tikzcd}$$

is homotopy cartesian. The fibrant objects in the positive global model structure are the positive global Ω-spectra. The model structure is monoidal with respect to the smash product of orthogonal spectra.

Proof. We start by establishing a *positive strong level model structure*. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *positive strong level equivalence* (respectively *positive strong level fibration*) if for every inner product space V with $V \neq 0$ the map $f(V) : X(V) \longrightarrow Y(V)$ is an $O(V)$-weak equivalence (respectively an $O(V)$-fibration). Then we claim that the positive strong level equivalences, positive strong level fibrations and positive cofibrations form a model structure on the category of orthogonal spectra.

The proof is another application of the general construction method for level model structures in Proposition A.3.27. Indeed, we let $\mathcal{C}(0)$ be the degenerate model structure on the category $\mathbf{T}_*$ of based spaces in which every morphism is a weak equivalence and a fibration, but only the isomorphisms are cofibrations. For $m \geq 1$ we let $\mathcal{C}(m)$ be the projective model structure (for the family of all closed subgroups) on the category of based $O(m)$-spaces. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition A.3.27 precisely become the positive strong level equivalences, positive strong level fibrations and positive cofibrations. The consistency condition (Definition A.3.26) is now strictly weaker than for the strong level model structure, so it holds. The verification that the model structure is proper and topological is the same as in Proposition IV.3.8.

The positive strong level model structure is cofibrantly generated: we can simply take the same sets of generating cofibrations and generating acyclic cofibrations as for the $\mathcal{A}ll$-level model structure in Proposition IV.3.8, *except* that we omit all morphisms freely generated in level 0.

We obtain the positive global model structure for orthogonal spectra by 'mixing' the positive strong level model structure with the global model structure of Theorem IV.3.26. Every positive strong level equivalence is a global equivalence and every positive cofibration is a flat cofibration. The global equivalences and the positive cofibrations are part of a model structure by Cole's mixing theorem [**37**, Thm. 2.1], which is our first claim. By [**37**, Cor. 3.7] (or rather its dual formulation), an orthogonal spectrum is fibrant in the positive global model structure if it is equivalent in the positive strong level model structure to a global Ω-spectrum; this is equivalent to being a positive global Ω-spectrum. The positive global model structure is again proper by Propositions 4.1 and 4.2 of [**37**]. The proof that this model structure topological is similar as for the global model structure. The proof of the pushout product property is as in the absolute global model structure (see Proposition IV.3.29); the only new ingredient is that the class of generators $F_{G,V}$ with $V \neq 0$ for the positive cofibrations is closed under the smash product of orthogonal spectra. □

We recall from [**60**, Def. 3] the notion of a *symmetrizable cofibration* respectively *symmetrizable acyclic cofibration*, see also Definition II.1.8. For a morphism $i : A \longrightarrow B$ of orthogonal spectra we let $K^n(i)$ denote the n-cube of orthogonal spectra whose value at a subset $S \subseteq \{1, 2, \ldots, n\}$ is

$$K^n(i)(S) \ = C_1 \wedge C_2 \wedge \ldots \wedge C_n$$

with

$$C_i \ = \begin{cases} A & \text{if } i \notin S \\ B & \text{if } i \in S. \end{cases}$$

All morphisms in the cube $K^n(i)$ are smash products of identities and copies of the morphism $i : A \longrightarrow B$. We let $Q^n(i)$ denote the colimit of the n-punctured cube, i.e., the cube $K^n(i)$ with the terminal vertex removed, and $i^{\Box n} : Q^n(i) \longrightarrow K^n(i)(\{1, \ldots, n\}) = B^{\wedge n}$ the canonical map. So the morphism $i^{\Box n}$ is an iterated pushout product morphism. The symmetric group Σ_n acts on $Q^n(i)$ and $B^{\wedge n}$ by permuting the smash factors, and the iterated pushout product morphism $i^{\Box n} : Q^n(i) \longrightarrow B^{\wedge n}$ is Σ_n-equivariant. The morphism $i : A \longrightarrow B$ of orthogonal spectra is a *symmetrizable cofibration* (respectively a *symmetrizable acyclic cofibration*) if the morphism

$$i^{\Box n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\wedge n}/\Sigma_n = \mathbb{P}^n(B)$$

is a cofibration (respectively an acyclic cofibration) for every $n \geq 1$. Since the morphism $i^{\square 1}/\Sigma_1$ is the original morphism i, every symmetrizable cofibration is in particular a cofibration, and similarly for acyclic cofibrations.

The next theorem says that in the category of orthogonal spectra, all cofibrations and acyclic cofibrations in the positive global model structure on orthogonal spectra are symmetrizable with respect to the monoidal structure given by the smash product.

THEOREM 1.3. (i) *Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spectra. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\wedge n}/\Sigma_n$$

is a flat cofibration. In other words, all cofibrations in the global model structure of orthogonal spectra are symmetrizable.

(ii) *Let $i : A \longrightarrow B$ be a positive flat cofibration of orthogonal spectra that is also a global equivalence. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\wedge n}/\Sigma_n$$

is a global equivalence. In other words, all acyclic cofibrations in the positive global model structure of orthogonal spectra are symmetrizable.

PROOF. (i) We recall from Theorem IV.3.21 (iv) the set

$$I_{\mathcal{A}ll} \ = \ \{ \ G_m((O(m)/H \times i_k)_+) \mid m, k \geq 0, H \leq O(m)\}$$

of generating flat cofibrations of orthogonal spectra, where $i_k : \partial D^k \longrightarrow D^k$ is the inclusion. The set $I_{\mathcal{A}ll}$ detects the acyclic fibrations in the strong level model structure of orthogonal spectra. In particular, every flat cofibration is a retract of an $I_{\mathcal{A}ll}$-cell complex. By [**60**, Cor. 9] it suffices to show that the generating flat cofibrations in $I_{\mathcal{A}ll}$ are symmetrizable.

For a space A, the orthogonal spectrum $G_m((O(m)/H \times A)_+)$ is isomorphic to $F_{H,\mathbb{R}^m} \wedge A_+$; so we show more generally that every morphism of the form

$$F_{G,V} \wedge (i_k)_+ \ : \ F_{G,V} \wedge \partial D^k_+ \ \longrightarrow \ F_{G,V} \wedge D^k_+$$

is a symmetrizable cofibration, where V is any representation of a compact Lie group G. The symmetrized iterated pushout product

$$(F_{G,V} \wedge (i_k)_+)^{\square n}/\Sigma_n \ : \ Q^n(F_{G,V} \wedge (i_k)_+)/\Sigma_n \ \longrightarrow \ (F_{G,V} \wedge B_+)^{\wedge n}/\Sigma_n \tag{1.4}$$

is isomorphic to

$$F_{\Sigma_n \wr G, V^n}((i_k^{\square n})_+) \ : \ F_{\Sigma_n \wr G, V^n}(Q^n(i_k)_+) \ \longrightarrow \ F_{\Sigma_n \wr G, V^n}(B^n_+) \ ,$$

where

$$i_k^{\square n} \ : \ Q^n(i_k) \ \longrightarrow \ B^n$$

is the n-fold pushout product of the inclusion $i_k : \partial D^k \longrightarrow D^k$, with respect to the cartesian product of spaces. The map $i_k^{\square n}$ is Σ_n-equivariant, and we showed in the proof of the analogous unstable result in Theorem II.1.11 (i) that $i_k^{\square n}$ is a cofibration of Σ_n-spaces. Proposition A.2.14 (i) then shows that $i_k^{\square n}$ is also a cofibration of $(\Sigma_n \wr G)$-spaces, with respect to the action by restriction along the projection $(\Sigma_n \wr G) \longrightarrow \Sigma_n$. So the morphism (1.4) is a flat cofibration.

(ii) Theorem IV.3.21 (iv) describes a set $J_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$ of generating acyclic cofibrations for the global model structure on the category of orthogonal spectra. From this we obtain a set $J^+ \cup K^+$ of generating acyclic cofibration for the *positive* global model structure of Proposition 1.2 by restricting to those morphisms in $J_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$ that are positive cofibrations, i.e., homeomorphisms in level 0; so explicitly, we set

$$J^+ \ = \ \{ \ G_m((O(m)/H \times j_k)_+) \mid m \geq 1, k \geq 0, H \leq O(m)\} \ ,$$

where $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion, and

$$K^+ \;=\; \bigcup_{G,V,W\,:\,V\neq 0} \mathcal{Z}(\lambda_{G,V,W})\ ,$$

the set of all pushout products of sphere inclusions i_k with the mapping cylinder inclusions of the global equivalences $\lambda_{G,V,W} : F_{G,V\oplus W}S^W \longrightarrow F_{G,V}$. Here (G,V,W) runs through a set of representatives of the isomorphism classes of triples consisting of a compact Lie group G, a non-zero faithful G-representation V and an arbitrary G-representation W. By [**60**, Cor. 9] it suffices to show that all morphisms in $J^+ \cup K^+$ are symmetrizable acyclic cofibrations.

We start with a morphism $G_m((O(m)/H \times j_k)_+)$ in J^+. For every $n \geq 1$, the morphism

$$(G_m((O(m)/H \times j_k)_+))^{\Box n}/\Sigma_n$$

is a flat cofibration by part (i), and a homeomorphism in level 0 because $m \geq 1$. Moreover, the morphism j_k is a homotopy equivalence of spaces, so $G_m((O(m)/H \times j_k)_+)$ is a homotopy equivalence of orthogonal spectra; the morphism $\mathbb{P}^n(G_m((O(m)/H \times j_k)_+))$ is then again a homotopy equivalence for every $n \geq 1$, by Proposition II.1.10 (i). Then [**60**, Cor. 23] shows that $G_m((O(m)/H \times j_k)_+)$ is a symmetrizable acyclic cofibration. This takes care of the set J^+.

Now we consider the morphisms in the set K^+. Since G acts faithfully on the non-zero inner product space V, the action of the wreath product $\Sigma_n \wr G$ on V^n is again faithful. So the morphism

$$\lambda_{\Sigma_n\wr G,V^n,W^n}\ :\ F_{\Sigma_n\wr G,V^n\oplus W^n}\,S^{W^n}\ \longrightarrow\ F_{\Sigma_n\wr G,V^n}$$

is a global equivalence by Theorem IV.1.31. The vertical morphisms in the commutative square

$$\begin{array}{ccc} F_{\Sigma_n\wr G,V^n\oplus W^n}\,S^{W^n} & \xrightarrow{\lambda_{\Sigma_n\wr G,V^n,W^n}} & F_{\Sigma_n\wr G,V^n} \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathbb{P}^n(F_{G,V\oplus W}S^W) & \xrightarrow[\mathbb{P}^n(\lambda_{G,V,W})]{} & \mathbb{P}^n(F_{G,V}) \end{array}$$

are isomorphisms; so the morphism $\mathbb{P}^n(\lambda_{G,V,W})$ is a global equivalence. Proposition II.1.10 (iii) then shows that all morphisms in $\mathcal{Z}(\lambda_{G,V,W})$ are symmetrizable acyclic cofibrations. $\Box$

Now we put all the pieces together. We call a morphism of ultra-commutative ring spectra a *global equivalence* (respectively *positive global fibration*) if the underlying morphism of orthogonal spectra is a global equivalence (respectively fibration in the positive global model structure of Proposition 1.2).

Theorem 1.5 (Global model structure for ultra-commutative ring spectra).

(i) *The global equivalences and positive global fibrations are part of a cofibrantly generated, topological model structure on the category of ultra-commutative ring spectra, the* global model structure.

(ii) *Let $j : R \longrightarrow S$ be a cofibration in the global model structure of ultra-commutative ring spectra.*
  (a) *The morphism of R-modules underlying j is a cofibration in the global model structure of R-modules of Corollary* IV.3.34 (i).
  (b) *The morphism of orthogonal spectra underlying j is an h-cofibration.*
  (c) *If the underlying orthogonal spectrum of R is flat, then j is a flat cofibration of orthogonal spectra.*

(iii) *The global model structure on ultra-commutative ring spectra is proper.*

Proof. The proof is completely parallel to the proof of the analogous unstable theorem, Theorem II.1.13.

(i) The positive global model structure of orthogonal spectra (Proposition 1.2) is monoidal and cofibrantly generated. The 'unit axiom' also holds: we let $f : \mathbb{S}^c \longrightarrow \mathbb{S}$ be any positive flat replacement of the orthogonal sphere spectrum. Then for every orthogonal spectrum X the induced morphism

$f \wedge X : \mathbb{S}^c \wedge X \longrightarrow \mathbb{S} \wedge X$ is a global equivalence by Theorem IV.3.32 (ii). The monoid axiom holds by Proposition IV.3.33. Cofibrations and acyclic cofibrations are symmetrizable by Theorem 1.3, so the model structure satisfies the 'commutative monoid axiom' of [**175**, Def. 3.1]. Theorem 3.2 of [**175**] thus shows that the positive global model structure of orthogonal spectra lifts to a cofibrantly generated model structure on the category of ultra-commutative ring spectra. The global model structure is topological by Proposition A.2.8, where we take $\mathcal{G}$ as the set of free ultra-commutative ring spectra $\Sigma^\infty_+\mathbb{P}(L_{H,\mathbb{R}^m})$ for all $m \geq 1$ and all closed subgroups H of $O(m)$.

Part (ii) is proved in exactly the same way as in the unstable case in Theorem II.1.13; whenever the unstable proof refers to the model category of R-modules in Corollary I.4.16, the stable proof instead refers to Corollary IV.3.34. For the symmetrizability of the cofibrations, the stable proof uses Theorem 1.3 (i) instead of Theorem II.1.11 (i). We refrain from repeating the remaining details.

(iii) Since weak equivalences, fibrations and pullbacks of ultra-commutative ring spectra are are created on underlying orthogonal spectra, right properness is inherited from the positive global model structure of orthogonal spectra (Proposition 1.2). A pushout square of ultra-commutative ring spectra has the form

$$\begin{tikzcd} R \arrow[r, "f", "\simeq"'] \arrow[d, "j"'] & T \arrow[d, "j\wedge_R T"] \\ S \arrow[r, "S\wedge_R f"'] & S \wedge_R T \end{tikzcd}$$

where S and T are considered as R-modules by restriction along i respectively f. For left properness we now suppose that j is a cofibration and f is a global equivalence. By part (a) of (ii), the morphism j is then a cofibration of R-modules in the global model structure of Corollary IV.3.34 (i). Since R is cofibrant in that model structure, also S is cofibrant as an R-module. Proposition IV.3.35 then shows that the functor $S \wedge_R -$ preserves global equivalences. So the cobase change $S \wedge_R f$ of f is a global equivalence. This shows that the global model structure of ultra-commutative ring spectra is left proper. □

There is a positive version of the $\mathcal{F}$-global model structure on the category of orthogonal spectra, for every global family $\mathcal{F}$; this positive model structure lifts to an *$\mathcal{F}$-global model structure* on the category of ultra-commutative ring spectra. We will not elaborate on this point.

Now we can show that the restriction maps, (additive) transfer maps and (multiplicative) power operations generate all natural operations between the 0-th equivariant homotopy groups of ultra-commutative ring spectra. The strategy is the one that we have employed several times before: the functor π_0^G from ultra-commutative ring spectra to sets is representable, namely by $\Sigma^\infty_+\mathbb{P}(B_{\mathrm{gl}}G)$, the unreduced suspension spectrum of the free ultra-commutative monoid generated by $B_{\mathrm{gl}}G$. (The orthogonal ring spectrum $\Sigma^\infty_+\mathbb{P}(B_{\mathrm{gl}}G)$ is isomorphic to the free ultra-commutative ring spectrum on the unreduced suspension spectrum of $B_{\mathrm{gl}}G$). So we have to determine the equivariant homotopy groups $\pi_0^K(\Sigma^\infty_+\mathbb{P}(B_{\mathrm{gl}}G))$, which just means assembling various results already proved.

Proposition 1.6. *Let G and K be compact Lie groups. The group of natural transformations $\pi_0^G \longrightarrow \pi_0^K$ of set valued functors on the category of ultra-commutative ring spectra is a free abelian group with basis the operations*

$$\mathrm{tr}_L^K \circ \alpha^* \circ P^m \ : \ \pi_0^G \ \longrightarrow \pi_0^K$$

for all $m \geq 0$ and all $(K \times (\Sigma_m \wr G))$-conjugacy classes of pairs (L, α) consisting of a subgroup L of K with finite Weyl group and a continuous homomorphism $\alpha : L \longrightarrow \Sigma_m \wr G$.

Proof. We let V be any faithful G-representation and write $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$ for the global classifying space of G based on V and $u_G = u_{G,V}$ for the associated tautological class. We denote by

$\hat{e}_G \in \pi_0^G(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ the class obtained by pushing forward the tautological class u_G along the adjunction unit $B_{\mathrm{gl}}G \longrightarrow \mathbb{P}(B_{\mathrm{gl}}G)$ and then applying the map σ^G (compare (1.10) of Chapter I) that makes an unstable homotopy class into a stable homotopy class of the suspension spectrum.

We apply the representability result of Proposition II.2.24 (ii) to the category of ultra-commutative ring spectra, the adjoint functor pair

$$spc \underset{U\circ\Omega^\bullet}{\overset{\Sigma_+^\infty\circ\mathbb{P}}{\rightleftarrows}} (\text{ultra-commutative ring spectra})$$

and the functor $\Psi = \pi_0^K$. We conclude that the evaluation at the tautological class is a bijection

$$\mathrm{Nat}^{\text{u-com}}(\pi_0^G, \pi_0^K) \ \longrightarrow\ \pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G)) \ , \quad \tau \ \longmapsto\ \tau(\hat{e}_G)$$

to the 0-th K-equivariant homotopy group of the ultra-commutative ring spectrum $\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G)$.

The identification (2.21) of Chapter II and Proposition I.5.16 (ii) show that the set $\pi_0^L(\mathbb{P}(B_{\mathrm{gl}}G))$ bijects with the set

$$\coprod_{m\geq 0} \mathrm{Rep}(L, \Sigma_m \wr G)$$

by sending the conjugacy class of $\alpha : L \longrightarrow \Sigma_m \wr G$ to $\alpha^*([m](u_G)) = \alpha^*(u_{\Sigma_m \wr G})$. Proposition IV.1.11 then implies that the group $\pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ is a free abelian group, and it specifies a basis consisting of the elements

$$\mathrm{tr}_L^K(\sigma^L(x))$$

where L runs through all conjugacy classes of subgroups of K with finite Weyl group and x runs through a set of representatives of the $W_K L$-orbits of the set $\pi_0^L(\mathbb{P}(B_{\mathrm{gl}}G))$. So together this shows that $\pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ is a free abelian group with basis the classes

$$\begin{aligned} \mathrm{tr}_L^K(\sigma^L(\alpha^*([m](u_G)))) \ &= \ \mathrm{tr}_L^K(\alpha^*(\sigma^{\Sigma_m \wr G}([m](u_G)))) \\ &= \ \mathrm{tr}_L^K(\alpha^*(P^m(\sigma^G(u_G)))) \ = \ \mathrm{tr}_L^K(\alpha^*(P^m(\hat{u}_G))) \end{aligned}$$

for all $m \geq 0$ and all $(K \times (\Sigma_m \wr G))$-conjugacy classes of pairs (L, α) consisting of a subgroup L of K with finite Weyl group and a homomorphism $\alpha : L \longrightarrow \Sigma_m \wr G$. □

Now we show that every global power functor is realized by an ultra-commutative ring spectrum. More is true: the next theorem effectively constructs a right adjoint functor

$$H \ : \ (\text{global power functors}) \ \longrightarrow\ \mathrm{Ho}^{\text{gl. conn.}}(\text{ultra-com ring spectra})$$

to the functor $\underline{\pi}_0$ such that the adjunction counit is an isomorphism $\underline{\pi}_0(HR) \cong R$ of global power functors. The analogous result in equivariant stable homotopy theory for a fixed finite group has been obtained by Ullman [**168**] by the same method of proof.

THEOREM 1.7. *Let R be a global power functor.*

(i) *There is an ultra-commutative ring spectrum HR such that $\underline{\pi}_k(HR) = 0$ for all $k \neq 0$ and an isomorphism of global power functors*

$$\underline{\pi}_0(HR) \ \cong\ R \ .$$

(ii) *For every globally connective ultra-commutative ring spectrum T, the functor $\underline{\pi}_0$ restricts to a bijection*

$$\underline{\pi}_0 \ : \ \mathrm{Ho}(\text{ultra-com ring spectra})(T, HR) \ \cong\ (\text{global power functors})(\underline{\pi}_0(T), R) \ .$$

PROOF. Proposition II.2.38 (iii) allows us to realize the multiplicative global power monoid underlying the global functor R by an ultra-commutative monoid. In other words, there is an ultra-commutative

monoid M and an isomorphism of global power monoids $\underline{\pi}_0(M) \cong R^\times$. The unreduced suspension spectrum $\Sigma^\infty_+ M$ is then a globally connective ultra-commutative ring spectrum, and Theorem V.3.8 identifies $\underline{\pi}_0(\Sigma^\infty_+ M)$ as the free global power functor generated by the global power monoid $\underline{\pi}_0(M)$. There is thus a unique morphism of global power functors

$$\epsilon \ : \ \underline{\pi}_0(\Sigma^\infty_+ M) \ \longrightarrow \ R$$

whose composite with the stabilization map $\sigma : \underline{\pi}_0(M) \longrightarrow \underline{\pi}_0(\Sigma^\infty_+ M)$ is the chosen isomorphism between $\underline{\pi}_0(M)$ and $R^\times$. So ϵ is in particular surjective.

Now we kill the kernel of ϵ and all higher homotopy groups of $\Sigma^\infty_+ M$ by attaching free commutative ring spectrum cells.

Claim: Let T be a globally connective ultra-commutative ring spectrum, $n \geq 0$ and $I \subseteq \underline{\pi}_n(T)$ a global subfunctor of the n-th homotopy group global functor. Then there is a morphism of ultra-commutative ring spectra $j : T \longrightarrow T'$ with the following properties:

- the induced morphism of global functors $j_* : \underline{\pi}_k(T) \longrightarrow \underline{\pi}_k(T')$ is bijective for $k < n$ and surjective for $k = n$, and
- the global functor I is contained in the kernel of $j_* : \underline{\pi}_n(T) \longrightarrow \underline{\pi}_n(T')$.

To prove the claim, we choose an index set J, compact Lie groups G_j and elements $y_j \in \pi_n^{G_j}(T)$, for $j \in J$, that altogether generate the global functor I. We represent each class y_j as a morphism of orthogonal spectra

$$f_j \ : \ S^n \wedge \Sigma^\infty_+ B_{\mathrm{gl}} G_j \ \longrightarrow \ T$$

that sends the n-fold suspension of the stable tautological class e_{G_j} to y_j; this involves an implicit choice of faithful G_j-representation. We form the wedge of all these morphisms and freely extend that to a morphism of ultra-commutative ring spectra

$$F \ : \ \mathbb{P}(S^n \wedge \bigvee_{j \in J} \Sigma^\infty_+ B_{\mathrm{gl}} G_j) \ \longrightarrow \ T \ .$$

There is a unique morphisms of ultra-commutative ring spectra from F to the sphere spectrum $\mathbb{S}$ that restricts to the trivial map on $S^n \wedge \bigvee_{j \in J} \Sigma^\infty_+ B_{\mathrm{gl}} G_j$; in other words, the morphism projects onto the 'constant' wedge summand in the free ultra-commutative ring spectrum. We let T' be the homotopy pushout, in the category of ultra-commutative ring spectra, of the diagram

$$\mathbb{S} \ \longleftarrow \ \mathbb{P}(S^n \wedge \bigvee_{j \in J} \Sigma^\infty_+ B_{\mathrm{gl}} G_j) \ \xrightarrow{F} \ T \ .$$

The resulting morphism of ring spectra $\psi : T \longrightarrow T'$ induces an isomorphism of homotopy group global functors below dimension n, and epimorphism in dimension n [...justify...]. The square

$$\begin{array}{ccc} \underline{\pi}_n(\mathbb{P}(S^n \wedge \bigvee_{j \in J} \Sigma^\infty_+ B_{\mathrm{gl}} G_j)) & \xrightarrow{F} & \underline{\pi}_n(T) \\ \downarrow & & \downarrow{\scriptstyle \psi_*} \\ \underline{\pi}_n(\mathbb{S}) & \longrightarrow & \underline{\pi}_n(T') \end{array}$$

commutes and the left vertical map sends the tautological classes $S^n \wedge e_{G_j}$ to zero. So the morphism ψ_* annihilates all the classes y_j, and hence the entire global subfunctor I.

Now we use the claim construct a sequence of closed embeddings of ultra-commutative ring spectra

$$\Sigma^\infty_+ M \ = \ T_0 \ \longrightarrow \ T_1 \ \longrightarrow \ T_2 \ \cdots \ . \tag{1.8}$$

We obtain T_1 by applying the claim to the connective ring spectrum $\Sigma^\infty_+ M$ and the global functor $I = \ker(\epsilon : \underline{\pi}_0(\Sigma^\infty_+ M) \longrightarrow R)$. The orthogonal spectrum T_1 is again globally connective, and ϵ factors through an isomorphism of global power functors between R and $\underline{\pi}_0(T_1)$. Now we continue by induction and

construct T_{n+1} from T_n by applying the claim in dimension n for $I = \underline{\pi}_n(T_n)$, the full homotopy group global functor in dimension n. Then T_{n+1} is globally connective, $\underline{\pi}_0(T_{n+1})$ is isomorphic to R, and the global functors $\underline{\pi}_k(T_{n+1})$ are trivial for all $1 \le k \le n$. Finally, we define HR as the colimit of the sequence (1.8) of ultra-commutative ring spectra. Since equivariant homotopy groups commute with colimits over closed embeddings, the colimit has the properties of an Eilenberg-Mac Lane spectrum for R. □

In Remark IV.4.13 we have constructed an Eilenberg-Mac Lane spectrum HM associated to a global functor M, an orthogonal spectrum satisfying

$$\underline{\pi}_k(HM) \;\cong\; \begin{cases} M & \text{for } k = 0, \text{ and} \\ 0 & \text{for } k \neq 0. \end{cases}$$

If M has the extra multiplicative structure of a global power functor, then Theorem 1.7 shows that HM can be realized as an ultra-commutative ring spectrum. In all these constructions, the global homotopy type of HM is determined by the algebraic input data up to preferred isomorphism (in the appropriate homotopy category), but the constructions are abstract versions of 'killing homotopy groups' and do not yield explicit pointset level models for Eilenberg-Mac Lane spectra. In this section we discuss a well-known pointset level construction that yields Eilenberg-Mac Lane spectra – at least when restricted to finite groups.

Construction 1.9. Let A be an abelian group. The *Eilenberg-Mac Lane spectrum* $\mathcal{H}A$ is defined at an inner product space V by

$$(\mathcal{H}A)(V) \;=\; A[S^V]\;, \tag{1.10}$$

the reduced A-linearization of the V-sphere. The orthogonal group $O(V)$ acts through the action on S^V and the structure map $\sigma_{V,W} : S^V \wedge (\mathcal{H}A)(W) \longrightarrow (\mathcal{H}A)(V \oplus W)$ is given by

$$S^V \wedge A[S^W] \;\longrightarrow\; A[S^{V\oplus W}]\;, \quad v \wedge \big(\textstyle\sum_i m_i \cdot w_i\big) \;\longmapsto\; \textstyle\sum_i m_i \cdot (v \wedge w_i)\;.$$

The underlying non-equivariant space of $A[S^V]$ is an Eilenberg-Mac Lane space of type (A, n), where $n = \dim(V)$. Hence the underlying non-equivariant homotopy type of $\mathcal{H}A$ is that of an Eilenberg-Mac Lane spectrum for A.

As we shall now discuss, the equivariant homotopy groups of $\mathcal{H}A$ are in general *not* concentrated in dimension 0, and hence $\mathcal{H}A$ is not the Eilenberg-Mac Lane spectrum of a global functor. However, on *finite* groups, the equivariant behavior of $\mathcal{H}A$ is as expected. We recall from Definition IV.5.7 that an orthogonal spectrum is *left induced* from the global family $\mathcal{F}in$ of finite groups if it is in the essential image of the left adjoint $L_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}in} \longrightarrow \mathcal{GH}$ from the $\mathcal{F}in$-global homotopy category. We let $\underline{A}$ denote the constant global functor with value A, compare Example IV.2.8 (iii). The inclusion of generators is a homeomorphism $A \cong A[S^0] = (\mathcal{H}A)(0)$ and this induces a bijection

$$A \;\cong\; \pi_0(A) \;=\; [S^0, (\mathcal{H}A)(0)]\;.$$

Since $\mathcal{H}A$ is in particular a non-equivariant Ω-spectrum (for example by Proposition 1.11), the composite of this bijection with the stabilization map is an isomorphism of abelian group

$$A \;\cong\; \pi_0(\mathcal{H}A)\;.$$

The restriction maps

$$\mathrm{res}^G_e \;:\; \pi^G_k(\mathcal{H}A) \;\longrightarrow\; \pi^e_0(\mathcal{H}A) \;\cong\; A \;=\; \underline{A}(G)$$

form a morphism of global functors $\underline{\pi}_0(\mathcal{H}A) \longrightarrow \underline{A}$. Since $\mathcal{H}A$ is globally connective, there is a unique morphism

$$\rho \;:\; \mathcal{H}A \;\longrightarrow\; H\underline{A}$$

in the global homotopy category that realizes the morphism on π^e_0.

PROPOSITION 1.11. *For every abelian group A the Eilenberg-Mac Lane spectrum $\mathcal{H}A$ is left induced from the global family $\mathcal{F}in$ of finite groups and $\mathcal{H}A$ is a $\mathcal{F}in$-Ω-spectrum. The morphism $\rho : \mathcal{H}A \longrightarrow H\underline{A}$ is a $\mathcal{F}in$-global equivalence.*

PROOF. The orthogonal spectrum $\mathcal{H}A$ is obtained by evaluation of a $\mathbf{\Gamma}$-space $\underline{A}$ on spheres, where $\underline{A}(S) = A[S]$ is the reduced A-linearization of a finite based set S. So the first claim is a special case of Proposition IV.5.17 (ii).

Dos Santos shows in [**131**] that for every finite group G and every G-representation V the G-space $\mathcal{H}A(V) = A[S^V]$ is an equivariant Eilenberg-Mac Lane space of type (A, V), i.e., the G-space $\mathrm{map}(S^V, A[S^V])$ has homotopically discrete fixed points for all subgroups H of G and the natural map

$$A \;\longrightarrow\; [S^V, A[S^V]]^H \;=\; \pi_0\left(\mathrm{map}^H(S^V, A[S^V])\right)$$

sending $m \in A$ to the homotopy class of $m \cdot - : S^V \longrightarrow A[S^V]$ is an isomorphism. This shows that $\mathcal{H}A$ is a $\mathcal{F}in$-Ω-spectrum for the constant global functor $\underline{A}$.

The result is also a special case of the earlier work of Segal and Shimakawa on $\mathbf{\Gamma}_G$-spaces [**143, 145**]. Indeed, we can view the $\mathbf{\Gamma}$-space $\underline{A}$ as a $\mathbf{\Gamma}_G$-space by letting G act trivially. If F is any $\mathbf{\Gamma}_G$-space and S a finite G-set, then we define the G-map

$$P_S \;:\; F(S^+) \;\longrightarrow\; \mathrm{map}(S, F(1^+)) \qquad \text{by} \qquad P_S(x)(s) \;=\; F(p_s)(x)\ ,$$

where $p_s : S_+ \longrightarrow 1^+$ sends s to 1 and all other element of S_+ to the basepoint. In the case of the Γ_G-space $\underline{A}$, the maps

$$P_S \;:\; \underline{A}(S) \;\longrightarrow\; \mathrm{map}(S, \underline{A}(1^+)) \;=\; A^S$$

are in fact homeomorphisms, so in particular G-homotopy equivalences, and $\underline{A}$ is a very special $\mathbf{\Gamma}_G$-space in the sense of Shimakawa [**145**, Def. 1.3]. Since $\pi_0(\underline{A}(1^+))$ is a group (as opposed to a monoid only), Shimakawa's Theorem B proves that the adjoint structure maps $\tilde{\sigma}_{V,W} : A[S^W] \longrightarrow \mathrm{map}(S^V, A[S^{V\oplus W}])$ are G-weak equivalences. □

Slightly more is true: the proof of Proposition 1.11 shows that for finite groups G, the orthogonal G-spectrum $(\mathcal{H}A)_G$ is a full fledged G-Ω-spectrum, i.e., the adjoint structure maps are equivariant weak equivalences starting at arbitrary G-representations (and not just at faithful representations).

The properties mentioned in the previous proposition do *not* generalize to compact Lie groups of positive dimension, i.e., contrary to what one may expect at first, $(\mathcal{H}A)_G$ is not generally a G-Ω-spectrum (see Example 1.12), not all restriction maps in dimension 0 are isomorphisms (see Example 1.16), and the groups $\pi_*^G(\mathcal{H}A)$ need not be concentrated in dimension 0 (see Theorem 1.18).

EXAMPLE 1.12. The equivariant Ω-spectrum property of $\mathcal{H}A$ already fails for the circle group $U(1)$. We consider the tautological $U(1)$-representation on $\mathbb{C}$, i.e., the action by scalar multiplication. Then the map

$$A \;\longrightarrow\; \left(A[S^{\mathbb{C}}]\right)^{U(1)}\ , \quad m \;\longmapsto\; m \cdot [0]$$

is an isomorphism, where we denote by $[0]$ is the generator corresponding to the fixed point $0 \in S^{\mathbb{C}}$. The composite map

$$A[S^0] \xrightarrow{\tilde{\sigma}^{U(1)}_{\mathbb{C},0}} \mathrm{map}^{U(1)}(S^{\mathbb{C}},\, A[S^{\mathbb{C}}]) \xrightarrow{\mathrm{ev}_0} \left(A[S^{\mathbb{C}}]\right)^{U(1)}$$

is thus a homeomorphism. On the other hand, evaluation at 0 is induced by the fixed point inclusion $(S^{\mathbb{C}})^{U(1)} \longrightarrow S^{\mathbb{C}}$ and it is a Serre fibration; its fiber over the basepoint is the mapping space

$$\mathrm{map}^{U(1)}(S^{\mathbb{C}}/S^0,\, A[S^{\mathbb{C}}]) \;\cong\; \mathrm{map}^{U(1)}(U(1)_+ \wedge S^1,\, A[S^{\mathbb{C}}]) \;\cong\; \Omega\left(A[S^{\mathbb{C}}]\right)\ .$$

Non-equivariantly $A[S^{\mathbb{C}}]$ is an Eilenberg-Mac Lane space of type $(A, 2)$, so the fiber is an Eilenberg-Mac Lane space of type $(A, 1)$, and hence the adjoint structure map

$$\tilde{\sigma}_{\mathbb{C},0} \;:\; A[S^0] \;\longrightarrow\; \mathrm{map}(S^{\mathbb{C}}, A[S^{\mathbb{C}}])$$

is *not* a $U(1)$-weak equivalence.

We now consider the important special case $A = \mathbb{Z}$. The equivariant homotopy group $\pi_0^G(\mathcal{H}\mathbb{Z})$ may be larger than a single copy of the integers, and we are now going to give a presentation of $\pi_0^G(\mathcal{H}\mathbb{Z})$. Before we do so, we compare $\mathcal{H}\mathbb{Z}$ to the 'infinite symmetric product of the sphere spectrum'.

Example 1.13 (Infinite symmetric product). There is no essential difference if we consider the infinite symmetric product Sp^∞ (i.e., the reduced free abelian monoid) instead of the reduced free abelian group $\mathbb{Z}[-]$ generated by representation spheres: the levelwise inclusions of the free abelian monoids into the free abelian groups provide a morphism of ultra-commutative ring spectra

$$Sp^\infty(\mathbb{S}) \;=\; \{Sp^\infty(S^n)\}_{n\geq 0} \;\longrightarrow\; \{\mathbb{Z}[S^n]\}_{n\geq 0} \;=\; \mathcal{H}\mathbb{Z}\;, \tag{1.14}$$

and this morphism is a global equivalence by part (ii) of the following proposition.

In his unpublished preprint [**143**], Segal argues that that for every finite group G and every G-representation V with $V^G \neq 0$ the map

$$Sp^\infty(S^V) \;\longrightarrow\; \mathbb{Z}[S^V]$$

is a G-weak equivalence. A published proof of this fact appears as Proposition A.6 in Dugger's paper [**45**]. This generalizes to compact Lie groups:

Proposition 1.15. *Let G be a compact Lie group.*

(i) *For every G-representation V such that $V^G \neq 0$, the natural map $Sp^\infty(S^V) \longrightarrow \mathbb{Z}[S^V]$ is a G-weak equivalence.*
(ii) *The morphism* (1.14) *is a global equivalence of orthogonal spectra from $Sp^\infty(\mathbb{S})$ to $\mathcal{H}\mathbb{Z}$.*

Proof. (i) It suffices to show (by application to all closed subgroups of G) that the fixed point map $(Sp^\infty(S^V))^G \longrightarrow (\mathbb{Z}[S^V])^G$ is a non-equivariant weak equivalence. We let $G^\circ \leq G$ denote the connected component of the identity element and $\bar{G} = G/G^\circ$ the finite group of components of G. To calculate G-fixed points we can first take G°-fixed points and then $\bar{G}$-fixed points. Proposition IV.5.17 (i) applied to the $\mathbf{\Gamma}$-spaces Sp^∞ and $\mathbb{Z}[-]$ provides homeomorphisms

$$(Sp^\infty(S^V))^G \;\cong\; (Sp^\infty(S^{V^{G^\circ}}))^{\bar{G}} \qquad \text{and} \qquad (\mathbb{Z}[S^V])^G \cong\; (\mathbb{Z}[S^{V^{G^\circ}}])^{\bar{G}}\;.$$

Since V^{G° is an orthogonal representation of the finite group $\bar{G}$ the map

$$(Sp^\infty(S^{V^{G^\circ}}))^{\bar{G}} \;\longrightarrow\; (\mathbb{Z}[S^{V^{G^\circ}}])^{\bar{G}}\;,$$

is a weak equivalence by [**45**, Prop. A.6]. Strictly speaking, Dugger's proposition is stated only for geometric realizations of $\bar{G}$-simplicial sets; since every $\bar{G}$-CW-complex, such as the representation sphere $S^{V^{G^\circ}}$, is $\bar{G}$-homotopy equivalent to the realization of a $\bar{G}$-simplicial set, we can also apply it in our situation. Part (ii) is then immediate from (i). □

The infinite symmetric product of any based space X has a natural filtration

$$X = Sp^1(X) \;\subseteq\; Sp^2(X) \;\subseteq\; \ldots \;\subseteq\; Sp^n(X) \;\subseteq\; \ldots$$

by the finite symmetric products. By evaluation on spheres, this provides a filtration

$$\mathbb{S} = Sp^1(\mathbb{S}) \;\subseteq\; Sp^2(\mathbb{S}) \;\subseteq\; \ldots \;\subseteq\; Sp^n(\mathbb{S}) \;\subseteq\; \ldots$$

of $Sp^\infty(\mathbb{S})$ by orthogonal subspectra. We study the equivariant and global properties of this filtration in [**137**]. The 0-th homotopy group global functor of the orthogonal spectrum $Sp^n(\mathbb{S})$ has a very compact description as follows. We recall that the Burnside ring $\mathbb{A}(G)$ of a compact Lie group G is freely generated by the classes $t_H^G = \mathrm{tr}_H^G(p_H^*(1))$ where H runs through a set of representatives of the conjugacy classes of subgroups of G with finite Weyl group. Theorem 3.13 of [**137**] shows that the global functor $\underline{\pi}_0(Sp^n(\mathbb{S}))$

is the quotient of the Burnside ring global functor by the global subfunctor generated by the element $n \cdot 1 - t^{\Sigma_n}_{\Sigma_{n-1}}$ in $\mathbb{A}(\Sigma_n)$,

$$\underline{\pi}_0(Sp^n(\mathbb{S})) \;\cong\; \mathbb{A}/\langle n \cdot 1 \;-\; t^{\Sigma_n}_{\Sigma_{n-1}} \rangle \;.$$

Letting n go to infinity and combining this with part (ii) of Proposition 1.15 gives the following calculation of the global functor $\underline{\pi}_0(\mathcal{H}\mathbb{Z})$. We let I_∞ denote the global subfunctor of the Burnside ring global functor $\mathbb{A}$ generated by the classes $n \cdot 1 - t^{\Sigma_n}_{\Sigma_{n-1}}$ for all $n \geq 1$. We show in [**137**, Thm. 3.13] that the unit map $\mathbb{S} \longrightarrow Sp^\infty$ induces an isomorphism of global functors

$$\mathbb{A}/I_\infty \;\cong\; \underline{\pi}_0(Sp^\infty) \;\cong\; \underline{\pi}_0(\mathcal{H}\mathbb{Z}) \;.$$

This calculation can be made even more explicit. Elementary algebra (see [**137**, Prop. 4.1]) identifies the value $I_\infty(G)$ at a compact Lie group G as the subgroup of $\mathbb{A}(G)$ generated by the classes

$$[H:K] \cdot t^G_H \;-\; t^G_K$$

for all nested sequences of closed subgroup $K \leq H \leq G$ such that $W_G H$ is finite and K has finite index in H. The possibility that K has infinite Weyl group in G is allowed here, in which case $t^G_K = 0$. This presents the equivariant homotopy group

$$\pi^G_0(\mathcal{H}\mathbb{Z}) \;\cong\; \pi^G_0(Sp^\infty) \;\cong\; \mathbb{A}(G)/I_\infty(G)$$

as an explicit quotient of the Burnside ring of G.

Example 1.16. For every finite group G, the group $\pi^G_0(\mathcal{H}\mathbb{Z})$ is free of rank 1, generated by the multiplicative unit, and the restriction map $p^*_G : \pi^e_0(\mathcal{H}\mathbb{Z}) \longrightarrow \pi^G_0(\mathcal{H}\mathbb{Z})$ is an isomorphism. This does *not* persist to general compact Lie groups of positive dimension. An explicit example for which $\pi^G_0(\mathcal{H}\mathbb{Z})$ has rank bigger than 1 is thus $G = SU(2)$. Here there are three conjugacy classes of connected subgroups: the trivial subgroup, the conjugacy class of the maximal torus T and the full group $SU(2)$. Among these, the maximal torus T and $SU(2)$ have finite Weyl groups, so the classes 1 and $\mathrm{tr}^{SU(2)}_{N_{SU(2)}T}(1)$ are a $\mathbb{Z}$-basis for $\pi^{SU(2)}_0(\mathcal{H}\mathbb{Z})$ modulo torsion.

To illustrate that the Eilenberg-Mac Lane spectrum $\mathcal{H}A$ is typically *not* an Eilenberg-Mac Lane spectrum of any global functor, we calculate the $U(1)$-equivariant group $\pi^{U(1)}_1(\mathcal{H}A)$ in dimension 1.

Construction 1.17. As we shall now discuss, all classes in the group $\pi^{U(1)}_1(\mathcal{H}A)$ arise in a systematic way via dimension shifting transfers from finite subgroups of $U(1)$. We organize these transfers into a natural homomorphism from $A \otimes \mathbb{Q}$ to $\pi^{U(1)}_1(\mathcal{H}A)$.

We let C be any finite subgroup of $U(1)$. We identify $\mathbb{R}$ with the tangent space of the distinguished coset C in $U(1)/C$ via the differential of the smooth curve

$$\mathbb{R} \;\longrightarrow\; U(1)/C \;, \quad t \;\longmapsto\; e^{2\pi i t} \cdot C \;.$$

The upshot of this identification $\mathbb{R} \cong T_C(U(1)/C)$ is that we can view the dimension shifting transfer (see (2.21) of Chapter IV) as a homomorphism

$$\mathrm{Tr}^{U(1)}_C \;:\; \pi^C_1(E \wedge S^1) \;\longrightarrow\; \pi^{U(1)}_1(E) \;.$$

We can then define an additive map from $\psi_C : A \longrightarrow \pi^{U(1)}_1(\mathcal{H}A)$ as the composite

$$A \;\cong\; \pi^e_0(\mathcal{H}A) \;\xrightarrow[\cong]{p^*_C}\; \pi^C_0(\mathcal{H}A) \;\xrightarrow[\cong]{-\wedge S^1}\; \pi^C_1(\mathcal{H}A \wedge S^1) \;\xrightarrow{\mathrm{Tr}^{U(1)}_C}\; \pi^{U(1)}_1(\mathcal{H}A) \;.$$

Here $p_C : C \longrightarrow e$ is the unique homomorphism and the map $- \wedge S^1$ is the suspension isomorphism.

Now we consider a subgroup C' of C of index m. Since the restriction of $\underline{\pi}_0(\mathcal{H}A)$ to finite groups is a constant global functor, the relation

$$\begin{aligned}\mathrm{Tr}_{C'}^{U(1)} \circ (- \wedge S^1) \circ p_{C'}^* &= \mathrm{Tr}_C^{U(1)} \circ \mathrm{tr}_{C'}^C \circ (- \wedge S^1) \circ p_{C'}^* \\ &= \mathrm{Tr}_C^{U(1)} \circ (- \wedge S^1) \circ \mathrm{tr}_{C'}^C \circ p_{C'}^* \ = \ m \cdot \mathrm{Tr}_C^{U(1)} \circ (- \wedge S^1) \circ p_C^*\end{aligned}$$

holds as homomorphisms $\pi_0^e(\mathcal{H}A) \longrightarrow \pi_1^{U(1)}(\mathcal{H}A)$. Hence $\psi_{C'} = m \cdot \psi_C$ and we obtain a well defined homomorphism

$$\psi \ : \ A \otimes \mathbb{Q} \ \longrightarrow \ \pi_1^{U(1)}(\mathcal{H}A) \qquad \text{by} \qquad \psi\left(a \otimes \frac{r}{s}\right) \ = \ r \cdot \psi_{C_s}(a) \ ,$$

where $C_s \subset U(1)$ is group of s-th roots of unity.

THEOREM 1.18. *For every abelian group A the homomorphism $\psi : A \otimes \mathbb{Q} \longrightarrow \pi_1^{U(1)}(\mathcal{H}A)$ is an isomorphism.*

PROOF. Since the orthogonal spectrum $\mathcal{H}A$ is obtained from a $\mathbf{\Gamma}$-space by evaluation on spheres, the inflation homomorphism

$$p^* \ : \ \pi_*^e(\mathcal{H}A) \ = \ \Phi_*^e(\mathcal{H}A) \ \longrightarrow \ \Phi_*^{U(1)}(\mathcal{H}A)$$

is an isomorphism of geometric fixed point homotopy groups, by Proposition IV.5.17 (ii). The non-equivariant homotopy groups of $\mathcal{H}A$ are concentrated in dimension 0, so the $U(1)$-equivariant geometric fixed point homotopy groups $\Phi_k^{U(1)}(\mathcal{H}A)$ vanish for all $k \neq 0$. The isotropy separation sequence (3.11) of Chapter IV thus shows that the map $E\mathcal{P} \longrightarrow *$ induces an isomorphism

$$\pi_1^{U(1)}(\mathcal{H}A \wedge E\mathcal{P}_+) \ \cong \ \pi_1^{U(1)}(\mathcal{H}A) \ ,$$

where $E\mathcal{P}$ is a universal $U(1)$-space for the family of proper (i.e., finite) subgroups.

We let X be mapping telescope of the sequence of projections

$$U(1) \ \longrightarrow \ U(1)/C_2 \ \longrightarrow \ U(1)/C_6 \ \longrightarrow \ \dots \ \longrightarrow \ U(1)/C_{n!} \ \longrightarrow \ \dots \ .$$

This is a $U(1)$-CW-complex without $U(1)$-fixed points and such that and the fixed point space X^C is path connected for every finite subgroup C of $U(1)$. So we can build a universal space $E\mathcal{P}$ by attaching equivariant cells with finite isotropy groups of dimension 2 and larger to X. The Wirthmüller isomorphisms

$$\pi_k^{U(1)}(\mathcal{H}A \wedge (U(1)/C)_+ \wedge S^n) \ \cong \ \pi_k^C(\mathcal{H}A \wedge S^1 \wedge S^n) \ \cong \ \pi_{k-1-n}^C(\mathcal{H}A)$$

and the fact that the C-equivariant homotopy groups of $\mathcal{H}A$ are concentrated in dimension 0 mean that the group $\pi_k^{U(1)}(\mathcal{H}A \wedge E\mathcal{P}/X)$ is trivial for all $k \leq 2$. Hence the inclusion $X \longrightarrow E\mathcal{P}$ induces an isomorphism

$$\pi_1^{U(1)}(\mathcal{H}A \wedge X_+) \ \xrightarrow{\cong} \ \pi_1^{U(1)}(\mathcal{H}A \wedge E\mathcal{P}_+) \ .$$

We let $f^n : U(1)/C_{n!} \longrightarrow X$ be the inclusion. The dimension shifting transfer map $\mathrm{Tr}_C^{U(1)}$ factors as the composite

$$\pi_1^C(\mathcal{H}A \wedge S^1) \ \xrightarrow{U(1) \ltimes_C -} \ \pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_{n!})_+) \ \xrightarrow{f_*^n} \ \pi_1^{U(1)}(\mathcal{H}A \wedge X_+) \ \longrightarrow \ \pi_1^{U(1)}(\mathcal{H}A) \ .$$

So the map ψ factors through $\pi_1^{U(1)}(\mathcal{H}A \wedge X_+)$. Since $\mathcal{H}A \wedge X_+$ is a mapping telescope, its equivariant homotopy groups can be calculated as the colimit of the sequence

$$\pi_1^{U(1)}(\mathcal{H}A \wedge U(1)_+) \ \longrightarrow \ \pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_2)_+) \ \longrightarrow \ \dots \ \longrightarrow \ \pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_{n!})_+) \ \longrightarrow \ \dots$$

We rewrite this using the Wirthmüller and suspension isomorphisms as

$$\pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_k)_+) \ \cong \ \pi_1^{C_k}(\mathcal{H}A \wedge S^1) \ \cong \ \pi_0^{C_k}(\mathcal{H}A) \ \cong \ A \ .$$

Example III.2.29 shows that for all $k, m \geq 1$ the following diagram commutes:

$$\begin{tikzcd}
\pi_1^{C_k}(\mathcal{H}A \wedge S^1) \arrow[r, "\mathrm{tr}_{C_k}^{C_{km}}"] \arrow[d, "U(1)\ltimes_{C_k} -"', "\cong"] & \pi_1^{C_{km}}(\mathcal{H}A \wedge S^1) \arrow[d, "U(1)\ltimes_{C_{km}} -", "\cong"'] \\
\pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_k)_+) \arrow[r, "(\mathcal{H}A\wedge \mathrm{proj})_1"'] & \pi_1^{U(1)}(\mathcal{H}A \wedge (U(1)/C_{km})_+)
\end{tikzcd}$$

Transfers commute with the suspension isomorphism and in the global functor $\underline{\pi}_0(\mathcal{H}A)$, the transfer $\mathrm{tr}_{C_k}^{C_{km}}$ is multiplication by the index $[C_{km} : C_k] = m$, so the above homotopy group sequence becomes the sequence

$$A \xrightarrow{\cdot 2} A \xrightarrow{\cdot 3} \cdots \longrightarrow A \xrightarrow{\cdot n} A \longrightarrow \cdots .$$

The colimit of this sequence is isomorphic to $A\otimes\mathbb{Q}$, compatible with the map $\bar{\psi} : A\otimes\mathbb{Q} \longrightarrow \pi_1^{U(1)}(\mathcal{H}A\wedge X_+)$. This proves the claim. □

2. Global Thom spectra

In this section we discuss two different global forms of the Thom spectrum MO that represents unoriented bordism, namely the ultra-commutative Thom ring spectrum **MO**, and a variation **mO** that is only E_∞-commutative. Both Thom spectra are the homogeneous degree 0 summands in certain $\mathbb{Z}$-graded periodic extensions **MOP** respectively **mOP**. All four orthogonal spectra are Thom spectra over certain orthogonal spaces defined in Section II.4. The partners are easy to identify from the notation: the relevant orthogonal spaces either have a **B** or **b** in their name, and in the corresponding Thom spectrum this letter is replaced by an **M** respectively **m**.

While the spectrum **mO** has less structure than **MO** (it is 'only' E_∞, not ultra-commutative), it is closely related to geometry. Indeed, the Thom spectrum **mO** is the natural target for the Thom-Pontryagin map from geometric equivariant bordism. The ultra-commutative ring spectrum **MO** is the $\mathbb{R}$-analog of equivariant homotopical unitary bordism, due to tom Dieck [**162**]; the unoriented version **MO** is studied in detail in [**28**]. The following commutative diagram gives a schematic overview over the relevant equivariant homology theories:

$$\begin{tikzcd}
\mathcal{N}_*^G \arrow[r, "\Theta^G"] \arrow[d] & \mathbf{mO}_*^G \arrow[r] \arrow[d] & \mathbf{mOP}_*^G \arrow[d] \\
\mathfrak{N}_*^{G:S} \arrow[r, "\cong", "\Theta^G"'] & \mathbf{MO}_*^G \arrow[r] & \mathbf{MOP}_*^G
\end{tikzcd}$$

The two theories in the middle column are the equivariant homology theories represented by the orthogonal Thom spectra **mO** and **MO**. The two vertical transformations $(b^{-1}a)_*$ are isomorphisms for $G = e$, but not in general. In fact, when G has more than one element, then $\pi_*^G(\mathbf{MO})$ has non-trivial elements in negative degrees, while **mO** is globally connective. The three vertical transformations are localizations, i.e., they become isomorphisms after inverting all inverse Thom classes, compare Corollary 2.37. The two theories **mOP** and **MOP** in the last column are the periodic versions of **mO** and **MO**; each is a wedge of all suspensions and desuspensions of the non-periodic version.

In the left column, $\mathcal{N}_*^G$ is geometrically defined equivariant bordism, and $\mathfrak{N}_*^{G:S}$ is *stable* equivariant bordism, a certain localization of $\mathcal{N}_*^G$. So the two theories in the left column are not represented by orthogonal spectra, but they are defined from bordism classes of G-manifolds; we will recall these geometric theories in Section 3. The transformations labeled Θ^G are the equivariant Thom-Pontryagin construction and its 'stabilization'. The upper Thom-Pontryagin map $\Theta^G : \mathcal{N}_*^G \longrightarrow \pi_*^G(\mathbf{mO})$ is an isomorphism whenever G is isomorphic to a product of a finite group and a torus; this result seems to have been folklore at some point, and we give a proof in Theorem 3.40. The upper Thom-Pontryagin map is not an isomorphism for more

general compact Lie groups; in fact the geometric bordism theory $\mathcal{N}_*^G$ cannot in general be represented by an orthogonal G-spectrum because the Wirthmüller map fails to be an isomorphism for all subgroups H of G that act non-trivially on the tangent space $T_H(G/H)$, compare Remark 3.15. The stabilized Thom-Pontryagin map $\Theta^G : \mathfrak{N}_*^{G:S} \longrightarrow \pi_*^G(\mathbf{MO})$ is an isomorphism in complete generality, by a theorem of Bröcker and Hook [**28**, Thm. 4.1]; we derive this fact in a different way, see Remark 3.46.

Example 2.1. We start with the ultra-commutative ring spectrum $\mathbf{MGr}$, the Thom spectrum over the additive Grassmannian $\mathbf{Gr}$, discussed in Example II.3.11. The value of $\mathbf{Gr}$ at an inner product space V is

$$\mathbf{Gr}(V) \;=\; \coprod_{n\geq 0} Gr_n(V)\ ,$$

the disjoint union of all Grassmannians in V. Over the space $\mathbf{Gr}(V)$ sits a tautological euclidean vector bundle (of non-constant rank) whose total space consisting of pairs $(x,U) \in V \times \mathbf{Gr}(V)$ such that $x \in U$. We define $\mathbf{MGr}(V)$ as the Thom space of this tautological vector bundle, i.e., the one-point compactification of the total space. The structure maps are given by

$$\mathbf{O}(V,W) \wedge \mathbf{MGr}(V) \;\longrightarrow\; \mathbf{MGr}(W)\ , \quad (w,\varphi)\wedge(x,U) \;\longmapsto\; (w+\varphi(x),\, \varphi^\perp + \varphi(U))\ ,$$

where $\varphi^\perp = W - \varphi(V)$ is the orthogonal complement of the image of $\varphi : V \longrightarrow W$. Multiplication maps are defined by direct sum, i.e.,

$$\mu_{V,W} \;:\; \mathbf{MGr}(V)\wedge\mathbf{MGr}(W) \;\longrightarrow\; \mathbf{MGr}(V\oplus W)\ , \quad (x,U)\wedge(x',U') \;\longmapsto\; ((x,x'),\, U\oplus U')\ .$$

Unit maps are defined by

$$S^V \;\longrightarrow\; \mathbf{MGr}(V)\ , \quad v \;\longmapsto\; (v,V)\ .$$

These multiplication maps are binatural, associative, commutative and unital, and all this structure makes $\mathbf{MGr}$ into an ultra-commutative ring spectrum. The orthogonal spectrum $\mathbf{MGr}$ is $\mathbb{Z}$-graded, with k-th homogeneous summand given by

$$\mathbf{MGr}^{[k]}(V) \;=\; Th(Gr_{|V|+k}(V))\ .$$

So $\mathbf{MGr}^{[k]}$ is concentrated in non-positive degrees, i.e., $\mathbf{MGr}^{[k]}$ is trivial for $k>0$. The unit morphism $\eta : \mathbb{S} \longrightarrow \mathbf{MGr}$ is an isomorphism onto the summand $\mathbf{MGr}^{[0]}$.

Now we let V be a representation of a compact Lie group G. We define the *inverse Thom class*

$$\tau_{G,V} \;\in\; \mathbf{MGr}_0^G(S^V) \tag{2.2}$$

as the class represented by the G-map

$$t_{G,V} \;:\; S^V \;\longrightarrow\; Th(Gr(V))\wedge S^V \;=\; \mathbf{MGr}(V)\wedge S^V\ , \quad v \;\longmapsto\; (0,\{0\})\wedge(-v)\ .$$

If V has dimension m, then the class $\tau_{G,V}$ has internal degree $-m$, i.e., it lies in the homogeneous summand $\mathbf{MGr}^{[-m]}$. The justification for the name 'inverse Thom class' is that in the theory $\mathbf{MOP}$, the image of the class $\tau_{G,V}$ becomes invertible, and its inverse is the Thom class of V (considered as G-vector bundle over a point). We explain this in more detail in Theorem 2.19 below. So while the theory $\mathbf{MGr}$ is not globally orientable, and does not have Thom isomorphisms for equivariant bundles, informally speaking the inverses of the prospective Thom classes are already present in $\mathbf{MGr}$.

Remark 2.3 ($\mathbf{MGr}$ as a wedge of free spectra). We recall from Construction IV.1.24 that the free orthogonal spectrum generated by the tautological $O(m)$-representation ν_m on $\mathbb{R}^m$ is given by

$$F_{O(m),\nu_m} \;=\; \mathbf{O}(\nu_m,-)/O(m)\ .$$

We claim that the spectrum $\mathbf{MGr}$ is isomorphic to the wedge of these free orthogonal spectra. Indeed, the maps

$$\mathbf{O}(\nu_m,V)/O(m) \;\longrightarrow\; \mathbf{MGr}^{[-m]}(V)\ , \quad (v,\varphi)\cdot O(m) \;\longmapsto\; (v, V-\varphi(\nu_m))$$

define an isomorphism of orthogonal spectra from $F_{O(m),\nu_m}$ to the homogeneous summand $\mathbf{MGr}^{[-m]}$. This isomorphism takes the equivariant homotopy class $a_{O(m),\nu_m}$ in $\pi_0^G(F_{O(m),\nu_m} \wedge S^{\nu_m})$ defined in (4.17) of Chapter IV to the inverse Thom class $\tau_{O(m),\nu_m}$ of the tautological $O(m)$-representation, by direct inspection of the definitions.

Now we suppose that $\varphi : V \longrightarrow W$ is an isomorphism of orthogonal G-representations. Then φ compactifies to a G-equivariant homeomorphism $S^\varphi : S^V \longrightarrow S^W$ and hence induces an isomorphism

$$(S^\varphi)_* \ : \ \mathbf{MGr}_0^G(S^V) \ \longrightarrow \ \mathbf{MGr}_0^G(S^W) \ .$$

The following properties of the inverse Thom classes $\tau_{G,V}$ are all straightforward from the definition.

PROPOSITION 2.4. *The inverse Thom classes $\tau_{G,V}$ have the following properties.*

(i) *For every isomorphism $\varphi : V \longrightarrow W$ of orthogonal G-representations, the induced isomorphism $(S^\varphi)_*$ takes the class $\tau_{G,V}$ to the class $\tau_{G,W}$.*

(ii) *The class $\tau_{G,0}$ of the trivial 0-dimensional G-representation is the multiplicative unit 1 in $\mathbf{MGr}_0^G(S^0) = \pi_0^G(\mathbf{MGr})$.*

(iii) *For every continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation*

$$\alpha^*(\tau_{G,V}) \ = \ \tau_{K,\alpha^*V}$$

holds in $\mathbf{MGr}_0^K(S^{\alpha^*(V)})$.

(iv) *For all orthogonal G-representations V and W the relation*

$$\tau_{G,V} \cdot \tau_{G,W} \ = \ \tau_{G,V\oplus W}$$

holds in $\mathbf{MGr}_0^G(S^{V\oplus W})$.

(v) *For all orthogonal G-representations V and all $k \geq 1$ the relation*

$$P^k(\tau_{G,V}) \ = \ \tau_{\Sigma_k \wr G,V^k}$$

holds in $\mathbf{MGr}_0^{\Sigma_k \wr G}(S^{V^k})$.

(vi) *For every closed subgroup H of G and all orthogonal H-representations W the relation*

$$N_H^G(\tau_{H,W}) \ = \ \tau_{G,\mathrm{Ind}_H^G W}$$

holds in $\mathbf{MGr}_0^G(S^{\mathrm{Ind}_H^G W})$.

The next proposition shows that multiplication by the inverse Thom class $\tau_{G,V}$ is realized by a certain morphism of orthogonal G-spectra $j^V_{\mathbf{MGr}} : \mathbf{MGr} \longrightarrow \mathrm{sh}^V \mathbf{MGr}$. The value at an inner product space U is the map

$$j^V_{\mathbf{MGr}}(U) \ : \ \mathbf{MGr}(U) \ \longrightarrow \ \mathbf{MGr}(U \oplus V) \ = \ (\mathrm{sh}^V \mathbf{MGr})(U) \ , \quad (x, L) \ \longmapsto \ (i(x), i(L)) \tag{2.5}$$

induced by the linear isometric embedding $i : U \longrightarrow U \oplus V$ as the first summand. If V has dimension m, then $j^V_{\mathbf{MGr}}$ is homogeneous of degree $-m$ in terms of the internal grading of $\mathbf{MGr}$, i.e., $j^V_{\mathbf{MGr}}$ takes the wedge summand $\mathbf{MGr}^{[k]}$ to the summand $\mathrm{sh}^V \mathbf{MGr}^{[k-m]}$.

PROPOSITION 2.6. *Let V be a representation of a compact Lie group G. The composite*

$$\pi_0^G(\mathbf{MGr} \wedge A) \ \xrightarrow{-\cdot\tau_{G,V}} \ \pi_0^G(\mathbf{MGr} \wedge A \wedge S^V) \ \xrightarrow{(\lambda^V_{\mathbf{MGr}\wedge A})_*} \ \pi_0^G(\mathrm{sh}^V \mathbf{MGr} \wedge A)$$

coincides with the effect of the morphism $j^V_{\mathbf{MGr}} \wedge A : \mathbf{MGr} \wedge A \longrightarrow \mathrm{sh}^V \mathbf{MGr} \wedge A$.

Proof. In (1.24) of Chapter IV we defined an isomorphism $\psi^V_X : \pi^G_k(\mathrm{sh}^V X) \longrightarrow \pi^G_k(X \wedge S^V)$, natural for orthogonal spectra X, essentially given by smashing with the identity of S^V. We also defined $\epsilon_V : \pi^G_k(X \wedge S^V) \longrightarrow \pi^G_k(X \wedge S^V)$ as the effect of the involution of $X \wedge S^V$ induced by the linear isometry $-\mathrm{Id}_V : V \longrightarrow V$. Now we observe that multiplication by the class $\tau_{G,V}$ factors as the composite

$$\begin{aligned} \pi^G_k(\mathbf{MGr} \wedge A) &\xrightarrow{(j^V_{\mathbf{MGr}} \wedge A)_*} \pi^G_k(\mathrm{sh}^V \mathbf{MGr} \wedge A) \\ &\xrightarrow{\psi^V_{\mathbf{MGr} \wedge A}} \pi^G_k(\mathbf{MGr} \wedge A \wedge S^V) \xrightarrow{\epsilon_V} \pi^G_k(\mathbf{MGr} \wedge A \wedge S^V) \ . \end{aligned}$$

Besides the definitions, this uses that the map $j^V_{\mathbf{MGr}}(U) \wedge S^V$ equals the composite

$$\mathbf{MGr}(U) \wedge S^V \xrightarrow{\mathbf{MGr}(U) \wedge t_{G,V}} \mathbf{MGr}(U) \wedge \mathbf{MGr}(V) \wedge S^V \xrightarrow{\mu_{U,V} \wedge S^{-\mathrm{Id}}} \mathbf{MGr}(U \oplus V) \wedge S^V \ .$$

The first claim then follows from the fact, established in Proposition III.1.25 (i), that $(\lambda^V_{\mathbf{MGr} \wedge A})_*$ is inverse to $\epsilon_V \circ \psi^V_{\mathbf{MGr} \wedge A}$. □

Example 2.7. We define two ultra-commutative ring spectra **MO** and **MOP**. The latter is a periodic version of the former, and the former is the homogeneous degree 0 summand with respect to a natural $\mathbb{Z}$-grading of the latter. Non-equivariantly, **MO** is an version of the unoriented Thom spectrum MO, and it is a global refinement of equivariant homotopical bordism, due to tom Dieck [**162**]; tom Dieck first considered the unitary version in [**162**], and the paper [**28**] by Bröcker and Hook studies the unoriented version.

The spectrum **MOP** is a Thom spectrum over the orthogonal space **BOP** discussed in Example II.4.2. The value of **BOP** at an inner product space V is

$$\mathbf{BOP}(V) \ = \ \coprod_{n \geq 0} Gr_n(V^2) \ ,$$

the disjoint union of all Grassmannians in V^2. Over the space $\mathbf{BOP}(V)$ sits a tautological euclidean vector bundle (again of non-constant rank) with total space consisting of pairs $(x, U) \in V^2 \times \mathbf{BOP}(V)$ such that $x \in U$. We define $\mathbf{MOP}(V)$ as the Thom space of this tautological vector bundle, i.e., the one-point compactification of the total space. The structure maps are given by

$$\mathbf{O}(V, W) \wedge \mathbf{MOP}(V) \ \longrightarrow \ \mathbf{MOP}(W) \ , \quad (w, \varphi) \wedge (x, U) \ \longmapsto \ ((w, 0) + \mathbf{BOP}(\varphi)(x), \mathbf{BOP}(\varphi)(U)) \ .$$

Multiplication maps

$$\mu_{V,W} \ : \ \mathbf{MOP}(V) \wedge \mathbf{MOP}(W) \ \longrightarrow \ \mathbf{MOP}(V \oplus W)$$

are defined by sending $(x, U) \wedge (x', U')$ to $(\kappa_{V,W}(x, x'), \kappa_{V,W}(U \oplus U'))$ where $\kappa_{V,W} : V^2 \oplus W^2 \cong (V \oplus W)^2$ is the preferred isometry defined by

$$\kappa_{V,W}((v, v'), (w, w')) \ = \ ((v, w), (v', w')) \ .$$

Unit maps are defined by

$$S^V \ \longrightarrow \ \mathbf{MOP}(V) \ , \quad v \ \longmapsto \ ((v, 0), V \oplus 0) \ .$$

These multiplication maps are binatural, associative, commutative and unital, and all this structure makes **MOP** into an ultra-commutative ring spectrum.

The orthogonal space **BOP** is $\mathbb{Z}$-graded, with k-th homogeneous summand given by

$$\mathbf{BOP}^{[k]}(V) \ = \ Gr_{|V|+k}(V^2) \ .$$

The spectrum **MOP** inherits a $\mathbb{Z}$-graded, where the summand $\mathbf{MOP}^{[k]}(V)$ of degree k is defined as the Thom space of the tautological $(|V| + k)$-plane bundle over $\mathbf{MOP}^{[k]}(V)$; then $\mathbf{MOP}(V)$ is the one-point union of the Thom spaces $\mathbf{MOP}^{[k]}(V)$ for $-|V| \leq k \leq |V|$, and thus

$$\mathbf{MOP} \ = \ \bigvee_{k \in \mathbb{Z}} \mathbf{MOP}^{[k]} \tag{2.8}$$

as orthogonal spectra.

We define $\mathbf{MO} = \mathbf{MOP}^{[0]}$ as the homogeneous degree wedge summand of $\mathbf{MOP}$; this is then an ultra-commutative ring spectrum in its own right. Explicitly, $\mathbf{MO}(V)$ is the Thom space of the tautological $\dim(V)$-plane bundle over $Gr_{|V|}(V^2)$.

Remark 2.9. Certain variations of the construction of $\mathbf{MO}$ and $\mathbf{MOP}$ are possible, and have been used at other places in the literature. Indeed, if U is any euclidean vector space, finite or infinite dimensional, and $u \in U$ a non-zero vector, we obtain an ultra-commutative ring spectrum $\mathbf{MO}_{U,u}$ in exactly the same way as above, with value at V given by the Thom space over the tautological vector bundle over $Gr_{|V|}(V \otimes U)$. The chosen vector u enters in the definition of the unit and structure maps. For $U = \mathbb{R}^2$ and $u = (1,0)$, the construction specializes to $\mathbf{MO}$ as above.

If the dimension of U is at least 2, then we always get the same global homotopy type. Indeed: any linear isometric embedding $\psi : U \longrightarrow U'$ such that $\psi(u) = u'$ induces a morphism of ultra-commutative ring spectra $\psi_* : \mathbf{MO}_{U,u} \longrightarrow \mathbf{MO}_{U',u'}$. If the dimension of U is at least 2, this morphism is a global equivalence.

Since $\mathbf{MOP}$ is an ultra-commutative ring spectrum, the collection of equivariant homotopy groups $\underline{\pi}_0(\mathbf{MOP})$ is a global power functor. The global power functor $\underline{\pi}_0(\mathbf{MOP})$ is an interesting algebraic structure, but a complete algebraic description does not seem to be known. Since $2 = 0$ in $\pi_0^e(\mathbf{MOP})$, the global power functor $\underline{\pi}_0(\mathbf{MOP})$ takes values in $\mathbb{F}_2$-vector spaces. In $\underline{\pi}_0(\mathbf{mO})$, and hence also in $\underline{\pi}_0(\mathbf{MOP})$, the stronger relation $\mathrm{tr}_e^{C_2}(1) = 0$ holds, compare Theorem 2.48.

The orthogonal spectrum underlying $\mathbf{MOP}$ comes with a $\mathbb{Z}$-grading, i.e., a wedge decomposition (2.8) into summands $\mathbf{MOP}^{[k]}$. The geometric splitting induces a direct sum decomposition of $\underline{\pi}_0^G(\mathbf{MOP})$ for every compact Lie group G and makes it into a commutative $\mathbb{Z}$-graded ring. The grading is multiplicative, and hence the m-th power operation takes the summand $\mathbf{MOP}^{[k]}$ to the summand $\mathbf{MOP}^{[mk]}$.

We move on to explain the periodicity of the ultra-commutative ring spectrum $\mathbf{MOP}$. We let $t \in \pi_{-1}(\mathbf{MOP}^{[-1]})$ be the class represented by the point

$$(0, \{0\}) \ \in \ Th(Gr_0(\mathbb{R}^2)) \ = \ \mathbf{MOP}^{[-1]}(\mathbb{R}) \ . \tag{2.10}$$

We let $\sigma \in \pi_1(\mathbf{MOP}^{[1]})$ be the class represented by the map

$$S^2 \ \longrightarrow \ Th(Gr_2(\mathbb{R}^2)) \ = \ \mathbf{MOP}^{[1]}(\mathbb{R}) \ , \quad x \ \longmapsto \ (x, \mathbb{R}^2) \ . \tag{2.11}$$

As the next proposition shows, $\mathbf{MOP}$ is periodic in the sense that t is a unit in the graded ring $\pi_*^e(\mathbf{MOP})$, with inverse σ.

The orthogonal spectrum $\mathbf{MOP}$ has an even stronger kind of '$RO(G)$-graded' periodicity, as we now explain. We define the *inverse Thom class*

$$\tau_{G,V} \ \in \ \mathbf{MOP}_0^G(S^V) \tag{2.12}$$

as the class represented by the G-map

$$t_{G,V} \ : \ S^V \ \longrightarrow \ Th(Gr(V^2)) \wedge S^V \ = \ \mathbf{MOP}(V) \wedge S^V \ , \quad v \ \longmapsto \ ((0,0), 0 \oplus 0) \wedge (-v) \ .$$

Here we abuse notation by denoting the inverse Thom class in $\mathbf{MOP}$-theory by the same symbol as the inverse Thom class in $\mathbf{MGr}$-theory defined in (2.2). The justification for this abuse is that the homomorphism $c : \mathbf{MGr} \longrightarrow \mathbf{MOP}$ introduced in (2.24) below takes one inverse Thom class to the other. If V has dimension m, then the class $\tau_{G,V}$ has internal degree $-m$, i.e., it lies in the homogeneous summand $\mathbf{MOP}^{[-m]}$. The justification for the name 'inverse Thom class' is that it is inverse to the Thom class $\sigma_{G,V}$ in $\mathbf{MOP}_G^0(S^V)$, defined in (2.18) below.

The periodicity class t of (2.10) is essentially the inverse Thom class of the 1-dimensional representation of the trivial group. More precisely,

$$t \wedge S^1 \ = \ \tau_{e,\mathbb{R}} \ ,$$

i.e., the suspension isomorphism

$$-\wedge S^1 \ : \ \pi^e_{-1}(\mathbf{MOP}) \ \xrightarrow{\cong} \ \pi^e_0(\mathbf{MOP}\wedge S^1) \ = \ \mathbf{MOP}^e_0(S^1)$$

takes the class periodicity class t to the inverse Thom class $\tau_{e,\mathbb{R}}$. Indeed, the suspension of the defining representative (2.10) for t differs from the defining representative for $\tau_{e,\mathbb{R}}$ by the inversion map $-\operatorname{Id} : S^1 \longrightarrow S^1$. So $t\wedge S^1 = -\tau_{e,\mathbb{R}}$; however, since $2=0$ in $\pi^e_0(\mathbf{MOP})$, this yields the claim.

The next proposition shows that multiplication by the inverse Thom classes is invertible in **MOP**-theory; moreover, multiplication by $\tau_{G,V}$ is realized, in a certain precise way, by a periodicity morphism of orthogonal G-spectra $j^V_{\mathbf{MOP}} : \mathbf{MOP} \longrightarrow \operatorname{sh}^V \mathbf{MOP}$ defined as follows. The value at an inner product space U is the map

$$j^V_{\mathbf{MOP}}(U) \ : \ \mathbf{MOP}(U) \ \longrightarrow \ \mathbf{MOP}(U\oplus V) \ = \ (\operatorname{sh}^V \mathbf{MOP})(U)\ , \quad (x,L) \ \longmapsto \ (i(x), i(L)) \tag{2.13}$$

induced by the linear isometric embedding $i : U\oplus U \longrightarrow U\oplus V\oplus U\oplus V$ with $i(u,u') = (u,0,u',0)$. The morphism $j^V_{\mathbf{MOP}}$ is even a homomorphism of left **MOP**-module spectra. If V has dimension m, then $j^V_{\mathbf{MOP}}$ is homogeneous of degree $-m$ in terms of the $\mathbb{Z}$-grading of **MOP**, i.e., it restricts to a morphism of orthogonal G-spectra

$$j^V_{\mathbf{MOP}} \ : \ \mathbf{MOP}^{[k+m]} \ \longrightarrow \ \operatorname{sh}^V \mathbf{MOP}^{[k]}$$

where k is any integer. In the special case $V=\mathbb{R}$ with trivial G-action, the map

$$j \ = \ j^{\mathbb{R}}_{\mathbf{MOP}} \ : \ \mathbf{MOP} \ \longrightarrow \ \operatorname{sh}\mathbf{MOP} \tag{2.14}$$

is a morphism of orthogonal spectra with (trivial G-action).

Theorem 2.15. (i) *For every representation V of a compact Lie group G, the morphism*

$$j^V_{\mathbf{MOP}} \ : \ \mathbf{MOP} \ \longrightarrow \ \operatorname{sh}^V \mathbf{MOP}$$

is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. In particular, the morphism $j = j^{\mathbb{R}}_{\mathbf{MOP}} : \mathbf{MOP} \longrightarrow \operatorname{sh}\mathbf{MOP}$ is a global equivalence.

(ii) *For every based G-space A, the composite*

$$\pi^G_0(\mathbf{MOP}\wedge A) \ \xrightarrow{-\cdot\tau_{G,V}} \ \pi^G_0(\mathbf{MOP}\wedge A\wedge S^V) \ \xrightarrow{(\lambda^V_{\mathbf{MOP}\wedge A})_*} \ \pi^G_0(\operatorname{sh}^V\mathbf{MOP}\wedge A)$$

coincides with the effect of the morphism $j^V_{\mathbf{MOP}}\wedge A : \mathbf{MOP}\wedge A \longrightarrow \operatorname{sh}^V\mathbf{MOP}\wedge A$. In particular, exterior multiplication by the inverse Thom class $\tau_{G,V}$ is invertible in equivariant **MOP***-homology.*

(iii) *The relation $t\cdot\sigma = 1$ holds in $\pi^e_0(\mathbf{MOP})$. Hence for every compact Lie group G, every based G-space A and all $k\in\mathbb{Z}$, the maps*

$$\mathbf{MOP}^G_{k+1}(A) \ \xrightarrow{\cdot p^*_G(t)} \mathbf{MOP}^G_k(A) \qquad \textit{and} \qquad \mathbf{MOP}^G_k(A) \ \xrightarrow{\cdot p^*_G(\sigma)} \mathbf{MOP}^G_{k+1}(A)$$

are mutually inverse isomorphisms.

Proof. (i) We show first that $j^V_{\mathbf{MOP}}$ induces an isomorphism on 0-th G-equivariant homotopy groups. To this end we define a map

$$\Phi \ : \ \pi^G_0(\operatorname{sh}^V\mathbf{MOP}) \ \longrightarrow \ \pi^G_0(\mathbf{MOP})$$

in the opposite direction as follows. We recall from (2.17) the G-map $s_{G,V} : S^{V\oplus V} \longrightarrow \mathbf{MOP}(V)$ given by $s_{G,V}(v,w) = ((v,w),V^2)$. If $f : S^U \longrightarrow \mathbf{MOP}(U\oplus V) = (\operatorname{sh}^V\mathbf{MOP})(U)$ represents a class in $\pi^G_0(\operatorname{sh}^V\mathbf{MOP})$, then we define $\Phi[f]$ as the class of the composite

$$S^{U\oplus V\oplus V} \ \xrightarrow{f\wedge s_{G,V}} \ \mathbf{MOP}(U\oplus V)\wedge\mathbf{MOP}(V) \ \xrightarrow{\mu_{U\oplus V,V}} \ \mathbf{MOP}(U\oplus V\oplus V)\ .$$

This recipe is compatible with stabilization, so Φ is indeed well-defined.

The composite

$$S^{V\oplus V} \ \xrightarrow{s_{G,V}} \ \mathbf{MOP}(V) \ \xrightarrow{j^V_{\mathbf{MOP}}(V)} \ \mathbf{MOP}(V\oplus V)$$

is G-equivariantly homotopic to the unit map $\eta(V \oplus V) : S^{V \oplus V} \longrightarrow \mathbf{MOP}(V \oplus V)$. So

$$\begin{aligned}
(j^V_{\mathbf{MOP}})_*(\Phi[f]) &= [j^V_{\mathbf{MOP}}(U \oplus V \oplus V) \circ \mu_{U \oplus V, V} \circ (f \wedge s_{G,V})] \\
&= [\mu_{U \oplus V, V \oplus V} \circ (\mathbf{MOP}(U \oplus V) \wedge j^V_{\mathbf{MOP}}(V)) \circ (f \wedge s_{G,V})] \\
&= [\mu_{U \oplus V, V \oplus V} \circ (f \wedge (j^V_{\mathbf{MOP}}(V) \circ s_{G,V}))] \\
&= [\mu_{U \oplus V, V \oplus V} \circ (f \wedge \eta(V \oplus V))] \;=\; [f \diamond V^2] \;=\; [f] \;.
\end{aligned}$$

The second equation is the fact that $j^V_{\mathbf{MOP}}$ is a homomorphism of left $\mathbf{MOP}$-modules. This proves that $(j^V_{\mathbf{MOP}})_* \circ \Phi$ is the identity.

On the other hand, the composite

$$\mathbf{MOP}(U) \wedge S^{V \oplus V} \xrightarrow{j^V_{\mathbf{MOP}}(U) \wedge s_{G,V}} \mathbf{MOP}(U \oplus V) \wedge \mathbf{MOP}(V) \xrightarrow{\mu_{U \oplus V, V}} \mathbf{mOP}(U \oplus V \oplus V)$$

agrees with the opposite structure map $\sigma^{\mathrm{op}}_{U, V \oplus V}$ of the spectrum $\mathbf{MOP}(U)$. So if $\varphi : S^U \longrightarrow \mathbf{MOP}(U)$ represents a class in $\pi_0^G(\mathbf{MOP})$, then

$$\begin{aligned}
\Phi((j^V_{\mathbf{MOP}})_*[\varphi]) &= [\mu_{U \oplus V, V} \circ (j^V_{\mathbf{MOP}}(U) \circ \varphi) \wedge s_{G,V}] \\
&= [\mu_{U \oplus V, V} \circ (j^V_{\mathbf{MOP}}(U) \wedge s_{G,V}) \circ (\varphi \wedge S^{V \oplus V})] \\
&= [\sigma^{\mathrm{op}}_{U, V \oplus V} \circ (\varphi \wedge S^{V \oplus V})] \;=\; [\varphi \diamond V^2] \;=\; [\varphi] \;.
\end{aligned}$$

This proves that $\Phi \circ (j^V_{\mathbf{MOP}})_*$ is the identity, and thus completes the proof that $\pi_0^G(j^V_{\mathbf{MOP}})$ is an isomorphism. Since $j^V_{\mathbf{MOP}}$ is a homomorphism of left $\mathbf{MOP}$-modules, its effect on homotopy groups is $\pi_*^G(\mathbf{MOP})$-linear. In particular, it commutes with the action of the invertible element $p_G^*(t)$, where $t \in \pi^e_{-1}(\mathbf{MOP})$ is the periodicity element defined in (2.10). Since $\pi_0^G(j^V_{\mathbf{MOP}})$ is an isomorphism, the map $\pi_k^G(j^V_{\mathbf{MOP}})$ is then an isomorphism for every integer k. Applying the above to a closed subgroup H of G and the underlying H-representation of V shows that $j^V_{\mathbf{MOP}}$ induces isomorphisms of equivariant stable homotopy groups for all closed subgroups of G. So $j^V_{\mathbf{MOP}}$ is a $\underline{\pi}_*$-isomorphism.

The proof of the first claim of part (ii) proceeds in the same way as its analog for $\mathbf{MGr}$ in Proposition 2.6. Since the morphisms $j^V_{\mathbf{MOP}}$ and $\lambda^V_{\mathbf{MOP} \wedge A}$ are both $\underline{\pi}_*$-isomorphism of orthogonal G-spectra, they induce isomorphisms on π_0^G. So exterior multiplication by the inverse Thom class $\tau_{G,V}$ is invertible.

(iii) The class $t \cdot \sigma$ is represented by the composite

$$S^2 \xrightarrow{x \mapsto (0, \{0\}) \wedge (x, \mathbb{R}^2)} \mathbf{MOP}^{[-1]}(\mathbb{R}) \wedge \mathbf{MOP}^{[1]}(\mathbb{R}) \xrightarrow{\mu_{\mathbb{R}, \mathbb{R}}} \mathbf{MOP}^{[0]}(\mathbb{R} \oplus \mathbb{R})$$

where the first map is the smash product of the defining representatives for t and σ. Expanding the multiplication of $\mathbf{MOP}$ identifies this composite as the map

$$S^2 \;\longrightarrow\; \mathbf{MOP}^{[0]}(\mathbb{R} \oplus \mathbb{R}) \;, \quad x \;\longmapsto\; ((\mathbb{R} \oplus \tau_{\mathbb{R}, \mathbb{R}} \oplus \mathbb{R})(0, 0, x), 0 \oplus \mathbb{R} \oplus 0 \oplus \mathbb{R}) \;.$$

This differs from the representative of the unit $1 \in \pi_0^G(\mathbf{MOP})$ by the action of the linear isometry

$$\mathbb{R}^4 \;\longrightarrow\; \mathbb{R}^4 \;, \quad (a, b, c, d) \;\longmapsto\; (b, d, c, a) \;.$$

This isometry has determinant 1, so we conclude that $t \cdot \sigma = 1$ in $\pi_0^G(\mathbf{MOP})$. □

The orthogonal spectra $\mathbf{MGr}$ and $\mathbf{MOP}$ both admit 'shift morphisms'

$$j^V_{\mathbf{MGr}} \;:\; \mathbf{MGr} \;\longrightarrow\; \mathrm{sh}^V \mathbf{MGr} \qquad \text{respectively} \qquad j^V_{\mathbf{MOP}} \;:\; \mathbf{MOP} \;\longrightarrow\; \mathrm{sh}^V \mathbf{MOP}$$

defined in essentially the same way in (2.5) respectively (2.13). However, the morphism $j^V_{\mathbf{MGr}}$ is *not* a $\underline{\pi}_*$-isomorphism, whereas the morphism $j^V_{\mathbf{MOP}}$ *is* a $\underline{\pi}_*$-isomorphism, by Theorem 2.15 (i). This is a reflection of the fact that the inverse Thom classes $\tau_{G,V}$ are *not* invertible in equivariant $\mathbf{MGr}$-homology, whereas their $\mathbf{MOP}$-counterparts are.

Construction 2.16 (Thom classes for representations). The Thom spectrum **MOP** comes with distinguished Thom classes for representations. We let G be a compact Lie group and V an orthogonal G-representation. We consider the G-map

$$s_{G,V} \ : \ S^{V^2} \ \longrightarrow \ Th(Gr(V \oplus V)) \ = \ \mathbf{MOP}(V) \ , \quad (v,w) \ \longmapsto \ ((v,w), V \oplus V) \ . \tag{2.17}$$

If V has dimension m, then $s_{G,V}$ is a homeomorphism onto the homogeneous summand $\mathbf{MOP}^{[m]}(V)$. The above based G-map represents a class

$$\sigma_{G,V} \ \in \ \mathbf{MOP}^0_G(S^V) \ = \ \mathrm{colim}_{U \in s(\mathcal{U}_G)} [S^{U \oplus V}, \mathbf{MOP}(U)]^G \ , \tag{2.18}$$

the *Thom class* in the G-equivariant **MOP**-cohomology of S^V. Here the maps in colimit system are formed by stabilization in much the same way as for the equivariant homotopy group $\pi_0^G(\mathbf{MOP})$; An abelian group structure on $\mathbf{MOP}^0_G(S^V)$ is defined in the same way as for $\pi_0^G(\mathbf{MOP})$.

The following theorem is a special case of a Thom isomorphism in the equivariant cohomology theory represented by **MOP**. It also makes precise in which way the inverse Thom class $\tau_{G,V}$ is inverse to the Thom class $\sigma_{G,V}$. This relation between $\tau_{G,V}$ and $\sigma_{G,V}$ is the ultimate justification for naming $\tau_{G,V}$ the 'inverse Thom class'.

Theorem 2.19. *Let V be a representation of a compact Lie group G. Then the composite*

$$\mathbf{MOP}^0_G(S^V) \ = \ \pi_0^G(\Omega^V \mathbf{MOP}) \ \xrightarrow{\cdot \tau_{G,V}} \ \pi_0^G((\Omega^V \mathbf{MOP}) \wedge S^V) \ \xrightarrow{(\epsilon^V_{\mathbf{MOP}})_*} \ \pi_0^G(\mathbf{MOP})$$

is inverse to multiplication by $\sigma_{G,V}$, where $\epsilon^V_{\mathbf{MOP}} : (\Omega^V \mathbf{MOP}) \wedge S^V \longrightarrow \mathbf{MOP}$ is the evaluation morphism. In particular, $\mathbf{MOP}^0_G(S^V)$ is a free module of rank 1 over the ring $\pi_0^G(\mathbf{MOP})$, and the Thom class $\sigma_{G,V}$ is a generator.

Proof. We consider a G-map $f : S^U \longrightarrow \Omega^V \mathbf{MOP}(U)$ that represents a class in $\mathbf{MOP}^0_G(S^V)$. Then $\epsilon^V_{\mathbf{MOP}}([f] \cdot \tau_{G,V})$ is represented by the following composite:

$$\begin{aligned} S^{U \oplus V} \ &\xrightarrow{f \wedge t_{G,V}} (\Omega^V \mathbf{MOP}(U)) \wedge \mathbf{MOP}(V) \wedge S^V \\ &\xrightarrow{\text{evaluate}} \mathbf{MOP}(U) \wedge \mathbf{MOP}(V) \ \xrightarrow{\mu^{\mathbf{MOP}}_{U,V}} \ \mathbf{MOP}(U \oplus V) \ , \end{aligned}$$

where $t_{G,V} : S^V \longrightarrow \mathbf{MOP}(V) \wedge S^V$ is the defining representative for the class $\tau_{G,V}$ from (2.12). If we let $f = \tilde{s}_{G,V} : S^V \longrightarrow \Omega^V \mathbf{MOP}(V)$ be adjoint to the defining representative for $\sigma_{G,V}$, then the composite comes out as the map

$$S^{V \oplus V} \ \longrightarrow \ \mathbf{MOP}(V \oplus V) \ , \quad (v,w) \ \longmapsto \ ((v,0,-w,0), V \oplus 0 \oplus V \oplus 0) \ .$$

This composite is equivariantly homotopic to the map

$$(v,w) \ \longmapsto \ ((v,w,0,0), \ V \oplus V \oplus 0 \oplus 0)$$

which represents the multiplicative unit. So we have shown the relation $\epsilon^V_{\mathbf{MOP}}(\sigma_{G,V} \cdot \tau_{G,V}) = 1$. All maps in sight are left $\pi_0^G(\mathbf{MOP})$-linear, so we deduce that

$$\epsilon^V_{\mathbf{MOP}}(x \cdot \sigma_{G,V} \cdot \tau_{G,V}) \ = \ x \cdot \epsilon^V_{\mathbf{MOP}}(\sigma_{G,V} \cdot \tau_{G,V}) \ = \ x$$

for every class $x \in \pi_0^G(\mathbf{MOP})$. On the other hand, the composite

$$\pi_0^G(\Omega^V \mathbf{MOP}) \ \xrightarrow{\cdot \tau_{G,V}} \ \pi_0^G((\Omega^V \mathbf{MOP}) \wedge S^V) \ \xrightarrow{(\lambda^V_{\Omega^V \mathbf{MOP}})_*} \ \pi_0^G(\mathrm{sh}^V \, \Omega^V \mathbf{MOP})$$

is the effect of the morphism $\Omega^V j^V_{\mathbf{MOP}} : \Omega^V \mathbf{MOP} \longrightarrow \mathrm{sh}^V \, \Omega^V \mathbf{MOP}$, by the same reasoning as in Theorem 2.15 (ii). Since $j^V_{\mathbf{MOP}}$ is a $\underline{\pi}_*$-isomorphism, so is $\Omega^V j^V_{\mathbf{MOP}}$ by Proposition III.1.41 (ii). On the other hand, $\lambda^V_{\Omega^V \mathbf{MOP}}$ is a $\underline{\pi}_*$-isomorphism by Proposition III.1.25 (ii). So multiplication by the class $\tau_{G,V}$ is an

isomorphism. The morphism $\epsilon^V_{\mathbf{MOP}}$ is a $\underline{\pi}_*$-isomorphism, also by Proposition III.1.25 (ii). So the composite $(\epsilon^V_{\mathbf{MOP}})_* \circ (- \cdot \tau_{G,V})$ is bijective. Since it is also left inverse to multiplication by $\sigma_{G,V}$, this proves the first claim. □

Construction 2.20 (Thom classes for equivariant vector bundles). The Thom spectrum **MOP** comes with a distinguished orientation, given by Thom classes for equivariant vector bundles. These Thom classes generalize the classes $\sigma_{G,V}$ defined in (2.18), when we view a G-representation as a G-vector bundle over a one-point G-space.

We recall the definition of the Thom classes. Given a compact Lie group, an orthogonal G-spectrum E and a compact based G-space A, we define the *G-equivariant E-cohomology group* of A as

$$E^0_G(A) \;=\; \mathrm{colim}_{V\in s(\mathcal{U}_G)}\,[S^V \wedge A, E(V)]^G \;.$$

We let B be a compact G-space and $\xi : E \longrightarrow B$ a G-equivariant vector bundle. The bundle has a classifying G-map $\psi : B \longrightarrow Gr(V)$ for some G-representation V, i.e., such that ξ is isomorphic to the pullback of the tautological G-vector bundle over the Grassmannian. We let $\bar\psi : E \longrightarrow V$ be a map that covers ψ, i.e., such that $\bar\psi(e) \in \psi(\xi(e))$. We define a based G-map

$$S^V \wedge Th(\xi) \;\longrightarrow\; \mathbf{MOP}(V) \;=\; Th(Gr(V^2)) \qquad \text{by} \qquad v\wedge e \;\longmapsto\; ((v,\bar\psi(e)), V\oplus\psi(\xi(e)))\;.$$

We denote the equivariant cohomology class represented by this map by

$$\sigma_G(\xi) \;\in\; \mathbf{MOP}^0_G(Th(\xi))$$

and refer to it as the *Thom class* of the G-vector bundle ξ. If the bundle ξ has constant rank n, then the image of the map lies in the wedge summand $\mathbf{MOP}^{[n]}$; so in that case the Thom class lies in the direct summand $(\mathbf{MOP}^{[n]})^0_G(Th(\xi))$. It is straightforward to see that the Thom classes just defined are natural for pullback of bundles, compatible with restriction along continuous homomorphisms, and the Thom class of an exterior product of bundles is the exterior product of the Thom classes.

The diagonal of the G-vector bundle $\xi : E \longrightarrow B$ is the map

$$\Delta \;:\; Th(\xi) \;\longrightarrow\; B_+ \wedge Th(\xi)\;, \quad e \;\longmapsto\; \xi(e)\wedge e\;.$$

This diagonal induces a action map of equivariant cohomology groups

$$\mathbf{MOP}^0_G(B_+)\times\mathbf{MOP}^0_G(Th(\xi)) \;\xrightarrow{\times}\; \mathbf{MOP}^0_G(B_+\wedge Th(\xi)) \;\xrightarrow{\Delta^*}\; \mathbf{MOP}^0_G(Th(\xi))\;.$$

Since the diagonal is coassociative and counital, the action map makes the group $\mathbf{MOP}^0_G(Th(\xi))$ into a left module over the commutative ring $\mathbf{MOP}^0_G(B_+)$. The Thom isomorphism then says that whenever B admits the structure of a finite G-CW-complex, then $\mathbf{MOP}^0_G(Th(\xi))$ is a free module of rank 1 over $\mathbf{MOP}^0_G(B_+)$, with the Thom class $\sigma_G(\xi)$ as a generator. Theorem 2.19 is the special case when the base consists of a single point. We refrain from giving the proof of the general Thom isomorphism in equivariant **MOP**-theory.

Construction 2.21 (Euler classes). We let G be a compact Lie group and V an m-dimensional G-representation. As usual, the Thom classes $\sigma_{G,V}$ give rise to Euler classes by 'restriction to the zero section', i.e.,

$$e(V) \;=\; i^*(\sigma_{G,V}) \;\in\; \mathbf{MOP}^0_G(S^0) \;=\; \pi^G_0(\mathbf{MOP})\;.$$

Here $i : S^0 \longrightarrow S^V$ is the inclusion of the fixed points 0 and ∞, and $i^* : \mathbf{MOP}^0_G(S^V) \longrightarrow \mathbf{MOP}^0_G(S^0)$ the induced map on equivariant cohomology groups given by precomposition. The Euler class is thus represented by the based G-map

$$S^V \;\longrightarrow\; \mathbf{MOP}(V)\;, \quad v \;\longmapsto\; ((v,0),\, V\oplus V)\;.$$

Since the Thom class lives in the homogeneous summand $\mathbf{MOP}^{[m]}$, so does the Euler class. If V has non-trivial G-fixed points, then the inclusion $i : S^0 \longrightarrow S^V$ is G-equivariantly null-homotopic, so $e(V) = 0$ whenever $V^G \neq 0$.

The above representative for the Euler class $e(V)$ is not to be confused with the unit map $\eta(V) : S^V \longrightarrow \mathbf{MOP}(V)$ of the ring spectrum structure, which is given by the similarly looking formula $\eta(V)(v) = ((v,0), V \oplus 0)$. The Euler class lies in the homogeneous summand $\mathbf{MOP}^{[m]}$, whereas the unit lies in the homogeneous summand $\mathbf{MOP}^{[0]} = \mathbf{MO}$.

Remark 2.22 (Shifted Thom and Euler classes in **MO**). The author thinks that the periodic theory **MOP** is the most natural home for the Thom classes, the Euler classes and the inverse Thom classes, but the more traditional place to host them is the degree 0 wedge summand $\mathbf{MO} = \mathbf{MOP}^{(0]}$. Indeed, for an m-dimensional G-representation V, the Thom class $\sigma_{G,V}$ and the Euler class $e(V)$ lie in the homogeneous summand $\mathbf{MOP}^{[m]}$, and we can use the periodicity of **MOP** to move the classes into **MO**, at the expense of shifting it from degree by m. In other words, by multiplying by a suitable power of the periodicity class $t \in \pi^e_{-1}(\mathbf{MOP}^{[-1]})$, we define

$$\bar\sigma_{G,V} \;=\; p_G^*(t^m)\cdot\sigma_{G,V} \in\; \mathbf{MO}^m_G(S^V) \qquad \text{and} \qquad \bar e(V) \;=\; p_G^*(t^m)\cdot e(V) \in\; \pi^G_{-m}(\mathbf{MO})\;,$$

where $p_G : G \longrightarrow e$ is the unique group homomorphism. The Thom isomorphism theorem for **MO** then says that $\mathbf{MO}^*_G(S^V)$ is a free graded module of rank 1 over the graded ring $\pi^G_*(\mathbf{MO})$, and the shifted Thom class $\bar\sigma_{G,V}$ is a generator. This version of the Thom isomorphism follows directly from Theorem 2.19 because **MOP** is globally equivalent to the wedge of all suspensions and desuspensions of **MO**. More precisely, the maps

$$\bigoplus_{n\in\mathbb{Z}} \pi^G_n(\mathbf{MO}) \;\longrightarrow\; \pi^G_0(\mathbf{MOP}) \qquad \text{and} \qquad \bigoplus_{n\in\mathbb{Z}} \mathbf{MO}^{-n}_G(S^V) \;\longrightarrow\; \mathbf{MOP}^0_G(S^V)\;,$$

given on the n-th summand by multiplication by $p_G^*(t^n)$, are isomorphisms; moreover the latter isomorphism takes the shifted Thom class $\bar\sigma_{G,V}$ to the original Thom class $\sigma_{G,V}$.

Similarly, we can use the periodicity of **MOP** to move the inverse Thom class $\tau_{G,V} \in \mathbf{MOP}^G_0(S^V)$ into **MO**, at the expense of shifting it from degree 0 to homological degree m. The class $\tau_{G,V}$ lies in the homogeneous summand $\mathbf{MOP}^{[-m]}$, so by multiplying by a suitable power of the periodicity class $\sigma \in \pi^e_1(\mathbf{MOP}^{[1]})$ we define

$$\bar\tau_{G,V} \;=\; p_G^*(\sigma^m)\cdot\tau_{G,V} \in\; \mathbf{MO}^G_m(S^V)\;. \tag{2.23}$$

Theorem 2.15 (ii) then implies that for every based G-space A, the map

$$-\cdot\bar\tau_{G,V} \;:\; \pi^G_*(\mathbf{MO}\wedge A) \;\longrightarrow\; \pi^G_{*+m}(\mathbf{MO}\wedge A\wedge S^V)$$

is an isomorphism.

Our next task is to show that the Thom spectrum **MOP** is a localization of the Thom spectrum **MGr**, obtained by formally inverting all inverse Thom classes. This result can be viewed as a 'thomification' of the fact that the morphism of ultra-commutative monoids $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ induces a group completion of abelian monoids $[A,i]^G : [A,\mathbf{Gr}]^G \longrightarrow [A,\mathbf{BOP}]^G$ for every compact Lie group G and every G-space A, compare Proposition II.4.6.

The ultra-commutative ring spectra **MGr** and **MOP** are connected by a homomorphism

$$a \;:\; \mathbf{MGr} \;\longrightarrow\; \mathbf{MOP} \tag{2.24}$$

whose value at an inner product space V is

$$a(V) \;:\; \mathbf{MGr}(V) \;\longrightarrow\; \mathbf{MOP}(V)\;, \quad (x,L) \;\longmapsto\; ((x,0), L\oplus 0)\;,$$

For varying V, these in turn form a morphism of $\mathbb{Z}$-graded ultra-commutative ring spectra. The morphism $a : \mathbf{MGr} \longrightarrow \mathbf{MOP}$ induces natural transformations of equivariant homology theories

$$(a\wedge A)_* \;:\; \mathbf{MGr}^G_*(A) \;\longrightarrow\; \mathbf{MOP}^G_*(A)$$

for all compact Lie groups G and all based G-spaces A. We observe that

$$(a \wedge S^V)_*(\tau_{G,V}) \;=\; \tau_{G,V} \;,$$

i.e., the morphism a takes the **MGr**-inverse Thom class to the **MOP**-inverse Thom class with the same name. The verification of this relation is immediate from the explicit representatives of the inverse Thom classes given in (2.2) respectively (2.12).

We define a localized version of equivariant **MGr**-homology by

$$\mathbf{MGr}_k^G(A)[1/\tau] \;=\; \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, \mathbf{MGr}_k^G(A \wedge S^V) \;;$$

for $V \subset W$ in $s(\mathcal{U}_G)$, the structure map in the colimit system is given by the multiplication

$$\mathbf{MGr}_k^G(A \wedge S^V) \;\xrightarrow{-\cdot\tau_{G,W-V}}\; \mathbf{MGr}_k^G(A \wedge S^V \wedge S^{W-V}) \;\cong\; \mathbf{MGr}_k^G(A \wedge S^W) \;.$$

In equivariant **MOP**-theory, the inverse Thom classes become invertible by Theorem 2.15 (ii). So for every G-representation V we can consider the maps

$$\mathbf{MGr}_k^G(A \wedge S^V) \;\xrightarrow{(a \wedge A \wedge S^V)_*}\; \mathbf{MOP}_k^G(A \wedge S^V) \;\xrightarrow{(-\cdot\tau_{G,V})^{-1}}\; \mathbf{MOP}_k^G(A) \;.$$

By the multiplicativity of a, these maps are compatible as V varies over the poset $s(\mathcal{U}_G)$, so they assemble into a natural transformation

$$a^\sharp \;:\; \mathbf{MGr}_k^G(A)[1/\tau] \;=\; \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, \mathbf{MGr}_k^G(A \wedge S^V) \;\longrightarrow\; \mathbf{MOP}_k^G(A) \;.$$

THEOREM 2.25. *For every compact Lie group G, every based G-space A and every integer k the map*

$$a^\sharp \;:\; \mathbf{MGr}_k^G(A)[1/\tau] \;\longrightarrow\; \mathbf{MOP}_k^G(A)$$

is an isomorphism.

PROOF. To simplify the exposition we prove the claim for $k = 0$ only, the argument for a general integer being essentially the same. Alternatively, we can observe that source and target of $a^\sharp$ are periodic, so it suffices to establish bijectivity in a single dimension. We show two separate statements that amount to the injectivity respectively surjectivity of the map $a^\sharp$.

(a) We show that for every class x is in the kernel of the map $(a \wedge A)_* : \mathbf{MGr}_0^G(A) \longrightarrow \mathbf{MOP}_0^G(A)$, there is a G-representation V such that $x \cdot \tau_{G,V} = 0$. Indeed, we can represent any such class x by a based G-map $f : S^V \longrightarrow \mathbf{MGr}(V) \wedge A$, for some G-representation V, such that the composite

$$S^V \;\xrightarrow{f}\; \mathbf{MGr}(V) \wedge A \;\xrightarrow{a(V) \wedge A}\; \mathbf{MOP}(V) \wedge A$$

is equivariantly null-homotopic. In (2.31) we defined a morphism of orthogonal G-spectra $j^V_{\mathbf{MGr}} : \mathbf{MGr} \longrightarrow \mathrm{sh}^V \mathbf{MGr}$. We observe that $\mathbf{MOP}(V) = \mathbf{MGr}(V \oplus V)$ and $a(V) = j^V_{\mathbf{MGr}}(V)$. So we can apply Proposition 2.6 and conclude that

$$(\lambda^V_{\mathbf{MGr} \wedge A})_*(x \cdot \tau_{G,V}) \;=\; (j^V_{\mathbf{MGr}} \wedge A)_*(x) \;=\; [(j^V_{\mathbf{MGr}}(V) \wedge A) \circ f] \;=\; [(a(V) \wedge A) \circ f] \;=\; 0 \;.$$

Since $\lambda^V_{\mathbf{MGr} \wedge A} : \mathbf{MGr} \wedge A \wedge S^V \longrightarrow \mathrm{sh}^V \mathbf{MGr} \wedge A$ is a $\underline{\pi}_*$-isomorphism (by Proposition III.1.25 (ii)), this implies the desired relation $x \cdot \tau_{G,V} = 0$.

(b) We show that for every class y in $\mathbf{MOP}_0^G(A)$, there is a G-representation V and a class x in $\mathbf{MGr}_0^G(A \wedge S^V)$ such that $y \cdot \tau_{G,V} = (a \wedge A \wedge S^V)_*(x)$. To this end we represent y by a based G-map $f : S^V \longrightarrow \mathbf{MOP}(V) \wedge A$. Because $\mathbf{MOP}(V) = \mathbf{MGr}(V^2) = (\mathrm{sh}^V \mathbf{MGr})(V)$, the map f also defines a class in $\pi_0^G(\mathrm{sh}^V \mathbf{MGr} \wedge A)$. Since $\lambda^V_{\mathbf{MGr} \wedge A} : \mathbf{MGr} \wedge A \wedge S^V \longrightarrow \mathrm{sh}^V \mathbf{MGr} \wedge A$ is a $\underline{\pi}_*$-isomorphism by Proposition III.1.25 (ii), there is a unique class $x \in \mathbf{MGr}_0^G(A \wedge S^V)$ such that

$$(\lambda^V_{\mathbf{MGr} \wedge A})_*(x) \;=\; [f] \;.$$

On the other hand, the map $a(V^2) : \mathbf{MGr}(V^2) \longrightarrow \mathbf{MOP}(V^2)$ is *equal* to the map $j^V_{\mathbf{MOP}}(V) : \mathbf{MOP}(V) \longrightarrow \mathbf{MOP}(V^2)$. So

$$\begin{aligned}(\lambda^V_{\mathbf{MOP}\wedge A})_*((a\wedge A\wedge S^V)_*(x)) &= (\mathrm{sh}^V a\wedge A)_*((\lambda^V_{\mathbf{MGr}\wedge A})_*(x))\\ &= (\mathrm{sh}^V a\wedge A)_*[f] \;=\; [(a(V^2)\wedge A)\circ f]\\ &= [(j^V_{\mathbf{MOP}}(V)\wedge A)\circ f] \;=\; (j^V_{\mathbf{MOP}}\wedge A)_*(y) \;=\; (\lambda^V_{\mathbf{MOP}\wedge A})_*(y\cdot\tau_{G,V})\end{aligned}$$

The sixth equation is Theorem 2.15 (ii), the others are either definitions or naturality properties. Since $\lambda^V_{\mathbf{MOP}'\wedge A}$ is a $\underline{\pi}_*$-isomorphism, we can conclude that

$$(a\wedge A\wedge S^V)_*(x) \;=\; y\cdot\tau_{G,V}\ . \qquad \square$$

Example 2.26 (The global Thom spectra **mO** and **mOP**). We define two E_∞ orthogonal ring spectra **mO** and **mOP**, the Thom spectrum over the orthogonal spaces **bO** and **bOP** defined in Example II.4.20 respectively Example II.4.33. The spectrum **mOP** is a periodic version of **mO**, and conversely **mO** is the homogeneous degree 0 summand with respect to a natural $\mathbb{Z}$-grading of **mOP**. Non-equivariantly, **mO** is another version of the unoriented Thom spectrum MO. The equivariant homology represented by **mO** is the natural target of the equivariant Thom-Pontryagin map from equivariant bordism, and that map is trying hard to be an isomorphism, see Theorem 3.40 below.

We recall that the value of **bOP** at an inner product space V is

$$\mathbf{bOP}(V) \;=\; \coprod_{n\geq 0} Gr_n(V\oplus\mathbb{R}^\infty)\ ,$$

disjoint union of all the Grassmannians in $V\oplus\mathbb{R}^\infty$. The map $\mathbf{bOP}(\varphi) : \mathbf{bOP}(V) \longrightarrow \mathbf{bOP}(W)$ induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is defined as

$$\mathbf{bOP}(\varphi)(L) \;=\; (\varphi\oplus\mathbb{R}^\infty)(L) \;+\; ((W-\varphi(V))\oplus 0)\ .$$

In other words: we apply the linear isometric embedding $\varphi\oplus\mathbb{R}^\infty : V\oplus\mathbb{R}^\infty \longrightarrow W\oplus\mathbb{R}^\infty$ to the subspace L and add the orthogonal complement of the image of φ (sitting in the first summand of $W\oplus\mathbb{R}^\infty$).

Over the space $\mathbf{bOP}(V)$ sits a tautological euclidean vector bundle (again of non-constant rank); the total space of this bundle consist of pairs $(x,U)\in(V\oplus\mathbb{R}^\infty)\times\mathbf{bO}(V)$ such that $x\in U$. We define $\mathbf{mOP}(V)$ as the Thom space of this tautological vector bundle. The structure maps are given by

$$\mathbf{O}(V,W)\wedge\mathbf{mOP}(V) \;\longrightarrow\; \mathbf{mOP}(W)\ ,\quad (w,\varphi)\wedge(x,U) \;\longmapsto\; ((w,0)+\mathbf{bOP}(\varphi)(x),\mathbf{bOP}(\varphi)(U))\ .$$

As we explained in Remark II.4.27, the orthogonal spaces **bO** and **bOP** have natural E_∞-structures. Correspondingly, the orthogonal spectra **mO** and **mOP** have natural E_∞-structures, by which we mean an action of the linear isometries operad. This multiplication is, however, *not* ultra-commutative. Multiplication maps

$$\mu_{V,W} \;:\; \mathbf{L}((\mathbb{R}^\infty)^2,\mathbb{R}^\infty)_+\wedge\mathbf{mOP}(V)\wedge\mathbf{mOP}(W) \;\longrightarrow\; \mathbf{mOP}(V\oplus W) \tag{2.27}$$

are defined by sending $\psi\wedge(x,U)\wedge(x',U')$ to $(\psi_\sharp(x,x'),\psi_\sharp(U\oplus U'))$, where $\psi_\sharp$ is the linear isometric embedding

$$V\oplus\mathbb{R}^\infty\oplus W\oplus\mathbb{R}^\infty \;\longrightarrow\; V\oplus W\oplus\mathbb{R}^\infty\ ,\quad \psi_\sharp(v,y,w,z) \;=\; (v,w,\psi(y,z))\ .$$

Unit maps are defined by

$$S^V \;\longrightarrow\; \mathbf{mOP}(V)\ ,\quad v \;\longmapsto\; ((v,0),\,V\oplus 0)\ .$$

All this structure makes **mOP** into an E_∞-orthogonal ring spectrum.

The orthogonal space **bOP** is $\mathbb{Z}$-graded, with k-th homogeneous summand given by

$$\mathbf{bOP}^{[k]}(V) \;=\; Gr_{|V|+k}(V\oplus\mathbb{R}^\infty)\ .$$

The spectrum $\mathbf{mOP}$ is $\mathbb{Z}$-graded, where the summand $\mathbf{mOP}^{[k]}(V)$ of degree k is defined as the Thom space of the tautological $(|V|+k)$-plane bundle over $\mathbf{bOP}^{[k]}(V)$; then $\mathbf{mOP}(V)$ is the one-point union of the Thom spaces $\mathbf{mOP}^{[k]}(V)$ for $|V|+k \geq 0$. So we have a wedge decomposition

$$\mathbf{mOP} \;=\; \bigvee_{k\in\mathbb{Z}} \mathbf{mOP}^{[k]} \tag{2.28}$$

as orthogonal spectra. We define $\mathbf{mO} = \mathbf{mOP}^{[0]}$ as the homogeneous wedge summand of degree 0.

In the rest of this section we will also use products on the equivariant homology theories represented by the Thom spectra $\mathbf{mO}$ and $\mathbf{mOP}$. As we just explained, the spectrum $\mathbf{mOP}$ comes with an E_∞ multiplication which is, however, neither strictly associative, nor strictly commutative. So we briefly explain how to define these products. We choose a linear isometric embedding $\psi : \mathbb{R}^\infty \oplus \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$. We define continuous maps

$$\begin{aligned} \psi_{V,W} \;:\; \mathbf{mOP}(V)\wedge\mathbf{mOP}(W) \;&\longrightarrow\; \mathbf{mOP}(V\oplus W) \qquad \text{by} \\ \psi_{V,W}((x,U),(x',U')) \;&=\; (\psi_\sharp(x,x'),\psi_\sharp(U\oplus U'))\ , \end{aligned} \tag{2.29}$$

where $\psi_\sharp$ is the linear isometric embedding

$$V\oplus\mathbb{R}^\infty\oplus W\oplus\mathbb{R}^\infty \;\longrightarrow\; V\oplus W\oplus\mathbb{R}^\infty\ , \quad \psi_\sharp(v,y,w,z) \;=\; (v,w,\psi(y,z))\ .$$

These maps form a bimorphism, which corresponds to a morphism of orthogonal spectra

$$\psi_\star \;:\; \mathbf{mOP}\wedge\mathbf{mOP} \;\longrightarrow\; \mathbf{mOP}$$

by the universal property of the smash product. Since the space $\mathbf{L}((\mathbb{R}^\infty)^2,\mathbb{R}^\infty)$ of linear isometric embeddings is contractible, the morphism $\psi_\star$ is independent up to homotopy of the choice of ψ. Even though the multiplication map $\psi_\star$ is neither associative nor commutative, the contractibility of the space $\mathbf{L}((\mathbb{R}^\infty)^3,\mathbb{R}^\infty)$ implies that the square

$$\begin{tikzcd} \mathbf{mOP}\wedge\mathbf{mOP}\wedge\mathbf{mOP} \arrow[r, "\mathbf{mOP}\wedge\psi_\star"] \arrow[d, "\psi_\star\wedge\mathbf{mOP}"'] & \mathbf{mOP}\wedge\mathbf{mOP} \arrow[d, "\psi_\star"] \\ \mathbf{mOP}\wedge\mathbf{mOP} \arrow[r, "\psi_\star"'] & \mathbf{mOP} \end{tikzcd}$$

commutes up to homotopy, and the contractibility of the space $\mathbf{L}((\mathbb{R}^\infty)^2,\mathbb{R}^\infty)$ implies that the composite

$$\mathbf{mOP}\wedge\mathbf{mOP} \xrightarrow{\tau_{\mathbf{mOP},\mathbf{mOP}}} \mathbf{mOP}\wedge\mathbf{mOP} \xrightarrow{\psi_\star} \mathbf{mOP}$$

is homotopic to $\psi_\star$. So whenever we pass to induced maps on equivariant homotopy groups, an E_∞-multiplication is as good as a strictly associative and commutative multiplication. However, an E_∞-multiplication does *not* entitle us to power operations.

Given a compact Lie group G and based G-space A and B, we define a multiplication

$$\cdot \;:\; \mathbf{mOP}^G_k(A)\times\mathbf{mOP}^G_l(B) \;\longrightarrow\; \mathbf{mOP}^G_{k+l}(A\wedge B)$$

as the composite

$$\begin{aligned} \pi^G_k(\mathbf{mOP}\wedge A)\times\pi^G_l(\mathbf{mOP}\wedge B) \;&\xrightarrow{\times}\; \pi^G_{k+l}(\mathbf{mOP}\wedge A\wedge\mathbf{mOP}\wedge B) \\ &\xrightarrow{(\text{twist})_*}\; \pi^G_{k+l}(\mathbf{mOP}\wedge\mathbf{mOP}\wedge A\wedge B) \\ &\xrightarrow{(\psi_\star\wedge A\wedge B)_*}\; \pi^G_{k+l}(\mathbf{mOP}\wedge A\wedge B)\ . \end{aligned}$$

Here the first map is the external pairing (5.4) defined in Construction III.5.3.

We move on to explain the periodicity property of **mOP**. As the theories **MGr** and **MOP**, the theory **mOP** also has its own inverse Thom classes and shift morphisms. We define the *inverse Thom class*

$$\tau_{G,V} \ \in \ \mathbf{mOP}_0^G(S^V) \tag{2.30}$$

as the class represented by the G-map

$$S^V \ \longrightarrow \ Th(Gr(V \oplus \mathbb{R}^\infty)) \wedge S^V \ = \ \mathbf{mOP}(V) \wedge S^V \ , \quad v \ \longmapsto \ ((0,0), 0 \oplus 0) \wedge (-v) \ .$$

Here we abuse notation one more time and also denote the inverse Thom class in **mOP**-theory by the same symbol as its counterparts in **MGr**-theory and **MOP**-theory defined in (2.2) respectively (2.12). The justification for this abuse is that the inverse Thom classes match up under certain homomorphism relating **MGr**, **MOP** and **mOP**. As usual, if V has dimension m, then the class $\tau_{G,V}$ lies in the homogeneous summand $\mathbf{MOP}^{[-m]}$. A shift morphism of orthogonal G-spectra $j^V_{\mathbf{mOP}} : \mathbf{mOP} \longrightarrow \mathrm{sh}^V \mathbf{mOP}$ is defined as for **MGr** and **MOP**: the value at an inner product space U is the map

$$j^V_{\mathbf{mOP}}(U) \ : \ \mathbf{mOP}(U) \ \longrightarrow \ \mathbf{mOP}(U \oplus V) \ = \ (\mathrm{sh}^V \mathbf{mOP})(U) \ , \quad (x, L) \ \longmapsto \ (i(x), i(L)) \tag{2.31}$$

induced by the linear isometric embedding $i : U \oplus \mathbb{R}^\infty \longrightarrow U \oplus V \oplus \mathbb{R}^\infty$ with $i(u, x) = (u, 0, x)$. If V has dimension m, then $j^V_{\mathbf{mOP}}$ is homogeneous of degree $-m$. In the special case $V = \mathbb{R}$ with trivial G-action, the map

$$j \ = \ j^{\mathbb{R}}_{\mathbf{mOP}} \ : \ \mathbf{mOP} \ \longrightarrow \ \mathrm{sh}\,\mathbf{mOP}$$

is a morphism of orthogonal spectra with (trivial G-action).

On the level of homotopy groups, the periodicity is realized by multiplication with a periodicity element $t \in \pi^e_{-1}(\mathbf{mOP}^{[-1]})$ represented by the point

$$(0, \{0\}) \ \in \ Th(Gr_0(\mathbb{R} \oplus \mathbb{R}^\infty)) \ = \ \mathbf{mOP}^{[-1]}(\mathbb{R}) \ . \tag{2.32}$$

Proposition 2.33. (i) *The morphism $j = j^{\mathbb{R}}_{\mathbf{mOP}} : \mathbf{mOP} \longrightarrow \mathrm{sh}\,\mathbf{mOP}$ is a homotopy equivalence of orthogonal spectra, hence a global equivalence. For every compact Lie group G and every based G-space A, the induced map*

$$(j \wedge A)_* \ : \ \pi^G_*(\mathbf{mOP} \wedge A) \ \longrightarrow \ \pi^G_*(\mathrm{sh}\,\mathbf{mOP} \wedge A)$$

is an isomorphism.

(ii) *For every compact Lie group G, every G-representation V and every based G-space A, the composite*

$$\pi^G_0(\mathbf{mOP} \wedge A) \ \xrightarrow{-\cdot \tau_{G,V}} \ \pi^G_0(\mathbf{mOP} \wedge A \wedge S^V) \ \xrightarrow{(\lambda^V_{\mathbf{mOP} \wedge A})_*} \ \pi^G_0(\mathrm{sh}^V \mathbf{mOP} \wedge A)$$

coincides with the effect of the morphism $j^V_{\mathbf{mOP}} \wedge A : \mathbf{mOP} \wedge A \longrightarrow \mathrm{sh}^V \mathbf{mOP} \wedge A$. In particular, exterior multiplication by the inverse Thom class $\tau_{G,\mathbb{R}}$ of the trivial 1-dimensional G-representation is invertible in equivariant **mOP***-homology.*

(iii) *For every compact Lie group G, every based G-space A and every integer k the multiplication map*

$$\mathbf{mOP}^G_{k+1}(A) \ \xrightarrow{\cdot p^*_G(t)} \ \mathbf{mOP}^G_k(A)$$

is an isomorphisms. In particular, the class $t \in \pi_{-1}(\mathbf{mOP})$ is a unit in the graded homotopy ring $\pi_*(\mathbf{mOP})$.

Proof. (i) The morphism j is based on the linear isometric embedding $i : \mathbb{R}^\infty \longrightarrow \mathbb{R} \oplus \mathbb{R}^\infty$ defined by $i(x) = (0, x)$. This linear isometric embedding is homotopic, through linear isometric embeddings, to the linear isometry $i' : \mathbb{R}^\infty \longrightarrow \mathbb{R} \oplus \mathbb{R}^\infty$ defined by $i'(x_1, x_2, x_3, \dots) = (x_1, (x_2, x_3, \dots))$. This homotopy induces a homotopy from the morphism i to an isomorphism between **mOP** and **mOP**. So i is homotopic to an isomorphism, hence a homotopy equivalence.

The first statement in (ii) is proved by the same argument as the analogous statement for **MGr** in Proposition 2.6. For $V = \mathbb{R}$, the trivial 1-dimensional G-representation the morphisms $j^{\mathbb{R}}_{\mathbf{MOP}}$ and $\lambda_{\mathbf{MOP} \wedge A}$

are both $\underline{\pi}_*$-isomorphism of orthogonal G-spectra, by part (i) respectively Proposition IV.1.4 (i). So they induce isomorphisms on π_0^G. So exterior multiplication by the inverse Thom class $\tau_{G,\mathbb{R}}$ is invertible.

(iii) The composite

$$\mathbf{mOP}^G_{k+1}(A) \xrightarrow{-\cdot p_G^*(t)} \mathbf{mOP}^G_k(A) \xrightarrow[\cong]{-\wedge S^1} \mathbf{mOP}^G_{k+1}(A\wedge S^1)$$

differs from multiplication by the inverse Thom class $\tau_{G,\mathbb{R}}$ by the effect of the involution $\mathbf{mOP}\wedge A\wedge S^{-\mathrm{Id}}$ of $\mathbf{mOP}\wedge A\wedge S^1$. Multiplication by $\tau_{G,\mathbb{R}}$ is an isomorphism by part (ii), and the suspension isomorphism is bijective by Proposition III.1.30. So multiplication by $p_G^*(t)$ is bijective as well. □

In the earlier Theorem 2.25 we showed that equivariant **MOP**-homology is obtained from equivariant **MGr**-homology by inverting all inverse Thom classes. Now we fit the theory **mOP** into this picture, which turns out to an intermediate localization. As we will now explain, **mOP**-theory is obtained from **MGr**-theory by inverting the inverse Thom classes of all *trivial* representation. Then **MOP**-theory is obtained from **mOP**-theory by inverting the remaining inverse Thom classes, i.e., the ones of non-trivial representations. Informally speaking, the first localization turns **MGr**-theory into a theory that is periodic in the $\mathbb{Z}$-graded sense; the second localization then turns **mOP**-theory into a theory that is periodic in the $RO(G)$-graded sense. Schematically:

$$\mathbf{MGr}^G_*(A) \xRightarrow{\text{invert } \tau_{G,\mathbb{R}}} \mathbf{mOP}^G_k(A) \xRightarrow{\text{invert all } \tau_{G,V}} \mathbf{MOP}^G_*(A)$$

The ring spectra **MGr** and **mOP** are connected by a homomorphism

$$b \ : \ \mathbf{MGr} \ \longrightarrow \ \mathbf{mOP} \tag{2.34}$$

whose value at an inner product space V is

$$b(V) \ : \ \mathbf{MGr}(V) \ \longrightarrow \ \mathbf{mOP}(V) \ , \quad (x,L) \ \longmapsto \ ((x,0), L\oplus 0) \ ;$$

in other words, the map is induced by the embedding $V \longrightarrow V\oplus\mathbb{R}^\infty$ as the first summand. For varying V, these in turn form a morphism of $\mathbb{Z}$-graded orthogonal E_∞-ring spectra. The morphism $b : \mathbf{MGr} \longrightarrow \mathbf{mOP}$ induces natural transformations of equivariant homology theories

$$(b\wedge A)_* \ : \ \mathbf{MGr}^G_*(A) \ \longrightarrow \ \mathbf{mOP}^G_*(A)$$

for all compact Lie groups G and all based G-spaces A. We observe that

$$(b\wedge S^V)_*(\tau_{G,V}) \ = \ \tau_{G,V} \ ,$$

i.e., the morphism b takes the **MGr**-inverse Thom class to the **mOP**-inverse Thom class with the same name. The verification of this relation is immediate from the explicit representatives of the inverse Thom classes given in (2.2) respectively (2.30).

We define a localized version of equivariant **MGr**-homology by

$$\mathbf{MGr}^G_k(A)[\tau_{G,\mathbb{R}}^{-1}] \ = \ \mathrm{colim}_{n\geq 0}\, \mathbf{MGr}^G_k(A\wedge S^n) \ ,$$

the colimit of the sequence

$$\mathbf{MGr}^G_k(A) \xrightarrow{-\cdot\tau_{G,\mathbb{R}}} \mathbf{MGr}^G_k(A\wedge S^1) \xrightarrow{-\cdot\tau_{G,\mathbb{R}}} \dots \ \longrightarrow \ \mathbf{MGr}^G_k(A\wedge S^n) \xrightarrow{-\cdot\tau_{G,\mathbb{R}}} \dots$$

given by multiplication by the inverse Thom class $\tau_{G,\mathbb{R}} \in \mathbf{MGr}^G_0(S^1)$. In equivariant **mOP**-theory, the class $\tau_{G,\mathbb{R}}$ becomes invertible by Theorem 2.33 (ii). So we can consider the maps

$$\mathbf{MGr}^G_k(A\wedge S^n) \xrightarrow{(b\wedge A\wedge S^n)_*} \mathbf{mOP}^G_k(A\wedge S^n) \xrightarrow{(-\cdot\tau^n_{G,V})^{-1}} \mathbf{mOP}^G_k(A) \ .$$

By the multiplicativity of b, these maps are compatible, so they assemble into a natural transformation

$$b^\sharp \ : \ \mathbf{MGr}^G_k(A)[\tau_{G,\mathbb{R}}^{-1}] \ \longrightarrow \ \mathbf{mOP}^G_k(A) \ .$$

THEOREM 2.35. *For every compact Lie group G, every based G-space A and every integer k the map*

$$b^\sharp \ : \ \mathbf{MGr}_k^G(A)[\tau_{G,\mathbb{R}}^{-1}] \ \longrightarrow \ \mathbf{mOP}_k^G(A)$$

is an isomorphism.

PROOF. The 'standard' linear isometric embedding

$$\mathbb{R}^n \ \longrightarrow \ \mathbb{R}^\infty \ , \quad (x_1,\dots,x_n) \ \longmapsto \ (x_1,\dots,x_n,0,0,\dots)$$

induces a continuous map

$$\psi^n(V) \ : \ (\mathrm{sh}^n \mathbf{MGr})(V) \ = \ Th(Gr(V\oplus\mathbb{R}^n)) \ \longrightarrow \ Th(Gr(V\oplus\mathbb{R}^\infty)) \ = \ \mathbf{mOP}(V) \ .$$

As V varies, these maps form a morphism of orthogonal spectra $\psi^n : \mathrm{sh}^n \mathbf{MGr} \longrightarrow \mathbf{mOP}$. Moreover, $\psi^n = \psi^{n+1}\circ(\mathrm{sh}^n j)$, so the morphisms ψ^n are compatible with the sequence of morphisms

$$\mathbf{MGr} \ \xrightarrow{j_{\mathbf{MGr}}} \ \mathrm{sh}\,\mathbf{MGr} \ \xrightarrow{\mathrm{sh}\, j_{\mathbf{MGr}}} \ \dots \ \mathrm{sh}^n \mathbf{MGr} \ \xrightarrow{\mathrm{sh}^n j_{\mathbf{MGr}}} \ \dots$$

Moreover, the morphisms ψ^n express $\mathbf{mOP}$ as the colimit of this sequence. So $\mathbf{mOP}\wedge A$ is the colimit of the sequence of orthogonal G-spectra $\mathrm{sh}^n \mathbf{MGr}\wedge A$. The map $j_{\mathbf{MGr}}(V)$ is an h-cofibration of based $O(V)$-spaces. So if G acts on V by linear isometries, then $j_{\mathbf{MGr}}(V)\wedge A$ is an h-cofibration of based G-spaces, hence a closed embedding. The morphism $j_{\mathbf{MGr}}\wedge A$ and all its shifts are thus levelwise closed embeddings.

Proposition III.1.42 (i) shows that equivariant homotopy groups commute with sequential colimits over levelwise closed embeddings; so the canonical map

$$\mathrm{colim}_{n\geq 0}\ \pi_k^G(\mathrm{sh}^n \mathbf{MGr}\wedge A) \ \longrightarrow \ \pi_k^G(\mathbf{mOP}\wedge A)$$

is an isomorphism. The diagram

$$\begin{array}{ccccc}
\pi_k^G(\mathbf{MGr}\wedge A\wedge S^{n-1}) & \xrightarrow{\cdot\tau_{G,\mathbb{R}}} & \pi_k^G(\mathbf{MGr}\wedge A\wedge S^n) & \xrightarrow{\pi_k^G(b\wedge A\wedge S^n)} & \pi_k^G(\mathbf{mOP}\wedge A\wedge S^n) \\
\lambda^{n-1}_{\mathbf{MGr}\wedge A}\Big\downarrow\cong & & \cong\Big\downarrow\lambda^n_{\mathbf{MGr}\wedge A} & & \cong\Big\downarrow(-\cdot\tau^n_{G,\mathbb{R}})^{-1} \\
\pi_k^G(\mathrm{sh}^{n-1}\mathbf{MGr}\wedge A) & \xrightarrow[\pi_k^G(\mathrm{sh}^{n-1} j_{\mathbf{MGr}}\wedge A)]{} & \pi_k^G(\mathrm{sh}^n \mathbf{MGr}\wedge A) & \xrightarrow[\pi_k^G(\psi^n\wedge A)]{} & \pi_k^G(\mathbf{mOP}\wedge A)
\end{array}$$

commutes by Proposition 2.6 and because $(\mathrm{sh}^{n-1} j_{\mathbf{MGr}})\circ\lambda^{n-1}_{\mathbf{MGr}} = \lambda^{n-1}_{\mathrm{sh}\,\mathbf{MGr}}\circ(j_{\mathbf{MGr}}\wedge S^{n-1})$. The three vertical maps are isomorphisms. So $\pi_k^G(\mathbf{MOP})$ is also a colimit of the sequence of multiplication maps by $\tau_{G.\mathbb{R}}$, with respect to the maps that define $b^\sharp$. □

There is one localization result left: it remains to exhibit $\mathbf{MOP}$-theory as the localization of $\mathbf{mOP}$-theory, by inverting the inverse Thom classes of arbitrary representations. This is in fact a formal consequence of Theorem 2.25 and Theorem 2.35 which exhibit both $\mathbf{MOP}_*^G(A)$ and $\mathbf{mOP}_*^G(A)$ as localizations of $\mathbf{MGr}_*^G(A)$, the former being a more drastic localization than the latter.

We spell this out in more detail. We define a localized version of equivariant $\mathbf{mOP}$-homology by

$$\mathbf{mOP}_0^G(A)[1/\tau] \ = \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}\ \mathbf{mOP}_0^G(A\wedge S^V)\ ;$$

for $V\subset W$ in $s(\mathcal{U}_G)$, the structure map in the colimit system is given by the multiplication

$$\mathbf{mOP}_0^G(A\wedge S^V) \ \xrightarrow{-\cdot\tau_{G,W-V}} \ \mathbf{mOP}_0^G(A\wedge S^V\wedge S^{W-V}) \ \cong \ \mathbf{mOP}_0^G(A\wedge S^W)\ .$$

In (2.24) we introduced a morphism of ultra-commutative ring spectra $a : \mathbf{MGr} \longrightarrow \mathbf{MOP}$. In (2.34) we introduced the morphism of E_∞-ring spectra $b : \mathbf{MGr} \longrightarrow \mathbf{mOP}$. These morphisms induce multiplicative natural transformations

$$(a\wedge A)_* \ : \ \mathbf{MGr}_*^G(A) \ \longrightarrow \ \mathbf{MOP}_*^G(A) \quad \text{respectively} \quad (b\wedge A)_* \ : \ \mathbf{MGr}_*^G(A) \ \longrightarrow \ \mathbf{mOP}_*^G(A)$$

for all compact Lie groups G and all based G-spaces A. Moreover, the morphisms match up the inverse Thom classes in the sense that

$$(a \wedge S^V)_*(\tau_{G,V}) \;=\; \tau_{G,V} \qquad \text{and} \qquad (b \wedge S^V)_*(\tau_{G,V}) \;=\; \tau_{G,V}\ ;$$

indeed, this is the justification for our abuse of notation of using the same name for the inverse Thom classes in **MGr**, **MOP** and **mOP**.

Theorem 2.35 says that the map $(b \wedge A)_*$ becomes an isomorphism after inverting the inverse Thom class $\tau_{G,\mathbb{R}}$ of the trivial 1-dimensional G-representation. So it also becomes an becomes after inverting the inverse Thom classes of all representations, i.e., the induced transformation

$$(b \wedge A)_*[1/\tau] \;:\; \mathbf{MGr}^G_*(A)[1/\tau] \;\longrightarrow\; \mathbf{mOP}^G_*(A)[1/\tau]$$

is an isomorphism. On the other hand, the transformation $(a \wedge A)_*$ induces an isomorphism $a^\sharp$ from $\mathbf{MGr}^G_*(A)[1/\tau]$ to $\mathbf{MOP}^G_*(A)$, by Theorem 2.25. So combining these two theorems yields:

Corollary 2.36. *For every compact Lie group G, every based G-space A and every integer k the map*

$$a^\sharp \circ (b \wedge A)_*[1/\tau] \;:\; \mathbf{mOP}^G_k(A)[1/\tau] \;\longrightarrow\; \mathbf{MOP}^G_k(A)$$

is an isomorphism.

While the authors thinks that the periodic theories **mOP** and **MOP** give the most convenient formulation of the localization result, the more traditional formulation is in terms of the degree 0 summands $\mathbf{mO} = \mathbf{MOP}^{[0]}$ and $\mathbf{MO} = \mathbf{MOP}^{[0]}$. So we also take the time to reformulate Corollary 2.36 in terms of **mO** and **MO**. Since the inverse Thom classes do *not* lie in the degree summands, we instead use the shifted inverse Thom classes

$$\bar\tau_{G,V} \;=\; p_G^*(\sigma^m) \cdot \tau_{G,V} \in\; \mathbf{MO}^G_m(S^V)$$

introduced in (2.23) and its **mOP**-analog

$$\bar\tau_{G,V} \;=\; p_G^*(\sigma^m) \cdot \tau_{G,V} \in\; \mathbf{mO}^G_m(S^V)\ .$$

In the **MOP**-case, the periodicity class $\sigma \in \pi_1^e(\mathbf{MOP}^{[1]})$ was defined in (2.11), and it is inverse to the class t. In the **mOP**-case, the periodicity class $\sigma \in \pi_1^e(\mathbf{mOP}^{[1]})$, was not yet defined, and we take it to be the inverse of the **mOP** periodicity class t from (2.32). Both theories **mOP** and **MOP** are periodic (in the $\mathbb{Z}$-graded sense), i.e., the maps

$$\bigoplus_{n\in\mathbb{Z}} \mathbf{MO}^G_n(A) \;\longrightarrow\; \mathbf{MOP}^G_0(A) \qquad \text{and} \qquad \bigoplus_{n\in\mathbb{Z}} \mathbf{mO}^G_n(A) \;\longrightarrow\; \mathbf{mOP}^G_0(A)$$

that multiply by the appropriate power of the periodicity classes are isomorphisms.

We define a localized version of equivariant **mO**-homology by

$$\mathbf{mO}^G_k(A)[1/\bar\tau] \;=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)} \mathbf{mO}^G_{k+|V|}(A \wedge S^V)\ ;$$

for $V \subset W$ in $s(\mathcal{U}_G)$, the structure map in the colimit system is given by the multiplication

$$\mathbf{mO}^G_{k+|V|}(A \wedge S^V) \;\xrightarrow{-\cdot\bar\tau_{G,W-V}}\; \mathbf{mO}^G_{k+|W|}(A \wedge S^V \wedge S^{W-V}) \;\cong\; \mathbf{mO}^G_{k+|W|}(A \wedge S^W)\ .$$

The periodicity of **mOP** is inherited by the localized theory, i.e., the map

$$\bigoplus_{n\in\mathbb{Z}} \mathbf{mO}^G_n(A)[1/\bar\tau] \;\longrightarrow\; \mathbf{mOP}^G_0(A)[1/\tau]$$

is an isomorphism. Moreover, the following square commutes:

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{Z}} \mathbf{mO}_n^G(A)[1/\bar{\tau}] & \longrightarrow & \mathbf{mOP}_0^G(A)[1/\tau] \\ \downarrow & & \downarrow{\scriptstyle a^\sharp\circ(b\wedge A)_*[1/\tau]} \\ \bigoplus_{n\in\mathbb{Z}} \mathbf{MO}_n^G(A) & \longrightarrow & \mathbf{MOP}_0^G(A) \end{array}$$

The right vertical map is an isomorphism by Corollary 2.36, and the two vertical maps are isomorphism by periodicity. So the left vertical map is an isomorphism. Since the left map is homogeneous with respect to the $\mathbb{Z}$-graded, it is an isomorphism in every degree. So we conclude:

Corollary 2.37. *For every compact Lie group G, every cofibrant based G-space A and every integer k the map*

$$a^\sharp \circ (b\wedge A)_*[1/\tau] \ : \ \mathbf{mO}_k^G(A)[1/\bar{\tau}] \ \longrightarrow \ \mathbf{MO}_k^G(A)$$

is an isomorphism.

Remark 2.38. The isomorphisms of Corollaries 2.36 and 2.37 that exhibit **MOP** and **MO** as localizations of **mOP** and **mO** were defined in an algebraic way. We explain now that these maps are actually induced by a weak morphism of $\mathbb{Z}$-graded orthogonal E_∞-ring spectra, i.e., a chain of morphism

$$\mathbf{mOP} \ \xrightarrow{a} \ \mathbf{MOP}' \ \underset{\sim}{\xleftarrow{b}} \ \mathbf{MOP} \ ,$$

such that b is a global equivalence.

The intermediate object $\mathbf{MOP}'$ is the Thom spectrum over the E_∞ orthogonal monoid space $\mathbf{BOP}'$ discussed in (4.36) of Chapter II, and $\mathbf{MOP}'$ combines the feature of **MOP** and **mOP** into one object. The value of $\mathbf{MOP}'$ at an inner product space V is the Thom space of the tautological bundle over

$$\mathbf{BOP}'(V) \ = \ Gr(V^2 \oplus \mathbb{R}^\infty) \ .$$

The structure maps and E_∞-multiplication are a mixture of the structure in **mOP** and **MOP**; the embeddings

$$V \oplus \mathbb{R}^\infty \ \xrightarrow{(v,x)\mapsto(v,0,x)} \ V^2 \oplus \mathbb{R}^\infty \ \xleftarrow{(v,w,0)\leftarrow\!\shortmid(v,w)} \ V^2$$

thomify to maps

$$\mathbf{mOP}(V) \ \xrightarrow{a(V)} \ \mathbf{MOP}'(V) \ \xleftarrow{\psi(V)} \ \mathbf{MOP}(V) \ .$$

For varying V, these in turn form morphisms of E_∞-orthogonal ring spectra

$$\mathbf{mOP} \ \xrightarrow{c} \ \mathbf{MOP}' \ \xleftarrow{\psi} \ \mathbf{MOP} \ . \tag{2.39}$$

The constructions respect the $\mathbb{Z}$-grading, so the morphisms restrict to morphisms of E_∞-ring spectra

$$\mathbf{mO} \ \xrightarrow{c} \ \mathbf{MO}' \ \xleftarrow{\psi} \ \mathbf{MO} \ .$$

The unstable global equivalence between **BO** and $\mathbf{BO}'$ of Proposition II.4.30 has a thomified stable analog, showing that the morphisms $\psi : \mathbf{MO} \longrightarrow \mathbf{MO}'$ and $\psi : \mathbf{MOP} \longrightarrow \mathbf{MOP}'$ are global equivalences of E_∞-orthogonal ring spectra. Not only the statement, but also the proof of Proposition II.4.30 thomifies, but we refrain from spelling it out since we don't need it.

The following square commutes by direct inspection:

$$\begin{array}{ccc} \mathbf{MGr} & \xrightarrow{a} & \mathbf{MOP} \\ {\scriptstyle b}\downarrow & & \sim\downarrow{\scriptstyle \psi} \\ \mathbf{mOP} & \xrightarrow[c]{} & \mathbf{MOP}' \end{array}$$

Here a is a morphism of ultra-commutative monoids, the other three maps are morphism of E_∞ ring spectra. So all four maps induce multiplicative transformations of equivariant homology theories. The global equivalence ψ induces an isomorphism, at least on cofibrant equivariant spaces, by Proposition 2.19 (ii). So we obtain a commutative diagram of multiplicative equivariant homology theories

$$\begin{tikzcd} \mathbf{MGr}_*^G(A) \arrow[r, "(a\wedge A)_*"] \arrow[d, "(b\wedge A)_*"'] & \mathbf{MOP}_*^G(A) \\ \mathbf{mOP}_*^G(A) \arrow[ur, bend right, "(\psi\wedge A)_*^{-1}\circ(c\wedge A)_*"'] & \end{tikzcd}$$

This shows that the weak homomorphism (2.39) from **mOP** to **MOP** induces the isomorphism of Corollary 2.36 between the localization of $\mathbf{mOP}_0^G(A)$ and $\mathbf{MOP}_*^G(A)$

Now we will investigate the global homotopy type of the Thom spectrum **mO** in more detail; the main tool is the 'rank filtration' that we discuss now. One reason for wanting to understand **mO** better is the close connection to equivariant bordism, compare Theorem 3.40 below. There is an analog of the rank filtration for **mOP**, but we refrain from making it explicit.

Construction 2.40 (Rank filtration of **mO**). The orthogonal spectrum **mO** is a global Thom spectrum over an orthogonal space **bO**; in Proposition II.4.26 we identified **bO** as a certain global homotopy colimit of the global classifying spaces $B_{\mathrm{gl}}O(m)$. More precisely, the filtration of $\mathbb{R}^\infty$ by the subspaces $\mathbb{R}^m$ induces a filtration of **bO** by orthogonal subspaces $\mathbf{bO}_{(m)}$, and $\mathbf{bO}_{(m)}$ receives a global equivalence from $B_{\mathrm{gl}}O(m) = \mathbf{L}_{O(m),\nu_m}$. We now define and study the corresponding orthogonal Thom spectrum $\mathbf{mO}_{(m)}$ over $\mathbf{bO}_{(m)}$, which turns out to be an m-fold suspension of the orthogonal spectrum $M_{\mathrm{gl}}T(m)$, the global refinement of the spectrum traditionally denoted $MT(m)$.

We recall from Construction IV.1.24 the definition of free orthogonal spectra $F_{G,V}$ generated by a G-representation V. We are interested in the tautological $O(m)$-representation ν_m with underlying inner product space $\mathbb{R}^m$, and to simplify the notation we abbreviate the corresponding free spectrum to

$$F_m \;=\; F_{O(m),\nu_m}\ .$$

The shift functor $\mathrm{sh}^m = \mathrm{sh}^{\mathbb{R}^m}$ by the inner product space $\mathbb{R}^m$ was defined in Construction III.1.21. We set

$$\mathbf{mO}_{(m)} \;=\; \mathrm{sh}^m F_m\ ,$$

the m-th shift of F_m. Unpacking this definition reveals the value of $\mathbf{mO}_{(m)}$ at an inner product space V as the space

$$\mathbf{mO}_{(m)}(V) \;=\; \mathbf{O}(\nu_m, V\oplus\mathbb{R}^m)/O(m)\ .$$

To justify the notation $\mathbf{mO}_{(m)}$ we clarify the connection to the orthogonal space $\mathbf{bO}_{(m)}$ defined in (4.25) of Chapter II. The value of $\mathbf{bO}_{(m)}$ at an inner product space V is

$$\mathbf{bO}_{(m)}(V) \;=\; Gr_{|V|}(V\oplus\mathbb{R}^m)\ ,$$

the Grassmannian of $|V|$-planes in $V\oplus\mathbb{R}^m$. Over the space $\mathbf{bO}_{(m)}(V)$ sits a tautological euclidean $|V|$-plane bundle, with total space consisting of pairs $(x,U)\in(V\oplus\mathbb{R}^m)\times\mathbf{bO}_{(m)}(V)$ such that $x\in U$. Passage to orthogonal complements provides a homeomorphism:

$$\begin{aligned} \mathbf{mO}_{(m)}(V) \;=\; \mathbf{O}(\nu_m, V\oplus\mathbb{R}^m)/O(m) \;&\cong\; Th(Gr_{|V|}(V\oplus\mathbb{R}^m)) \\ [x,\varphi] \quad&\longmapsto\; (x,\varphi(\nu_m)^\perp) \end{aligned}$$

In this sense, $\mathbf{mO}_{(m)}(V)$ 'is' the Thom space over $\mathbf{bO}_{(m)}(V)$. Just as the orthogonal spaces $\mathbf{bO}_{(m)}$ form an exhaustive filtration of **bO**, the orthogonal spectra $\mathbf{mO}_{(m)}$ form an exhaustive filtration of the Thom spectrum **mO**, compare Proposition 2.42 below.

As we explained in Remark IV.1.26, the free orthogonal spectrum $F_m = F_{O(m),\nu_m}$ is a global refinement of the Thom spectrum $MT(m)$ of the negative of the tautological m-plane bundle over $Gr_m(\mathbb{R}^\infty)$. Since shift and suspension are globally equivalent (by Proposition IV.1.4 (i)), $\mathbf{mO}_{(m)}$ is globally equivalent to the m-fold suspension of the orthogonal spectrum $M_{\mathrm{gl}}T(m) = F_{O(m),\nu_m}$:

$$\mathbf{mO}_{(m)} \;=\; \mathrm{sh}^m F_m \;\simeq_{\mathrm{gl}}\; F_m \wedge S^m \;=\; M_{\mathrm{gl}}T(m) \wedge S^m \;.$$

We define a morphism of orthogonal spectra

$$i \;:\; F_m \;\longrightarrow\; \mathrm{sh}\, F_{m+1} \;;$$

the value at an inner product space V is the closed embedding

$$\begin{array}{rcl} i(V) \;:\; F_m(V) = \mathbf{O}(\nu_m, V)/O(m) & \xrightarrow{-\oplus\mathbb{R}} & \mathbf{O}(\nu_{m+1}, V\oplus\mathbb{R})/O(m+1) = (\mathrm{sh}\, F_{m+1})(V) \\ [x,\varphi] & \longmapsto & [(x,0), \varphi\oplus\mathbb{R}] \end{array}$$

where we identify $\nu_m \oplus \mathbb{R}$ with ν_{m+1} by sending $((x_1,\dots,x_m),y)$ to $(x_1,\dots,x_m,y)$. In fact, there are not many morphisms of orthogonal spectra from F_m to $\mathrm{sh}\, F_{m+1}$: by the representing property of the free spectrum F_m, such morphisms biject with $O(m)$-fixed points of $(\mathrm{sh}\, F_{m+1})(\nu_m) = \mathbf{O}(\nu_{m+1}, \nu_m\oplus\mathbb{R})/O(m+1)$; this space only has two elements, and the morphism i corresponds to the non-basepoint element. We define

$$j^m \;=\; \mathrm{sh}^m i \;:\; \mathbf{mO}_{(m)} = \mathrm{sh}^m F_m \;\longrightarrow\; \mathrm{sh}^m(\mathrm{sh}\, F_{m+1}) \;=\; \mathrm{sh}^{m+1} F_{m+1} \;=\; \mathbf{mO}_{(m+1)} \;.$$

The rank filtration expresses the orthogonal spectrum $\mathbf{mO}$ as the colimit of the sequence of closed embeddings

$$\mathbb{S} \;\cong\; \mathbf{mO}_{(0)} \xrightarrow{j^0} \mathbf{mO}_{(1)} \xrightarrow{j^1} \dots \longrightarrow \mathbf{mO}_{(m)} \xrightarrow{j^m} \dots \;. \tag{2.41}$$

We define a morphism

$$\psi^m \;:\; \mathbf{mO}_{(m)} \;=\; \mathrm{sh}^m F_m \;\longrightarrow\; \mathbf{mO}$$

at an inner product space V as the map

$$\begin{array}{rcl} \psi^m(V) \;:\; \mathbf{O}(\nu_m, V\oplus\mathbb{R}^m)/O(m) & \longrightarrow & Th(Gr_{|V|}(V\oplus\mathbb{R}^\infty)) \\ [x,\varphi] & \longmapsto & (i(x), i(\varphi^\perp)) \;, \end{array}$$

where $\varphi^\perp = (V\oplus\mathbb{R}^m) - \varphi(\nu_m)$ is the orthogonal complement of the image of φ and $i : V\oplus\mathbb{R}^m \longrightarrow V\oplus\mathbb{R}^\infty$ is the 'standard' embedding given by

$$i(v, x_1, \dots, x_m) \;=\; (v, x_1, \dots, x_m, 0, 0, \dots) \;.$$

The following proposition is straightforward from the definitions. The periodic analog was already used in the proof of Theorem 2.35, since it expresses $\mathbf{mOP}$ as a sequential colimit of the spectra $\mathrm{sh}^m \mathbf{MGr}$. We omit the proof.

Proposition 2.42. *For every $m \geq 0$ the morphism $j^m :\; \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$ is levelwise a closed embedding. These morphisms satisfy $\psi^{m+1} \circ j^m = \psi^m$. With respect to the morphisms $\psi^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}$, the orthogonal spectrum $\mathbf{mO}$ is a colimit of the sequence* (2.41).

Since colimits along sequences of closed embeddings are invariant under global equivalences (Proposition IV.1.4 (v)), Proposition 2.42 says that $\mathbf{mO}$ is also a homotopy colimit of the sequence (2.41), The underlying non-equivariant statement, i.e., that MO is a homotopy colimit of the spectra $\Sigma^m MT(m)$, can for example be found in [**56**, Sec. 3]. The identification of $\mathbf{mO}$ as a homotopy colimit of free orthogonal spectra now allows an algebraic description of $[\![\mathbf{mO}, E]\!]$, the group of morphisms in the global stable homotopy category $\mathcal{GH}$, into any orthogonal spectrum E.

We define a distinguished class

$$\tau_m \;\in\; \pi_m^{O(m)}(\mathbf{mO}_{(m)} \wedge S^{\nu_m})$$

in the m-th $O(m)$-equivariant $\mathbf{mO}_{(m)}$-homology group of S^{ν_m} as the class represented by the based $O(m)$-map

$$\begin{aligned} S^{\nu_m \oplus \mathbb{R}^m} \ &\longrightarrow\ \mathbf{O}(\nu_m, \nu_m \oplus \mathbb{R}^m)/O(m) \wedge S^{\nu_m} \ = \ \mathbf{mO}_{(m)}(\nu_m) \wedge S^{\nu_m} \\ (v,x) \ &\longmapsto \qquad [(0,x),i] \wedge (-v) \ , \end{aligned} \tag{2.43}$$

where $i : \nu_m \longrightarrow \nu_m \oplus \mathbb{R}^m$ is the embedding as the first summand. We recall that the inverse Thom class $\tau_{O(m),\nu_m}$ in $\mathbf{mOP}_0^{O(m)}(S^{\nu_m})$ was defined in (2.30). By multiplying with the m-th power of the periodicity class $\sigma \in \pi_1^e(\mathbf{mOP})$, we obtain the shifted inverse Thom class

$$\bar{\tau}_{O(m),\nu_m} \ = \ \tau_{O(m),\nu_m} \cdot p^*_{O(m)}(\sigma^m) \ \in \ \mathbf{mO}_m^{O(m)}(S^{\nu_m}) \ .$$

PROPOSITION 2.44. *Let $m \geq 0$ be a natural number.*

(i) *The pair $(\mathbf{mO}_{(m)}, \tau_m)$ represents the functor*

$$\mathcal{GH} \longrightarrow (\text{sets}) \ , \quad E \ \longmapsto \ E_m^{O(m)}(S^{\nu_m}) \ = \ \pi_m^{O(m)}(E \wedge S^{\nu_m}) \ .$$

(ii) *The morphism $j^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$ satisfies the relation*

$$(j^m \wedge S^{\nu_m})_*(\tau_m) \wedge S^1 \ = \ \text{res}_{O(m)}^{O(m+1)}(\tau_{m+1})$$

in the group $\pi_{m+1}^{O(m)}(\mathbf{mO}_{(m+1)} \wedge S^{\nu_m} \wedge S^1)$.

(iii) *The morphism $\psi^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}$ sends τ_m to the shifted inverse Thom class $\bar{\tau}_{O(m),\nu_m}$.*

PROOF. (i) In (4.17) of Section IV we defined a distinguished equivariant homology class

$$a_m \ = \ a_{O(m),\nu_m} \ \in \ \pi_0^{O(m)}(F_m \wedge S^{\nu_m}) \ .$$

Expanding the definitions of a_m and of the morphism

$$\lambda^m_{F_m \wedge S^{\nu_m}} \ : \ F_m \wedge S^{\nu_m} \wedge S^m \ \longrightarrow \ \text{sh}^m F_m \wedge S^{\nu_m} \ = \ \mathbf{mO}_{(m)} \wedge S^{\nu_m}$$

shows that

$$\tau_m \ = \ (\lambda^m_{F_m \wedge S^{\nu_m}})_*(a_m \wedge S^m) \ .$$

By Theorem IV.4.19, the pair (F_m, a_m) represents the functor

$$\mathcal{GH} \longrightarrow (\text{sets}) \ , \quad E \ \longmapsto \ E_0^{O(m)}(S^{\nu_m}) \ = \ \pi_0^{O(m)}(E \wedge S^{\nu_m}) \ .$$

We claim that the following composite

$$\begin{aligned} [\![\mathbf{mO}_{(m)}, E]\!] \ = \ [\![\text{sh}^m F_m, E]\!] \ &\xrightarrow{[\![\lambda^m_{F_m}, E]\!]} \ [\![F_m \wedge S^m, E]\!] \\ &\xrightarrow{\text{adjunction}} \ [\![F_m, \Omega^m E]\!] \\ &\xrightarrow{\text{eval at } a_m} \ \pi_0^{O(m)}((\Omega^m E) \wedge S^{\nu_m}) \\ &\xrightarrow{\text{assembly}_*} \ \pi_0^{O(m)}(\Omega^m(E \wedge S^{\nu_m})) \ \xrightarrow{\alpha^m} \ \pi_m^{O(m)}(E \wedge S^{\nu_m}) \end{aligned}$$

coincides with evaluation at the class τ_m. Here the fourth map is induced by the assembly morphism

$$(\Omega^m E) \wedge S^{\nu_m} \ \longrightarrow \ \Omega^m(E \wedge S^{\nu_m}) \ ,$$

and

$$\alpha^m \ : \ \pi_0^{O(m)}(\Omega^m Y) \ \cong \ \pi_m^{O(m)}(Y)$$

is the analog of the loop isomorphism. Since all the maps are natural in E, it suffices to check this claim for the identity of the universal example $E = \mathbf{mO} = \mathrm{sh}^m F_m$. Since the square

$$\begin{tikzcd} E \wedge S^{\nu_m} \arrow[r, "\eta"] \arrow[d, "\tilde{\lambda}^m_E \wedge S^{\nu_m}"'] & \Omega^m(E \wedge S^{\nu_m} \wedge S^m) \arrow[d, "\Omega^m(\lambda^m_{E \wedge S^{\nu_m}})"] \\ (\Omega^m \mathrm{sh}^m E) \wedge S^{\nu_m} \arrow[r, "\text{assembly}"'] & \Omega^m(\mathrm{sh}^m E \wedge S^{\nu_m}) \end{tikzcd}$$

commutes, we obtain

$$\begin{aligned} \alpha^m(\text{assembly}_*((\tilde{\lambda}^m_{F_m} \wedge S^{\nu_m})_*(a_m))) &= \alpha^m(\Omega^m(\lambda^m_{F_m \wedge S^{\nu_m}})_*(\eta_*(a_m))) \\ &= (\lambda^m_{F_m \wedge S^{\nu_m}})_*(\alpha^m((\eta_*(a_m)))) \\ &= (\lambda^m_{F_m \wedge S^{\nu_m}})_*(a_m \wedge S^m) \ = \ \tau_m \ . \end{aligned}$$

This verifies the relation in the universal example. Since all the individual maps in the above composite are bijective, so is the composite, which proves the representability property of the pair $(\mathbf{mO}_{(m)}, \tau_m)$.

(ii) The class $(j^m \wedge S^{\nu_m})_*(\tau_m) \wedge S^1$ is represented by the composite

$$\begin{aligned} S^{\nu_m \oplus \mathbb{R}^{m+1}} &\xrightarrow{t_m \wedge S^1} \mathbf{O}(\nu_m, \nu_m \oplus \mathbb{R}^m)/O(m) \wedge S^{\nu_m} \wedge S^1 \\ &\xrightarrow{j^m(\nu_m \oplus \mathbb{R}^m) \wedge S^{\nu_m} \wedge S^1} \mathbf{O}(\nu_{m+1}, \nu_m \oplus \mathbb{R}^{m+1})/O(m+1) \wedge S^{\nu_m} \wedge S^1 \end{aligned}$$

$$(v, x, s) \ \longmapsto \ ((0, x), i) \cdot O(m) \wedge (-v) \wedge s \ \longmapsto \ ((0, x, 0), i \oplus \mathbb{R}) \cdot O(m+1) \wedge (-v) \wedge s$$

If we stabilize this representative along the linear isometric embedding

$$j \ : \ \nu_m \oplus \mathbb{R}^{m+1} \ \longrightarrow \ \mathrm{res}^{O(m+1)}_{O(m)}(\nu_{m+1}) \oplus \mathbb{R}^{m+1} \ , \quad (v, x, s) \ \longmapsto \ (v, 0, x, s)$$

we obtain another representative, namely

$$S^{\nu_m \oplus \mathbb{R} \oplus \mathbb{R}^{m+1}} \ \longrightarrow \ \mathbf{O}(\nu_{m+1}, \mathrm{res}^{O(m+1)}_{O(m)}(\nu_{m+1}) \oplus \mathbb{R}^{m+1})/O(m+1) \wedge S^{\nu_m} \wedge S^1$$

$$(v, u, x, s) \ \longmapsto \ ((0, u, x, 0), i \oplus \mathbb{R}) \cdot O(m+1) \wedge (-v) \wedge s$$

Here $i \oplus \mathbb{R} : \nu_{m+1} \longrightarrow \mathrm{res}^{O(m+1)}_{O(m)}(\nu_{m+1}) \oplus \mathbb{R}^{m+1}$ sends (v, u) to $(v, 0, 0, u)$.

On the other hand, $\mathrm{res}^{O(m+1)}_{O(m)}(\tau_{m+1})$ is represented by the underlying $O(m)$-map of the $O(m+1)$-map

$$S^{\nu_{m+1} \oplus \mathbb{R}^{m+1}} \ \xrightarrow{t_{m+1}} \ \mathbf{O}(\nu_{m+1}, \nu_{m+1} \oplus \mathbb{R}^{m+1})/O(m+1) \wedge S^{\nu_{m+1}}$$

$$(v, u, x, s) \ \longmapsto \ ((0, 0, x, s), i') \cdot O(m+1) \wedge (-v, -u) \ ,$$

where $i' : \nu_{m+1} \longrightarrow \mathrm{res}^{O(m+1)}_{O(m)}(\nu_{m+1}) \oplus \mathbb{R}^{m+1}$ sends (v, u) to $(v, u, 0, 0)$.

The two representatives differ by conjugation with the $O(m)$-equivariant linear isometry

$$\nu_m \oplus \mathbb{R} \oplus \mathbb{R}^m \oplus \mathbb{R} \ \longrightarrow \ \nu_m \oplus \mathbb{R} \oplus \mathbb{R}^m \oplus \mathbb{R} \ , \quad (v, u, x, s) \ \longmapsto \ (v, -s, x, u) \ ,$$

so they represent the same class in the group $\pi^{O(m)}_{m+1}(\mathbf{mO}_{(m+1)} \wedge S^{\nu_m} \wedge S^1)$.

(iii) Substituting the definitions of τ_m and of the morphism ψ^m shows that $(\psi^m \wedge S^{\nu_m})_*(\tau_m)$ is represented by the map

$$S^{\nu_m \oplus \mathbb{R}^m} \ \longrightarrow \ \mathbf{mO}(\nu_m) \wedge S^{\nu_m} \ , \quad (v, x) \ \longmapsto \ [(0, i(x)), 0 \oplus i(\mathbb{R}^m)] \wedge (-v) \ ,$$

where $i : \mathbb{R}^m \longrightarrow \mathbb{R}^\infty$ is the 'standard' embedding as the leading m coordinates. The periodicity class σ is represented by the map

$$s \ : \ S^1 \ \longrightarrow \ Th(Gr(\mathbb{R}^\infty)) \ = \ \mathbf{mOP}(0) \ , \quad x \ \longmapsto \ ((x, 0, 0, \dots), \mathbb{R} \oplus 0) \ .$$

The multiplication of **mOP** is an E_∞-multiplication, so as explained in (2.29), multiplication of classes in **mOP**-theory involves choices of linear isometric embeddings. We choose a linear isometric embedding

$$\psi \ : \ (\mathbb{R}^\infty)^m \ \longrightarrow \ \mathbb{R}^\infty$$

that satisfies

$$\psi((x_1,0,0,\dots),\dots,(x_m,0,0,\dots)) \ = \ (x_1,\dots,x_m,0,0,\dots) \ .$$

If we base the multiplication of **mOP** on such a choice ψ, then the product of the defining representative for $\tau_{O(m),\nu_m}$ and m factors of the map s above is precisely the previous representative for the class $(\psi^m \wedge S^{\nu_m})_*(\tau_m)$. This proves the relation $(\psi^m \wedge S^{\nu_m})_*(\tau_m) = \tau_{O(m),\nu_m} \cdot p^*_{O(m)}(\sigma^m)$ in $\mathbf{mO}^{O(m)}_m(S^{\nu_m})$. □

The fact that **mO** is the global homotopy colimit of the sequence of orthogonal spectra $\mathbf{mO}_{(m)}$ (see Proposition 2.42) has the following consequence.

Corollary 2.45. *For every orthogonal spectrum E the following sequence is short exact:*

$$0 \ \longrightarrow \ \lim_m{}^1 \, E^{O(m)}_{m+1}(S^{\nu_m}) \ \longrightarrow \ [\![\mathbf{mO},E]\!] \ \longrightarrow \ \lim_m \, E^{O(m)}_m(S^{\nu_m}) \ \longrightarrow \ 0$$

Here the inverse limit and derived limit are formed along the maps

$$E^{O(m+1)}_{m+1}(S^{\nu_{m+1}}) \ \xrightarrow{\mathrm{res}^{O(m+1)}_{O(m)}} \ E^{O(m)}_{m+1}(S^{\nu_m} \wedge S^1) \ \xrightarrow{(-\wedge S^1)^{-1}} \ E^{O(m)}_m(S^{\nu_m})$$

and the right map is given by evaluation at the shifted inverse Thom classes $\bar\tau_{O(m),\nu_m}$.

Proof. Since **mO** is the sequential homotopy colimit, in the triangulated category global stable homotopy category, of the sequence of orthogonal spectra (2.41), the Milnor exact sequence takes the form:

$$0 \ \longrightarrow \ \lim_m{}^1 \, [\![S^1 \wedge \mathbf{mO}_{(m)},E]\!] \ \longrightarrow \ [\![\mathbf{mO},E]\!] \ \longrightarrow \ \lim_m [\![\mathbf{mO}_{(m)},E]\!] \ \longrightarrow \ 0$$

By Proposition 2.44 the pair $(\mathbf{mO}_{(m)},\tau_m)$ then represents the functor

$$\mathcal{GH} \longrightarrow (\text{sets}) \ , \quad E \ \longmapsto \ E^{O(m)}_m(S^{\nu_m}) \ ,$$

and the morphism $j^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$ has the correct behavior on the universal classes. □

Now that we recognized **mO** as the global homotopy colimit of the rank filtration (2.41), we study how one filtration term $\mathbf{mO}_{(m)}$ is obtained from the previous one. The answer given by Theorem 2.46 below takes the form of a distinguished triangle in the global stable homotopy category, witnessing that the mapping cone of $j^m : \mathbf{mO}_{(m-1)} \longrightarrow \mathbf{mO}_{(m)}$ 'is' the m-fold suspension of the suspension spectrum of the global classifying space $B_{\mathrm{gl}}O(m)$.

We define a morphism of orthogonal spectra

$$T_m \ = \ T^{O(m+1)}_{O(m)} \ : \ \Sigma^\infty_+ B_{\mathrm{gl}}O(m+1) \ \longrightarrow \ F_m$$

as the adjoint of the $O(m+1)$-equivariant map

$$r \ : \ S^{\nu_{m+1}} \ \longrightarrow \ \mathbf{O}(\nu_m,\nu_{m+1})/O(m) \ = \ F_m(\nu_{m+1})$$
$$r(A\cdot(0,\dots,0,t)) \ = \ A\cdot((0,\dots,0,(t^2-1)/t),\mathrm{incl})\cdot O(m) \ ,$$

where $A \in O(m+1)$ and $t \in [0,\infty]$. Theorem IV.4.22 (ii) shows that the morphism T_m represents the dimension shifting transfer from $O(m)$ to $O(m+1)$, in the sense of the relation

$$(T_m)_*(e_{O(m+1)}) \ = \ \mathrm{Tr}^{O(m+1)}_{O(m)}(a_m)$$

between the tautological classes. From T_m we define another morphism of orthogonal spectra

$$\partial \ = \ \lambda^m_{F_m} \circ (T_m \wedge S^m) \ = \ (\mathrm{sh}^m T_m) \circ \lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}}O(m+1)} \ : \ \Sigma^\infty_+ B_{\mathrm{gl}}O(m+1) \ \longrightarrow \ \mathrm{sh}^m F_m \ = \ \mathbf{mO}_{(m)} \ .$$

The space

$$((\Sigma^\infty_+ \mathbf{L}_{O(m+1),\nu_{m+1}})(\nu_{m+1}))^{O(m+1)} \;=\; (S^{\nu_{m+1}} \wedge \mathbf{L}(\nu_{m+1},\nu_{m+1})/O(m+1))^{O(m+1)}$$

has two points, the basepoint and $0 \wedge \mathrm{Id}_{\nu_{m+1}} \cdot O(m+1)$. So there is a unique non-trivial morphism of orthogonal spectra

$$a \;:\; F_{m+1} \;\longrightarrow\; \Sigma^\infty_+ \mathbf{L}_{O(m+1),\nu_{m+1}} = \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)\ .$$

For every orthogonal spectrum E the morphism $\lambda^{m+1}_E : E \wedge S^{m+1} \longrightarrow \mathrm{sh}^{m+1} E$ is a global equivalence (by an iteration of Proposition IV.1.4 (i)), so it becomes invertible in the global stable homotopy category. So we can define a morphism in $\mathcal{GH}$ as

$$q \;=\; (\lambda^{m+1}_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)})^{-1} \circ (\mathrm{sh}^{m+1} a) \;:\; \mathbf{mO}_{(m+1)} \;\longrightarrow\; \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^{m+1}\ .$$

Theorem 2.46. *The sequence*

$$\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^m \;\xrightarrow{\partial}\; \mathbf{mO}_{(m)} \;\xrightarrow{j^m}\; \mathbf{mO}_{(m+1)} \;\xrightarrow{q}\; \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^{m+1}$$

is a distinguished triangle in the global stable homotopy category. The behavior of the first morphism on the stable tautological class is given by

$$\partial_*(e_{O(m+1)} \wedge S^m) \;=\; \mathrm{Tr}^{O(m+1)}_{O(m)}(\tau_m)\ .$$

Proof. The tautological $O(m+1)$-representation ν_{m+1} is faithful, and the action of $O(m+1)$ on the unit sphere $S(\nu_{m+1})$ is transitive. The stabilizer group of the unit vector $(0,\dots,0,1)$ identifies with the group $O(m)$, and the orthogonal complement of this vector becomes the tautological representation ν_m. We can thus apply Theorem IV.4.22 and obtain a distinguished triangle:

$$F_{m+1} \;\xrightarrow{a}\; \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \;\xrightarrow{T_m}\; F_m \;\xrightarrow{-\lambda^{-1}_{F_{m+1}} \circ i}\; F_{m+1} \wedge S^1$$

By Example IV.4.1 shifting preserves distinguished triangles; so the following sequence is also distinguished:

$$\mathrm{sh}^m F_{m+1} \;\xrightarrow{\mathrm{sh}^m a}\; \mathrm{sh}^m \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \;\xrightarrow{\mathrm{sh}^m T_m}\; \mathrm{sh}^m F_m \;\xrightarrow{-\,\mathrm{sh}^m(\lambda^{-1}_{F_{m+1}} \circ i)}\; \mathrm{sh}^m F_{m+1} \wedge S^1$$

The rotation of this triangle is the lower sequence in the following diagram:

$$\begin{tikzcd}
\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^m \arrow[r, "\partial"] \arrow[d, "\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)}"'] & \mathbf{mO}_{(m)} \arrow[r, "j^m"] \arrow[d, equal] & \mathbf{mO}_{(m+1)} \arrow[r, "q"] \arrow[d, "\mathrm{sh}^m(\lambda^{-1}_{F_{m+1}})"] & \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^{m+1} \arrow[d, "\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \wedge S^1"] \\
\mathrm{sh}^m \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \arrow[r, "\mathrm{sh}^m T_m"'] & \mathrm{sh}^m F_m \arrow[r, "\mathrm{sh}^m(\lambda^{-1}_{F_{m+1}} \circ i)"'] & \mathrm{sh}^m F_{m+1} \wedge S^1 \arrow[r, "\mathrm{sh}^m a \wedge S^1"'] & \mathrm{sh}^m \Sigma^\infty_+ B_{\mathrm{gl}} O(m+1) \wedge S^1
\end{tikzcd}$$

The left and middle squares commute by definition of the morphisms ∂ respectively j^m; the right square commutes by the relation

$$\begin{aligned}
(\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \wedge S^1) \circ q \;&=\; (\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \wedge S^1) \circ (\lambda^{m+1}_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)})^{-1} \circ (\mathrm{sh}^{m+1} a) \\
&=\; (\mathrm{sh}^m \lambda_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)})^{-1} \circ (\mathrm{sh}^{m+1} a) \\
&=\; \mathrm{sh}^m(\lambda^{-1}_{\Sigma^\infty_+ B_{\mathrm{gl}} O(m+1)} \circ (\mathrm{sh}\, a)) \;=\; \mathrm{sh}^m((a \wedge S^1) \circ \lambda^{-1}_{F_{m+1}})\ .
\end{aligned}$$

So the upper sequence is a distinguished triangle.

The relation is a formal consequence of various other previously established relations:

$$\begin{aligned}
\partial_*(e_{O(m+1)} \wedge S^m) &= (\lambda^m_{F_m})_*(T_m \wedge S^m)_*(e_{O(m+1)} \wedge S^m) \\
&= (\lambda^m_{F_m})_*((T_m)_*(e_{O(m+1)}) \wedge S^m) \\
&= (\lambda^m_{F_m})_*(\mathrm{Tr}^{O(m+1)}_{O(m)}(a_m) \wedge S^m) \\
&= (\lambda^m_{F_m})_*(\mathrm{Tr}^{O(m+1)}_{O(m)}((F_m \wedge \tau_{\nu_m,\mathbb{R}^m})_*(a_m \wedge S^m))) \\
&= \mathrm{Tr}^{O(m+1)}_{O(m)}((\lambda^m_{F_m} \wedge S^{\nu_m})_*((F_m \wedge \tau_{\nu_m,\mathbb{R}^m})_*(a_m \wedge S^m))) \\
&= \mathrm{Tr}^{O(m+1)}_{O(m)}((\lambda^m_{F_m \wedge S^{\nu_m}})_*(a_m \wedge S^m)) \;=\; \mathrm{Tr}^{O(m+1)}_{O(m)}(\tau_m) \ .
\end{aligned}$$

The second and fifth equations are naturality. The fourth equation is the compatibility of transfer and suspension isomorphism, see Proposition III.2.25. □

For some calculations of equivariant homotopy groups of **mO** we also need to understand the composite:

$$\Sigma^\infty_+ B_{\mathrm{gl}}O(m+1) \wedge S^m \xrightarrow{\partial} \mathbf{mO}_{(m)} \xrightarrow{q} \Sigma^\infty_+ B_{\mathrm{gl}}O(m) \wedge S^m$$

We start from the calculation

$$\begin{aligned}
(a \circ T_m)_*(e_{O(m+1)}) &= a_*\left(\mathrm{Tr}^{O(m+1)}_{O(m)}(a_m)\right) \;=\; \mathrm{Tr}^{O(m+1)}_{O(m)}(a_*(a_m)) \\
&= \mathrm{Tr}^{O(m+1)}_{O(m)}\left(a \cdot e_{O(m)})\right) \;=\; \mathrm{tr}^{O(m+1)}_{O(m)}(e_{O(m)}) \ .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
q \circ \partial &= (\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}}O(m)})^{-1} \circ (\mathrm{sh}^m a) \circ \lambda^m_{F_m} \circ (T_m \wedge S^m) \\
&= (\lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}}O(m)})^{-1} \circ \lambda^m_{\Sigma^\infty_+ B_{\mathrm{gl}}O(m)} \circ (a \wedge S^m) \circ (T_m \wedge S^m) \;=\; (a \circ T_m) \wedge S^m \ ,
\end{aligned}$$

by definition. Combining these two facts gives

$$\begin{aligned}
(q \circ \partial)_*(e_{O(m+1)} \wedge S^m) &= ((a \circ T_m) \wedge S^m)_*(e_{O(m+1)} \wedge S^m) \\
&= (a \circ T_m)_*(e_{O(m+1)}) \wedge S^m \;=\; \mathrm{tr}^{O(m+1)}_{O(m)}(e_{O(m)}) \wedge S^m \ .
\end{aligned} \tag{2.47}$$

In other words, the composite $q \circ \partial$ represents the degree zero transfer $\mathrm{tr}^{O(m+1)}_{O(m)}$.

Now we can easily show that **mO** is globally connective and describe the global functor $\underline{\pi}_0(\mathbf{mO})$. We denote by $\langle \mathrm{tr}^{O(1)}_e \rangle$ the global subfunctor of the Burnside ring functor $\mathbb{A}$ generated by $\mathrm{tr}^{O(1)}_e \in \mathbb{A}(O(1))$.

THEOREM 2.48. *The orthogonal spectrum* **mO** *is globally connective and the action of the Burnside ring global functor on the unit element* $1 \in \pi^e_0(\mathbf{mO})$ *induces an isomorphism of global functors*

$$\mathbb{A}/\langle \mathrm{tr}^{O(1)}_e \rangle \;\cong\; \underline{\pi}_0(\mathbf{mO}) \ .$$

PROOF. The suspension spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}O(m+1)$ is globally connective; so the distinguished triangle of Theorem 2.46 implies that the morphism $j^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$ induces an isomorphism of global functors

$$\underline{\pi}_k(\mathbf{mO}_{(m)}) \;\cong\; \underline{\pi}_k(\mathbf{mO}_{(m+1)})$$

for $k \le m-1$ and an exact sequence of global functors

$$\mathbf{A}(O(m+1), -) \;\longrightarrow\; \underline{\pi}_m(\mathbf{mO}_{(m)}) \;\longrightarrow\; \underline{\pi}_m(\mathbf{mO}_{(m+1)}) \;\longrightarrow\; 0 \ .$$

Here we used the isomorphisms

$$\mathbf{A}(O(m+1), -) \xrightarrow{\tau \mapsto \tau(e_{O(m+1)})} \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}O(m+1)) \xrightarrow[\cong]{-\wedge S^m} \underline{\pi}_m(\Sigma^\infty_+ B_{\mathrm{gl}}O(m+1) \wedge S^m) \ .$$

Since $\mathbf{mO}_{(0)}$ is isomorphic to the global sphere spectrum, which is globally connective, we conclude inductively that $\mathbf{mO}_{(m)}$ is globally connective for all $m \geq 0$, and that the inclusion $\mathbf{mO}_{(1)} \longrightarrow \mathbf{mO}_{(m)}$ induces an isomorphism on $\underline{\pi}_0$ for all $m \geq 1$. Since $\mathbf{mO}$ is a colimit of the sequence of closed embeddings $j^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}_{(m+1)}$, the map

$$\operatorname{colim}_m \, \underline{\pi}_k(\mathbf{mO}_{(m)}) \;\longrightarrow\; \underline{\pi}_k(\mathbf{mO})$$

induced by the morphisms $\psi^m : \mathbf{mO}_{(m)} \longrightarrow \mathbf{mO}$ is an isomorphism of global functors for every integer k. So $\mathbf{mO}$ is globally connective and the inclusion induces an isomorphism

$$\underline{\pi}_0(\mathbf{mO}_{(1)}) \;\cong\; \underline{\pi}_0(\mathbf{mO}) \;.$$

The unit morphism identifies $\mathbf{mO}_{(0)}$ with the global sphere spectrum, so the action on the class $1 \in \mathbf{mO}_{(0)}$ is an isomorphism of global functors $\mathbb{A} \cong \underline{\pi}_0(\mathbf{mO}_{(0)})$. For $m = 1$, the exact sequence thus becomes an exact sequence of global functors

$$\mathbf{A}(O(1), -) \;\xrightarrow{\mathbf{A}(\mathrm{tr}_e^{O(1)}, -)}\; \mathbf{A}(e, -) \;\longrightarrow\; \underline{\pi}_0(\mathbf{mO}_{(1)}) \;\longrightarrow\; 0 \;;$$

here we used (2.47) to identify the first morphism as the one induced by the transfer $\mathrm{tr}_e^{O(1)}$. This proves the claim about $\underline{\pi}_0(\mathbf{mO})$. □

Theorem 2.48 gives a nice compact description of the 0-th equivariant homotopy groups of the Thom spectrum $\mathbf{mO}$, but we may still ask for a more explicit calculation of the group $\underline{\pi}_0^G(\mathbf{mO})$ for an individual compact Lie group G. Given the presentation of $\underline{\pi}_0(\mathbf{mO})$ as the quotient of $\mathbb{A}$ by the global subfunctor generated by $\mathrm{tr}_e^{O(1)}$, this is a purely algebraic exercise. Since the element $\mathrm{tr}_e^{O(1)}(1)$ is trivial in the group $\pi_0^{O(1)}(\mathbf{mO})$, also

$$2 \;=\; \mathrm{res}_e^{O(1)}(\mathrm{tr}_e^{O(1)}(1)) \;=\; 0$$

in $\pi_0^e(\mathbf{mO})$; thus all equivariant homotopy groups $\pi_*^G(\mathbf{mO})$ are $\mathbb{F}_2$-vector spaces. Then next proposition pins down an $\mathbb{F}_2$-basis of $\pi_*^G(\mathbf{mO})$ in terms of the subgroup structure of G.

In the next proposition we use the familiar notation

$$t_H^G \;=\; \mathrm{tr}_H^G(p_H^*(1)) \;\in\; \pi_0^G(\mathbf{mO}) \;,$$

where H is a closed subgroup of G and $p_H : H \longrightarrow e$ is the unique homomorphism.

Proposition 2.49. *For every compact Lie group G, an $\mathbb{F}_2$-basis of $\pi_0^G(\mathbf{mO})$ is given by the classes t_H^G, indexed by conjugacy classes of those closed subgroups H of G whose Weyl group is finite and of odd order.*

Proof. We abbreviate $C = O(1)$. The group $\mathbb{A}(G)$ is free abelian with basis the classes t_H^G for all conjugacy classes of subgroups H with finite Weyl group. So the claim follows if we can show that the value $\langle \mathrm{tr}_e^C \rangle(G)$ of the global functor $\langle \mathrm{tr}_e^C \rangle$ at G is the subgroup of $\mathbb{A}(G)$ generated by $2 \cdot \mathbb{A}(G)$ and the classes t_H^G for those closed subgroups H whose Weyl group is finite of even order.

By Theorem IV.2.6, the group $\mathbf{A}(C, G)$ is freely generated by the elements $\mathrm{tr}_L^G \circ \alpha^*$ indexed by $(G \times C)$-conjugacy classes of pairs (L, α) where L is a closed subgroup of G with finite Weyl group and $\alpha : L \longrightarrow C$ is a continuous group homomorphism. So $\langle \mathrm{tr}_e^C \rangle(G)$ is generated as an abelian group by the elements $\mathrm{tr}_L^G \circ \alpha^* \circ \mathrm{tr}_e^C$.

A homomorphisms to C is either trivial or surjective, and the generating elements come in two flavors. If α is the trivial homomorphism, then

$$\mathrm{tr}_L^G \circ \alpha^* \circ \mathrm{tr}_e^C \;=\; \mathrm{tr}_L^G \circ p_L^* \circ \mathrm{res}_e^C \circ \mathrm{tr}_e^C \;=\; 2 \cdot \mathrm{tr}_L^G \circ p_L^* \;=\; 2 \cdot t_L^G \;.$$

These elements generate the subgroup $2 \cdot \mathbb{A}(G)$. If α is surjective with kernel H, then

$$\mathrm{tr}_L^G \circ \alpha^* \circ \mathrm{tr}_e^C \;=\; \mathrm{tr}_L^G \circ \mathrm{tr}_H^L \circ p_H^* \;=\; \mathrm{tr}_H^G \circ p_H^* \;=\; t_H^G.$$

The group H that arises in this way is normal of index 2 in L. So L is contained in the normalizer $N_G H$, and 2 divides $[N_G H : H]$. So the order of the Weyl group $W_G H$ is even. Conversely, if H is a subgroup of G with finite Weyl group of even order, then we can choose a subgroup $C \leq W_G H$ of order 2. The preimage L of C under the projection $N_G H \longrightarrow W_G H$ then contains H as an index 2 subgroup. By the above, the class t_H^G is then one of the generating elements of $\langle \mathrm{tr}_e^C \rangle(G)$. □

Example 2.50 (Geometric fixed points of **mO** and **mOP**). We give a description of the geometric fixed points of **mO** and **mOP**. The geometric fixed points of other global Thom spectra such as $\mathbf{mO}_{(m)}$, **MO** or **MOP** can be worked out in a similar fashion, but we leave that to the interested reader. We let V be a representation of a compact Lie group G. We write $V^\perp = V - V^G$ for the orthogonal complement of the G-fixed subspace. A point $((v,x),U) \in \mathbf{mOP}(V) = Th(Gr(V \oplus \mathbb{R}^\infty))$ is G-fixed if and only if the subspace U of $V \oplus \mathbb{R}^\infty$ is G-invariant and the vector $v \in V$ is G-fixed. The first condition guarantees that $U = U^G \oplus (U \cap V^\perp)$, and U^G is a subspace of $V^G \oplus \mathbb{R}^\infty$. So we obtain a homeomorphism

$$\mathbf{mOP}(V^G) \wedge \mathbf{Gr}(V^\perp)^G \xrightarrow{\cong} \mathbf{mOP}(V)^G \qquad \text{by} \qquad (x,U) \wedge W \longmapsto (x, U \oplus (V^\perp - W)) \ . \tag{2.51}$$

Under this identification, the structure map

$$(\sigma_{V,W})^G \ : \ S^{V^G} \wedge \mathbf{mOP}(W)^G \longrightarrow \mathbf{mOP}(V \oplus W)^G$$

becomes the smash product of the structure map

$$\sigma_{V^G,W^G} \ : \ S^{V^G} \wedge \mathbf{mOP}(W^G) \longrightarrow \mathbf{mOP}(V^G \oplus W^G)$$

with the map

$$\mathbf{Gr}(i)^G \ : \ \mathbf{Gr}(W^\perp)^G \longrightarrow \mathbf{Gr}(V^\perp \oplus W^\perp)^G$$

induced by the embedding $W^\perp \longrightarrow V^\perp \oplus W^\perp$ as the second summand. So in the colimit over $V \in s(\mathcal{U}_G)$ this gives an isomorphism

$$\Phi_*^G(\mathbf{mOP}) \ \cong \ \mathbf{mOP}_* \left(\mathbf{Gr}(\mathcal{U}_G^\perp)_+^G\right) \tag{2.52}$$

to the non-equivariant **mOP**-homology groups of the G-fixed point space of $\mathbf{Gr}(\mathcal{U}_G^\perp)$, the disjoint union of all Grassmannians in $\mathcal{U}_G^\perp$, the orthogonal complement of the fixed points in the complete G-universe.

The formula (2.52) is a very convenient and compact way to express the geometric fixed points of **mOP**, but it can be decomposed and rewritten further, thereby making it more explicit. The orthogonal spectrum **mOP** is a $\mathbb{Z}$-indexed wedge of homogeneous summands $\mathbf{mOP}^{[k]}$. The space $\mathbf{Gr}(\mathcal{U}_G^\perp)$, and hence also its G-fixed points, is the disjoint union indexed by the dimension of the subspaces, i.e.,

$$\mathbf{Gr}(\mathcal{U}_G^\perp)^G \ = \ \coprod_{j \geq 0} \left(Gr_j(\mathcal{U}_G^\perp)\right)^G \ = \ \coprod_{j \geq 0} Gr_j^{G,\perp} \ .$$

The two decomposition induce a direct sum decomposition of the right hand side of the isomorphism (2.52) as

$$\mathbf{mOP}_* \left(\mathbf{Gr}(\mathcal{U}_G^\perp)_+^G\right) \ = \ \bigoplus_{k \in \mathbb{Z}} \bigoplus_{j \geq 0} \mathbf{mOP}_*^{[k]} \left((Gr_j^{G,\perp})_+\right) \ \cong \ \bigoplus_{k \in \mathbb{Z},\ j \geq 0} \mathbf{mO}_{*-k} \left((Gr_j^{G,\perp})_+\right) \ . \tag{2.53}$$

The second isomorphism uses the periodicity of **mOP** to identify $\mathbf{mOP}^{[k]}$ with $S^k \wedge \mathbf{mO}$.

The condition $\dim(U \oplus (V^\perp - W)) = \dim(V)$ is equivalent to $\dim(U) = \dim(V^G) + \dim(W)$. So the homeomorphism (2.51) identifies $\mathbf{mO}(V)^G$ with the wedge of the spaces $\mathbf{mOP}^{[j]}(V^G) \wedge Gr_j(V^\perp)$ for $j \geq 0$. The composite of the isomorphism (2.52) and the isomorphism (2.53) thus takes the wedge summand of $\Phi_*^G(\mathbf{mOP})$ corresponding to $\mathbf{mO} = \mathbf{mOP}^{[0]}$ to the sum of the terms with $k = j$. So the isomorphisms restrict to an isomorphism

$$\Phi_*^G(\mathbf{mO}) \ \cong \ \bigoplus_{j \geq 0} \mathbf{mO}_{*-j} \left((Gr_j^{G,\perp})_+\right) \ .$$

The space $Gr_j^{G,\perp}$ can be decomposed further: every G-invariant subspace of $\mathcal{U}_G^\perp$ is the direct sum of its isotypical components, indexed by the non-trivial irreducible G-representations. The irreducibles come in three flavors (real, complex or quaternionic), and so the space $Gr_j^{G,\perp}$ is a disjoint union of products of classifying spaces of the groups $O(n)$, $U(n)$, and $Sp(n)$ for various n.

The reader may want to compare the previous description of the geometric G-fixed points of $\mathbf{mO}$ with Proposition II.4.23 (i), which identifies the G-fixed points of the orthogonal space $\mathbf{bO}$ as

$$\mathbf{bO}(\mathcal{U}_G)^G \;\simeq\; \coprod_{j\geq 0} BO \times Gr_j^{G,\perp} \;.$$

This illustrates the general phenomenon that 'geometric G-fixed points of a global Thom spectrum are the Thom spectrum over the G-fixed points'.

Example 2.54. The Thom spectrum $\mathbf{mO}$ has an oriented analog. For $m \geq 0$ we define an orthogonal spectrum $\mathbf{mSO}_{(m)}$ by

$$\mathbf{mSO}_{(m)} \;=\; \mathrm{sh}^m\, F_{SO(m),\nu_m} \;,$$

the m-th shift of the free orthogonal spectrum generated by the $SO(m)$-representation ν_m. In much the same way in Construction 2.40, $\mathbf{mSO}_{(m)}(V)$ 'is' (by passage to orthogonal complements) the Thom space over the tautological bundle over the oriented Grassmannian $Gr^+_{|V|}(V \oplus \mathbb{R}^m)$ of oriented $|V|$-planes in $V \oplus \mathbb{R}^m$.

We define a morphism

$$i \;:\; F_{SO(m),\nu_m} \;\longrightarrow\; \mathrm{sh}\, F_{SO(m+1),\nu_{m+1}}$$

at an inner product space V as the closed embedding

$$\begin{array}{rcl} i(V) \;:\; F_m(V) = \mathbf{O}(\nu_m, V)/SO(m) & \xrightarrow{-\oplus\mathbb{R}} & \mathbf{O}(\nu_{m+1}, V\oplus\mathbb{R})/SO(m+1) = (\mathrm{sh}\, F_{m+1})(V) \\ [x,\varphi] & \longmapsto & [(x,0), \varphi\oplus\mathbb{R}] \;. \end{array}$$

Then we set

$$j^m \;=\; \mathrm{sh}^m\, i \;:\; \mathbf{mSO}_{(m)} = \mathrm{sh}^m\, F_{SO(m),\nu_m} \;\longrightarrow\; \mathrm{sh}^{m+1}\, F_{SO(m+1),\nu_{m+1}} \;=\; \mathbf{mSO}_{(m+1)}$$

and we define $\mathbf{mSO}$ as the colimit of the sequence of closed embeddings of orthogonal spectra

$$\mathbf{mSO}_{(0)} \xrightarrow{j^0} \mathbf{mSO}_{(1)} \xrightarrow{j^1} \ldots \longrightarrow \mathbf{mSO}_{(m)} \xrightarrow{j^m} \ldots \;.$$

The orthogonal spectrum $\mathbf{mSO}$ supports inverse Thom classes for *oriented* representations, i.e., oriented inner product spaces V equipped with an orientation preserving isometric action of a compact Lie group G.

Now we mention the unitary analogs $\mathbf{mU}$ and $\mathbf{MU}$ of the Thom spectra $\mathbf{mO}$ and $\mathbf{MO}$. Beside the complexification, there is an extra twist to the unitary definitions, because we need to 'loop by imaginary spheres' to really get orthogonal spectra. The unitary Thom spectra have periodic version $\mathbf{mUP}$ respectively $\mathbf{MUP}$, but we will not go into any details about those. The ultra-commutative ring spectrum $\mathbf{MU}$ is a global refinement of equivariant homotopical unitary bordism, due to tom Dieck [**162**]; its underlying non-equivariant homotopy type is the spectrum MU. For a compact Lie group G, the G-equivariant spectrum $\mathbf{MU}_G$ is a model for tom Dieck's homotopical equivariant bordism [**162**]. When the group G is non-trivial, this is different from the geometric theory of unitary bordism of G-manifolds, and it represents *stable equivariant unitary bordism*. Closely related, strictly commutative ring spectrum models for these homotopy types have been discussed in various places, see for example [**107**], [**64**, Ex. 5.8], [**153**, App. A] or [**31**, Sec. 8].

Example 2.55. We define an orthogonal spectrum $\mathbf{mU}$ in analogy with $\mathbf{mO}$. Non-equivariantly, $\mathbf{mU}$ the unitary Thom spectrum MU. The spectrum $\mathbf{mU}$ is essentially a Thom spectrum over the orthogonal

space $\mathbf{bU}$, the complex analog of the E_∞-orthogonal monoid space $\mathbf{bO}$ discussed in Example II.4.20. The value of $\mathbf{bU}$ at an inner product space V is

$$\mathbf{bU}(V) \;=\; Gr^{\mathbb{C}}_{|V|}(V_{\mathbb{C}} \oplus \mathbb{C}^\infty)\ ,$$

the Grassmannian of $\mathbb{C}$-linear subspaces of $V_{\mathbb{C}} \oplus \mathbb{C}^\infty$ of the same dimension as V. The map $\mathbf{bU}(\varphi) : \mathbf{bU}(V) \longrightarrow \mathbf{bU}(W)$ induced by a linear isometric embedding $\varphi : V \longrightarrow W$ is defined as

$$\mathbf{bU}(\varphi)(L) \;=\; (\varphi_{\mathbb{C}} \oplus \mathbb{C}^\infty)(L) \;+\; ((W - \varphi_{\mathbb{C}}(V)) \oplus 0)\ .$$

In other words: we apply the linear isometric embedding $\varphi_{\mathbb{C}} \oplus \mathbb{C}^\infty : V_{\mathbb{C}} \oplus \mathbb{C}^\infty \longrightarrow W_{\mathbb{C}} \oplus \mathbb{C}^\infty$ to the subspace L and add the orthogonal complement of the image of $\varphi_{\mathbb{C}}$ (sitting in the first summand of $W_{\mathbb{C}} \oplus \mathbb{C}^\infty$).

Over the space $\mathbf{bU}(V)$ sits a tautological hermitian vector bundle with total space consisting of the pairs $(U, x) \in \mathbf{bU}(V) \times (V_{\mathbb{C}} \oplus \mathbb{C}^\infty)$ such that $x \in U$. We define $\mathbf{mU}(V)$ as

$$\mathbf{mU}(V) \;=\; \mathrm{map}(S^{iV}, Th(\mathbf{bU}(V)))\ ,$$

the space of based maps from the 'imaginary sphere' S^{iV} to the Thom space of this tautological vector bundle. The structure map

$$\mathbf{mU}(V) \wedge \mathbf{O}(V, W) \;\longrightarrow\; \mathbf{mU}(W)$$

is adjoint to the map

$$\begin{aligned} S^{iW} \wedge \mathbf{mU}(V) \wedge \mathbf{O}(V, W) \;&\longrightarrow\; Th(\mathbf{bU}(W)) \\ x \wedge f \wedge (w, \varphi) \;&\longmapsto\; (\mathbf{bU}(\varphi)(U), \mathbf{bU}(\varphi)(x) + (w, 0))\ . \end{aligned}$$

Looping by S^{iV} is essential for obtaining an orthogonal spectrum; without it, we would end with a structure one may call a 'unitary spectrum'.

Most of our results about $\mathbf{mO}$ have analogues for $\mathbf{mU}$. The orthogonal spectrum $\mathbf{mU}$ comes with an E_∞-structure, which is, however, *not* ultra-commutative. There are unitary versions of the shifted inverse Thom classes

$$\bar{\tau}^U_{G,V} \;\in\; \mathbf{mU}^G_{2n}(S^V)\ ,$$

in the G-equivariant $\mathbf{mU}$-homology groups of S^V, defined for *unitary* representations V of a compact Lie group G. Here n is the complex dimension of V (so that $2n$ is its real dimension). The orthogonal spectrum $\mathbf{mU}$ is the union of a sequence of orthogonal subspectra

$$\mathbf{mU}_{(0)} \;\subset\; \mathbf{mU}_{(1)} \;\subset\; \ldots \;\subset\; \mathbf{mU}_{(m)} \;\subset\; \ldots$$

and $\mathbf{mU}$ is also the global homotopy colimit of this sequence. The unit morphism is a global equivalence $\mathbb{S} \simeq \mathbf{mU}_{(0)}$ and $\mathbf{mU}_{(m)}$ is globally equivalent to the $2m$-th suspension of the free orthogonal spectrum generated by the tautological unitary representation ν^U_m of $U(m)$ on $\mathbb{C}^m$:

$$\mathbf{mU}_{(m)} \;\simeq\; F_{U(m),\nu^U_m} \wedge S^{2m}\ .$$

There are distinguished triangles in the global stable homotopy category:

$$\Sigma^\infty_+ B_{\mathrm{gl}} U(m+1) \wedge S^{2m+1} \;\longrightarrow\; \mathbf{mU}_{(m)} \;\longrightarrow\; \mathbf{mU}_{(m+1)} \;\longrightarrow\; \Sigma^\infty_+ B_{\mathrm{gl}} U(m+1) \wedge S^{2m+2}$$

and the first map is classified by the $U(m+1)$-equivariant homotopy class

$$\mathrm{Tr}^{U(m+1)}_{U(m)}(\tau_{U(m),\nu^U_m} \wedge S^1) \qquad \text{in} \quad \pi^{U(m+1)}_{2m+1}(\mathbf{mU}_{(m)}).$$

(This uses that the tangent $U(m)$-representation of $U(m+1)/U(m)$ is isomorphic to $\mathbb{R} \oplus \nu^U_m$.) So loosely speaking, $\mathbf{mU}_{(m+1)}$ is obtained from $\mathbf{mU}_{(m)}$ by coning off this transfer class. A consequence is then that all $\mathbf{mU}_{(m)}$ and $\mathbf{mU}$ are globally connective.

Corollary 2.45 has a unitary analog that describes morphisms in the global stable homotopy category out of $\mathbf{mU}$: for every orthogonal spectrum E the sequence

$$0 \longrightarrow \lim_m{}^1 E^{U(m)}_{2m+1}(S^{\nu^U_m}) \longrightarrow [\![\mathbf{mU}, E]\!] \longrightarrow \lim_m E^{U(m)}_{2m}(S^{\nu^U_m}) \longrightarrow 0$$

is short exact. This time the inverse and derived limits are formed along the maps

$$E^{U(m+1)}_{2m+2}(S^{\nu^U_{m+1}}) \xrightarrow{\mathrm{res}^{U(m+1)}_{U(m)}} E^{U(m)}_{2m+2}(S^{\nu^U_m} \wedge S^2) \xrightarrow{(-\wedge S^2)^{-1}} E^{U(m)}_{2m}(S^{\nu^U_m})$$

and the right map is given by evaluation at the unitary shifted inverse Thom classes.

Since the first 'basic unitary relation' $\mathrm{Tr}^{U(1)}_e(S^1 \wedge 1) = 0$ in $\pi^{U(1)}_1(\mathbf{mU})$ lives in a positive dimension, the description of the global functor $\underline{\pi}_0(\mathbf{mU})$ is easier than the corresponding calculation of $\underline{\pi}_0(\mathbf{mO})$ in Theorem 2.48. Indeed, the action of the Burnside ring global functor on the element $1 \in \pi^e_0(\mathbf{mU})$ induces an isomorphism of global functors

$$\mathbb{A} \cong \underline{\pi}_0(\mathbf{mU}) .$$

Moreover, there is an exact sequence of global functors

$$\mathbf{A}(U(1), -) \longrightarrow \underline{\pi}_1(\mathbb{S}) \longrightarrow \underline{\pi}_1(\mathbf{mU}) \longrightarrow 0 ,$$

where the first map is the action on the class

$$\mathrm{Tr}^{U(1)}_e(S^1 \wedge 1) \qquad \text{in} \quad \pi^{U(1)}_1(\mathbb{S}).$$

We leave it to the interested reader to formulate the analogous properties of the rank filtrations for the special unitary and symplectic global Thom spectra $\mathbf{mSU}$ and $\mathbf{mSp}$.

Example 2.56 (Unitary global Thom spectrum). We define the *unitary global Thom spectrum* $\mathbf{MU}$, an ultra-commutative ring spectrum. Non-equivariantly, $\mathbf{MU}$ is another version of the complex bordism spectrum MU. For an inner product space V we consider the complex Grassmannian

$$\mathbf{BU}(V) = Gr^{\mathbb{C}}_{|V|}(V^2_{\mathbb{C}})$$

where as before $V_{\mathbb{C}} = \mathbb{C} \otimes V$ is the complexification of V. Over the space $\mathbf{BU}(V)$ sits a tautological hermitian vector bundle and we set

$$\mathbf{MU}(V) = \mathrm{map}(S^{iV}, Th(\mathbf{BU}(V))) ,$$

the iV-loop space of the Thom space of this tautological vector bundle. The structure maps are defined in a similar way as for $\mathbf{mU}$, and a commutative multiplication is given by

$$\mu_{V,W} : \mathbf{MU}(V) \wedge \mathbf{MU}(W) \longrightarrow \mathbf{MU}(V \oplus W) , \quad f \wedge g \longmapsto \kappa_{V,W} \circ (f \cdot g) \wedge \kappa_{W,V} . \tag{2.57}$$

Here $f : S^{iV} \longrightarrow Th(\mathbf{bU}(V))$, $g : S^{iW} \longrightarrow Th(\mathbf{BU}(W))$, and $f \cdot g$ denotes the composite

$$S^{iV} \wedge S^{iW} \xrightarrow{f \wedge g} Th(\mathbf{BU}(V)) \wedge Th(\mathbf{BU}(W)) \xrightarrow{(x,U)\wedge(x',U') \mapsto (\kappa_{V,W}(x,x'), \kappa_{V,W}(U \oplus U'))} Th(\mathbf{BU}(V \oplus W)) .$$

Here $\kappa_{V,W}$ is the preferred isometry from $V^2_{\mathbb{C}} \oplus W^2_{\mathbb{C}} \cong (V \oplus W)^2_{\mathbb{C}}$ sending $((v, v'), (w, w'))$ to $((v, w), (v', w'))$. Unit maps are defined by

$$S^V \longrightarrow \mathbf{MU}(V) , \quad v \longmapsto [v' \mapsto (V_{\mathbb{C}} \oplus 0, (v + v', 0))] .$$

These multiplication maps unital, associative and commutative, and make $\mathbf{MU}$ into an ultra-commutative ring spectrum.

The global Thom spectrum $\mathbf{MU}$ comes with distinguished Thom classes

$$\sigma^U_{G,V} \in \mathbf{MU}^{2n}_G(S^V) , \tag{2.58}$$

for *unitary* representations V of compact Lie groups G, where n is the complex dimension of V. As in the orthogonal situation in Theorem 2.19 the Thom class $\sigma^U_{G,V}$ is inverse to the image of the inverse Thom class $\tau^U_{G,V}$ in $\mathbf{mU}$, and $\sigma^U_{G,V}$ restricts to a unitary Euler class $e^U(V)$ in $\pi^G_{-2n}(\mathbf{MU})$.

There is a unitary version of the weak morphism from $\mathbf{mO}$ to $\mathbf{MO}$ defined in (2.39); the analog is a chain of morphisms of E_∞-orthogonal ring spectra

$$\mathbf{mU} \xrightarrow{c} \mathbf{MU}' \xleftarrow{\psi} \mathbf{MU}$$

such that ψ is a global equivalence. The proof of Corollary 2.37 generalizes to the unitary situation and proves that the G-equivariant homology represented by $\mathbf{MU}$ is the localization at the unitary inverse Thom classes of the theory represented by $\mathbf{mU}$.

Some known general facts about the equivariant homotopical bordism $\mathbf{MU}$ are known for *abelian* compact Lie groups A. In that case, $\pi^A_*(\mathbf{MU})$ is a free module on even dimensional generators over the non-equivariant homotopy group ring $\pi^e_*(\mathbf{MU})$; this calculation was announced by Löffler in [**100**], and a proof by Comezaña can be found in [**109**, XXVIII Thm. 5.3]. Since the graded ring $\pi^e_*(\mathbf{MU})$ is concentrated in even degrees and $\mathbf{MUP}$ is a wedge of even suspensions of $\mathbf{MU}$ the groups $\pi^A_*(\mathbf{MUP})$ are concentrated in even degrees. Since these groups are also 2-periodic, for abelian compact Lie groups all the information is concentrated in the ring $\pi^A_0(\mathbf{MUP})$. The non-equivariant homotopy groups $\pi^e_0(\mathbf{MUP})$ are a polynomial ring in countably many generators. For cyclic groups of prime order, Kriz [**91**] has described $\pi^{C_p}_0(\mathbf{MUP})$ as a pullback of two explicit ring homomorphisms. For the cyclic group of order 2, Strickland [**152**] has turned this into an explicit presentation of $\pi^{C_2}_0(\mathbf{MUP})$ as an algebra over $\pi^e_0(\mathbf{MUP})$.

3. Equivariant bordism

In this section we recall equivariant bordism groups and their relationship to the equivariant homology groups of $\mathbf{mO}$. The main result here is Theorem 3.40 which says that when G is isomorphic to a product of a finite group and a torus, the Thom-Pontryagin map is an isomorphism from G-equivariant bordism to G-equivariant $\mathbf{mO}$-homology. Theorem 3.40 is usually credited to Wasserman because it can be derived from his equivariant transversality theorem [**172**, Thm. 3.11]; as far as I know, the only place where the translation is spelled out in detail is the unpublished part of Costenoble's thesis, see [**39**, Thm. 11.1]. Wasserman's theorem is heavily based on equivariant differential topology; for finite groups, tom Dieck [**164**, Satz 5] gives a different proof of Theorem 3.40 based on the geometric and homotopy theoretic isotropy separation sequences. We generalize tom Dieck's proof, translated into our present language, with an emphasis on global aspects.

In Theorem 3.45 we also present a localized version of this result: the Thom-Pontryagin map is an isomorphism from *stable* equivariant bordism to $\mathbf{mO}[1/\tau]$-theory, without any restriction on the compact Lie group. Given that $\mathbf{mO}[1/\tau]$-theory is isomorphic to $\mathbf{MO}$-theory (by Corollary 2.37), this is equivalent to a result of Bröcker and Hook [**28**, Thm. 4.1] that identifies stable equivariant bordism with equivariant $\mathbf{MO}$-homology.

The serious study of equivariant bordism groups was initiated by the work [**38**] of Conner and Floyd. We recall equivariant bordism as a homology theory for G-spaces X, where G is a compact Lie group. A *singular G-manifold* over X is a pair (M, h) consisting of a closed smooth G-manifold M and a continuous G-map $h : M \longrightarrow X$. Two singular G-manifolds (M, h) and (M', h') are bordant if there is a triple (B, H, ψ) consisting of a compact smooth G-manifold B, a continuous G-map $H : B \longrightarrow X$ and an equivariant diffeomorphism

$$\psi \ : \ M \cup M' \ \cong \ \partial B$$

such that $(H \circ \psi)|_M = h$ and to $(H \circ \psi)|_{M'} = h'$.

Bordism of singular G-manifolds over X is an equivalence relation. Reflexivity and symmetry are straightforward; transitivity is established by gluing two bordisms along a common piece of the boundary.

To get a smooth structure on the glued bordism that is compatible with the G-action one needs smooth equivariant collars; the existence of such collars is guaranteed by [**38**, Thm. 21.2].

We denote by $\mathcal{N}_n^G(X)$ the set of bordism classes of n-dimensional singular G-manifolds over X. This set becomes an abelian group under disjoint union. Every element x of $\mathcal{N}_n^G(X)$ satisfies $2x = 0$. The groups $\mathcal{N}_n^G(X)$ are covariantly functorial in continuous G-maps, by postcomposition. When $X = *$ consists of a single point the reference maps are uniquely determined, and whenever convenient we will identify $\mathcal{N}_n^G(*)$ with the coefficient group $\mathcal{N}_n^G$ by forgetting the redundant reference map.

Proposition 3.1. *Let G be a compact Lie group.*

(i) *Let $\varphi, \varphi' : X \longrightarrow Y$ be equivariantly homotopic continuous G-maps. Then $\varphi_* = \varphi'_*$ as homomorphisms from $\mathcal{N}_n^G(X)$ to $\mathcal{N}_n^G(Y)$.*

(ii) *For every G-weak equivalence $\varphi : X \longrightarrow Y$ the induced homomorphism $\varphi_* : \mathcal{N}_n^G(X) \longrightarrow \mathcal{N}_n^G(Y)$ is an isomorphism.*

(iii) *Let $\{X_i\}_{i\in I}$ be a family of G-spaces. Then the canonical map*

$$\bigoplus_{i\in I} \mathcal{N}_n^G(X_i) \;\longrightarrow\; \mathcal{N}_n^G\left(\coprod_{i\in I} X_i\right)$$

is an isomorphism.

Proof. (i) We let $H : X \times [0,1] \longrightarrow Y$ be an equivariant homotopy from φ to φ' and (M,h) a singular G-manifold over X. Then $(M \times [0,1], H \circ (h \times [0,1]), \psi)$ is a bordism from $(M, \varphi h)$ to $(M, \varphi' h)$, where $\psi : M \cup M \longrightarrow \partial(M \times [0,1])$ identifies one copy of M with $M \times \{0\}$ and the other copy with $M \times \{1\}$. So $\varphi_*[M,h] = [M, \varphi \circ h] = [M, \varphi' \circ h] = \varphi'_*[M,h]$.

(ii) For surjectivity of φ_* we consider any singular G-manifold (M,g) over Y. Illman's theorem [**82**, Cor. 7.2] shows that M admits the structure of a G-CW-complex. Since φ is a G-weak equivalence there exists a continuous G-map $h : M \longrightarrow X$ such that φh is equivariantly homotopic to g. So $\varphi_*[M,h'] = [M, \varphi g] = [M,g]$.

The argument for injectivity is similar. We consider a singular G-manifold (M,h) over X that represents an element in the kernel of φ_*. There is then a null-bordism (B, H, ψ) of $(M, \varphi h)$. By Illman's theorem [**82**, Cor. 7.2] and the discussion immediately preceding it there is a G-CW-structure on B for which the boundary is an equivariant subcomplex. So the map $\psi : M \longrightarrow B$ that identifies M with the boundary of B is a G-cofibration. Since φ is a G-weak equivalence, there exists a continuous G-map $H' : B \longrightarrow X$ such that $H' \circ \psi = h$. The triple (B, H', ψ) thus witnesses that $[M,h] = 0$. Since φ_* is a group homomorphism, it is injective.

Property (iii) holds because compact manifolds only have finitely many connected components, so all continuous reference maps from singular manifolds or bordisms have image in a finite union. □

Now we state the key exactness property of equivariant bordism in the form of a Mayer-Vietoris sequence. The definition of the boundary map needs the existence of G-invariant separating functions as provided by the following lemma. The construction employs averaging via an invariant integral for compact Lie groups, a concept that nowadays often goes under the name of 'Haar measure'. I am not sure who deserves credit for the invariant integral for compact Lie groups, but in that special the construction seems to be quite old: in the introduction to his fundamental paper [**67**] on invariant integrals for locally compact topological groups, Haar credits the invariant integral for Lie groups to Hurwitz, as far back as 1897. Soon after Haar's construction for locally compact topological groups, von Neumann [**119**] has given a simpler construction of the invariant integral for *compact* topological groups.

Lemma 3.2. *Let M be a compact smooth G-manifold, C and C' two disjoint, closed, G-invariant subsets of M, and*

$$s \;:\; \partial M \;\longrightarrow\; \mathbb{R}$$

a smooth G-invariant map such that

$$C \cap \partial M \subseteq s^{-1}(0) \qquad \text{and} \qquad C' \cap \partial M \subseteq s^{-1}(1)\ .$$

Then there exists a smooth G-invariant extension $r : M \longrightarrow [0,1]$ of s such that $C \subseteq r^{-1}(0)$ and $C' \subseteq r^{-1}(1)$.

Proof. Since M is compact, hence normal, the Tietze extension theorem provides a continuous map $r_0 : M \longrightarrow \mathbb{R}$ that extends s and satisfies $C \subseteq r_0^{-1}(0)$ and $C' \subseteq r_0^{-1}(1)$. We average r_0 to make it G-invariant, i.e., we define $r_1 : M \longrightarrow \mathbb{R}$ by

$$r_1(x) \ = \ \int_G r_0(g \cdot x)\, dg\ .$$

The integral is taken with respect to the normalized invariant measure ('Haar measure') on G. The new map r_1 is again continuous, compare [**27**, Ch. 0 Prop. 3.2]. Since r_0 is already G-invariant on $C \cup C' \cup \partial M$, the new map r_1 coincides with r_0 on this subset. In particular, r_1 is smooth on $C \cup C' \cup \partial M$. By [**27**, VI Thm. 4.2] we can then find a smooth G-invariant map $r : M \longrightarrow \mathbb{R}$ that coincides with r_1 on $C \cup C' \cup \partial M$; this is the desired separating function. □

Construction 3.3 (Boundary map in equivariant bordism). We define a boundary homomorphism for a Mayer-Vietoris sequence. We let X be a G-space and $A, B \subset X$ open G-invariant subsets with $X = A \cup B$. Then a homomorphism

$$\partial \ : \ \mathcal{N}_n^G(X) \ \longrightarrow \ \mathcal{N}_{n-1}^G(A \cap B)$$

is defined as follows.

We let (M, h) be a singular G-manifold that represents a class in $\mathcal{N}_n^G(X)$. The sets $h^{-1}(X - A)$ and $h^{-1}(X - B)$ are G-invariant, disjoint closed subsets of M; we let $r : M \longrightarrow \mathbb{R}$ be a G-invariant smooth separating function as provided by Lemma 3.2, i.e., such that $h^{-1}(X - A) \subseteq r^{-1}(0)$ and $h^{-1}(X - B) \subseteq r^{-1}(1)$. We let $t \in (0,1)$ be any regular value of r. Then

$$M_t \ = \ r^{-1}(t)$$

is a smooth closed G-submanifold of M of dimension $n-1$ (possibly empty), and $h_t = h|_{M_t}$ lands in $A \cap B$; so (M_t, h_t) is a singular G-manifold over $A \cap B$.

Proposition 3.4. *In the situation above, the bordism class $[M_t, h_t]$ is independent of the choice of regular value t, of the choice of separating function and of the representative for the given class in $\mathcal{N}_n^G(X)$. The resulting map*

$$\partial \ : \ \mathcal{N}_n^G(X) \ \longrightarrow \ \mathcal{N}_{n-1}^G(A \cap B)\ , \quad [M, h] \ \longmapsto \ [M_t, h_t]$$

is a group homomorphism.

Proof. We let $t < t'$ be two regular values in $(0,1)$ of the separating function r. Then

$$(r^{-1}[t, t'],\ h|_{r^{-1}[t,t']},\ \text{incl})$$

is a bordism from $(r^{-1}(t), h|_{r^{-1}(t)})$ to $(r^{-1}(t'), h|_{r^{-1}(t')})$, so the bordism class does not depend on the regular value.

Now we let (M, h) and (N, g) be two singular G-manifolds over X in the same bordism class, and we let (B, H, ψ) be a G-bordism from (M, h) to (N, g). We let $r : M \longrightarrow [0,1]$ and $\bar{r} : N \longrightarrow [0,1]$ be two G-invariant separating functions. Lemma 3.2 lets us extend this data to a smooth G-invariant separating function

$$\Psi \ : \ B \ \longrightarrow \ \mathbb{R}$$

such that $\Psi \circ \psi|_M = r$, $\Psi \circ \psi|_N = \bar{r}$,

$$H^{-1}(X - A) \subseteq \Psi^{-1}(0) \quad \text{and} \quad H^{-1}(X - B) \subseteq \Psi^{-1}(1)\ .$$

We choose a $t \in (0,1)$ that is simultaneously a regular value for Ψ, r and $\bar{r}$. Then

$$(\Psi^{-1}(t),\, H|_{\Psi^{-1}(t)},\, \psi|_{r^{-1}(t)\cup\bar{r}^{-1}(t)})$$

is a bordism from $(r^{-1}(t), h|_{r^{-1}(t)})$ to $(\bar{r}^{-1}(t), g|_{\bar{r}^{-1}(t)})$. This shows at the same time that the bordism class is independent of the choice of separating function and of the choice of representing singular G-manifold. Additivity of the resulting boundary map is then clear: a separating function for a disjoint union can be taken as the union of separating functions for each summand. □

Now we formulate the properties that make equivariant bordism a homology theory for G-spaces. There does not seem to be any published reference of the following proposition in the generality of compact Lie groups; it is a tradition in this subject to proclaim that one can adapt the non-equivariant proof. We honor this tradition and refrain from giving a proof.

Proposition 3.5. *Let G be a compact Lie group, X a G-space and $A, B \subset X$ open G-invariant subsets with $X = A \cup B$. Let $i^A : A \cap B \longrightarrow A$, $i^B : A \cap B \longrightarrow B$, $j^A : A \longrightarrow X$ and $j^B : B \longrightarrow X$ denote the inclusions. Then the following sequence of abelian groups is exact:*

$$\ldots \longrightarrow \mathcal{N}_n^G(A\cap B) \xrightarrow{(i_*^A, i_*^B)} \mathcal{N}_n^G(A) \oplus \mathcal{N}_n^G(B) \xrightarrow{\binom{j_*^A}{-j_*^B}} \mathcal{N}_n^G(X) \xrightarrow{\partial} \mathcal{N}_{n-1}^G(A\cap B) \longrightarrow \ldots$$

The proof of Proposition 3.5 is somewhat involved, but it proceeds along the lines of the non-equivariant argument as given for example in [**167**, Prop. 21.1.7]. As input one needs that certain basic tools from differential topology generalize to the G-equivariant context, such as for example the existence of equivariant collars and bicollars, and that non-equivariant smoothing of corners is compatible with G-actions.

We now introduce a reduced version of equivariant bordism and show that the reduced G-bordism groups form a G-homology theory. We define the *reduced bordism group* of a based G-space X as

$$\widetilde{\mathcal{N}}_n^G(X) \;=\; \operatorname{coker}\big(\mathcal{N}_n^G \longrightarrow \mathcal{N}_n^G(X)\big)\ ,$$

the cokernel of the homomorphism induced by the basepoint inclusion. The unique map $u : X \longrightarrow *$ is retraction to the basepoint inclusion, so the map

$$(\mathrm{proj}, u_*) \;:\; \mathcal{N}_n^G(X) \;\longrightarrow\; \widetilde{\mathcal{N}}_n^G(X) \oplus \mathcal{N}_n^G$$

is an isomorphism. On the other hand, if we add a disjoint G-fixed basepoint to an unbased G-space X, then the composite

$$\mathcal{N}_n^G(X) \xrightarrow{\mathrm{incl}_*} \mathcal{N}_n^G(X_+) \xrightarrow{\mathrm{proj}} \widetilde{\mathcal{N}}_n^G(X_+)$$

is an isomorphism.

Construction 3.6. We consider a continuous G-map $f : X \longrightarrow Y$ and let

$$Cf \;=\; CX \cup_f Y$$

denote its unreduced mapping cone. The two open sets

$$A \;=\; X \times [0,1)/X \times 0 \qquad \text{and} \qquad B \;=\; X \times (0,1] \cup_f Y$$

are G-invariant and together cover the mapping cone. The intersection $A \cap B$ is homeomorphic to $X \times (0,1)$, so the open covering has an associated boundary homomorphism

$$\partial \;:\; \mathcal{N}_n^G(Cf) \;\longrightarrow\; \mathcal{N}_{n-1}^G(X \times (0,1))$$

as in Construction 3.3. We take the cone point as the basepoint of Cf; this is contained in the subset A, so the map $\iota : \mathcal{N}_* \longrightarrow \mathcal{N}_*(Cf)$ induced by the basepoint inclusion factors through $j_*^A : \mathcal{N}_n^G(A) \longrightarrow \mathcal{N}_n^G(Cf)$,

and the composite $\partial \circ \iota$ is trivial by exactness of the excision sequence. The boundary map thus factors over the reduced bordism group. We define a 'reduced boundary map' $\bar{\partial}$ as the composite

$$\widetilde{\mathcal{N}}_n^G(Cf) \xrightarrow{\partial} \mathcal{N}_{n-1}^G(X \times (0,1)) \xrightarrow{\mathrm{proj}_*} \mathcal{N}_{n-1}^G(X) \ .$$

PROPOSITION 3.7. *Let G be a compact Lie group and $f : X \longrightarrow Y$ a continuous G-map between G-spaces. Then the following sequence of abelian groups is exact:*

$$\ldots \longrightarrow \mathcal{N}_n^G(X) \xrightarrow{f_*} \mathcal{N}_n^G(Y) \xrightarrow{i_*} \widetilde{\mathcal{N}}_n^G(Cf) \xrightarrow{\bar{\partial}} \mathcal{N}_{n-1}^G(X) \longrightarrow \ldots$$

PROOF. We use the open covering of the mapping cone Cf as in the definition of the boundary map. In the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x\mapsto(x,1/2)"'] & Y \arrow[r, "i"] \arrow[d, "i"'] & Cf \arrow[d, equal] \\
A \cap B \arrow[r, "\mathrm{incl}"'] & B \arrow[r, "\mathrm{incl}"'] & Cf
\end{tikzcd}$$

the right square commutes and the left square commutes up to equivariant homotopy. Moreover, all vertical maps are equivariant homotopy equivalences, so they induce isomorphisms in equivariant bordism, by Proposition 3.1. So the resulting diagram of bordism groups commutes:

$$\begin{tikzcd}
\mathcal{N}_n^G(X) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \mathcal{N}_n^G(Y) \arrow[r, "i_*"] \arrow[d, "\cong"', "i_*"] & \mathcal{N}_n^G(Cf) \arrow[d, equal] \\
\mathcal{N}_n^G(A \cap B) \arrow[r] & \mathcal{N}_n^G(B) \arrow[r] & \mathcal{N}_n^G(Cf)
\end{tikzcd}$$

Moreover, all vertical maps in this diagram are isomorphisms, so we can substitute $\mathcal{N}_n^G(X)$ and $\mathcal{N}_n^G(Y)$ into the long exact excision sequence of Proposition 3.5. Since A is equivariantly contractible to the cone point, we can also replace the corresponding summand by the coefficient group, and the result is an exact sequence

$$\ldots \longrightarrow \mathcal{N}_n^G(X) \xrightarrow{(f_*,u_*)} \mathcal{N}_n^G(Y) \oplus \mathcal{N}_n^G \longrightarrow \mathcal{N}_n^G(Cf) \xrightarrow{\partial} \mathcal{N}_{n-1}^G(X) \longrightarrow \ldots$$

The sequence then remains exact if we divide out the summand $\mathcal{N}_n^G$ and replace the absolute bordism group of Cf by the reduced group $\widetilde{\mathcal{N}}_n^G(Cf)$, so this shows the claim. □

REMARK 3.8. If $f : A \longrightarrow B$ is a cofibration of G-spaces, then the projection $q : Cf \longrightarrow B/A$ from the mapping cone to the quotient is a based equivariant homotopy equivalence. So we can substitute $\widetilde{\mathcal{N}}_*^G(B/A)$ into the long exact mapping cone sequence of Proposition 3.7 and a long exact sequence of abelian groups:

$$\ldots \longrightarrow \mathcal{N}_n^G(A) \xrightarrow{f_*} \mathcal{N}_n^G(B) \xrightarrow{q_*} \widetilde{\mathcal{N}}_n^G(B/A) \longrightarrow \mathcal{N}_{n-1}^G(A) \longrightarrow \ldots$$

The next proposition says, loosely speaking, that in a reduced bordism group the part of a singular G-manifold that sits over the basepoint can be ignored.

The next proposition says, loosely speaking, that in a reduced bordism group the part of a singular G-manifold that sits over the basepoint can be ignored.

PROPOSITION 3.9. *Let G be a compact Lie group, X be a based G-space and $h : N \longrightarrow X$ a singular G-manifold over X. Let V be a G-representation and M a closed smooth G-manifold such that* $\dim(M) + \dim(V) = \dim(N)$. *Let $j : M \times D(V) \longrightarrow N$ be a smooth G-equivariant embedding. Suppose that h sends $N - j(M \times \mathring{D}(V))$ to the basepoint. Then*

$$[\![N, h]\!] \ = \ [\![M \times S(V \oplus \mathbb{R}), f]\!] \quad \textit{in } \widetilde{\mathcal{N}}_n^G(X),$$

where $f : M \times S(V \oplus \mathbb{R}) \longrightarrow X$ *is defined by*

$$f(m,(v,\lambda)) \;=\; \begin{cases} h(j(m,v)) & \text{if } \lambda \le 0, \text{ and} \\ \quad * & \text{if } \lambda \ge 0. \end{cases}$$

Proof. We define a G-space by

$$B \;=\; (N \times [-1,0] \;\cup\; N \times [0,1])/\sim\,,$$

where the equivalence relation identifies the two copies of $(n,0)$ for every $n \in N - j(M \times \mathring{D}(V))$. The group G acts by $g \cdot [n,t] = [gn,t]$. The space B is a topological $(n+1)$-manifold whose boundary consists of three disjoint parts that we now parametrize. There are two obvious embeddings

$$\psi\;,\;\psi' \;:\; N \;\longrightarrow\; B \qquad \text{by} \qquad \psi(n) = [n,-1] \qquad \text{and} \qquad \psi'(n) = [n,1]$$

as the two endpoints in direction of the internal $[-1,1]$. We define another embedding

$$i \;:\; M \times S(V \oplus \mathbb{R}) \;\longrightarrow\; B \qquad \text{by}$$

$$i(m,(v,\lambda)) \;=\; \begin{cases} [(j(m,v),0]^{\text{left}} & \text{for } \lambda \le 0, \text{ and} \\ [(j(m,v),0]^{\text{right}} & \text{for } \lambda \ge 0. \end{cases}$$

Here the superscripts 'left' and 'right' indicate whether we refer to the point $[n,0]$ as the image of $(n,0)$ in $N \times [-1,0]$ or in $N \times [0,1]$. The manifold boundary of B is then the disjoint union of the images of ψ, ψ' and i.

The topological manifold B admits a smooth structure for which the given G-action is smooth and such that the embeddings ψ, ψ' and i are smooth; the construction involves 'smoothing of corners' (also called 'straightening of angles') near the image of $j(M \times S(V)) \times 0$ and is explained for example in Construction 15.10.3 of tom Dieck's textbook [**167**]. Tom Dieck has no group actions around, but we also have to ensure that the given G-action on B is smooth. This can be arranged by insisting that the collars used in [**167**, 15.10.3] are G-equivariant collars, which is possible for example by [**38**, Thm. 21.2].

Now we define a continuous G-map $H : B \longrightarrow X$ by $H(n,t) = h(n)$ on $N \times [-1,0]$ and as the constant map to the base point of X on $N \times [0,1]$. Then

$$H \circ \psi \;=\; h \qquad \text{and} \qquad H \circ i \;=\; f\;,$$

so the bordism $(B, H, \psi + \psi' + i)$ witnesses the relation

$$[N,h] \;=\; [N, H \circ \psi'] \;+\; [M \times S(V \oplus \mathbb{R}), f]$$

in the unreduced bordism group $\mathcal{N}_n^G(X)$. Since $H \circ \psi'$ is constant to the basepoint of X, the class $[N, H \circ \psi']$ vanishes in the reduced bordism group $\widetilde{\mathcal{N}}_n^G(X)$; this proves the claim. □

The next proposition shows that the distinguished bordism classes $d_{G,V}$ measure the failure of the Wirthmüller isomorphism in equivariant bordism, see Remark 3.15 below. We consider a closed subgroup H of a compact Lie group G and write

$$L \;=\; T_H(G/H)$$

for the tangent H-representation. For an H-space Y the H-equivariant continuous map

$$l_Y \;:\; G \times_H Y \;\longrightarrow\; Y_+ \wedge S^L$$

was defined in Construction III.2.2.

Proposition 3.10. *For every closed subgroup H of a compact Lie group G and every H-space Y, the composite*

$$\mathcal{N}_n^H(Y) \;\xrightarrow{G \times_H}\; \mathcal{N}_{n+d}^G(G \times_H Y) \;\xrightarrow{\text{res}_H^G}\; \mathcal{N}_{n+d}^H(G \times_H Y) \;\xrightarrow{(l_Y)_*}\; \widetilde{\mathcal{N}}_{n+d}^H(Y_+ \wedge S^L)$$

is exterior multiplication by the class $d_{H,L} \in \widetilde{\mathcal{N}}_d^H(S^L)$, where $d = \dim(G/H)$.

PROOF. We let (M,h) be a singular H-manifold that represents a class in $\mathcal{N}_*^H(Y)$. The class $G\times_H[M,h]$ is then represented by $G\times_H h : G\times_H M \longrightarrow G\times_H Y$. As in the construction of the collapse map l_Y we choose a 'slice' around $1\in G$ orthogonal to H, i.e., a smooth embedding

$$s \ : \ D(L) \ \longrightarrow \ G$$

satisfying

$$s(0)=1 \ , \quad s(h\cdot v) \ = \ h\cdot s(v)\cdot h^{-1}$$

for all $(h,v)\in H\times D(L)$ and such that the differential at $0\in D(L)$ of the composite

$$D(L) \ \xrightarrow{s} \ G \ \xrightarrow{\text{proj}} \ G/H$$

is the identity of L. The slice property implies that the map

$$\bar{s} \ : \ D(L)\times H \ \longrightarrow \ G \ , \quad (v,h) \ \longmapsto \ s(v)\cdot h$$

is a tubular neighborhood of H inside G. Moreover, this embedding is equivariant for the action of H^2, acting on source and target by

$$(h_1,h_2)\cdot(v,h) \ = \ (h_1v,\, h_1hh_2^{-1}) \qquad \text{respectively} \qquad (h_1,h_2)\cdot g \ = \ h_1gh_2^{-1} \ .$$

The map

$$l_H^G \ : \ G_+ \ \longrightarrow \ S^L\wedge H_+$$

was defined as the H^2-equivariant collapse map with respect to the tubular neighborhood $\bar{s}$. So explicitly,

$$l_H^G(g) \ = \ \begin{cases} (v/(1-|v|))\wedge h & \text{if } g=s(v)\cdot h \text{ with } (v,h)\in D(L)\times H, \text{ and} \\ \ \ \infty & \text{if } g \text{ is not in the image of } \bar{s}. \end{cases}$$

We obtains a smooth H-equivariant embedding

$$j \ : \ M\times D(L) \ \longrightarrow \ G\times_H M \qquad \text{by} \qquad j(m,v) \ = \ [s(v),m] \ ,$$

where H acts diagonally on the left. The composite

$$G\times_H M \ \xrightarrow{G\times_H h} \ G\times_H Y \ \xrightarrow{l_H^G\wedge_H Y_+} \ (S^L\wedge H_+)\wedge_H Y_+ \ \cong \ Y_*\wedge S^L$$

sends the complement of $j(D(V)\times M)$ to the basepoint at infinity. Proposition 3.9 (for (H,L) instead of (G,V)) shows that then

$$((l_Y)_*\circ \mathrm{res}_H^G\circ \mathrm{Ind}_H^G)[M,h] \ = \ [\![G\times_H M,\ l_Y\circ(G\times_H h)]\!] \ = \ [\![M\times S(L\oplus\mathbb{R}), f]\!] \ ,$$

in the reduced bordism group of $Y_+\wedge S^L$, where $f : M\times S(V\oplus\mathbb{R}) \longrightarrow Y_+\wedge S^L$ is defined by

$$f(m,(v,\lambda)) \ = \ \begin{cases} (l_Y\circ(G\times_H h))(j(m,v)) & \text{if } \lambda\le 0, \text{ and} \\ \qquad \infty & \text{if } \lambda\ge 0. \end{cases}$$

So f equals the composite

$$M\times S(L\oplus\mathbb{R}) \ \xrightarrow{h\times\Psi} \ Y\times S^L \ \xrightarrow{q} \ Y_+\wedge S^L$$

where

$$\Psi(v,\lambda) \ = \ v/(1-|v|)$$

for $\lambda\le 0$, and $f(v,\lambda)=\infty$ for $\lambda\ge 0$. The map $\Psi : S(L\oplus\mathbb{R}) \longrightarrow S^L$ is homotopic, in the equivariant based sense, to the stereographic projection Π_L. We can thus conclude that

$$((l_Y)_*\circ \mathrm{res}_H^G\circ \mathrm{Ind}_H^G)[M,h] \ = \ [\![M\times S(L\oplus\mathbb{R}), f]\!] \ = \ [M,h]\wedge[\![S(L\oplus\mathbb{R}),\Psi]\!] \ = \ [M,h]\wedge d_{H,L}$$

in the group $\widetilde{\mathcal{N}}_{*+d}^H(Y_+\wedge S^L)$. $\square$

The equivariant bordism groups come with natural products, given by the biadditive maps

$$\times \ : \ \mathcal{N}_m^G(X) \ \times \ \mathcal{N}_n^G(Y) \ \longrightarrow \ \mathcal{N}_{m+n}^G(X \times Y) \ , \quad [M,h] \times [N,g] \ = \ [M \times N, h \times g] \ .$$

These products are suitably associative, commutative and unital. The product pairing descends to a pairing on reduced bordism groups if the G-spaces X and Y are based. Indeed, the composite

$$\mathcal{N}_m^G(X) \otimes \mathcal{N}_n^G(Y) \ \xrightarrow{\times} \ \mathcal{N}_{m+n}^G(X \times Y) \ \xrightarrow{q_*} \ \mathcal{N}_{m+n}^G(X \wedge Y) \ \xrightarrow{\text{proj}} \ \widetilde{\mathcal{N}}_{m+n}^G(X \wedge Y)$$

annihilates the image of $\mathcal{N}_m^G \otimes \mathcal{N}_n^G(Y)$ and the image of $\mathcal{N}_m^G(X) \otimes \mathcal{N}_n^G$, where $q : X \times Y \longrightarrow X \wedge Y$ is the quotient map; so the composite factors uniquely over a homomorphism

$$\wedge \ : \ \widetilde{\mathcal{N}}_m^G(X) \ \otimes \ \widetilde{\mathcal{N}}_n^G(Y) \ \longrightarrow \ \widetilde{\mathcal{N}}_{m+n}^G(X \wedge Y) \ .$$

We will frequently use certain distinguished bordism classes associated to representations. We let V be an n-dimensional representation of a compact Lie group G. Stereographic projection is a G-equivariant homeomorphism

$$\Pi_V \ : \ S(V \oplus \mathbb{R}) \ \xrightarrow{\cong} \ S^V \ , \quad (v, \lambda) \ \longmapsto \ \frac{v}{1-\lambda}$$

from the unit sphere of $V \oplus \mathbb{R}$ to the one-point compactification S^V; this homeomorphism takes the point $(0,1)$ to the basepoint at infinity. We define a reduced G-bordism class over S^V by

$$d_{G,V} \ = \ [\![S(V \oplus \mathbb{R}), \Pi_V]\!] \quad \in \ \widetilde{\mathcal{N}}_n^G(S^V) \ . \tag{3.11}$$

The following multiplicativity property is maybe expected, but not completely obvious.

Proposition 3.12. *Let V and W be two representations of a compact Lie group G. Then the relation*

$$d_{G,V} \wedge \ d_{G,W} \ = \ d_{G,V \oplus W}$$

holds in $\widetilde{\mathcal{N}}_{n+m}^G(S^{V \oplus W})$.

Proof. We define the 'distorted' version $\tau_V : S(V \oplus \mathbb{R}) \longrightarrow S^V$ of the stereographic projection as the composite

$$S(V \oplus \mathbb{R}) \ \xrightarrow{\Pi_V} \ S^V \ \xrightarrow{J} \ S^V \ ,$$

where the second map J is given by

$$J(v) \ = \ \begin{cases} \frac{v}{1-|v|} & \text{for } |v| < 1, \text{ and} \\ \infty & \text{for } |v| \geq 1. \end{cases}$$

The map J is equivariantly based homotopic to the identity of S^V, so $(S(V \oplus \mathbb{R}), \tau_V)$ is another representative for the bordism class $d_{G,V}$. The map

$$\begin{aligned} j \ : \ V \oplus W \ &\longrightarrow \ S(V \oplus \mathbb{R}) \times S(W \oplus \mathbb{R}) \\ j(v,w) \ &= \ \left(\frac{2v}{|v|^2+1}, \frac{|v|^2-1}{|v|^2+1}, \frac{2w}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1} \right) \end{aligned}$$

is a smooth G-equivariant embedding. The map j is the product of the inverse stereographic projections $\Pi_V^{-1} \times \Pi_W^{-1} : S^V \times S^W \longrightarrow S(V \oplus \mathbb{R}) \times S(W \oplus \mathbb{R})$, restricted to $V \oplus W$.

If a quadruple (v, λ, w, μ) is in the image of the unit disc $j(D(V \oplus W))$, then in particular $\lambda \leq 0$ and $\mu \leq 0$. Equivalently, the points (v, λ, w, μ) of $S(V \oplus \mathbb{R}) \times S(W \oplus \mathbb{R})$ that are in the complement of $j(D(V \oplus W))$ have $\lambda \geq 0$ or $\mu \geq 0$. So the map

$$q \circ (\tau_V \times \tau_W) \ : \ S(V \oplus \mathbb{R}) \times S(W \oplus \mathbb{R}) \ \longrightarrow \ S^{V \oplus W}$$

sends the complement of $j(\mathring{D}(V \oplus W))$ to the basepoint at infinity, where $q : S^V \times S^W \longrightarrow S^{V \oplus W}$ is the projection. Proposition 3.9 thus applies and shows that

$$\begin{aligned} q_*(d_{G,V} \wedge d_{G,W}) &= q_*([\![S(V \oplus \mathbb{R}), \tau_V]\!] \wedge [\![S(W \oplus \mathbb{R}), \tau_W]\!]) \\ &= [\![S(V \oplus \mathbb{R}) \times S(W \oplus \mathbb{R}),\ q \circ (\tau_V \times \tau_W)]\!] \ = \ [\![S(V \oplus W \oplus \mathbb{R}),\ f]\!] \ , \end{aligned}$$

where $f : S(V \oplus W \oplus \mathbb{R}) \longrightarrow S^{V \oplus W}$ is defined by

$$f(v, w, \lambda) \ = \ \begin{cases} q(\tau_V \times \tau_W(j(v, w))) & \text{if } \lambda \leq 0, \text{ and} \\ \infty & \text{if } \lambda \geq 0. \end{cases}$$

We claim that for all $(v, w, \lambda) \in S(V \oplus W \oplus \mathbb{R})$ the relation

$$f(v, w, \lambda) \ \neq \ \Pi_{V \oplus W}(-v, -w, -\lambda)$$

holds in $S^{V \oplus W}$. Assuming this claim, we can finish the proof as follows: since the G-map $\Pi_{V \oplus W}^{-1} \circ f$ never takes a point to its antipode, the linear homotopy between $\Pi_{V \oplus W}^{-1} \circ f$ and the identity in the ambient vector space $V \oplus W \oplus \mathbb{R}$ can be normalized to land in the unit sphere; this yields an equivariant based homotopy between $\Pi_{V \oplus W}^{-1} \circ f$ and the identity of $S(V \oplus W \oplus \mathbb{R})$. Hence f is equivariantly based homotopic to the stereographic projection $\Pi_{V \oplus W}$, and so $(S(V \oplus W \oplus \mathbb{R}), f)$ represents the bordism class $d_{G,V \oplus W}$.

It remains to prove the claim. The only point of $S(V \oplus W \oplus \mathbb{R})$ that $\Pi_{V \oplus W}$ sends to the point at infinity is $(0, 0, 1)$. Since $f(0, 0, -1) = (0, 0)$, the claim is true for all (v, w, λ) such that $f(v, w, \lambda) = \infty$. It remains to consider those tuples for which $f(v, w, \lambda) \neq \infty$, which means that $\lambda < 0$ and $|v| < 1$ and $|w| < 1$. On such points the map f is given by

$$f(v, w, \lambda) \ = \ q\left(\tau_V(\frac{2v}{|v|^2 + 1}, \frac{|v|^2 - 1}{|v|^2 + 1}),\ \tau_W(\frac{2w}{|w|^2 + 1}, \frac{|w|^2 - 1}{|w|^2 + 1})\right) \ = \ q(J(v), J(w)) \ ,$$

whereas

$$\Pi_{V \oplus W}(-v, -w, -\lambda) \ = \ \left(\frac{-v}{1 + \lambda},\quad \frac{-w}{1 + \lambda}\right) \ .$$

If these two expressions were the same, then

$$\frac{v}{1 - |v|} \ = \ \frac{-v}{1 + \lambda} \qquad \text{and} \qquad \frac{w}{1 - |w|} \ = \ \frac{-w}{1 + \lambda} \ .$$

For $v \neq 0$ this implies

$$|v| - 1 \ = \ 1 + \lambda \ ,$$

which is impossible since $|v| < 1$ and $\lambda \geq -1$. If $w \neq 0$ we obtain the same kind of contradiction. The final case is $(v, w, \lambda) = (0, 0, -1)$, in which case

$$f(0, 0, -1) \ = \ (0, 0) \ \neq \ \infty \ = \ \Pi_{V \oplus W}(0, 0, 1) \ . \qquad \square$$

We shall now recall that if G acts trivially on V, then exterior multiplication by $d_{G,V}$ is an isomorphism, the suspension isomorphism in reduced equivariant bordism. In general, however, the class $d_{G,V}$ is not invertible and exterior multiplication by $d_{G,V}$ need not be an isomorphism. The theory obtained by formally inverting all the classes $d_{G,V}$ is *stable equivariant bordism*, to which we return in Remark 3.43 below.

PROPOSITION 3.13. *If G acts trivially on V and X is a cofibrant based G-space, then the exterior product map*

$$- \wedge d_{G,V} \ : \ \widetilde{\mathcal{N}}_n^G(X) \ \longrightarrow \ \widetilde{\mathcal{N}}_{n+|V|}^G(X \wedge S^V)$$

is an isomorphism. For every continuous G-map $f : X \longrightarrow Y$ between based G-spaces the connecting homomorphism in the mapping cone sequence equals the composite

$$\widetilde{\mathcal{N}}_n^G(Cf) \ \xrightarrow{p_*} \ \widetilde{\mathcal{N}}_n^G(X \wedge S^1) \ \xrightarrow{(-\wedge d_{G,\mathbb{R}})^{-1}} \ \widetilde{\mathcal{N}}_{n-1}^G(X) \ .$$

Proof. We start with the special case $V = \mathbb{R}$. We apply Proposition 3.7 to the embedding $- \wedge 1 : X \longrightarrow CX = [0,1] \wedge X$ of X into its cone. Since the cone is equivariantly contractible and X has a G-fixed point, the map $(- \wedge 1)_* : \mathcal{N}_*^G(X) \longrightarrow \mathcal{N}_*^G(CX)$ is a split epimorphism. So the long exact sequence provided by Proposition 3.7 reduces to a short exact sequence:

$$0 \;\longrightarrow\; \widetilde{\mathcal{N}}_{n+1}^G(Ci) \;\xrightarrow{\bar\partial}\; \mathcal{N}_n^G(X) \;\xrightarrow{u_*}\; \mathcal{N}_n^G \;\longrightarrow\; 0$$

The space $Ci = CX \cup_X CX$ is the double cone of X, and hence isomorphic to the unreduced suspension. Since X is cofibrant in the based sense, the projection

$$p \;:\; CX \cup_X CX \;\longrightarrow\; X \wedge S^1$$

that collapses the second cone is an equivariant homotopy equivalence. We let ∂' denote the composite

$$\widetilde{\mathcal{N}}_{n+1}^G(X \wedge S^1) \;\xrightarrow{p_*^{-1}}\; \widetilde{\mathcal{N}}_{n+1}^G(CX \cup_X CX) \;\xrightarrow{\bar\partial}\; \mathcal{N}_n^G(X)\ .$$

Then we conclude altogether that the map

$$(\iota, \partial') \;:\; \mathcal{N}_n^G \oplus \widetilde{\mathcal{N}}_{n+1}^G(X \wedge S^1) \;\longrightarrow\; \mathcal{N}_n^G(X)$$

is an isomorphism. The composite

$$\widetilde{\mathcal{N}}_{n+1}^G(X \wedge S^1) \;\xrightarrow{\partial'}\; \mathcal{N}_n^G(X) \;\xrightarrow{\mathrm{proj}_*}\; \widetilde{\mathcal{N}}_n^G(X)$$

is then an isomorphism as well.

We claim that the relation

$$\mathrm{proj}_*(\partial'(x \wedge d_{G,\mathbb{R}})) \;=\; x$$

holds for all classes $x \in \widetilde{\mathcal{N}}_n^G(X)$. This relation, in turn, is a consequence of the geometric origin of the class $d_{G,\mathbb{R}}$, the product in bordism and the boundary map. In more detail, we suppose that $x = [\![M, h]\!]$ for a singular G-manifold (M, h) over X. We use the smooth separating function

$$M \times S(\mathbb{R} \oplus \mathbb{R}) \;\xrightarrow{\mathrm{proj}}\; S(\mathbb{R} \oplus \mathbb{R}) \;\xrightarrow{r}\; [0,1]$$

where $r(x,y) = (y+1)/2$. Then $1/2$ is a regular value of this separating function, and the preimage over this regular value consists of two disjoint copies of M. So we obtain

$$\partial(x \wedge d_{G,\mathbb{R}}) \;=\; \partial[M \times S(\mathbb{R} \oplus \mathbb{R}), h \circ \mathrm{proj}] \;=\; [\![M \amalg M, h + h]\!] \;=\; [\![M, h]\!]\ .$$

The last equation exploits that one of the copies of M sits over $M \vee S^1$, so it does not contribute to the reduced group of the smash product.

The general case now follows easily. Since the claim is true for $V = \mathbb{R}$, it also holds for $V = \mathbb{R}^n$ by the associativity of the smash product pairing and the classes $d_{G,\mathbb{R}}$, compare Proposition 3.12. If G acts trivially on V, then it is equivariantly isomorphic to $\mathbb{R}^n$ for some n. □

The bordism theories $\mathcal{N}_*^G$ for different compact Lie groups are related by geometrically defined restriction and induction maps. Every continuous group homomorphism $\alpha : K \longrightarrow G$ is automatically smooth (see for example [**29**, Prop. I.3.12]), and thus induces a restriction homomorphism

$$\alpha^* \;:\; \mathcal{N}_n^G(X) \;\longrightarrow\; \mathcal{N}_n^K(\alpha^*(X))\ , \quad [M, h] \;\longmapsto\; [\alpha^*(M), \alpha^*(h)]$$

by restricting all actions along α. Restriction maps preserve the distinguished classes (3.11) in the sense that

$$\alpha^*(d_{G,V}) \;=\; d_{K,\alpha^*(V)}\ .$$

The product in equivariant bordism is compatible with restriction maps in the sense that

$$\alpha^*(x \times y) \;=\; \alpha^*(x) \times \alpha^*(y)\ .$$

For every closed subgroup H of G and every closed smooth H-manifold M of dimension n, the induced space

$$G \times_H M$$

is a smooth closed G-manifold of dimension $d+n$ with $d = \dim(G/H) = \dim(G) - \dim(H)$. We can also apply $G \times_H -$ to bordisms, so this gives a well-defined induction homomorphism

$$G \times_H - \ : \ \mathcal{N}_n^H(Y) \ \longrightarrow \ \mathcal{N}_{d+n}^G(G \times_H Y) \ , \quad [M,h] \ \longmapsto \ [G \times_H M, G \times_H h] \ . \tag{3.14}$$

For $Y = *$ we can compose the induction map with the effect of the projection $G \times_H * \longrightarrow *$ and arrive at an induction homomorphism on coefficient groups

$$\mathrm{ind}_H^G \ : \ \mathcal{N}_n^H \ \longrightarrow \ \mathcal{N}_{d+n}^G \ , \quad [M] \ \longmapsto \ [G \times_H M] \ .$$

The induction map ind_H^G is compatible with inflations. If H has finite index in G, then the induction ind_H^G preserves the dimension, and then it satisfies the double coset formula. So for fixed $n \geq 0$ the coefficient groups $\mathcal{N}_n^G$ *almost* form a global functor; the only missing structure are the transfer maps for closed inclusions that are not of finite index.

Multiplication, restriction and induction satisfy a reciprocity relation. We let H be a closed subgroup of G, X an H-space and Y a G-space. Then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{N}_m^H(X) \times \mathcal{N}_n^G(Y) \arrow[r, "(G\times_H -)\times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \mathrm{res}_H^G"'] & \mathcal{N}_{d+m}^G(G \times_H X) \times \mathcal{N}_n^G(Y) \arrow[d, "\times"] \\
\mathcal{N}_m^H(X) \times \mathcal{N}_n^H(\mathrm{res}_H^G(Y)) \arrow[d, "\times"'] & \mathcal{N}_{d+m+n}^G((G \times_H X) \times Y) \arrow[d, "\chi_*"] \\
\mathcal{N}_{m+n}^H(X \times \mathrm{res}_H^G(Y)) \arrow[r, "G\times_H -"'] & \mathcal{N}_{d+m+n}^G(G \ltimes_H (X \times \mathrm{res}_H^G(Y))
\end{tikzcd}$$

Here

$$\chi \ : \ (G \times_H X) \times Y \ \longrightarrow \ G \ltimes_H (X \times \mathrm{res}_H^G(Y)) \ , \quad ([g,x],y) \ \longmapsto \ [g,(x,g^{-1}y)]$$

is the G-equivariant shearing isomorphism. The proof of the commutativity is straightforward from the definitions, using the shearing diffeomorphism for equivariant manifolds. If we specialize to $X = Y = *$ and postcompose with the projection to the one-point G-space, the commutativity becomes the reciprocity formula

$$\mathrm{ind}_H^G(x \times \mathrm{res}_H^G(y)) \ = \ \mathrm{ind}_H^G(x) \times y$$

for classes $x \in \mathcal{N}_m^H$ and $y \in \mathcal{N}_n^G$ in the coefficient groups.

Remark 3.15 (Failure of the Wirthmüller isomorphism in equivariant bordism). As we now explain, there is no general Wirthmüller isomorphism in equivariant bordism, i.e., the Wirthmüller map

$$\mathrm{Wirth}_H^G \ = \ (l_Y)_* \circ \mathrm{res}_H^G \ : \ \mathcal{N}_*^G(G \times_H Y) \ \longrightarrow \ \mathcal{N}_*^H(Y_+ \wedge S^L)$$

is *not* in general bijective for closed subgroups H of G. Here Y is an H-space and

$$l_Y \ : \ G \times_H Y \ \longrightarrow \ Y_+ \wedge S^L$$

is the H-equivariant collapse map defined in Construction III.2.2. This immediately shows that there *cannot* be a natural isomorphism, compatible with restriction to subgroups, between equivariant bordism and the equivariant homology theory represented by any orthogonal G-spectrum.

The equivariant bordism theories come with induction homomorphisms $G \times_H -$ defined in (3.14) that increase the dimension by $\dim(G/H) = \dim(G) - \dim(H)$. For every H-space Y, the composite

$$\mathcal{N}_m^H(Y) \ \xrightarrow[\cong]{G\times_H -} \ \mathcal{N}_{m+d}^G(G \times_H Y) \ \xrightarrow{\mathrm{res}_H^G} \ \mathcal{N}_{m+d}^H(G \times_H Y) \ \xrightarrow{(l_Y)_*} \ \widetilde{\mathcal{N}}_{m+d}^H(Y_+ \wedge S^L)$$

is exterior multiplication by the class $d_{H,L} \in \widetilde{\mathcal{N}}_d^H(S^L)$, by Proposition 3.10. Since the induction map $G \times_H -$ is an isomorphism, the Wirthmüller map is bijective if any only if the class $d_{H,L}$ acts invertibly on $\mathcal{N}_*^H(Y)$. In this sense, the distinguished bordism class $d_{H,L}$ precisely measures the failure of the Wirthmüller isomorphism in equivariant bordism.

As we will now show, the class $d_{H,L}$ is *not* invertible as soon as H acts non-trivially on L, i.e., in this situation the class $d_{H,L}$ does *not* generate $\widetilde{\mathcal{N}}_*^H(S^L)$ as an $\mathcal{N}_*^H$-module. We set $k = \dim(L^H)$, which is then strictly less than the dimension of L. We let $i : L^H \longrightarrow L$ denote the inclusion of the fixed points and consider the commutative square

$$\begin{tikzcd}
\mathcal{N}_0^H \arrow[r, "-\wedge d_{H,L^H}", "\cong"'] \arrow[d] & \widetilde{\mathcal{N}}_k^H(S^{L^H}) \arrow[r, "i_*"] \arrow[d] & \widetilde{\mathcal{N}}_k^H(S^L) \arrow[d] \\
\widetilde{\mathcal{N}}_0^H(\tilde{E}\mathcal{P}) \arrow[r, "\cong", "-\wedge d_{H,L^H}"'] & \widetilde{\mathcal{N}}_k^H(\tilde{E}\mathcal{P} \wedge S^{L^H}) \arrow[r, "\cong", "i_*"'] & \widetilde{\mathcal{N}}_k^H(\tilde{E}\mathcal{P} \wedge S^L)
\end{tikzcd}$$

Since H acts trivially on L^H, exterior multiplication by the class d_{H,L^H} is an isomorphism by Proposition 3.13. Since the map $i : S^{L^H} \longrightarrow S^L$ is a homeomorphism on H-fixed points, the map $i \wedge \tilde{E}\mathcal{P} : S^{L^H} \wedge \tilde{E}\mathcal{P} \longrightarrow S^L \wedge \tilde{E}\mathcal{P}$ is an H-equivariant homotopy equivalence, and so the lower right horizontal map is an isomorphism. Since the multiplicative unit $1 \in \mathcal{N}_0^H$ maps to a non-trivial element in $\widetilde{\mathcal{N}}_0^H(\tilde{E}\mathcal{P})$, the class $i_*(d_{H,L^H})$ is non-zero in the group $\widetilde{\mathcal{N}}_k^H(S^L)$. Since the dimension of the class $d_{H,L}$ is strictly larger than k and the graded ring $\mathcal{N}_*^H$ is concentrated in non-negative dimensions, $d_{H,L}$ does not generate $\widetilde{\mathcal{N}}_k^H(S^L)$ as an $\mathcal{N}_*^H$-module.

Now we let H be a closed subgroup of G. We let $k = \dim(L^H)$ be the dimension of the H-fixed points of the tangent representation $L = T_H(G/H)$. Then by the previous paragraph, the class

$$i_*(d_{H,L^H}) \ \in \ \widetilde{\mathcal{N}}_k^H(S^L)$$

is non-trivial. If the H-action on L is non-trivial, then $k < d = \dim(L) = \dim(G/H)$. Since bordism groups are concentrated in non-negative dimensions, the class $i_*(d_{H,L^V})$ is not in the image of the composite

$$\mathcal{N}_*^G(G/H) \ \xrightarrow{\mathrm{res}_H^G} \ \mathcal{N}_*^H(G/H) \ \xrightarrow{(l_H^G/H)_*} \ \widetilde{\mathcal{N}}_{*+d}^H(S^L) \ .$$

So the geometric bordism theory has no Wirthmüller isomorphism for subgroups H of G with non-trivial tangent representation.

If G is a product of a finite group and a torus, then H acts trivially on L for every closed subgroup H of G, and hence the Wirthmüller isomorphism *does hold* for such compact Lie groups G. And in fact, for this class of groups, the Thom-Pontryagin construction provides an natural isomorphism between equivariant bordism and the equivariant homology theory represented by the Thom spectrum $\mathbf{mO}$, compare Theorem 3.40 below.

Our argument to compare the geometric bordism theory with the equivariant homotopy groups of the Thom spectrum $\mathbf{mO}$ is based on the isotropy separation sequence. We will now identify the 'geometric fixed point term' in the isotropy separation sequence for the geometric theory.

Construction 3.16. We recall the 'geometric fixed point' homomorphism

$$\Phi_{\mathrm{geom}} \ : \ \widetilde{\mathcal{N}}_n^G(\tilde{E}\mathcal{P}) \ \longrightarrow \ \bigoplus_{j \geq 0} \mathcal{N}_{n-j}(Gr_j^{G,\perp}) \ ;$$

here, as before, we use the abbreviation

$$Gr_j^{G,\perp} \ = \ \big(Gr_j(\mathcal{U}_G^\perp)\big)^G$$

for the space of j-dimensional G-invariant subspaces of $\mathcal{U}_G^\perp = \mathcal{U}_G - (\mathcal{U}_G)^G$. The G-space $\tilde{E\mathcal{P}}$ has exactly two fixed points 0 and ∞, with ∞ being the basepoint. Loosely speaking, the class $\Phi_{\text{geom}}[\![M,h]\!]$ remembers the bordism class of the part of the fixed point manifold that lies over the fixed point 0, together with the normal data of the embedding into M.

Here is the construction in more detail. We first define a fixed point map into the unreduced bordism group

$$\bar{\Phi} \ : \ \mathcal{N}_n^G(\tilde{E\mathcal{P}}) \ \longrightarrow \ \bigoplus_{j\geq 0} \mathcal{N}_{n-j}(Gr_j^{G,\perp})$$

as follows. We let $(M, h : M \longrightarrow \tilde{E\mathcal{P}})$ be an n-dimensional singular G-manifold. Since G acts smoothly, M^G is a disjoint union of smooth submanifolds of varying dimensions, compare [**27**, VI Cor. 2.5]. Since $\tilde{E\mathcal{P}}$ has exactly two fixed points, and both are isolated, h must map every path component of the fixed point manifold M^G to either 0 or ∞. We denote by

$$M_0^G \ = \ M^G \cap h^{-1}(0)$$

the union of those components of M^G that lie over the point 0.

We let $M^{(j)}$ be the union of all $(n-j)$-dimensional components of M_0^G. The Mostow-Palais embedding theorem [**115, 122**] provides a smooth G-equivariant embedding $i : M \longrightarrow V$, for some G-representation V; we can assume that V is in fact a subrepresentation of the complete universe $\mathcal{U}_G$. Then for every fixed point $x \in M^{(j)}$,

$$(di)(T_x(M^{(j)})) \ = \ ((di)(T_xM))^G \ ,$$

i.e., the tangent space inside $M^{(j)}$ 'is' the G-fixed part of the tangent space in M. So we can define a continuous map

$$\nu_j \ : \ M^{(j)} \ \longrightarrow \ (Gr_j(V^\perp))^G \ \xrightarrow{\text{incl}} \ Gr_j^{G,\perp}$$

by sending a fixed point $x \in M^{(j)}$ to the orthogonal complement of $(di)(T_x(M^{(j)}))$ inside $(di)(T_xM)$. By its very construction, the map ν_j classifies the normal bundle of the inclusion $M^{(j)} \longrightarrow M$. The geometric fixed point map is then given by

$$\bar{\Phi}[M,h] \ = \ \sum_{j=0}^{n} [M^{(j)}, \nu_j] \ .$$

Since the image of the map $i_* : \mathcal{N}_*^G \longrightarrow \mathcal{N}_*^G(\tilde{E\mathcal{P}})$ is concentrated over the basepoint ∞, it is annihilated by the map $\bar{\Phi}$. So $\bar{\Phi}$ factors over the reduced bordism group of $\tilde{E\mathcal{P}}$ as a homomorphism

$$\Phi_{\text{geom}} \ : \ \widetilde{\mathcal{N}}_n^G(\tilde{E\mathcal{P}}) \ \longrightarrow \ \bigoplus_{j\geq 0} \mathcal{N}_{n-j}(Gr_j^{G,\perp}) \ . \tag{3.17}$$

Proposition 3.18. *For every compact Lie group G and every $n \geq 0$ the geometric fixed point map* (3.17) *is an isomorphism.*

Proof. The proof is by explicit geometric constructions. We start with surjectivity. We consider a non-equivariant closed smooth $(n-j)$-manifold N and a continuous map $f : N \longrightarrow Gr_j^{G,\perp}$. The space $Gr_j^{G,\perp}$ is the filtered colimit of its closed subspaces $(Gr_j(V))^G$ for $V \in s(\mathcal{U}_G^\perp)$. Since N is compact, the image of f lands in $(Gr_j(V))^G$ for some finite dimensional G-representation V with $V^G = 0$. The space $(Gr_j(V))^G$ is a smooth manifold, and by smooth approximation we can assume without loss of generality that f is a smooth map. We define a closed smooth n-dimensional manifold by

$$M \ = \ \{(x, v, \lambda) \in N \times S(V \oplus \mathbb{R}) \mid v \in f(x)\} \ .$$

Another way to say this is that M is a double of the unit disc bundle of the pullback of the tautological bundle along $f : N \longrightarrow (Gr_j(V))^G$. The group G acts smoothly on this by

$$g \cdot (x, v, \lambda) \ = \ (x, gv, \lambda) \ .$$

At this point it will be convenient to use a specific model for the space $\tilde{E}\mathcal{P}$, namely

$$\tilde{E}\mathcal{P} \;=\; S(\mathcal{U}_G^{\perp} \oplus \mathbb{R}) \;,$$

the unit sphere in $\mathcal{U}_G^{\perp} \oplus \mathbb{R}$, compare Example III.3.8. We can then define a continuous G-map,

$$h \;:\; M \;\longrightarrow\; E\mathcal{P}^{\diamond} \qquad \text{by} \qquad h(x,v,\lambda) \;=\; (v,\lambda) \;.$$

So the pair (M,h) is a singular G-manifold over $\tilde{E}\mathcal{P}$, and it represents a bordism class

$$[M,h] \quad \in\; \mathcal{N}_n^G(\tilde{E}\mathcal{P}) \;.$$

Since $V^G = 0$, the G-fixed points of M are a disjoint union of two copies of N embedded as

$$N \times (0,-1) \qquad \text{respectively} \qquad N \times (0,1) \;.$$

and h maps one copy to each of the two fixed point of $E\mathcal{P}^{\diamond}$. The normal bundle of the copy of N over the non-base point is the bundle classified by the original map f, we obtain

$$\Phi_{\text{geom}}[M,h] \;=\; [N,f] \;.$$

This shows that every class in $\mathcal{N}_{n-j}(Gr_j^{G,\perp})$ is in the image of the geometric fixed point map, and so the map (3.17) is surjective.

Now we consider a reduced equivariant bordism class $[\![M,h]\!]$ over $\tilde{E}\mathcal{P}$ in the kernel of the fixed point map (3.17). Being in the kernel of Φ_{geom} means that for every $0 \le j \le n$ there is a non-equivariant null-bordism W_j with $\partial W_j = M^{(j)}$ and a continuous maps $F_j : W \longrightarrow Gr_j^{G,\perp}$ whose restriction to $M^{(j)}$ classifies the normal bundle of $M^{(j)}$ inside M. As in the first part of the proof we can compress F_j to a G-map $W \longrightarrow (Gr_j(V))^G$ for some finite dimensional G-subrepresentation V of $\mathcal{U}_G^{\perp}$ and replace this factorization by a homotopic smooth map $F : W \longrightarrow (Gr_j(V))^G$. We use this data to 'cut out' the fixed points M_0^G from M by replacing a tubular neighborhood by the sphere bundles of the maps F_j; this produces a new singular G-manifold over $\tilde{E}\mathcal{P}$, bordant to (M,h), that has no more fixed points over 0.

The construction is done separately and disjointly over each of the components $M^{(j)}$ of the fixed points M_0^G. To simplify the exposition we restrict to the special case where the fixed points M_0^G are of constant dimension $n-j$ (i.e., all other components $M^{(i)}$ for $i \neq j$ are empty). Then we proceed as follows. We let ν denote the normal bundle of $M^{(j)}$ inside M. The equivariant tubular neighborhood theorem provides a smooth G-equivariant embedding

$$\psi \;:\; D(\nu) \;\longrightarrow\; M$$

of the unit disc bundle of ν whose composite with the zero section $s : M^{(j)} \longrightarrow D(\nu)$ is the inclusion, see for example [**27**, VI Thm. 2.2]. In particular, the image of ψ is a closed G-invariant tubular neighborhood of $M^{(j)}$. By shrinking the neighborhood, if necessary, we can make its image disjoint from all other components of M^G except $M^{(j)}$. The bundle arising by pulling back the tautological bundle along $F : W \longrightarrow (Gr_j(V))^G$ has its own disc bundle with total space

$$D(F) \;=\; \{(x,v) \in W \times V \mid v \in F(x), |v| \le 1\} \;.$$

The sphere bundle $S(F)$ is then a smooth compact G-manifold with boundary

$$\partial(S(F)) \;=\; S(F|_{\partial W}) \;=\; S(\nu) \;.$$

Now we form the G-manifold

$$\bar{M} \;=\; (M - \psi(\mathring{D}(\nu)) \cup_{S(\nu)} S(F) \;,$$

where the gluing uses the restriction $\psi : S(\nu) \longrightarrow M$ of the tubular neighborhood to the sphere bundle. A smooth structure on $\bar{M}$ is provided by choices of G-equivariant collars of $\psi(S(\nu))$ inside $M - \psi(\mathring{D}(\nu))$ and of $S(\nu)$ inside $S(F)$.

The boundary of the disc bundle of F decomposes as

$$\partial(D(F)) \;=\; S(F) \cup_{S(F|_{\partial W})} D(F|_{\partial W}) \;=\; S(F) \cup_{S(\nu)} D(\nu)\ .$$

By equivariant smoothing of corners the disc bundle $D(F)$ can be given a smooth structure such that the given G-action is smooth and that the embeddings of $D(\nu)$ and $S(F)$ into $D(F)$ are smooth. We define a bordism as the G-space

$$B \;=\; (M \times [0,1]) \cup_{D(\nu)} D(F)$$

where the gluing is along $(\psi, 1) : D(\nu) \longrightarrow M \times [0,1]$. The space B is a topological $(n+1)$-manifold whose boundary is the disjoint union of two disjoint parts that we now parametrize. An obvious embedding is given by

$$\psi \;:\; M \;\longrightarrow\; B\ , \quad \psi(m) = [m, 0]\ .$$

A second embedding

$$i \;:\; \bar{M} \;=\; (M - \psi(\mathring{D}(\nu))) \cup_{S(\nu)} S(F) \;\longrightarrow\; B$$

identifies $M - \psi(\mathring{D}(\nu))$ with $M - \psi(\mathring{D}(\nu)) \times 1$ and includes the sphere bundle $S(F)$ into the disc bundle $D(F)$. The boundary ∂B is then the disjoint union of the images of ψ and i.

The topological manifold B admits a smooth structure for which the given G-action is smooth and such that the embeddings ψ and i are smooth; in the non-equivariant version, the construction is explained for example in Construction 15.10.3 of [**167**]. To ensure that the given G-action on B is smooth, we must insist that the collars involved in the construction are G-equivariant collars, which is possible for example by [**38**, Thm. 21.2].

Now we have to arrange the equivariant reference maps to $\tilde{E}\mathcal{P}$. In fact, this extra data goes along for the ride, as the only homotopical information it encodes is a decomposition of the G-fixed points into two disjoint subspaces, the preimages of the two fixed point of $\tilde{E}\mathcal{P}$. In more detail, we let W be a compact smooth G-manifold, possibly with boundary. Then W admits the structure of a finite G-CW-complex by Illman's triangulation theorem [**82**, Thm. 7.1]. By the proof of Proposition III.3.9, the fixed point map

$$(-)^G \;:\; \mathrm{map}^G(W, \tilde{E}\mathcal{P}) \;\longrightarrow\; \mathrm{map}(W^G, \{0, \infty\})$$

is a weak equivalence and Serre fibration. A continuous map to the discrete space $\{0, \infty\}$ is equivalent to a decomposition into two disjoint open subsets. So altogether we conclude that for every disjoint union decomposition $W^G = A \cup B$ there is a G-map $h : W \longrightarrow \tilde{E}\mathcal{P}$ with $h(A) = \{0\}$ and $h(B) = \{\infty\}$, and any two such maps are equivariantly homotopic. We use this property three times, namely for the G-manifolds B, M and $\bar{M}$.

The G-fixed points of B are a disjoint union of $M^G_\infty \times [0,1]$ and $(M^{(j)} \times [0,1]) \cup_{M^{(j)} \times 1} W$. So there is a G-map $H : B \longrightarrow \tilde{E}\mathcal{P}$ such that $H(M^G_\infty \times [0,1]) = \{\infty\}$ and $H((M^{(j)} \times [0,1]) \cup_{M^{(j)} \times 1} W) = \{0\}$. The triple $(B, H, \psi + i)$ is then a bordism that witnesses the relation $[\![M, H\psi]\!] = [\![\bar{M}, Hi]\!]$ in the group $\widetilde{\mathcal{N}}^G_n(\tilde{E}\mathcal{P})$. The map $Hi : \bar{M} \longrightarrow \tilde{E}\mathcal{P}$ sends all of $\bar{M}^G$ to the fixed point ∞, so Hi is equivariantly homotopic to the constant map with value ∞. By homotopy invariance, the class $[\![\bar{M}, Hi]\!]$ thus vanishes in the reduced bordism group. On the other hand, the G-map $H\psi$ agrees with the original map h on the fixed points M^G. So $H\psi$ is equivariantly homotopic to h, and homotopy invariance yields

$$[\![M, h]\!] \;=\; [\![M, H\psi]\!] \;=\; [\![\bar{M}, H|_{\bar{M}}]\!] \;=\; 0$$

in the group $\widetilde{\mathcal{N}}^G_n(\tilde{E}\mathcal{P})$. This shows that the map Φ_{geom} is injective. □

Remark 3.19. The fact that the geometric fixed point map (3.17) is an isomorphism is ubiquitous in calculations of equivariant bordism groups, and it goes back to Conner and Floyd [**38**]. However, in the classical literature, the reduced equivariant bordism group of $\tilde{E}\mathcal{P}$ usually appears in a different guise,

namely as the group $\mathcal{N}_*^G[\mathcal{A}ll, \mathcal{P}]$ of bordism classes of smooth compact G-manifolds with boundary, but where there are no fixed points on the boundary. A homomorphism

$$\mathcal{N}_*^G[\mathcal{A}ll, \mathcal{P}] \;\longrightarrow\; \widetilde{\mathcal{N}}_n^G(\tilde{E}\mathcal{P}) \tag{3.20}$$

is defined as follows. We let M be a smooth compact G-manifold without G-fixed points on the boundary. The double of M is the smooth closed G-manifold

$$DM \;=\; M \cup_{\partial M} M$$

obtained by gluing two copies of M along their boundary. A smooth structure in the neighborhood of the gluing locus is provided by a choice of G-equivariant collar (see [**38**, Thm. 21.2]). By Illman's theorem [**82**, Cor. 7.2], DM admits the structure of a finite G-CW-complex. Since the original manifold M had no G-fixed points on the boundary, $(DM)^G$ is the disjoint union of two copies of M^G, one from each of the two copies of M in the double. There is thus a continuous G-map $h : DM \longrightarrow \tilde{E}\mathcal{P}$, unique up to equivariant homotopy, that takes the 'left' copy of M^G to the fixed point 0 and the 'right' copy of M^G to the fixed point ∞. The pair (DM, h) is then a singular G-manifold over $\tilde{E}\mathcal{P}$, and it represents a bordism class

$$[DM, h] \quad \in\; \mathcal{N}_n^G(\tilde{E}\mathcal{P})\ .$$

The proof that the map (3.20) is an isomorphism can be found as Satz 3 in [**163**].

For finite groups, Stong shows in [**151**, Cor. 5.1] that the geometric fixed point map

$$\mathcal{N}_*^G[\mathcal{A}ll, \mathcal{P}] \;\longrightarrow\; \bigoplus_{j\geq 0} \mathcal{N}_{n-j}(Gr_j^{G,\perp})$$

is an isomorphism. Combined with tom Dieck's isomorphism (3.20) this gives a different proof of Proposition 3.18 in the special case of finite groups. I was unable to find a convenient reference for Proposition 3.18 in the generality of compact Lie groups, which is the main reason for including a proof.

Example 3.21 (Bordism of manifolds with involution). We look at the geometric isotropy separation sequence in the simplest non-trivial case of the two-element group $G = C_2 = C$, the case originally considered by Conner and Floyd [**38**, Thm. 28.1]. In this case $\mathcal{P} = \{e\}$ consists only of the trivial subgroup, and so $\mathcal{N}_*^C(E\mathcal{P}) = \mathcal{N}_*^C(EC)$ is the bordism ring of manifolds with free C-action. If C acts freely and smoothly on M, then we can form the smooth manifold

$$[-1, 1] \times_C M \;=\; ([-1, 1] \times M)/(x, m) \sim (-x, \tau m)$$

with C-action by $\tau \cdot [x, m] = [-x, \tau m]$; the boundary of this manifold is equivariantly diffeomorphic to the original manifold M. So every C-manifold with free action is null-bordant, and thus the forgetful map $\mathcal{N}_*^C(E\mathcal{P}) \longrightarrow \mathcal{N}_*^C$ is zero.

For every compact Lie group G, the map

$$\mathcal{N}_n^G(EG) \;\longrightarrow\; \mathcal{N}_n(BG)\ , \quad [M, h] \;\longmapsto\; [G\backslash M, G\backslash h]$$

is an isomorphism from the bordism group of G-manifolds with free action to the non-equivariant bordism group of the classifying space $BG = G\backslash EG$. So in the case at hand, $\mathcal{N}_*^C(E\mathcal{P}) = \mathcal{N}_*^C(EC)$ is isomorphic to $\mathcal{N}_*(\mathbb{R}P^\infty)$.

Since there is only one non-trivial irreducible C-representation, the sign representation σ, every linear subspace of $\mathcal{U}_C^\perp$ is C-invariant. Hence $Gr_j^{C,\perp}$ is just a Grassmannian of j-planes in an infinite dimensional $\mathbb{R}$-vector space, hence a classifying space of the orthogonal group $BO(j)$. The long exact mapping cone sequence of the C-map $EC \longrightarrow *$ decomposes into a short exact sequence

$$0 \;\longrightarrow\; \mathcal{N}_*^C \;\xrightarrow{\Phi_{\mathrm{geom}}}\; \bigoplus_{j\geq 0} \mathcal{N}_{*-j}(BO(j)) \;\xrightarrow{J}\; \mathcal{N}_{*-1}(\mathbb{R}P^\infty) \;\longrightarrow\; 0\ .$$

Here we used Proposition 3.18 to replace the group $\widetilde{\mathcal{N}}_*^C(\tilde{E}C)$ by the direct sum of non-equivariant bordism groups. The map J sends $[\eta : F \longrightarrow BO(k)]$ to the class of $P(\eta)$, the projectivized bundle, equipped with the map to $\mathbb{R}P^\infty$ that classifies the tautological line bundle over $P(\eta)$.

One can deduce from the above short exact sequence that $\mathcal{N}_*^C$ is free as a module over the non-equivariant bordism ring $\mathcal{N}_*$. This is in fact true much more generally: Stong shows in [**151**, Prop. 9.4] that $\mathcal{N}_*^G(X)$ is free as an $\mathcal{N}_*$-module for every finite group G and every G-space X. The paper [**4**] by Alexander exhibits an explicit $\mathcal{N}_*$-basis of $\mathcal{N}_*^C$. Some basic C-bordism classes are represented by the projective spaces $\mathbb{R}P^n$ equipped with the involution given by

$$\tau \cdot [x_0 : x_1 : \ldots : x_n] \;=\; [-x_0 : x_1 : \ldots : x_n] \;.$$

The fixed points of this involution are a copy of $\mathbb{R}P^{n-1}$ (formed by the points of the form $[0 : x_1 : \ldots : x_n]$) and the isolated fixed point $[1 : 0 : \ldots : 0]$. We denote by

$$y_n \;=\; [\mathbb{R}P^n, \tau] \;\in\; \mathcal{N}_n^C$$

the bordism class of this C-manifold. An $\mathcal{N}_*$-linear map

$$\Gamma \;:\; \mathcal{N}_*^C \;\longrightarrow\; \mathcal{N}_{*+1}^C$$

of degree 1 is given by sending the class of a manifold M with involution $\tau : M \longrightarrow M$ to the manifold

$$S(\mathbb{C}) \times_C M \;=\; (S(\mathbb{C}) \times M)/(z, m) \sim (-z, \tau m) \;.$$

So $S(\mathbb{C}) \times_C M$ is diffeomorphic to the mapping torus of the involution τ. The involution on this manifold is by

$$\tau \;:\; S(\mathbb{C}) \times_C M \;\longrightarrow\; S(\mathbb{C}) \times_C M \;, \quad \tau \cdot [z, m] \;=\; [\bar{z}, \tau m] \;.$$

In our present notation, the operator can also be expressed as

$$\Gamma \;=\; \mathrm{res}_C^{O(2)} \circ \mathrm{ind}_{O(1)\times O(1)}^{O(2)} \circ p^* \;,$$

where we embed C into $O(2)$ as a reflection, and where $p : O(1) \times O(1) \longrightarrow C$ is the epimorphism with kernel $e \times O(1)$.

Alexander shows in [**4**, Thm. 1.1] that the multiplicative unit 1 together with the classes

$$\Gamma^n(y_{i_1} \cdot \ldots \cdot y_{i_r})$$

for all $n \geq 0$, $r \geq 1$ and $i_j \geq 2$ form a basis of $\mathcal{N}_*^C$ as a module over $\mathcal{N}_*$. Besides the trivial group and C, the equivariant bordism groups have been calculated for various finite abelian groups, see [**12, 13, 14, 52**].

Now we work our way towards the equivariant Thom-Pontryagin construction that assigns to every equivariant bordism class over a G-space X an equivariant homology class in $\mathbf{mO}_*^G(X_+)$. We break the construction up into two steps, and we first discuss the *normal class*, a basic invariant associated to a closed smooth G-manifold.

Construction 3.22 (Normal class of a G-manifold). To every smooth closed G-manifold M we associate a *normal class*

$$\langle M \rangle \;\in\; \mathbf{MGr}_0^G(M_+) \;.$$

This class records the equivariant homotopical information in the stable normal bundle of M, and it is the geometric input for the Thom-Pontryagin map to equivariant $\mathbf{mO}$-homology. If M has dimension m, then the class lives in the homogeneous summand $\mathbf{MGr}^{[-m]}$ of $\mathbf{MGr}$.

The construction starts from the Mostow-Palais embedding theorem [**115, 122**] that provides a smooth G-equivariant embedding $i : M \longrightarrow V$, for some G-representation V. We can assume without loss of generality that V is a subrepresentation of the chosen complete G-universe $\mathcal{U}_G$. We use the inner product on V to the define the normal bundle ν of the embedding at $x \in M$ by

$$\nu_x \;=\; V - (di)(T_x M) \;,$$

the orthogonal complement of the image of the tangent space T_xM in V. By multiplying with a suitably large scalar, if necessary, we can assume that the embedding is *wide* in the sense that the exponential map

$$D(\nu) \longrightarrow V\ , \quad (x,v) \longmapsto i(x) + v$$

is injective on the unit disc bundle of the normal bundle, and hence a closed G-equivariant embedding. The image of this map is a tubular neighborhood of radius 1 around $i(M)$, and it determines a G-equivariant Thom-Pontryagin collapse map

$$c_M \ : \ S^V \longrightarrow Th(Gr(V)) \wedge M_+ \ = \ \mathbf{MGr}(V) \wedge M_+ \tag{3.23}$$

as follows: every point outside of the tubular neighborhood is sent to the basepoint, and a point $i(x) + v$, for $(x, v) \in D(\nu)$, is sent to

$$c_M(i(x) + v) \ = \ \left(\frac{v}{1 - |v|}, \nu_x \right) \wedge x\ .$$

The normal class $\langle M \rangle$ is now defined as the homotopy class of the collapse map c_M.

Proposition 3.24. *The normal class of a smooth closed G-manifold is independent of the choice of wide embedding into a G-representation.*

Proof. If we enlarge the embedding $i : M \longrightarrow V$ by postcomposition with a direct summand embedding $(0, -) : V \longrightarrow U \oplus V$, then the collapse map associated to the composite embedding $(0, -) \circ i$ is equivariantly homotopic to the composite

$$S^{U \oplus V} \xrightarrow{S^U \wedge c_M} S^U \wedge \mathbf{MGr}(V) \wedge M_+ \xrightarrow{\sigma_{U,V}} \mathbf{MGr}(U \oplus V) \wedge M_+\ .$$

So the resulting class in $\mathbf{MGr}_0^G(M_+)$ does not change. Two classes based on two different wide embeddings $i : M \longrightarrow V$ and $j : M \longrightarrow W$ can be compared by passing to $V \oplus W$; in this larger representation, the map

$$M \times [0,1] \longrightarrow V \oplus W\ , \quad (m,t) \longmapsto (t \cdot i(m),\ (1-t) \cdot j(m))$$

is a smooth isotopy through wide embeddings. This isotopy induces a homotopy between the two collapse maps and shows that altogether the normal class $\langle M \rangle$ is independent of the wide embedding. □

Part (ii) of the following proposition refers to an external multiplication morphism

$$\begin{aligned} \mu_{A,B} \ : \ (\mathbf{MGr} \wedge A_+) \wedge (\mathbf{MGr} \wedge B_+) \ &\cong (\mathbf{MGr} \wedge \mathbf{MGr}) \wedge (A \times B)_+ \\ &\xrightarrow{\mu \wedge (A \times B)_+} \mathbf{MGr} \wedge (A \times B)_+ \end{aligned}$$

where A and B are two G-spaces and $\mu : \mathbf{MGr} \wedge \mathbf{MGr} \longrightarrow \mathbf{MGr}$ the multiplication morphism. In part (iv) we consider the k-th power M^k of a G-manifold M as a $(\Sigma_k \wr G)$-manifold via the action

$$(\sigma;\, g_1, \dots, g_k) \cdot (x_1, \dots, x_k) \ = \ (g_{\sigma^{-1}(1)} x_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(k)} x_{\sigma^{-1}(k)})\ .$$

Proposition 3.25. *Let G be a compact Lie group and M and N be smooth closed G-manifolds.*

(i) *Let $i^1 : M \longrightarrow M \cup N$ and $i^2 : N \longrightarrow M \cup N$ denote the inclusions into a disjoint union. Then the relation*

$$\langle M \cup N \rangle \ = \ i^1_* \langle M \rangle \ + \ i^2_* \langle N \rangle$$

holds in the group $\mathbf{MGr}_0^G((M \cup N)_+)$.

(ii) *The relation*

$$\langle M \times N \rangle \ = \ (\mu_{M,N})_* (\langle M \rangle \times \langle N \rangle)$$

holds in the group $\mathbf{MGr}_0^G((M \times N)_+)$.

(iii) *For every continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups, every smooth closed G-manifold M the relation*

$$\langle \alpha^* M \rangle \;=\; \alpha^* \langle M \rangle$$

holds in $\mathbf{MGr}_0^K(\alpha^*(M)_+)$.

(iv) *For every $k \geq 0$, the relation*

$$\langle M^k \rangle \;=\; P^k \langle M \rangle$$

holds in the group $\mathbf{MGr}_0^{\Sigma_k \wr G}(M^k)$.

(v) *Let B be a compact smooth G-manifold with boundary ∂B. Then the class $\langle \partial B \rangle$ is in the kernel of the homomorphism*

$$(j \wedge \iota_+)_* \;:\; \mathbf{MGr}_0^G(\partial B_+) \;\longrightarrow\; (\mathrm{sh}\,\mathbf{MGr})_0^G(B_+)\ ,$$

where $\iota : \partial B \longrightarrow B$ is the inclusion.

Proof. (i) We let $p^1 : (M \cup N)_+ \longrightarrow M_+$ and $p^2 : (M \cup N)_+ \longrightarrow N_+$ denote the two projections. Our first claim is the relation

$$p^1_* \langle M \cup N \rangle \;=\; \langle M \rangle\ .$$

To see this we choose any wide smooth equivariant embedding $i : M \cup N \longrightarrow V$ and observe that the composite

$$S^V \;\xrightarrow{c_{M \cup N}}\; \mathbf{MGr}(V) \wedge (M \cup N)_+ \;\xrightarrow{\mathbf{MGr}(V) \wedge p^1}\; \mathbf{MGr}(V) \wedge M$$

is on the nose the collapse map for M based on the restriction of the embedding i to M. We then obtain

$$p^1_* \langle M \cup N \rangle \;=\; \langle M \rangle \;=\; p^1_*(i^1_* \langle M \rangle + i^2_* \langle N \rangle)\ ;$$

the second relation uses that $p^1 \circ i^1$ is the identity, $p^2 \circ i^1$ is the trivial map and p^1_* is additive. The analogous argument shows that $p^2_* \langle M \cup N \rangle \;=\; p^2_*(i^1_* \langle M \rangle + i^2_* \langle N \rangle)$. Since equivariant homotopy groups are additive on wedges, the map

$$(p^1_*, p^2_*) \;:\; \mathbf{MGr}_0^G((M \cup N)_+) \;\longrightarrow\; \mathbf{MGr}_0^G(M_+) \times \mathbf{MGr}_0^G(N_+)$$

is bijective, and this proves the claim.

(ii) We choose smooth equivariant wide embeddings

$$i \;:\; M \;\longrightarrow\; V \qquad \text{and} \qquad j \;:\; N \;\longrightarrow\; W$$

into G-representations. The product map

$$i \times j \;:\; M \times N \;\longrightarrow\; V \oplus W$$

is then another smooth equivariant wide embedding that we use for the Thom-Pontryagin construction of $M \times N$. The normal bundle of $i \times j$ is the exterior direct sum of the normal bundles of i and j. The unit disc $D(V \oplus W)$ of the direct sum is contained in the product $D(V) \times D(W)$ of the unit discs, so the exponential tubular neighborhood for $i \times j$ is contained in the product of the exponential tubular neighborhoods for i and j. The collapse map

$$S^{V \oplus W} \;\xrightarrow{c_{M \times N}}\; \mathbf{MGr}(V \oplus W) \wedge (M \times N)_+$$

is equivariantly homotopic to the composite

$$S^V \wedge S^W \;\xrightarrow{c_M \wedge c_N}\; (\mathbf{MGr}(V) \wedge M_+) \wedge (\mathbf{MGr}(W) \wedge N_+) \;\xrightarrow{\mu^{V,W}_{M,N}}\; \mathbf{MGr}(V \oplus W) \wedge (M \times N)_+\ .$$

This shows the desired relation. Part (iii) is straightforward from the definitions.

(iv) We choose a wide smooth equivariant embedding $i : M \longrightarrow V$ into a G-representation. Then

$$i^k \;:\; M^k \;\longrightarrow\; V^k$$

is a $(\Sigma_k \wr G)$-equivariant wide smooth embedding that we use to calculate the class $\langle M^k \rangle$. The collapse map

$$c_{M^k} \;:\; S^{V^k} \;\longrightarrow\; Th(Gr(V^k)) \wedge M^k_+$$

based on i^k is $(\Sigma_k \wr G)$-equivariantly homotopic to the composite

$$\begin{aligned} S^{V^k} &\xrightarrow{(c_M)^{\wedge k}} (\mathbf{MGr}(V) \wedge M_+)^{\wedge k} \\ &\xrightarrow{\text{shuffle}} \mathbf{MGr}(V)^{\wedge k} \wedge M_+^k \xrightarrow{\mu^{\mathbf{MGr}}_{V,\dots,V} \wedge M_+^k} \mathbf{MGr}(V^k) \wedge M_+^k \ . \end{aligned}$$

This latter composite represents the power operation $P^k\langle M\rangle$, so altogether this shows the desired relation.

(v) We let $C(j \wedge B_+)$ denote the mapping cone of the morphism $j \wedge B_+ : \mathbf{MGr} \wedge B_+ \longrightarrow \operatorname{sh} \mathbf{MGr} \wedge B_+$. We define a relative normal class

$$\langle B\rangle^{\text{rel}} \ \in \ \pi_1^G(C(j \wedge B_+))$$

such that the relation

$$\partial(\langle B\rangle^{\text{rel}}) \ = \ (\mathbf{MGr} \wedge \iota_+)_*\langle \partial B\rangle \tag{3.26}$$

holds in the group $\pi_0^G(\mathbf{MGr} \wedge B_+)$, where ∂ is the connecting homomorphism in the long exact homotopy group sequence of the mapping cone (see (1.34) of Chapter III). Because two consecutive maps in the long exact homotopy group sequence compose to zero, we can then conclude that

$$(j \wedge \iota_+)_*\langle \partial B\rangle \ = \ (j \wedge B_+)_*(\partial(\langle B\rangle^{\text{rel}})) \ = \ 0 \ .$$

The second relation is the additivity of part (i).

It remains to construct the class $\langle B\rangle^{\text{rel}}$ and establish the relation (3.26). We choose an equivariant collar, i.e., a smooth G-equivariant embedding

$$c \ : \ \partial B \times [0,1) \ \longrightarrow \ B$$

such that $c(-,0) : \partial B \longrightarrow B$ is the inclusion and the image of c is an open neighborhood of the boundary inside B. Then we choose a smooth function

$$\kappa \ : \ [0,1] \ \longrightarrow \ [0,1]$$

that is the identity on $[0,1/3]$, identically 1 on $[2/3,1]$ and whose restriction to $[0,2/3)$ is injective. We define the smooth function

$$\psi \ : \ [0,1) \ \longrightarrow \ [0,1) \qquad \text{by} \qquad \psi(t) \ = \ \frac{\kappa(t)-1}{t-1} \ ;$$

then $\psi(t) = 1$ for $t \in [0,1/3]$ and $\kappa(t) = 0$ for $t \in [2/3,1)$.

The Mostow-Palais embedding theorem provides a wide smooth G-equivariant embedding $j : B \longrightarrow V$, for some G-representation V. Then the smooth G-map

$$i \ : \ B \ \longrightarrow \ V \oplus \mathbb{R}$$

defined by

$$i(b) \ = \ \begin{cases} (\psi(t) \cdot j(x) + (1-\psi(t)) \cdot j(b),\, \kappa(t)) & \text{for } b = c(x,t) \text{ with } (x,t) \in \partial B \times [0,1), \text{ and} \\ (j(b),1) & \text{for } b \notin c(B \times [0,1)). \end{cases}$$

is a new wide smooth equivariant embedding which satisfies

$$i(\partial B) \ \subset \ V \times \{0\}$$

and which is 'orthogonal to V near the boundary', i.e., the set $U = c(\partial B \times [0,1/3))$ is an open neighborhood of ∂B in B, and

$$i(U) \ = \ i(\partial B) \times [0,1) \ . \tag{3.27}$$

Since the embedding i is wide, the exponential map

$$D(\nu) \ \longrightarrow \ V \oplus \mathbb{R} \ , \quad (x,v,t) \ \longmapsto \ i(x) \ + \ (v,t)$$

is injective on the unit disc bundle of the normal bundle ν of j, and hence a closed G-equivariant embedding. We define a continuous map

$$\kappa \ : \ D(\nu) \ \longrightarrow \ C\,(j(V)\colon \mathbf{MGr}(V) \longrightarrow \mathbf{MGr}(V \oplus \mathbb{R})) \wedge B_+$$

to the reduced mapping cone of the embedding $j(V)$ of $\mathbf{MGr}(V) \wedge B_+$ into $\mathbf{MGr}(V \oplus \mathbb{R}) \wedge B_+$ as follows. We consider $(b, v, t) \in D(\nu)$ where $b \in B$ and $(v, t) \in V \oplus \mathbb{R}$ is normal to $i(B)$ at $i(b)$. If $b \in U$, then the normal vector must lie in $V \oplus 0$, i.e., $t = 0$. The map κ then takes $(b, v, 0)$ to

$$\left[\left(\frac{v}{1-|v|}, \nu_b\right), i_2(b)\right] \wedge b$$

in the cone of $\mathbf{MGr}(V) \wedge B_+$, where $i_2(b) \in [0, 1)$ is the second component of $i(b)$. For $b \notin U$, the map κ sends (b, v, t) to $\left(\frac{(v,t)}{1-|v,t|}, \nu_b\right) \wedge b$ in $\mathbf{MGr}(V \oplus \mathbb{R}) \wedge B_+$.

The total space of the disc bundle $D(\nu)$ is a topological manifold with boundary, and its boundary is the union of the sphere bundle $S(\nu)$ and the subspace $D(\nu)|_{\partial B}$, the part sitting over the boundary of B. The map κ sends the subspace $D(\nu)|_{\partial B}$ to the cone point in the mapping cone, and it sends the sphere bundle $S(\nu)$ to the basepoint. So κ sends the entire boundary of $D(\nu)$ to the basepoint of the reduced mapping cone. So we can extend κ continuously to $S^{V \oplus \mathbb{R}}$ by sending the complement of $D(\nu)$ to the basepoint. The result is a continuous based G-map

$$\tilde{c}_B \ : \ S^{V \oplus \mathbb{R}} \ \longrightarrow \ C\big(j(V) \wedge B_+ : \mathbf{MGr}(V) \wedge B_+ \longrightarrow (\operatorname{sh} \mathbf{MGr})(V) \wedge B_+\big) \ .$$

The map $\tilde{c}_B$ represents the relative normal class $\langle B \rangle^{\mathrm{rel}}$ in the group $\pi_1^G(C(j \wedge B_+))$.

It remains to establish the relation (3.26). The composite

$$S^{V \oplus \mathbb{R}} \ \xrightarrow{\tilde{c}_B} \ C(i(V) \wedge B_+) \ \xrightarrow{p} \ \mathbf{MGr}(V) \wedge B_+ \wedge S^1$$

is equivariantly homotopic to the map $((\mathbf{MGr}(V) \wedge \iota_+) \circ c_{\partial B}) \wedge S^1$, where

$$c_{\partial B} \ : \ S^V \ \longrightarrow \ \mathbf{MGr}(V) \wedge (\partial B)_+$$

is the collapse map for ∂B based on the restriction of i to an embedding $\partial B \longrightarrow V$. Thus

$$p_*(\langle B \rangle^{\mathrm{rel}}) \ = \ \iota_* \langle \partial B \rangle \wedge S^1$$

in the group $\pi_1^G(\mathbf{MGr} \wedge B_+ \wedge S^1)$. The relation (3.26) thus follows from the definition of the connecting homomorphism (compare (1.34) of Chapter IV) as the composite of p_* and the inverse suspension isomorphism. □

The inverse Thom class

$$\tau_{H,W} \ \in \ \mathbf{MGr}_0^H(S^W)$$

of an H-representation W was defined in (2.2). The next theorem shows how the normal class of an induced equivariant manifold $G \times_H M$ is determined by the normal class of the H-manifold M and the inverse Thom class of the tangent H-representation $L = T_H(G/H)$. The Wirthmüller isomorphism

$$\mathrm{Wirth}_H^G \ : \ \mathbf{MGr}_0^G((G \times_H M)_+) \ \xrightarrow{\cong} \ \mathbf{MGr}_0^H(M_+ \wedge S^L)$$

was established in Theorem III.2.14.

THEOREM 3.28. *Let H be a closed subgroup of a compact Lie group G and M a closed smooth H-manifold. Then the relation*

$$\mathrm{Wirth}_H^G \langle G \times_H M \rangle \ = \ \langle M \rangle \wedge \tau_{H,L}$$

holds in the group $\mathbf{MGr}_0^H(M_+ \wedge S^L)$, where $L = T_H(G/H)$ is the tangent H-representation.

Proof. The Wirthmüller map is defined as the composite

$$\mathbf{MGr}_0^G((G\times_H M)_+) \xrightarrow{\mathrm{res}_H^G} \mathbf{MGr}_0^H((G\times_H M)_+) \xrightarrow{(\mathbf{MGr}\wedge l_M)_*} \mathbf{MGr}_0^H(M_+\wedge S^L)\ .$$

Here $l_H^G : G \longrightarrow S^L\wedge H_+$ is the H^2-equivariant collapse maps for the embedding of H into G, and $l_M : (G\times_H M)_+ \longrightarrow M_+\wedge S^L$ is the composite

$$(G\times_H M)_+ \xrightarrow{l_H^G\wedge_H M} S^L\wedge M_+ \ \cong\ M_+\wedge S^L\ ,$$

compare Construction III.2.2. So the claim is equivalent to the relation

$$(l_H^G\wedge_H M)_*(\mathrm{res}_H^G\langle G\times_H M\rangle) \ =\ \tau_{H,L}\wedge\langle M\rangle$$

in the group $\mathbf{MGr}_0^H(S^L\wedge M_+)$.

We choose a G-equivariant wide smooth embedding $i : G/H \longrightarrow V$ into a G-representation V; we also choose an H-equivariant wide smooth embedding $j : M \longrightarrow W$ into an H-representation underlying some other G-representation. Then the map

$$\psi\ :\ G\times_H M \ \longrightarrow\ V\oplus W\ ,\quad [g,m]\ \longmapsto\ (i(gH), g\cdot j(m))$$

is a G-equivariant wide smooth embedding. We base the collapse map for the G-manifold $G\times_H M$ on the embedding ψ.

The differential at the coset H of the embedding i is a linear embedding

$$L\ =\ T_H(G/H)\ \xrightarrow{di}\ V\ ;$$

we define a scalar product on L so that this embedding becomes isometric. Now we choose a slice as in the construction of the map l_H^G, i.e., a smooth embedding

$$s\ :\ D(L)\ \longrightarrow\ G$$

of the unit disc of L with $s(0)=1$, such that

$$s(h\cdot l)\ =\ h\cdot s(l)\cdot h^{-1}$$

for all $(h,l)\in H\times D(L)$, and such that the differential at 0 of the composite

$$D(L)\ \xrightarrow{s}\ G\ \xrightarrow{\mathrm{proj}}\ G/H$$

is the identity of L. After scaling the slice, if necessary, the map

$$\bar s\ :\ D(L)\times H\ \longrightarrow\ G\ ,\quad (v,h)\ \longmapsto\ s(v)\cdot h\ ;$$

is a smooth embedding whose image is an H^2-equivariant tubular neighborhood of H inside G. Here the group H^2 acts on the source by

$$(h',h)\cdot(v,\bar h)\ =\ (h'v,\, h'\bar h h^{-1})\ .$$

The map $l_H^G : G\longrightarrow S^L\wedge H_+$ is then the collapse map for this tubular neighborhood.

The following H-equivariant composite is thus a representative for the class $(l_H^G\wedge_H M)_*(\mathrm{res}_H^G\langle G\times_H M\rangle)$

$$S^{V\oplus W}\ \xrightarrow{c_{G\times_H M}}\ \mathbf{MGr}(V\oplus W)\wedge(G\times_H M_+) \xrightarrow{\mathrm{Id}\wedge(l_H^G\wedge_H M)}\ \mathbf{MGr}(V\oplus W)\wedge S^L\wedge M_+\ .$$

The map l_H^G takes the complement of the tubular neighborhood $\bar s(D(L)\times H)$ to the basepoint. So the map

$$l_H^G\wedge_H M_+\ :\ (G\times_H M)_+\ \longrightarrow\ S^L\wedge M_+$$

takes the complement of the subset $\bar S(D(L)\times H)\times_H M$ to the basepoint. The composite

$$D(L)\times M\ \xrightarrow{\bar s}\ G\ \xrightarrow{\psi}\ V\oplus W$$

is given by the formula

$$(l,m)\ \longmapsto\ (i(s(l)\cdot H),\, s(l)\cdot j(m))\ .$$

We define a homotopy of smooth wide H-equivalence embeddings

$$[0,1] \times D(L) \times M \;\longrightarrow\; V \oplus W \quad \text{by} \quad (t,l,m) \;\longmapsto\; (i(s(l)\cdot H),\, s(tl)\cdot j(m))\ .$$

For every time $t \in [0,1]$, the corresponding embedding has an associated collapse map

$$S^{V\oplus W} \;\longrightarrow\; \mathbf{MGr}(V\oplus W) \wedge S^L \wedge M_+\ ,$$

where we identified S^L with $D(L)/S(L)$ by scaling. So the homotopy induces a homotopy of based H-equivariant maps between the above representative for $(l_H^G \wedge_H M)_*(\mathrm{res}_H^G\langle G \times_H M\rangle)$ and the collapse map for the product embedding

$$D(L) \times M \;\longrightarrow\; V \oplus W \quad \text{by} \quad (l,m) \;\longmapsto\; (i(s(l)\cdot H),\, j(m))\ . \qquad \square$$

Example 3.29. We identify the normal classes of some equivariant manifolds in terms of classes that were previously defined. In the case where $M = *$ is a 0-dimensional G-manifold with a single point, Theorem 3.28 specializes to the relation

$$\mathrm{Wirth}_H^G\langle G/H\rangle \;=\; \tau_{H,L}$$

in the group $\mathbf{MGr}_0^H(S^L)$, where $L = T_H(G/H)$ is the tangent H-representation. In other words, the normal class of the homogeneous G-manifold G/H is the inverse, under the Wirthmüller isomorphism, of the inverse Thom class.

We let V be a representation of a compact Lie group G. Then the unit sphere $S(V\oplus\mathbb{R})$ in $V\oplus\mathbb{R}$ is a smooth closed G-manifold. We recall that

$$\Pi_V \;:\; S(V\oplus\mathbb{R}) \;\longrightarrow\; S^V\ , \quad (v,\lambda) \;\longmapsto\; \frac{v}{1-\lambda}$$

is the G-equivariant stereographic projection. We claim that

$$\langle S(V\oplus\mathbb{R})\rangle \;=\; (\mathbf{MGr}\wedge\Pi_V^{-1})_*(\tau_{G,V})\ . \tag{3.30}$$

Since both sides of equation (3.30) commute with restriction along continuous homomorphisms, it suffices to show the relation for the tautological m-dimensional $O(m)$-representation ν_m. The composite

$$O(m+1)/O(m) \;\xrightarrow{\psi}\; S(\nu_m\oplus\mathbb{R}) \;\xrightarrow{\Pi_{\nu_m}}\; S^{\nu_m}$$

is $O(m)$-equivariantly homotopic to the map $l_{O(m)}^{O(m+1)} : O(m+1)/O(m) \longrightarrow S^{\nu_m}$ that appears in the Wirthmüller isomorphism, where $\psi(A\cdot O(m)) = A\cdot(0,\dots,0,1)$. So we can argue:

$$\begin{aligned}
(\mathbf{MGr}\wedge\Pi_{\nu_m})_*\langle S(\nu_m\oplus\mathbb{R})\rangle \;&=\; (\mathbf{MGr}\wedge\Pi_{\nu_m})_*((\mathbf{MGr}\wedge\psi_*)\langle\mathrm{res}_{O(m)}^{O(m+1)}\, O(m+1)/O(m)\rangle)\\
&=\; (\mathbf{MGr}\wedge l_{O(m)}^{O(m+1)})_*(\mathrm{res}_{O(m)}^{O(m+1)}\langle O(m+1)/O(m)\rangle)\\
&=\; \mathrm{Wirth}_{O(m)}^{O(m+1)}\langle O(m+1)/O(m)\rangle \;=\; \tau_{O(m),\nu_m}\ .
\end{aligned}$$

Inverting $(\mathbf{MGr}\wedge\Pi_{\nu_m})_*$ gives the desired relation (3.30) in the universal example.

Construction 3.31 (Equivariant Thom-Pontryagin construction). The equivariant Thom-Pontryagin construction defines a natural transformation of G-homology theories

$$\Theta^G \;=\; \Theta^G(X) \;:\; \widetilde{\mathcal{N}}_*^G(X) \;\longrightarrow\; \mathbf{mO}_*^G(X)\ , \tag{3.32}$$

as we now recall. We let (M,h) be an m-dimensional singular G-manifold over a based G-space X. The way we have set things up, all the geometry is already encoded in the normal class $\langle M\rangle \in \mathbf{MGr}_0^G(M_+)$; the rest is a formal procedure: we pushing the normal class forward along the morphism $b : \mathbf{MGr} \longrightarrow \mathbf{mOP}$ and use the periodicity of $\mathbf{mOP}$ to move into the homogeneous summand $\mathbf{mO}$ of degree 0. While the normal class is *not* yet a bordism invariant, pushing it forward to $\mathbf{mOP}$ makes it so, see Proposition 3.34 below.

The periodicity class $t \in \pi^e_{-1}(\mathbf{mOP}^{[-1]})$ was defined in (2.32), and we let $\sigma \in \pi^e_1(\mathbf{mOP}^{[1]})$ be its inverse. We define

$$\Theta^G[M,h] \;=\; (b \wedge h)_*\langle M\rangle \cdot p_G^*(\sigma^m) \;\in\; \mathbf{mO}^G_m(X)\ , \tag{3.33}$$

i.e., we take the image of the normal class of M under the homomorphism

$$(b \wedge h)_* \;:\; \mathbf{MGr}^G_0(M_+) \;\longrightarrow\; \mathbf{mOP}^G_0(X)$$

and multiply by the unit $p_G^*(\sigma^m)$ in $\pi^G_m(\mathbf{mOP}^{[m]})$. Since m has dimension m, the normal class lies in the homogeneous summand $\mathbf{MGr}^{[-m]}$, whereas σ^m lies in the summand $\mathbf{mOP}^{[m]}$; so the product indeed lies in the homogeneous degree 0 summand $\mathbf{mO} = \mathbf{mOP}^{[0]}$.

Proposition 3.34. *The class $\Theta^G[M,h]$ in $\mathbf{mO}^G_m(X)$ only depends on the bordism class of the singular G-manifold (M,h).*

Proof. With start by considering the singular G-manifold $(\partial B, \iota : \partial B \longrightarrow B)$, where B is some smooth compact G-manifold with boundary ∂B. The square

$$\begin{tikzcd}
\mathbf{MGr} \arrow[r, "j_{\mathbf{MGr}}"] \arrow[d, "d"'] & \operatorname{sh}\mathbf{MGr} \arrow[d, "\operatorname{sh} d"] \\
\mathbf{mOP} \arrow[r, "\simeq", "j_{\mathbf{mOP}}"'] & \operatorname{sh}\mathbf{mOP}
\end{tikzcd}$$

commutes and the lower horizontal morphism is a homotopy equivalence of orthogonal spectra; so

$$(j_{\mathbf{mOP}} \wedge B_+)_*((b \wedge \iota_+)_*\langle \partial B\rangle) \;=\; (\operatorname{sh} b \wedge B_+)_*((j_{\mathbf{MGr}} \wedge \iota_+)_*\langle \partial B\rangle) \;=\; 0\ ,$$

using Proposition 3.25 (v). Since $j_{\mathbf{mOP}}$ is a homotopy equivalence, this implies $(b \wedge \iota_+)_*\langle \partial B\rangle = 0$. Now we let $(M, h : M \longrightarrow X)$ be any singular G-manifold that is null-bordant. We choose a null-bordism $(B, H : B \longrightarrow X, \psi : M \cong \partial B)$, so that $H \circ \iota \circ \psi = h$.

$$(b \wedge h)_*\langle M\rangle \;=\; (b \wedge (H \circ \iota \circ \psi))_*\langle M\rangle \;=\; (\mathbf{mOP} \wedge H)_*((b \wedge \iota_+)_*\langle \partial B\rangle) \;=\; 0\ .$$

Multiplying by $p_G^*(\sigma^m)$ gives that $\Theta^G[M,h] = 0$. Since the normal class is additive on disjoint unions (Proposition 3.25 (i)), naturality then implies that $\Theta^G[M,h]$ only depends on the bordism class of (M,h). □

Example 3.35. We let G be a compact Lie group and V a G-representation of dimension m. We claim that then

$$\Theta^G(d_{G,V}) \;=\; \tilde{\tau}_{G,V} \tag{3.36}$$

in the group $\mathbf{mO}^G_m(S^V)$. In other words, the Thom-Pontryagin construction matches the distinguished geometric bordism class $d_{G,V}$ in $\widetilde{\mathcal{N}}^G_m(S^V)$ with the shifted inverse Thom class in the Thom spectrum $\mathbf{mO}$.

Indeed, the class $d_{G,V}$ is represented by the singular G-manifold $(S(V \oplus \mathbb{R}), \Pi_V)$, where $\Pi_V : S(V \oplus \mathbb{R}) \longrightarrow S^V$ is the stereographic projection. So

$$\begin{aligned}
\Theta^G(d_{G,V}) \;&=\; (b \wedge \Pi_V)_*\langle S(V \oplus \mathbb{R})\rangle \cdot p_G^*(\sigma^m) \\
&\overset{(3.30)}{=}\; (b \wedge \Pi_V)_*((\mathbf{MGr} \wedge \Pi_V^{-1})_*(\tau_{G,V})) \cdot p_G^*(\sigma^m) \\
&=\; (b \wedge S^V)_*(\tau_{G,V}) \cdot p_G^*(\sigma^m) \;=\; \tilde{\tau}_{G,V}\ .
\end{aligned}$$

The next theorem says, roughly speaking, that the refined Thom-Pontryagin construction is compatible 'with all global structure'. There is one caveat, though, namely in how the geometric induction in equivariant bordism compares with the homotopy theoretic transfer. Indeed, the geometric induction increases the dimension by the dimension of G/H, whereas the Wirthmüller isomorphism increases the dimension in a

twisted way, namely by the sphere of the tangent representation $L = T_H(G/H)$ of H in G. Multiplication by the inverse Thom class $\tau_{H,L}$ is needed to compensate this 'twist' on the homotopy theory side. In the special case where H has finite index in G, then the tangent representation is zero, so in this special case $\tau_{H,L} = 1$ and part (v) of the following theorem specializes to the simpler relation

$$\Theta^G(G \times_H y) \;=\; G \ltimes_H \Theta^H(y)\ .$$

THEOREM 3.37. (i) *The Thom-Pontryagin map Θ^G is additive.*

(ii) *The Thom-Pontryagin map is multiplicative, i.e., for all classes $x \in \widetilde{\mathcal{N}}_m^G(X)$ and $y \in \widetilde{\mathcal{N}}_n^G(Y)$, the relation*

$$\Theta^G(x \wedge y) \;=\; \Theta^G(x) \wedge \Theta^G(y)$$

holds in $\mathbf{mO}_{m+n}^G(X \wedge Y)$.

(iii) *The Thom-Pontryagin map is compatible with the boundary maps in mapping cone sequences in equivariant bordism and equivariant $\mathbf{mO}$-homology, i.e., Θ^G is a transformation of equivariant homology theories.*

(iv) *For every continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups, every based G-space X and all $x \in \widetilde{\mathcal{N}}_m^G(X)$ the relation*

$$\Theta^K(\alpha^*(x)) \;=\; \alpha^*(\Theta^G(x))$$

holds in $\mathbf{mO}_m^K(\alpha^*(X))$.

(v) *If H is a closed subgroup of G, then for every H-space Y and all $y \in \mathcal{N}_m^G(Y)$, the relation*

$$\Theta^G(G \times_H y) \;=\; G \ltimes_H (\Theta^H(y) \wedge \bar{\tau}_{H,L})$$

holds in $\mathbf{mO}_{m+d}^G((G \times_H Y)_+)$, where $L = T_H(G/H)$ is the tangent H-representation and $d = \dim(G/H)$.

PROOF. (i) The two functors

$$X \;\longmapsto\; \widetilde{\mathcal{N}}_n^G(X) \qquad \text{and} \qquad X \longmapsto\; \mathbf{mO}_n^G(X)$$

from the category of based G-spaces to the category of abelian groups are reduced and additive. Proposition II.2.13 thus shows that the Thom-Pontryagin map is additive.

Part (ii) is a formal consequence of the multiplicativity of the normal classes formulated in Proposition 3.25 (ii). We consider singular G-manifolds $(M, h : M \longrightarrow X)$ and $(N, g : N \longrightarrow Y)$. The class $[M, h] \wedge [N, g]$ is then represented by the singular G-manifolds $(M \times N, q \circ (h \times g))$, where $q : X \times Y \longrightarrow X \wedge Y$ is the quotient map. Proposition 3.25 (ii) provides the relation

$$\langle M \times N \rangle \;=\; (\mu_{M,N})_*(\langle M \rangle \times \langle N \rangle)$$

in the group $\mathbf{MGr}_0^G((M \times N)_+)$. Because $b : \mathbf{MGr} \longrightarrow \mathbf{mOP}$ is a homomorphism of E_∞-orthogonal ring spectra we can deduce that

$$\begin{aligned}
(b \wedge (q \circ (h \times g)))_* \langle M \times N \rangle \;&=\; (b \wedge (q \circ (h \times g)))_*((\mu_{M,N})_*(\langle M \rangle \times \langle N \rangle)) \\
&=\; (\mu_{M,N})_*((b \wedge h)_* \langle M \rangle \times (b \wedge g)_* \langle N \rangle) \\
&=\; (b \wedge h)_* \langle M \rangle \wedge (b \wedge g)_* \langle N \rangle
\end{aligned}$$

in the group $\mathbf{mOP}_0^G(X \wedge Y)$. Now we multiply with the class $p_G^*(\sigma^{m+n})$ and obtain the desired relation

$$\begin{aligned}
\Theta^G[M \times N, q \circ (h \times g)] \;&=\; (b \wedge (q \circ (h \times g)))_* \langle M \times N \rangle \cdot p_G^*(\sigma^{m+n}) \\
&=\; ((b \wedge h)_* \langle M \rangle \wedge (b \wedge g)_* \langle N \rangle) \cdot p_G^*(\sigma^{m+n}) \\
&=\; ((b \wedge h)_* \langle M \rangle \cdot p_G^*(\sigma^m)) \wedge ((b \wedge g)_* \langle N \rangle \cdot p_G^*(\sigma^n)) \;=\; \Theta^G[M, h] \wedge \Theta^G[N, g]\ .
\end{aligned}$$

(iii) We let $f : X \longrightarrow Y$ be a continuous G-map. Compatibility of the Thom-Pontryagin construction with the boundary homomorphism amounts to the commutativity of the following square:

$$\begin{tikzcd}
\widetilde{\mathcal{N}}^G_{m+1}(Cf) \arrow[r, "p_*"] \arrow[d, "\Theta^G"'] & \widetilde{\mathcal{N}}^G_{m+1}(X_+ \wedge S^1) \arrow[r, "(-\wedge d_{G,\mathbb{R}})^{-1}", "\cong"'] \arrow[d, "\Theta^G"'] & \mathcal{N}^G_m(X) \arrow[d, "\Theta^G"] \\
\mathbf{mO}^G_{m+1}(Cf) \arrow[r, "p_*"'] & \mathbf{mO}^G_{m+1}(X_+ \wedge S^1) \arrow[r, "\cong", "(-\wedge \bar{\tau}_{G,\mathbb{R}})^{-1}"'] & \mathbf{mO}^G_m(X_+)
\end{tikzcd}$$

Here $p : Cf \longrightarrow X_+ \wedge S^1$ is the projection defined in (1.32) of Chapter IV. Indeed, the upper composite agrees with the boundary map in equivariant bordism by Proposition 3.13; the lower composite is the homotopy theoretic boundary map by the definition in Construction III.1.31 and the fact that the suspension isomorphism in G-equivariant $\mathbf{mO}$-homology is exterior multiplication with the class $\bar{\tau}_{G,\mathbb{R}} \in \mathbf{mO}^G_1(S^1)$.

The Thom-Pontryagin construction is natural for continuous G-maps, so it remains to show the commutativity of the right square above. However, multiplicativity and (3.36) give

$$\Theta^G(x \wedge d_{G,\mathbb{R}}) \;=\; \Theta^G(x) \wedge \Theta^G(d_{G,\mathbb{R}}) \;=\; \Theta^G(x) \wedge \bar{\tau}_{G,\mathbb{R}}$$

for all $x \in \mathcal{N}^G_n(X)$.

Part (iv) is straightforward from the definitions.

(v) This is a formal consequence of the formula for the normal class of an induced manifold in Theorem 3.28. We consider a singular H-manifold $(M, h : M \longrightarrow Y)$. The class $G \times_H [M, h]$ is then represented by the singular G-manifold $(G \times_H M, G \times_H h)$. Then

$$\begin{aligned}
\mathrm{Wirth}^G_H((b \wedge (G \ltimes_H h))_*\langle G \times_H M\rangle) \;&=\; (b \wedge h \wedge S^L)_*(\mathrm{Wirth}^G_H\langle G \times_H M\rangle) \\
&=\; (b \wedge h \wedge S^L)_*(\langle M\rangle \wedge \tau_{H,L}) \\
&=\; (b \wedge h)_*\langle M\rangle \wedge (b \wedge S^L)_*(\tau_{H,L}) \;=\; (b \wedge h)_*\langle M\rangle \wedge \epsilon_L(\tau_{H,L})
\end{aligned}$$

Now we multiply with the class $p^*_G(\sigma^{m+n})$ and obtain

$$\begin{aligned}
\mathrm{Wirth}^G_H(\Theta^G(G \times_H y)) \;&=\; \mathrm{Wirth}^G_H\left((b \wedge (G \ltimes_H h))_*\langle G \times_H M\rangle \cdot p^*_G(\sigma^{m+d})\right) \\
&=\; \mathrm{Wirth}^G_H((b \wedge (G \ltimes_H h))_*\langle G \times_H M\rangle) \cdot p^*_H(\sigma^{m+d}) \\
&=\; ((b \wedge h)_*\langle M\rangle \wedge \epsilon_L(\tau_{H,L})) \cdot p^*_H(\sigma^{m+d}) \\
&=\; ((b \wedge h)_*\langle M\rangle \cdot p^*_H(\sigma^m)) \wedge (\epsilon_L(\tau_{H,L}) \cdot p^*_H(\sigma^d)) \\
&=\; \Theta^H(y) \wedge \epsilon_L(\bar{\tau}_{H,L})
\end{aligned}$$

The Wirthmüller isomorphism is inverse to the composite of ϵ_L and the exterior transfer. So this last relation is equivalent to the desired relation $\Theta^G(G \times_H y) = G \ltimes_H (\Theta^H(y) \wedge \bar{\tau}_{H,L})$. □

Remark 3.38 (Power operations are not bordism invariant). Theorem 3.37 does not mention power operations; the main reason for this is that neither the source nor the target of the Thom-Pontryagin map Θ^G have any natural power operations, as we shall now explain.

For the source of the Thom-Pontryagin map the issue is that the geometrically defined power operations of smooth manifolds are *not* bordism invariant. A G-manifold M gives rise to a $(\Sigma_k \wr G)$-manifold M^k, and the normal class matches this geometric power operation with the homotopy theoretic power operation in the ultra-commutative ring spectrum $\mathbf{MGr}$, by Proposition 3.25 (iv). However, the operation $M \mapsto M^k$ does *not* descend to a well-defined map from $\mathcal{N}^G_m$ to $\mathcal{N}^{\Sigma_k \wr G}_{km}$. We illustrate this in the simplest non-trivial case: we consider a non-equivariant closed smooth manifold M of dimension m and view M^2 as a Σ_2-manifold

by swapping coordinates. As we recalled in Example 3.21, the geometric fixed point map

$$\Phi_{\text{geom}} \ : \ \mathcal{N}_{2m}^{\Sigma_2} \ \longrightarrow \bigoplus_{j\geq 0} \mathcal{N}_{2m-j}(BO(j))$$

is injective by [**38**, Thm. 28.1]. The Σ_2-fixed points of M^2 are given by the diagonal copy of M, and the normal bundle of the diagonal inside M^2 is isomorphic to the tangent bundle of M. So when applied to the Σ_2-bordism class of M^2, the geometric fixed point map returns

$$\Phi_{\text{geom}}[M^2] \ = \ [M,\tau] \ \in \mathcal{N}_m(BO(m)) \ ,$$

the singular bordism class of the original manifold M equipped with a map $\tau : M \longrightarrow BO(m)$ that classifies its tangent bundle. There are null-bordant manifolds M for which the pair (M,τ) is *not* null-bordant as a singular manifold over $BO(m)$; the 2-sphere is an example [...justify...]. For such manifolds the class $[M^2]$ is thus non-zero in the equivariant bordism group $\mathcal{N}_{2m}^C$, so this shows that the power construction of smooth manifolds does *not* descend to bordism classes.

On the one hand, the Thom spectrum $\mathbf{mO}$ comes to us as an E_∞ orthogonal ring spectrum, and it does not even admit a chain of global equivalences of E_∞-orthogonal ring spectra to any ultra-commutative ring spectrum. While an E_∞-structure is enough to provide commutative multiplications on the associated equivariant homology theories, it does *not* entitle us to power operations.

As before we let $E\mathcal{P}$ be a universal G-space for the family of proper subgroups of G. So $E\mathcal{P}$ is a cofibrant G-space without G-fixed points, and $(E\mathcal{P})^H$ is contractible for every proper subgroup of G. We let $\tilde{E}\mathcal{P}$ denote the unreduced suspension of $E\mathcal{P}$. Then $(\tilde{E}\mathcal{P})^G = S^0$, consisting of the two cone points, and $(\tilde{E}\mathcal{P})^H$ is contractible for every proper subgroup of G.

Proposition 3.39. *For every compact Lie group G the Thom-Pontryagin map*

$$\Theta^G \ : \ \widetilde{\mathcal{N}}_*^G(\tilde{E}\mathcal{P}) \ \longrightarrow \ \mathbf{mO}_*^G(\tilde{E}\mathcal{P})$$

is an isomorphism.

Proof. We claim that the following diagram commutes:

$$\begin{array}{ccccc}
\widetilde{\mathcal{N}}_n^G(\tilde{E}\mathcal{P}) & & \xrightarrow[\cong]{\Phi_{\text{geom}}} & & \bigoplus_{j\geq 0} \mathcal{N}_{n-j}(Gr_j(\mathcal{U}_G^\perp)^G) \\
\downarrow{\scriptstyle \Theta^G} & & & & \downarrow{\scriptstyle \oplus\, \Theta^e} \\
\mathbf{mO}_n^G(\tilde{E}\mathcal{P}) & \xrightarrow[\Phi]{\cong} & \Phi_n^G(\mathbf{mO}) & \xrightarrow[\cong]{} & \bigoplus_{j\geq 0} \mathbf{mO}_{n-j}(Gr_j(\mathcal{U}_G^\perp)^G)
\end{array}$$

To see this we let (M,h) be an n-dimensional singular G-manifold over $\tilde{E}\mathcal{P}$. We choose a smooth G-equivariant wide embedding $i : M \longrightarrow V$ into a G-representation and let

$$c_M \ : \ S^V \ \longrightarrow \ Th(Gr_{|V|-n}(V)) \wedge \tilde{E}\mathcal{P}_+$$

be the associated collapse map. As before we let $M^{(j)}$ denote the $(n-j)$-dimensional component of the fixed point manifold M_0^G over the point 0, and

$$\nu_j \ : \ M^{(j)} \ \longrightarrow \ (Gr_j(V^\perp))^G$$

the classifying map for the morphism bundle of $M^{(j)}$ inside M, sending a fixed point $x \in M^{(j)}$ to the orthogonal complement of $(di)(T_x(M^{(j)}))$ inside $(di)(T_xM)$.

The trick is now to base the Thom-Pontryagin construction for $M^{(j)}$ on the non-equivariant embedding

$$M^{(j)} \ \xrightarrow{\text{incl}} \ M_0^G \ \xrightarrow{i^G} \ V^G$$

into the G-fixed points of V. As explained in Example 2.50, the G-fixed points of the Thom space that is the target of c_M decompose as a wedge. The composite

$$S^{V^G} \xrightarrow{(c_M)^G} (Th(Gr_{|V|-n}(V)))^G$$

with the projection to the j-th summand is then on the nose the map

$$S^{V^G} \xrightarrow{\nu_j \times c_{M^{(j)}}} Th(Gr_{\dim(V^G)+j}(V^G))) \wedge (Gr_j(V^\perp))^G \ ,$$

the product of ν_j and the collapse map for the non-equivariant manifold $M^{(j)}$, based on the embedding i^G. This shows that $\Phi(\Theta^G[M,h])$ is the non-equivariant Thom-Pontryagin construction applied to $\Phi_{\text{geom}}[M,h]$.

The upper map is an isomorphism by Proposition 3.18. The lower left homotopical geometric fixed point map Φ identifies the equivariant homotopy groups of $\mathbf{mO} \wedge \tilde{E}\mathcal{P}$ with the geometric fixed point groups $\Phi^G_*(\mathbf{mO})$, as shown in Proposition III.3.9. The lower right isomorphism is the calculation of these geometric fixed point groups in Example 2.50. The right vertical map is a direct sum of non-equivariant Thom-Pontryagin maps, hence an isomorphism by Thom's theorem [**160**, Thm. IV.8]. Since the diagram commutes, the left vertical map is also an isomorphism. □

THEOREM 3.40 (Wasserman). *Let G be a compact Lie group that is isomorphic to a product of a finite group and a torus. Then for every cofibrant G-space X, the Thom-Pontryagin map*

$$\Theta^G(X) \ : \ \mathcal{N}^G_*(X) \ \longrightarrow \ \mathbf{mO}^G_*(X_+)$$

is an isomorphism.

PROOF. We adapt tom Dieck's proof given in [**164**, Satz 5] to our setting. Tom Dieck considers only finite groups, where homotopy theoretic transfer and geometric induction match up under the Thom-Pontryagin construction. The general case has a new ingredient, namely the observation that for compact Lie groups of positive dimensions the difference between homotopy theoretic transfer and geometric induction is controlled by the inverse Thom class of the tangent representation, compare Theorem 3.28 and Theorem 3.37 (v).

We prove the statement by double induction over the dimension and the number of path components of G. The induction starts with the trivial group, i.e., the non-equivariant statement, which is Thom's celebrated theorem [**160**, Thm. IV.8]. Now we let G be a non-trivial compact Lie group that is a product of a finite group and a torus, and we assume that the theorem has been established for all such groups of smaller dimension, and for all groups of the same dimension but with fewer path components.

Every cofibrant G-space is equivariantly homotopy equivalent to a G-CW-complex. We can thus assume without loss of generality that X is a G-CW-complex. To show that Θ^G is an isomorphism, we exploit that $\mathcal{N}^G_*$ and $\mathbf{mO}^G_*$ are both equivariant homology theories and Θ^G is a morphism of homology theories. This reduces the claim to the special case $X = G/H$ of an orbit for a subgroup H of G. The argument for an orbit falls into two cases, depending on whether H is a proper subgroup or $H = G$.

When H is a proper subgroup of G, the following diagram commutes by Proposition 3.37 (v) for the one-point H-space:

$$\begin{tikzcd}
\mathcal{N}^H_m \arrow[rr, "G\times_H -"] \arrow[d, "\Theta^H"'] & & \mathcal{N}^G_{m+d}(G/H) \arrow[d, "\Theta^G"] \\
\mathbf{mO}^H_m(S^0) \arrow[r, "-\cdot\bar{\tau}_{H,L}"'] & \mathbf{mO}^H_{m+d}(S^L) \arrow[r, "G\ltimes_H -"'] & \mathbf{mO}^G_{m+d}(G/H_+)
\end{tikzcd}$$

In that diagram, the left vertical map is an isomorphism by the inductive hypothesis. The induction map $G \times_H - \ : \mathcal{N}^H_m \longrightarrow \mathcal{N}^G_{m+d}(G/H)$ is an isomorphism, with inverse given by sending a singular G-manifold $f : M \longrightarrow G/H$ to the fiber over the coset H of an equivariant smooth approximation of f. The

lower right horizontal map is an isomorphism by Theorem III.2.14. Now we use the hypothesis that the group G is a product of a finite group and a torus. In this situation the group H acts trivially on the tangent representation L, so multiplication by the class $\bar{\tau}_{H,L}$ in $\mathbf{mO}_d^H(S^L)$ is the suspension isomorphism, hence bijective, by Proposition 3.13. Hence the right vertical Thom-Pontryagin map $\Theta^G(G/H)$ is an isomorphism.

Now we treat the case $H = G$. We let $E\mathcal{P}$ be a universal G-space for the family of proper subgroups of G. So $E\mathcal{P}$ is a G-CW-complex without G-fixed points, and $(E\mathcal{P})^H$ is contractible for every proper subgroup of G. The space $\tilde{E}\mathcal{P}$ is the unreduced mapping cone of the unique map $E\mathcal{P} \longrightarrow *$. We thus get compatible long exact 'isotropy separation sequences', i.e., exact sequence associated to this mapping cone:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{N}_*^G(E\mathcal{P}) \arrow[r, "p_*"] \arrow[d, "\Theta^G"] & \mathcal{N}_*^G \arrow[r, "i_*"] \arrow[d, "\Theta^G"] & \widetilde{\mathcal{N}}_*^G(\tilde{E}\mathcal{P}) \arrow[r, "\partial"] \arrow[d, "\Theta^G"] & \mathcal{N}_{*-1}^G(E\mathcal{P}) \arrow[r] \arrow[d, "\Theta^G"] & \cdots \\
\cdots \arrow[r] & \mathbf{mO}_*^G(E\mathcal{P}_+) \arrow[r, "p_*"'] & \mathbf{mO}_*^G(S^0) \arrow[r, "i_*"'] & \mathbf{mO}_*^G(\tilde{E}\mathcal{P}) \arrow[r, "\partial"'] & \mathbf{mO}_{*-1}^G(E\mathcal{P}_+) \arrow[r] & \cdots
\end{tikzcd}$$

The map Θ^G is an isomorphism for the fixed point free G-space $E\mathcal{P}$ by the previous paragraph, and Θ^G is an isomorphism for $\tilde{E}\mathcal{P}$ by Proposition 3.39. Since the Thom-Pontryagin map is an isomorphism for the G-space $E\mathcal{P}$ and for the based G-space $\tilde{E}\mathcal{P}$, the five lemma shows that the Thom-Pontryagin map for the one-point G-space is an isomorphism. This completes the inductive step, and hence the proof of the theorem. □

Remark 3.41. In dimension 0, Theorem 2.48 gives an explicit description of the global functor $\underline{\pi}_0(\mathbf{mO})$. Equivariant manifolds of dimension 0 are easy to understand, and this allows us to present the groups $\mathcal{N}_0^G$ in a global fashion very similar to (but different from) the presentation of $\underline{\pi}_0(\mathbf{mO})$ in Theorem 2.48. We use this description to give a direct verification that the map

$$\Theta^G \ : \ \mathcal{N}_0^G \ \longrightarrow \ \pi_0^G(\mathbf{mO})$$

is an isomorphism for G finite or abelian. We will also use the calculation of $\underline{\pi}_0(\mathbf{mO})$ to show that the map Θ^G is *not* is isomorphism in general.

As we mentioned above, the groups $\mathcal{N}_0^G$ enjoy the structure of a restricted global functor, i.e., a group-like global power monoid (but written additively). Moreover, the interval $[-1, 1]$ with C_2-action by reflection at the origin is a C_2-equivariant null-bordism of the free transitive C_2-set. This shows that

$$\mathrm{ind}_e^{C_2}(1) \ = \ 0 \quad \text{in} \quad \mathcal{N}_0^{C_2} \ ,$$

where $1 \in \mathcal{N}_0^e$ is the bordism class of a point. The action on the class 1 thus factors over a morphism of restricted global functors

$$\mathbb{A}^{\mathrm{res}}/\langle \mathrm{ind}_e^{C_2} \rangle^{\mathrm{res}} \ \longrightarrow \ \underline{\mathcal{N}}_0 \ , \tag{3.42}$$

from the quotient of the represented restricted global functor $\mathbb{A}^{\mathrm{res}} = \mathbf{A}^{\mathrm{res}}(e, -)$ by the restricted global subfunctor generated by $\mathrm{ind}_e^{C_2} \in \mathbb{A}^{\mathrm{res}}(C_2)$. We claim that this morphism is an isomorphism. Indeed, a smooth G-manifold of dimension 0 is just a finite G-set, so the map $\mathbb{A}^{\mathrm{res}}(G)/\langle \mathrm{ind}_e^{C_2} \rangle^{\mathrm{res}}(G) \longrightarrow \mathcal{N}_0^G$ is surjective.

By the same algebraic argument as for unrestricted global functors in Proposition 2.49, the value of the restricted global functor $\langle \mathrm{ind}_e^{C_2} \rangle^{\mathrm{res}}$ at G is the subgroup of $\mathbb{A}^{\mathrm{res}}(G)$ generated by $2 \cdot \mathbb{A}^{\mathrm{res}}(G)$ and the classes $\mathrm{ind}_H^G \circ p_H^*$ for those finite index subgroups H with Weyl group of even order. So the source of the map (3.42) at a group G is an $\mathbb{F}_2$-vector space with basis the classes $\mathrm{ind}_H^G \circ p_H^*$ for those finite index subgroups H with Weyl group of odd order. The same classes form a basis for the bordism group $\mathcal{N}_0^G$; this is shown for finite groups in [**151**, Prop. 13.1], and the general case follows because restriction along the projection $p : G \longrightarrow \pi_0(G)$ induces an isomorphism $p^* : \mathcal{N}_0^{\pi_0(G)} \longrightarrow \mathcal{N}_0^G$ for bordism of 0-manifolds.

Summing up, we have calculated both sides of the Thom-Pontryagin map Θ in dimension 0; under these isomorphisms Θ becomes the map

$$\mathbb{A}^{\mathrm{res}}/\langle \mathrm{ind}_e^{C_2}\rangle^{\mathrm{res}} \;\longrightarrow\; \mathbb{A}/\langle \mathrm{tr}_e^{C_2}\rangle \;,$$

i.e., the same kind of quotient in the category of restricted versus unrestricted global functors. So the only difference between $\underline{\mathcal{N}}_0$ and $\underline{\pi}_0(\mathbf{mO})$ is that the left side only has finite index transfers, whereas the right hand side also has transfers for infinite index inclusions with finite Weyl group. In finite and abelian compact Lie groups, every subgroup inclusion with finite Weyl group is necessarily of finite index, so for finite and abelian compact Lie groups, there is no difference in the two kinds of quotients. This is an independent verification of Theorem 3.40 in dimension 0. Moreover, we conclude that the Thom-Pontryagin map $\Theta^G : \mathcal{N}_0^G \longrightarrow \pi_0^G(\mathbf{mO})$ in dimension 0 is always injective.

On the other hand, the map Θ^G is *not* generally surjective in dimension 0. A specific example is the group $G = SU(2)$: the normalizer $N = N_{SU(2)}T$ of a maximal torus T of $SU(2)$ is self-normalizing, so N has trivial Weyl group in $SU(2)$. So the classes 1 and $\mathrm{tr}_N^{SU(2)}(1)$ are linearly independent in $\pi_0^{SU(2)}(\mathbf{mO})$. On the other hand $\mathcal{N}_0^{SU(2)} = \mathbb{Z}/2$ because $SU(2)$ is connected.

Remark 3.43 (Stable equivariant bordism and localized $\mathbf{mO}$). We showed in Corollary 2.37 that the localized equivariant $\mathbf{mO}$-theory $\mathbf{mO}_*^G[1/\tau]$ is isomorphic to the theory $\mathbf{MO}_*^G$. Our next task is to show that $\mathbf{mO}[1/\tau]$, and hence also $\mathbf{MO}$, has a geometric interpretation as *stable equivariant bordism*. In contrast to Theorem 3.40, this stable interpretation works for all compact Lie groups, not only for products of finite groups and tori.

In [**28**], Bröcker and Hook define the *stable equivariant bordism groups* $\tilde{\mathfrak{N}}_*^{G:S}(X)$ of a based G-space X as the localization of the geometric bordism group $\tilde{\mathcal{N}}_*^G(X)$ by formally inverting all the classes $d_{G,V}$. More precisely, their definition comes down to

$$\tilde{\mathfrak{N}}_m^{G:S}(X) \;=\; \mathrm{colim}_{V\in s(\mathcal{U}_G)}\, \widetilde{\mathcal{N}}_{m+|V|}^G(X\wedge S^V)\;; \tag{3.44}$$

for $V \subset W$ in $s(\mathcal{U}_G)$, the structure map in the colimit system is given by multiplication

$$\widetilde{\mathcal{N}}_{m+|V|}^G(X\wedge S^V) \;\xrightarrow{-\wedge d_{G,W-V}}\; \widetilde{\mathcal{N}}_{m+|W|}^G(X\wedge S^V\wedge S^{W-V}) \;\cong\; \widetilde{\mathcal{N}}_{m+|W|}^G(X\wedge S^W)\;.$$

As we explained in Example 3.35, the Thom-Pontryagin construction takes the distinguished geometric bordism class $d_{G,V}$ to the shifted inverse Thom class, i.e., $\Theta^G(d_{G,V}) = \bar{\tau}_{G,V}$ in $\mathbf{mO}_n^G(S^V)$. Since the Thom-Pontryagin maps take the geometric product in equivariant bordism to the homotopy theoretic product in $\mathbf{mO}$, we conclude that for every compact Lie group G, every G-representation V and every based G-space X, the following square commutes:

$$\begin{array}{ccc}
\widetilde{\mathcal{N}}_n^G(X) & \xrightarrow{\quad\Theta^G\quad} & \mathbf{mO}_n^G(X) \\
{\scriptstyle -\wedge d_{G,V}}\big\downarrow & & \big\downarrow{\scriptstyle -\cdot\bar{\tau}_{G,V}} \\
\widetilde{\mathcal{N}}_{n+|V|}^G(X\wedge S^V) & \xrightarrow[\Theta^G]{} & \mathbf{mO}_{n+|V|}^G(X\wedge S^V)
\end{array}$$

The Thom-Pontryagin maps thus assemble into a natural transformation

$$\Theta^G \;:\; \tilde{\mathfrak{N}}_n^{G:S}(X) \;\longrightarrow\; \mathbf{mO}_n^G(X)[1/\tau]$$

between the localized theories; we denote this transformation by the same letter.

If G is a product of a finite group and a torus, then the next theorem is a direct consequence of Theorem 3.40. The point, however, is that the following localized version holds without any restriction on the compact Lie group G. Morally, the reason for this is that formally inverting the classes $d_{G,V}$ forces the Wirthmüller isomorphism to hold, so in stable equivariant bordism this potential obstruction

to representability by a global homotopy type vanishes. And indeed, Bröcker and Hook show in [**28**, Thm. 4.1] that stable equivariant bordism is represented by tom Dieck's homotopical equivariant bordism spectrum **MO**.

THEOREM 3.45. *For every compact Lie group G and every cofibrant based G-space X, the map*

$$\Theta^G \ : \ \tilde{\mathfrak{N}}_*^{G:S}(X) \ \longrightarrow \ \mathbf{mO}_*^G(X)[1/\tau]$$

is an isomorphism of graded abelian groups.

PROOF. Since filtered colimits of abelian groups are exact and preserve direct sums, the localized theories $\tilde{\mathfrak{N}}_*^{G:S}(-)$ and $\mathbf{mO}_*^G(-)[1/\tau]$ are both equivariant homology theories for every fixed group G. We can run the same inductive argument as in the proof of Proposition 3.40, i.e., show the claim by double induction over the dimension and the number of path components of G. But this time the case of an orbit $X = G/H$ for a proper subgroup H of G works without any restriction on G and H, because in the commutative diagram

$$\begin{tikzcd}
\mathfrak{N}_m^{H:S} \arrow[rr, "G\times_H -"] \arrow[d, "\Theta^H"'] & & \mathfrak{N}_{m+d}^{G:S}(G/H) \arrow[d, "\Theta^G"] \\
\mathbf{mO}_m^H[1/\tau] \arrow[r, "-\cdot\bar{\tau}_{H,L}"'] & \mathbf{mO}_{m+d}^H(S^L)[1/\tau] \arrow[r, "G\ltimes_H -"'] & \mathbf{mO}_{m+d}^G(G/H_+)[1/\tau]
\end{tikzcd}$$

In the localized theory $\mathbf{mO}_*^G[1/\tau]$ multiplication by the class $\bar{\tau}_{G,V}$ is now invertible for every G-representation V. So all horizontal maps in the diagram are isomorphisms. Since the left vertical map is an isomorphism by induction, the right vertical map is an isomorphism as well.

The case $X = G/G$ is essentially taken care of by Proposition 3.39. We let V be any G-representation, and observe that the fixed point inclusion $i : V^G \longrightarrow V$ induces a G-homotopy equivalence

$$\tilde{E\mathcal{P}} \wedge i \ : \ \tilde{E\mathcal{P}} \wedge S^{V^G} \ \longrightarrow \ \tilde{E\mathcal{P}} \wedge S^V \ .$$

In the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{N}}_{m+|V^\perp|}^G(\tilde{E\mathcal{P}}) \arrow[r, "\Theta^G"] \arrow[d, "-\wedge d_{G,V^G}"'] & \mathbf{mO}_{m+|V^\perp|}^G(\tilde{E\mathcal{P}}) \arrow[d, "-\cdots\tau_{G,V^G}"] \\
\widetilde{\mathcal{N}}_{m+|V|}^G(\tilde{E\mathcal{P}} \wedge S^{V^G}) \arrow[r, "\Theta^G"] \arrow[d, "(\tilde{E\mathcal{P}}\wedge i)_*"'] & \mathbf{mO}_{m+|V|}^G(\tilde{E\mathcal{P}} \wedge S^{V^G}) \arrow[d, "(\tilde{E\mathcal{P}}\wedge i)_*"] \\
\widetilde{\mathcal{N}}_{m+|V|}^G(\tilde{E\mathcal{P}} \wedge S^V) \arrow[r, "\Theta^G"'] & \mathbf{mO}_{m+|V|}^G(\tilde{E\mathcal{P}} \wedge S^V)
\end{tikzcd}$$

the lower vertical maps are thus isomorphisms. The upper vertical maps are suspension isomorphisms, hence both vertical composites are isomorphisms. The upper horizontal map is an isomorphism by Proposition 3.39, hence so is the lower horizontal map. As a colimit of isomorphisms, the localized Thom-Pontryagin map

$$\Theta^G \ : \ \mathfrak{N}_*^{G:S}(\tilde{E\mathcal{P}}) \ \longrightarrow \ \mathbf{mO}_*^G(\tilde{E\mathcal{P}})[1/\tau]$$

is also an isomorphism. Now we finish the argument as in Proposition 3.40: we compare the two isotropy separation sequences for $\tilde{\mathfrak{N}}_*^{G:S}(-)$ and $\mathbf{mO}_*^G(-)[1/\tau]$, and the five lemma concludes the inductive step. $\square$

Remark 3.46 (Stable equivariant bordism and **MO**). Corollary 2.37 and Theorem 3.45 together provide an alternative proof that stable equivariant bordism agrees with equivariant **MO**-homology, which is the main result of the paper [**28**] by Bröcker and Hook. Indeed, by these two results, the two maps

$$\tilde{\mathfrak{N}}_*^{G:S}(X) \xrightarrow{\Theta^G} \mathbf{mO}_*^G(X)[1/\tau] \xrightarrow{\Psi^G} \mathbf{MO}_*^G(X)$$

are isomorphisms for every cofibrant based G-space X, hence so is the composite; this reproves Theorem 4.1 of [**28**]. Strictly speaking there is a bit more work involved in the translation, because our group $\mathbf{MO}_*^G(X)$ is not literally the same as the homotopy theoretic equivariant bordism group $\tilde{N}_n^G(X)$ in [**28**]; we invite the reader to spell out an isomorphism, which boils down to a certain rewriting of colimits, and verify that under this implicit isomorphism our composite $\Psi^G \circ \Theta^G$ is the map Φ^S considered by Bröcker and Hook.

Remark 3.47 ($RO(G)$-modeled equivariant bordism groups). There is also a notion of '$RO(G)$-modeled equivariant bordism groups' which has to be carefully distinguished from the $\mathbb{Z}$-graded bordism groups $\mathcal{N}_*^G$ discussed above. In fact, in the published literature, these theories are called $RO(G)$-*graded* theories, but that terminology is bound to create confusion here, because in equivariant stable homotopy theory, the term '$RO(G)$-graded' is usually reserved for 'genuine' G-equivariant homology theories that are endowed with suspension isomorphism for arbitrary G-representations. So to reduce the risk of such a confusion, we follow Costenoble's thesis [**39**, Ch. 13] and use the term '$RO(G)$-modeled' instead (even though this terminology has not caught on).

The $RO(G)$-modeled bordism groups were first introduced by Pulikowski [**124**] and later studied by Kosniowski [**89**] and Waner [**170**], among others. For a compact Lie group G and a class $\gamma \in RO(G)$, a smooth compact G-manifold M is called a γ-*manifold* if for every point $x \in M$ the relation $[T_xM] = \mathrm{res}_{G_x}^G(\gamma)$ holds in the group $RO(G_x)$, where G_x is the stabilizer group of x and T_xM the tangent G_x-representation at x. A bordism between closed γ-manifolds M and M' is a $(\gamma+1)$-manifold W whose boundary is equivariantly diffeomorphic to the disjoint union of M and M'. The set B_γ^G of bordism classes of closed γ-manifolds is then a group under disjoint union. Similarly, one defines singular γ-manifolds over a G-space X as closed smooth γ-manifolds equipped with a continuous G-equivariant reference map to X; then we get a group $B_\gamma^G(X)$ of singular bordism classes. Pulikowski states in [**124**] that for fixed $\gamma \in RO(G)$, the groups $B_{\gamma+n}^G(X)$, for $n \in \mathbb{Z}$, form a G-equivariant homology theory (and follows the prevailing tradition of not giving any details).

The product of a β-manifold and a γ-manifold is a $(\beta+\gamma)$-manifold, so product of G-manifolds induces an $RO(G)$-graded product on $RO(G)$-modeled bordism groups. The unit sphere of a G-representation V is a $([V]-1)$-manifold; so the unit sphere of $V \oplus \mathbb{R}$ represents a tautological reduced bordism class

$$d_{G,V} \;=\; [\![S(V \oplus \mathbb{R}), \Pi_V]\!] \;\in\; \tilde{B}_{[V]}^G(S^V)\ .$$

Product with this class is a suspension homomorphism

$$\sigma(V) \;:\; \tilde{B}_\gamma^G(X) \;\longrightarrow\; \tilde{B}_{\gamma+[V]}^G(X \wedge S^V)\ .$$

If G acts trivially on V, then $\sigma(V)$ is an isomorphism, but *not* in general. An orthogonal G-spectrum E defines equivariant homology groups $E_W^G(X)$ indexed by G-representations W, by

$$E_W^G(X) \;=\; \mathrm{colim}_{V \in s(\mathcal{U}_G)}\, [S^{V \oplus W}, E(V) \wedge X]^G\ .$$

These groups come with natural isomorphisms

$$E_W^G(X) \;\cong\; E_{W \oplus V}^G(X \wedge S^V)$$

that in turn satisfy some coherence conditions. With some extra care, one can even arrange an honest $RO(G)$-grading, i.e., an indexing by the group $RO(G)$. Because the geometric theories $B_\star^G$ do not have

suspension isomorphisms for general representations, they cannot be represented by a genuine G-spectrum. In particular, there *cannot* be an orthogonal G-spectrum E and consistent natural isomorphisms between

$$\tilde{B}^G_{[W]-[V]}(X) \qquad \text{and} \qquad E^G_W(X \wedge S^V)$$

for all G-representations V and W. This is why the usual terminology as '$RO(G)$-graded equivariant homology theory' can be misleading from the perspective of equivariant homotopy theory.

Nevertheless, for finite groups G, Waner [**170**, Thm. 3.7 (a)] does give an identification of $\tilde{B}^G_\gamma(X)$ with the 0-th equivariant homotopy group of a G-spectrum ${}_\gamma E$ that is defined in [**170**, Sec. 5]:

$$\tilde{B}^G_\gamma(X) \;\cong\; ({}_\gamma E)^G_0(X)$$

The potentially confusing point is, however, that the right hand side is the 0-th equivariant homotopy group of a G-spectrum that depends on γ, and *not* the γ-graded homotopy group of a G-spectrum that is independent of γ. The collection of G-spectra $\{{}_\gamma E\}$ for varying G and γ certainly has a global feeling to it, but it does *not* define a global homotopy type in our sense.

4. Connective global K-theory

In this section we define and discuss various global forms of topological K-theory. We start with a *connective global K-theory* **ku**, an elaboration of a model of non-equivariant connective K-theory by Segal [**142**], constructed from certain equivariant $\mathbf{\Gamma}$-spaces of 'orthogonal subspaces in the symmetric algebra'. Then we recall a model **KU** for *periodic global K-theory*, due to Joachim [**85**]. A certain homotopy pullback of the periodic theory **KU**, its associated global Borel theory, and the global Borel theory of connective K-theory define *global connective K-theory*, a global refinement of Greenlees 'equivariant connective K-theory' $\mathbf{ku}^c$ [**63**]. One should note the different order of the adjectives 'global' and 'connective', indicating that **ku** and $\mathbf{ku}^c$ are quite different global homotopy types (with the same underlying non-equivariant homotopy type).

Construction 4.1. We let $\mathcal{U}$ be a complex vector space of countable dimension (finite or infinite) equipped with a hermitian inner product. We recall a certain $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ of 'orthogonal subspaces in $\mathcal{U}$', due to Segal [**142**, Sec. 1]. The $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ is special whenever $\mathcal{U}$ is infinite dimensional; so the associated orthogonal spectrum obtained by evaluating $\mathscr{C}$ on spheres is a positive Ω-spectrum and a (non-equivariant) model for connective complex topological K-theory.

For a finite based set A we let $\mathscr{C}(\mathcal{U}, A)$ be the space of tuples (V_a), indexed by the non-basepoint elements of A, of finite dimensional, pairwise orthogonal $\mathbb{C}$-subspaces of $\mathcal{U}$. The topology on $\mathscr{C}(\mathcal{U}, A)$ is that of a disjoint union of subspaces of a product of Grassmannians. The basepoint of $\mathscr{C}(\mathcal{U}, A)$ is the tuple where each V_a is the zero subspace. For a based map $\alpha : A \longrightarrow B$ the induced map $\mathscr{C}(\mathcal{U}, \alpha) : \mathscr{C}(\mathcal{U}, A) \longrightarrow \mathscr{C}(\mathcal{U}, B)$ sends (V_a) to (W_b) where

$$W_b \;=\; \bigoplus_{\alpha(a)=b} V_a \;.$$

Then $\mathscr{C}(\mathcal{U})$ is a $\mathbf{\Gamma}$-space whose underlying space is

$$\mathscr{C}(\mathcal{U}, 1^+) \;=\; \coprod_{n\geq 0} Gr_n(\mathcal{U})$$

the disjoint union of the different Grassmannians of $\mathcal{U}$. Of course, if $\mathcal{U}$ is finite dimensional, then $Gr_n(\mathcal{U})$ is empty when n exceeds the dimension of $\mathcal{U}$.

Every $\mathbf{\Gamma}$-space can be evaluated on a based space by a coend construction, compare (5.15) of Chapter IV. We write $\mathscr{C}(\mathcal{U}, K) = \mathscr{C}(\mathcal{U})(K)$ for the value of the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ on a based space K. Elements of $\mathscr{C}(\mathcal{U}, K)$ can be interpreted as 'labeled configurations': a point is represented by an unordered tuple

$$[E_1, \dots, E_n;\, k_1, \dots, k_n]$$

where $(E_1, \dots, E_n)$ is an n-tuple of finite dimensional, pairwise orthogonal subspaces of $\mathcal{U}$, and $k_1, \dots, k_n$ are points of K, for some n. The topology is such that, informally speaking, the labels are summed up whenever two points collide and a label disappears whenever a point approaches the basepoint of K.

REMARK 4.2. When K is compact, the space $\mathscr{C}(\mathcal{U}, K)$ can be described differently, compare again [**142**, Sec. 1], namely as the space

$$C^*(C_0(K), \mathcal{F}(\mathcal{U}))$$

of C^*-algebra homomorphisms, with the subspace topology of the compact open topology on the space of all continuous maps. Here $C_0(K)$ is the C^*-algebra of continuous $\mathbb{C}$-valued functions on K that vanish at the point at infinity, $\mathcal{F}(\mathcal{U})$ is the C^*-algebra of finite rank operators on $\mathcal{U}$ (i.e., $\mathbb{C}$-linear endomorphisms with finite dimensional image); the basepoint is the zero map. [$\mathcal{F}(\mathcal{U})$ need not be complete, hence not a C^*-algebra...] A homeomorphism

$$\mathscr{C}(\mathcal{U}, K) \ \longrightarrow \ C^*(C_0(K), \mathcal{F}(\mathcal{U}))$$

is given by sending a configuration $[E_1, \dots, E_n;\, k_1, \dots, k_n]$ in $\mathscr{C}(\mathcal{U}, K)$ to the homomorphism that takes a function $\varphi \in C_0(K)$ to

$$\sum_{i=1}^{n} \varphi(k_i) \cdot p_{E_i} \ ,$$

where $p_{E_i} : \mathcal{U} \longrightarrow \mathcal{U}$ is the orthogonal projection onto the subspace E_i.

If $\mathcal{U}$ is infinite dimensional, then the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ is special, compare Theorem 4.17 below (with G the trivial group). The orthogonal spectrum $\mathscr{C}(\mathcal{U})(\mathbb{S})$ is then a positive Ω-spectrum by the general theory. In particular, the space $\mathscr{C}(\mathcal{U})(\mathbb{S})_1 = \mathscr{C}(\mathcal{U}, S^1)$ is an infinite loop space. In fact, $\mathscr{C}(\mathcal{U}, S^1)$ is a familiar space, namely the infinite unitary group $U(\mathcal{U})$, i.e., the group of linear self-isometries of $\mathcal{U}$ that are the identity on the orthogonal complement of some finite dimensional subspace. To construct a homeomorphism from $\mathscr{C}(\mathcal{U}, S^1)$ to $U(\mathcal{U})$ we identify S^1 with the unit circle $U(1)$ in the complex numbers via the 'Cayley transform'.

$$S^1 \ \cong \ U(1) \ , \quad x \ \longmapsto \ \frac{x-i}{x+i} \ .$$

This homeomorphism sends the basepoint at infinity to 1. Given a tuple $(E_1, \dots, E_n)$ of pairwise orthogonal subspaces of $\mathcal{U}$ and a point $(\lambda_1, \dots, \lambda_n) \in U(1)^n$ we let $\psi(E_1, \dots, E_n;\, \lambda_1, \dots, \lambda_n)$ be the isometry of $\mathcal{U}$ that is multiplication by λ_i on V_i and the identity on the orthogonal complement of $\bigoplus_{i=1}^n V_i$. In other words: E_i is the eigenspace of $\psi(E_1, \dots, E_n;\, \lambda_1, \dots, \lambda_n)$ for eigenvalue λ_i. As n varies, these maps are compatible with the equivalence relation and so they assemble into a continuous map

$$\begin{aligned} \mathscr{C}(\mathcal{U}, U(1)) \ = \ \int^{n^+ \in \mathbf{\Gamma}} \mathscr{C}(\mathcal{U}, n^+) \times U(1)^n \ &\longrightarrow \ U(\mathcal{U}) \\ [E_1, \dots, E_n;\, \lambda_1, \dots, \lambda_n] \ &\longmapsto \ \psi(E_1, \dots, E_n;\, \lambda_1, \dots, \lambda_n) \ . \end{aligned} \tag{4.3}$$

This map is bijective because every unitary transformation is diagonalizable with eigenvalues in $U(1)$ and pairwise orthogonal eigenspaces.

We let $\mathcal{U}$ and $\mathcal{U}'$ be two hermitian vector spaces (of countable dimension, but possibly finite dimensional). We endow the tensor product $\mathcal{U} \otimes \mathcal{U}'$ with a scalar product by declaring

$$\langle u \otimes u',\, v \otimes v' \rangle \ = \ \langle u, v \rangle \cdot \langle u', v' \rangle \tag{4.4}$$

on elementary tensors and extending biadditively. If V, W are orthogonal subspaces of $\mathcal{U}$ and V' is a subspace of $\mathcal{U}'$, then $V \otimes V'$ and $W \otimes V'$ are orthogonal subspaces of $\mathcal{U} \otimes \mathcal{U}'$ (and similarly in the second variable). For finite based sets A and A' we can thus define a multiplication map

$$\mathscr{C}(\mathcal{U}, A) \wedge \mathscr{C}(\mathcal{U}', A') \ \longrightarrow \ \mathscr{C}(\mathcal{U} \otimes \mathcal{U}', A \wedge A') \ , \quad (V_a) \wedge (W_b) \ \longmapsto \ (V_a \otimes W_b)_{a \wedge b} \ . \tag{4.5}$$

These multiplication maps are associative, and commutative in the sense that the following square commute:

$$\begin{tikzcd} \mathscr{C}(\mathcal{U},A)\wedge\mathscr{C}(\mathcal{U}',A') \arrow[r] \arrow[d, "\tau_{\mathscr{C}(\mathcal{U},A),\mathscr{C}(\mathcal{U}',A')}"'] & \mathscr{C}(\mathcal{U}\otimes\mathcal{U}',A\wedge A') \arrow[d, "\mathscr{C}(\tau_{\mathcal{U},\mathcal{U}'},\tau_{A,A'})"] \\ \mathscr{C}(\mathcal{U}',A')\wedge\mathscr{C}(\mathcal{U},A) \arrow[r] & \mathscr{C}(\mathcal{U}'\otimes\mathcal{U},A'\wedge A) \end{tikzcd}$$

Our construction of connective global K-theory needs induced inner products on symmetric powers. We explain the complex version; the real version works in much the same way. For a $\mathbb{C}$-vector space V we denote by

$$\mathrm{Sym}^n(V) \;=\; V^{\otimes n}/\Sigma_n$$

the n-th symmetric power of V and by $\mathrm{Sym}(V) = \bigoplus_{n\geq 0}\mathrm{Sym}^n(V)$ the symmetric algebra of V. If W is another $\mathbb{C}$-vector space, then the two direct summand embeddings of V and W into $V\oplus W$ induce homomorphisms of symmetric algebras that combine (by multiplying in the target) into a natural $\mathbb{C}$-algebra isomorphism

$$\mathrm{Sym}(V)\otimes\mathrm{Sym}(W) \;\cong\; \mathrm{Sym}(V\oplus W)\;. \tag{4.6}$$

If V is equipped with a hermitian inner product, then the symmetric powers inherit a preferred inner product:

Proposition 4.7. *For every hermitian inner product space V there is a unique inner product on $\mathrm{Sym}^n(V)$ that satisfies*

$$(v_1\cdot\ldots\cdot v_n,\,\bar v_1\cdot\ldots\cdot\bar v_n) \;=\; \sum_{\sigma\in\Sigma_n}(v_1,\bar v_{\sigma(1)})\cdot\ldots\cdot(v_n,\bar v_{\sigma(n)}) \tag{4.8}$$

for all $v_i,\bar v_i\in V$. This inner product on $\mathrm{Sym}^n(V)$ is natural for $\mathbb{C}$-linear isometric embeddings and it makes the algebra isomorphism (4.6) *into an isometry.*

Proof. Uniqueness of the scalar product follows from the fact that the symmetric products $v_1\cdot\ldots\cdot v_n$ generate $\mathrm{Sym}^n(V)$ as a $\mathbb{C}$-vector space. The tensor product $V\otimes W$ of two hermitian inner product spaces V and W has a preferred inner product characterized by

$$\langle v\otimes w,\bar v\otimes\bar w\rangle \;=\; \langle v,\bar v\rangle\cdot\langle w,\bar w\rangle$$

for all $v,\bar v\in V$ and $w,\bar w\in W$. By iteration, the n-fold tensor product $V^{\otimes n}$ inherits an inner product. To construct the induced inner product we consider the normalized $\mathbb{C}$-linear 'symmetrization' embedding

$$\mathrm{Sym}^n(V)\;\longrightarrow\;V^{\otimes n}\;,\quad v_1\cdot\ldots\cdot v_n\;\longmapsto\;\frac{1}{\sqrt{n!}}\cdot\sum_{\sigma\in\Sigma_n}v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)} \tag{4.9}$$

that identifies $\mathrm{Sym}^n(V)$ with a linear subspace of the n-fold tensor product. There is thus a unique inner product $\langle -,-\rangle'$ on $\mathrm{Sym}^n(V)$ that makes (4.9) a linear isometric embedding. This inner product satisfies

$$\begin{aligned}\langle v_1\cdot\ldots\cdot v_n,\,\bar v_1\cdot\ldots\cdot\bar v_n\rangle' \;&=\; \frac{1}{n!}\cdot\sum_{(\kappa,\tau)\in\Sigma_n^2}\langle v_{\kappa(1)}\otimes\ldots\otimes v_{\kappa(n)},\,\bar v_{\tau(i)}\otimes\ldots\otimes\bar v_{\tau(n)}\rangle\\ &=\; \frac{1}{n!}\cdot\sum_{(\kappa,\tau)\in\Sigma_n^2}\langle v_{\kappa(1)},\bar v_{\tau(1)}\rangle\cdot\ldots\cdot\langle v_{\kappa(n)},\bar v_{\tau(n)}\rangle\\ &=\; \frac{1}{n!}\cdot\sum_{(\kappa,\tau)\in\Sigma_n^2}\langle v_1,\bar v_{\tau(\kappa^{-1}(1))}\rangle\cdot\ldots\cdot\langle v_n,\bar v_{\tau(\kappa^{-1}(n))}\rangle\\ &=\; \sum_{\sigma\in\Sigma_n}\langle v_1,\bar v_{\sigma(1)}\rangle\cdot\ldots\cdot\langle v_n,\bar v_{\sigma(n)}\rangle\;.\end{aligned}$$

The naturality for linear isometric embeddings is straightforward.

The algebra isomorphism (4.6) is the sum of the embeddings

$$\begin{array}{rcl} \mathrm{Sym}^n(V) \otimes \mathrm{Sym}^m(W) & \longrightarrow & \mathrm{Sym}^{n+m}(V \oplus W) \\ (v_1 \cdot \ldots \cdot v_n) \otimes (w_1 \cdot \ldots \cdot w_m) & \longmapsto & (v_1, 0) \cdot \ldots \cdot (v_n, 0) \cdot (0, w_1) \cdot \ldots \cdot (0, w_m) \ . \end{array}$$

The relation

$$\begin{aligned} &\langle (v_1 \cdot \ldots \cdot v_n) \otimes (w_1 \cdot \ldots \cdot w_m),\, (\bar{v}_1 \cdot \ldots \cdot \bar{v}_n) \otimes (\bar{w}_1 \cdot \ldots \cdot \bar{w}_m) \rangle \\ &\qquad = \ \langle v_1 \cdot \ldots \cdot v_n,\, \bar{v}_1 \cdot \ldots \cdot \bar{v}_n \rangle \cdot \langle w_1 \cdot \ldots \cdot w_m,\, \bar{w}_1 \cdot \ldots \cdot \bar{w}_m \rangle \\ &\qquad = \ \sum_{(\sigma,\tau) \in \Sigma_n \times \Sigma_m} \langle v_1, \bar{v}_{\sigma(1)} \rangle \cdot \ldots \cdot \langle v_n, \bar{v}_{\sigma(n)} \rangle \cdot \langle w_1, \bar{w}_{\tau(1)} \rangle \cdot \ldots \cdot \langle w_m, \bar{w}_{\tau(n)} \rangle \\ &\qquad = \ \langle (v_1, 0) \cdot \ldots \cdot (v_n, 0) \cdot (0, w_1) \cdot \ldots \cdot (0, w_m),\, (\bar{v}_1, 0) \cdot \ldots \cdot (\bar{v}_n, 0) \cdot (0, \bar{w}_1) \cdot \ldots \cdot (0, \bar{w}_m) \rangle \end{aligned}$$

then proves that (4.6) preserves the inner product. The last equation uses that $\langle (v, 0), (0, w) \rangle = 0$ for all $v \in V$ and $w \in W$. □

Example 4.10. To given a better idea of the induced inner product on $\mathrm{Sym}^n(V)$ we consider the case $n = 2$. We let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of V. Then

$$\langle e_i^2, e_i^2 \rangle \ = \ \langle e_i, e_i \rangle \cdot \langle e_i, e_i \rangle + \langle e_i, e_i \rangle \cdot \langle e_i, e_i \rangle \ = \ 2$$

and

$$\langle e_i \cdot e_j,\, e_i \cdot e_j \rangle \ = \ \langle e_i, e_i \rangle \cdot \langle e_j, e_j \rangle + \langle e_j, e_i \rangle \cdot \langle e_j, e_i \rangle \ = \ 1$$

for $i \neq j$. Moreover, for $\{i, k\} \neq \{j, l\}$ we have

$$\langle e_i \cdot e_k,\, e_j \cdot e_l \rangle \ = \ \langle e_i, e_j \rangle \cdot \langle e_k, e_l \rangle + \langle e_i, e_l \rangle \cdot \langle e_k, e_j \rangle \ = \ 0 \ .$$

So the vectors

$$\frac{1}{\sqrt{2}} \cdot e_i^2 \quad (1 \leq i \leq k) \qquad \text{and} \qquad e_i \cdot e_j \quad (1 \leq i < j \leq k)$$

form an orthonormal basis of $\mathrm{Sym}^2(V)$.

Construction 4.11 (Connective global K-theory). We can now define an ultra-commutative ring spectrum $\mathbf{ku}$, the *connective global K-theory spectrum*. The proof that $\mathbf{ku}$ is indeed globally connective is postponed until the end of this section in Proposition 4.44. The value of $\mathbf{ku}$ on a real inner product space V is

$$\mathbf{ku}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ , \tag{4.12}$$

the value of the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ on the one-point compactification of V. Here $V_{\mathbb{C}}$ is the complexification of V with the induced hermitian inner product, and the inner product on the symmetric algebra described in Proposition 4.7. The action of $O(V)$ on V then extends to a unitary action on $\mathrm{Sym}(V_{\mathbb{C}})$.

We let the orthogonal group $O(V)$ act diagonally, via the action on the sphere S^V and the action on the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$. Explicitly, given an orthogonal automorphism $\varphi : V \longrightarrow V$, pairwise orthogonal subspaces $E_1, \ldots, E_n$ of $\mathrm{Sym}(V_{\mathbb{C}})$ and elements $v_1, \ldots, v_n$ of S^V, we set

$$\varphi \cdot [E_1, \ldots, E_n;\, v_1, \ldots, v_n] \ = \ [\mathrm{Sym}(\varphi_{\mathbb{C}})(E_1), \ldots, \mathrm{Sym}(\varphi_{\mathbb{C}})(E_n);\, \varphi(v_1), \ldots, \varphi(v_n)] \ .$$

Using the tensor product pairing (4.5) we define an $O(V) \times O(W)$-equivariant multiplication map

$$\begin{aligned} \mu_{V,W} \ : \ \mathbf{ku}(V) \wedge \mathbf{ku}(W) \ &= \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \wedge \mathscr{C}(\mathrm{Sym}(W_{\mathbb{C}}), S^W) \\ &\longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathrm{Sym}(W_{\mathbb{C}}), S^V \wedge S^W) \\ &\overset{(4.6)}{\cong} \ \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^{V \oplus W}) = \mathbf{ku}(V \oplus W) \ . \end{aligned}$$

The maps $\mu_{V,W}$ are associative and commutative. An $O(V)$-equivariant unit map is given by

$$\iota_V \ : \ S^V \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) = \mathbf{ku}(V) \ , \quad v \ \longmapsto \ [\mathbb{C}{\cdot}1;\, v] \ ,$$

where $\mathbb{C}{\cdot}1$ is the homogeneous summand of degree 0 in the symmetric algebra, i.e., the line spanned by the multiplicative unit. This structure makes $\mathbf{ku}$ into a commutative orthogonal ring spectrum.

The space $\mathbf{ku}_0 = \mathscr{C}(\mathbb{C}\cdot 1, S^0)$ consists of all subspaces of $\mathrm{Sym}(0) = \mathbb{C}\cdot 1$, so it has two points, the basepoint 0 and the point $\mathbb{C}{\cdot}1$. The unit map $\iota_0 : S^0 \longrightarrow \mathbf{ku}_0$ is thus an isomorphism.

Construction 4.13 (Complex conjugation on $\mathbf{ku}$). The ultra-commutative ring spectrum $\mathbf{ku}$ comes with an involution by 'complex conjugation' that preserves all the structure available. Indeed, for every real inner product space V the complex symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ of the complexification is canonically $O(V)$-equivariantly isomorphic to $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Sym}_{\mathbb{R}}(V)$, the complexification of the real symmetric algebra of V. So $\mathrm{Sym}(V_{\mathbb{C}})$ comes with an involution $\psi_{\mathrm{Sym}(V)}$ that is $\mathbb{C}$-semilinear and preserves the orthogonality relation. Applying this involution elementwise to tuples of orthogonal subspaces given an involution $\mathscr{C}(\psi_{\mathrm{Sym}(V)}) : \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}})) \longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ of the $\mathbf{\Gamma}$-space and hence a homeomorphism

$$\psi(V) \ = \ \mathscr{C}(\psi_{\mathrm{Sym}(V)}, S^V) \ : \ \mathbf{ku}(V) = \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) = \mathbf{ku}(V)$$

of order 2. As V varies, the maps $\psi(V)$ form an automorphism

$$\psi \ : \ \mathbf{ku} \ \longrightarrow \ \mathbf{ku} \tag{4.14}$$

of the ultra-commutative ring spectrum $\mathbf{ku}$.

Now we will justify that for every finite group G the underlying orthogonal G-spectrum of $\mathbf{ku}$ represents connective G-equivariant topological K-theory. Every linear isometric embedding $u : \mathcal{U} \longrightarrow \bar{\mathcal{U}}$ of complex inner product spaces induces a morphism of $\mathbf{\Gamma}$-space $\mathscr{C}(u) : \mathscr{C}(\mathcal{U}) \longrightarrow \mathscr{C}(\bar{\mathcal{U}})$ by applying u elementwise to a tuple of orthogonal subspaces. So if a compact Lie group G acts on $\mathcal{U}$ by linear isometries (for example if $\mathcal{U}$ is a G-universe), then the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ inherits a G-action, so it becomes a $\mathbf{\Gamma}$-G-space.

Proposition 4.15. *let G be a compact Lie group and $\mathcal{U}$ and $\bar{\mathcal{U}}$ two isomorphic complex G-universes. Then for every G-equivariant linear isometric embedding $u : \mathcal{U} \longrightarrow \bar{\mathcal{U}}$ and every based G-space K the map*

$$\mathscr{C}(u, K) \ : \ \mathscr{C}(\mathcal{U}, K) \ \longrightarrow \ \mathscr{C}(\bar{\mathcal{U}}, K)$$

is a G-homotopy equivalence.

Proof. We start with the special case where $\bar{\mathcal{U}} = \mathcal{U}$. The space of G-equivariant linear isometric embeddings from $\mathcal{U}$ to itself is contractible. A homotopy from u to the identity then induces a G-homotopy from $\mathscr{C}(u, K)$ to the identity of the G-space $\mathscr{C}(\mathcal{U}, K)$. In the general case we choose a G-equivariant linear isometry $v : \bar{\mathcal{U}} \cong \mathcal{U}$. By the previous paragraph the two G-maps

$$\mathscr{C}(vu, K) \ : \ \mathscr{C}(\mathcal{U}, K) \ \longrightarrow \ \mathscr{C}(\mathcal{U}, K) \qquad \text{and} \qquad \mathscr{C}(uv, K) \ : \ \mathscr{C}(\bar{\mathcal{U}}, K) \ \longrightarrow \ \mathscr{C}(\bar{\mathcal{U}}, K)$$

are G-homotopic to the respective identity maps. □

If F is any $\mathbf{\Gamma}$-space and S a finite set, then we define the map

$$P_S \ : \ F(S^+) \ \longrightarrow \ \mathrm{map}(S, F(1^+))$$

by

$$P_S(x)(s) \ = \ F(p_s)(x) \ ,$$

where $p_s : S_+ \longrightarrow 1^+$ sends s to 1 and all other elements of S_+ to the basepoint. The map P_S is natural for bijections in S; so whenever a group G acts on S, then P_S is G-equivariant.

Definition 4.16. Let G be a compact Lie group. A $\mathbf{\Gamma}$-G-space F is *special* if for every finite G-set S the map

$$P_S \ : \ F(S^+) \ \longrightarrow \ \mathrm{map}(S, F(1^+))$$

is a G-weak equivalence.

Theorem 4.17. *Let G be a compact Lie group. Then for every complete complex G-universe $\mathcal{U}$ the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathcal{U})$ is special.*

Proof. For all closed subgroups H of G the underlying H-universe of a complete G-universe is again complete, so it suffices to show that for every finite G-set S the map

$$(P_S)^G \ : \ \mathscr{C}(\mathcal{U}, S^+)^G \ \longrightarrow \ \mathrm{map}^G(S, \mathscr{C}(\mathcal{U}, 1^+)) \tag{4.18}$$

is a weak equivalence. We may assume that $S = \{1, \dots, m\}$ with some action of G, which is then specified by a continuous homomorphism $\alpha : G \longrightarrow \Sigma_m$. The G-fixed points above are the same as the fixed points of the graph subgroup of α with respect to the $(G \times \Sigma_m)$-action on $\mathscr{C}(\mathcal{U}, m^+)$ and $\mathrm{map}(m^+, \mathscr{C}(\mathcal{U}, 1^+)) = \mathscr{C}(\mathcal{U}, 1^+)^m$. In other words, it suffices to show that the map

$$P_m \ : \ \mathscr{C}(\mathcal{U}, m^+) \ \longrightarrow \ \mathscr{C}(\mathcal{U}, 1^+)^m$$

is an $\mathcal{F}(G; \Sigma_m)$-weak equivalence, where $\mathcal{F}(G; \Sigma_m)$ is the family of graph subgroups of $G \times \Sigma_m$.

We define a morphism of $(G \times \Sigma_m)$-spaces

$$\lambda_m \ : \ \mathscr{C}(\mathcal{U}, 1^+)^m \ \longrightarrow \ \mathscr{C}(\mathbb{C}^m \otimes \mathcal{U}, m^+) \ ;$$

here the Σ_m-action on the target is diagonally, from the permutation action on m^+ and on the tensor factor $\mathbb{C}^m$. The map λ_m sends a tuple $(V_1, \dots, V_m)$ of finite dimensional subspace of $\mathcal{U}$ to the tuple

$$(e_1 \otimes V_1, \dots, e_m \otimes V_m)$$

of pairwise orthogonal subspaces of $\mathbb{C}^m \otimes \mathcal{U}$, where $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{C}^m$.

Now we consider the two composites $\lambda_m \circ P_m$ and $P'_m \circ \lambda_m$, where P'_m is the map P_m for the universe $\mathbb{C}^m \otimes \mathcal{U}$ (as opposed to $\mathcal{U}$):

$$\begin{tikzcd}
\mathscr{C}(\mathcal{U}, m^+) \arrow[r, "P_m"'] \arrow[rrr, bend left, "\simeq"] & \mathscr{C}(\mathcal{U}, 1^+)^m \arrow[r, "\lambda_m"] \arrow[rr, bend right, "\simeq"'] & \mathscr{C}(\mathbb{C}^m \otimes \mathcal{U}, m^+) \arrow[r, "P'_m"] & \mathscr{C}(\mathbb{C}^m \otimes \mathcal{U}, 1^+)^m
\end{tikzcd}$$

We start by investigating the composite $\lambda_m \circ P_m : \mathscr{C}(\mathcal{U}, m^+) \longrightarrow \mathscr{C}(\mathbb{C}^m \otimes \mathcal{U}, m^+)$. For $1 \le i \le m$, we define a 1-parameter family of unit vectors

$$u_i \ : \ [0,1] \ \longrightarrow \ \mathbb{C}^m \qquad \text{by} \qquad u_i(t) \ = \ t \cdot e_i + \sqrt{\frac{1-t^2}{n-1}} \cdot \sum_{j \neq i} e_i \ ,$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i-th vector of the standard basis of $\mathbb{C}^m$. This provides a homotopy

$$H \ : \ \mathscr{C}(\mathcal{U}, m^+) \times [0,1] \ \longrightarrow \ \mathscr{C}(\mathbb{C}^m \otimes \mathcal{U}, m^+)$$

by

$$H(V_1, \dots, V_m; t) \ = \ (u_1(t) \otimes V_1, \dots, u_m(t) \otimes V_m) \ .$$

We emphasize that because $(V_1, \dots, V_m)$ are pairwise orthogonal in $\mathcal{U}$, then $H(V_1, \dots, V_m; t)$ is indeed another tuple of pairwise orthogonal subspaces, now in $\mathbb{C}^m \otimes \mathcal{U}$. We have

$$u_1(1/\sqrt{n}) \ = \ \dots \ = \ u_m(1/\sqrt{n}) \ = \ \frac{1}{\sqrt{n}} \cdot (1, \dots, 1)$$

and

$$H(V_1, \dots, V_m; 1) \ = \ (e_1 \otimes V_1, \dots, e_m \otimes V_m) \ = \ (\lambda_m \circ P_m)(V_1, \dots, V_m) \ .$$

Moreover, each of the maps $H(-,t)$ is $(G\times\Sigma_m)$-equivariant, so the composite $\lambda_m\circ P_m:\mathscr{C}(\mathcal{U},m^+)\longrightarrow\mathscr{C}(\mathbb{C}^m\otimes\mathcal{U},m^+)$ is $(G\times\Sigma_m)$-equivariantly homotopic to the morphism $\mathscr{C}(j,m^+)$, where $j:\mathcal{U}\longrightarrow\mathbb{C}^m\otimes\mathcal{U}$ is the $(G\times\Sigma_m)$-equivariant linear isometric embedding given by

$$j(v) \;=\; \frac{1}{\sqrt{n}}\cdot(1,\dots,1)\otimes v\ .$$

Now we let $\alpha:H\longrightarrow\Sigma_m$ be a continuous homomorphism defined on a closed subgroup H of G. Since $\mathcal{U}$ is a complete complex G-universe, both $\mathcal{U}$ and $\alpha^*(\mathbb{C}^m)\otimes\mathcal{U}$ are complete complex H-universes, and j is an H-equivariant linear isometric embedding

$$j\ :\ \mathrm{res}^G_H(\mathcal{U})\ \longrightarrow\ \alpha^*(\mathbb{C}^m)\otimes\mathrm{res}^G_H(\mathcal{U})\ .$$

Interpreted in this way, the map $\mathscr{C}(j,m^+)^H$ is a homotopy equivalence by Proposition 4.15. This shows that the morphism $\mathscr{C}(j,m^+)$ is an $\mathcal{F}(G;\Sigma_m)$-weak equivalence, hence so is the morphism $\lambda_m\circ P_m$.

Now we show that the composite $P'_m\circ\lambda_m$ is an $\mathcal{F}(G;\Sigma_m)$-weak equivalence. We let H be a closed subgroup of G, $\alpha:H\longrightarrow\Sigma_m$ be a continuous homomorphism, and $\Gamma=\{(h,\alpha(h))\mid h\in H\}\le G\times\Sigma_m$ the graph of α. We let $a_1,\dots,a_k\in\{1,\dots,m\}$ be a set of representatives of the orbits of the H-action on $\{1,\dots,m\}$ through α. We let $H_i\le H$ be the stabilizer group of a_i. Then projection to the factors indexed by $a_1,\dots,a_k$ provides homeomorphisms

$$(\mathscr{C}(\mathcal{U},1^+)^m)^\Gamma\ =\ \prod_{j=1}^k\mathscr{C}(\mathcal{U},1^+)^{H_i}\qquad\text{and}\qquad(\mathscr{C}(\mathbb{C}^m\otimes\mathcal{U},1^+)^m)^\Gamma\ =\ \prod_{j=1}^k\mathscr{C}(\mathbb{C}^m\otimes\mathcal{U},1^+)^{\Gamma_i}\ ,$$

where $\Gamma_i=(H_i\times\Sigma_m)\cap\Gamma$ is the graph of the restriction of α to H_i. Under these identifications, the morphism $(P'_m\circ\lambda_m)^\Gamma$ becomes the product, for $i=1,\dots,k$, of the morphisms

$$(\mathscr{C}(\mathcal{U},1^+))^{H_i}\ \longrightarrow\ \mathscr{C}(\mathbb{C}^m\otimes\mathcal{U},1^+)^{\Gamma_i}\ =\ \mathscr{C}(\alpha^*(\mathbb{C}^m)\otimes\mathcal{U},1^+)^{H_i}\tag{4.19}$$

induced by the H_i-equivariant linear isometric embeddings

$$\mathcal{U}\ \longrightarrow\ \alpha^*(\mathbb{C}^m)\otimes\mathcal{U}\ ,\quad v\ \longmapsto\ e_{a_i}\otimes v\ .$$

Since $\mathcal{U}$ is a complete complex G-universe, both $\mathcal{U}$ and $\alpha^*(\mathbb{C}^m)\otimes\mathcal{U}$ are complete complex H_i-universes. So the map (4.19) is a homotopy equivalence by Proposition 4.15.

The $\mathcal{F}(G;\Sigma_m)$-weak equivalences satisfy the 2-out-of-6 property; since $\lambda_m\circ P_m$ and $P'_m\circ\lambda_m$ are $\mathcal{F}(G;\Sigma_m)$-weak equivalences, so is the morphism $P_m:\mathscr{C}(\mathcal{U},m^+)\longrightarrow\mathscr{C}(\mathcal{U},1^+)^m$. □

Proposition 4.20. *For every finite group G and every G-representation V (possibly infinite dimensional), the $\mathbf{\Gamma}$-G-space $\mathscr{C}(V)$ is Reedy cofibrant.*

Proof. [...fill in...] □

The orthogonal spectrum **ku** is trying to be a 'positive global Ω-spectrum'. However, the global Ω-spectrum condition on the adjoint structure maps only hold in certain situations.

Definition 4.21. Let G be a compact Lie group. An orthogonal G-representation is *ample* if the complexified symmetric algebra $\mathrm{Sym}(V)$ is a complete complex G-universe.

Remark 4.22. The author does not know a good characterization of ample G-representations a necessary condition is clearly that V is non-zero and G acts faithfully on V. A class of representations that qualify in this context are non-empty faithful permutation representations. Indeed, we let A be a faithful finite G-set, which means in particular that the group G is finite, and we let $\mathbb{R}A$ denote the associated permutation representation. Then $(\mathbb{R}A)_{\mathbb{C}}=\mathbb{C}A$ is the complex permutation representation and its symmetric algebra $\mathrm{Sym}(\mathbb{C}A)$ is also a complex permutation representation, namely of the infinite G-set $\mathbb{N}^A=\mathrm{map}(A,\mathbb{N})$ of functions from A to $\mathbb{N}$. Since G acts faithfully on A, every *injective* map $A\longrightarrow\mathbb{N}$ generates a free G-orbit in the G-set $\mathbb{N}^A$. Since A is non-empty, there are infinitely many injections from A to $\mathbb{N}$ with pairwise disjoint

images, and these generate infinitely many distinct free G-orbits in $\mathbb{N}^A$. So $\mathrm{Sym}(\mathbb{C}A)$ contains infinitely many copies of the complex regular G-representation, and is thus a complete G-universe.

THEOREM 4.23. *Let G be a finite group and W an ample orthogonal G-representation. Then for every G-representation V the adjoint structure map*

$$\tilde{\sigma}_{V,W} \ : \ \mathbf{ku}(W) \ \longrightarrow \ \mathrm{map}(S^V, \mathbf{ku}(V \oplus W))$$

is a G-weak equivalence.

PROOF. The adjoint structure map $\tilde{\sigma}_{V,W}$ factors as the composite

$$\begin{aligned} \mathscr{C}(\mathrm{Sym}(W_{\mathbb{C}}), S^W) \ &\xrightarrow{\mathscr{C}(\mathrm{Sym}(i_{\mathbb{C}}), S^W)} \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^W) \\ &\xrightarrow{\tilde{\alpha}} \qquad \mathrm{map}(S^V, \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^{V \oplus W})) \ ; \end{aligned}$$

here $i : W \longrightarrow V \oplus W$ is the inclusion of the second summand and $\tilde{\alpha}$ is the adjoint of the assembly map for the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}))$. By the hypothesis on W the map $\mathrm{Sym}(i_{\mathbb{C}}) : \mathrm{Sym}(W_{\mathbb{C}}) \longrightarrow \mathrm{Sym}((V \oplus W)_{\mathbb{C}})$ is an equivariant linear isometric embedding between complete complex G-universes. So the first map is a G-homotopy equivalence by Proposition 4.15. Since $\mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}))$ is a special $\mathbf{\Gamma}$-G-space (see Theorem 4.17), the second map is a G-weak equivalence by Shimakawa's theorem [**145**, Thm. B]. We are using here without proof here that $\mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^{V \oplus W})$ (which is a categorical coend) is G-weakly equivalent to the bar construction used by Shimakawa (which is the corresponding homotopy coend). In the context of Γ-G-spaces of simplicial sets (as opposed to topological spaces), the analogous comparison can be found in [**121**, Lemma 5.3]. □

CONSTRUCTION 4.24. We define a morphism of orthogonal spaces

$$c \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \Omega^{\bullet}(\mathbf{ku}) \ . \tag{4.25}$$

The value at an inner product space V is the continuous map

$$c(V) \ : \ \mathbf{Gr}^{\mathbb{C}}(V) \ \longrightarrow \ \mathrm{map}(S^V, \mathbf{ku}(V)) \ = \ (\Omega^{\bullet}\mathbf{ku})(V)$$

that sends a complex subspace $L \subset V_{\mathbb{C}}$ to the continuous based map

$$[L; -] \ : \ S^V \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}(V) \ , \quad v \ \longmapsto \ [L; v] \ .$$

Here we consider $L \subset V_{\mathbb{C}}$ as sitting in the linear summand of the symmetric algebra of $V_{\mathbb{C}}$.

It will be useful to also define a delooping of the morphism c, namely a morphism of orthogonal spaces

$$\mathrm{eig} \ : \ \mathbf{U} \ \longrightarrow \ \Omega^{\bullet}(s\mathbf{ku}) \ , \tag{4.26}$$

where $\mathbf{U}$ is the ultra-commutative monoid of unitary groups (compare Example II.3.7), and $s\mathbf{ku}$ is essentially a shift of $\mathbf{ku}$. More precisely, $s\mathbf{ku}$ is defined in level V as

$$s\mathbf{ku}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^{V \oplus \mathbb{R}}) \ . \tag{4.27}$$

The only difference between $s\mathbf{ku}$ and the shift $\mathrm{sh}\,\mathbf{ku}$ as defined in (1.22) of Chapter IV is that we take configurations of subspaces in $\mathrm{Sym}(V_{\mathbb{C}})$, as opposed to $\mathrm{Sym}((V \oplus \mathbb{R})_{\mathbb{C}})$, The name 'eig' refers to the fact that the morphism records the eigenspace decomposition of a unitary isomorphism. The definition uses the homeomorphism

$$h \ : \ U(1) \ \cong \ S^1 \ , \quad h(\lambda) \ = \ i \cdot \frac{\lambda + 1}{\lambda - 1} \ ,$$

the inverse of the 'Cayley transform'. Every unitary automorphism of a finite dimensional hermitian inner product space is diagonalizable with eigenvalues in $U(1)$ and pairwise orthogonal eigenspaces. So given an inner product space V we define

$$\mathrm{eig}(V) \ : \ \mathbf{U}(V) \ = \ U(V_{\mathbb{C}}) \ \longrightarrow \ \mathrm{map}(S^V, s\mathbf{ku}(V)) = (\Omega^{\bullet}(s\mathbf{ku}))\,(V)$$

by

$$\mathrm{eig}(V)(A)(v) \;=\; [E(\lambda_1),\dots,E(\lambda_n);\,(v,h(\lambda_1)),\dots,(v,h(\lambda_n))]\ .$$

Here $\lambda_1,\dots\lambda_n \in U(1)$ are the eigenvalues of A and $E(\lambda_i)$ is the eigenspace of λ_i. Strictly speaking, $E(\lambda_i)$ is a subspace of $V_{\mathbb{C}}$, which we identify with the linear summand of $\mathrm{Sym}(V_{\mathbb{C}})$.

THEOREM 4.28. *The morphism*

$$\mathrm{eig}\ :\ \mathbf{U}\ \longrightarrow\ \Omega^\bullet(s\mathbf{ku})$$

is a $\mathcal{F}in$-global equivalence of orthogonal spaces.

PROOF. We let $\bar{\mathbf{U}}$ denote the orthogonal space with

$$\bar{\mathbf{U}}(V)\ =\ U(\mathrm{Sym}(V_{\mathbb{C}}))\ .$$

The structure map induced by $\varphi : V \longrightarrow W$ is given by extending by the identity on the orthogonal complement of $\mathrm{Sym}(\varphi_{\mathbb{C}}) : \mathrm{Sym}(V_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(W_{\mathbb{C}})$. The eigenspace decomposition map then factors as the composite

$$\mathbf{U}\ \longrightarrow\ \bar{\mathbf{U}}\ \xrightarrow{\overline{\mathrm{eig}}}\ \Omega^\bullet(s\mathbf{ku})\ ,$$

where $\overline{\mathrm{eig}}$ is defined in the same way as eig, recording the set of eigenvalues and eigenspaces. Since $\bar{\mathbf{U}}(V)$ is the colimit, along closed embeddings, of the spaces $U(\mathrm{Sym}^{[n]}(V_{\mathbb{C}}))$, the morphism $\mathbf{U} \longrightarrow \bar{\mathbf{U}}$ is a global equivalence of orthogonal spaces by Theorem I.1.11 and Proposition I.1.9 (viii). We may thus show that the morphism

$$\overline{\mathbf{U}}\ \xrightarrow{\overline{\mathrm{eig}}}\ \Omega^\bullet(s\mathbf{ku})$$

is a $\mathcal{F}in$-global equivalence. We show the stronger statement that for every finite group G and every ample G-representation V the map

$$\overline{\mathrm{eig}}(V)\ :\ U(\mathrm{Sym}(V_{\mathbb{C}}))\ =\ \bar{\mathbf{U}}(V)\ \longrightarrow\ (\Omega^\bullet(s\mathbf{ku}))(V)\ =\ \mathrm{map}(S^V,\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^{V\oplus\mathbb{R}}))$$

is a G-weak equivalence. Indeed, the map $\overline{\mathrm{eig}}(V)$ factors as the composite

$$U(\mathrm{Sym}(V_{\mathbb{C}}))\ \xrightarrow{\cong}\ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^1)\ \xrightarrow{\sim}\ \mathrm{map}(S^V,\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^{V\oplus\mathbb{R}}))\ .$$

The first map is the eigenspace decomposition, hence a homeomorphism. The second map is adjoint to the assembly map

$$\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^1)\wedge S^V\ \longrightarrow\ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^{V\oplus\mathbb{R}})$$

of the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$. Since V is ample, $\mathrm{Sym}(V_{\mathbb{C}})$ is a complete complex G-universe, and so $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ is a special $\mathbf{\Gamma}$-G-space by Theorem 4.17. So the adjoint assembly map is a G-weak equivalence by Shimakawa's theorem [**145**, Thm. B]. □

The set $[A,\mathbf{Gr}]^G$ has an abelian monoid structure arising from the ultra-commutative multiplication of $\mathbf{Gr}$ as explained in (4.5) of Chapter II. The set $[A,\Omega^\bullet\mathbf{ku}]^G$ has an abelian group structure as an equivariant stable homotopy group, i.e., through the adjunction bijection

$$[A,\Omega^\bullet\mathbf{ku}]^G\ \cong\ \pi_0^G(\mathrm{map}(A,\mathbf{ku}))\ ;$$

unraveling this reveals this group structure as arising from concatenation of loops. The morphism of orthogonal spaces $c\ :\ \mathbf{Gr}^{\mathbb{C}}\ \longrightarrow\ \Omega^\bullet(\mathbf{ku})$ is *not* a homomorphism of ultra-commutative monoids, nor is it a loop map; so it is not a priori clear whether the induced map on equivariant homotopy sets is a homomorphism of monoids.

THEOREM 4.29. *Let G be a compact Lie group G and A a finite G-CW-complex. Then the map*

$$[A,\mathbf{Gr}]^G\ \xrightarrow{[A,c]^G}\ [A,\Omega^\bullet\mathbf{ku}]^G$$

is a monoid homomorphism. If all isotropy groups of A are finite, then $[A,c]^G$ is a group completion of abelian monoids.

Proof. To show that $[A,c]^G$ is a monoid homomorphism we exhibit a 'delooping' of c, namely the eigenspace morphism (4.26). In (5.39) of Chapter II we defined a morphism of ultra-commutative monoids

$$\beta' \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \Omega\mathbf{U}$$

that is a global group completion by Corollary II.5.40. For every compact Lie group G and every finite G-CW-complex A, the map

$$[A,\beta']^G \ : \ [A,\mathbf{Gr}^{\mathbb{C}}]^G \ \longrightarrow \ [A,\Omega\mathbf{U}]^G$$

is a group completion of abelian monoids, also by Corollary II.5.40. Now we link the monoid homomorphism $[A,\beta']^G$ to the set map $[A,c]^G : [A,\mathbf{Gr}]^G \longrightarrow [A,\Omega^\bullet\mathbf{ku}]^G$. We define a morphism of orthogonal spectra

$$\omega \ : \ \mathbf{ku} \ \longrightarrow \ \Omega(s\mathbf{ku})$$

in level V as the map

$$\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^V) \ \longrightarrow \ \mathrm{map}(S^1,\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^{V\oplus\mathbb{R}}))$$

adjoint to the assembly map

$$\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^V)\wedge S^1 \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^{V\oplus\mathbb{R}}) \ .$$

We claim that the morphism $\Omega^\bullet\omega$ is a global equivalence of orthogonal spaces. Indeed, the embedding $i_V : V \longrightarrow V\oplus\mathbb{R}$ as the first summand induces a continuous map

$$s\mathbf{ku}(V) \ = \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}),S^{V\oplus\mathbb{R}}) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}),S^{V\oplus\mathbb{R}}) \ = \ (\mathrm{sh}\,\mathbf{ku})(V) \ ;$$

as V varies, these maps form a morphism of orthogonal spectra $j : s\mathbf{ku} \longrightarrow \mathrm{sh}\,\mathbf{ku}$. If V is an ample orthogonal G-representation, then $\mathrm{Sym}(i_{\mathbb{C}}) : \mathrm{Sym}(V_{\mathbb{C}}) \longrightarrow \mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}})$ is a linear isometric embedding between complete complex G-universes, so the map $j(V) : s\mathbf{ku}(V) \longrightarrow (\mathrm{sh}\,\mathbf{ku})(V)$ is a G-homotopy equivalence by Proposition 4.15. Hence for ample G-representations, the map

$$(\Omega^\bullet\Omega j)(V) \ : \ (\Omega^\bullet\Omega(s\mathbf{ku}))(V) \ \longrightarrow \ (\Omega^\bullet\Omega(\mathrm{sh}\,\mathbf{ku}))(V)$$

is a G-homotopy equivalence. Since the ample G-representations are cofinal in the poset $s(\mathcal{U}_G)$, this shows that $\Omega^\bullet(\Omega j) : \Omega^\bullet(\Omega(s\mathbf{ku})) \longrightarrow \Omega^\bullet(\Omega(\mathrm{sh}\,\mathbf{ku}))$ is a global equivalence of orthogonal spaces. On the other hand, the adjunction isomorphisms

$$(\Omega^\bullet(\Omega(\mathrm{sh}\,\mathbf{ku})))(V) \ = \ \mathrm{map}(S^V,\Omega\mathbf{ku}(V\oplus\mathbb{R})) \ \cong \ \mathrm{map}(S^{V\oplus\mathbb{R}},\mathbf{ku}(V\oplus\mathbb{R})) \ = \ (\mathrm{sh}_\oplus(\Omega^\bullet\mathbf{ku}))(V)$$

provide an isomorphism of orthogonal spaces between $\Omega^\bullet(\Omega(\mathrm{sh}\,\mathbf{ku}))$ and $\mathrm{sh}_\oplus(\Omega^\bullet\mathbf{ku})$, where $\mathrm{sh}_\oplus = \mathrm{sh}_\oplus^{\mathbb{R}}$ is the additive shift as defined in Example I.1.12. Under this isomorphism, the composite

$$\Omega^\bullet\mathbf{ku} \ \xrightarrow{\Omega^\bullet\omega} \ \Omega^\bullet(\Omega(s\mathbf{ku})) \ \xrightarrow{\Omega^\bullet(\Omega j)} \ \Omega^\bullet(\Omega(\mathrm{sh}\,\mathbf{ku}))$$

becomes the morphism

$$(\Omega^\bullet\mathbf{ku})\circ i \ : \ \Omega^\bullet\mathbf{ku} \ \longrightarrow \ \mathrm{sh}_\oplus(\Omega^\bullet\mathbf{ku})$$

given by precomposition with the embeddings $i_V : V \longrightarrow V\oplus\mathbb{R}$. This morphism is a global equivalence by Theorem I.1.11. Since the composite and $\Omega^\bullet(\Omega j)$ are global equivalences, so is the morphism $\Omega^\bullet\omega : \Omega^\bullet\mathbf{ku} \longrightarrow \Omega(\Omega^\bullet(s\mathbf{ku}))$.

The rigorous statement of the delooping property of the eigenspace morphism is the following commutative square of orthogonal spaces:

$$\begin{tikzcd}
\mathbf{Gr}^{\mathbb{C}} \arrow[r, "c"] \arrow[d, "\beta'"'] & \Omega^\bullet\mathbf{ku} \arrow[d, "\Omega^\bullet\omega", "\simeq"'] \\
\Omega\mathbf{U} \arrow[r, "\Omega\,\mathrm{eig}"'] & \Omega(\Omega^\bullet(s\mathbf{ku}))
\end{tikzcd}$$

This square induces a commutative square of set maps

$$\begin{tikzcd} {[A,\mathbf{Gr}^{\mathbb{C}}]G} \arrow[r, "{[A,c]^G}"] \arrow[d, "{[A,\beta']^G}"', "\cong"] & {[A,\Omega^\bullet\mathbf{ku}]^G} \arrow[d, "{[A,\Omega^\bullet\omega]^G}", "\cong"'] \\ {[A,\Omega\mathbf{U}]^G} \arrow[r, "{[A,\Omega\,\mathrm{eig}]^G}"'] & {[A,\Omega\,(\Omega^\bullet(s\mathbf{ku}))]^G} \end{tikzcd} \tag{4.30}$$

The set $[A,\Omega\mathbf{U}]^G$ can be endowed with a monoid structure in two ways, via the ultra-commutative multiplication as in (4.5) of Chapter II, and by concatenation of loops. Since the ultra-commutative monoid structure of $\Omega\mathbf{U}$ is 'pointwise' (i.e., induced by the ultra-commutative monoid structure of $\mathbf{U}$), these two monoid structures satisfy the interchange law, so they coincide, and both are abelian group structures. The morphism β' is a homomorphism of ultra-commutative monoids, so it induces an additive map on $[A,-]^G$. The morphisms $\Omega\,\mathrm{eig}$ and $\Omega^\bullet\omega$ are loop maps, so they induce homomorphisms with respect to the group structure by concatenation of loops. This shows that three of the four maps in (4.30) are homomorphisms of abelian monoids. Since the vertical maps are isomorphisms, this also shows that the map $[A,c]^G$ is a homomorphism of abelian monoids.

For the rest of the proof we suppose that the isotropy groups of A are finite. The morphism $\Omega\,\mathrm{eig}$ is a $\mathcal{F}in$-global equivalence (by Theorem 4.28), so the map $[A,\Omega\,\mathrm{eig}]^G$ is then bijective, and hence an isomorphism of abelian groups, by Proposition I.5.2 (ii). The morphism $\Omega^\bullet\omega : \Omega^\bullet\mathbf{ku} \longrightarrow \Omega(\Omega^\bullet(s\mathbf{ku}))$ is a global equivalence or orthogonal spaces, so the map $[A,\Omega^\bullet\omega]^G$ is bijective, and hence an isomorphism of abelian groups, again by Proposition I.5.2 (ii). Since $[A,\beta']^G$ is a group completion of abelian monoids, the commutativity of the square (4.30) shows at the same time that $[A,c]^G$ is a homomorphism and a group completion of abelian monoids. □

We draw an important consequence of Theorem 4.29, namely that the equivariant cohomology theory represented by the connective global K-theory spectrum $\mathbf{ku}$ is essentially equivariant K-theory. There is a caveat, however, as this is *not* true on arbitrary finite G-CW-complexes, but only under the hypothesis of finite stabilizer groups.

We let G be a compact Lie group and A a compact G-space. We define the 0-th G-equivariant $\mathbf{ku}$-cohomology group of A as

$$\mathbf{ku}^0_G(A_+) \;=\; [A,\Omega^\bullet\mathbf{ku}]^G\ ,$$

the equivariant homotopy set into the orthogonal space $\Omega^\bullet\mathbf{ku}$. This set has an abelian group structure by concatenation of loops, i.e., via the adjunction bijection

$$[A,\Omega^\bullet\mathbf{ku}]^G \;\cong\; \pi_0^G(\mathrm{map}(A,\mathbf{ku}))$$

to an equivariant stable homotopy group. The set also has a multiplication, i.e., another commutative binary operation arising from the ring spectrum structure of $\mathbf{ku}$, which turns $\Omega^\bullet\mathbf{ku}$ into a 'multiplicative' ultra-commutative monoid as in Example IV.1.16. A conjugation involution of the ultra-commutative ring spectrum $\mathbf{ku}$ was defined in Construction 4.13.

THEOREM 4.31. *Let G be a compact Lie group and A a finite G-CW-complex.*

(i) *There is a unique homomorphism of abelian groups*

$$[-]\ :\ \mathbf{K}_G(A)\ \longrightarrow\ \mathbf{ku}^0_G(A_+)$$

from the G-equivariant K-group of A such that the composite

$$[A,\mathbf{Gr}^{\mathbb{C}}]^G\ \xrightarrow{\langle -\rangle}\ \mathbf{K}_G(A)\ \xrightarrow{[-]}\ \mathbf{ku}^0_G(A_+)\ =\ [A,\Omega^\bullet\mathbf{ku}]^G$$

is the map induced by the morphism of orthogonal spaces $c : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega^\bullet\mathbf{ku}$.

(ii) *For every G-representation V and every continuous G-map $f : A \longrightarrow Gr^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))$, the homomorphism $[-]$ sends the class of the G-vector bundle $f^\star(\gamma^{\mathbb{C}}_{\mathrm{Sym}(V_{\mathbb{C}})})$ to the homotopy class of the G-map*

$$A \xrightarrow{[f(-);-]} \mathrm{map}(S^V, \mathbf{ku}(V)) \ , \quad a \longmapsto \{v \longmapsto [f(a);v]\} \ .$$

(iii) *The additive map $[-]$ is a ring homomorphism, natural for G-maps in A, natural for restriction homomorphisms in G, and compatible with complex conjugation.*

(iv) *If A has finite isotropy groups, then the homomorphism $[-]$ is an isomorphism.*

Proof. (i) The map

$$\langle - \rangle \ : \ [A, \mathbf{Gr}^{\mathbb{C}}]^G \ \longrightarrow \ \mathbf{K}_G(A) \ , \quad [f] \ \longmapsto \ [f^\star(\gamma^{\mathbb{C}}_V)]$$

is a group completion of abelian monoids by Theorem II.4.11 (or rather its complex analog). Since $[A,c]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow [A, \Omega^\bullet \mathbf{ku}]^G$ is a monoid homomorphism (by Theorem 4.29) to an abelian group, the universal property of group completion provides a unique additive extension to $\mathbf{K}_G(A)$.

(ii) We recall from Example II.3.18 the multiplicative Grassmannian $\mathbf{Gr}_\otimes$ with values

$$\mathbf{Gr}_\otimes(V) \ = \ \coprod_{n \geq 0} Gr_n(\mathrm{Sym}(V)) \ ,$$

the disjoint union of all Grassmannians in the symmetric algebra of V. For an inner product space V we let $i : V \longrightarrow \mathrm{Sym}(V)$ be the embedding as the linear summand of the symmetric algebra. Then as V varies, the maps

$$i(V) \ : \ \mathbf{Gr}(V) \ \longrightarrow \ \mathbf{Gr}_\otimes(V) \ , \quad L \ \longmapsto \ i(L)$$

form a global equivalence $\mathbf{Gr} \longrightarrow \mathbf{Gr}_\otimes$ of orthogonal spaces, see Example II.3.18. The morphism of orthogonal spaces $c : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \Omega^\bullet(\mathbf{ku})$ defined in (4.25) has an extension

$$c_\otimes \ : \ \mathbf{Gr}^{\mathbb{C}}_\otimes \ \longrightarrow \ \Omega^\bullet(\mathbf{ku}) \ ,$$

defined by the same formula as for c, namely

$$c_\otimes(V) \ : \ \mathbf{Gr}^{\mathbb{C}}_\otimes(V) \ \longrightarrow \ \mathrm{map}(S^V, \mathbf{ku}(V)) \ = \ (\Omega^\bullet \mathbf{ku})(V) \ , \quad L \ \longmapsto \ [L;-]$$

with

$$[L;-] \ : \ S^V \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}(V) \ , \quad v \ \longmapsto \ [L;v] \ .$$

The 'pullback bundle' map $\langle - \rangle : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$ also has a straightforward extension

$$\langle - \rangle_\otimes \ : \ [A, \mathbf{Gr}^{\mathbb{C}}_\otimes]^G \ \longrightarrow \ \mathbf{K}_G(A)$$

again given by the same recipe: we send the class represented by a G-map $f : A \longrightarrow Gr^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))$ to the class of the pullback $f^\star(\gamma^{\mathbb{C}}_{\mathrm{Sym}(V_{\mathbb{C}})})$ of the tautological bundle over $Gr^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))$. In the diagram

$$\begin{tikzcd}
{[A, \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "\langle - \rangle"] \arrow[d, "{[A,i]^G}"', "\cong"] & \mathbf{K}_G(A) \arrow[d, "{[-]}"] \\
{[A, \mathbf{Gr}^{\mathbb{C}}_\otimes]^G} \arrow[r, "{[A,c^{\mathbb{C}}_\otimes]^G}"'] \arrow[ur, "\langle - \rangle_\otimes"] & {[A, \Omega^\bullet \mathbf{ku}]^G}
\end{tikzcd}$$

the outer square commutes because $c_\otimes \circ i = c$ and $[A,c]^G = [-] \circ \langle - \rangle$ by part (i). The upper left triangle commutes because the tautological bundle on $\mathbf{Gr}^{\mathbb{C}}_\otimes(V)$ restricts to the tautological bundle on $\mathbf{Gr}^{\mathbb{C}}(V)$ along the map $i(V) : \mathbf{Gr}^{\mathbb{C}}(V) \longrightarrow \mathbf{Gr}^{\mathbb{C}}_\otimes(V)$. The left vertical map is bijective by Proposition I.5.2 (ii), because $i : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{Gr}^{\mathbb{C}}_\otimes$ is a global equivalence. So the lower right triangle commutes as well. The map

$$[A, c_\otimes]^G \ : \ [A, \mathbf{Gr}^{\mathbb{C}}_\otimes]^G \ \longrightarrow \ [A, \Omega^\bullet \mathbf{ku}]^G \ = \ \mathbf{ku}^0_G(A_+)$$

sends the homotopy class of a continuous G-map $f : A \longrightarrow Gr^{\mathbb{C}}(\mathrm{Sym}(V_{\mathbb{C}}))$ to the class of the map $[f(-); -] : A \longrightarrow \mathrm{map}(S^V, \mathbf{ku}(V))$, by the very definition of $c_{\otimes}$. So this proves the claim.

(iii) Naturality in G-maps is straightforward. For a G-map $h : A' \longrightarrow A$ and every G-map $f : A \longrightarrow \mathbf{Gr}^{\mathbb{C}}(V)$, the two G-vector bundles $h^\star(f^\star(\gamma_V^{\mathbb{C}}))$ and $(fh)^\star(\gamma_V^{\mathbb{C}})$ over A' are isomorphic, so the following square commutes:

$$\begin{tikzcd} {[A, \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "\langle - \rangle"] \arrow[d, "{[h, \mathbf{Gr}^{\mathbb{C}}]^G}"'] & \mathbf{K}_G(A) \arrow[d, "\mathbf{K}_G(h)"] \\ {[A', \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "\langle - \rangle"'] & \mathbf{K}_G(A') \end{tikzcd}$$

Together with the characterizing property of the homomorphism $[-]$ of part (i), this implies that the two group homomorphisms

$$[-] \circ \mathbf{K}_G(h) \ , \ \mathbf{ku}_G^0(h_+) \circ [-] \ : \ \mathbf{K}_G(A) \ \longrightarrow \ \mathbf{ku}_G^0(A'_+)$$

coincide after precomposition with $\langle - \rangle : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$. Since this map is a group completion of abelian monoids, already the homomorphisms $[-] \circ \mathbf{K}_G(h)$ and $\mathbf{ku}_G^0(h_+) \circ [-]$ already agree, by the universal property of group completions.

The compatibility with complex conjugation and restriction along group homomorphisms follow the same pattern. The conjugation morphism of orthogonal spectra $\psi : \mathbf{ku} \longrightarrow \mathbf{ku}$ deloops the conjugation morphism of ultra-commutative monoids $\psi : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{Gr}^{\mathbb{C}}$, in the sense that the square of orthogonal spaces commutes:

$$\begin{tikzcd} \mathbf{Gr}^{\mathbb{C}} \arrow[r, "c"] \arrow[d, "\psi"'] & \Omega^\bullet \mathbf{ku} \arrow[d, "\Omega^\bullet \psi"] \\ \mathbf{Gr}^{\mathbb{C}} \arrow[r, "c"'] & \Omega^\bullet \mathbf{ku} \end{tikzcd}$$

In particular, the homomorphism $[A, c]^G : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{ku}_G^+(A_+)$ commutes with complex conjugation. On the other hand, every G-map $f : A \longrightarrow \mathbf{Gr}^{\mathbb{C}}(V)$, the bundle $(\psi(V) \circ f)^\star(\gamma_V^{\mathbb{C}})$ is isomorphic to the complex conjugate of the bundle $f^\star(\gamma_V^{\mathbb{C}})$. So the following square commutes:

$$\begin{tikzcd} {[A, \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "\langle - \rangle"] \arrow[d, "{[A, \psi]^G}"'] & \mathbf{K}_G(A) \arrow[d, "\psi"] \\ {[A, \mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "\langle - \rangle"'] & \mathbf{K}_G(A) \end{tikzcd}$$

Thus the two group homomorphisms

$$[-] \circ \psi \ , \ \psi \circ [-] \ : \ \mathbf{K}_G(A) \ \longrightarrow \ \mathbf{ku}_G^0(A_+)$$

coincide after precomposition with a group completion, hence they coincide. The analogous argument works for restriction along a continuous homomorphism $\alpha : K \longrightarrow G$, using that the underlying K-vector bundle of $f^\star(\gamma_V^{\mathbb{C}})$ equals the bundle $\alpha^*(f)(\gamma_{\alpha^*(V)}^{\mathbb{C}})$, and the effect of c – being a morphism of orthogonal space – commutes with restriction.

Now we show that the additive map $[-] : \mathbf{K}_G(A) \longrightarrow \mathbf{ku}_G^0(A_+)$ is a ring homomorphism. The multiplicative unit of $\mathbf{K}_G(A)$ is the class of the trivial 1-dimensional line bundle $A \times \mathbb{C}$. This bundle is isomorphic to $f^\star(\gamma_{\mathrm{Sym}(0)}^{\mathbb{C}})$ for the constant map $f : A \longrightarrow Gr^{\mathbb{C}}(\mathrm{Sym}(0))$ with value $\mathbb{C}$, the constant (and only non-trivial) summand in the symmetric algebra associated to the 0-dimensional G-representation. The associated G-map $[f(-); -] : A \longrightarrow \mathrm{map}(S^0, \mathbf{ku}(0))$ is constant with image the unit map $\iota_0 : S^0 \longrightarrow \mathbf{ku}(0)$;

so by part (ii) the class $\langle A \times \mathbb{C}\rangle$ of the trivial line bundle is the multiplicative unit in the ring $\mathbf{ku}^0_G(A_+)$. So the map $[-]$ preserves multiplicative units.

It remains to show that the map $[-]$ preserves products. Because $[-]$ is additive and the group $\mathbf{K}_G(A)$ is generated by classes of actual vector bundles (as opposed to virtual bundles), it suffices to show multiplicativity for two classes in $\mathbf{K}_G(A)$ represented by G-vector bundles. We may assume that the two bundles are classified by continuous G-maps

$$f \ : \ A \ \longrightarrow \ Gr^{\mathbb{C}}(V_{\mathbb{C}}) \qquad \text{respectively} \qquad \bar{f} \ : \ A \ \longrightarrow \ Gr^{\mathbb{C}}(W_{\mathbb{C}}) \ ,$$

where V and W are two G-representations. The tensor product bundle $f^\star(\gamma^{\mathbb{C}}_V) \otimes \bar{f}^\star(\gamma^{\mathbb{C}}_W)$ is then classified by the composite

$$A \ \xrightarrow{(f,\bar{f})} Gr^{\mathbb{C}}(V_{\mathbb{C}}) \times Gr^{\mathbb{C}}(W_{\mathbb{C}}) \ \xrightarrow{-\otimes-} \ Gr^{\mathbb{C}}(V_{\mathbb{C}} \otimes W_{\mathbb{C}}) \ \xrightarrow{Gr^{\mathbb{C}}(j)} \ Gr^{\mathbb{C}}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}})) \ ,$$

where

$$j \ : \ V_{\mathbb{C}} \otimes W_{\mathbb{C}} \ \longrightarrow \ \mathrm{Sym}((V \oplus W)_{\mathbb{C}}) \qquad \text{is defined by} \qquad j(v \otimes w) = (v,0) \cdot (0,w) \ .$$

By part (ii) the associated homotopy class $[f^\star(\gamma^{\mathbb{C}}_V) \otimes \bar{f}^\star(\gamma^{\mathbb{C}}_W)]$ is represented by the G-map

$$[(j \circ (f \otimes \bar{f}))(-); -] \ : \ A \ \longrightarrow \ \mathrm{map}(S^{V \oplus W}, \mathbf{ku}(V \oplus W)) \ .$$

This map is the adjoint of the composite

$$A_+ \wedge S^{V \oplus W} \ \xrightarrow{a \wedge v \wedge w \mapsto [f(a);v] \wedge [\bar{f}(a);w]} \ \mathbf{ku}(V) \wedge \mathbf{ku}(W) \ \xrightarrow{\mu_{V,W}} \ \mathbf{ku}(V \oplus W) \ .$$

This composite represents the product of the classes $[f^\star(\gamma^{\mathbb{C}}_V)]$ and $[\bar{f}^\star(\gamma^{\mathbb{C}}_W)]$, so this establishes the relation

$$[f^\star(\gamma^{\mathbb{C}}_V)] \ \cdot \ [\bar{f}^\star(\gamma^{\mathbb{C}}_W)] \ = \ [f^\star(\gamma^{\mathbb{C}}_V) \otimes \bar{f}^\star(\gamma^{\mathbb{C}}_W)]$$

in the group $\mathbf{ku}^0_G(A)$.

(iv) If A is a finite G-CW-complex with finite isotropy groups, then the map $[A,c]^G : [A, \mathbf{Gr}]^G \longrightarrow \mathbf{ku}^0_G(A_+)$ is a group completion of abelian monoids by Theorem 4.29. The map $\langle - \rangle : [A, \mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$ is also a group completion, so the unique extension $[-] : \mathbf{K}_G(A) \longrightarrow \mathbf{ku}^0_G(A_+)$ is an isomorphism. □

We specialize Theorem 4.31 to the case $A = *$, i.e., when the base is a single point. In this case the bundle projection is no information, G-vector bundles over a point specialize to G-representations, and the ring $\mathbf{K}_G(*)$ becomes the unitary representation ring $\mathbf{RU}(G)$. On the other hand, $\mathbf{ku}^0_G(S^0)$ specializes to $\pi^G_0(\mathbf{ku})$. The ring homomorphism

$$[-] \ : \ \mathbf{RU}(G) \ \longrightarrow \ \pi^G_0(\mathbf{ku})$$

is then determined by its effect on the classes of actual representations; if W is a unitary G-representation, we can spell out an explicit representative for the class $\langle W \rangle$ as follows. The map

$$j_W \ : \ W \ \longrightarrow \ (uW)_{\mathbb{C}} \ , \quad j_W(w) \ \longmapsto \ 1/\sqrt{2} \cdot (1 \otimes w - i \otimes (iw)) \tag{4.32}$$

is a G-equivariant $\mathbb{C}$-linear isometric embedding into the complexification of the underlying orthogonal G-representation of W. So $[W]$ is the homotopy class of the G-map

$$S^{uW} \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}((uW)_{\mathbb{C}}), S^{uW}) \ = \ \mathbf{ku}(uW) \ , \quad w \ \longmapsto \quad [j_W(W); w] \ .$$

Both $\mathbf{RU}(G)$ and $\pi^G_0(\mathbf{ku})$ have restriction maps, transfers and multiplicative power operations in G, i.e., they are global power functors in the sense of Definition 1.5. The maps $[-] : \mathbf{RU}(G) \longrightarrow \pi^G_0(\mathbf{ku})$ preserve most of this additional structure, but that is not apparent from what we discussed so far.

THEOREM 4.33. *For every compact Lie group G, the map*

$$[-] \ : \ \mathbf{RU}(G) \ \longrightarrow \ \pi^G_0(\mathbf{ku}) \tag{4.34}$$

is a ring homomorphism. As G varies over all compact Lie groups, the homomorphisms (4.34) *are compatible with restriction maps, with complex conjugation, with finite index transfers, and with multiplicative power operations. Moreover, the map $[-]$ is an isomorphism whenever the group G is finite.*

Proof. The fact that the map is a ring homomorphism, compatible with restrictions, compatible with complex conjugation and an isomorphism for finite groups is a special case of Theorem 4.31 for a one-point G-space.

Now we show that the maps $[-]$ are compatible with finite index transfers. We let T be a finite G-set. We define an orthogonal G-spectrum $\mathbf{ku}[T]$ by

$$\mathbf{ku}[T](V) \;=\; \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \wedge T_+)\ .$$

The structure maps of $\mathbf{ku}[T]$ are defined in much the same way as for $\mathbf{ku}$, with the extra smash factor T_+ acting as a dummy. We define a morphism of orthogonal G-spectra

$$\Psi \;:\; \mathbf{ku}[T] \;\longrightarrow\; \mathrm{map}(T, \mathbf{ku})$$

at an inner product space V as the map

$$\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \wedge T_+) \;\longrightarrow\; \mathrm{map}(T, \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V))$$

whose s-th component is induced by $S^V \wedge p_s : S^V \wedge T_+ \longrightarrow S^V$.

We claim that for every compact Lie group G and every finite G-set T the morphism Ψ is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra. To prove the claim we show that Ψ induces an isomorphism of geometric H-fixed point homotopy groups for every closed subgroup H of G. Since the situation is stable under passage to subgroups, we may assume that $H = G$. If V is ample, then the map

$$\Psi(V)^G \;:\; \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V \wedge T_+)^G \;\longrightarrow\; \mathrm{map}^G(T, \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V))$$

is a weak equivalence since $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V)$ is a special $\mathbf{\Gamma}$-G-space. [...generalize Theorem 4.17 to show this...] Since the ample G-representations are cofinal in the poset $s(\mathcal{U}_G)$, the morphism Ψ induces an isomorphism on Φ^G_*.

Now we let H be a finite index subgroup of G. The composite

$$\mathbf{ku} \wedge G/H_+ \;\xrightarrow{\alpha}\; \mathbf{ku}[G/H] \;\xrightarrow{\Psi}\; \mathrm{map}(G/H, \mathbf{ku})$$

is a $\underline{\pi}_*$-isomorphism of orthogonal G-spectra by the Wirthmüller isomorphism. So the assembly morphism $\alpha : \mathbf{ku} \wedge G/H_+ \longrightarrow \mathbf{ku}[G/H]$ induces an isomorphism on G-equivariant stable homotopy groups. We let V be an orthogonal H-representation. Then the following diagram

$$\begin{tikzcd}
\pi_0(\mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), 1^+)^H) \arrow[rrr, "\kappa"] & & & \pi_0^H(\mathbf{ku}) \\
\pi_0(\mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), G/H_+)^G) \arrow[u, "(\mathrm{ev}_H)_*"] \arrow[r, "\kappa"] \arrow[d, "\mathrm{proj}_*"'] & \pi_0^G(\mathbf{ku}[G/H]) \arrow[urr, "(\mathrm{ev}_H)_*"] \arrow[drr, "\mathrm{proj}_*"'] & & \pi_0^G(\mathbf{ku} \wedge G/H_+) \arrow[ul, "\cong"', "\alpha_*"] \arrow[u, "\mathrm{Wirth}_H^G"'] \arrow[d, "\mathrm{proj}_*"] \\
\pi_0(\mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), 1^+)^G) \arrow[rrr, "\kappa"'] & & & \pi_0^G(\mathbf{ku})
\end{tikzcd}$$

commutes by naturality of the κ-maps and direct inspection of the definitions. The map $(\mathrm{ev}_H)_*$ projects to the part of a configuration that sits over the distinguished coset H of G/H; more formally, it is defined as the composite of the effect of the inclusion

$$\mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), G/H_+)^G \;\longrightarrow\; \mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), G/H_+)^H$$

and the map induces by the H-map $p_H : G/H_+ \longrightarrow 1^+$ with $p_H(H) = 1$ and $p_H(gH) = 0$ for $g \notin H$. The three horizontal maps labeled κ are induced by the G-map

$$\mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), S_+) \;\longrightarrow\; \mathrm{map}(S^{G \otimes_H V}, \mathbf{ku}[S_+](G \otimes_H V))$$

adjoint to the assembly map (see (5.16) of Chapter IV)

$$S^{G \otimes_H V} \wedge \mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), S_+) \;\longrightarrow\; \mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), S^{G \otimes_H V} \wedge S_+) \;=\; \mathbf{ku}[S_+](G \otimes_H V)\ ,$$

followed by the canonical maps to the colimit

$$\pi_0\left(\mathrm{map}(S^{G \otimes_H V}, \mathbf{ku}[S_+](G \otimes_H V))^G\right) \;=\; [S^{G \otimes_H V}, \mathbf{ku}[S_+](G \otimes_H V)]^G \;\longrightarrow\; \pi_0^G(\mathbf{ku}[S_+])\ ,$$

and similarly for H instead of G.

Since the transfer $\mathrm{tr}_H^G : \pi_0^H(\mathbf{ku}) \longrightarrow \pi_0^G(\mathbf{ku})$ is defined as the composite of the inverse Wirthmüller isomorphism and the effect of the projection $G/H \longrightarrow *$, we conclude that the following diagram of abelian monoids also commutes:

$$\begin{tikzcd}
\pi_0(\mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), 1^+)^H) \arrow[r, "\kappa"] & \pi_0^H(\mathbf{ku}) \arrow[dd, "\mathrm{tr}_H^G"] \\
\pi_0(\mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), G/H_+)^G) \arrow[u, "(\mathrm{ev}_H)_*"] \arrow[d, "\mathrm{proj}_*"'] & \\
\pi_0(\mathscr{C}(\mathrm{Sym}((G \otimes_H V)_{\mathbb{C}}), 1^+)^G) \arrow[r, "\kappa"'] & \pi_0^G(\mathbf{ku})
\end{tikzcd} \tag{4.35}$$

Now we can prove the compatibility of the ring homomorphisms $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ with finite index transfers. We let W be a unitary H-representation and $g_1, \dots, g_m$ be a set of coset representatives for G/H. We consider the configuration

$$[g_1 \otimes W, \dots, g_m \otimes W;\, g_1 H, \dots, g_m H] \;\in\; \mathscr{C}(\mathrm{Sym}(G \otimes_H W), G/H_+)\ .$$

This configuration is G-invariant, so its path component is an element of the monoid $\pi_0(\mathscr{C}(\mathrm{Sym}(G \otimes_H W), G/H_+)^G)$. We have

$$\mathrm{proj}_*[g_1 \otimes W, \dots, g_m \otimes W;\, g_1 H, \dots, g_m H] \;=\; [g_1 \otimes W, \dots, g_m \otimes W; 1, \dots, 1] \;=\; [G \otimes_H W;\, 1]$$

in $\pi_0(\mathscr{C}(\mathrm{Sym}(G \otimes_H W), 1^+)^G)$. So the composite through the lower left corner of the diagram (4.35) takes the class of this configuration to $\langle G \otimes_H W \rangle$ in $\pi_0^H(\mathbf{ku})$. On the other hand,

$$(\mathrm{ev}_H)_*[g_1 \otimes W, \dots, g_m \otimes W;\, g_1 H, \dots, g_m H] \;=\; [g_i \otimes W; 1] \;=\; [1 \otimes W; 1]$$

in $\pi_0(\mathscr{C}(\mathrm{Sym}(G \otimes_H W), 1^+)^H)$, where g_i is the representative of the preferred coset H. So the composite through the upper right corner of (4.35) takes the class to $\mathrm{tr}_H^G\langle W \rangle$. This proves the desired relation $\langle G \otimes_H W \rangle = \mathrm{tr}_H^G\langle W \rangle$ for classes of actual representations. Since transfer maps are additive, the relation persists to classes of virtual representations.

Now we treat the compatibility with multiplicative power operations, i.e., that the following square commutes for every compact Lie group G and all $m \geq 1$:

$$\begin{tikzcd}
\mathbf{RU}(G) \arrow[r, "{[-]}"] \arrow[d, "P^m"'] & \pi_0^G(\mathbf{ku}) \arrow[d, "P^m"] \\
\mathbf{RU}(\Sigma_m \wr G) \arrow[r, "{[-]}"'] & \pi_0^{\Sigma_m \wr G}(\mathbf{ku})
\end{tikzcd}$$

We consider a unitary G-representation W. The class $\langle W \rangle$ is represented by the G-map

$$[j_W(W); -] \;:\; S^W \;\longrightarrow\; \mathbf{ku}(W)\ ,$$

where $j = j_W : W \longrightarrow (uW)_{\mathbb{C}}$ was defined in (4.32). The map

$$\begin{array}{rcl} J \ : \ W^{\otimes m} & \longrightarrow & \mathrm{Sym}((uW)^m_{\mathbb{C}}) \\ w_1 \otimes \cdots \otimes w_m & \longmapsto & (j_W(w_1), 0, \ldots, 0) \cdot (0, j_W(w_2), 0, \ldots, 0) \cdot \ldots \cdot (0, \ldots, 0, j_W(w_m)) \ . \end{array}$$

is a $(\Sigma_m \wr G)$-equivariant linear isometric embedding. So by Theorem 4.31 (ii), the class $[W^{\otimes m}] = [P^m\langle W\rangle]$ is represented by the G-map

$$[J(W^{\otimes m}); -] \ : \ S^{W^m} \ \longrightarrow \ \mathbf{ku}(W^m) \ .$$

This map coincides with the composite

$$S^{W^m} \ \xrightarrow{[j(W);-]^{\wedge m}} \ \mathbf{ku}(W)^{\wedge m} \ \xrightarrow{\mu_{W,\ldots,W}} \ \mathbf{ku}(W^m)$$

which represents the power operation $P^m[W]$, so we have shown the relation

$$P^m[W] \ = \ [W^{\otimes m}] \ = \ [P^m\langle W\rangle]$$

in the group $\pi_0^{\Sigma_m \wr G}(\mathbf{ku})$. Since the map $[-]$ is additive, compatible with finite index transfers and the classes of actual representations generate $\mathbf{RU}(G)$ as an abelian group, the additivity formulas for power operations

$$P^m(x+y) \ = \ \sum_{i=0}^{m} \ \mathrm{tr}_{i,m-i}(P^i(x) \times P^{m-i}(y))$$

imply that the map $[-]$ is compatible with power operations of virtual representations. □

Remark 4.36. It is a formal consequence of Theorem 4.33 that the maps (4.34) are also compatible with external multiplication and with norm maps ('multiplicative transfers'). Indeed, the external product

$$\otimes \ : \ \mathbf{RU}(G) \times \mathbf{RU}(K) \ \longrightarrow \ \mathbf{RU}(G \times K)$$

can be obtained from the internal product as

$$x \otimes y \ = \ p_G^*(x) \times p_K^*(Y) \ ,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the projections. Similarly, the norm

$$N_H^G \ : \ \mathbf{RU}(H) \ \longrightarrow \ \mathbf{RU}(G)$$

is the composite of the power operation P^m with a certain restriction map, where $m = [G : H]$. Since the maps $[-]$ commute with restrictions, internal product and power operations, they also commute with external products and norms.

Remark 4.37. Segal [**144**, §2] discusses a 'smooth transfer' $i_! : \mathbf{RU}(H) \longrightarrow \mathbf{RU}(G)$ where $i : H \longrightarrow G$ is the inclusion of a closed subgroup of a compact Lie group G. One could hope that the maps $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ take the smooth transfer to the homotopy theoretic transfer

$$\mathrm{tr}_H^G \ : \ \pi_0^H(\mathbf{ku}) \ \longrightarrow \ \pi_0^G(\mathbf{ku}) \ .$$

This (false!) expectation is suggested by the fact that the smooth transfer has all the right formal properties, and the complex representations rings do form a global power functor.

However, contrary to the obvious guess, the maps $[-]$ do *not* in general transform the smooth transfer into the homotopy theoretic transfers in $\underline{\pi}_0(\mathbf{ku})$; a specific example will be discussed below. In other words, the collection of ring homomorphisms $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ comes very close to being a morphism of global functors, but it fails to commute with infinite index transfers. For *periodic* global K-theory $\mathbf{KU}$, discussed below in Construction 5.7, the two transfers do correspond also for infinite index inclusions; in fact, $\underline{\pi}_0(\mathbf{KU})$ is isomorphic, as a global power functor, to the representation ring global functor, see Theorem 5.17 below.

To illustrate the issue we discuss a specific example and recall the smooth transfer to $G = SU(2)$ from its diagonal maximal torus

$$T \;=\; \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \;:\; \lambda \in U(1) \right\} \;.$$

The representation ring $\mathbf{RU}(T) = \mathbb{Z}[x, x^{-1}]$ is a Laurent polynomial ring generated by the class x of the tautological T-representation on $\mathbb{C}$. The representation ring $\mathbf{RU}(SU(2)) = \mathbb{Z}[s]$ is a polynomial ring generated by the class s of the tautological 2-dimensional representation. The restriction map

$$\mathrm{res}_T^{SU(2)} \;:\; \mathbf{RU}(SU(2)) \;\longrightarrow\; \mathbf{RU}(T)$$

identifies $\mathbf{RU}(SU(2))$ with the polynomial subring of $\mathbf{RU}(T)$ generated by $\mathrm{res}_T^{SU(2)}(s) = x + x^{-1}$.

We use Segal's character formula [**144**, p. 119] to calculate the character of the smooth transfer $i_!(1)$ of the trivial 1-dimensional T-representation. The normalizer of T in $SU(2)$ is generated by T and the matrix $t = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Conjugation by t is the involution of T that interchanges the diagonal entries. So the space of conjugacy classes

$$T/N_{SU(2)}T \;\cong\; SU(2)/\mathrm{conj}$$

is the quotient of T by $\lambda \simeq \bar{\lambda}$. An element $\lambda \in T$ is regular if and only if λ has infinite order. In particular, a regular λ is different from 1 and -1, and the fixed set of a regular λ is thus $F_\lambda = \{T, tT\}$ (independent of λ). So

$$\chi_{i_!(1)}(\lambda) \;=\; \chi_1(\lambda) \;+\; \chi_1(t^{-1}\lambda t) \;=\; 2 \;.$$

Since the character is continuous and determines the representation, we have $i_!(1) = 2$.

We claim that in contrast to this relation in $\mathbf{RU}(SU(2))$, the elements $\mathrm{tr}_T^{SU(2)}(1)$ and 2 are linearly independent (so in particular different) in $\pi_0^{SU(2)}(\mathbf{ku})$. We can detect this through the *dimension homomorphism*, a homomorphism of ultra-commutative ring spectra

$$\dim \;:\; \mathbf{ku} \;\longrightarrow\; \mathcal{H}\mathbb{Z} \tag{4.38}$$

from connective global K-theory to the Eilenberg-Mac Lane spectrum of the integers (in the sense of Construction 1.9). The value of this homomorphism at an inner product space V is the map

$$\begin{aligned} \dim(V) \;:\; \mathbf{ku}(V) = \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \;&\longrightarrow\; \mathbb{Z}[S^V] \;=\; (\mathcal{H}\mathbb{Z})(V) \\ [E_1, \dots, E_n;\, v_1, \dots, v_n] \;&\longmapsto\; \sum_{i=1}^n \dim(E_i) \cdot v_i \;, \end{aligned}$$

i.e., a configuration of vector spaces is mapped to the configurations of its dimensions. The map is multiplicative because dimension is multiplicative on tensor products.

As we discuss in Example 1.16, the elements $\mathrm{tr}_T^{SU(2)}(1)$ and 2 are linearly independent in $\pi_0^{SU(2)}(H\mathbb{Z})$, so they must also be linearly independent in $\pi_0^{SU(2)}(\mathbf{ku})$. The dimension homomorphism can also be used to detect some odd-dimensional classes in the coefficient ring $\pi_*^{U(1)}(\mathbf{ku})$: we show in Theorem 1.18 that the group $\pi_1^{U(1)}(\mathcal{H}\mathbb{Z})$ is isomorphic to $\mathbb{Q}$, and that the dimension shifting transfer from the trivial group

to $U(1)$, applied to the suspension of the multiplicative unit $1 \in \pi_0^e(\mathcal{H}\mathbb{Z})$, is non-zero in $\pi_1^{U(1)}(\mathcal{H}\mathbb{Z})$. So the class

$$\mathrm{Tr}_e^{U(1)}(S^1 \wedge 1) \ \in \ \pi_1^{U(1)}(\mathbf{ku})$$

must also be non-zero.

Remark 4.39. For finite groups G, the ring homomorphism $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ is an isomorphism. It will follow from Theorem 5.17 below that the map $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ is always a split monomorphism, also when G has positive dimension. On the other hand, Remark 4.37 shows that the map is not always surjective, as the class $\mathrm{tr}_T^{SU(2)}(1)$ in $\pi_0^{SU(2)}(\mathbf{ku})$ is not in the image. In [**73**], Hausmann and Ostermayr give a complete calculation of $\underline{\pi}_0(\mathbf{ku})$ as global functor. The strategy of [**73**] is to identify the global homotopy types of the subquotients of the rank filtration (see Construction 4.42 below) and deduce from it a presentation $\underline{\pi}_0(\mathbf{ku})$ by generators an relations. The final answer is that $\underline{\pi}_0(\mathbf{ku})$ is generated as a global functor by the classes

$$x_n \ = \ [\mathbb{C}^n] \ \in \ \pi_0^{U(n)}(\mathbf{ku}) \ ,$$

where $\mathbb{C}^n$ denotes the tautological $U(n)$-representation. There are two kinds of relations; on the one hand, the relations

$$\mathrm{res}_{U(n)\times U(m)}^{U(n+m)}(x_{n+m}) \ = \ p^*(x_n) + q^*(x_m) \quad \in \pi_0^{U(n)\times U(m)}(\mathbf{ku})$$

for all $n, m \geq 1$, where $p : U(n) \times U(m) \longrightarrow U(n)$ and $q : U(n) \times U(m) \longrightarrow U(m)$ are the two projections. Theses relations follow from the fact that the maps $[-]$ are additive and compatible with restrictions. The other set of relations equates finite index transfers of representations with the corresponding finite index transfer in homotopy theory.

Remark 4.40 (Connective real global K-theory). There is a straightforward real analog $\mathbf{ko}$ of the complex connective global K-theory spectrum $\mathbf{ku}$. The value of $\mathbf{ko}$ on a real inner product space V is

$$\mathbf{ko}(V) \ = \ \mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V), S^V) \ ,$$

where now $\mathrm{Sym}(V)$ is the symmetric algebra, over the real numbers, of the real inner product space V, and $\mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V))$ is the $\mathbf{\Gamma}$-space of tuples of pairwise orthogonal, finite dimensional (real) subspaces of $\mathrm{Sym}(V)$. As V varies, the spaces $\mathbf{ko}(V)$ again come with the structure of a commutative orthogonal ring spectrum. There is a morphism of orthogonal spaces

$$c \ : \ \mathbf{Gr} \ \longrightarrow \ \Omega^\bullet(\mathbf{ko})$$

defined in much the same way as its complex analog (4.25). Derived from this are ring homomorphisms

$$[-] \ : \ \mathbf{KO}_G(B) \ \longrightarrow \ \mathbf{ko}_G^0(B_+)$$

for every compact G-space B, analogous to the homomorphism (4.34) in the complex case. For a one-point G-space this specializes to a ring homomorphism

$$[-] \ : \ \mathbf{RO}(G) \ \longrightarrow \ \pi_0^G(\mathbf{ko}) \ ,$$

where $\mathbf{RO}(G)$ is the orthogonal representation ring of G. The analog of Theorem 4.33 then says that these homomorphisms are compatible with restriction, finite index transfer and multiplicative power operations, and an isomorphism for finite groups G. However, in the real situation there is no eigenspace decomposition, hence no direct analog of the delooping $\mathrm{eig} : \mathbf{U} \longrightarrow \Omega^\bullet(\mathrm{sh}\,\mathbf{ku})$ of the morphism c. So to prove the real analog of Theorem 4.29 (which is used in the proof of Theorem 4.33), one has to use a different proof.

Complexification defines a morphism of ultra-commutative ring spectra

$$c \ : \ \mathbf{ko} \ \longrightarrow \ \mathbf{ku} \ . \tag{4.41}$$

In more detail: if V is a real inner product space, then a continuous, $O(V)$-equivariant based map

$$c(V) \ : \ \mathbf{ko}(V) = \mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V), S^V) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) = \mathbf{ku}(V)$$

is defined by sending a configuration $[E_1, \dots, E_n;\, v_1, \dots, v_n]$ to the 'complexified' configuration

$$[(E_1)_{\mathbb{C}}, \dots, (E_n)_{\mathbb{C}};\, v_1, \dots, v_n]\ .$$

As V varies, these maps form the morphism (4.41) of ultra-commutative ring spectra. The complexification of a real subspace of $\mathrm{Sym}(V)$ is invariant under the complex conjugation involution $\psi_{\mathrm{Sym}(V)}$ of $\mathrm{Sym}(V_{\mathbb{C}})$, so the image of the complexification morphism is invariant under the complex conjugation involution ψ of $\mathbf{ku}$ defined in (4.14),

$$\psi \circ c \ = \ c \ : \ \mathbf{ko} \ \longrightarrow \ \mathbf{ku}\ .$$

Even more is true: a complex subspace of $\mathrm{Sym}(V_{\mathbb{C}})$ is ψ-invariant if and only if it is the complexification of its real part (the $+1$ eigenspaces of the restriction of $\psi_{\mathrm{Sym}(V)}$). This means that $\mathbf{ko}$ 'is' the ψ-fixed orthogonal ring subspectrum of $\mathbf{ku}$; more formally, the complexification morphism is an isomorphism

$$c \ : \ \mathbf{ko} \ \cong \ \mathbf{ku}^{\psi}$$

to the ψ-fixed orthogonal ring subspectrum of $\mathbf{ku}$.

Construction 4.42 (Rank filtrations). The connective global K-theory spectra also come with exhaustive multiplicative filtrations

$$\mathbf{ko}^{[1]} \longrightarrow \mathbf{ko}^{[2]} \longrightarrow \dots \longrightarrow \mathbf{ko}^{[m]} \longrightarrow \dots \ \longrightarrow \ \mathbf{ko}$$

respectively

$$\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}^{[2]} \longrightarrow \dots \longrightarrow \mathbf{ku}^{[m]} \longrightarrow \dots \ \longrightarrow \ \mathbf{ku}$$

by orthogonal subspectra. We define the rank filtration for $\mathbf{ku}$, the case of $\mathbf{ko}$ being similar. For $m \geq 1$ we define an orthogonal subspectrum $\mathbf{ku}^{[m]}$ of $\mathbf{ku}$ by

$$\mathbf{ku}^{[m]}(V) \ = \ \mathscr{C}^{[m]}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ \subset \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}(V)$$

Here $\mathbf{ku}^{[m]}(V)$ is the subspace of those configurations $[E_1, \dots, E_n;\, v_1, \dots, v_n]$ such that

$$\sum_{i=1}^{n} \dim(E_i) \ \leq \ m\ .$$

As V varies, the spaces $\mathbf{ku}^{[m]}(V)$ form an orthogonal subspectrum $\mathbf{ku}^{[m]}$ of $\mathbf{ku}$.

The dimension function is multiplicative on tensor products, so the multiplication of $\mathbf{ku}$ restricts to associative and unital pairings

$$\mathbf{ku}^{[m]} \wedge \mathbf{ku}^{[n]} \ \longrightarrow \ \mathbf{ku}^{[mn]}\ .$$

For $m = 1$ this gives $\mathbf{ku}^{[1]}$ the structure of an ultra-commutative ring spectrum and it gives $\mathbf{ku}^{[n]}$ a module structure over $\mathbf{ku}^{[1]}$. The inclusion $\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}$ is multiplicative, i.e., a morphism of ultra-commutative ring spectra.

The first pieces $\mathbf{ko}^{[1]}$ and $\mathbf{ku}^{[1]}$ are multiplicative models for the suspension spectra of global classifying space of the cyclic group $C_2 = O(1)$ respectively the circle group $U(1)$. Indeed, a configuration of points labeled by vector spaces of total dimension 1 has to be concentrated on at most one point. So the map

$$S^V \wedge \mathbf{P}^{\mathbb{C}}(V)_+ \ = \ S^V \wedge P(\mathrm{Sym}(V_{\mathbb{C}}))_+ \ \longrightarrow \ \mathscr{C}^{[1]}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}^{[1]}(V)\ , \quad v \wedge L \ \longmapsto \ [L; v]$$

is a homeomorphism. As V varies through real inner product spaces, these maps form an isomorphism of orthogonal ring spectra

$$\Sigma^{\infty}_{+}\mathbf{P}^{\mathbb{C}} \ \cong \ \mathbf{ku}^{[1]}$$

from the unreduced suspension spectrum of the complex global projective space, introduced in Example II.3.18. Since $\mathbf{P}^{\mathbb{C}}$ is a multiplicative model of the global classifying space of the circle group $U(1)$, the ultra-commutative ring spectrum $\mathbf{ku}^{[1]}$ is globally a suspension spectrum of a global classifying space of the circle group. Similarly, the ultra-commutative ring spectrum $\mathbf{ko}^{[1]}$ is globally a suspension spectrum of a global classifying space of the group C_2.

The underlying non-equivariant spectrum of $\Sigma^\infty_+ B_{\mathrm{gl}}U(1)$, hence of $\mathbf{ku}^{[1]}$, has the stable homotopy type of the suspension spectrum of $\mathbb{C}P^\infty$; moreover, on underlying non-equivariant spectra, the morphism

$$\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku} \tag{4.43}$$

is a rational stable equivalence. However, the morphism (4.43) is *not* a rational global equivalence, which can be seen already on the level of $\underline{\pi}_0$ for the group C_2. Indeed, by Proposition IV.2.5 the homotopy group global functor $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}U(1))$, and hence also the global functor $\underline{\pi}_0(\mathbf{ku}^{[1]})$, is the represented global functor $\mathbf{A}(U(1),-)$. The inclusion $\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}$ induces a morphism of global power functors $\mathbf{A}(U(1),-) \cong \underline{\pi}_0(\mathbf{ku}^{[1]}) \longrightarrow \underline{\pi}_0(\mathbf{ku})$. An element of infinite order in the kernel of the map $\mathbf{A}(U(1),C_2) \longrightarrow \pi_0^{C_2}(\mathbf{ku})$ is

$$\mathrm{tr}_e^{C_2} \circ \mathrm{res}_e^{U(1)} - z^* - \mathrm{res}_{C_2}^{U(1)} \ \in \ \mathbf{A}(U(1),C_2)\ ,$$

where $z : U(1) \longrightarrow C_2$ is the trivial homomorphism. This element maps trivially to $\underline{\pi}_0(\mathbf{ku})$ because the regular representation of C_2 splits as the sum of the 1-dimensional trivial and sign representations.

Even though we have attached the adjective 'connective' to the orthogonal spectrum $\mathbf{ku}$, we still owe the proof that it actually is globally connective. We give a proof now, based on the rank filtration.

Proposition 4.44. *The orthogonal spectra* $\mathbf{ku}$ *and* $\mathbf{ko}$ *are globally connective.*

Proof. The arguments for $\mathbf{ku}$ and $\mathbf{ko}$ are completely parallel, and we concentrate on the complex case. We use some of the analysis of the rank filtration given by Hausmann and Ostermayr in [**73**]. By [**73**, App. 1] all inclusions $\mathbf{ku}^{[n-1]}(V) \longrightarrow \mathbf{ku}^{[n]}(V)$ in the rank filtrations are levelwise $O(V)$-equivariant h-cofibrations. This has several useful consequences:

- The projection from the mapping cone of the inclusion $\mathbf{ku}^{[n-1]} \longrightarrow \mathbf{ku}^{[n]}$ to the subquotient $\mathbf{ku}^{[n]}/\mathbf{ku}^{[n-1]}$ is a global equivalence. So the Proposition III.1.37 provides a long exact sequence relating the equivariant homotopy groups of $\mathbf{ku}^{[n-1]}$, $\mathbf{ku}^{[n]}$ and the subquotient $\mathbf{ku}^{[n]}/\mathbf{ku}^{[n-1]}$;
- the equivariant homotopy groups of $\mathbf{ku}$ are the sequential colimit of the equivariant homotopy groups of the spectra $\mathbf{ku}^{[n]}$;
- the space $(\mathbf{ku}^{[n]}/\mathbf{ku}^{[n-1]})(V)$ is well-pointed as an $O(V)$-space.

Theorem 3.6 of [**73**] establishes an isomorphism between the subquotient $\mathbf{ku}^{[n]}/\mathbf{ku}^{[n-1]}$ of the rank filtration and the reduced suspension spectrum on a well-pointed based equivariant space. So all subquotients of the rank filtration of $\mathbf{ku}$ are globally connective by Proposition III.1.45. We emphasize that this includes the case $n = 1$, as $\mathbf{ku}^{[1]}$ and $\mathbf{ko}^{[1]}$ are themselves suspension spectra of orthogonal spaces. Each filtration step $\mathbf{ku}^{[n]}$ is then globally connective, by induction over the long exact homotopy groups sequence. So $\mathbf{ku}$ is also globally connective. □

5. Periodic global K-theory

Our next goal is to define *periodic global K-theory* $\mathbf{KU}$, an ultra-commutative ring spectrum whose G-homotopy type realizes G-equivariant periodic K-theory. The model we use is due to M. Joachim [**85**], and is made of spaces of homomorphisms of $\mathbb{Z}/2$-graded C^*-algebras, with source the graded C^*-algebra $s = C_0(\mathbb{R})$ of continuous $\mathbb{C}$-valued functions on $\mathbb{R}$ that vanish at infinity. Similar constructions of orthogonal or symmetric spectra that represent topological K-theory of spaces, and generalizations to K-theory of C^*-algebras or equivariant C^*-algebras can be found in [**84**, Sec. 4.1] or [**41**, Sec. 3.3]. In the Greenlees-May context of global homotopy theory, it was also observed by Bohmann ([**21**, Sec. 4.3], [**22**, Sec. 4]) that Joachim's model 'is global'.

Construction 5.1 (The C^*-algebra s). The construction of the orthogonal spectrum $\mathbf{KU}$ is based on a certain C^*-algebra and on spaces of $*$-homomorphisms out of it. We recall some basic facts. We consider *graded* C^*-algebras, i.e., C^*-algebras A equipped with a $*$-automorphism $\alpha : A \longrightarrow A$ such that $\alpha^2 = \mathrm{Id}$. We

can then decompose A into the ± 1 eigenspaces of α and obtain a $\mathbb{Z}/2$-grading of the underlying $\mathbb{C}$-algebra by setting

$$A^{\mathrm{ev}} \;=\; \{a \in A \mid \alpha(a) = a\} \qquad \text{and} \qquad A^{\mathrm{odd}} \;=\; \{a \in A \mid \alpha(a) = -a\}\ .$$

The conjugation of A preserves the grading into even and odd parts.

We let s denote the C^*-algebra of complex valued continuous functions on $\mathbb{R}$ vanishing at infinity; this is a $\mathbb{Z}/2$-graded C^*-algebra with involution $\alpha : s \longrightarrow s$ defined by

$$\alpha(f)(t) \;=\; f(-t)\ .$$

With respect to this grading, 'even' and 'odd' have their usual meaning: a function $f \in s$ is even (respectively odd) if and only if $f(-t) = f(t)$ (respectively $f(-t) = -f(t)$) for all $t \in \mathbb{R}$.

Graded $*$-morphisms out of s correspond to special elements in the target C^*-algebra. The continuous function

$$r(t) \;=\; \frac{2i}{t-i} \;:\; \mathbb{R} \;\longrightarrow\; \mathbb{C}$$

is an element of s that satisfies

$$rr^* \;+\; r \;+\; r^* \;=\; 0 \qquad \text{and} \qquad \alpha(r) \;=\; r^*\ . \tag{5.2}$$

Moreover, s is the universal $\mathbb{Z}/2$-graded C^*-algebra generated by such an element, i.e., the assignment $\varphi \mapsto \varphi(r)$

$$C^*_{\mathrm{gr}}(s, B) \;\cong\; \{x \in B \;:\; xx^* + x + x^* = 0,\ \alpha(x) = x^*\}$$

is a bijection for every graded C^*-algebra B. In fact, after adjoining a unit, the relations (5.2) are equivalent to saying that $1 + r$ is a unitary element with $\alpha(1 + r) = 1 + r^*$. Adjoining a unit to the algebra s gives the unital C^*-algebra $C(S^1)$ of continuous functions on $S^1 = \mathbb{R} \cup \{\infty\}$, so the universal property of s follows from the fact that $C(S^1)$ is freely generated, as an ungraded, unital C^*-algebra, by the unitary element $1+r$ [ref]. The element r is not homogeneous; its even respectively odd components are given by

$$r_+(t) \;=\; \frac{-2}{t^2+1} \qquad \text{respectively} \qquad r_-(t) \;=\; \frac{2ti}{t^2+1}\ .$$

The algebra s has another important piece of extra structure, namely a comultiplication $\Delta : s \longrightarrow s \hat{\otimes} s$ in the category of $\mathbb{Z}/2$-graded C^*-algebras. Here, and in what follows, $\hat{\otimes}$ denotes the *minimal tensor product* of graded C^*-algebras, see [**75**, Def. 1.10]

While the ungraded C^*-algebra underlying s is commutative, the multiplication of s is *not* commutative in the graded sense The underlying C^*-algebra of the graded tensor product $s \hat{\otimes} s$ is *not* commutative anymore. In particular, $s \hat{\otimes} s$ does not embed into the algebra of continuous functions on any space.

Construction 5.3 (Clifford algebras). We let V be a euclidean inner product space and denote by

$$\mathrm{Cl}_{\mathbb{C}}(V) \;=\; \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(V)$$

the complexification of the real Clifford algebra $\mathrm{Cl}(V)$ considered in the definition of the ultra-commutative monoids **Pin** and **Spin** in Example II.3.9. Unpacking this definition yields

$$\mathrm{Cl}_{\mathbb{C}}(V) \;=\; (TV)_{\mathbb{C}}/(v \otimes v + \langle v, v\rangle \cdot 1)\ ,$$

the quotient of the complexified tensor algebra of V by the ideal generated by the elements $v \otimes v + |v|^2 \cdot 1$ for all $v \in V$. The composite

$$V \;\xrightarrow{\text{linear summand}}\; TV \;\longrightarrow\; \mathrm{Cl}(V) \;\xrightarrow{1 \otimes -}\; \mathrm{Cl}_{\mathbb{C}}(V)$$

is $\mathbb{R}$-linear and injective, and we will identify elements of V with their images in $\mathrm{Cl}_{\mathbb{C}}(V)$. The Clifford algebra becomes a C^*-algebra by declaring $v^* = -v$ for all $v \in V$. This makes the elements of $S(V)$ into unitaries.

The formal properties of the Clifford algebra $\mathrm{Cl}(V)$ carry over to the complexification $\mathrm{Cl}_{\mathbb{C}}(V)$. The Clifford algebra construction is functorial for $\mathbb{R}$-linear isometric embeddings, so in particular $\mathrm{Cl}_{\mathbb{C}}(V)$ inherits an action of the orthogonal group $O(V)$. The Clifford algebra is $\mathbb{Z}/2$-graded, coming from the grading of the tensor algebra by even and odd tensor powers. We recall here that the tensor product of $\mathbb{Z}/2$-graded algebras A and B involves a sign in the formula for multiplication: if $a, a' \in A$ and $b, b' \in B$ are homogeneous elements, then the multiplication in $A \otimes B$ is defined by

$$(a \otimes b) \cdot (a' \otimes b') \;=\; (-1)^{|b||a'|} \cdot (aa') \otimes (bb') \;.$$

The Clifford algebra functor sends orthogonal direct sum to graded tensor product, in the following sense. The $\mathbb{R}$-linear map

$$V \oplus W \;\longrightarrow\; \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(W) \;, \quad (v, w) \;\longmapsto\; v \otimes 1 + 1 \otimes w$$

satisfies

$$(v \otimes 1 + 1 \otimes w)^2 \;=\; v^2 \otimes 1 + 1 \otimes w^2 \;=\; (|v|^2 + |w|^2) \cdot 1 \otimes 1$$

because the elements $v \otimes 1$ and $1 \otimes w$ anti-commute in $\mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(W)$. The universal property then provides a unique extension to a morphism of graded $\mathbb{C}$-algebras

$$\mu_{V,W} \;:\; \mathrm{Cl}_{\mathbb{C}}(V \oplus W) \;\cong\; \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(W) \;, \tag{5.4}$$

and this morphism is an isomorphism. The square

$$\begin{array}{ccc} \mathrm{Cl}_{\mathbb{C}}(V \oplus W) & \xrightarrow{\mu_{V,W}} & \mathrm{Cl}(V)_{\mathbb{C}} \otimes \mathrm{Cl}_{\mathbb{C}}(W) \\ {\scriptstyle \mathrm{Cl}(\tau_{V,W})}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathrm{Cl}_{\mathbb{C}}(W \oplus V) & \xrightarrow[\mu_{W,V}]{} & \mathrm{Cl}(W)_{\mathbb{C}} \otimes \mathrm{Cl}_{\mathbb{C}}(V) \end{array}$$

commutes, where the right vertical map is the symmetry isomorphism for $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras (which involves a sign whenever two odd degree elements interchange places).

Remark 5.5. Some sources (e.g. [**85**]) define the complex Clifford algebra in a different (but isomorphic) way as

$$\mathrm{Cl}'_{\mathbb{C}}(V) \;=\; (TV)_{\mathbb{C}}/(v \otimes v - \langle v, v\rangle \cdot 1) \;,$$

the quotient of the complexified tensor algebra of V by the ideal generated by the elements $v \otimes v - |v|^2 \cdot 1$ for all $v \in V$. Over the field $\mathbb{R}$ it makes a difference whether we impose the relations $v \otimes v = |v| \cdot 1$ or $v \otimes v = -|v| \cdot 1$, but the presence of the imaginary unit makes the difference disappear over $\mathbb{C}$. Indeed, the $\mathbb{R}$-linear map

$$V \;\longrightarrow\; \mathrm{Cl}_{\mathbb{C}}(V) \;, \quad v \;\longmapsto\; i \cdot v$$

extends to an morphism of $\mathbb{C}$-algebras $\mathrm{Cl}'_{\mathbb{C}}(V) \longrightarrow \mathrm{Cl}_{\mathbb{C}}(V)$ by the universal property of the former, and this morphism is an isomorphism. Since the two complex versions of the Clifford algebra are naturally isomorphic, it is a matter of convenience with which one to work. We made the choice so as to be consistent with the discussion in Example II.3.9.

The complex Clifford algebra is in fact a $\mathbb{Z}/2$-graded $O(V)$-C^*-algebra. It remains to define the norm on $\mathrm{Cl}_{\mathbb{C}}(V)$, which arises from an embedding into the endomorphism algebra of the exterior algebra $\Lambda^*(V_{\mathbb{C}})$. More precisely, if W is a complex hermitian inner product space we use the hermitian inner product on the exterior algebra $\Lambda^*(W)$ characterized by the formula

$$(v_1 \wedge \ldots \wedge v_n,\, w_1 \wedge \ldots \wedge w_n) \;=\; \det((v_i, w_j)_{i,j})$$

for all $v_i, w_j \in W$. Another way to say this is that if $(e_i)_{i=1,\ldots,k}$ is an orthonormal basis of W, then the vectors

$$e_{i_1} \wedge \ldots \wedge e_{i_n}$$

form an orthonormal basis of $\Lambda^*(W)$ as the indices run through all tuples with $1 \leq i_1 < \cdots < i_n \leq k$. An anti-involution $(-)^* : \mathrm{End}(\Lambda^*(W)) \longrightarrow \mathrm{End}(\Lambda^*(W))$ is given by adjoint operator with respect to this inner product, i.e., A^* is characterized by

$$(x, A^*y) \;=\; (Ax, y)$$

for all $x, y \in \Lambda^*(W)$. Together with the operator norm

$$|A| \;=\; \sup\{|Ax| \;:\; x \in \Lambda^*(W), |x| = 1\}$$

this makes $\mathrm{End}(\Lambda^*(W))$ a $\mathbb{Z}/2$-graded C^*-algebra.

For a euclidean inner product space V, an algebra monomorphism is given by

$$\delta \;:\; \mathrm{Cl}_{\mathbb{C}}(V) \;\longrightarrow\; \mathrm{End}(\Lambda^*(V_{\mathbb{C}}))\;, \quad v \;\longmapsto\; (v \wedge -) + (v \wedge -)^*\;.$$

We endow $\mathrm{Cl}_{\mathbb{C}}(V)$ with the norm induced from the operator norm on the endomorphism algebra. [...is the embedding $V \longrightarrow \mathrm{Cl}_{\mathbb{C}}(V)$ isometric?...] The $*$-involution

$$(-)^* \;:\; \mathrm{Cl}_{\mathbb{C}}(V) \;\longrightarrow\; \mathrm{Cl}_{\mathbb{C}}(V)$$

is the unique $\mathbb{C}$-semilinear anti-involution characterized by $v^* = v$ for all $v \in V$.

A key piece of structure is a continuous based map

$$\mathrm{fc} \;:\; S^V \;\longrightarrow\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V))\;, \quad v \;\longmapsto\; (-)[v]\;, \tag{5.6}$$

often referred to as 'functional calculus'. For $v \in V$ the $*$-homomorphism $\mathrm{fc}(v)$ is given on homogeneous elements of s by

$$f[v] \;=\; \mathrm{fc}[v](f) \;=\; \begin{cases} f(|v|) \cdot 1 & \text{when } f \text{ is even, and} \\ f(|v|)/|v| \cdot [v] & \text{when } f \text{ is odd.} \end{cases}$$

For $v = 0$ the second formula for odd functions f is to be interpreted as $f[0] = 0$; this is continuous because for $v \neq 0$, the norm of $f(|v|)/|v| \cdot v$ is $f(|v|)$, which tends to $f(0) = 0$ if v tends to 0. If the norm of v tends to infinity, then $f(|v|)$ tends to 0, so $\mathrm{fc}(v)$ tends to the constant $*$-homomorphism with value 0, the basepoint of $C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V))$ So fc extends to a continuous map on the one-point compactification S^V. In terms of the homeomorphism

$$C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V)) \;\cong\; \{u \in \mathrm{Cl}_{\mathbb{C}}(V) \mid uu^* = u^*u = 1,\; \alpha(u) = u^*\}\;, \quad \varphi \;\longmapsto\; 1 + \varphi(r)$$

the functional calculus is given by

$$v \;\longmapsto\; 1 + r[v] \;=\; \frac{|v|^2 - 1 + 2i[v]}{|v|^2 + 1}\;.$$

Another way to say this is that the composite

$$S(V \oplus \mathbb{R}) \;\xrightarrow[\cong]{\Pi_V}\; S^V \;\xrightarrow{\mathrm{fc}}\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V)) \;\xrightarrow{\mathrm{ev}_z}\; U'(\mathrm{Cl}_{\mathbb{C}}(V))\;,$$

where Π_V is the stereographic projection, takes a pair (v, λ) with $|v|^2 + \lambda^2 = 1$ to the element $\lambda \cdot 1 + [v]$. The functional calculus map is $O(V)$-equivariant for the $O(V)$-action on the mapping space $C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V))$ through the action on the target.

Construction 5.7 (Periodic global K-theory). We let V be a euclidean inner product space. As we explained in Proposition 4.7, the symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ of the complexification inherits a hermitian inner product and an action of $O(V)$ is by $\mathbb{C}$-linear isometries. The inner product space $\mathrm{Sym}(V_{\mathbb{C}})$ is usually infinite dimensional (unless $V = 0$), but it is not complete. We denote by $\mathcal{H}_V$ the Hilbert space completion of $\mathrm{Sym}(V_{\mathbb{C}})$. Since the action of $O(V)$ on $\mathrm{Sym}(V_{\mathbb{C}})$ is by linear isometries, it extends to an analogous action on the completion $\mathcal{H}_V$. So $\mathcal{H}_V$ becomes a complex Hilbert space representation of the orthogonal group $O(V)$. We denote by $\mathcal{K}_V$ the C^*-algebra of compact operators on the Hilbert space $\mathcal{H}_V$.

The orthogonal spectrum $\mathbf{KU}$ assigns to a euclidean inner product space V the space

$$\mathbf{KU}(V) \;=\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V)$$

of $\mathbb{Z}/2$-graded $*$-homomorphisms from s to the tensor product, over $\mathbb{C}$, of $\mathrm{Cl}_{\mathbb{C}}(V)$ and $\mathcal{K}_V$. Here we consider $\mathcal{K}_V$ as evenly graded, so the grading comes entirely from the grading of the Clifford algebra. The topology is the topology of pointwise convergence in the operator norm of $\mathcal{K}_V$; the basepoint is the zero $*$-homomorphism.

The continuous, graded, action of the orthogonal group $O(V)$ by linear isometries of $\mathcal{H}_V$ induces a graded action on the algebra $\mathcal{K}_V$ by conjugation, and together with the action on $\mathrm{Cl}_{\mathbb{C}}(V)$ it gives an $O(V)$-action on $\mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V$ by $*$-automorphisms. This induces an $O(V)$-action on the mapping space $\mathbf{KU}(V)$.

The multiplication of the spectrum $\mathbf{KU}$ starts from the natural, graded, $(O(V) \times O(W))$-equivariant isometric isomorphism

$$\mathrm{Sym}(V_{\mathbb{C}}) \otimes \mathrm{Sym}(W_{\mathbb{C}}) \;\cong\; \mathrm{Sym}(V_{\mathbb{C}} \oplus W_{\mathbb{C}}) \;\cong\; \mathrm{Sym}((V \oplus W)_{\mathbb{C}}) \;,$$

compare (4.6). This extends to an isometry of the Hilbert space completions

$$\mathcal{H}_V \hat{\otimes} \mathcal{H}_W \;=\; \widehat{\mathrm{Sym}}(V_{\mathbb{C}}) \hat{\otimes} \widehat{\mathrm{Sym}}(W_{\mathbb{C}}) \;\cong\; (\mathrm{Sym}(V_{\mathbb{C}}) \otimes \mathrm{Sym}(W_{\mathbb{C}}))^{\wedge} \;\cong\; \widehat{\mathrm{Sym}}((V \oplus W)_{\mathbb{C}}) \;=\; \mathcal{H}_{V \oplus W} \;.$$

Tensor product of compact operators and conjugation with this isometry are two graded $*$-isomorphisms

$$\mathcal{K}_V \hat{\otimes} \mathcal{K}_W \;\cong\; \mathcal{K}(\mathcal{H}_V \hat{\otimes} \mathcal{H}_W) \;\cong\; \mathcal{K}(\mathcal{H}_{V \oplus W}) = \mathcal{K}_{V \oplus W} \;.$$

[...explain the tensor product here...] On the other hand, an isomorphism between $\mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(W)$ and $\mathrm{Cl}_{\mathbb{C}}(V \oplus W)$ was specified in (5.4). The multiplication map

$$\mu_{V,W} \;:\; \mathbf{KU}(V) \wedge \mathbf{KU}(W) \;\longrightarrow\; \mathbf{KU}(V \oplus W)$$

is now defined as the composite

$$\begin{aligned} C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V) \wedge C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(W) \otimes \mathcal{K}_W) \;&\xrightarrow{\hat{\otimes}}\; C^*_{\mathrm{gr}}(s \hat{\otimes} s,\; (\mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V) \hat{\otimes} (\mathrm{Cl}_{\mathbb{C}}(W) \otimes \mathcal{K}_W)) \\ &\xrightarrow{C^*_{\mathrm{gr}}(\Delta, \cong)}\; C^*_{\mathrm{gr}}(s, (\mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(W)) \otimes (\mathcal{K}_V \hat{\otimes} \mathcal{K}_W)) \\ &\cong\quad C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V \oplus W) \otimes \mathcal{K}_{V \oplus W}) \;. \end{aligned}$$

These multiplication maps are associative and commutative.

Next we define $O(V)$-equivariant continuous based maps

$$j(V) \;:\; \mathbf{ku}(V) \;=\; \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \;\longrightarrow\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V) \;=\; \mathbf{KU}(V) \tag{5.8}$$

that form a morphism of ultra-commutative ring spectra. We consider a configuration

$$[E_1, \dots, E_n;\, v_1, \dots, v_n] \;\in\; \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V)$$

of pairwise orthogonal, finite dimensional subspaces on S^V. The associated $*$-homomorphism

$$j(V)[E_1, \dots, E_n;\, v_1, \dots, v_n] \;:\; s \;\longrightarrow\; \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V$$

is then defined on a function $f \in C_0(\mathbb{R})$ by

$$j(V)[E_1, \dots, E_n;\, v_1, \dots, v_n](f) \;=\; \sum_{i=1}^{n} f[v_i] \otimes p_{E_i} \;,$$

where $f[v_i] = \mathrm{fc}(f)(v_i)$ is the functional calculus (5.6) and p_E denotes the orthogonal projection onto a subspace E. We check that the map $j(V)$ is $O(V)$-equivariant:

$$\begin{aligned} j(V)(\varphi_*[E_1,\dots,E_n;\, v_1,\dots,v_n])(f) \;&=\; j(V)[\varphi(E_1),\dots,\varphi(E_n);\, \varphi(v_1),\dots,\varphi(v_n)](f) \\ &=\; \sum_{i=1}^n f[\varphi(v_i)] \otimes p_{\varphi(E_i)} \;=\; \sum_{i=1}^n \mathrm{Cl}_{\mathbb{C}}(\varphi)(f[v_i]) \otimes {}^{\varphi}(p_{E_i}) \\ &=\; \varphi_*\left(\sum_{i=1}^n f[v_i] \otimes p_{E_i}\right) \\ &=\; \varphi_*(j(V)[E_1,\dots,E_n;\, v_1,\dots,v_n])(f)\ . \end{aligned}$$

The unit map

$$\eta_V \;:\; S^V \;\longrightarrow\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V) \;=\; \mathbf{KU}(V)$$

is then defined as the composite

$$S^V \;\xrightarrow{\eta_V}\; \mathbf{ku}(V) \;\xrightarrow{j(V)}\; \mathbf{KU}(V)$$

of the unit map for the connective global K-theory spectrum and the map $j(V)$ just defined. Unraveling the definitions shows that

$$\eta_V(v)(f) \;=\; f[v] \otimes p_{\mathbb{C}}\ , \tag{5.9}$$

where $p_{\mathbb{C}} \in \mathcal{K}_V$ is the orthogonal projection to the constant summand in the symmetric algebra. For the time being, we omit the verifications needed to show that the maps $j(V)$ form a morphism of ultra-commutative ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$.

Remark 5.10. We have defined $\mathbf{KU}(V)$ in a slightly different way, compared to the presentation of Joachim [**85**]. We use the Hilbert space completion of $\mathrm{Sym}(V_{\mathbb{C}})$, instead of the Hilbert space $L^2(V)$ of $\mathbb{C}$-valued square integrable functions on V. These two Hilbert spaces are naturally isomorphic, as follows. We use the inner product on V to identify it with its dual space V^*, and hence the symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ with $\mathrm{Sym}(V^*_{\mathbb{C}})$. Elements of $\mathrm{Sym}(V^*_{\mathbb{C}})$ are complex valued polynomial functions on V. We make them square integrable by multiplying with the rapidly decaying function $v \mapsto \exp(-\langle v, v\rangle)$. This provides an $O(V)$-equivariant linear isometric embedding

$$\mathrm{Sym}(V^*_{\mathbb{C}}) \;\longrightarrow\; L^2(V)\ , \quad f \;\longmapsto\; f \cdot \exp(-\langle -, -\rangle)$$

with dense image. Altogether this exhibits $L^2(V)$ as a Hilbert space completion of $\mathrm{Sym}(V_{\mathbb{C}})$.

Remark 5.11 (Homotopy type of $\mathbf{KU}(V)$). Up to isomorphism, there are only three different graded C^*-algebras of the form $\mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V$, and only three different homeomorphism types of spaces $\mathbf{KU}(V)$. The case $V = 0$ is degenerate in that the inner product space $\mathrm{Sym}(0)$ is just the copy of $\mathbb{C}$ generated by 1. This is already complete, so $\mathcal{H}_0 = \mathrm{Sym}(0) = \mathbb{C}$ and $\mathrm{Cl}_{\mathbb{C}}(0) = \mathcal{K}_0 = \mathbb{C}$. The space $\mathbf{KU}(0) = C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(0) \otimes \mathcal{H}_0)$ is thus discrete with two points: the zero homomorphism as basepoint and the augmentation

$$s \;\longrightarrow\; \mathrm{Cl}_{\mathbb{C}}(0) \otimes \mathcal{H}_0\ , \quad f \;\longmapsto\; f(0) \otimes 1\ .$$

This non-basepoint is a unit for the multiplication maps $\mu_{V,W}$.

We can also identify the next space $\mathbf{KU}(\mathbb{R})$ as follows. We claim that the map

$$j(\mathbb{R}) \;:\; \mathbf{ku}(\mathbb{R}) \;\longrightarrow\; \mathbf{KU}(\mathbb{R})$$

is a homotopy equivalence. Indeed, if A is an (ungraded) C^*-algebra, then $*$-homomorphisms from s to A (in the ungraded sense) biject with graded $*$-homomorphisms from s to to $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes A$, via

$$C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes A) \;\cong\; C^*(s, A)\ , \quad f \;\longmapsto\; p \circ f\ ,$$

where

$$p \;:\; \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes A \;\longrightarrow\; A$$

is the $*$-homomorphism that sends $1 \otimes a$ and $e \otimes a$ to a, with $e = [1] \in \mathrm{Cl}_{\mathbb{C}}(\mathbb{R})$. The inverse is given by

$$C^*(s, A) \;\cong\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes A)\;, \quad f \;\longmapsto\; \bar{f}$$

where

$$\bar{f}(\lambda) \;=\; p \otimes f(\lambda) \;+\; q \otimes f(\bar{\lambda})\;.$$

Here $p = (1+e)/2$ and $q = (1-e)/2$ are the two self-adjoint orthogonal idempotents of $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R})$. The composite

$$S^1 \;\xrightarrow{\mathrm{fc}}\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(\mathbb{R})) \;\xrightarrow[\cong]{p \circ -}\; C^*(s, \mathbb{C})$$

sends $t \in S^1$ to evaluation at t, or equivalently,

$$p(f[t]) \;=\; f(t)$$

for all functions f in $s = C_0(\mathbb{R})$. This implies that the following square commutes:

$$\begin{tikzcd}
\mathscr{C}(\mathrm{Sym}(\mathbb{C}), S^1) \;=\; \mathbf{ku}(\mathbb{R}) \arrow[r, "j(\mathbb{R})"] \arrow[d, "\cong"'] & \mathbf{KU}(\mathbb{R}) = C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes \mathcal{K}_{\mathbb{R}}) \arrow[d, "\cong"] \\
\mathrm{colim}_n\, C^*(s, \mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^{[n]}(\mathbb{C}))) \arrow[r] & C^*(s, \mathcal{K}_{\mathbb{R}})
\end{tikzcd}$$

Here the lower horizontal map is induced by the $*$-homomorphisms $\mathrm{End}_{\mathbb{C}}(\mathrm{Sym}^{[n]}) \longrightarrow \mathcal{K}_{\mathbb{R}}$ that extends an endomorphism by zero on the orthogonal complement, and the left vertical map sends a configuration $[E_1, \dots, E_n; t_1, \dots, t_n]$ to the $*$-homomorphism

$$f \;\longmapsto\; \sum_{i=1}^{n} f(t_i) \cdot p_{E_i}\;.$$

The two vertical maps are homeomorphisms (compare Remark 4.2) and the lower horizontal map is a homotopy equivalence by [**142**, Prop. 1.2] or [**78**, Prop. 4.6] (or rather its complex analog). So the map $j(\mathbb{R})$ is a homotopy equivalence.

The spectrum $\mathbf{KU}$ is positive Ω-spectrum (in the non-equivariant sense) by Theorem 5.13 below. So the adjoint structure map $\mathbf{KU}(\mathbb{R}) \longrightarrow \Omega\mathbf{KU}(\mathbb{R}^2)$ is a weak equivalence. By Bott periodicity, $\mathbf{KU}(\mathbb{R}^2)$ has the weak homotopy type of $\mathbb{Z} \times BU$.

For the rest of this remark we consider a non-zero inner product space V. Then

$$\mathcal{H}_V \qquad \text{and} \qquad \Lambda^*(\mathbb{C}) \otimes \mathcal{H}_{V \oplus u\mathbb{C}}$$

are two graded Hilbert spaces whose even and odd summands are infinite dimensional and separable, so they are isometrically isomorphic as graded Hilbert spaces. Conjugation with a graded isometric isomorphism induces an isomorphism of graded C^*-algebras

$$\mathcal{K}_V \;=\; \mathcal{K}(\mathcal{H}_V) \;\cong \mathcal{K}(\Lambda^*(\mathbb{C}) \otimes \mathcal{H}_{V \oplus u\mathbb{C}}) \;\cong\; \mathrm{End}_{\mathbb{C}}(\Lambda^*(\mathbb{C}^2)) \otimes \mathcal{K}_{V \oplus u\mathbb{C}} \;\cong\; \mathrm{Cl}_{\mathbb{C}}(u\mathbb{C}) \otimes \mathcal{K}_{V \oplus u\mathbb{C}}\;.$$

This in turn induces an isomorphism of graded C^*-algebras

$$\mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V \;\cong\; \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(u\mathbb{C}) \otimes \mathcal{K}_{V \oplus u\mathbb{C}} \;\cong\; \mathrm{Cl}_{\mathbb{C}}(V \oplus u\mathbb{C}) \otimes \mathcal{K}_{V \oplus u\mathbb{C}}\;.$$

Applying $C^*_{\mathrm{gr}}(s, -)$ provides a homeomorphism

$$\mathbf{KU}(V) \;=\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V) \otimes \mathcal{K}_V) \;\cong\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V \oplus u\mathbb{C}) \otimes \mathcal{K}_{V \oplus u\mathbb{C}}) \;=\; \mathbf{KU}(V \oplus u\mathbb{C})\;.$$

So for $V \neq 0$, the space $\mathbf{KU}(V)$ is either homeomorphic to $\mathbf{KU}(\mathbb{R})$ or to $\mathbf{KU}(\mathbb{R}^2)$, depending on whether the dimension of V is odd or even, a shadow of Bott periodicity. Of course, these identifications don't yet

contain any equivariant information because they do not pay attention to actions of the isometry groups of the inner product spaces.

The analysis of the global homotopy type of **KU** depends on the operator theoretic formulation of equivariant Bott periodicity that we now recall. We let G be a compact Lie group and V an orthogonal G-representation. We denote by

$$\mathscr{C}(V) \;=\; C_0(V, \mathrm{Cl}_{\mathbb{C}}(V))$$

the G-C^*-algebra of continuous $\mathrm{Cl}_{\mathbb{C}}(V)$-valued functions on V that vanish at infinity. Functional calculus (5.6) provides a distinguished graded $*$-homomorphism

$$\beta_V \;:\; s \;\longrightarrow\; \mathscr{C}(V)\;, \quad \beta_V(f)(v) \;=\; f[v]\;.$$

Since the functional calculus map is G-equivariant, the $*$-homomorphism β_V is a G-fixed point of $\mathscr{C}(V)$.

We let $\mathcal{H}_G$ be any complete G-Hilbert space universe, i.e., a Hilbert G-representation that is isometrically isomorphic to the completion of a complete unitary G–universe. We let $\mathcal{K}_G$ be the G-C^*-algebra of (not necessarily equivariant) compact operators on $\mathcal{H}_G$, with G acting by conjugation.

Theorem 5.12 (Equivariant Bott periodicity). *Let G be a compact Lie group and V an orthogonal G-representation. Then for every graded G-C^*-algebra A, the map*

$$\beta_V \cdot - \;:\; C^*_{\mathrm{gr}}(s, A \otimes \mathcal{K}_G) \;\longrightarrow\; C^*_{\mathrm{gr}}(s, \mathscr{C}(V) \otimes A \otimes \mathcal{K}_G)$$

is a G-weak equivalence.

When $G = e$ is a trivial group, $\mathcal{K}_G$ reduces to the C^*-algebra of compact operators on a separable Hilbert space, and then this formulation of Bott periodicity can be found in [**75**, Thm. 1.14]. Unfortunately, we are lacking a reference in the generality of compact Lie groups. If one specializes [**74**, Sec. 3, Thm. 5] to finite dimensional representations of finite groups, one obtains a formulation very close to (but not exactly the same as) Theorem 5.12. Since the unit maps $\eta_V : S^V \longrightarrow \mathbf{KU}(V)$ is essentially the functional calculus map, a direct consequence of the formulation of equivariant Bott periodicity in Theorem 5.12 is that **KU** is 'eventually' a global Ω-spectrum.

We recall from Definition 4.21 that an orthogonal G-representation is *ample* if its complexified symmetric algebra is a complete complex G-universe.

Theorem 5.13. *For every orthogonal G-representation V and every ample G-representation W, the adjoint structure map*

$$\tilde{\sigma}_{V,W} \;:\; \mathbf{KU}(W) \;\longrightarrow\; \mathrm{map}(S^V, \mathbf{KU}(V \oplus W))$$

is a G-weak equivalence.

Proof. The adjoint structure map factors as the composite

$$\begin{aligned}
\mathbf{KU}(W) \;&= C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(W) \otimes \mathcal{K}_W)\\
&\xrightarrow{\beta_V \cdot -} C^*_{\mathrm{gr}}(s, \mathscr{C}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(W) \otimes \mathcal{K}_W)\\
&\xrightarrow{- \otimes p_{\mathbb{C}}} C^*_{\mathrm{gr}}(s, C_0(V, \mathrm{Cl}_{\mathbb{C}}(V)) \otimes \mathrm{Cl}_{\mathbb{C}}(W) \otimes \mathcal{K}_{V \oplus W})\\
&\cong\; C^*_{\mathrm{gr}}(s, C_0(V) \otimes \mathrm{Cl}_{\mathbb{C}}(V \oplus W) \otimes \mathcal{K}_{V \oplus W})\\
&\cong\; \mathrm{map}(S^V, C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V \oplus W) \otimes \mathcal{K}_{V \oplus W})) \;=\; \mathrm{map}(S^V, \mathbf{KU}(V \oplus W))
\end{aligned}$$

Since W is ample, $\mathcal{H}_W$ is a Hilbert G-universe and so $\mathcal{K}_W$ is a $\mathcal{K}_G$. Bott periodicity (Theorem 5.12) for the G-C^*-algebra $\mathrm{Cl}_{\mathbb{C}}(W)$ shows that the first map is a G-weak equivalence. The second map is a G-homotopy equivalence because the map $\mathcal{H}_W \longrightarrow \mathcal{H}_{V \oplus W}$ induced by the direct summand embedding $W \longrightarrow V \oplus W$ is an equivariant linear isometric embedding between complete G-Hilbert space universes; so the induced map $p_{\mathbb{C}} \otimes - : \mathcal{K}_W \longrightarrow \mathcal{K}_{V \oplus W}$ is a G-equivariant homotopy equivalence of C^*-algebras. Together this shows that $\tilde{\sigma}_{V,W}$ is a G-weak equivalence. $\square$

The orthogonal spectrum $s\mathbf{ku}$ was defined in (4.27), and is essentially a shift of the connective global K-theory $\mathbf{ku}$. We now introduce the periodic analog: we define an orthogonal spectrum $s\mathbf{KU}$ at an inner product space V by

$$(s\mathbf{KU})(V) \;=\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V \oplus \mathbb{R}) \otimes \mathcal{K}_V)\ .$$

The $O(V)$-action and structure maps are defined in much the same way as for $\mathbf{KU}$. The spectrum $s\mathbf{KU}$ is essentially a shift of $\mathbf{KU}$; the only difference between $s\mathbf{KU}$ and the shift $\mathrm{sh}\,\mathbf{KU}$ as defined in (1.22) of Chapter IV is that we take compact operators in the Hilbert space $\mathcal{H}_V$, as opposed to $\mathcal{H}_{V\oplus\mathbb{R}}$. We define a morphism of orthogonal spectra

$$sj \;:\; s\mathbf{ku} \;\longrightarrow\; s\mathbf{KU}$$

in an analogous fashion to the morphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ in (5.8).

THEOREM 5.14. *The composite*

$$\mathbf{U} \;\xrightarrow{\mathrm{eig}}\; \Omega^\bullet(s\mathbf{ku}) \;\xrightarrow{\Omega^\bullet(sj)}\; \Omega^\bullet(s\mathbf{KU})$$

is a global equivalence of orthogonal spaces.

PROOF. As in the proof of Theorem 4.28 we let $\bar{\mathbf{U}}$ denote the orthogonal space with

$$\bar{\mathbf{U}}(V) \;=\; U(\mathrm{Sym}(V_{\mathbb{C}}))\ ,$$

and we factor the eigenspace decomposition morphism as the composite

$$\mathbf{U} \;\longrightarrow\; \bar{\mathbf{U}} \;\xrightarrow{\bar{\mathrm{eig}}}\; \Omega^\bullet(s\mathbf{ku})\ ,$$

where the first morphism is the global equivalence of orthogonal spaces induced by the natural linear isometric embedding of V as the linear summand in $\mathrm{Sym}(V)$.

We claim that composite

$$\bar{\mathbf{U}} \;\xrightarrow{\bar{\mathrm{eig}}}\; \Omega^\bullet(s\mathbf{ku}) \;\xrightarrow{\Omega^\bullet sj}\; \Omega^\bullet(s\mathbf{KU})$$

is an 'eventually strong level equivalence'. More precisely, we show that for every ample G-representation V the map

$$U(\mathrm{Sym}(V_{\mathbb{C}})) \;=\; \bar{\mathbf{U}}(V) \;\longrightarrow\; (\Omega^\bullet(s\mathbf{KU}))(V) \;=\; \mathrm{map}(S^V, C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V \oplus \mathbb{R}) \otimes \mathcal{K}_V))$$

is a G-weak equivalence. This map factors through the G-equivariant map

$$U(\mathrm{Sym}(V_{\mathbb{C}})) \cong \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), U(1)) \;\longrightarrow\; \hat{\mathscr{C}}(\mathcal{H}_V, S^1) \;\cong\; C^*(s, \mathcal{K}_V) \;\cong\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes \mathcal{K}_V)\ .$$

Here $\hat{\mathscr{C}}(\mathcal{H}_V, S^1)$ is the space of configurations, not necessarily finite, of pairwise orthogonal subspaces of $\mathcal{H}_V$, that are allowed to accumulate around ∞. This map is a G-equivariant homotopy equivalence; the non-equivariant argument (or rather its real analog) can be found in [**142**, Prop. 1.2] or [**78**, Prop. 4.6], and the explicit homotopies in [**78**, Prop. 4.6] work just the same way in the presence of an isometric action of a compact Lie group.

By the operator theoretic equivariant Bott periodicity theorem (Theorem 5.12), multiplication by the graded $*$-homomorphism $\beta_V : s \longrightarrow \mathscr{C}(V)$ is a G-weak equivalence

$$\beta_V \cdot - \;:\; C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes \mathcal{K}_V) \;\longrightarrow\; C^*_{\mathrm{gr}}(s, \mathscr{C}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes \mathcal{K}_V)$$

We use here that $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes \mathcal{K}_V$ is a $\mathcal{K}_G$ because the G-representation V is ample. The target is G-equivariantly homeomorphic to

$$\begin{aligned} C^*_{\mathrm{gr}}(s, \mathscr{C}(V) \otimes \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}) \otimes \mathcal{K}_V) \;&\cong\; C^*_{\mathrm{gr}}(s, C_0(V) \otimes \mathrm{Cl}_{\mathbb{C}}(V \oplus \mathbb{R}) \otimes \mathcal{K}_V) \\ &\cong\; \mathrm{map}(S^V, C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V \oplus \mathbb{R}) \otimes \mathcal{K}_V)) \;=\; \mathrm{map}(S^V, s\mathbf{KU}(V))\ . \end{aligned}$$

□

We emphasize that in contrast to Theorem 4.28, the composite in Theorem 5.14 is a global equivalence for all compact Lie groups (as opposed to only a $\mathcal{F}in$-global equivalence). Another key difference is that the spectrum $\mathbf{ku}$ is globally connective; in contrast, we will see in Theorem 5.31 below that $\mathbf{KU}$ is Bott periodic, so applying $\Omega^\bullet$ is losing a substantial amount of information.

Remark 5.15 ($\mathbf{KU}$ globally deloops $\mathbf{BUP}$). A corollary of the previous theorem is that $\mathbf{KU}$ globally deloops the orthogonal space $\mathbf{BUP}$, the complex analog of the ultra-commutative monoid $\mathbf{BOP}$ defined in Example II.4.2. Indeed, the global formulation of Bott periodicity in Theorem II.4.39 provides a global equivalence of ultra-commutative monoids $\beta : \mathbf{BUP} \longrightarrow \Omega(\mathrm{sh}_\otimes \mathbf{U})$. Combined with Theorem 5.14 this yields a chain of global equivalences of orthogonal spaces

$$\mathbf{BUP} \xrightarrow[\simeq]{\beta} \Omega(\mathrm{sh}_\otimes \mathbf{U}) \xleftarrow[\simeq]{} \Omega\mathbf{U} \xrightarrow[\simeq]{\Omega((\Omega^\bullet sj)\circ \mathrm{eig})} \Omega(\Omega^\bullet(s\mathbf{KU})) \xleftarrow[\simeq]{} \Omega^\bullet\mathbf{KU} \ .$$

We spell out another corollary of Theorem 5.14. We recall that $\mathbf{K}_G(A)$ denotes the equivariant K-group of a G-space A, i.e., the Grothendieck group of isomorphism classes of G-vector bundles over A. A ring homomorphism $[-]$ from this Grothendieck group to the equivariant cohomology group $\mathbf{ku}^0_G(A_+)$ was defined in Theorem 4.31.

Corollary 5.16. *For every compact Lie group G and every finite G-CW-complex A the composite*

$$\mathbf{K}_G(A) \xrightarrow{[-]} \mathbf{ku}^0_G(A_+) \xrightarrow{j_*} \mathbf{KU}^0_G(A_+)$$

is an isomorphism.

Proof. We contemplate the commutative diagram of orthogonal spaces:

$$\begin{tikzcd}
\mathbf{Gr}^{\mathbb{C}} \arrow[r, "c"] \arrow[d, "\beta'"'] & \Omega^\bullet\mathbf{ku} \arrow[r, "\Omega^\bullet j"] \arrow[d, "\simeq"', "\Omega^\bullet\omega"] & \Omega^\bullet\mathbf{KU} \arrow[d, "\simeq"', "\Omega^\bullet\omega"] \\
\Omega\mathbf{U} \arrow[r, "\Omega\,\mathrm{eig}"'] & \Omega\left(\Omega^\bullet(s\mathbf{ku})\right) \arrow[r, "\Omega(\Omega^\bullet sj)"'] & \Omega(\Omega^\bullet(s\mathbf{KU}))
\end{tikzcd}$$

The morphism $(\Omega^\bullet sj)\circ \mathrm{eig}$ is a global equivalence by Theorem 5.14, so the lower horizontal composite is also a global equivalence.

We claim that the morphism $\Omega^\bullet\omega : \Omega^\bullet\mathbf{KU} \longrightarrow \Omega^\bullet(\Omega(s\mathbf{KU}))$ is a global equivalence of orthogonal spaces. To see that, we adapt the argument employed for the $\mathbf{ku}$-analog in the proof of Theorem 4.29. The embedding $i_V : V \longrightarrow V\oplus\mathbb{R}$ as the first summand induces a morphism of C^*-algebras $i'_V : \mathcal{K}_V \longrightarrow \mathcal{K}_{V\oplus\mathbb{R}}$, and hence a continuous map

$$j(V) \ : \ s\mathbf{KU}(V) \ = \ C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V\oplus\mathbb{R})\otimes\mathcal{K}_V) \ \longrightarrow \ C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(V\oplus\mathbb{R})\otimes\mathcal{K}_{V\oplus\mathbb{R}}) \ = \ (\mathrm{sh}\,\mathbf{KU})(V) \ ;$$

as V varies, these maps form a morphism of orthogonal spectra $j : s\mathbf{KU} \longrightarrow \mathrm{sh}\,\mathbf{KU}$.

If V is an ample orthogonal G-representation, then $i'_V : \mathcal{K}_V \longrightarrow \mathcal{K}_{V\oplus\mathbb{R}}$ is a G-equivariant homotopy equivalence of C^*-algebras. Indeed, in this case both $\mathrm{Sym}(V_{\mathbb{C}})$ and $\mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}})$ are complete complex G-universes. So we can choose an equivariant linear isometry $\lambda : \mathrm{Sym}((V\oplus\mathbb{R})_{\mathbb{C}}) \cong \mathrm{Sym}(V_{\mathbb{C}})$ that completes to an equivariant linear isometry

$$\hat\lambda \ : \ \mathcal{H}_{V\oplus\mathbb{R}} \ \cong \ \mathcal{H}_V \ .$$

Conjugation with $\hat\lambda$ is an isomorphism of $\mathbb{Z}/2$-graded C^*-algebras

$$\tilde\lambda \ : \ \mathcal{K}_{V\oplus\mathbb{R}} \ \cong \ \mathcal{K}_V \ .$$

The composite

$$\lambda\circ\mathrm{Sym}((i_V)_{\mathbb{C}}) \ : \ \mathrm{Sym}(V_{\mathbb{C}}) \ \longrightarrow \ \mathrm{Sym}(V_{\mathbb{C}})$$

is homotopic, through equivariant linear isometric embeddings, to the identity map. After conjugation with completions, these homotopies induce homotopies of C^*-algebra homomorphisms, from

$$\tilde{\lambda} \circ i'_V \ : \ \mathcal{K}_V \ \longrightarrow \ \mathcal{K}_V$$

to the identity. The same argument applies to the other composite, so i'_V is homotopy equivalence.

Now we know that the map $j(V) : s\mathbf{KU}(V) \longrightarrow (\operatorname{sh}\mathbf{KU})(V)$ is a G-homotopy equivalence for every ample G-representations. Hence the map

$$(\Omega^\bullet \Omega j)(V) \ : \ (\Omega^\bullet(\Omega(s\mathbf{KU})))(V) \ \longrightarrow \ (\Omega^\bullet(\Omega(\operatorname{sh}\mathbf{KU})))(V)$$

is a G-homotopy equivalence. Since the ample G-representations are cofinal in the poset $s(\mathcal{U}_G)$, this shows that $\Omega^\bullet(\Omega j) : \Omega^\bullet(\Omega(s\mathbf{KU})) \longrightarrow \Omega^\bullet(\Omega(\operatorname{sh}\mathbf{KU}))$ is a global equivalence of orthogonal spaces. On the other hand, the adjunction isomorphisms

$$(\Omega^\bullet(\Omega(\operatorname{sh}\mathbf{KU})))(V) \ = \ \operatorname{map}(S^V, \Omega\mathbf{KU}(V \oplus \mathbb{R})) \ \cong \ \operatorname{map}(S^{V\oplus\mathbb{R}}, \mathbf{KU}(V \oplus \mathbb{R})) \ = \ (\operatorname{sh}_\oplus(\Omega^\bullet\mathbf{KU}))(V)$$

provide an isomorphism of orthogonal spaces between $\Omega^\bullet(\Omega(\operatorname{sh}\mathbf{KU}))$ and $\operatorname{sh}_\oplus(\Omega^\bullet\mathbf{KU})$, where $\operatorname{sh}_\oplus = \operatorname{sh}_\oplus^{\mathbb{R}}$ is the additive shift as defined in Example I.1.12. Under this isomorphism, the composite

$$\Omega^\bullet\mathbf{KU} \ \xrightarrow{\Omega^\bullet\omega} \ \Omega^\bullet(\Omega(s\mathbf{KU})) \ \xrightarrow{\Omega^\bullet(\Omega j)} \ \Omega^\bullet(\Omega(\operatorname{sh}\mathbf{KU}))$$

becomes the morphism

$$(\Omega^\bullet\mathbf{KU}) \circ i \ : \ \Omega^\bullet\mathbf{KU} \ \longrightarrow \ \operatorname{sh}_\oplus(\Omega^\bullet\mathbf{KU})$$

given by precomposition with the embeddings $i_V : V \longrightarrow V \oplus \mathbb{R}$. This morphism is a global equivalence by Theorem I.1.11. Since the composite and $\Omega^\bullet(\Omega j)$ are global equivalences, so is the morphism $\Omega^\bullet\omega : \Omega^\bullet\mathbf{KU} \longrightarrow \Omega(\Omega^\bullet(s\mathbf{KU}))$.

In the commutative square of abelian monoids

$$\begin{tikzcd}
{[A,\mathbf{Gr}^{\mathbb{C}}]^G} \arrow[r, "{[A,c]^G}"] \arrow[d, "{[A,\beta']^G}"'] & {[A,\Omega^\bullet\mathbf{ku}]^G} \arrow[r, "{[A,\Omega^\bullet j]^G}"] & {[A,\Omega^\bullet\mathbf{KU}]^G} \arrow[r, equal] \arrow[d, "{[A,\Omega^\bullet\omega]^G}", "\cong"'] & \mathbf{KU}^0_G(A_+) \\
{[A,\Omega\mathbf{U}]^G} \arrow[rr, "\cong", "{[A,\Omega((\Omega^\bullet sj)\circ\mathrm{eig})]^G}"'] & & {[A,\Omega(\Omega^\bullet(s\mathbf{KU}))]^G} &
\end{tikzcd}$$

the lower horizontal and right vertical maps are thus isomorphisms by Proposition I.5.2. The map

$$[A,\beta']^G \ : \ [A,\mathbf{Gr}^{\mathbb{C}}]^G \ \longrightarrow \ [A,\Omega\mathbf{U}]^G$$

is a group completion of abelian monoids by Corollary II.5.40. So the upper horizontal composite $[A,\Omega^\bullet j]^G \circ [A,c]^G : [A,\mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{KU}^0_G(A_+)$ is also a group completion of abelian monoids. On the other hand, the homomorphism $\langle - \rangle : [A,\mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$ is yet another group completion of abelian monoids, by the complex analog of Theorem II.4.11. The universal property of group completions yields a unique homomorphism of abelian groups

$$\Psi \ : \ \mathbf{K}_G(A) \ \longrightarrow \ \mathbf{KU}^0_G(A_+)$$

such that $\Psi \circ \langle - \rangle = [A,\Omega^\bullet j]^G \circ [A,c]^G : [A,\mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{KU}^0_G(A_+)$, and Ψ is an isomorphism. The relation $[A,c]^G = [-] \circ \langle - \rangle$ from Theorem 4.31 (i) then yields

$$\Psi \circ \langle - \rangle \ = \ [A,\Omega^\bullet j]^G \circ [A,c]^G \ = \ [A,\Omega^\bullet j]^G_* \circ [-] \circ \langle - \rangle \ .$$

Since $\langle - \rangle : [A,\mathbf{Gr}^{\mathbb{C}}]^G \longrightarrow \mathbf{K}_G(A)$ is a group completion, this forces the relation $\Psi = [A,\Omega^\bullet j]^G_* \circ [-]$. $\square$

The special case $A = *$, a one-point G-space, of the previous corollary is worth spelling out explicitly. In this case the group $\mathbf{K}_G(*)$ becomes the unitary representation ring $\mathbf{RU}(G)$ and $\mathbf{KU}^0_G(*)$ becomes the 0-th equivariant homotopy group $\pi^G_0(\mathbf{KU})$.

THEOREM 5.17. *As G ranges of all compact Lie groups, the composite maps*

$$\mathbf{RU}(G) \xrightarrow{[-]} \pi_0^G(\mathbf{ku}) \xrightarrow{j_*} \pi_0^G(\mathbf{KU})$$

form an isomorphism of global power functors between $\mathbf{RU}$ *and* $\underline{\pi}_0(\mathbf{KU})$.

PROOF. By Corollary 5.16 for $B = *$ the composite is a ring isomorphism for every compact Lie group G. The second map is induced by a homomorphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ of ultra-commutative ring spectra, so they form a morphism of global power functors.

The maps $[-] : \mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ are ring homomorphisms, compatible with restriction homomorphisms, with finite index transfers and multiplicative power operations by Theorem 4.33. There is still something to show, though, because the maps $\langle - \rangle$ are definitely *not* compatible with general transfers (i.e., of infinite index). So an additional argument is needed to see that the composite *does* commute with all transfers.

We consider a closed subgroup H of a compact Lie group G, not necessarily of finite index, and want to show that

$$\mathrm{tr}_H^G(j_*[x]) \;=\; j_*[\mathrm{tr}_H^G(x)] \tag{5.18}$$

in $\pi_0^G(\mathbf{KU})$ for all classes $x \in \mathbf{RU}(H)$. Since representations are detected by characters, two classes in $\mathbf{RU}(G)$ are equal already if their restrictions to all closed abelian subgroups of G coincide. Since the composite maps $\mathbf{RU}(G) \longrightarrow \pi_0^G(\mathbf{KU})$ are all isomorphisms and compatible with restriction, the analogous property holds for the global functor $\underline{\pi}_0(\mathbf{KU})$. So it suffices to show that the relation (5.18) holds after restriction to any abelian closed subgroup A of G. The double coset formula

$$\mathrm{res}_A^G \circ \mathrm{tr}_H^G \;=\; \sum_{[M]} \chi^\sharp(M) \cdot \mathrm{tr}_{A \cap {}^g H}^A \circ g_\star \circ \mathrm{res}_{A^g \cap H}^H$$

holds for $\mathbf{RU}$ by [**147**, Thm. 2.4], and it holds in the homotopy global functor of every orthogonal spectrum. Since A is abelian, all its closed subgroups either have finite index or infinite Weyl group in A. Transfers from subgroups with infinite Weyl groups are zero, so after dropping those terms, the right hand side of the double coset formula only contains finite index transfers. Since the maps from $\mathbf{RU}(G)$ to $\pi_0^G(\mathbf{KU})$ under consideration do commute with finite index transfers, they commute with the right hand side of the double coset formula, hence also with the left hand side. This shows that (5.18) holds after restriction to every closed abelian subgroup, so it holds altogether. □

REMARK 5.19. In Construction 4.42 we discussed the rank filtration of the connective global K-theory spectrum $\mathbf{ku}$, and we identified the first stage $\mathbf{ku}^{[1]}$ with the suspension spectrum of the ultra-commutative monoid $\mathbf{P}^{\mathbb{C}}$. On $\underline{\pi}_0$, the morphisms of ultra-commutative ring spectra

$$\Sigma_+^\infty \mathbf{P}^{\mathbb{C}} \;\cong\; \mathbf{ku}^{[1]} \;\longrightarrow\; \mathbf{ku} \;\longrightarrow\; \mathbf{KU}$$

induce morphisms of global power functors. Since $\mathbf{P}^{\mathbb{C}}$ is a global classifying space of the circle group $U(1)$, the global functor $\underline{\pi}_0(\Sigma_+^\infty \mathbf{P}^{\mathbb{C}})$ is representable by $U(1)$, by Proposition IV.2.5. On the other hand, $\underline{\pi}_0(\mathbf{KU})$ is isomorphic to the representation ring global functor $\mathbf{KU}$, by Theorem 5.17. Under these identifications, the composite morphism $\Sigma_+^\infty \mathbf{P}^{\mathbb{C}} \longrightarrow \mathbf{KU}$ of ultra-commutative ring spectra becomes the morphism

$$\mathbf{A}(U(1), -) \;\longrightarrow\; \mathbf{KU}$$

that sends the generator $\mathrm{Id}_{U(1)} \in \mathbf{A}(U(1), U(1))$ to the class of the tautological $U(1)$-representation on $\mathbb{C}$. As we recalled in Remark 4.12, this morphism is surjective, and 'explicit Brauer induction' provides a specific section.

Our next aim is to establish 'equivariant Bott periodicity' for the global K-theory spectrum $\mathbf{KU}$. We formulate this by defining, for every G-Spin^c-representation W, explicit classes

$$\beta_{G,W} \;\in\; \mathbf{ku}_G^0(S^W) \qquad \text{and} \qquad \lambda_{G,W} \;\in\; \mathbf{KU}_0^G(S^W)$$

that become multiplicative inverses to each other in the ring $\pi_0^G(\mathbf{KU})$, compare Theorem 5.31. We call $\beta_{G,W}$ the *Bott class* and $\lambda_{G,W}$ the *inverse Bott class* of W. We emphasize that the Bott classes already 'live' in the connective theory $\mathbf{ku}$, but they only become invertible in the periodic theory $\mathbf{KU}$.

Construction 5.20. The Bott classes are special cases of Thom classes associated to G-Spin^c-vector bundles, so we first recall the construction of these Thom classes. The construction depends heavily on a certain canonical isomorphism, for every hermitian inner product space W, between $\mathrm{Cl}_{\mathbb{C}}(uW)$, the complexified Clifford algebra of the underlying euclidean vector space of W, and $\mathrm{End}_{\mathbb{C}}(\Lambda^*(W))$, the endomorphism algebra of the exterior algebra of W. We recall this isomorphism now.

The underlying $\mathbb{R}$-vector space uW of the given hermitian inner product space has a euclidean inner product defined by

$$\langle w, w' \rangle \;=\; \mathrm{Re}(w, w')\ ,$$

the real part of the complex inner product. For $w \in W$ we let

$$d_w \;:\; \Lambda^*(W) \;\longrightarrow\; \Lambda^*(W)\ , \quad d_w(x) \;=\; w \wedge x$$

denote left exterior multiplication by w. We let $d_w^* : \Lambda^*(W) \longrightarrow \Lambda^*(W)$ denote the adjoint of d_w, characterized by

$$\langle x, d_w^*(y) \rangle \;=\; \langle d_w(x), y \rangle \;=\; \langle w \wedge x, y \rangle$$

for all $x, y \in \Lambda^*(W)$. Exterior multiplication by w takes $\Lambda^n(W)$ to $\Lambda^{n+1}(W)$, so its adjoint takes $\Lambda^{n+1}(W)$ to $\Lambda^n(W)$. So

$$\delta_w \;=\; d_w - d_w^*\ ,$$

is an odd operator on the exterior algebra. It is straightforward to check that the $\mathbb{R}$-linear map

$$uW \;\longrightarrow\; \mathrm{End}_{\mathbb{C}}(\Lambda^*(W))\ , \quad w \;\longmapsto\; \delta_w$$

satisfies

$$\delta_w \circ \delta_w \;=\; -|w|^2 \cdot \mathrm{Id}\ ;$$

so the universal property of the Clifford algebra provides a unique homomorphism of $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras

$$\delta_W \;:\; \mathrm{Cl}_{\mathbb{C}}(uW) \;\longrightarrow\; \mathrm{End}_{\mathbb{C}}(\Lambda^*(W)) \tag{5.21}$$

sending $w \in W$ to δ_w. This homomorphism makes the exterior algebra $\Lambda^*(W)$ into a $\mathbb{Z}/2$-graded $\mathrm{Cl}_{\mathbb{C}}(uW)$-module. We recall that the homomorphism δ_W is in fact an isomorphism. Indeed, given two hermitian inner product spaces V and W, the following square commutes:

$$\begin{tikzcd}
\mathrm{Cl}_{\mathbb{C}}(uV \oplus uW) \arrow[r, "\delta_{V\oplus W}"] \arrow[d, "(5.4)"', "\cong"] & \mathrm{End}_{\mathbb{C}}(\Lambda^*(V \oplus W)) \arrow[d, "\cong"'] \\
\mathrm{Cl}_{\mathbb{C}}(uV) \otimes \mathrm{Cl}_{\mathbb{C}}(uW) \arrow[r, "\delta_V \otimes \delta_W"'] & \mathrm{End}_{\mathbb{C}}(\Lambda^*(V)) \otimes \mathrm{End}_{\mathbb{C}}(\Lambda^*(W)) \arrow[u, "\cong"', "\otimes"] \\
 & \mathrm{End}_{\mathbb{C}}(\Lambda^*(V) \otimes \Lambda^*(W))
\end{tikzcd}$$

The upper right vertical map is induced by the isomorphism of $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras

$$\Lambda^*(V \oplus W) \;\xrightarrow{\cong}\; \Lambda^*(V) \otimes \Lambda^*(W)$$

that extends the linear map

$$V \oplus W \;\longrightarrow\; \Lambda^*(V) \otimes \Lambda^*(W)\ , \quad (v, w) \;\longmapsto\; v \otimes 1 \;+\; 1 \otimes w\ .$$

The vertical maps in the above diagram are isomorphisms, and every hermitian inner product space is isometrically isomorphic to $\mathbb{C}^n$ with standard inner product; so this reduces the claim to the special

case $W = \mathbb{C}$, which can be checked directly from the definitions by working out the images of the $\mathbb{C}$-basis $\{1, [1], [i], [1] \cdot [i]\}$ of $\mathrm{Cl}_{\mathbb{C}}(u\mathbb{C})$ under $\delta_{\mathbb{C}}$.

We claim that the composite

$$U(W) \xrightarrow{l(W)} \mathrm{Spin}^c(uW) \xrightarrow{\mathrm{incl}} \mathrm{Cl}_{\mathbb{C}}(uW) \xrightarrow{\cong} \mathrm{End}_{\mathbb{C}}(\Lambda^* W)$$

is the functoriality of the exterior algebra, i.e., it sends a unitary automorphism A to $\Lambda^*(A)$ [...]

Construction 5.22 (Thom class in equivariant K-theory). We recall the construction, due to Atiyah-Bott-Shapiro [**8**], of the Thom class of an equivariant spin^c-vector bundle. We let G be a compact Lie group, W a hermitian inner product space, and $\bar{\xi} : P \longrightarrow B$ a G-equivariant principal $\mathrm{Spin}^c(uW)$-bundle over a compact G-space B. In other words, $\bar{\xi}$ is a $(G, \mathrm{Spin}^c(uW))$-bundle in the sense of Remark I.2.13. The bundle has an associated G-vector bundle $\xi : E \longrightarrow B$ over the same base, with total space

$$E \;=\; P \times_{\mathrm{Spin}^c(uW)} W \;,$$

where the spin^c group acts on W through the adjoint representation $\mathrm{ad} : \mathrm{Spin}^c(uW) \longrightarrow SO(uW)$.

Despite the fact that W is a complex vector space, this associated vector bundle is in general only an $\mathbb{R}$-vector bundle; to make it a $\mathbb{C}$-vector bundle we would have to reduce the structure group further to $U(W)$ along the monomorphism $l(W) : U(W) \longrightarrow \mathrm{Spin}^c(uW)$.

One thinks of the $(G, \mathrm{Spin}^c(uW))$-bundle $\bar{\xi}$ as additional structure on the G-vector bundle ξ, and this additional spin^c-structure gives rise to a *Thom class*

$$u_G(\bar{\xi}) \;\in\; \tilde{\mathbf{K}}_G(Th(\xi))$$

in the reduced G-equivariant K-group of the Thom G-space

$$Th(\xi) \;=\; P_+ \wedge_{\mathrm{Spin}^c(uW)} S^W \;.$$

The Thom class arises first as a specific element in $\mathbf{K}_G(D(\xi), S(\xi))$, the G-equivariant relative K-group of the unit disc bundle relative to the sphere bundle of ξ. Elements in this relative K-group are equivalence classes of triples (E_0, E_1, ψ) where E_0 and E_1 are G-vector bundles over the disc bundle $D(\xi)$ and $\alpha : (E_0)|_{S(\xi)} \cong (E_1)|_{S(\xi)}$ is an equivariant isomorphism between the restrictions of the two bundles to the sphere bundle. The Thom class $u_G(\bar{\xi})$ is then the element

$$[P \times_{\mathrm{Spin}^c(uW)} (D(uW) \times \Lambda^{\mathrm{ev}}(W)), P \times_{\mathrm{Spin}^c(uW)} (D(W) \times \Lambda^{\mathrm{odd}}(W)), \alpha] \;\in\; \mathbf{K}_G(D(\xi), S(\xi)) \;,$$

where the equivariant bundle isomorphism α is associated, via $P \times_{\mathrm{Spin}^c(uW)} -$ to the Clifford action of Construction 5.20:

$$\alpha \;:\; S(W) \times \Lambda^{\mathrm{ev}}(W) \;\longrightarrow\; S(W) \times \Lambda^{\mathrm{odd}}(W) \;, \quad (w, x) \;\longmapsto\; (w, \delta_w(x)) \;.$$

Since the G-action is entirely through the action on P, all this data is G-equivariant.

Remark 5.23. If V is a euclidean (as opposed to hermitian) inner product space and $\bar{\xi} : P \longrightarrow B$ a G-equivariant principal $\mathrm{Spin}^c(V)$-bundle over a compact G-space B, then we can define another Thom class

$$u_G(\bar{\xi}) \;\in\; \tilde{\mathbf{K}}_G(S^V \wedge Th(\xi))$$

in the reduced G-equivariant K-group of $S^V \wedge Th(\xi)$; here G acts trivially on V. To define this class, we form the product $V \times \xi$ of ξ with V; the $\mathrm{Spin}^c(V)$-structure on ξ induces a $\mathrm{Spin}^c(V \oplus V)$-structure on $V \times \xi$. We identify $V \oplus V = u(V_{\mathbb{C}})$, so that we get a $\mathrm{Spin}^c(uW)$-structure with $W = V_{\mathbb{C}}$ the complexification of V. The Thom class for this $(G, \mathrm{Spin}^c(u(V_{\mathbb{C}}))$-bundle is then a class in the Thom space of $V \times \xi$, which is naturally homeomorphic to $S^V \wedge Th(\xi)$.

Construction 5.24 (Equivariant Bott classes). When we specialize the Thom class to the situation when the base is a one-point G-space, we arrive at the the definition of the equivariant Bott classes as given, for example, in [**86**, III p.44]. A G-vector bundle over a one-point space amounts to a G-representation, and an equivariant spin^c-structure of a G-representation amounts to a lift $G \longrightarrow \mathrm{Spin}^c(V)$ of the representation homomorphism $G \longrightarrow O(V)$. To get a Thom class via Construction 5.22 we thus start with a hermitian inner product space W and a continuous homomorphism $\rho : G \longrightarrow \mathrm{Spin}^c(uW)$. If we let G act on $\mathrm{Spin}^c(uW)$ through ρ, the map $\mathrm{Spin}^c(uW) \longrightarrow *$ is a $(G, \mathrm{Spin}^c(uW))$-bundle whose associated G-vector bundle is the underlying orthogonal G-representation W. We write

$$\beta_{G,W} \ \in \ \mathbf{K}_G(D(W), S(W)) \ \cong \ \tilde{\mathbf{K}}_G(S^W)$$

for the associated Thom class and refer to is as the *Bott class* of the spin^c-representation W. By construction, then Bott class is represented by the triple

$$(D(W) \times \Lambda^{\mathrm{ev}}(W), D(W) \times \Lambda^{\mathrm{odd}}(W), \alpha)$$

consisting of the trivial vector bundles over $D(W)$ with fibers $\Lambda^{\mathrm{ev}}(W)$ respectively $\Lambda^{\mathrm{odd}}(W)$, and the equivariant bundle isomorphism α is given by the Clifford action of Construction 5.20:

$$\alpha \ : \ S(W) \times \Lambda^{\mathrm{ev}}(W) \ \longrightarrow \ S(W) \times \Lambda^{\mathrm{odd}}(W) \ , \quad (w, x) \ \longmapsto \ (w, \delta_w(x)) \ .$$

The group G acts on $D(W)$ through its action on W, and on $\Lambda^*(W)$ through the composite

$$G \ \xrightarrow{\rho} \ \mathrm{Spin}^c(uW) \ \xrightarrow{\mathrm{incl}} \ \mathrm{Cl}_{\mathbb{C}}(uW) \ \cong \ \mathrm{End}_{\mathbb{C}}(\Lambda^*(W)) \ . \tag{5.25}$$

Equivariant Bott periodicity [**7**, Thm. 4.3] says that for every compact G-space B and every G-spin^c-representation W as above, exterior product with the Bott class is an isomorphism

$$- \times \beta_{G,W} \ : \ \mathbf{K}_G(B) \ \longrightarrow \ \mathbf{K}_G(B \times D(W), B \times S(W)) \ .$$

To represent the Bott class in $\mathbf{ku}^0_G(S^W)$ we interpret the Clifford action as a clutching function for a vector bundle. We define a G-vector bundle $\xi(W)$ over S^W by gluing the trivial bundle with fiber $\Lambda^{\mathrm{ev}}(W)$ over $D(W)$ with the trivial bundle with fiber $\Lambda^{\mathrm{odd}}(W)$ over $S^W - \mathring{D}(W)$ using the map

$$S(W) \ \longrightarrow \ \mathbf{L}^{\mathbb{C}}(\Lambda^{\mathrm{ev}}(W), \Lambda^{\mathrm{odd}}(W)) \ , \quad w \ \longmapsto \ \delta_w \ .$$

The homomorphism

$$[-] \ : \ \mathbf{K}_G(S^W) \ \longrightarrow \ \mathbf{ku}^0_G(S^W_+)$$

defined in Theorem 4.31 turns this G-vector bundle into an unreduced equivariant **ku**-cohomology class. By construction, the fiber of $\xi(W)$ over the basepoint at infinity is the G-representation $\Lambda^{\mathrm{odd}}(W)$, with G acting via the composite (5.25). So by subtracting the class of the trivial vector bundle with fiber $\Lambda^{\mathrm{odd}}(W)$ we obtained a reduced equivariant **ku**-cohomology class, the *Bott class*

$$\beta_{G,W} \ = \ \langle \xi(W) \rangle - \langle S^W \times \Lambda^{\mathrm{odd}}(W) \rangle \ \in \ \mathbf{ku}^0_G(S^W)$$

of the G-spin^c-representation W.

Example 5.26. We look at the Bott class of the tautological $\mathrm{Spin}^c(u\mathbb{C})$. The group $\mathrm{Spin}^c(u\mathbb{C})$ is isomorphic to $U(1) \times U(1)$ via the map

$$U(1) \times U(1) \ \longrightarrow \ \mathrm{Spin}^c(u\mathbb{C}) \ , \quad (\lambda, \mu) \ \longmapsto \ l(\mathbb{C})(\lambda) \cdot \mu \ ,$$

where the monomorphism $l(W) : U(W) \longrightarrow \mathrm{Spin}^c(uW)$ was defined in Example II.3.9. Under this identification, the action of $\mathrm{Spin}^c(u\mathbb{C})$ on $\Lambda^*(\mathbb{C})$ becomes the map

$$U(1) \times U(1) \ \longrightarrow \ M_2(\mathbb{C}) \ , \quad (\lambda, \mu) \ \longmapsto \ \begin{pmatrix} \mu & 0 \\ 0 & \lambda\mu \end{pmatrix} \ .$$

In other words, the action on Λ^0 has weight $(0,1)$ and the action on Λ^1 has weight $(1,1)$. The action on $\mathbb{C}$ has weight $(1,0)$. The Bott class $\beta \in \mathbf{ku}^0_{U(1)\times U(1)}(S^{(1,0)})$ is thus represented by the triple

$$(D(1,0)\times\mathbb{C}(0,1),\ D(1,0)\times\mathbb{C}(1,1),\ \alpha)$$

where α is the $(U(1)\times U(1))$-equivariant bundle isomorphism

$$\alpha\ :\ S(1,0)\times\mathbb{C}(0,1)\ \longrightarrow\ S(1,0)\times\mathbb{C}(1,1)\ ,\quad (w,x)\ \longmapsto\ (w,wx)\ .$$

We claim that the vector over $S^{(1,0)}$ obtained by gluing $D(1,0)\times\mathbb{C}(0,1)$ to $D(1,0)\times\mathbb{C}(1,1)$ via α is isomorphic to the tautological line bundle over the projective space of the T^2-representation $\mathbb{C}(0,1)\oplus\mathbb{C}(1,1)$. Indeed, the two T^2-equivariant maps

$$D(1,0)\times\mathbb{C}(0,1)\ \longrightarrow\ \gamma(\mathbb{C}(0,1)\oplus\mathbb{C}(1,1))\ ,\quad (w,x)\ \longmapsto\ ([1:w],(x,xw))$$

and

$$D(1,0)\times\mathbb{C}(1,1)\ \longrightarrow\ \gamma(\mathbb{C}(0,1)\oplus\mathbb{C}(1,1))\ ,\quad (w,x)\ \longmapsto\ ([\bar{w}:1],(x\bar{w},x))$$

make the following diagram commute:

$$\begin{tikzcd}
S(1,0)\times\mathbb{C}(0,1) \arrow[r, "\text{incl}"] \arrow[d, "\alpha"'] & D(1,0)\times\mathbb{C}(0,1) \arrow[dr] & \\
 & & \gamma(\mathbb{C}(0,1)\oplus\mathbb{C}(1,1)) \\
S(1,0)\times\mathbb{C}(1,1) \arrow[r, "\text{incl}"'] & D(1,0)\times\mathbb{C}(1,1) \arrow[ur] &
\end{tikzcd}$$

PROPOSITION 5.27. *The Bott classes $\beta_{G,W}$ have the following properties.*

(i) *For every isomorphism $\varphi : W \longrightarrow W'$ of G-Spin^c-representations, the induced isomorphism $(S^\varphi)^* : \mathbf{ku}^0_G(S^{W'}) \longrightarrow \mathbf{ku}^0_G(S^W)$ takes the class $\beta_{G,W'}$ to the class $\beta_{G,W}$.*

(ii) *The Bott class $\beta_{G,0}$ of the trivial 0-dimensional G-Spin^c-representation is the unit element $1 \in \mathbf{ku}^0_G(S^0)$*

(iii) *For every continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation*

$$\alpha^*(\beta_{G,W})\ =\ \beta_{K,\alpha^*W}$$

holds in $\mathbf{ku}^0_K(S^{\alpha^*(W)})$.

(iv) *For all G-spin^c-representations W and W' the relation*

$$\beta_{G,W}\times\beta_{G,W'}\ =\ \beta_{G,\,W\oplus W'}\qquad \textit{holds in } \mathbf{ku}^0_G(S^{W\oplus W'})\ .$$

CONSTRUCTION 5.28 (Inverse Bott classes). We let W be a hermitian inner product space. The isomorphism (5.21)

$$\mathrm{Cl}_{\mathbb{C}}(uW)\ \cong\ \mathrm{End}_{\mathbb{C}}(\Lambda^*(W))$$

provides an action of the group $\mathrm{Spin}^c(uW)$ on $\Lambda^*(W)$ by $\mathbb{C}$-linear automorphisms. We denote by $q_W \in \mathrm{Cl}_{\mathbb{C}}(uW)$ the self-adjoint idempotent in $\mathrm{Cl}_{\mathbb{C}}(uW)$ that corresponds under the isomorphism (5.21)

$$\mathrm{Cl}_{\mathbb{C}}(uW)\ \cong\ \mathrm{End}_{\mathbb{C}}(\Lambda^*(W))$$

to the projection onto $\Lambda^{\mathrm{ev}}(W)$, the even summand of the exterior algebra. Since $\Lambda^{\mathrm{ev}}(W)$ is invariant under the action of the group $\mathrm{Spin}^c(uW)$, the projector q_W commutes with all elements of $\mathrm{Spin}^c(uW)$.

Moreover, we denote by $p_{\mathbb{C}} \in \mathcal{K}_{uW}$ the orthogonal projection onto the subspace $\mathbb{C}\cdot 1$ in $\mathcal{H}_{uW} = \widehat{\mathrm{Sym}}((uW)_{\mathbb{C}})$. Then the element

$$q_W\otimes p_{\mathbb{C}}\ \in\ \mathrm{Cl}_{\mathbb{C}}(uW)\otimes\mathcal{K}_{uW}\ .$$

is an even self-adjoint idempotent, and it is invariant under the action of the group $\mathrm{Spin}^c(uW)$ by conjugation on $\mathrm{Cl}_{\mathbb{C}}(uW)$ and via the adjoint action to $SO(uW)$ on $\mathbf{KU}(uW)$. So the map

$$s\ \longrightarrow\ \mathrm{Cl}_{\mathbb{C}}(uW)\otimes\mathcal{K}_{uW}\ ,\quad f\ \longmapsto\ f(0)\cdot q_W\otimes p_{\mathbb{C}}$$

is a $\mathbb{Z}/2$-graded $*$-homomorphism and a $\mathrm{Spin}^c(uW)$-fixed point in

$$C^*_{\mathrm{gr}}(s, \mathrm{Cl}_{\mathbb{C}}(uW) \otimes \mathcal{K}_{uW}) \;=\; \mathbf{KU}(uW)\ .$$

We denote by

$$\lambda_W \;\in\; \mathbf{KU}_0^{U(W)}(S^W)$$

as the equivariant homotopy class defined by this fixed point.

Now we let G be a compact Lie group and we let W be a unitary G-representation with representation homomorphism $\rho : G \longrightarrow U(W)$. We define the *inverse Bott class* as

$$\lambda_{G,W} \;=\; \rho^*(\lambda_W) \;\in\; \mathbf{KU}_0^G(S^W)\ ,$$

the restriction of the inverse Bott class λ_W along the homomorphism ρ.

PROPOSITION 5.29. *The inverse Bott classes $\lambda_{G,W}$ have the following properties.*

(i) *For every isomorphism $\varphi : W \longrightarrow W'$ of unitary G-representations, the induced isomorphism $(S^\varphi)^* : \mathbf{KU}_G^0(S^{W'}) \longrightarrow \mathbf{KU}_G^0(S^W)$ takes the class $\lambda_{G,W'}$ to the class $\lambda_{G,W}$.*

(ii) *The inverse Bott class $\lambda_{G,0}$ of the trivial 0-dimensional G-representation is the unit element $1 \in \mathbf{KU}_G^0(S^0)$*

(iii) *For every continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation*

$$\alpha^*(\lambda_{G,W}) \;=\; \lambda_{K,\alpha^*W}$$

holds in $\mathbf{KU}_K^0(S^{\alpha^*(W)})$.

(iv) *For all unitary G-representations W and W' the relation*

$$\lambda_{G,W} \times \lambda_{G,W'} \;=\; \lambda_{G,\,W\oplus W'} \qquad \textit{holds in } \mathbf{KU}_G^0(S^{W\oplus W'})\ .$$

PROOF. □

REMARK 5.30 (Bott classes for unitary representations). We let W be a hermitian inner product space. In (3.10) of Chapter II we reviewed a preferred continuous homomorphism $U(W) \longrightarrow \mathrm{Spin}^c(uW)$ that can be used to turn a unitary G-representation into a G-Spin^c-representation; so this in particular assigns equivariant Bott classes and inverse Bott classes to all unitary representations of compact Lie groups.

Now we come to the key relation between the Bott and inverse Bott classes, which also explains the terminology. In Construction 5.7 we defined a homomorphism of ultra-commutative ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$. The next theorem shows that the image of the Bott class under the induced homomorphism

$$j_* \;:\; \mathbf{ku}_G^0(S^W) \;\longrightarrow\; \mathbf{KU}_G^0(S^W)$$

is invertible in the periodic K-theory spectrum $\mathbf{KU}$, and that the inverse Bott class is its inverse (whence the name). The product of $\beta_{G,W}$ and $\lambda_{G,W}$ is to be interpreted in the '$RO(G)$-graded' sense, which we realize as a special case of the Kronecker pairing:

$$\langle -,- \rangle \;:\; \mathbf{KU}_0^G(A) \times \mathbf{KU}_G^0(A) \;\longrightarrow\; \pi_0^G(\mathbf{KU})\ ,$$

where A is a based G-CW-complex. The pairing sends a pair

$$([f : S^U \longrightarrow \mathbf{KU}(U) \wedge A,\ g : A \wedge S^V \longrightarrow \mathbf{KU}(V)]) \quad \in \quad \mathbf{KU}_0^G(A) \times \mathbf{KU}_G^0(A)$$

of equivariant (co-)homology classes to the class represented by the composite

$$S^{U\oplus V} \xrightarrow{f\wedge S^V} \mathbf{KU}(U) \wedge A \wedge S^V \xrightarrow{\mathbf{KU}(U)\wedge g} \mathbf{KU}(U) \wedge \mathbf{KU}(V) \xrightarrow{\mu_{U,V}} \mathbf{KU}(U \oplus V)\ .$$

THEOREM 5.31. *For every compact Lie group G and every unitary G-representation W the relation*

$$\langle j_*(\beta_{G,W}),\ \lambda_{G,W} \rangle \;=\; 1$$

holds in $\pi_0^G(\mathbf{KU})$.

Proof. We proceed in a sequence of steps.

Step 1: We denote by $\mathbb{C}(1)$ the tautological $\mathrm{Spin}^c(u\mathbb{C})$-spin^c-representation on the hermitian inner product space $\mathbb{C}$. Then $\lambda_{\mathrm{Spin}^c(u\mathbb{C}),\mathbb{C}} = \lambda_{\mathbb{C}}$. We let $a \in \mathbb{S}_0^{\mathrm{Spin}^c(u\mathbb{C})}(S^{\mathbb{C}(1)})$ denote the class represented by the fixed point inclusion

$$i \ : \ S^0 \ \longrightarrow \ S^{\mathbb{C}} \ .$$

We use the same symbol to denote the image of a in $\mathbf{ku}_0^{\mathrm{Spin}^c(u\mathbb{C})}(S^{\mathbb{C}(1)})$ and in $\mathbf{KU}_0^{\mathrm{Spin}^c(u\mathbb{C})}(S^{\mathbb{C}(1)})$ under the unit morphism $\mathbb{S} \longrightarrow \mathbf{ku}$ respectively $\mathbb{S} \longrightarrow \mathbf{KU}$. The class a is obtained by inflating the Euler class along the epimorphism $\mathrm{Spin}^c(u\mathbb{C}) \longrightarrow SO(u\mathbb{C})$. We claim that the relation

$$\lambda_{\mathbb{C}} \cdot (\langle \mathbb{C}(1) \rangle - 1) \ = \ a \tag{5.32}$$

holds in $\pi_0^{\mathrm{Spin}^c(u\mathbb{C})}(\mathbf{KU})$.

The class a sends the non-basepoint 0 to the $*$-homomorphism

$$s \longrightarrow \mathrm{Cl}_{\mathbb{C}}(u\mathbb{C}(1)) \otimes \mathcal{K}_{u\mathbb{C}(1)} \ , \quad f \ \longmapsto \ f[0] \otimes p_{\mathbb{C}} \ = \ f(0) \cdot 1 \otimes p_{\mathbb{C}} \ .$$

This is the same as the composite

$$s \ \xrightarrow{\epsilon} \ \mathbb{C} \ \longrightarrow \ \mathrm{Cl}_{\mathbb{C}}(u\mathbb{C}) \otimes \mathcal{K}_{u\mathbb{C}}$$

where the second map is given by the orthogonal idempotent $1 \otimes p_{\mathbb{C}}$. Under $\mathrm{Cl}_{\mathbb{C}}(u\mathbb{C}(1)) \cong \mathrm{End}_{\mathbb{C}}(\Lambda^*(\mathbb{C}(1)))$ the unit 1 corresponds to the identity, which is the sum of the projections to the exterior powers. Since the even powers are even and the odd powers are odd, the claim follows.

Step 2: We prove the tautological $\mathrm{Spin}^c(u\mathbb{C})$-representation on $\mathbb{C}$. Step 1 then shows that

$$\begin{aligned} a \cdot \langle j_*(\beta), \ \lambda_{\mathbb{C}} \rangle \ &= \ \langle a \cdot j_*(\beta), \ \lambda_{\mathbb{C}} \rangle \\ &= \ \langle j_*(a \cdot \beta), \ \lambda_{\mathbb{C}} \rangle \\ &= \ \langle \langle \mathbb{C}(0,1) \rangle - \langle \mathbb{C}(1,1) \rangle, \ \lambda_{\mathbb{C}} \rangle \ = \ (\langle \mathbb{C}(0,1) \rangle - \langle \mathbb{C}(1,1) \rangle) \cdot \lambda_{\mathbb{C}} \ = \ a \ . \end{aligned}$$

In other words, the desired relation holds after multiplication by the class $a \in \mathbf{KU}_0^{\mathrm{Spin}^c(u\mathbb{C})}(S^{\mathbb{C}(1)})$.

The cofiber sequence of based $\mathrm{Spin}^c(u\mathbb{C})$-spaces

$$S(\mathbb{C}(1,0))_+ \ \longrightarrow \ S^0 \ \xrightarrow{i} \ S^{\mathbb{C}(1,0)} \ \longrightarrow \ S^1 \wedge S(\mathbb{C}(1,0))_+$$

induces a long exact sequence in $\mathrm{Spin}^c(u\mathbb{C})$-equivariant $\mathbf{KU}$-homology. The Wirthmüller isomorphism (see Theorem III.2.14) shows that

$$\mathbf{KU}_0^{\mathrm{Spin}^c(u\mathbb{C})}(S(\mathbb{C}(1,0))_+) \ \cong \ \mathbf{KU}_0^{T^2}((T^2/e \times T)_+) \ \cong \ \mathbf{KU}_0^{U(1)}(S^1) \ = \ 0 \ .$$

[...justify the vanishing...] So the map

$$i_* \ : \ \pi_0^{\mathrm{Spin}^c(u\mathbb{C})}(\mathbf{KU}) = \mathbf{KU}_0^{\mathrm{Spin}^c(u\mathbb{C})}(S^0) \ \longrightarrow \ \mathbf{KU}_0^{\mathrm{Spin}^c(u\mathbb{C})}(S^{\mathbb{C}(1,0)})$$

is injective. Since this map equals multiplication by the class a, this proves the claim.

Step 3: We prove the claim for the tautological unitary $U(n)$-representation on $\mathbb{C}^n$. We let T^n denote the diagonal maximal torus inside $U(n)$. Step 2 and the multiplicativity properties of Bott classes, inverse Bott classes and the Kronecker pairing give

$$\begin{aligned} \mathrm{res}_{T^n}^{U(n)} \langle j_*(\beta_{U(n),\mathbb{C}^n}), \ \lambda_{U(n),\mathbb{C}^n} \rangle \ &= \ \langle j_*(\beta_{T^n,\mathbb{C}^n}), \ \lambda_{T^n,\mathbb{C}^n} \rangle \\ &= \ \langle j_*(\beta_{U(1),\mathbb{C}(1)}) \times \cdots \times j_*(\beta_{U(1),\mathbb{C}(1)}), \ \lambda_{U(1),\mathbb{C}(1)} \times \cdots \times \lambda_{U(1),\mathbb{C}(1)} \rangle \\ &= \ \langle j_*(\beta_{U(1),\mathbb{C}(1)}), \lambda_{U(1),\mathbb{C}(1)} \rangle \times \cdots \times \langle j_*(\beta_{U(1),\mathbb{C}(1)}), \lambda_{U(1),\mathbb{C}(1)} \rangle \ = \ 1 \ . \end{aligned}$$

In other words, the desired relation holds after restriction to the maximal torus. The horizontal maps in the commutative square

$$\begin{array}{ccc} \mathbf{RU}(U(n)) & \xrightarrow{j_*\circ\langle -\rangle} & \pi_0^{U(n)}(\mathbf{KU}) \\ {\scriptstyle \mathrm{res}_{T^n}^{U(n)}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{T^n}^{U(n)}} \\ \mathbf{RU}(T^n) & \xrightarrow[j_*\circ\langle -\rangle]{} & \pi_0^{T^n}(\mathbf{KU}) \end{array}$$

are isomorphisms by Theorem 5.17. Representations of connected compact Lie groups are detected on a maximal torus, i.e., the left vertical map is injective. So the right vertical map is injective, which shows that $\langle j_*(\beta_{U(n),\mathbb{C}^n}),\ \lambda_{U(n),\mathbb{C}^n}\rangle = 1$.

Step 4: A naturality argument now proves the claim in general. Indeed, for every unitary G-representation W there is a continuous homomorphism $\alpha : G \longrightarrow U(n)$ and an isomorphism $\varphi : W \cong \alpha^*(\mathbb{C}^n)$. Thus

$$\begin{aligned} \langle j_*(\beta_{G,W}),\ \lambda_{G,W}\rangle &= \langle j_*(\beta_{G,\alpha^*(\mathbb{C}^n)}),\ \lambda_{G,\alpha^*(\mathbb{C}^n)}\rangle \\ &= \langle j_*(\alpha^*(\beta_{U(n),\mathbb{C}^n})),\ \alpha^*(\lambda_{U(n),\mathbb{C}^n})\rangle \\ &= \alpha^*\langle j_*(\beta_{U(n),\mathbb{C}^n}),\ \lambda_{U(n),\mathbb{C}^n}\rangle \ = \ \alpha^*(1) \ = \ 1\ , \end{aligned}$$

using step 3 and various naturality properties of the Kronecker pairing. □

Example 5.33. Our language allows a reformulation of the generalization, due to Adams, Haeberly, Jackowski and May [**1**], of the Atiyah-Segal completion theorem: the global K-theory spectrum $\mathbf{KU}$ is right induced from the global family *cyc* of finite cyclic groups.

If G is a compact Lie group then a virtual G-representation that restricts to zero on every finite cyclic subgroup is already zero. In other words, the intersection of the kernels of all restriction maps $\mathrm{res}_C^G : R(G) \longrightarrow R(C)$ for all finite cyclic subgroups C of G, is trivial. Then by [**1**, Cor. 2.1], the projection $A \times E(cyc \cap G) \longrightarrow A$ induces an isomorphism

$$\mathbf{K}_G^*(A) \ \cong\ \mathbf{K}_G^*(A \times E(cyc \cap G))$$

on equivariant K-groups for every finite G-CW-complex A, where $E(cyc \cap G)$ is a universal G-space for the family of finite cyclic subgroups of G. The Milnor short exact sequence lets us extend this to infinite G-CW-complexes, so the criterion provided by Proposition IV.5.18 for being right induced from the global family *cyc* is satisfied.

Construction 5.34 (Global connective K-theory). Now we define *global connective K-theory* $\mathbf{ku}^c$, an ultra-commutative ring spectrum whose associated G-homotopy type, for compact Lie groups G, is that of *G-equivariant connective K-theory* in the sense of Greenlees [**63**]. This is *not* a connective equivariant theory, i.e., the equivariant homotopy groups $\pi_k^G(\mathbf{ku}^c)$ do *not* generally vanish in negative dimensions, as soon as the group G is non-trivial. Hence the order of the adjectives 'global' and 'connective' matters, i.e., 'global connective' K-theory is different from 'connective global' K-theory. Two of the key advantages of $\mathbf{ku}^c$ over $\mathbf{ku}$ are that $\mathbf{ku}^c$ is equivariantly (and in fact globally) orientable, and that $\mathbf{ku}^c$ satisfies a completion theorem in the sense that for every compact Lie group G, the completion of the graded ring $\pi_*^G(\mathbf{ku}^c)$ at the augmentation ideal is the connective $\mathbf{ku}$-cohomology of the classifying space BG, see [**63**, Prop. 2.2 (i)].

Our construction of $\mathbf{ku}^c$ is a direct 'globalization' of Greenlees' definition in [**63**, Def. 3.1]. We define $\mathbf{ku}^c$ as the homotopy pullback in the square of ultra-commutative ring spectra

$$\begin{tikzcd} \mathbf{ku}^c \arrow[r] \arrow[d] & b(\mathbf{ku}) \arrow[d, "bj"] \\ \mathbf{KU} \arrow[r, "i_{\mathbf{KU}}"'] & b(\mathbf{KU}) \end{tikzcd} \tag{5.35}$$

The morphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ from connective to periodic global K-theory was defined in Construction 5.7; the Borel theory functor b and the natural transformation $i : \mathrm{Id} \longrightarrow b$ were defined in Construction IV.5.24. So more explicitly, we set

$$\mathbf{ku}^c \;=\; \mathbf{KU} \times_{b(\mathbf{KU})} b(\mathbf{KU})^{[0,1]} \times_{b(\mathbf{KU})} b(\mathbf{ku})\ .$$

As a homotopy pullback, the square (5.35) does *not* commute, but the construction comes with a preferred homotopy between the two composites around the square.

Since the spectra $\mathbf{KU}$, $b(\mathbf{KU})$ and $b(\mathbf{ku})$ are ultra-commutative ring spectra and the two morphisms $i_{\mathbf{KU}} : \mathbf{KU} \longrightarrow b(\mathbf{KU})$ and $bj : b(\mathbf{ku}) \longrightarrow b(\mathbf{KU})$ are homomorphisms, the homotopy pullback is canonically an ultra-commutative ring spectrum and the two morphisms from $\mathbf{ku}^c$ to $\mathbf{KU}$ and $b(\mathbf{ku})$ are morphisms of ultra-commutative ring spectra.

Naturality of the morphism $i_{\mathbf{ku}} : \mathbf{ku} \longrightarrow b(\mathbf{ku})$ provides a morphism of ultra-commutative ring spectra

$$\mathbf{ku} \;\longrightarrow\; \mathbf{ku}^c$$

from connective global K-theory to global connective K-theory. For finite groups, this morphisms induces an isomorphism on homotopy global functors in non-negative dimensions. The construction of $\mathbf{ku}^c$ endows it with a morphism of ultra-commutative ring spectra $\mathbf{ku}^c \longrightarrow \mathbf{KU}$. As explained by Greenlees in [**63**, Thm. 2.1 (iv)], this morphism become a global equivalence after inverting the Bott class $v \in \pi_2^e(\mathbf{ku}^c)$, defined as the image of the Bott class $\beta \in \pi_2^e(\mathbf{ku})$. Indeed, for every compact Lie group G, the morphism $j : \mathbf{ku} \longrightarrow \mathbf{KU}$ induces a morphism of graded rings

$$j^*(BG) \;:\; \mathbf{ku}^*(BG) \;\longrightarrow\; \mathbf{KU}^*(BG)$$

that becomes an isomorphism after inverting the Bott class β by [**33**, Lemma 1.1.1]. The natural isomorphisms

$$\pi^G_{-*}(b(\mathbf{ku})) \;\longrightarrow\; \mathbf{ku}^*(BG) \qquad \text{and} \qquad \pi^G_{-*}(b(\mathbf{KU})) \;\longrightarrow\; \mathbf{KU}^*(BG)$$

of Proposition IV.5.27 then show that the map

$$\underline{\pi}_*(bj) \;:\; \underline{\pi}_*(b(\mathbf{ku})) \;\longrightarrow\; \underline{\pi}_*(b(\mathbf{KU}))$$

becomes an isomorphism of graded global functors after inverting v. Since the right vertical morphism in the defining global homotopy pullback (5.35) becomes a global equivalence after inverting v, the same is true for the left vertical morphism.

The composite

$$\dim^c \;:\; \mathbf{ku}^c \;\longrightarrow\; b(\mathbf{ku}) \;\xrightarrow{b(\dim)}\; b(\mathcal{H}\mathbb{Z})$$

is a morphism of ultra-commutative ring spectra, where $\mathcal{H}\mathbb{Z}$ is the Eilenberg-Mac Lane spectrum of the integers (see Construction 1.9) and the dimension homomorphism $\dim : \mathbf{ku} \longrightarrow \mathcal{H}\mathbb{Z}$ was defined in (4.38). We claim that there is a global homotopy cofiber sequence

$$S^2 \wedge \mathbf{ku}^c \;\xrightarrow{\beta}\; \mathbf{ku}^c \;\xrightarrow{\dim^c}\; b(\mathcal{H}\mathbb{Z})\ . \tag{5.36}$$

Indeed, the sequence

$$S^2 \wedge \mathbf{ku} \;\xrightarrow{\beta}\; \mathbf{ku} \;\xrightarrow{\dim}\; \mathcal{H}\mathbb{Z}$$

is a non-equivariant homotopy fiber sequence; the Borel theory functor b takes this non-equivariant homotopy fiber sequence to the global homotopy fiber sequence

$$S^2 \wedge b(\mathbf{ku}) \xrightarrow{b(\beta)} b(\mathbf{ku}) \xrightarrow{b(\dim)} b(\mathcal{H}\mathbb{Z}) \ .$$

Since the spectra $\mathbf{KU}$ and $b(\mathbf{KU})$ are Bott periodic, the two sequences

$$S^2 \wedge b(\mathbf{KU}) \xrightarrow{b(\beta)} b(\mathbf{KU}) \longrightarrow *$$

and

$$S^2 \wedge \mathbf{KU} \xrightarrow{\beta} \mathbf{KU} \longrightarrow *$$

are global homotopy fiber sequences. Passing to homotopy pullbacks gives the desired global homotopy fiber sequence (5.36). The global homotopy cofiber sequence (5.36) and the isomorphism

$$\pi_n^G(b(\mathcal{H}\mathbb{Z})) \ \cong \ H^{-n}(BG, \mathbb{Z})$$

of Proposition IV.5.27 (iii) give rise to a long exact sequence of global functors

$$\cdots \longrightarrow \underline{\pi}_{n+1}(b(\mathcal{H}\mathbb{Z})) \xrightarrow{\partial} \underline{\pi}_n(\mathbf{ku}^c) \xrightarrow{\beta\cdot} \underline{\pi}_n(\mathbf{ku}^c) \xrightarrow{\dim_*^c} \underline{\pi}_n(b(\mathcal{H}\mathbb{Z})) \longrightarrow \cdots \ .$$

This way Greenlees calculates some of the homotopy group global functors of equivariant connective K-theory in [**63**, Prop. 2.6].

We review Greenlees' calculations in our language. The group $H^{-k}(BG, \mathbb{Z})$ vanishes for $k > 0$. So multiplication by the Bott class

$$\beta \cdot - \ : \ \underline{\pi}_{k-2}(\mathbf{ku}^c) \longrightarrow \underline{\pi}_k(\mathbf{ku}^c)$$

is an isomorphism for $k > 0$ and a monomorphism for $k = 0$. In particular, we conclude that

$$\underline{\pi}_k(\mathbf{ku}^c) \ \cong \ \begin{cases} \mathbf{RU} & \text{for } k \geq 0 \text{ and } k \text{ even, and} \\ 0 & \text{for } k \geq -1 \text{ and } k \text{ odd.} \end{cases}$$

The global functor $G \mapsto \pi_0^G(b(\mathcal{H}\mathbb{Z})) = H^0(BG, \mathbb{Z})$ is constant with value $\mathbb{Z}$ and the morphism $\dim_*^c : \underline{\pi}_0(\mathbf{ku}^c) \longrightarrow \underline{\pi}_0(b(\mathcal{H}\mathbb{Z}))$ is isomorphic to the augmentation morphism $\dim : \mathbf{RU} \longrightarrow \underline{\mathbb{Z}}$ of global functors. This is an epimorphism, so the sequence of global functors

$$0 \longrightarrow \underline{\pi}_{-2}(\mathbf{ku}^c) \xrightarrow{\beta\cdot} \underline{\pi}_0(\mathbf{ku}^c) \xrightarrow{\dim_*^c} \underline{\pi}_0(b(\mathcal{H}\mathbb{Z})) \longrightarrow 0$$

is short exact and

$$\underline{\pi}_{-2}(\mathbf{ku}^c) \ \cong \ JU = \ker(\dim : \mathbf{RU} \longrightarrow \underline{\mathbb{Z}})$$

is the augmentation ideal global functor. Again since the map $\dim_*^c : \underline{\pi}_0(\mathbf{ku}^c) \longrightarrow \underline{\pi}_0(b(\mathcal{H}\mathbb{Z}))$ surjective, the global functor $\underline{\pi}_{-3}(\mathbf{ku}^c)$ injects into $\underline{\pi}_{-1}(\mathbf{ku}^c)$ which is trivial by the above. So we conclude that also

$$\underline{\pi}_{-3}(\mathbf{ku}^c) \ = \ 0 \ .$$

The group $\pi_{-1}^G(b(\mathcal{H}\mathbb{Z})) \cong H^1(BG, \mathbb{Z})$ is isomorphic to $\mathrm{Hom}(\pi_1(BG), \mathbb{Z})$ by the universal coefficient theorem; since G is compact, the group $\pi_1(BG) \cong \pi_0(G)$ is finite, and so $H^1(BG, \mathbb{Z})$ vanishes for all compact Lie groups G; so the global functor $\underline{\pi}_{-1}(b(\mathcal{H}\mathbb{Z}))$ vanishes. The long exact sequence splits off an exact sequence of global functors

$$0 \longrightarrow \underline{\pi}_{-4}(\mathbf{ku}^c) \xrightarrow{\beta\cdot} \underline{\pi}_{-2}(\mathbf{ku}^c) \xrightarrow{\dim_*^c} \underline{\pi}_{-2}(b(\mathcal{H}\mathbb{Z})) \xrightarrow{\partial} \underline{\pi}_{-5}(\mathbf{ku}^c) \longrightarrow 0 \ . \tag{5.37}$$

The square

$$\begin{tikzcd} \pi^G_{-2}(\mathbf{ku}^c) \arrow[r, "\dim^c_*"] \arrow[d, "\cong"'] & \pi^G_{-2}(b(\mathcal{H}\mathbb{Z})) \arrow[d, "\cong"] \\ JU(G) \arrow[r, "c_1 \circ \det"'] & H^2(BG, \mathbb{Z}) \end{tikzcd}$$

commutes, where the lower horizontal map sends a virtual representation $[V] - [V']$ of dimension zero to the first Chern class of the determinant line bundle $\det(V) \otimes \det(V')^{-1}$ [show]. Every class in $H^2(BG, \mathbb{Z})$ is the Chern class of a line bundle induced by a 1-dimensional representation V of G. The class of $[V] - 1$ in $JU(G)$ then hits the given cohomology class. So the map $c_1 \circ \det : JU(G) \longrightarrow H^2(BG, \mathbb{Z})$, and hence the map $\dim^c_* : \pi^G_{-2}(\mathbf{ku}^c) \longrightarrow \pi^G_{-2}(b(\mathcal{H}\mathbb{Z}))$ is surjective. The exact sequence (5.37) above then shows that

$$\underline{\pi}_{-4}(\mathbf{ku}^c) \;\cong\; JSU(G) \;=\; \{x \in JU(G) \mid \det(x) = 0\}$$

and that $\underline{\pi}_{-5}(\mathbf{ku}^c) = 0$. The next piece below in the long exact sequence is:

$$0 \;\longrightarrow\; \underline{\pi}_{-3}(b(\mathcal{H}\mathbb{Z})) \;\xrightarrow{\partial}\; \underline{\pi}_{-6}(\mathbf{ku}^c) \;\xrightarrow{\beta\cdot}\; \underline{\pi}_{-4}(\mathbf{ku}^c) \;\xrightarrow{\dim^c_*}\; \underline{\pi}_{-4}(b(\mathcal{H}\mathbb{Z})) \;\xrightarrow{\partial}\; \underline{\pi}_{-7}(\mathbf{ku}^c) \;\longrightarrow\; 0$$

Using the already established isomorphism, this translates into an exact sequence

$$0 \;\longrightarrow\; H^2(BG, \mathbb{Z}) \;\xrightarrow{\partial}\; \pi^G_{-6}(\mathbf{ku}^c) \;\longrightarrow\; JSU(G) \;\xrightarrow{\dim^c_*}\; H^4(BG, \mathbb{Z}) \;\xrightarrow{\partial}\; \pi^G_{-7}(\mathbf{ku}^c) \;\longrightarrow\; 0$$

The group $H^2(BG, \mathbb{Z})$ classifies central extensions of the Lie group G by the circle group $U(1)$. This shows that the global functor $\underline{\pi}_{-6}(\mathbf{ku}^c)$ can have additive torsion and $\underline{\pi}_{-7}(\mathbf{ku}^c)$ may be non-zero. In fact, the global functor $\underline{\pi}_k(\mathbf{ku}^c)$ is supposedly non-zero for every integer $k \leq -6$. After this point things become less explicit.

APPENDIX A

Miscellaneous tools

1. Compactly generated spaces

In this section we recall some background material about compactly generated spaces, our basic category to work in. Compactly generated spaces are in particular 'k-spaces', a notion that seems to go back to Kelley's book [**87**, p.230]. Compactly generated spaces were popularized by Steenrod in his paper [**149**] as a 'convenient category of spaces'; however, in contrast to our usage of the term, Steenrod includes the Hausdorff property in his definition of 'compactly generated'. But in fact, Steenrod already say that '(...) *The Hausdorff property is imposed to ensure that compact subsets are closed.* (...)'; the *weak* Hausdorff condition it thus the next logical step, as it isolates this relevant property. The weak Hausdorff condition first appears in print in McCord's paper [**112**], who credits the idea to J. C. Moore. McCord also was the first to use the terminology 'compactly generated spaces' for weak Hausdorff k-spaces. The fact that much of the recent literature in equivariant and stable homotopy theory uses compactly generated spaces can be taken as evidence that the weak Hausdorff condition is even more convenient than the actual Hausdorff separation property. Two influential – but unpublished – sources about compactly generated spaces are the Appendix A of Gaunce Lewis's thesis [**94**] and Neil Strickland's preprint [**154**].

I want to emphasize that this appendix does not contain any new mathematics and makes no claim to originality. I have decided to include it because I found it cumbersome to collect proofs of all the relevant properties of compactly generated spaces from the scattered literature. An additional complication stems from the fact that the basic references [**87, 149, 112**] all work in slightly different categories; so in the interest of a self-contained treatment, I also felt obliged to fill in arguments where a reference confines itself to the statement that '(...) the proof is analogous to that of (...)'.

Let us fix some terminology. A topological space is *compact* if it is quasi-compact (i.e., every open cover has a finite subcover) and satisfies the Hausdorff separation property (i.e., every pair of distinct points can be separated by disjoint open subsets).

Definition 1.1. Let X be a topological space.

- A subset A of X is *compactly closed* if for every compact space K and every continuous map $f : K \longrightarrow X$, the inverse image $f^{-1}(A)$ is closed in K.
- X is a *k-space* if every compactly closed subset is closed.
- X is *weak Hausdorff* if for every compact space K and every continuous map $f : K \longrightarrow X$ the image $f(K)$ is closed in X.
- X is a *compactly generated* space if it is a k-space and weak Hausdorff.

As already mentioned, we follow the terminology used by McCord in [**112**]. We warn the reader that the usage of the term 'compactly generated' is not consistent throughout the literature. There are sources that require a compactly generated space to be Hausdorff (as opposed to only weak Hausdorff), and several recent references do not include the weak Hausdorff condition in 'compactly generated'.

Every closed subset is also compactly closed. One can similarly define *compactly open* subsets of X by demanding that for every compact space K and every continuous map $f : K \longrightarrow X$, the inverse image

is open in K. A subset is then compactly open if and only if its complement is compactly closed. Thus k-spaces can equivalently be defined by the property that all compactly open subsets are open.

Here is a list of useful properties, mostly straightforward from the definitions. We recall that a subset of a topological space is *locally closed* if it is open in its closure; equivalently, a subset is locally closed if and only if it is the intersection of an open and a closed subset.

PROPOSITION 1.2. (i) *Every quotient space of a k-space is again a k-space.*
(ii) *Every locally closed subset of a k-space is a k-space with respect to the subspace topology.*
(iii) *If $i : B \longrightarrow X$ is a continuous injection and X is weak Hausdorff, then B is also weak Hausdorff. In particular, every subspace of a weak Hausdorff space is again weak Hausdorff.*
(iv) *Every Hausdorff space is also weak Hausdorff.*
(v) *Every finite subset of a weak Hausdorff space is closed.*
(vi) *Every continuous bijection from a compact space to weak Hausdorff space is a homeomorphism.*
(vii) *Let $f : K \longrightarrow X$ be a continuous map from a compact space to a weak Hausdorff space. Then the image $f(K)$ is compact in the subspace topology.*
(viii) *Every locally closed subset of a compactly generated space is compactly generated with respect to the subspace topology.*

PROOF. (i) Let $p : X \longrightarrow Y$ be a quotient projection and B a compactly closed subset of Y. We claim that $p^{-1}(B)$ is compactly closed in X. Indeed, if $f : K \longrightarrow X$ is a continuous map from a compact space, then $pf : K \longrightarrow Y$ is continuous, so $f^{-1}(p^{-1}(B)) = (pf)^{-1}(B)$ is closed because B is compactly closed. Since $p^{-1}(B)$ is compactly closed and X is a k-space, the set $p^{-1}(B)$ is in fact closed in X. So B is closed in Y by definition of the quotient topology.

(ii) In a first step we let Y be a closed subset of a k-space X. We let A be a compactly closed subset of Y with respect to the subspace topology; and we claim that then A is also compactly closed as a subset of X. Indeed, if $f : K \longrightarrow X$ is a continuous map from a compact space, then $L = f^{-1}(Y)$ is closed in K, and hence compact. The restriction $f|_L : L \longrightarrow Y$ is then continuous and so $(f|_L)^{-1}(A) = f^{-1}(A)$ is closed in L because A was assumed to be compactly closed in Y. Since L is closed in K, the set $f^{-1}(A)$ is closed in K. This shows that A is compactly closed as a subset of X. Since X is a k-space, A is closed in X. But then A is also closed in Y in the subspace topology, so this concludes the proof that Y is a k-space.

The case of an open subset is slightly more involved. We reproduce the argument from [**154**, Lemma 2.26]. We let U be an open subset of a k-space X. We let V be a compactly open subset of U with respect to the subspace topology, We claim that then V is compactly open in X. To this end we let $f : K \longrightarrow X$ is a continuous map from a compact space. Consider $k \in f^{-1}(V)$. Since U is open in X, the set $f^{-1}(U)$ is an open neighborhood of the point k in K. Since K is compact, it is regular, and so there exists another open neighborhood N of k in K whose closure $\bar{N}$ is contained in $f^{-1}(U)$. Then $\bar{N}$ is itself compact, and f restricts to a continuous map $f|_{\bar{N}} : \bar{N} \longrightarrow U$. Since V was assumed as compactly open in U, the set $(f|_{\bar{N}})^{-1}(V) = f^{-1}(V)$ is open in $\bar{N}$. But then $f^{-1}(V) \cap N$ is open in N, and hence also in K. This shows that the set $f^{-1}(V)$ contains an open neighborhood of every of its points. So $f^{-1}(V)$ is open in K.

Altogether this shows that the set V is compactly open in X. Since X is a a k-space, this means that V is open in X, and hence also in its subspace U. This concludes the proof that U is a k-space.

(iii) Let $f : K \longrightarrow B$ be a continuous map from a compact space. Then $if : K \longrightarrow X$ is also continuous, hence $(if)(K)$ is closed in X. Since i is injective we have $f(K) = i^{-1}((if)(K)))$, which is thus closed in B. This shows that B is a weak Hausdorff space.

(iv) If K is compact and $f : K \longrightarrow X$ continuous, then the image $f(K)$ is always quasi-compact; if X is Hausdorff, then any quasi-compact subset such as $f(K)$ is closed. So X is weak Hausdorff.

(v) Every one point space is compact, so every point of any space is the continuous image of a compact space. So in weak Hausdorff spaces, all points and thus all finite subsets are closed.

(vi) Let $f : X \longrightarrow Y$ be a continuous bijection from a compact space to weak Hausdorff space. Every closed subset A of X is compact in the subspace topology, so $f(A)$ is closed in Y by the weak Hausdorff property. This shows that f is also a closed map, hence a homeomorphism.

(vii) The property of being quasi-compact is automatically inherited under continuous surjections, so the main issue is the Hausdorff property of $f(K)$; the proof can be found in [**112**, Lemma 2.1].

Part (viii) is the combination of parts (ii) and (iii). □

A topological space is *locally compact* if every point has a compact neighborhood; so compact spaces are in particular locally compact. A topological space is *first countable* if every point has a countable basis of neighborhoods. The following proposition goes all the way back to Kelley [**87**, Ch. 7, Thm. 13].

PROPOSITION 1.3. (i) *Every locally compact Hausdorff space, and hence every compact space, is compactly generated.*
(ii) *Every first countable space is a k-space.*
(iii) *Every metric space is first countable, and hence compactly generated.*

PROOF. (i) Since Hausdorff spaces are in particular weak Hausdorff by Proposition 1.2 (iv), it remains to show that every locally compact Hausdorff space X is a k-space. Let A be a compactly closed subset of X. Let $\bar{A}$ be the closure of A in X and $x \in \bar{A}$. Since X is locally compact, the point x has a compact neighborhood K. Since A is compactly closed and the inclusion $K \longrightarrow X$ is continuous, the set $K \cap A$ is closed inside K. Since K is compact and X is Hausdorff, K is closed in X. So $K \cap A$ is closed in X.

Now we claim that $x \in K \cap A$. We argue by contraction and suppose that $x \notin K \cap A$. Then $X - (K \cap A)$ is an open neighborhood of x, and hence $K \cap (X - (K \cap A)) = K \cap (X - A)$ is another neighborhood of x. Let U be an open subset of X with $x \in U \subset K \cap (X - A)$. Then $A \subset X - U$, and hence $x \in \bar{A} \subset X - U$ because $X - U$ is closed. But his contradicts the hypothesis $x \in U$. Altogether this proves that claim that $x \in K \cap A \subset A$. Hence $A = \bar{A}$, and so A is closed.

(ii) We let X be a first countable space and A a compactly closed subset of X. We let $z \in \bar{A}$ be a point in the closure of A. The point z has a countable basis of open neighborhoods $\{U_n\}_{n \geq 1}$, which we can moreover take to be nested, i.e.,

$$U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots .$$

If the intersection of U_n and A were empty, then $\bar{A} \subset X - U_n$ which contradicts the fact that $z \in \bar{A} \cap U_n$. So for every $n \geq 1$ there is a point $x_n \in U_n \cap A$. We define a map

$$g \ : \ K \ = \ \{0\} \cup \{1/n \mid n \geq 1\} \ \longrightarrow \ X$$

by $g(0) = z$ and $g(1/n) = x_n$. The hypotheses imply that the map g is continuous if we give the source K the subspace topology of the interval $[0, 1]$. In this topology the space K is compact, so $g^{-1}(A)$ is closed since A was assumed to be compactly closed. On the other hand, all the points $1/n$ are contained in $g^{-1}(A)$, and the closure of the set of these point is all of K. So $0 \in g^{-1}(A)$, which means that $z = g(0) \in A$. So A coincides with its closure, i.e., A is closed in X.

(iii) Every metric space is Hausdorff, hence weak Hausdorff. In a metric space the ϵ-balls for all positive rational numbers ϵ form a countable neighborhood basis of a given point. So metric spaces are first countable, hence compactly generated by (ii). □

We denote by **Spc** the category of topological spaces and continuous maps, by **K** its full subcategory of k-spaces and by **T** the full subcategory of **Spc** and **K** of compactly generated spaces. If X is any topological space we let kX be the space which has the same underlying set as X, but such that the open subsets of kX are the compactly open subsets of X. This indeed defines a topology which makes kX into a k-space and such that the identity $\mathrm{Id} : kX \longrightarrow X$ is continuous. Moreover, any continuous map $Y \longrightarrow X$ whose source Y is a k-space is also continuous when viewed as a map to kX. In more fancy language, the assignment $X \mapsto kX$ extends to a functor $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ that is right adjoint to the inclusion of the full

subcategory of k-spaces. Since the inclusion $\mathbf{K} \longrightarrow \mathbf{Spc}$ has a right adjoint, the category $\mathbf{K}$ of k-spaces has small limits and colimits. Colimits can be calculated in the ambient category of all topological spaces; equivalently, any colimit of k-spaces is again a k-space. To construct limits, we can first take a limit in the ambient category of all topological spaces; this 'ambient limit' need not be a k-space, but applying the functor $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ yields a limit in $\mathbf{K}$. Since k does not change the underlying set, the categories $\mathbf{K}$ and $\mathbf{Spc}$ share the property that the forgetful functor to sets preserves all limits and colimits. More loosely speaking, the underlying set of a limit or colimit in $\mathbf{K}$ is what one first thinks of.

The discussion about limits above applies in particular to products, and the product of two k-spaces need *not* be a k-space in the usual product topology. In the following we shall denote by $X \times_0 Y$ the cartesian product of two space X and Y, endowed with the product topology (which makes it a categorical product in the category $\mathbf{Spc}$ of all spaces). We denote by $X \times Y = k(X \times_0 Y)$ the Kelleyfication of the product topology; if X and Y are k-spaces, then $X \times Y$ is a categorical product in the category $\mathbf{K}$. The next proposition says that Kelleyfication is unnecessary if one of the factors is locally compact Hausdorff.

Proposition 1.4. (i) *If X is a weak Hausdorff space, then kX is again weak Hausdorff.*
(ii) *If X is a k-space and Y a locally compact Hausdorff space, then the product $X \times_0 Y$ is a k-space in the product topology.*

Proof. (i) Every continuous map $f : K \longrightarrow kX$ from a compact space is also continuous with respect to the original topology, so $f(K)$ is closed in the original topology, hence also in kX.

(ii) This argument goes back to Steenrod [**149**, Thm. 4.3], and we reproduce it for the convenience of the reader. Let A be a compactly closed subset of $X \times_0 Y$. We let $(x_0, y) \in (X \times Y) - A$ be a point in the complement. We set

$$A_0 \;=\; \{y \in Y \mid (x_0, y) \in A\}$$

by the 'slice' of A over $x \in X$. Let N be a compact neighborhood of y in Y. Then $A_0 \cap N$ is closed in N because A is compactly closed. Since Y is Hausdorff, N is closed in Y, and hence $A_0 \cap N$ is closed in Y. Since $y \notin A_0$, the set $Y - A_0$ is an open neighborhood of y. Since Y is locally compact Hausdorff, there is a compact neighborhood K of y with $K \subset Y - A_0$. We let

$$B \;=\; \{x \in X \mid (\{x\} \times K) \cap A \neq \emptyset\}$$

be the projection of $(X \times K) \cap A$ to X. The condition $K \subset Y - A_0$ is then equivalent to $x_0 \notin B$.

If $f : C \longrightarrow X$ be a continuous map from a compact space. Then $C \times K$ is compact, and so $(f \times K)^{-1}(A)$ is closed in $C \times K$ since A is compactly closed. Then $(C \times K) \cap A$ is compact in the subspace topology. Since

$$f^{-1}(B) \;=\; \{c \in C \mid (\{f(c)\} \times K) \cap A \neq \emptyset\}$$

is the projection of $(f \times K)^{-1}(A)$ onto C, the set $f^{-1}(B)$ is closed in C. Altogether this shows that the set B is compactly closed. Since X is a k-space, B must be closed in X. Since $x_0 \notin B$, the set $(X - B) \times K$ is a neighborhood of (x_0, y). Moreover, $(X - B) \times K$ is disjoint from A by definition of the set B. This shows that the complement of the original set A is open, hence A is closed. This completes the proof that $X \times_c Y$ is a k-space in the product topology. □

The following proposition is essentially due to Steenrod, who states it in [**149**, Thm. 4.4] for k-spaces which are also Hausdorff spaces. The following slight generalization is stated without proof as Proposition 2.2 of [**112**].

Proposition 1.5. *Let X and Z be k-spaces and $p : X \longrightarrow X'$ a proclusion. Then the map $p \times Z : X \times Z \longrightarrow X' \times Z$ is a proclusion.*

Proof. We adapt Steenrod's argument from [**149**, Thm. 4.4] to the slightly more general context. [...fill in...] □

An important example where a limit in **K** can differ from the limit in **Spc** is the product of two CW-complexes X and Y. All cofibrant objects in the Quillen model structure on **Spc** are compactly generated, compare Proposition 1.11 below; in particular, every CW-complex is compactly generated. The product $X \times_0 Y$ with the usual product topology is a Hausdorff space which comes with a natural filtration $(X \times_0 Y)_{(n)} = \cup_{p+q=n} X_{(p)} \times_0 Y_{(q)}$, where $X_{(p)}$ is the p-skeleton of the CW structure on X. If X or Y is locally finite, then the product topology is compactly generated, and then the above filtration makes $X \times_0 Y$ into a CW-complex. In general, however, $X \times_0 Y$ may not be a k-space, and hence cannot have a CW structure. But the product in the category **K**, i.e., the space $X \times Y = k(X \times_0 Y)$, is always compactly generated and a CW-complex via the above filtration.

There is a useful criterion, due to McCord [**112**, Prop. 2.3], for when a k-space is weak Hausdorff (and hence compactly generated): a k-space X is weak Hausdorff if and only if the diagonal subset in $X \times X$ is closed in the k-topology (i.e., compactly closed in the usual product topology). Now we consider a k-space X and a continuous map $f : X \longrightarrow Y$ to a compactly generated space. The equivalence relation

$$E \;=\; \{(x, x') \in X \times X \mid f(x) = f(x')\}$$

is the preimage of the diagonal under the continuous map

$$f \times f \;:\; X \times X \;\longrightarrow\; Y \times Y \;.$$

Since the diagonal is closed in $Y \times Y$, the equivalence relation E is closed in $X \times X$.

Proposition 1.6. *Let X be a k-space and $E \subset X \times X$ an equivalence relation. Then the quotient space X/E is compactly generated if and only if E is closed in the k-topology of $X \times X$.*

Proof. Any quotient space of a k-space is automatically a k-space. The relation E can be recovered as the relation associated to the quotient map $p : X \longrightarrow X/E$. So if X/E is weak Hausdorff, then E is closed in $X \times X$ be the remark preceding this proposition.

Conversely, suppose that E is closed in $X \times X$. Since X and X/E are k-spaces, the map

$$p \times p \;:\; X \times X \;\longrightarrow\; (X/E) \times (X/E)$$

is again a quotient map (i.e., the target has the quotient topology) by [**112**, Prop. 2.2]. Since

$$E \;=\; (p \times p)^{-1}(\Delta_{X/E})$$

is closed by hypothesis, $\Delta_{X/E}$ is closed in $(X/E) \times (X/E)$. So X/E is weak Hausdorff. □

Corollary 1.7. *Let X be a compactly generated space and A a closed subset of X. Then the quotient topology on X/A is again compactly generated.*

Proof. Any quotient space of a k-space is again a k-space, so the issue is the weak Hausdorff property of the quotient topology on X/A. Since A is closed in X, $A \times A$ is closed in $X \times X$. Since X is weak Hausdorff, the diagonal Δ_X is closed $X \times X$. So the equivalence relation $E = (A \times A) \cup \Delta_X$ is closed in $X \times X$, and so the quotient space $X/A = X/E$ is compactly generated by Proposition 1.6. □

Proposition 1.6 also suggests how to construct a 'maximal compactly generated quotient' of a k-space:

Proposition 1.8. *Let X be a k-space. Let $E_{\min}$ be the intersection of all equivalence relations on X that are closed in the k-topology of $X \times X$. Then $X/E_{\min}$ with the quotient topology is a compactly generated space and the quotient map $X \longrightarrow X/E_{\min}$ is the initial example of a continuous map from X to a compactly generated space.*

Proof. The intersection $E_{\min}$ is again an equivalence relation, and the quotient space $X/E_{\min}$ is a coequalizer, hence a colimit, so it is again a k-space. Moreover, $E_{\min}$ is closed in $X \times X$ as an intersection of closed subsets, so the quotient topology is also weak Hausdorff by Proposition 1.6.

If Y is compactly generated and $f : X \longrightarrow Y$ continuous, then by the remark immediately preceding the proposition the equivalence relation $E = \{(x, x') \mid f(x) = f(x')\}$ is closed, so $E_{\min} \subseteq E$. So f factors uniquely over a continuous map from $X/E_{\min}$, by the universal property of the quotient space. □

The previous proposition implies that the assignment

$$X \longmapsto X/E_{\min} = w(X)$$

extends canonically to a functor

$$w : \mathbf{K} \longrightarrow \mathbf{T}$$

that is left adjoint to the inclusion of compactly generated spaces into k-spaces. Moreover, if X is already compactly generated, then the diagonal is closed in $X \times X$; every equivalence relation contains the diagonal, so $E_{\min} = \Delta_X$ whenever X is compactly generated. In this situation the quotient map

$$X \longrightarrow X/E_{\min} = w(X)$$

is a homeomorphism, and we will pretend in the following that $w(X) = X$ for compactly generated spaces.

It follows formally from the existence of the left adjoint $w : \mathbf{K} \longrightarrow \mathbf{T}$ to the inclusion that the category $\mathbf{T}$ of compactly generated spaces has small limits and colimits; limits can be calculated in the category $\mathbf{K}$ of k-spaces. To construct a colimit of a diagram in $\mathbf{T}$, we can first take a colimit in the category $\mathbf{K}$ of k-spaces (or equivalently in $\mathbf{Spc}$); while a k-space, this colimit need not be weak Hausdorff, but applying the functor $w : \mathbf{K} \longrightarrow \mathbf{T}$ yields a colimit in $\mathbf{T}$.

If the colimit, taken in the category $\mathbf{K}$ of k-spaces, of a diagram of compactly generated spaces is not already weak Hausdorff, then the minimal closed equivalence relation on it is strictly larger than the diagonal, so the 'maximal weak Hausdorff quotient' identifies at least two distinct points and thus changes the underlying set. So one has to be especially careful with general colimits in $\mathbf{T}$: unlike for $\mathbf{Spc}$ of $\mathbf{K}$, the forgetful functor from $\mathbf{T}$ to sets need not preserve colimits. More loosely speaking, the underlying set of colimit in $\mathbf{T}$ may be smaller than one first thinks.

The fact that colimits in $\mathbf{T}$ may be hard to identify may seem like a problem at first. However, the issue is largely irrelevant for purposes of homotopy theory because we don't expect to have homotopical control over arbitrary colimits anyhow. The colimits that we do care about turn out to be 'as expected'; in particular for wedges, pushouts along closed embeddings, sequential colimits along closed embeddings and orbits by actions of compact topological groups it makes no difference if we calculate the colimit in the category $\mathbf{T}$ or in $\mathbf{K}$ respectively $\mathbf{Spc}$.

We call a continuous map $f : X \longrightarrow Y$ between topological spaces a *closed embedding* if f is injective and a closed map; equivalently, the image $f(X)$ is closed in Y and f is a homeomorphism onto its image. The base change, in $\mathbf{Spc}$ or $\mathbf{K}$, of a closed embedding is again a closed embedding.

There is an ambiguity with the meaning of 'embedding' in general, due to the fact that a general subset of a k-space, endowed with the subspace topology, need not be a k-space, and so one may or may not want to apply 'Kelleyfication' $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ to the subspace topology. However, closed subsets of k-spaces are again k-spaces with the usual subspace topology, so there is no such ambiguity with the notion of 'closed embedding'.

A *partially ordered set* is a set P equipped with a binary relation '$\leq$' which is reflexive (i.e., $x \leq x$ for all $x \in P$), anti-symmetric (i.e., $x \geq y$ and $y \leq x$ imply $x = y$) and transitive (i.e., $x \geq y$ and $y \leq z$ imply $x \leq z$). The partially ordered set P is *filtered* if for every pair of elements $x, y \in P$ there exists an element $z \in P$ such that $x \leq z$ and $y \leq z$. We will routinely interpret a partially ordered set P as a category without change in notation. In the associated category, the objects are the elements of P and there is a unique morphism from x to y if $x \leq y$, and no morphism from x to y otherwise. Via this interpretation we can consider functors defined on partially ordered sets.

PROPOSITION 1.9. (i) *Given a pushout in the category* **K** *of k-spaces*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, "g"'] & Y \cup_f Z \end{tikzcd}$$

such that f is a closed embedding, then g is also a closed embedding. If moreover, X, Y and Z are weak Hausdorff, then so is $Y \cup_f Z$, and hence the diagram is a pushout in **T**.

(ii) *Let P be a filtered partially ordered set and $F : P \longrightarrow \mathbf{T}$ a functor from the associated poset category. Let F_∞ be a colimit of F in the category* **K** *of k-spaces (or, equivalently, in* **Spc***) and $\kappa_i : F(i) \longrightarrow F_\infty$ the canonical map. If for every $i \leq j$ in P the map $F(i) \longrightarrow F(j)$ is injective, then the maps κ_i are also injective and the colimit F_∞ is weak Hausdorff, thus a colimit of F in the category* **T**. *If moreover, all maps $F(i) \longrightarrow F(j)$ are closed embedding, then so are the maps $\kappa_i : F(i) \longrightarrow F_\infty$.*

PROOF. Part (i) is proved in [**112**, Prop. 2.5] and [**94**, Prop. 7.5]. Part (ii) is [**94**, Prop. 9.3]. □

The following example of Lewis [**94**, App. A, p.168] shows that a pushout in **T** along a non-closed embedding need not even be injective. We consider the diagram

$$\{-1, 1\} \longleftarrow [-1, 0) \cup (0, 1] \xrightarrow{\text{inclusion}} [-1, 1]$$

where all three spaces have the subspace topology of $\mathbb{R}$, and the left map takes $[-1, 0)$ to -1 and it takes $(0, 1]$ to 1. All three spaces are compactly generated, and the pushout P in the categories **Spc** or **K** has three points, only one of which is closed. Any continuous map from P to a weak Hausdorff space must be constant, so the space $w(P)$ (which is a pushout in **T**) has only one point.

The following proposition says that compact spaces are 'small with respect to closed embeddings'. Note that if all spaces in the λ-sequence are compactly generated, then by Proposition 1.9 (ii) it makes no difference whether the colimit is calculated in the category **Spc**, **K** or **T**.

PROPOSITION 1.10. [**79**, Prop. 2.4.2] *Let λ be an ordinal and $X : \lambda \longrightarrow \mathbf{T}$ a λ-sequence of closed embeddings. Then for every compact space K the natural map*

$$\operatorname{colim}_{i>\lambda} C(K, X_i) \longrightarrow C(K, \operatorname{colim}_\lambda X))$$

is bijective.

It is not in general true that $C(K, -)$ commutes with filtered colimits over closed embeddings. An example from Lewis' thesis (which he credits to Myles Tierney) is the unit interval $[0, 1]$. A subset of $[0, 1]$ is closed if and only if its intersection with every countable closed subset J of $[0, 1]$ is closed in J, which implies that $[0, 1]$ is the filtered colimit of its countable closed subsets, ordered by inclusion. Since $[0, 1]$ is uncountable, the identity map of $[0, 1]$ does not factor through any of the spaces in the filtered system.

PROPOSITION 1.11. *Every cofibrant topological space is compactly generated.*

PROOF. Every cofibrant space is a retract of a space B that can be written as the sequential colimit

$$\emptyset = B_0 \longrightarrow B_1 \longrightarrow \dots B_m \longrightarrow \dots$$

such that each B_m is obtained from B_{m-1} by attaching discs along their boundaries. □

PROPOSITION 1.12. *Let $\{X_i\}_{i \in I}$ be a family of based compactly generated spaces. Then the wedge (one-point union) $\bigvee_{i \in I} X_i$ is compactly generated, thus the coproduct of the family in* **T**. *Moreover, for every compact space K and every continuous map $f : K \longrightarrow \bigvee_{i \in I} X_i$ there is a finite subset J of I such that f factors through the sub-wedge $\bigvee_{i \in J} X_i$.*

PROOF. The disjoint union $\coprod_{i\in I} X_i$ is again compactly generated, and the equivalence relation that identifies all the basepoints is closed in $(\coprod_{i\in I} X_i) \times (\coprod_{i\in I} X_i)$. So the quotient space $\bigvee_{i\in I} X_i$ is compactly generated by Proposition 1.6.

Now we let $f : K \longrightarrow \bigvee_{i\in I} X_i$ be a continuous map from a compact space. Then $L = f(K)$ is a compact subset of $\bigvee_{i\in I} X_i$ in the subspace topology, by Proposition 1.2 (vii). If L is not contained in any finite sub-wedge of $\bigvee_{i\in I} X_i$, then we can find indices $j_1, j_2, j_3, \dots$ and elements $a_n \in X_{j_n}$, different from the basepoint of X_{j_n}, such that $a_n \in L$. We claim that the set $S = \{a_1, a_2, \dots\}$ is discrete in the subspace topology of $\bigvee_{i\in I} X_i$. Indeed, if $T \subset S$ is any subset, then the inverse image T under the quotient map

$$q \ : \ \coprod_{i\in I} X_i \ \longrightarrow \ \bigvee_{i\in I} X_i$$

meets every summand in at most one point. Since points in weak Hausdorff spaces are closed, the inverse image $q^{-1}(T)$ is closed, hence so is T itself. So S is a discrete subspace of the compact space L, which is a contradiction. So $L = f(K)$ must be contained in $\bigvee_{i\in J} X_i$ for some finite subset J of I. □

One key advantage of the categories **K** and **T** over the category of all topological spaces is that they are cartesian closed, i.e., categorical product with a fixed object is a left adjoint. We review this for the category **T** of compactly generated spaces, which is the main category of interest for us; in this category, the internal function space is simply the set of all continuous maps with the Kelleyfied compact-open topology. For topological spaces X and Y, we let $C(X, Y)$ denote the set of continuous maps from X to Y. We recall that a subbasis for the compact-open topology is given by all sets $W(K, U)$ where K a compact subset of X and U is an open subset of Y; the set $W(K, U)$ consists of all those continuous maps $\varphi : X \longrightarrow Y$ such that $\varphi(K) \subset U$.

PROPOSITION 1.13. *If Y is a weak Hausdorff space and X is any topological space, then the compact-open topology on $C(X, Y)$ is weak Hausdorff.*

PROOF. We adapt the proof of [**94**, Lemma A.5.2]. We recall that for every point $x \in X$ the evaluation map

$$\mathrm{ev}_x \ : \ C(X, Y) \ \longrightarrow \ Y \ , \quad f \ \longmapsto \ f(x)$$

is continuous. Indeed, for an open set U of Y we have $\mathrm{ev}_x^{-1}(U) = W(\{x\}, U)$ which is open because every one-point space is compact.

We let $Z = \prod^0_{x\in X} Y$ be a product of copies of Y indexed by the elements of X, with the usual product topology. Since weak Hausdorff spaces are closed under products [...justify...], Z is again weak Hausdorff. The evaluation map

$$C(X, Y) \ \longrightarrow \ Z \ , \quad f \ \longmapsto \ \{f(x)\}_{x\in X}$$

is injective, and it is is continuous by the above. So the space $C(X, Y)$ is weak Hausdorff by Proposition 1.2 (iii). □

Now we let X and Y be compactly generated spaces. Then the compact-open topology on the set $C(X, Y)$ of continuous maps is weak Hausdorff by Proposition 1.13, but $C(X, Y)$ need not be a k-space. We arrange for this by Kelleyfication, i.e., we define the internal mapping space in **T** as

$$\mathrm{map}(X, Y) \ = \ kC(X, Y) \ .$$

The space $\mathrm{map}(X, Y)$ is then compactly generated by Proposition 1.4 (i). The spaces $C(X, Y)$ and $\mathrm{map}(X, Y)$ are contravariantly functorial in X and covariantly functorial in Y for all continuous maps.

The proof of the following theorem can be found in [**94**, Theorem A.5.5].

THEOREM 1.14. *The category of compactly generated spaces is cartesian closed, i.e., the natural map*

$$\Psi \ : \ \mathrm{map}(X \times Y, Z) \ \longrightarrow \ \mathrm{map}(X, \mathrm{map}(Y, Z)) \ , \quad \Psi(f)(x)(y) \ = \ f(x, y)$$

is a homeomorphism for all compactly generated spaces X, Y and Z.

PROOF. We start by checking that the two maps

$$\eta_X \ : \ X \ \longrightarrow \ \mathrm{map}(Y, X \times Y) \ , \quad \eta(x)(y) \ = \ (x, y)$$

and

$$\epsilon_Z \ : \ \mathrm{map}(Y, Z) \times Y \ \longrightarrow \ Z \ , \quad \epsilon(f, y) \ = \ f(y)$$

are continuous for all compactly generated spaces X, Y and Z. [...]

We observe that

$$\mathrm{map}(Y, Z) \times Y \ = \ k(\mathrm{map}(Y, Z) \times_0 Y) \ = \ k(C(X, Y) \times_0 Y) \ ,$$

so it suffices to show that ϵ_Z is continuous on $C(X, Y) \times_0 Y$, i.e., with respect to the usual product topology and the compact-open topology on $C(X, Y)$. If U is any open subset of Z, then

$$\epsilon_Z^{-1}(U) \ = \ \{(f, y) \ : \ f(y) \in U\}$$

□

Now we let X and Y be compactly generated topological spaces equipped with basepoint $x_0 \in X$ and $y_0 \in Y$. We define the *smash product*

$$X \wedge Y \ = \ X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y) \ ,$$

i.e., the quotient space of the product (with the k-topology) by the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$- Given $(x, y) \in X \times Y$, we write $x \wedge y$ for the class of (x, y) in the quotient space.

The next proposition is another important reason for working in the category **T** of compactly generated spaces. Indeed, in the larger category *spc* of all topological spaces, the smash product is not generally associative, in the sense that the canonical map from $(X \wedge Y) \wedge Z$ to $X \wedge (Y \wedge Z)$ need not be continuous.

PROPOSITION 1.15. *Let (X, x_0), (Y, y_0), and (Z, z_0) be based compactly generated spaces.*

(i) *The space $X \wedge Y$ is compactly generated in the quotient topology of $X \times Y$.*

(ii) *The map*

$$X \wedge Y \ \longrightarrow \ Y \wedge X \ , \quad x \wedge y \ \longmapsto \ y \wedge x$$

is a homeomorphism.

(iii) *The map*

$$(X \wedge Y) \wedge Z \ \longrightarrow \ X \wedge (Y \wedge Z) \ , \quad (x \wedge y) \wedge z \ \longmapsto \ x \wedge (y \wedge z)$$

is a homeomorphism.

PROOF. (i) In weak Hausdorff spaces all points are closed (Proposition 1.2 (v)), hence $X \times \{y_0\} \cup \{x_0\} \times Y$ is a closed subspace of $X \times_0 Y$, hence also in the k-topology on $X \times Y = k(X \times_0 Y)$. So the quotient topology is compactly generated by Corollary 1.7.

Part (ii) is straightforward from the fact that the twist homeomorphism $X \times Y \longrightarrow Y \times X$ sending (x, y) to (y, x) takes the subspace $X \times \{y_0\} \cup \{x_0\} \times Y$ to the subspace $Y \times \{x_0\} \cup \{y_0\} \times X$ and hence descends to a homeomorphism on quotient spaces.

(iii) Since the projection $p_{X,Y} : X \times Y \longrightarrow X \wedge Y$ is a proclusion, so is $p_{X,Y} \times Z : X \times Y \times Z \longrightarrow (X \wedge Y) \times Z$, by Proposition 1.5. Hence the map

$$p_{X \wedge Y, Z} \circ (p_{X,Y} \times Z) \ : \ X \times Y \times Z \ \longrightarrow \ (X \wedge Y) \wedge Z \ , \quad (x, y, z) \ \longmapsto \ (x \wedge y) \wedge z$$

is a proclusion as the composite of two proclusions. The composite

$$X \times Y \times Z \ \xrightarrow{p_{X \wedge Y, Z} \circ (p_{X,Y} \times Z)} \ (X \wedge Y) \wedge Z \ \xrightarrow{(x \wedge y) \wedge z \mapsto x \wedge (y \wedge z)} \ X \wedge (Y \wedge Z)$$

is the map $p_{X, Y \wedge Z} \circ (X \times p_{Y,Z})$, and hence continuous. Since $p_{X \wedge Y, Z} \circ (p_{X,Y} \times Z)$ is a proclusion, the second map $(X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z)$ is continuous. By the same reasoning the inverse map $X \wedge (Y \wedge Z) \longrightarrow (X \wedge Y) \wedge Z$ is continuous. Hence the two maps are mutually inverse homeomorphisms. □

h-cofibrations. The *h-cofibrations* are the morphisms with the homotopy extension property. We will use this concept in various categories, for example in the category of G-spaces, orthogonal spaces and orthogonal spectra. So we recall some basic properties of h-cofibrations in the context of categories enriched over the category of spaces. The arguments are all standard and well-known, and we include them for completeness and convenient reference.

For the discussion of h-cofibrations we work in a cocomplete category $\mathcal{C}$ that is tensored and cotensored over the category $\mathbf{T}$ of compactly generated spaces. We write '$\times$' for the pairing and X^K for the cotensor of an object X with a compact Hausdorff space K. A *homotopy* is then a morphism $H : [0,1] \times A \longrightarrow X$ defined on the pairing of the unit interval with a $\mathcal{C}$-object. For a homotopy and any $t \in [0,1]$ we denote by $H_t : A \longrightarrow X$ the composite morphism

$$A \;\cong\; \{t\} \times A \;\xrightarrow{\mathrm{incl} \times A}\; [0,1] \times A \;\xrightarrow{H}\; X \;.$$

Definition 1.16. Let $\mathcal{C}$ be a category tensored over the category of spaces. A $\mathcal{C}$-morphism $f : A \longrightarrow B$ is an *h-cofibration* if it has the homotopy extension property, i.e., given a morphism $\varphi : B \longrightarrow X$ and a homotopy $H : [0,1] \times A \longrightarrow X$ such that $H_0 = \varphi f$, there is a homotopy $\bar{H} : [0,1] \times B \longrightarrow X$ such that $\bar{H} \circ ([0,1] \times f) = H$ and $\bar{H}_0 = \varphi$.

There is a universal test case for the homotopy extension problem, namely when X is the pushout:

$$\begin{tikzcd} A \arrow[r, "0\wedge -"] \arrow[d, "f"'] & {[0,1] \times A} \arrow[d, "H"] \\ B \arrow[r, "\varphi"'] & B \cup_f ([0,1] \times A) \end{tikzcd}$$

So a morphism $f : A \longrightarrow B$ is an h-cofibration if and only if the canonical morphism

$$B \cup_f ([0,1] \times A) \;\longrightarrow\; [0,1] \times B \tag{1.17}$$

has a retraction. Also, the adjunction between $[0,1] \times -$ and $(-)^{[0,1]}$ lets us rewrite any homotopy extension data (φ, H) in adjoint form as a commutative square:

$$\begin{tikzcd} A \arrow[r, "\hat{H}"] \arrow[d, "f"'] & X^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ B \arrow[r, "\hat{\varphi}"'] & X \end{tikzcd}$$

A solution to the homotopy extension problem is adjoint to a lifting, i.e., a morphism $\lambda : B \longrightarrow X^{[0,1]}$ such that $\lambda f = \hat{H}$ and $\mathrm{ev}_0\, \lambda = \hat{\varphi}$. So a morphism $f : A \longrightarrow B$ is an h-cofibration if and only if it has the left lifting property with respect to the morphisms $\mathrm{ev}_0 : X^{[0,1]} \longrightarrow X$ for all objects in $\mathcal{C}$.

The three equivalent characterizations of h-cofibrations quickly imply various closure properties.

Corollary 1.18. *Let $\mathcal{C}$ be a cocomplete category tensored and cotensored over the category of spaces.*

(i) *The class of h-cofibrations in $\mathcal{C}$ is closed under retracts, cobase change, coproducts, sequential compositions and transfinite compositions.*
(ii) *Let $\mathcal{C}'$ be another category tensored and cotensored over the category of spaces, and $F : \mathcal{C} \longrightarrow \mathcal{C}'$ a continuous functor that commutes with colimits and tensors with $[0,1]$. Then F takes h-cofibrations in $\mathcal{C}$ to h-cofibrations in $\mathcal{C}'$.*
(iii) *If $\mathcal{C}$ is a topological model category in which every object is fibrant, then every cofibration is an h-cofibration.*

PROOF. (i) Every class of morphisms that can be characterized by the left lifting property with respect to some other class has the closure properties listed.

(ii) Let $f : A \longrightarrow B$ be a cofibration in $\mathcal{C}$ and $r : [0,1] \times B \longrightarrow B \cup_f ([0,1] \times A)$ a retraction to the canonical morphism. The composite

$$[0,1] \times FB \;\cong\; F([0,1] \times B) \;\xrightarrow{Fr}\; F(B \cup_f ([0,1] \times A)) \;\cong\; FB \cup_{Ff} ([0,1] \times FA)$$

is then a retraction to the canonical morphism for $Ff : FA \longrightarrow FB$. So Ff is an h-cofibration.

(iii) Since the model structure is topological, for every cofibration $f : A \longrightarrow B$ the canonical morphism (1.17) is an acyclic cofibration. Since every object is fibrant, this morphism has a retraction, and so f is an h-cofibration. □

PROPOSITION 1.19. (i) *Let $j : X \longrightarrow Y$ and $r : Y \longrightarrow X$ be continuous maps of compactly generated spaces such that $rj = \mathrm{Id}_X$. Then j is a closed embedding and r is a quotient map.*

(ii) *Every h-cofibration between compactly generated spaces is a closed embedding.*

PROOF. (i) This is [**94**, Lemma 8.1].

(ii) This is [**94**, Lemma 8.2]. Since the map $i_0 : A \longrightarrow A \times [0,1]$ given by $i_0(a) = (a,0)$ is a closed embedding, Proposition 1.9 (i) says that the pushout $B \cup_i A \times [0,1]$ in the category $\mathbf{T}$ can in fact be calculated in $\mathbf{K}$ or $\mathbf{Spc}$. In particular, the map $i_1 : A \longrightarrow B \cup_f A \times [0,1]$ sending a to the image of $(a,1)$ is a closed embedding.

The universal example of the homotopy extension property provides a retraction to the canonical map

$$j \;=\; (-,0) \cup (f \times [0,1]) \;:\; B \cup_f A \times [0,1] \;\longrightarrow\; B \times [0,1]\ .$$

So j is a closed embedding by part (i). Hence the map

$$j \circ i_1 \;:\; A \;\longrightarrow\; B \times [0,1]\ , \quad a \;\longmapsto\; (i(a),1)$$

is a closed embedding as the composite of two closed embeddings. Since this map factors as

$$A \;\xrightarrow{f}\; B \;\xrightarrow{b \mapsto (b,1)}\; B \times [0,1]$$

and the second map is a closed embedding, so is f. □

We recall that the unreduced suspension $A^\diamond$ of a space A is the quotient space of $A \times [0,1]$ with $A \times \{0\}$ and $A \times \{1\}$ each identified to a point. A based space is *well-pointed* if the basepoint inclusion has the homotopy extension property in the category of unbased spaces. Equivalently, a based space (A, a_0) is well-pointed if and only if the inclusion of $A \times \{0\} \cup \{a_0\} \times [0,1]$ into $A \times [0,1]$ has a continuous retraction.

PROPOSITION 1.20. *The unreduced suspension functor takes m-connected maps to $(m+1)$-connected maps. The functor $- \wedge S^1$ takes m-connected maps between well-pointed based spaces to $(m+1)$-connected maps.*

PROOF. The unreduced suspension $A^\diamond$ is the union of the open cones $C^+A = A^\diamond - (A \times \{0\})$ and $C^-A = A^\diamond - (A \times \{1\})$, both of which are contractible to the respective cone point. Moreover, the intersection $C^+A \cap C^-A$ is homeomorphic to $A \times (0,1)$. So if $f : A \longrightarrow B$ is m-connected, then $f^\diamond : A^\diamond \longrightarrow B^\diamond$ restricts to weak equivalences on the two open cones and to an m-connected map on the intersections of the cones; so $f^\diamond$ is a $(m+1)$-connected, compare [**167**, Thm. 6.7.9].

If (A, a_0) is well-pointed, then the inclusion

$$A \times \{0,1\} \cup \{a_0\} \times [0,1] \;\longrightarrow\; A \times [0,1]$$

is an h-cofibration, see for example [**125**, Satz 2] or [**167**, Prop. 5.1.6]. Hence the cobase change

$$\{a_0\} \times [0,1] \;\longrightarrow\; A^\diamond\ , \quad t \;\longmapsto\; [a_0, t]\ .$$

is an h-cofibration as well. Since $\{a_0\} \times [0,1]$ is contractible, the quotient map

$$A^\diamond \longrightarrow A^\diamond/(\{a_0\} \times [0,1])$$

is a homotopy equivalence. But the target is the quotient of $A \times [0,1]$ with the subspace $A \times \{0,1\} \cup \{a_0\} \times [0,1]$ collapsed to a point, hence homeomorphic to $A \wedge S^1$. So altogether this shows that $- \wedge S^1$ increases the connectivity of continuous maps between well-pointed spaces. □

2. Model structures for equivariant spaces

The 'classical' model structure on the category of all topological spaces was established by Quillen in [**126**, II.3 Thm. 1]. We use the straightforward adaptation of this model structure to the category of compactly generated spaces, which is described for example in [**79**, Thm. 2.4.25]. In this model structure on the category **T**, the weak equivalences are the weak homotopy equivalences and fibrations are the Serre fibrations. The cofibrations are the retracts of generalized CW-complexes, i.e., cell complexes in which cells can be attached in any order and not necessarily to cells of lower dimensions.

We let G be a topological group, which we take to mean a group object in the category **T** of compactly generated spaces. So for us a topological group is a compactly generated space equipped with an associative and unital multiplication

$$\mu \ : \ G \times G \longrightarrow G$$

that is continuous with respect to the compactly generated product topology, and such that the shearing map

$$G \times G \longrightarrow G \times G \ , \quad (g,h) \longmapsto (g, gh)$$

is a homeomorphism (again for the compactly generated product topology). This implies in particular that inverse exist in G, and that the inverse map $g \mapsto g^{-1}$ is continuous. A G-space is then a compactly generated space X equipped with an associative and unital action

$$\alpha \ : \ G \times X \longrightarrow X$$

that is continuous with respect to the compactly generated product topology.

For a subgroup H of G we denote by

$$X^H \ = \ \{x \in X \mid nx = x \text{ for all } n \in H\}$$

the subspace of H-fixed points. For an individual element $h \in H$ the h-fixed subspace $\{x \in X \mid hx = x\}$ is the preimage of the diagonal under the continuous map $(\mathrm{Id}, h \cdot -) : X \longrightarrow X \times X$, so it is a closed subspace of X by the weak Hausdorff condition. The H-fixed points X^H are then closed in X as an intersection of closed subsets. This means that the subspace topology on X^H is again compactly generated (see Proposition 1.2 (ii)) and so

$$X^H \xrightarrow{\text{incl}} X \rightrightarrows \mathrm{map}(H, X) \tag{2.1}$$

is an equalizer diagram in the category of compactly generated spaces, where the two maps on the right are adjoint to the projection $H \times X \longrightarrow X$ respectively the composite

$$H \times X \xrightarrow{\text{incl} \times X} G \times X \xrightarrow{\alpha} X \ .$$

Proposition 2.2. *Let G be a compact topological group and X a G-space, not necessarily compactly generated.*

(i) *The projection $\Pi_X : X \longrightarrow G\backslash X$ is both open and closed.*
(ii) *For every G-invariant subset Y of X, the tautological map*

$$G\backslash Y \longrightarrow \Pi_X(Y) \ , \quad Gy \longmapsto Gy$$

is a homeomorphism with respect to the subspace topology on the target.
(iii) *If X is compactly generated, then so is the orbit space $G\backslash X$, equipped with the quotient topology.*

PROOF. (i) If O is an open subset of X, then gO is open for every $g \in G$, since left translation by g is a homeomorphism. So

$$\Pi_X^{-1}(\Pi_X(O)) \;=\; \bigcup_{g\in G} gO$$

is open as a union of open subsets. Hence $\Pi_X(O)$ is open in the quotient topology. The map

$$\chi \;:\; G\times X \;\longrightarrow\; G\times X\;, \quad (g,x) \;\longmapsto\; (g,gx)$$

is a homeomorphism, and the composite

$$G\times X \;\xrightarrow{\chi}\; G\times X \;\xrightarrow{\text{proj}}\; X$$

is the action map $\alpha : G\times X \longrightarrow X$. Since G is compact, projection away from G is a closed map; so the action map is also a closed map. If A is closed in X, then $G\times A$ is closed in $G\times X$. So the set

$$\Pi_X^{-1}(\Pi_X(A)) \;=\; \alpha(G\times A)$$

is closed in X. Hence $\Pi_X(A)$ is closed in the quotient topology.

(ii) The tautological map $G\backslash Y \longrightarrow \Pi_X(Y)$ is continuous by the universal property of the quotient topology on $G\backslash Y$, and set-theoretically bijective. We show that the map is also open. We denote by $\Pi_Y : Y \longrightarrow G\backslash Y$ the quotient map. We let $O \subset G\backslash Y$ be any open subset. Then $\Pi_Y^{-1}(O)$ is open in Y, so there is an open subset U of X with $\Pi_Y^{-1}(O) = Y\cap U$. The relation $\Pi_X(Y\cap U) = \Pi_X(Y)\cap\Pi_X(U)$ holds because Y is G-invariant. Since Π_X is an open map, $\Pi_X(U)$ is open in $G\backslash X$; so

$$\Pi_X(\Pi_Y^{-1}(O)) \;=\; \Pi_X(Y\cap U) \;=\; \Pi_X(Y)\cap\Pi_X(U)$$

is open in the subspace topology of $\Pi_X(Y)$. This proves the claim.

(iii) Proposition 1.6 reduces us to showing that the orbit equivalence relation

$$E \;=\; \{(gx,x) \mid g\in G,\; x\in X\}$$

is closed in $X\times X$. Since G is compact, projection away from G is a closed map; so the composite

$$G\times X\times X \;\xrightarrow{(g,x,y)\mapsto(g,gx,y)}\; G\times X\times X \;\xrightarrow{\text{proj}}\; X\times X$$

is a closed map since the first map is a homeomorphism. Since X is weak Hausdorff, the diagonal Δ_X is closed in $X\times X$. So $G\times\Delta_X$ is closed in $G\times X\times X$. The orbit relation E is the image of $G\times\Delta_X$ under the above composite, so E it is closed in $X\times X$. □

Now we show that the functor sending a G-space X to the set of H-fixed points is representable by an 'orbit space' G/H. We denote by G/H a coequalizer in the category of G-spaces

$$G\times H \;\underset{\mu'}{\overset{\text{proj}}{\rightrightarrows}}\; G \;\xrightarrow{\;q\;}\; G/H\;, \tag{2.3}$$

where $\mu' = \mu\circ(G\times\text{incl})$. Since the forgetful functor creates colimits, we could equivalently take a coequalizer in the underlying category of compactly generated spaces, and that inherits a unique G-action that makes the projection $q : G \longrightarrow G/H$ a morphism of G-spaces.

Now we let K be any compactly generated space. Since product with K is a left adjoint, the diagram

$$G\times H\times K \;\underset{\mu'\times K}{\overset{\text{proj}\times K}{\rightrightarrows}}\; G\times K \;\xrightarrow{q\times K}\; G/H\times K\;.$$

is another coequalizer of G-spaces, where G acts trivially on K. So for every G-space X, precomposition with $q\times K$ is a bijection from $G\mathbf{T}(G/H\times K, X)$ to the equalizer of the two maps

$$G\mathbf{T}(G\times K, X) \;\underset{G\mathbf{T}(\mu'\times K,X)}{\overset{G\mathbf{T}(\text{proj}\times K,X)}{\rightrightarrows}}\; G\mathbf{T}(G\times H\times K, X)\;.$$

The free-forgetful adjunction and the adjunction between $H \times -$ and $\mathrm{map}(H, -)$ identifies this with the set of those continuous maps $f : K \longrightarrow X$ that are equalized by the two right maps in the equalizer diagram (2.1). Since the inclusion of X^H into X is an equalizer, we have shown altogether that evaluation at the class of the identity element is a bijection

$$G\mathbf{T}(G/H \times K, X) \longrightarrow \mathbf{T}(K, X^H) \tag{2.4}$$

from the set of continuous G-maps from $G/H \times K$ to X to the set of continuous maps from K to the H-fixed points of X.

Remark 2.5. As we explained in Proposition 2.2 (iii), orbit spaces of compactly generated spaces by actions of compact topological groups behave as expected, i.e., the usual quotient topology on the orbit set is compactly generated. On the other hand if H is a closed subgroup of a topological group G (not necessarily compact), then the quotient topology on the orbit space G/H is also compactly generated (without further hypotheses on H). Indeed, the equivalence relation that gives rise to G/H is the inverse image of H under the continuous map

$$G \times G \longrightarrow G\ , \quad (g, \bar{g}) \longmapsto \bar{g}g^{-1}$$

(where $G \times G$ has the compactly generated product topology). Because H is closed, so is the equivalence relation, and hence the quotient topology is compactly generated by Proposition 1.6.

Lemma 2.6. *For every topological group G, every subgroup H and every compact space K the G-space $G/H \times K$ is finite with respect to sequences of closed embeddings of G-spaces.*

Proof. We let

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$

be a sequence of morphisms of G-spaces that are closed embeddings of underlying spaces, and

$$f : G/H \times K \longrightarrow \mathrm{colim}_{i \geq 0}\, X_i$$

a morphism of G-spaces. The composite

$$K \xrightarrow{(H,-)} G/H \times K \xrightarrow{f} \mathrm{colim}_{i \geq 0}\, X_i$$

factors through a continuous map $g : K \longrightarrow X_i$ for some $i \geq 0$ by [**79**, Prop. 2.4.2] (this uses that singletons in weak Hausdorff spaces are closed). Sequential colimits of compactly generated spaces along injective continuous maps are given by the colimits of underlying sequence of sets [ref to Appendix of Lewis' thesis], so the canonical map $X_i \longrightarrow \mathrm{colim}_{i \geq 0}\, X_i$ is injective. Since the map $f(H, -) : K \longrightarrow \mathrm{colim}_{i \geq 0}\, X_i$ lands in the H-fixed points of the colimit, the factorization g lands in the H-fixed points, so it extends uniquely to a morphism of G-spaces $\tilde{g} : G/H \times K \longrightarrow X_i$ by the adjunction (2.4). Since morphisms out of $G/H \times K$ are determined by their restriction to K, the morphism $\tilde{g}$ is the desired factorization of the original morphism f. □

Another useful consequence of working with compactly generated spaces is that fixed points commute with smash products, as we shall now explain. We consider a topological group G acting continuously on X and Y. We consider basepoints $x_0 \in X$ and $y_0 \in Y$ that are fixed by the G-action. Then we can form smash product and fixed points in two different orders: the composite

$$X^G \times Y^G \xrightarrow{\text{incl}} X \times Y \xrightarrow{\text{proj}} X \wedge Y$$

takes values in the subspace $(X \wedge Y)^G$, the G-fixed points with respect to the diagonal action. The composite also sends the subspace $X^G \times \{y_0\} \cup \{x_0\} \times Y^G$ to one point, so it factors through a continuous map

$$X^G \wedge Y^G \longrightarrow (X \wedge Y)^G\ .$$

PROPOSITION 2.7. *Let (X, x_0) and (Y, y_0) be based compactly generated spaces. Let G be a topological group acting continuously on X and Y fixing the basepoints. Then the canonical map*

$$(X^G) \wedge (Y^G) \ \longrightarrow \ (X \wedge Y)^G$$

is a homeomorphism.

PROOF. In the constructions of $(X^G) \wedge (Y^G)$ and $(X \wedge Y)^G$ we never had to force the weak Hausdorff property anywhere, so both sides of the map have the 'expected' underlying sets. The map is thus a continuous bijection, and we show that it is also a closed map. The two maps

$$X^G \times Y^G \ \xrightarrow{\text{incl}} \ X \times Y \ \xrightarrow{\text{proj}} \ X \wedge Y$$

are closed maps. Hence the composite is a closed map, and it remains closed when considered as a map $X^G \times Y^G \longrightarrow (X \wedge Y)^G$ to the subspace of G-fixed points. So the induced map on the quotient space is a closed map. □

An argument that we need several times in this book proves that a certain model structure is topological. To avoid repeating the same kind of argument several times, we axiomatize it. We consider a model category $\mathcal{M}$ that is also enriched, tensored and cotensored over the category $\mathbf{T}$ of compactly generated spaces. We denote the tensor by $\times$. Given a continuous map of spaces $f : A \longrightarrow B$ and a morphism $g : X \longrightarrow Y$ in $\mathcal{M}$, we denote by $f \square g$ the *pushout product* morphism defined as

$$f \square g = (f \times Y) \cup (A \times g) \ : \ A \times Y \cup_{A \times X} B \times X \ \longrightarrow \ B \times Y \ .$$

We recall that the model structure is called *topological* if the following two conditions hold:

- if f is a cofibration of spaces and g is a cofibration in $\mathcal{M}$, then the pushout product morphism $f \square g$ is also a cofibration;
- if in addition f or g is a weak equivalence, then so is the pushout product morphism $f \square g$.

In the next proposition, we denote by

$$i_k \ : \ \partial D^k \ \longrightarrow \ D^k \qquad \text{and} \qquad j_k \ : \ D^k \times \{0\} \ \longrightarrow \ D^k \times [0,1]$$

the inclusions. Then $\{i_k\}_{k \geq 0}$ is the standard set of generating cofibrations for the Quillen model structure on the category of spaces, and $\{j_k\}_{k \geq 0}$ is the standard set of generating acyclic cofibrations, compare Theorem [**79**, Thm. 2.4.25]. The pushout product condition can also be stated in two different, but equivalent, adjoint forms, compare [**79**, Lemma 4.2.2].

PROPOSITION 2.8. *Let $\mathcal{M}$ be a model category that is also enriched, tensored and cotensored over the category $\mathbf{T}$ of spaces. Suppose that there is a set of objects $\mathcal{G}$ of $\mathcal{M}$ with the following properties:*

(a) *The acyclic fibrations are characterized by the right lifting property with respect to the morphisms of the form $i_k \times K$ for all $k \geq 0$ and $K \in \mathcal{G}$.*
(b) *The fibrations are characterized by the right lifting property with respect to the morphisms of the form $j_k \times K$ for all $k \geq 0$ and $K \in \mathcal{G}$.*

Then the model structure is topological.

PROOF. The hypothesis are saying that $\{i_k \times K\}_{k \geq 0, K \in \mathcal{G}}$ is a set of generating cofibrations for the given model structure on $\mathcal{M}$, and that $\{j_k \times K\}_{k \geq 0, K \in \mathcal{G}}$ is a set of generating acyclic cofibrations. Since the tensor bifunctor $\times$ has an adjoint in each variable, it preserves colimits in each variable. So it suffices to the check the pushout product properties when the maps f and g are from the sets of generating (acyclic) cofibrations, compare [**79**, Cor. 4.2.5].

The set of inclusions of spheres into discs is closed under pushout product, in the sense that $i_k \square i_l$ is homeomorphic to i_{k+l}. So pushout product with i_k preserves the set $\{i_k \times K\}_{k \geq 0, K \in \mathcal{G}}$ of generating cofibrations (up to isomorphism). Similarly, the pushout product of a sphere inclusion i_k with the inclusion j_l isomorphic to j_{k+l}. So pushout product with i_k preserves the set $\{j_l \times K\}_{l \geq 0, K \in \mathcal{G}}$ of generating cofibrations;

and pushout product with j_k takes the set $\{i_l \times K\}_{l\geq 0, K\in\mathcal{G}}$ of generating cofibrations to the set of generating acyclic cofibrations. □

Now we let $\mathcal{C}$ be a collection of closed subgroups of a topological group G, i.e., a set of closed subgroups that is stable under conjugacy. We call a morphism $f : X \longrightarrow Y$ of G-spaces a *$\mathcal{C}$-equivalence* (respectively *$\mathcal{C}$-fibration*) if the restriction $f^H : X^H \longrightarrow Y^H$ to H-fixed points is a weak equivalence (respectively Serre fibration) of spaces for all subgroups H of G that belong to the collection $\mathcal{C}$. A *$\mathcal{C}$-cofibration* is a morphism with the right lifting property with respect to all morphisms that are simultaneously $\mathcal{C}$-equivalences and $\mathcal{C}$-fibrations.

Proposition 2.9. *Let G be a topological group and $\mathcal{C}$ a collection of closed subgroups of G. Consider a commutative diagram of G-spaces*

$$\begin{tikzcd} C \arrow[d, "\gamma"'] & A \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ C' & A' \arrow[l, "g'"] \arrow[r, "f'"'] & B' \end{tikzcd}$$

such that g and g' are h-cofibrations of G-spaces. Suppose that the maps α, β and γ are $\mathcal{C}$-equivalences. Then the induced map of pushouts

$$\gamma \cup \beta \ : \ C \cup_A B \ \longrightarrow \ C' \cup_{A'} B' \ .$$

is a $\mathcal{C}$-equivalence.

Proof. We let H be a closed subgroup from the collection $\mathcal{C}$, and we contemplate the commutative diagram of fixed points:

$$\begin{tikzcd} C^H \arrow[d, "\gamma^H"'] & A^H \arrow[l, "g^H"'] \arrow[r, "f^H"] \arrow[d, "\alpha^H"] & B^H \arrow[d, "\beta^H"] \\ (C')^H & (A')^H \arrow[l, "(g')^H"] \arrow[r, "(f')^H"'] & (B')^H \end{tikzcd}$$

Since g and g' are h-cofibrations of G-spaces, g^H and $(g')^H$ are h-cofibrations of spaces. The three vertical maps are weak equivalences by hypothesis. The gluing lemma for weak equivalences and pushout along h-cofibrations shows that then the induced map on horizontal pushouts

$$\gamma^H \cup \beta^H \ : \ C^H \cup_{A^H} B^H \ \longrightarrow \ (C')^H \cup_{(A')^H} (B')^H$$

is a weak equivalence, see for example [**20**, Appendix, Prop. 4.8 (b)]. Since g and g' are h-cofibrations of G-spaces, they are in particular h-cofibrations of underlying spaces, and hence closed embeddings (Proposition 1.19 (ii)). So taking H-fixed points commutes with the horizontal pushout, and we conclude that also the map

$$(\gamma \cup \beta)^H \ : \ (C \cup_A B)^H \ \longrightarrow \ (C' \cup_{A'} B')^H$$

is a weak equivalence. This proves the claim. □

The following $\mathcal{C}$-projective model structure is well known, mentioned in various places in the literature (for example in [**50**, Prop. 2.11], [**105**, III Thm. 1.8]) and fairly standard. However, I do not know a reference that is both self-contained and complete, so I provide the proof.

Proposition 2.10. *Let G be a topological group and $\mathcal{C}$ a collection of closed subgroups of G. Then the $\mathcal{C}$-equivalences, $\mathcal{C}$-cofibrations and $\mathcal{C}$-fibrations form a model structure, the* $\mathcal{C}$-projective model structure *on the category of G-spaces. This model structure is proper, cofibrantly generated and topological.*

Proof. We refer the reader to [**47**, 3.3] for the numbering of the model category axioms. The forgetful functor from the category of G-spaces to the category of compactly generated spaces has a left adjoint free functor $G \times -$ and a right adjoint cofree functor $\mathrm{map}(G, -)$; so the category of G-spaces is complete and cocomplete and all limits and colimits are created in the underlying category of compactly generated spaces.

Model category axioms MC2 (2-out-of-3) and MC3 (closure under retracts) are clear. One half of MC4 (lifting properties) holds by the definition of $\mathcal{C}$-cofibrations. The proof of the remaining axioms uses Quillen's small object argument, originally given in [**126**, II p. 3.4], and later axiomatized in various places, for example in [**47**, 7.12] or [**79**, Thm. 2.1.14]. We recall the 'standard' set of generating cofibrations and generating acyclic cofibrations. In the category of (non-equivariant) spaces, the set $\{i_k : \partial D^k \longrightarrow D^k\}_{k\geq 0}$ of inclusions of spheres into discs detects Serre fibrations that are simultaneously weak equivalences (where ∂D^0 is the empty set). By adjointness (i.e., the bijection (2.4)), the set

$$I_{\mathcal{C}} \;=\; \{G/H \times i_k \;:\; G/H \times \partial D^k \;\longrightarrow\; G/H \times D^k\}_{k\geq 0, H\in\mathcal{C}} \tag{2.11}$$

then detects acyclic fibrations in the $\mathcal{C}$-projective model structure on G-spaces. Similarly, the set of inclusions $\{j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]\}_{k\geq 0}$ detects Serre fibrations; so by adjointness, the set

$$J_{\mathcal{C}} \;=\; \{G/H \times j_k\}_{k\geq 0, H\in\mathcal{C}} \tag{2.12}$$

detects fibrations in the $\mathcal{C}$-projective model structure on G-spaces.

All morphisms in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ are closed embeddings, and this property is preserved by coproducts, cobase change and sequential colimits in the category of G-spaces. Lemma 2.6 guarantees that sources and targets of all morphisms in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ are finite (sometimes called 'finitely presented') with respect to sequences of closed embeddings of G-spaces. In particular, the sources of all these morphisms are finite with respect to sequences of morphisms in $I_{\mathcal{C}}$-cell and $J_{\mathcal{C}}$-cell.

Now we can prove the factorization axiom MC5. Every morphism in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ is a $\mathcal{C}$-cofibration by adjointness. Hence every $I_{\mathcal{C}}$-cofibration or $J_{\mathcal{C}}$-cofibration is a $\mathcal{C}$-cofibration of G-spaces. The small object argument applied to the set $I_{\mathcal{C}}$ gives a (functorial) factorization of any morphism of G-spaces as a $\mathcal{C}$-cofibration followed by a morphism with the right lifting property with respect to $I_{\mathcal{C}}$. Since $I_{\mathcal{C}}$ detects the $\mathcal{C}$-acyclic $\mathcal{C}$-fibrations, this provides the factorizations as cofibrations followed by acyclic fibrations.

For the other half of the factorization axiom MC5 we apply the small object argument to the set $J_{\mathcal{C}}$; we obtain a (functorial) factorization of any morphism of G-spaces as a $J_{\mathcal{C}}$-cell complex followed by a morphism with the right lifting property with respect to $J_{\mathcal{C}}$. Since $J_{\mathcal{C}}$ detects the $\mathcal{C}$-fibrations, it remains to show that every $J_{\mathcal{C}}$-cell complex is a $\mathcal{C}$-equivalence. To this end we observe that the morphisms in $J_{\mathcal{C}}$ are inclusions of deformation retracts internal to the category of G-spaces. This property is inherited by coproducts and cobase changes, so every morphisms obtained by cobase changes of coproducts of morphisms in $J_{\mathcal{C}}$ is a homotopy equivalence of G-spaces, hence also a $\mathcal{C}$-equivalence. We also need to pass to sequential colimits, which is fine because $J_{\mathcal{C}}$-cell complexes are closed embeddings, and taking H-fixed points commutes with sequential colimits over closed embeddings.

It remains to prove the other half of MC4, i.e., that every $\mathcal{C}$-acyclic $\mathcal{C}$-cofibration $f : A \longrightarrow B$ has the left lifting property with respect to $\mathcal{C}$-fibrations. In other words, we need to show that the $\mathcal{C}$-acyclic $\mathcal{C}$-cofibrations are contained in the $J_{\mathcal{C}}$-cofibrations. The small object argument provides a factorization

$$A \xrightarrow{j} W \xrightarrow{q} B$$

as a $J_{\mathcal{C}}$-cell complex j followed by a $\mathcal{C}$-fibration q. In addition, q is a $\mathcal{C}$-equivalence since f is. Since f is a $\mathcal{C}$-cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus f is a retract of a morphism q that has the left lifting property for $\mathcal{C}$-fibrations. So f itself has the left lifting property for $\mathcal{C}$-fibrations. The model structure is topological by Proposition 2.8.

Right properness of the model structure is straightforward from right properness of the model structure on spaces, since the H-fixed point functor, for $H \in \mathcal{C}$, preserves pullbacks and takes $\mathcal{C}$-fibrations to Serre fibrations. Since the projective $\mathcal{C}$-model structure is topological and all objects are fibrant, every cofibration is an h-cofibration by Corollary A.1.18 (iii). So left properness follows from the gluing lemma for $\mathcal{C}$-equivalences (Proposition 2.9). □

We want to clarify the relationship between $\mathcal{F}$-cofibrations and relative $\mathcal{F}$-CW-complexes. A *relative $\mathcal{F}$-CW-complex* is a relative G-CW-complex where all relative cells are of orbit type G/H for H in the family $\mathcal{F}$. Equivalently, the isotropy group of all points in the complement of the image are in $\mathcal{F}$. Relative $\mathcal{F}$-CW-complexes are built from the generating cofibrations (2.11) by coproducts, cobase change and countable compositions, so every relative $\mathcal{F}$-CW-complex is an $\mathcal{F}$-cofibration. Conversely, every $\mathcal{F}$-cofibration is G-homotopy equivalent to a relative $\mathcal{F}$-CW-complex, in the following precise sense:

Proposition 2.13. *Let $\mathcal{F}$ be a family of closed subgroups of a compact Lie group G and $i : A \longrightarrow B$ an $\mathcal{F}$-cofibration of G-spaces. Then there is a relative $\mathcal{F}$-CW-complex $j : A \longrightarrow B'$ and G-maps $f : B \longrightarrow B'$ and $g : B' \longrightarrow B$ such that both gf and fg are G-homotopic, relative A, to the respective identity maps. In particular, every $\mathcal{F}$-cofibrant G-space is G-homotopy equivalent to a G-CW-complex with all isotropy groups in $\mathcal{F}$.*

Proof. By attaching $\mathcal{F}$-cells we can factor the map $i : A \longrightarrow B$ as a relative $\mathcal{F}$-CW-complex $j : A \longrightarrow B'$ followed by G-map $g : B' \longrightarrow B$ that is an $\mathcal{F}$-equivalence [elaborate]. The category of G-spaces under A inherits a model structure in which a morphism (of G-spaces under A) is a weak equivalence, cofibration or fibration if the underlying G-map (i.e., after forgetting all references to A) is an $\mathcal{F}$-equivalence, $\mathcal{F}$-cofibration respectively $\mathcal{F}$-fibration. The morphisms $i : A \longrightarrow B$ and $j : A \longrightarrow B'$ are then cofibrant-fibrant objects in this under category, and g is a weak equivalence from j to i. By general model category theory, every weak equivalence between cofibrant-fibrant objects is a homotopy equivalence, and in the case of the under category at hand, this means precisely that g has a G-homotopy inverse relative A. □

If H and K are closed subgroups of a compact Lie group G and the dimension of K is strictly smaller than that of G, then the homogeneous space G/H typically has no preferred K-CW-structure. Hence the underlying K-spaces of G-CW-complexes cannot be made into K-CW-complexes in any natural way. The next proposition is usually sufficient to remedy this, because it says in particular that the restriction functor takes cofibrant G-spaces (for example G-CW-complexes) to cofibrant K-spaces. A *G-cofibration* is a morphism with the right lifting property with respect to all morphisms that are simultaneously weak equivalences and Serre fibrations on the fixed points for all subgroups of G. Equivalently, G-cofibrations are the $\mathcal{F}$-cofibrations for the maximal family of all subgroups of G. Thus we have the following implications between the various kinds of 'nice equivariant embeddings':

$$\text{relative } G\text{-CW-complex} \implies \text{relative } G\text{-cell complex} \implies G\text{-cofibration} \implies \text{h-cofibration of } G\text{-spaces}$$

All these implications are strict. However, Proposition 2.13 (for the family of all subgroups) makes precise that relative G-CW-complex, relative G-cell complex and G-cofibration are all equally good 'up to G-homotopy'.

Proposition 2.14. *Let G be a compact Lie group.*

(i) *For every compact Lie group K and every continuous homomorphism $\alpha : K \longrightarrow G$ the restriction functor $\alpha^* : G\mathbf{T} \longrightarrow K\mathbf{T}$ takes G-cofibrations to K-cofibrations.*
(ii) *For every closed subgroup H of G the induction functor $G \times_H - : H\mathbf{T} \longrightarrow G\mathbf{T}$ takes H-cofibrations to G-cofibrations.*

(iii) *For every closed normal subgroup N of G the quotient functor $N\backslash - : G\mathbf{T} \longrightarrow (G/N)\mathbf{T}$ takes G-cofibrations to G/N-cofibrations.*

Proof. (i) The restriction functor α^* preserves all colimits, so it suffices to show that it takes the generating G-cofibrations $G/H \times \partial D^n \longrightarrow G/H \times D^n$ to K-cofibrations, for any subgroup H of G. We treat two special cases separately.

If α is surjective, then $\alpha^*(G/H)$ is K-equivariantly homeomorphic to K/L for the closed subgroup $L = \alpha^{-1}(H)$ of K. So in this case α^* takes the generating G-cofibration to a K-map isomorphic to $K/L \times \partial D^n \longrightarrow K/L \times D^n$, which is then a K-cofibration.

If α is the inclusion of a closed subgroup K of G, then the left translation action is a smooth K-action on the smooth compact manifold G/H. Illman's theorem [**82**, Cor. 7.2] provides a finite K-CW-structure on G/H, so in particular G/H is cofibrant as a K-space. Since the projective model structure on K-spaces (for the family of all subgroups) is topological, the map $G/H \times \partial D^n \longrightarrow G/H \times D^n$ is a K-cofibration.

Any continuous homomorphism factors as a continuous epimorphism onto its image, followed by the inclusion of the image, so the two special cases combine to show that α^* takes G-cofibrations to K-cofibrations.

(ii) Since $G \times_H -$ preserves all colimits, it suffices to show that it takes the generating H-cofibrations $H/J \times \partial D^n \longrightarrow H/J \times D^n$ to G-cofibrations, for any closed subgroup J of H. This in turn is clear since $G \times_H (H/J)$ is G-homeomorphic to G/J.

(iii) Since $N\backslash -$ preserves all colimits, it suffices to show that it takes the generating G-cofibrations $G/H \times \partial D^n \longrightarrow G/H \times D^n$ to G/N-cofibrations, for any closed subgroup H of G. This in turn is clear since $N\backslash(G/H)$ is (G/N)-homeomorphic to $(G/N)/(H/H \cap N)$. □

Proposition 2.15. *Let N be a closed normal subgroup of a compact Lie group G. Then for every G-cofibration $i : A \longrightarrow B$ the maps*

$$i^N \ : \ A^N \ \longrightarrow \ B^N \qquad \textit{and} \qquad i \cup \mathrm{incl} \ : \ A \cup_{A^N} B^N \ \longrightarrow \ B$$

are G-cofibrations.

Proof. The class of G-cofibrations for which the claim holds is clearly closed under coproducts and retracts. We consider a pushout square of G-spaces on the left

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "j"'] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} A^N \arrow[r, "i^N"] \arrow[d] & B^N \arrow[d] \\ C^N \arrow[r, "j^N"'] & D^N \end{tikzcd}$$

such that i is a G-cofibration for which the claim holds. Then i is in particular a closed embedding, and the square on the right is also a pushout of G-spaces. In particular, j^N is a G-cofibration whenever i^N is. This means that $C \cup_{C^N} D^N$ is also a pushout of the maps of C and B^N over A^N, and hence the square

$$\begin{tikzcd} A \cup_{A^N} B^N \arrow[r, "i \cup \mathrm{incl}"] \arrow[d] & B \arrow[d] \\ C \cup_{C^N} D^N \arrow[r, "j \cup \mathrm{incl}"'] & D \end{tikzcd}$$

is another pushout of G-spaces. The upper horizontal map is a G-cofibration by hypothesis, hence so is the lower horizontal map. So the class of G-cofibrations satisfying the claim is closed under cobase change. Because N-fixed points preserve sequential colimits along closed embeddings, the class of G-cofibrations satisfying the claim is also closed under sequential composites.

Given these closure properties, it suffices to verify the claim for the generating G-cofibrations of the form

$$G/H \times i_k \ : \ G/H \times \partial D^k \ \longrightarrow \ G/H \times D^k \ .$$

Because $(G/H)^N$ is either empty (whenever N is not contained in H) or all of G/H (whenever $N \leq H$), the map in question is either the map $G/H \times i_k$ or the identity of $G/H \times D^k$; in either case it is a G-cofibration. This proves the claim for the generating cofibrations, and thus concludes the proof. □

Now we discuss when taking cartesian product preserves equivariant cofibrations. There are two related questions, namely 'external products' of a G-space and a K-space and 'internal products' of two G-spaces with diagonal action.

Proposition 2.16. *Let G and K be topological groups.*

(i) *The pushout product of a G-cofibration with a K-cofibration is a $(G \times K)$-cofibration.*
(ii) *If G is a compact Lie group, then the pushout product of two G-cofibrations is a G-cofibration with respect to the diagonal G-action.*

Proof. (i) The product functor

$$\times \ : \ G\mathbf{T} \times K\mathbf{T} \ \longrightarrow \ (G \times K)\mathbf{T}$$

preserves colimits in each variable, so it suffices to check the pushout product of a generating G-cofibration $G/H \times i_m$ with a generating K-cofibration $K/L \times i_n$, where $i_n : \partial D^n \longrightarrow D^n$ is the inclusion. The pushout product of these is isomorphic to

$$(G \times K)/(H \times L) \times (i_n \Box i_m)$$

and hence a cofibration of $(G \times K)$-spaces.

(ii) By (i) the pushout product of two G-cofibrations is a $(G \times G)$-cofibration. Since G is compact Lie and the diagonal is a closed subgroup, restriction along the diagonal embedding $G \longrightarrow G \times G$ preserves cofibrations by Proposition 2.14 (i). □

Definition 2.17. Let G be a topological group and $\mathcal{C}$ a collection of closed subgroups of G. A *universal space* for the collection $\mathcal{C}$ is an G-space E with the following properties:

(i) for every subgroup $H \in \mathcal{C}$ the fixed point space E^H is weakly contractible;
(ii) E is G-homotopy equivalent to a $\mathcal{C}$-cofibrant G-space.

Any cofibrant replacement of the one-point G-space in the $\mathcal{C}$-projective model structure is a universal space for the family $\mathcal{C}$, so universal spaces exist for any collection. Moreover, the following proposition shows that any two universal spaces for the same collection are G-equivariantly homotopy equivalent.

Proposition 2.18. *Let G be a topological group, $\mathcal{C}$ a collection of closed subgroups of G, and E a universal G-space for the collection $\mathcal{C}$.*

(i) *Every $\mathcal{C}$-cofibrant G-space admits a homomorphism to E, and any two such morphisms are homotopic as G-maps.*
(ii) *If E' is another universal G-space then every homomorphism from E' to E is an G-equivariant homotopy equivalence.*

In the rest of this section we collect some standard facts about spaces of homomorphisms between compact Lie groups.

Proposition 2.19. *Let X be a compact space and Y any topological space. Then for every open subset O of $X \times Y$ the set*

$$\{f \in \mathrm{map}(X, Y) \mid \Gamma_f \subset O\}$$

is open in the compact-open topology.

PROOF. We let $g : X \longrightarrow Y$ be a continuous map such that $\Gamma_g \subset O$. We let $x \in X$. Since O is open and $(x, g(x)) \in O$, there are open subsets $U_x \subset X$ and $V_X \subset Y$ with

$$(x, g(x)) \ \subseteq \ U_x \times V_x \ \subseteq \ O \ .$$

Since g is continuous, the set $g^{-1}(V_x)$ is open in X, hence so is $U_x \cap g^{-1}(V_x)$; moreover, $x \in U_x \cap g^{-1}(V_x)$. Since X is compact, it is also locally compact, so there is a compact neighborhood K_x of x that is contained in $U_x \cap g^{-1}(V_x)$. In particular, we have $g(K_x) \subseteq V_x$ and

$$(x, g(x)) \ \subseteq \ K_x \times V_x \ \subseteq \ O \ .$$

Since K_x is a neighborhood of x and X is compact, there are finitely many points $x_1, \dots, x_n$ such that $X = K_{x_1} \cup \dots \cup K_{x_n}$. Since $g(K_{x_i}) \subset V_{x_i}$ for all $1 \le i \le n$, the map g is contained in the set

$$W \ = \ \bigcap_{i=1,\dots,n} W(K_{x_i}, V_{x_i})$$

which is an open subset of $\mathrm{map}(X, Y)$ by the definition of the compact-open topology. So we are done if we can show that every $f \in W$ satisfies $\Gamma_f \subset O$.

So we consider a continuous map $f : X \longrightarrow Y$ such that $f(K_{X_i}) \subseteq V_{x_i}$ for all $1 \le i \le n$. Given $x \in X$, there is an $i \in \{1, \dots, n\}$ with $x \in K_{x_i}$. Thus $f(x) \in V_{x_i}$ by hypothesis on f, and so

$$(x, f(x)) \ \in \ K_{x_i} \times V_{x_i} \ \subseteq \ O \ .$$

Since x was arbitrary, this proves that $\Gamma_f \subseteq O$. □

We let K and G be locally compact Hausdorff topological groups. This hypothesis guarantees in particular that the underlying topological spaces are compactly generated (by Proposition 1.3 (iii)) and for every compactly generated space X the product topology on $X \times K$ is compactly generated (by Proposition 1.4 (ii)), i.e., $X \times_0 K = X \times K$, and similarly for G. We let $\mathrm{hom}(K, G)$ denote the set of continuous homomorphisms with the subspace topology of the compact-open topology on the set of all continuous maps. The following proposition shows that $\mathrm{hom}(K, G)$ is a closed subset of $\mathrm{map}(K, G)$, so the subspace topology is again compactly generated. For $g \in G$ we write

$$c_g \ : \ G \ \longrightarrow \ G \ , \quad c_g(x) \ = \ g^{-1} \cdot x \cdot g$$

for the conjugation automorphism. Then $c_g \circ c_h = c_{hg}$; so the group G acts from the right on the set of continuous homomorphisms into G by postcomposition with conjugation maps.

PROPOSITION 2.20. *Let K and G be topological groups.*

(i) *The set* $\mathrm{hom}(K, G)$ *is a closed subset of* $\mathrm{map}(K, G)$.
(ii) *The conjugation action of G on* $\mathrm{hom}(K, G)$ *is continuous.*

PROOF. Part (i) can be shown by the formal argument given by Lück and Uribe in [**101**, Lemma A.2]; for the convenience of the reader, and because Lück and Uribe work in a slightly different category (Hausdorff k-spaces versus *weak* Hausdorff k-spaces), we reproduce their proof in our context. The map

$$u \ : \ \mathrm{map}(K, G) \times K \times K \ \longrightarrow \ G \ , \quad u(k, k', \alpha) \ = \ \alpha(k) \cdot \alpha(k') \cdot \alpha(kk')^{-1}$$

is continuous since it can be written as a composite of diagonal maps, evaluation maps $K \times \mathrm{map}(K, G) \longrightarrow G$ and data from the group structure of G (i.e., multiplication and inverse). Each of the constituent maps is continuous, hence so is u. Since u is continuous, so is its adjoint

$$\bar{u} \ : \ \mathrm{map}(K, G) \ \longrightarrow \ \mathrm{map}(K \times K, G) \ .$$

The set $\mathrm{hom}(K, G)$ is the inverse image under $\bar{u}$ of the constant map with value 1. Since $\mathrm{map}(K \times K, G)$ is a Hausdorff space, singletons are closed, hence the set $\mathrm{hom}(K, G)$ is closed inside $\mathrm{map}(K, G)$.

(ii) This is another 'formal' proof. The map

$$v \ : \ \mathrm{map}(K,G) \times G \times K \ \longrightarrow \ G \ , \quad (\varphi, g, k) \ \longmapsto \ g^{-1} \cdot \varphi(k) \cdot g$$

is continuous since it can be written as a composite of a diagonal map, an evaluation map and data from the group structure. The adjoint

$$\mathrm{map}(K,G) \times G \ \longrightarrow \ \mathrm{map}(K,G) \ , \quad (\varphi, g) \ \longmapsto \ c_g \circ \varphi$$

is thus continuous as well. Since $\hom(K,G)$ has the subspace topology of $\mathrm{map}(K,G)$, the restriction of the conjugation action to $\hom(K,G)$ is then continuous as well. □

We recall some properties of the space of continuous homomorphisms between Lie groups. If (Y,d) is a metric space and K a compact topological space, then the supremum metric

$$d(f,g) \ = \ \sup_{x \in K} \{d(f(x), g(x))\}$$

is a metric on the set $C(K,Y)$ of continuous maps. Moreover, the topology induced by the supremum metric agrees with the compact-open topology, see for example [**68**, Prop. A.13]. Since the compact-open topology on $C(K,Y)$ is metrizable, it is in particular compactly generated, by Proposition 1.3 (vi). The topology of every second smooth manifold is metrizable, so the previous discussion applies in particular when K is a compact Lie group and G is a Lie group whose set of path components is countable. So in that case the compact-open topology on the space $C(K,G)$ of continuous maps is metrizable, and hence compactly generated. Hence $C(K,G) = \mathrm{map}(K,G)$, i.e., the compact-open topology is the topology of the internal mapping space in the category $\mathbf{T}$ of compactly generated spaces. All these facts carry over to The restriction of the compact-open topology to the closed subspace $\hom(K,G)$ of continuous homomorphisms is then also metrizable, and hence compactly generated.

The next proposition in particular shows that for a compact Lie group K and any Lie group G the space $\hom(K,G)$ is the topological disjoint union of the orbits, under conjugation, of the identity component group G°. The key input is a theorem of Montgomery and Zippin [**113**, Thm. 1 and Corollary] from 1942 which roughly says that in a Lie group 'nearby subgroups are conjugate'. More precisely, if H is any compact subgroup of a Lie group G, then there exists an open subset O of G, containing H, with the following property: for every closed subgroup K of G with $K \subseteq O$ there is an element $g \in G^\circ$ in the identity component of G such that $g^{-1} \cdot K \cdot g \subseteq H$. The following consequence of Montgomery and Zippin's result appears, for example, in [**38**, III, Lemma 38.1].

Proposition 2.21. *Let K and G be Lie groups, and suppose that K is compact. Then every orbit of the conjugation action by G° on $\hom(K,G)$ is an open subset of the space $\hom(K,G)$. In particular, the connected components of the space $\hom(K,G)$ coincide with its path components, and with the G°-orbits under the conjugation action.*

Proof. We reproduce the argument from [**38**, III, Lemma 38.1], in somewhat expanded form. We start by showing that the G°-orbits are open subsets of $\hom(K,G)$. Let $\alpha : K \longrightarrow G$ be a continuous homomorphism. Its graph $\Gamma_\alpha = \{(k, \alpha(k)) \mid k \in K\}$ is then a compact subgroup of the Lie group $K \times G$. By the above mentioned theorem of Montgomery and Zippin [**113**, Thm. 1 and Corollary], there is an open subset O of $K \times G$ containing Γ_α with the following property: for every closed subgroup Δ of $K \times G$ with $\Delta \subseteq O$ there is an element $(k,g) \in K^\circ \times G^\circ$ such that $(k,g)^{-1} \cdot \Delta \cdot (k,g) \subseteq \Gamma_\alpha$. We set

$$U \ = \ \{\beta \in \hom(K,G) \mid \Gamma_\beta \subseteq O\} \ ;$$

then $\alpha \in U$, and U is contained in the G°-orbit of α because

$$(k,g)^{-1} \cdot \Gamma_\beta \cdot (k,g) \ = \ \Gamma_{c_g \circ c_{\beta(k)^{-1}} \circ \beta} \ .$$

Then U is an open subset of $\hom(K,G)$ by Proposition 2.19 and the fact that $\hom(K,G)$ carries the subspace topology. This shows that the G°-orbit of α is open in $\mathrm{map}(K,G)$.

Now we prove the second claim. The component group G° is path connected; since the conjugation action is continuous, every G°-orbit is path connected, so in particularly connected. The G°-orbits are open by the previous paragraph. Since the complement of an orbit is a union of other orbits, the G°-orbits are also closed. So the G°-orbits are open, closed and path connected; hence they coincide with the path components and the connected components. □

For every G-space B and every closed subgroup K of G, the fixed points B^K are invariant under the normalizer $N_K G$. The normalizer contains the centralizer $C_G K$, so the fixed points B^K still come with an action of $C_K G$.

Proposition 2.22. *Let H and K be two closed subgroups of a compact Lie group G. Then the group $(C_G K)^\circ$, the identity component of the centralizer of K in G, acts transitively on every path component of the space $(G/H)^K$. Hence for every $gH \in (G/H)^K$ the map*

$$(C_G K)^\circ/((C_G K)^\circ \cap {}^g H) \ \longrightarrow\ (G/H)^K \ , \quad x \cdot ((C_G K)^\circ \cap {}^g H) \ \longmapsto\ xgH$$

is a homeomorphism onto the path component of gH.

Proof. We consider a continuous path $\omega : [0,1] \longrightarrow (G/H)^K$. The projection $G \longrightarrow G/H$ is a locally trivial fiber bundle, so the path from the unit in $W_G H$ can be lifted to a path $\bar\omega : [0,1] \longrightarrow G$. all $t \in [0,1]$. Since $\bar\omega(t)H = \omega(t)$ is K fixed, we have $K^{\omega(t)} \leq H$ for all $t \in [0,1]$. So conjugation by the elements $\omega(t)$ is a path in the space $\hom(K,H)$ of continuous homomorphisms. So by Proposition 2.21 there is an element $h \in H$ such that $c_{\omega(0)} = c_h \circ c_{\omega(1)} = c_{\omega(1)h}$. So the element $\omega(1)h\omega(0)^{-1}$ centralizes K. Since $\omega(1)H = (\omega(1)h\omega(0)^{-1}) \cdot (\omega(0)H)$, this shows that the image of the action map

$$a \ : \ C_G K \ \longrightarrow\ (G/H)^K \ , \quad x \longmapsto\ x\omega(0)H$$

contains the entire path component of $\omega(0)H$.

We let $N = (C_G K) \cap {}^{\omega(0)}H$ denote the stabilizer of the coset $\omega(0)H$. Then the map a factors over a continuous injective map

$$(C_G K)/N \ \longrightarrow\ (G/H)^K \ ;$$

the source is compact, so this map is also closed. The source is the disjoint union of its components, so only one of these components can intersect the component of $\omega(0)H$ non-trivially; since $a(1) = \omega(0)H$, this must by the component of the identity coset. So the map a factors over a homeomorphism from the component of $a((C_G K)/N)$ of the identity coset onto the component of $\omega(0)H$.

The projection $C_G K \longrightarrow (C_G K)/N$ is a locally trivial fiber bundle, so every path starting with the identity coset in $(C_G H)/N$ can be lifted to a path in $C_G K$ starting at the identity. Hence every element in the identity component of $(C_G K)/N$ is represented by an element in $(C_G K)^\circ$. This completes the proof that the group$(C_G K)^\circ$ acts transitively on the path component of $\omega(0)H$. □

Proposition 2.23. *Let $K \leq H \leq G$ be a nested sequence of closed subgroups inside a compact Lie group G. Then the dimension of the homogeneous space $N_G K/N_H K$ is equal to the dimension of the vector space $(T_H(G/H))^K$, the K-fixed points inside the tangent H-representation.*

Proof. The translation action of K on the homogeneous space G/H is smooth, so the fixed points $(G/H)^K$ are a disjoint union of finitely many smooth submanifolds, of possibly varying dimensions, compare [**27**, VI Cor. 2.5]. The distinguished coset eH is K-fixed, and we let $\langle eH \rangle$ denote the connected component of eH in $(G/H)^K$. By the differentiable slice theorem, the K-fixed point eH has an open K-invariant neighborhood inside G/H that is K-equivariantly diffeomorphic to the tangent space $T_{eH}(G/H)$, compare [**123**, Thm. 1.6.5] or [**27**, VI Cor. 2.4]. So the manifold dimension of $\langle eH \rangle$ agrees with the dimension of the vector space $(T_{eH}(G/H))^K$.

The two tautological maps

$$(C_G K)^\circ/((C_G K)^\circ \cap H) \ \longrightarrow\ (N_G K)/(N_H K) \ \longrightarrow\ (G/H)^K$$

are both injective, and the composite is a homeomorphism onto $\langle eH\rangle$ by Proposition 2.22. So

$$\dim((C_GK)^\circ/((C_GK)\circ\cap H) \;\le\; \dim(N_G(K)/N_H(K)) \;\le\; \dim(\langle eH\rangle)\ .$$

Since the first and third numbers coincide, this proves the claim. □

Example 2.24. We let K be a closed subgroup of a compact Lie group G and we let $N = N_GK$ denote the normalizer of K in G. Then $N_GK = N_NK = N$ and Proposition 2.23 shows that $(T_N(G/N))^K$ is 0-dimensional. In other words, 0 is the only K-fixed vector in the tangent N-representation $T_N(G/N)$.

Proposition 2.25. *Let $K \le H \le G$ be a nested sequence of closed subgroups inside a compact Lie group G. Then the following are equivalent:*

(i) *The Weyl group of K in G is finite.*
(ii) *The Weyl group of K in H is finite and 0 is the only K-fixed point of the tangent H-representation on $T_H(G/H)$.*

Proof. Since K and its two normalizers form a nested sequence of subgroups $K \le N_HK \le N_GK$, the group K has finite index N_GK if and only if K has finite index N_HK and N_HK has finite index N_GK. On the other hand, N_HG has finite index N_GK if and only if the homogeneous space N_GK/N_HK is 0-dimensional. So Proposition 2.23 shows that K has finite index N_GK if and only if K has finite index N_HK and 0 is the only K-fixed point of the tangent H-representation on $T_H(G/H)$. This proves the claim. □

Now we prove a decomposition result for certain kinds of fixed points. We let G and K be topological groups and X a $(K\times G)$-space. We want to describe the K-fixed points $(G\backslash X)^K$ of the G-orbit space $G\backslash X$. For a continuous homomorphism $\alpha : K \longrightarrow G$ we set

$$X^\alpha \;=\; \{x\in X \mid (k,\alpha(k))x = x \text{ for all } k\in K\}\ .$$

Equivalently, X^α is the fixed point space of the graph of α, which is a closed subgroup of $K\times G$. The subspace X^α of X is stabilized by the centralizer of the image of α, i.e., the group

$$C(\alpha) \;=\; \{g\in G \mid g\alpha(k) = \alpha(k)g \text{ for all } k\in K\}\ .$$

The inclusion $X^\alpha \longrightarrow X$ then passes to a continuous map

$$\alpha^\flat \;:\; C(\alpha)\backslash X^\alpha \;\longrightarrow\; G\backslash X$$

on orbit spaces. For $a\in X^\alpha$ and $k\in K$ the relation

$$k(Ga) \;=\; G(ka) \;=\; G(\alpha(k)^{-1}a) \;=\; Ga$$

shows that $\alpha^\flat$ takes values in the K-fixed points $(G\backslash X)^K$. Moreover, for $g\in G$ we have

$$g\cdot X^\alpha \;=\; X^{c_g\circ\alpha} \tag{2.26}$$

as subspaces of X. So the maps $\alpha^\flat$ and $(c_g\circ\alpha)^\flat$ arising from conjugate homomorphisms have the same image in the orbit space $G\backslash X$. It is relatively straightforward to see that the coproduct of all the maps $\alpha^\flat$ is bijective; some pointset topology is involved in showing that this continuous bijection is in fact a homeomorphism.

Proposition 2.27. *Let G and K be Lie groups, and assume that K is compact. Let X be a $(K\times G)$-space such that G-action is free. Then the map*

$$\coprod \alpha^\flat \;:\; \coprod_{\langle\alpha\rangle} C(\alpha)\backslash X^\alpha \;\longrightarrow\; (G\backslash X)^K$$

is a homeomorphism. Here the disjoint union runs over conjugacy classes of continuous homomorphisms $\alpha : K\longrightarrow G$ and $C(\alpha)$ is the centralizer, in G, of the image of α.

PROOF. We let $\Pi : X \longrightarrow G\backslash X$ denote the quotient map. We set

$$\bar{X} \;=\; \Pi^{-1}((G\backslash X)^K)\ .$$

Since X is compactly generated, so is $G\backslash X$ by Proposition 2.2 (iii). Since $(G\backslash X)^K$ a closed subset of the orbit space, $\bar{X}$ is a $(K \times G)$-invariant closed subspace of X. In particular, $\bar{X}$ is compactly generated in the subspace topology of X.

For a given continuous homomorphism $\alpha : K \longrightarrow G$, we set

$$X^{(\alpha)} \;=\; G \cdot X^\alpha\ ,$$

the smallest G-subspace of $\bar{X}$ containing X^α. The relation (2.26) shows that $X^{(\alpha)}$ is the union of the subsets $X^{\alpha'}$ as α' runs over all conjugates of α. We factor the map in question as a composite

$$\coprod_{\langle\alpha\rangle} C(\alpha)\backslash X^\alpha \;\longrightarrow\; \coprod_{\langle\alpha\rangle} G\backslash X^{(\alpha)} \;\longrightarrow\; G\backslash\bar{X} \;\longrightarrow\; (G\backslash X)^K\ ,$$

induced by the inclusions $X^\alpha \longrightarrow X^{(\alpha)} \longrightarrow \bar{X} \longrightarrow X$; we show that each of the three maps is a homeomorphism. The third map $G\backslash\bar{X} \longrightarrow (G\backslash X)^K$ is a homeomorphism by Proposition 2.2 (iii).

For every $x \in \bar{X}$ and every $k \in K$ we have $k \cdot (Gx) = G(kx) = Gx$, so there exists a $g \in G$ such that $kx = g^{-1}x$. Since the G-action is free, the element g is uniquely determined by this property. So we can define a map $\beta_x : K \longrightarrow G$ by the property $kx = \beta_x(k)^{-1}x$; equivalently, the characterizing condition for β_x is that $(k, \beta_x(k)) \cdot x = x$. It is straightforward to see that β_x is a group homomorphism.

By definition, the stabilizer group of $x \in \bar{X}$ inside $K \times G$ is precisely the graph of the homomorphism $\beta_x : K \longrightarrow G$. Since X is compactly generated, this stabilizer group is a closed subset of $K \times G$, which means that the homomorphism β_x is continuous.

Our next claim is that the assignment

$$\beta \;:\; \bar{X} \;\longrightarrow\; \hom(K, G)\ , \quad x \;\longmapsto\; \beta_x$$

is continuous. Since $\hom(K, G)$ has the subspace topology of $\mathrm{map}(K, G)$, it suffices to show that the adjoint map

$$\bar{\beta} \;:\; \bar{X} \times K \;\longrightarrow\; G\ , \quad (x, k) \;\longmapsto\; \beta_x(k)$$

is continuous. This, in turn, is equivalent to the claim that the graph of $\bar{\beta}$ is closed as a subset of $\bar{X} \times K \times G$. The graph of $\bar{\beta}$ is the inverse image of the diagonal under the continuous map

$$\bar{X} \times K \times G \;\longrightarrow\; \bar{X} \times \bar{X}\ , \quad (x, k, g) \;\longmapsto\; (kx, gx)\ .$$

So the graph of $\bar{\beta}$ is closed, hence $\bar{\beta}$ and β are continuous.

By Proposition 2.21 the space $\hom(K, G)$ is the topological disjoint union of the G°-orbits under the conjugation action. In particular, every G°-conjugacy class is open and closed. A G-conjugacy class is a finite union of G°-conjugacy classes, so every G-conjugacy class in $\hom(K, G)$ is also open and closed. Since $\beta : \bar{X} \longrightarrow \hom(K, G)$ is continuous, $\bar{X}$ is the topological disjoint union of the subsets $X^{(\alpha)}$ as α ranges over all conjugacy classes in $\hom(K, G)$. Taking orbits commutes with disjoint unions, so the canonical map

$$\textstyle\coprod_{(\alpha)} G\backslash X^{(\alpha)} \;\longrightarrow\; G\backslash\bar{X}$$

is a homeomorphism.

The final step is to show that for every continuous homomorphism $\alpha : K \longrightarrow G$ the canonical continuous map

$$C(\alpha)\backslash X^\alpha \;\longrightarrow\; G\backslash X^{(\alpha)} \tag{2.28}$$

is a homeomorphism. The map is surjective because $X^{(\alpha)} = G \cdot X^\alpha$. The map is also injective; if $x, y \in X^\alpha$ satisfy $Gx = Gy$, then $x = gy$ for some $g \in G$, and so $x \in X^\alpha \cap X^{c_g \circ \alpha}$. This implies that $\alpha = c_g \circ \alpha$, and so $g \in C(\alpha)$. Hence $C(\alpha)x = C(\alpha)gy = C(\alpha)y$. Finally, the map (2.28) is closed: if $O \subset C(\alpha)\backslash X^\alpha$ is a closed

subset, then $\Pi_{X^\alpha}^{-1}(O)$ is closed in X^α. Since X^α is closed in X, hence also in X^α, the set $\Pi_{X^\alpha}^{-1}(O)$ is also closed in $X^{(\alpha)}$. Since the projection $X^{(\alpha)} \longrightarrow G\backslash X^{(\alpha)}$ is a closed map, the image of O in $G\backslash X^{(\alpha)}$ is closed. This completes the proof that the map (2.28) is a homeomorphism. □

We recall that $\mathcal{F}(K;G)$ denotes the family of graph subgroups of the product $K \times G$, i.e., the family of closed subgroups $\Gamma \leq K \times G$ that intersect $1 \times G$ only in the neutral element $(1,1)$. These are precisely the graphs of all continuous homomorphisms $L \longrightarrow G$ for all closed subgroups L of K.

Corollary 2.29. *Let G and K be compact Lie groups and $\mathcal{F}$ a family of closed subgroups of K. Let $\tilde{\mathcal{F}}$ denote the family of those graph subgroups of $K \times G$ whose projection to K belongs to $\mathcal{F}$. Let A be a G-free cofibrant $(K \times G)$-space. Then the functor $A \times_G -$ takes $\tilde{\mathcal{F}}$-weak equivalences of $(K \times G)$-spaces to $\mathcal{F}$-weak equivalences of K-spaces.*

Proof. We let L be a closed subgroup of K in the family $\mathcal{F}$. Since G acts freely on A, Proposition 2.27 provides a homeomorphism

$$\coprod_{\langle\alpha\rangle} A^{\Gamma(\alpha)} \times_{C(\alpha)} X^{\Gamma(\alpha)} \;\cong\; (A \times_G X)^L$$

that is natural for $(K \times G)$-maps in X. Here the disjoint union runs over conjugacy classes of continuous homomorphisms $\alpha : L \longrightarrow G$, $\Gamma(\alpha)$ is the graph of α and $C(\alpha)$ is the centralizer, in G, of the image of α. Since $L \in \mathcal{F}$, the group $\Gamma(\alpha)$ belongs to the family $\tilde{\mathcal{F}}$.

Now we let $f : X \longrightarrow Y$ be a $\tilde{\mathcal{F}}$-weak equivalences of $(K \times G)$-spaces. Then for every L and $\alpha : L \longrightarrow G$ as in the previous paragraph, the map $f^{\Gamma(\alpha)} : X^{\Gamma(\alpha)} \longrightarrow Y^{\Gamma(\alpha)}$ is a $C(\alpha)$-weak equivalence. Since A is a cofibrant G-free $(K \times G)$-space, the fixed point $A^{\Gamma(\alpha)}$ form a cofibrant free $C(\alpha)$-space. So the functor$A^{\Gamma}(\alpha) \times_{C(\alpha)} -$ takes $C(\alpha)$-weak equivalences to weak equivalences [...ref...]. Since the disjoint union of weak equivalences is a weak equivalence, the previous decomposition of $(A \times_G X)^L$ shows that $(A \times_G f)^L : (A \times_G X)^L \longrightarrow (A \times_G Y)^L$ is a weak equivalence. □

3. Enriched functor categories

In this section we review some general definitions, properties and constructions involving categories of enriched functors. The general setup consists of the following data:

- a complete and cocomplete symmetric monoidal category $\mathcal{V}$ (the 'base category'); we denote the monoidal product in $\mathcal{V}$ by $\otimes$;
- a skeletally small $\mathcal{V}$-category $\mathcal{D}$ (the 'index category') [...spell out...].

We denote by $\mathcal{D}^*$ the category of covariant $\mathcal{V}$-functors $F : \mathcal{D} \longrightarrow \mathcal{V}$ from the index category to the base category.

We are mostly interested in the following special cases of such functor categories:

(i) orthogonal spaces (where $\mathcal{V} = \mathbf{T}$ and $\mathcal{D} = \mathbf{L}$);
(ii) orthogonal spectra (where $\mathcal{V} = \mathbf{T}_*$ and $\mathcal{D} = \mathbf{O}$);
(iii) global functors (where $\mathcal{V} = \mathcal{A}b$ and $\mathcal{D} = \mathbf{A}$).

Abuse of notation. In the following we will deal a lot with symmetric monoidal categories and their actions on other categories. Part of the data are then typically associativity isomorphisms that satisfy a pentagon coherence conditions. We will generally denote all associativity isomorphisms by the letter α, decorated with three indices that indicate the objects involved. So in the same diagram, different occurrences of α may refer to different kinds of associativity isomorphism, but the context always makes it clear which one is intended. Various times, symmetry isomorphisms will also come up that satisfy a hexagon condition with respect to the associativity isomorphisms. We will generally use the letter τ, decorated with two indices that indicated the objects involved, to denote symmetry isomorphisms.

Convolution product. We review a general method, due to B. Day [**40**] for constructing symmetric monoidal structures on certain functor categories. We make the additional assumption that the 'index category' $\mathcal{D}$ is a *symmetric monoidal* $\mathcal{V}$-category $\mathcal{D}$ [...spell out...]. We denote the monoidal product on $\mathcal{D}$ by $\oplus$.

We get the cases of interest above by specializing as follows:

(i) For $\mathcal{V} = \mathbf{T}$ the category of spaces under cartesian product and $\mathcal{D} = \mathbf{L}$ the topological category of inner product spaces under orthogonal direct sum, $\mathcal{D}^* = spc$ is the category of orthogonal spaces. This yields the box product of orthogonal spaces.

(ii) For $\mathcal{V} = \mathbf{T}_*$ the category of based spaces under smash product and $\mathcal{D} = \mathbf{O}$ the Thom space category of the orthogonal complement bundles, under orthogonal direct sum, $\mathcal{D}^* = \mathcal{S}p$ is the category of orthogonal spectra. This yields the smash product of orthogonal spectra.

(iii) For $\mathcal{V} = \mathcal{A}b$ the category of abelian groups under tensor product and $\mathcal{D} = \mathbf{A}$ the pre-additive Burnside category with monoidal structure constructed in Theorem IV.2.16, $\mathcal{D}^* = \mathcal{GF}$ is the category of global functors. This yields the box product of global functors.

Given two $\mathcal{V}$-functors $X, Y : \mathcal{D} \longrightarrow \mathcal{V}$, the composite

$$X \otimes Y \ : \ \mathcal{D} \times \mathcal{D} \xrightarrow{X \times Y} \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$$

is a $\mathcal{V}$-functor on the product $\mathcal{V}$-category.

Definition 3.1. A *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of objects (X, Y) of $\mathcal{D}^*$ to another object Z of $\mathcal{D}^*$ is a $\mathcal{V}$-natural transformation $b : X \otimes Y \longrightarrow Z \circ \oplus$ of $\mathcal{V}$-functors $\mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{V}$.

So a bimorphism $b : (X, Y) \longrightarrow Z$ consists of $\mathcal{V}$-morphisms

$$b(d, e) \ : \ X(d) \otimes Y(e) \ \longrightarrow \ Z(d \oplus e)$$

for all objects d, e of $\mathcal{D}$, that form a $\mathcal{V}$-natural transformation.

We can then define a box product of X and Y as a universal example of an object of $\mathcal{D}$ with a bimorphism from X and Y.

Definition 3.2. A *box product* for objects X and Y of $\mathcal{D}^*$ is a pair $(X \Box Y, i)$ consisting of a $\mathcal{D}^*$-object $X \Box Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \Box Y$, i.e., a bimorphism such that for every $\mathcal{D}^*$-object Z the map

$$\mathcal{D}^*(X \Box Y, Z) \ \longrightarrow \ \mathrm{Bimor}((X, Y), Z) \ , \quad f \longmapsto f \circ i \tag{3.3}$$

is bijective.

Remark 3.4 (Uniqueness of box products). The universal property makes box products of two functors unique up to preferred isomorphism. Indeed, if $(X \Box Y, i)$ and $(X \Box' Y, i')$ are two box products, then the universal properties provide unique morphisms $f : X \Box Y \longrightarrow X \Box' Y$ and $g : X \Box' Y \longrightarrow X \Box Y$ that satisfy

$$f \circ i \ = \ i' \qquad \text{and} \qquad g \circ i' \ = \ i \ .$$

Then $(g \circ f) \circ i = g \circ i' = i = \mathrm{Id}_{X \Box Y} \circ i$, so $g \circ f = \mathrm{Id}_{X \Box Y}$ by the uniqueness part of the universal property. Reversing the roles gives $f \circ g = \mathrm{Id}_{X \Box' Y}$.

Proposition 3.5. *Every pair of objects of $\mathcal{D}^*$ has a box product.*

Proof. The universal property of the box product precisely means that $(X \Box Y, i)$ is an enriched left Kan extension of the functor $X \otimes Y : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{V}$ along the $\mathcal{V}$-functor $\oplus : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$. Such a Kan extension exists because $\mathcal{V}$ is cocomplete and $\mathcal{D}$ is skeletally small. $\square$

Remark 3.6. The box product is constructed as an enriched Kan extension along the functor $\oplus : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$. We can make this a little more explicit. Given $X, Y \in \mathcal{D}^*$, a box product $X \Box Y$ is a coequalizer of the two morphisms of functors

$$\coprod_{e,e',d,d'} \mathcal{D}(e \oplus e', -) \otimes \mathcal{D}(d,e) \otimes \mathcal{D}(d',e') \otimes X(d) \otimes Y(d') \rightrightarrows \coprod_{d,d'} \mathcal{D}(d \oplus d', -) \otimes X(d) \otimes Y(d') \ .$$

The left coproduct is indexed over all quadruples (e, e', d, d') in a set of representatives of the isomorphism classes of $\mathcal{D}$-objects. The right coproduct is indexed over pairs (d, d') of such representatives. One of the two morphisms to be coequalized is the coproduct of the products of

$$\mathcal{D}(e \oplus e', -) \otimes \mathcal{D}(d,e) \otimes \mathcal{D}(d',e') \ \longrightarrow \ \mathcal{D}(d \oplus d', -) \ , \quad (\varphi, \tau, \tau') \ \longmapsto \ \varphi \circ (\tau \oplus \tau')$$

and the identity on $X(d) \otimes Y(d')$. The other morphism is the coproduct of the products of the identity on $\mathcal{D}(e \oplus e', -)$ and the action maps $\mathcal{D}(d,e) \otimes X(d) \longrightarrow X(e)$ respectively $\mathcal{D}(d',e') \otimes Y(d') \longrightarrow Y(e')$.

Example 3.7. We start with a toy example of the box product, the tensor product of modules over a commutative ring R. For this purpose we take the base category $\mathcal{V} = \mathcal{A}b$ as the category of abelian groups under tensor product. We view the ring as a pre-additive category $\underline{R}$ with one object $*$ whose abelian endomorphism group is R; composition is the multiplication in R.

A symmetric monoidal structure on $\underline{R}$ is defined as follows: the monoidal product $\mu : \underline{R} \times \underline{R} \longrightarrow \underline{R}$ is $\mu(*,*) = *$ on objects (there is no room for anything else) and also given by the multiplication

$$\underline{R}(*,*) \otimes \underline{R}(*,*) \ = \ R \otimes R \ \longrightarrow \ R \ = \ \underline{R}(*,*)$$

on morphisms. The hypothesis that R is commutative is needed to make this a functor. The symmetric monoidal structure is (very) strict, i.e., all coherence isomorphisms for unit, associativity and even symmetry are the identities.

The category of (left, say) R-modules is isomorphic to the category of additive functors from $\underline{R}$ to abelian groups, and we leave it as an exercise to the reader that under the usual isomorphism the tensor product of R-modules corresponds to the box product of enriched functors.

In fact, this example works more generally in any base symmetric monoidal category $\mathcal{V}$, not just in the special case $\mathcal{V} = \mathcal{A}b$. Every commutative monoid R in $\mathcal{V}$ gives rise to a symmetric monoidal $\mathcal{V}$-category $\underline{R}$ such that the category of R-modules is isomorphic to the category of $\mathcal{V}$-functors $\underline{R} \longrightarrow \mathcal{V}$. The 'tensor' product of R-objects in $\mathcal{V}$ corresponds to the box product of enriched functors.

It is a fact of life that Kan extensions such as a box product are not unique, but only unique up to preferred isomorphism. In the following we *choose* a box product $(X \Box Y, i_{X,Y})$ for every pair of objects X and Y of $\mathcal{D}^*$. The universal bimorphism i is often omitted from the notation, but one should remember that it is the pair $(X \Box Y, i)$, and not the object $X \Box Y$ alone, that has a universal property. It will be convenient later to make the constant functor 0^* into a strict unit for the box product (as opposed to a unit up to coherent isomorphisms). So we make the following **conventions:**

- (Right unit) We choose $X \Box 0^* = X$ with universal bimorphism $i : (X, 0^*) \longrightarrow X$ given by the maps

$$\begin{aligned} X(d) \otimes 0^*(e) \ = \ X(d) \otimes \mathcal{D}(0,e) \ &\xrightarrow{X(d) \otimes (d \oplus -)} X(d) \otimes \mathcal{D}(d \oplus 0, d \oplus e) \\ &\cong \ X(d) \otimes \mathcal{D}(d, d \oplus e) \ \xrightarrow{\circ} \ X(d \oplus e) \ . \end{aligned}$$

- (Left unit) We choose $0^*\Box Y = Y$ with universal bimorphism $i : (0^*, Y) \longrightarrow Y$ given by the composite

$$\begin{aligned} 0^*(d) \otimes Y(e) \;=\; \mathcal{D}(0,d) \otimes Y(e) \;&\xrightarrow{(-\oplus e)\otimes Y(e)} \mathcal{D}(0\oplus e, d\oplus e) \otimes Y(e) \\ &\cong\; \mathcal{D}(e, d\oplus e) \otimes Y(e) \;\xrightarrow{\circ}\; Y(d\oplus e)\ . \end{aligned}$$

The unnamed isomorphisms are induced by the right respective left unit isomorphisms in $\mathcal{D}$, and '$\circ$' is the functor structure of X respectively Y. For $X = Y = 0^*$ these two bimorphisms $(0^*, 0^*) \longrightarrow 0^*$ are equal, so there is no ambiguity. While the box products are choices, the good new is that these are the only choices that need to be made – the rest of the structure is canonically determined by the choices of box product.

Construction 3.8 (Functoriality). For example, with these choices made, the box product $X\Box Y$ automatically becomes a functor in both variables: if $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are morphisms in $\mathcal{D}^*$, then the collection of $\mathcal{V}$-maps

$$\left\{ X(d) \otimes Y(e) \xrightarrow{f(d)\otimes Y(e)} X'(d) \otimes Y'(e) \xrightarrow{i'_{d,e}} (X'\Box Y')(d\oplus e) \right\}_{d,e}$$

forms a bimorphism $(X, Y) \longrightarrow X'\Box Y'$. So there is a unique morphism $f\Box g : X\Box Y \longrightarrow X'\Box Y'$ such that $(f\Box g)(d\oplus e) \circ i_{d,e} = i'_{d,e} \circ (f(d) \otimes g(e))$ for all objects d, e of $\mathcal{D}$. The uniqueness part of the universal property implies that this is compatible with identities and composition in both variables.

Construction 3.9 ($\mathcal{V}$-enrichment). The category $\mathcal{D}^*$ is tensored over the base category $\mathcal{V}$. In more detail we consider an object A of $\mathcal{V}$ and a functor $Y \in \mathcal{D}^*$. We define a new functor $A \otimes Y \in \mathcal{D}^*$ by objectwise product in $\mathcal{V}$, i.e., $A \otimes Y$ is the composite $\mathcal{V}$-functor

$$\mathcal{D} \xrightarrow{Y} \mathcal{V} \xrightarrow{A\otimes -} \mathcal{V}$$

This construction is an action of $\mathcal{V}$ on $\mathcal{D}^*$, which means preferred natural associativity and unit isomorphisms

$$(A \otimes B) \otimes Y \;\cong\; A \otimes (B \otimes Y) \qquad \text{and} \qquad 1 \otimes Y \;\cong\; Y\ .$$

Indeed, the value of the above isomorphisms at an object $d \in \mathcal{D}$ is simply the associativity (respectively unit) isomorphism in the monoidal structure of $\mathcal{V}$, for the triple of objects $(A, B, Y(d))$.

The box product commutes with such $\mathcal{V}$-tensors up to distinguished isomorphism

$$\alpha_{A,Y,Z} \;:\; (A \otimes Y)\Box Z \;\cong\; A \otimes (Y\Box Z)$$

that we now construct. The family of morphisms

$$(A \otimes Y(d)) \otimes Z(e) \xrightarrow{\alpha_{A,Y(d),Z(e)}} A \otimes (Y(d) \otimes Z(e)) \xrightarrow{A\otimes i_{d,e}} A \otimes (Y\Box Z)(d\oplus e)$$

forms a bimorphism from $((A \otimes Y), Z)$ to $A \otimes (Y\Box Z)$. So the universal property provides a distinguished morphism $\alpha_{A,Y,Z} : (A\otimes Y)\Box Z \longrightarrow A\otimes(Y\Box Z)$. Since the monoidal structure on $\mathcal{V}$ is closed, the functor $A\otimes -$ is a left adjoint, so it preserves coends. So the morphism $\alpha_{A,Y,Z}$ is an isomorphism. A minimally more complicated construction that also involves the symmetry constraint in $\mathcal{V}$ to move the object A past the functor Y provides an isomorphism

$$Y\Box(A \otimes Z) \;\cong\; A \otimes (Y\Box Z)\ .$$

The two types of associativity isomorphisms $(A\otimes B)\otimes Y \cong (A\otimes(B\otimes Y)$ and $(A\otimes Y)\Box Z \cong A\otimes(Y\Box Z)$ are compatible in the sense of a commuting pentagon

$$\begin{tikzcd}
 & ((A\otimes B)\otimes Y)\Box Z \arrow[dl, "\alpha_{A,B,Y}\Box Z"'] \arrow[dr, "\alpha_{A\otimes B,Y,Z}"] & \\
(A\otimes(B\otimes Y))\Box Z \arrow[d, "\alpha_{A,B\otimes Y,Z}"'] & & (A\otimes B)\otimes(Y\Box Z) \arrow[d, "\alpha_{A,B,Y\Box Z}"] \\
A\otimes((B\otimes Y)\Box Z) \arrow[rr, "A\otimes\alpha_{B,Y,Z}"'] & & A\otimes(B\otimes(Y\Box Z))
\end{tikzcd}$$

Construction 3.10 (Coherence isomorphisms). Now that we have constructed a box product functor

$$\Box \ : \ \mathcal{D}^* \times \mathcal{D}^* \ \longrightarrow \ \mathcal{D}^* \ ,$$

we can show it 'is' automatically symmetric monoidal. Since 'symmetric monoidal' is extra data, and not a property, we are obliged to construct associativity isomorphisms

$$\alpha_{X,Y,Z} \ : \ (X\Box Y)\Box Z \ \longrightarrow \ X\Box(Y\Box Z)$$

and symmetry isomorphisms

$$\tau_{X,Y} \ : \ X\Box Y \ \longrightarrow \ Y\Box X \ .$$

We have arranged things so that the constant functor $\mathbf{1}$ is a strict unit, so the left and right unit isomorphisms which are part of a symmetric monoidal structure are the identity maps and don't have to be explicitly mentioned.

We can obtain $\alpha_{X,Y,Z}$ and $\tau_{X,Y}$ just from the universal property of the box products. For the associativity isomorphism we notice that the family

$$\left\{ X(d)\times \ Y(e)\times Z(f) \ \xrightarrow{i_{d,e}\times Z(f)} \ (X\Box Y)(d\oplus e)\times Z(f) \ \xrightarrow{i_{d\oplus e,f}} \ ((X\Box Y)\Box Z)((d\oplus e)\oplus f) \right\}_{d,e,f}$$

and the family

$$\left\{ X(d)\times \ Y(e)\times Z(f) \ \xrightarrow{X(d)\times i_{e,f}} \ X(d)\times(Y\Box Z)(e\oplus f) \ \xrightarrow{i_{e,d\oplus f}} \ (X\Box(Y\Box Z))(d\oplus(e\oplus f)) \right\}_{p,q,r\geq 0}$$

both have the universal property of a *tri*morphism (whose definition is hopefully clear) out of X, Y and Z. The uniqueness of representing objects gives a unique isomorphism of enriched functors

$$\alpha_{X,Y,Z} \ : \ (X\Box Y)\Box Z \ \cong X\Box(Y\Box Z)$$

such that $(\alpha_{X,Y,Z})_{p+q+r}\circ i_{p+q,r}\circ(i_{p,q}\Box Z_r) = i_{p,q+r}\circ(X_p\Box i_{q,r})$.

The symmetry isomorphism $\tau_{X,Y} : X\Box Y \longrightarrow Y\Box X$ corresponds to the bimorphism from (X,Y) to $Y\Box X$ with components

$$X(d)\Box Y(e) \ \xrightarrow{\tau_{X(d),Y(e)}} \ Y(e)\Box X(d) \ \xrightarrow{i_{e,d}} \ (Y\Box X)(e\oplus d) \ \xrightarrow{(Y\Box X)(\tau_{e,d})} \ (Y\Box X)(e\oplus d) \ . \tag{3.11}$$

The following theorem is due to B. Day. In fact, Theorems 3.3 and 4.1 of [**40**] together imply the 'monoidal' part, and Theorem 3.6 of [**40**] deals with the symmetries.

Theorem 3.12. [**40**] *The associativity and symmetry isomorphisms make the box product into a symmetric monoidal product on the category $\mathcal{D}^*$ with the functor $\mathbf{1}$ as a strict unit object.*

For objects a, b of $\mathcal{D}$ the represented functors of a, b and $a \oplus b$ are related by a preferred isomorphism. Indeed, the $\mathcal{V}$-morphisms

$$a^*(d) \times b^*(e) \;=\; \mathcal{D}(a,d) \times \mathcal{D}(b,e) \;\xrightarrow{\oplus}\; \mathcal{D}(a \oplus b, d \oplus e) \;=\; (a \oplus b)^*(d \oplus e)$$

form a bimorphism as d and e vary. So there is a unique morphism

$$j_{a,b} \;:\; a^* \Box b^* \;\cong\; (a \oplus b)^*$$

such that $j_{a,b} \circ i_{a^*,b^*} = \oplus$.

Proposition 3.13. *For all objects a, b of $\mathcal{D}$ the morphism $j_{a,b} : a^* \Box b^* \longrightarrow (a \oplus b)^*$ is an isomorphism. As a and b vary, these isomorphism form a strong symmetric monoidal structure on the Yoneda functor*

$$Y \;:\; \mathcal{D}^{\mathrm{op}} \;\longrightarrow\; \mathcal{D}^* \;, \quad a \;\longmapsto\; a^* \;.$$

Construction 3.14 (Internal function object). The box product is a *closed* monoidal product in the sense that for all objects Y and Z of $\mathcal{D}^*$ the functor

$$\mathcal{D}^*(-\Box Y, Z) \;:\; \mathcal{D}^* \;\longrightarrow\; \mathcal{V} \;, \quad X \;\longmapsto\; \mathcal{D}^*(X \Box Y, Z)$$

is representable. A representing object can be constructed as follows. Given $Z \in \mathcal{D}^*$ and $d \in \mathcal{D}$, we define the *d-shift* Z_d as the composite $\mathcal{V}$-functor

$$\mathcal{D} \;\xrightarrow{d \oplus -}\; \mathcal{D} \;\xrightarrow{Z}\; \mathcal{V} \;.$$

Then we define an object $\mathrm{Hom}(Y, Z)$ of $\mathcal{D}^*$ by

$$\mathrm{Hom}(Y,Z)(d) \;=\; \mathcal{D}^*(Y, Z_d) \;.$$

We define a morphism

$$\epsilon_{Y,Z} \;:\; \mathrm{Hom}(Y,Z) \Box Y \;\longrightarrow\; Z$$

from the bimorphism from $(\mathrm{Hom}(Y,Z), Y)$ to Z with components

$$\mathrm{Hom}(Y,Z)(d) \times Y(e) \;=\; \mathcal{D}^*(Y, Z_d) \times Y(e) \;\longrightarrow\; Z(d \oplus e)$$

defined as the adjoint to evaluation at e

$$\mathcal{D}^*(Y, Z_d) \;\longrightarrow\; \mathcal{V}(Y(e), Z_d(e)) \;=\; \mathcal{V}(Y(e), Z(d \oplus e)) \;.$$

[...ref or proof that this represents $\mathcal{D}^*(-\Box Y, Z)$...] We choose, for all $Y, Z \in \mathcal{D}^*$, a representing object, i.e., a pair $(\mathrm{Hom}(Y,Z), \epsilon)$ consisting of an object $\mathrm{Hom}(Y,Z)$ of $\mathcal{D}^*$ and a morphism $\epsilon_{Y,Z} : \mathrm{Hom}(Y,Z) \Box Y \longrightarrow Z$ such that for all $X \in \mathcal{D}^*$ the map

$$\mathcal{D}^*(X, \mathrm{Hom}(Y,Z)) \;\longrightarrow\; \mathcal{D}^*(X \Box Y, Z) \;, \quad f \;\mapsto\; \epsilon_{Y,Z}(f \Box Y)$$

is bijective. As for the box product, there is generally no way to avoid the choice here, but one these choices have been made, the rest of the structure is determined. In particular, there is now a preferred way to make the choice $\mathrm{Hom}(Y,Z)$ the object part of a functor

$$\mathrm{Hom}(-,-) \;:\; (\mathcal{D}^*)^{\mathrm{op}} \times \mathcal{D}^* \;\longrightarrow\; \mathcal{D}^* \;.$$

Once this is done, the morphisms $\epsilon_{Y,Z}$ are automatically the counits of an adjunction between the functor $-\Box Y$ and the functor $\mathrm{Hom}(Y,-)$.

We recall that a *monoid* in the symmetric monoidal category $\mathcal{D}^*$ with respect to the box product is an object $M \in \mathcal{D}^*$ equipped with a multiplication morphism $\mu : M\Box M \longrightarrow M$ and a unit morphism $\eta : 0^* \longrightarrow M$ such that the following associativity and unit diagrams commute:

$$\begin{tikzcd} (M\Box M)\Box M \arrow[r, "\mu\Box M"] \arrow[d, "\cong"', "\alpha_{M,M,M}"] & M\Box M \arrow[d, "\mu"] \\ M\Box(M\Box M) \arrow[r, "M\Box\mu"'] & M\Box M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M\Box 0^* \arrow[r, "M\Box\eta"] \arrow[dr, equal] & M\Box M \arrow[d, "\mu"] & 0^*\Box M \arrow[l, "\eta\Box M"'] \arrow[dl, equal] \\ & M & \end{tikzcd} \tag{3.15}$$

A monoid M is *commutative* if also $\mu \circ \tau_{M,M} = \mu$, i.e., the following triangle commutes:

$$\begin{tikzcd} M\Box M \arrow[d, "\tau_{M,M}"'] \arrow[dr, "\mu"] & \\ M\Box M \arrow[r, "\mu"'] & M \end{tikzcd}$$

A *morphism* of monoids is a morphism $f : R \longrightarrow S$ in $\mathcal{D}^*$ such that $f \circ \mu^R = \mu^S \circ (f\Box f)$ and $f\eta^R = \eta^S$.

Skeleton filtration. For the rest of this section we assume that the index category $\mathcal{D}$ comes with a dimension function $\dim : \mathrm{ob}(\mathcal{D}) \longrightarrow \mathbb{N}$ to the natural numbers, satisfying the following conditions:

(i) If two objects d, e satisfy $\dim(e) < \dim(d)$, then $\mathcal{D}(d, e)$ is an initial object of the base category $\mathcal{V}$.
(ii) If two objects d, e satisfy $\dim(e) = \dim(d)$, then d and e are isomorphic.

The two examples that we care about in this book are:

- $\mathcal{D} = \mathbf{L}$ the topological category of inner product spaces, $\mathcal{V} = \mathbf{T}$, when $\mathcal{D}^* = spc$ is the category of orthogonal spaces;
- $\mathcal{D} = \mathbf{O}$ the Thom space category of the orthogonal complement bundles, $\mathcal{V} = \mathbf{T}_*$, when $\mathcal{D}^* = \mathcal{S}p$ is the category of orthogonal spectra.

In both cases the dimension function is the dimension as an $\mathbb{R}$-vector space.

We denote by $\mathcal{D}_{\le m}$ the full enriched subcategory of $\mathcal{D}$ spanned by all objects of dimension at most m. We denote by $\mathcal{D}^*_{\le m} = (\mathcal{D}_{\le m})^*$ the category of enriched functors from $\mathcal{D}_{\le m}$ to $\mathcal{V}$. The restriction functor

$$\mathcal{D}^* \ \longrightarrow\ \mathcal{D}^*_{\le m}\ , \quad Y \ \longmapsto\ Y^{\le m} \ =\ Y|_{\mathcal{D}^{\le m}}$$

has a left adjoint

$$l_m \ :\ \mathcal{D}^*_{\le m} \ \longrightarrow\ \mathcal{D}^*$$

given by an enriched Kan extension as follows. For every $k \ge 0$ we choose an object $\mathbf{k}$ of $\mathcal{D}$ of dimension k. The extension $l_m(Z)$ of an enriched functor $Z : \mathcal{D}_{\le m} \longrightarrow \mathcal{V}$ is a coequalizer of the two morphisms in $\mathcal{D}^*$

$$\coprod_{0\le j\le k\le m} \mathcal{D}(\mathbf{k}, -) \times \mathcal{D}(\mathbf{j}, \mathbf{k}) \times Z(\mathbf{k}) \underset{V}{\overset{U}{\rightrightarrows}} \coprod_{0\le i\le m} \mathcal{D}(\mathbf{i}, -) \times Z(\mathbf{i}) \longrightarrow l_m(Z) \tag{3.16}$$

The morphism U arises from the composition morphisms

$$\mathcal{D}(\mathbf{k}, -) \times \mathcal{D}(\mathbf{j}, \mathbf{k}) \ \longrightarrow\ \mathcal{D}(\mathbf{j}, -)$$

and the identity on $Z(\mathbf{k})$; the morphism V arises from the action maps

$$\mathcal{D}(\mathbf{j}, \mathbf{k}) \times Z(\mathbf{j}) \ \longrightarrow\ Z(\mathbf{k})$$

and the identity on the represented functor $\mathcal{D}(\mathbf{k}, -)$. Colimits in the functor category $\mathcal{D}^*$ are created objectwise, so the value $l_m(Z)(d)$ at an object d can be calculated by plugging d into the variable slot in the coequalizer diagram (3.16).

It is a general property of Kan extensions along a fully faithful functor (such as the inclusion $\mathcal{D}_{\leq m} \longrightarrow \mathcal{D}$) that the values do not change on the given subcategory; more precisely, the adjunction unit

$$Z \;\longrightarrow\; (l_m Z)^{\leq m}$$

is an isomorphism for every functor $Z : \mathcal{D}_{\leq m} \longrightarrow \mathcal{V}$, see for example [**88**, Prop. 4.23].

Definition 3.17. The *m-skeleton*, for $m \geq 0$, of an enriched functor $Y : \mathcal{D} \longrightarrow \mathcal{V}$ is the functor

$$\mathrm{sk}^m Y \;=\; l_m(Y^{\leq m}) \;,$$

the extension of the restriction of Y to $\mathcal{D}_{\leq m}$. It comes with a natural morphism $i_m : \mathrm{sk}^m Y \longrightarrow Y$, the counit of the adjunction $(l_m, (-)^{\leq m})$. We set $\mathcal{D}(m) = \mathcal{D}(\mathbf{m}, \mathbf{m})$, a $\mathcal{V}$-monoid under composition. The *m-th latching object* of Y is the $\mathcal{D}(m)$-object

$$L_m Y \;=\; (\mathrm{sk}^{m-1} Y)(\mathbf{m}) \;;$$

it comes with a natural $\mathcal{D}(m)$-equivariant morphism

$$\nu_m = i_{m-1}(\mathbf{m}) \;:\; L_m Y \;\longrightarrow\; Y(\mathbf{m}) \;,$$

the *m-th latching morphism.*

We also agree to set $\mathrm{sk}^{-1} Y = \emptyset$, an initial object of $\mathcal{D}^*$; then $L_0 Y = \emptyset$, an initial $\mathcal{V}$-object. The value

$$i_m(d) \;:\; (\mathrm{sk}^m Y)(d) \;\longrightarrow\; Y(d)$$

of this morphism is an isomorphism for all objects d of dimension at most m.

Now we consider $0 \leq l \leq m$. The two morphisms $i_l : \mathrm{sk}^l Y \longrightarrow Y$ and $i_m : \mathrm{sk}^m Y \longrightarrow Y$ both restrict to isomorphisms on $\mathcal{D}^{\leq l}$, so there is a unique morphism $j_{l,m} : \mathrm{sk}^l Y \longrightarrow \mathrm{sk}^m Y$ such that $i_m \circ j_{l,m} = i_l$. These morphisms satisfy

$$j_{l,m} \circ j_{k,l} \;=\; j_{k,m}$$

for all $0 \leq k \leq l \leq m$. The sequence of skeleta stabilizes to Y in a very strong sense. For every object d of $\mathcal{D}$, the morphisms $j_{m,m+1}(d)$ and $i_m(d)$ are isomorphisms as soon as $m \geq \dim(d)$. In particular, $Y(d)$ is a colimit, with respect to the morphisms $i_m(d)$, of the sequence of maps $j_{m,m+1}(d)$, Since colimits in the functor category $\mathcal{D}^*$ are created objectwise, we deduce that Y is a colimit, with respect to the morphisms i_m, of the sequence of morphisms $j_{m,m+1}$.

Example 3.18 (Latching objects of represented functors). Let d be an object of $\mathcal{D}$ of dimension $n = \dim(d)$, and A an object of $\mathcal{V}$. Then the functor $\mathcal{D}(d, -) \otimes A$ is 'purely n-dimensional' in the following sense. The evaluation functor

$$\mathrm{ev}_d \;:\; \mathcal{D}^* \;\longrightarrow\; \mathcal{V}$$

factors through the category $\mathcal{D}_{\leq n}$ as the composite

$$\mathcal{D}^* \;\longrightarrow\; \mathcal{D}_{\leq n} \;\xrightarrow{\mathrm{ev}_d}\; \mathcal{V} \;.$$

So the left adjoint free functor $A \mapsto \mathcal{D}(d, -) \otimes A$ can be chosen as the composite of the two individual left adjoints

$$\mathcal{D}(d, -) \otimes - \;=\; l_n \circ (\mathcal{D}_{\leq n}(d, -) \otimes -) \;.$$

The object $\mathcal{D}(d, e) \otimes A$ is initial for $\dim(e) < n$, by hypothesis on the dimension function, and hence the latching object $L_m(\mathcal{D}(d, -) \otimes A)$ is initial for $m \leq n$. For $m > n$ the latching morphism $\nu_m : L_m(\mathcal{D}(d, -) \otimes A) \longrightarrow \mathcal{D}(d, \mathbf{m}) \otimes A$ is an isomorphism. So the skeleton $\mathrm{sk}^m(\mathcal{D}(d, -) \otimes A)$ is initial for $m < n$ and $\mathrm{sk}^m(\mathcal{D}(d, -) \otimes A) = \mathcal{D}(d, -) \otimes A$ is the entire functor for $m \geq n$.

We denote by

$$G_m \;:\; \mathcal{D}(m)\mathcal{V} \;\longrightarrow\; \mathcal{D}^*$$

the left adjoint to the functor $Y \mapsto Y(\mathbf{m})$. As a special case [...a consequence?...] of the previous example, the orthogonal space $G_m A = \mathcal{D}(\mathbf{m}, -) \otimes_{\mathcal{D}(m)} A$ is purely m-dimensional for every $\mathcal{D}(m)$-object A.

PROPOSITION 3.19. *For every enriched functor $Y : \mathcal{D} \longrightarrow \mathcal{V}$ and every $m \geq 0$ the commutative square*

$$\begin{tikzcd} G_m L_m Y \arrow[r, "G_m \nu_m"] \arrow[d] & G_m Y(\mathbf{m}) \arrow[d] \\ \mathrm{sk}^{m-1} Y \arrow[r, "j_{m-1,m}"'] & \mathrm{sk}^m Y \end{tikzcd} \tag{3.20}$$

is a pushout in the category $\mathcal{D}^*$. The two vertical morphisms are adjoint to the identity of $L_m Y$ respectively $Y(\mathbf{m})$.

PROOF. All four functors are 'm-dimensional', i.e., isomorphic to the extensions of their restrictions to $\mathcal{D}_{\leq m}$; for $G_m L_m Y$ and $G_m Y(\mathbf{m})$ this follows from Example 3.18. So it suffices to check that the square is a pushout of spaces when evaluated at objects d of dimension at most m. When the dimension of d is strictly less than m, then $(G_m L_m Y)(d)$ and $(G_m Y_m)(d)$ are initial objects and the map $j_{m-1,m}(V) : (\mathrm{sk}^{m-1} Y)(d) \longrightarrow (\mathrm{sk}^m Y)(d)$ is an isomorphism (both source and target are isomorphic to $Y(d)$). So for $\dim(d) < m$ both horizontal maps are isomorphisms at d. When $\dim(d) = m$, then d is isomorphic to $\mathbf{m}$, so we may suppose that $d = \mathbf{m}$. Then the square (3.20) evaluates to

$$\begin{tikzcd} \mathcal{D}(\mathbf{m},\mathbf{m}) \otimes_{\mathcal{D}(m)} (L_m Y) \arrow[r, "\mathcal{D}(\mathbf{m},\mathbf{m}) \otimes_{\mathcal{D}(m)} \nu_m"] \arrow[d] & \mathcal{D}(\mathbf{m},\mathbf{m}) \otimes_{\mathcal{D}(m)} Y(\mathbf{m}) \arrow[d] \\ L_m Y \arrow[r, "j_{m-1,m}(\mathbf{m})"'] & Y(\mathbf{m}) \end{tikzcd}$$

So for $\dim(d) = m$, both vertical maps are isomorphisms at d. □

EXAMPLE 3.21. As an illustration of the definition, we describe the skeleta and latching objects for small values of m. By hypothesis, all objects of dimension 0 are isomorphic, so $\mathcal{D}_{\leq 0}$ is equivalent to the category with one object $\mathbf{0}$ and endomorphism monoid $\mathcal{D}(\mathbf{0},\mathbf{0})$. If X is an object of $\mathcal{V}$ equipped with a left action of the $\mathcal{V}$-monoid $\mathcal{D}(\mathbf{0},\mathbf{0})$, then the left Kan extension $l_0 : \mathcal{D}^*_{\leq 0} \longrightarrow \mathcal{D}^*$ takes X to the enriched functor whose value at $d \in \mathcal{D}$ is the coend

$$(l_0 X)(d) \;=\; \mathcal{D}(\mathbf{0}, d) \otimes_{\mathcal{D}(\mathbf{0},\mathbf{0})} X \;.$$

So in particular, the 0-skeleton of a $\mathcal{D}$-functor Y is given by

$$\mathrm{sk}^0 Y \;=\; \mathcal{D}(\mathbf{0}, -) \otimes_{\mathcal{D}(\mathbf{0},\mathbf{0})} Y(\mathbf{0}) \;.$$

The latching morphism

$$\nu_1 \;:\; L_1 Y \;=\; (\mathrm{sk}^0 Y)(\mathbf{1}) \;=\; \mathcal{D}(\mathbf{0},\mathbf{1}) \otimes_{\mathcal{D}(\mathbf{0},\mathbf{0})} Y(\mathbf{0}) \;\longrightarrow\; Y(\mathbf{1})$$

is induced by the morphism $\circ : \mathcal{D}(\mathbf{0},\mathbf{1}) \otimes Y(\mathbf{0}) \longrightarrow Y(\mathbf{1})$ which is part of the functoriality of Y.

We make this formula more explicit in the two cases we mostly care about, namely for $\mathcal{D} = \mathbf{L}$ and $\mathcal{D} = \mathbf{O}$. In those cases the monoid $\mathcal{D}(\mathbf{0},\mathbf{0})$ is the initial $\mathcal{V}$-monoid (the monoidal unit in $\mathcal{V}$); so in those cases an action of $\mathcal{D}(\mathbf{0},\mathbf{0})$ is no additional data. Moreover, for $\mathcal{D} = \mathbf{L}$, the orthogonal space $\mathbf{L}(0,-)$ is constant with value one point, so for orthogonal space the situation specializes to

$$\mathrm{sk}^0 Y \;=\; \mathrm{const}(Y(\mathbf{0})) \;,$$

the constant functor with value $Y(0)$. For $\mathcal{D} = \mathbf{O}$, the orthogonal spectrum $\mathbf{O}(0,-)$ is the sphere spectrum, so for orthogonal spectra the formula specializes to

$$\mathrm{sk}^0 Y \;=\; \Sigma^\infty Y(0) \;,$$

the suspension spectrum of the based space $Y(0)$.

Now we evaluate the pushout square (3.20) for $m = 1$ at an object d; the results is a pushout square of $\mathcal{D}(\mathbf{1},\mathbf{1})$-objects

$$\begin{tikzcd}
\mathcal{D}(\mathbf{1},d) \times_{\mathcal{D}(\mathbf{1},\mathbf{1})} \times(\mathcal{D}(\mathbf{0},\mathbf{1}) \times_{\mathcal{D}(\mathbf{0},\mathbf{0})} Y(\mathbf{0})) \arrow[r] \arrow[d, "\mathrm{proj}"'] & \mathcal{D}(\mathbf{1},d) \times_{\mathcal{D}(\mathbf{1},\mathbf{1})} Y(\mathbf{1}) \arrow[d] \\
\mathcal{D}(\mathbf{0},\mathbf{d}) \times_{\mathcal{D}(\mathbf{0},\mathbf{0})} Y(\mathbf{0}) \arrow[r] & (\mathrm{sk}^1 Y)(d)
\end{tikzcd}$$

Given any morphism $f : A \longrightarrow B$ of enriched functors in $\mathcal{D}^*$, we define a relative skeleton filtration as follows. We have a commutative square of enriched functors

$$\begin{tikzcd}
\mathrm{sk}^m A \arrow[r, "\mathrm{sk}^m f"] \arrow[d, "i_m"'] & \mathrm{sk}^m B \arrow[d, "i_m"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

The *relative m-skeleton* of f is the pushout

$$\mathrm{sk}^m[f] \;=\; A \cup_{\mathrm{sk}^m A} \mathrm{sk}^m B \;; \tag{3.22}$$

it comes with a unique morphism $i_m : \mathrm{sk}^m[f] \longrightarrow B$ which restricts to $f : A \longrightarrow B$ respectively to $i_m : \mathrm{sk}^m B \longrightarrow B$.

Since $L_m A = (\mathrm{sk}^{m-1} A)(\mathbf{m})$ we have

$$(\mathrm{sk}^{m-1}[f])(\mathbf{m}) \;=\; A(\mathbf{m}) \cup_{L_m A} L_m B \;,$$

the m-th relative latching object. A morphism $j_m[f] : \mathrm{sk}^{m-1}[f] \longrightarrow \mathrm{sk}^m[f]$ is obtained from the commutative diagram

$$\begin{tikzcd}
A \arrow[d, equal] & \mathrm{sk}^{m-1} A \arrow[l, "i_{m-1}"'] \arrow[r, "\mathrm{sk}^{m-1} f"] \arrow[d, "j^A_{m-1,m}"'] & \mathrm{sk}^{m-1} B \arrow[d, "j^B_{m-1,m}"] \\
A & \mathrm{sk}^m A \arrow[l, "i_m"] \arrow[r, "\mathrm{sk}^m f"'] & \mathrm{sk}^m B
\end{tikzcd}$$

by taking pushouts. The square

$$\begin{tikzcd}
G_m(A_m \cup_{L_m A} L_m B) \arrow[r, "G_m(\nu_m f)"] \arrow[d] & G_m B_m \arrow[d] \\
\mathrm{sk}^{m-1}[f] \arrow[r, "j_m[f]"'] & \mathrm{sk}^m[f]
\end{tikzcd} \tag{3.23}$$

is a pushout [deduce this from Proposition 3.19] and the original morphism $f : A \longrightarrow B$ factors as the composite of the countable sequence

$$A = \mathrm{sk}^{-1}[f] \xrightarrow{j_0[f]} \mathrm{sk}^0[f] \xrightarrow{j_1[f]} \mathrm{sk}^1[f] \longrightarrow \cdots \xrightarrow{j_m[f]} \mathrm{sk}^m[f] \longrightarrow \cdots \;. \tag{3.24}$$

If d has dimension n, then the sequence stabilizes to the identity map of $B(d)$ from $(\mathrm{sk}^n[f])(d)$ on; in particular, the compatible morphisms $j_m : \mathrm{sk}^m[f] \longrightarrow B$ exhibit B as a colimit of the sequence.

The following proposition is an immediate application of the relative skeleton filtration. It is the key ingredient to the lifting properties of the various level model structures in this book. We recall that a pair $(i : A \longrightarrow B, f : X \longrightarrow Y)$ of morphisms in some category has the *lifting property* if for all morphism $\varphi : A \longrightarrow X$ and $\psi : B \longrightarrow Y$ such that $f\varphi = \psi i$ there exists a *lifting*, i.e., a morphism $\lambda : B \longrightarrow Y$ such

that $\lambda i = \varphi$ and $f\lambda = \psi$. Instead of saying that the pair (i, f) has the lifting property we may equivalently say 'i has the left lifting property with respect to f' or 'f has the right lifting property with respect to i'.

Given any morphism $f : A \longrightarrow B$ of enriched functors in $\mathcal{D}^*$ and $m \geq 0$, we have a commutative square of $\mathcal{D}(m)$-objects

$$\begin{tikzcd} L_m A \arrow[r, "L_m f"] \arrow[d, "\nu_m"'] & L_m B \arrow[d, "\nu_m"] \\ A(\mathbf{m}) \arrow[r, "f(\mathbf{m})"'] & B(\mathbf{m}) \end{tikzcd}$$

PROPOSITION 3.25. *Let $i : A \longrightarrow B$ and $f : X \longrightarrow Y$ be morphisms of enriched functors in $\mathcal{D}^*$. If the pair*

$$(\nu_m i : A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow B(\mathbf{m}), \quad f(\mathbf{m}) : X(\mathbf{m}) \longrightarrow Y(\mathbf{m}))$$

has the lifting property in the category of $\mathcal{D}(m)$-objects for every $m \geq 0$, then the pair (i, f) has the lifting property in the functor category $\mathcal{D}^*$.

PROOF. We consider the class f-cof of all morphisms in $\mathcal{D}^*$ that have the left lifting property with respect to f; this class is closed under cobase change and countable composition. Since the pair $(\nu_m i, f_m)$ has the lifting property in the category of $\mathcal{D}(m)$-objects, the morphism $G_m(\nu_m i)$ belongs to the class f-cof by adjointness. The relative skeleton filtration (3.22) shows that i is a countable composite of cobase changes of the morphisms $G_m(\nu_m i)$, so i belongs to the class f-cof. □

Now we discuss a general recipe for constructing level model structures on the functor category $\mathcal{D}^*$. As input we need, for every $m \geq 0$, a model structure $\mathcal{C}(m)$ on the category of $\mathcal{D}(m)$-objects. We call a morphism $f : X \longrightarrow Y$ in $\mathcal{D}^*$

- a *level equivalence* if $f(\mathbf{m}) : X(\mathbf{m}) \longrightarrow Y(\mathbf{m})$ is a weak equivalence in the model structure $\mathcal{C}(m)$ for all $m \geq 0$;
- a *level fibration* if the morphism $f(\mathbf{m}) : X(\mathbf{m}) \longrightarrow Y(\mathbf{m})$ is a fibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$;
- a *cofibration* if the latching morphism $\nu_m f : X(\mathbf{m}) \cup_{L_m X} L_m Y \longrightarrow Y(\mathbf{m})$ is a cofibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$.

Proposition 3.27 below shows that if the various model structures $\mathcal{C}(m)$ satisfy the following 'consistency condition', then the level equivalences, level fibrations and cofibrations define a model structure on the functor category $\mathcal{D}^*$.

DEFINITION 3.26 (Consistency condition). For all $m, n \geq 0$ and every acyclic cofibration $i : A \longrightarrow B$ in the model structure $\mathcal{C}(m)$ on $\mathcal{D}(m)$-objects, every cobase change, in the category of $\mathcal{D}(m+n)$-objects, of the morphism

$$\mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \times_{\mathcal{D}(m)} i \ : \ \mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \times_{\mathcal{D}(m)} A \ \longrightarrow \ \mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \times_{\mathcal{D}(m)} B$$

is a weak equivalence in the model structure $\mathcal{C}(m+n)$.

PROPOSITION 3.27. *Let $\mathcal{C}(m)$ be a model structure on the category of $\mathcal{D}(m)$-objects, for $m \geq 0$, such that the consistency condition holds.*

(i) *The classes of level equivalences, level fibrations and cofibrations define a model structure on the functor category $\mathcal{D}^*$.*
(ii) *A morphism $i : A \longrightarrow B$ in $\mathcal{D}^*$ is simultaneously a cofibration and a level equivalence if and only if for all $m \geq 0$ the latching morphism $\nu_m i : A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow B(\mathbf{m})$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$.*

(iii) *Suppose that the fibrations in the model structure $\mathcal{C}(m)$ are detected by a set of morphisms $J(m)$; then the level fibrations of orthogonal spaces are detected by the set of morphisms*

$$\{G_m j \mid m \geq 0,\, j \in J(m)\}\ .$$

Similarly, if the acyclic fibrations in the model structure $\mathcal{C}(m)$ are detected by a set of morphisms $I(m)$, then the level acyclic fibrations of orthogonal spaces are detected by the set of morphisms

$$\{G_m i \mid m \geq 0,\, i \in I(m)\}\ .$$

Proof. We start by showing one of the directions of part (ii): we let $i : A \longrightarrow B$ be a morphism such that the latching morphism $\nu_m i : A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow B(\mathbf{m})$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$; we show that then i is a level equivalence.

The map $i(\mathbf{n}) : A(\mathbf{n}) \longrightarrow B(\mathbf{n})$ is the finite composite

$$A(\mathbf{n}) = (\mathrm{sk}^{-1}[i])(\mathbf{n}) \xrightarrow{(j_0[i])(\mathbf{n})} (\mathrm{sk}^0[i])(\mathbf{n}) \xrightarrow{(j_1[i])(\mathbf{n})} \dots \xrightarrow{(j_{n-1}[i])(\mathbf{n})} (\mathrm{sk}^{n-1}[i])(\mathbf{n}) \xrightarrow{(j_n[i])(\mathbf{n})} (\mathrm{sk}^n[i])(\mathbf{n}) = B(\mathbf{n})\ ,$$

so it suffices to show that $j_k[i]$ is a level equivalence for all $k \geq 0$. The pushout square (3.23) in level $m+n$ is a pushout of $\mathcal{D}(m+n)$-objects

$$\begin{tikzcd}
\mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \times_{\mathcal{D}(m)} (A(\mathbf{m}) \cup_{L_m A} L_m B) \arrow[r, "\mathcal{D}(\mathbf{m},\mathbf{m}+\mathbf{n}) \times_{\mathcal{D}(m)} (\nu_m i)"] \arrow[d] & \mathcal{D}(\mathbf{m}, \mathbf{m}+\mathbf{n}) \times_{\mathcal{D}(m)} B(\mathbf{m}) \arrow[d] \\
(\mathrm{sk}^{m-1}[i])(\mathbf{m}+\mathbf{n}) \arrow[r, "(j_m[i])(\mathbf{m}+\mathbf{n})"'] & (\mathrm{sk}^m[i])(\mathbf{m}+\mathbf{n})
\end{tikzcd}$$

The consistency condition guarantees that the lower horizontal morphism is a $\mathcal{C}(m+n)$-weak equivalence.

(i) Several of the axioms are straightforward: the functor category $\mathcal{D}^*$ inherits all small limits and colimits and colimits from the base category $\mathcal{V}$; the level equivalences satisfy the 2-out-of-3 property; the classes of level equivalences, cofibrations and fibrations are closed under retracts.

Now we prove the factorization axiom, i.e., we show that every morphism $f : A \longrightarrow X$ of orthogonal spaces can be factored as $f = qi$ where q is a level acyclic fibration and i a cofibration; and it can be factored as $f = pj$ where p is a level fibration and j a cofibration and level equivalence. We start with the first factorization and construct an enriched functor B and morphisms $i : A \longrightarrow B$ and $q : B \longrightarrow X$ by induction over the dimension. [...factor successively in $\mathcal{D}^*_{\leq m}$...] In level 0 we choose a factorization

$$A(\mathbf{0}) \xrightarrow{i(\mathbf{0})} B(\mathbf{0}) \xrightarrow{q(\mathbf{0})} X(\mathbf{0})$$

of $f(\mathbf{0})$ in the category of $\mathcal{D}(0)$-objects such that $i(\mathbf{0})$ is a cofibration and $q(\mathbf{0})$ is an acyclic fibration in the model structure $\mathcal{C}(0)$. Now we suppose that the enriched functor B and the morphisms i and q have already been constructed up to dimension $m-1$. Then we have all the data necessary to define the m-th latching object $L_m B$; moreover, the 'partial morphism' $q : B \longrightarrow X$ provides a $\mathcal{D}(m)$-morphism $L_m B \longrightarrow X(\mathbf{m})$ such that the square

$$\begin{tikzcd}
L_m A \arrow[r, "L_m i"] \arrow[d, "\nu_m"'] & L_m B \arrow[d] \\
A(\mathbf{m}) \arrow[r, "f(\mathbf{m})"'] & X(\mathbf{m})
\end{tikzcd}$$

commutes. We factor the resulting morphism $A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow X(\mathbf{m})$ in the category of $Dc(m)$-objects

$$A(\mathbf{m}) \cup_{L_m A} L_m B \xrightarrow{\nu_m i} B(\mathbf{m}) \xrightarrow{q(\mathbf{m})} X(\mathbf{m}) \tag{3.28}$$

such that $\nu_m i$ is a cofibration and $q(\mathbf{m})$ is an acyclic fibration in the model structure $\mathcal{C}(m)$. The intermediate $\mathcal{D}(m)$-object $B(\mathbf{m})$ defines the value of B at the object $\mathbf{m}$, and the second morphism $q(\mathbf{m})$ is the m-th level of the morphism q. For $0 \leq i < m$, the structure morphism is the composite

$$\mathcal{D}(\mathbf{i},\mathbf{m}) \times B(\mathbf{i}) \;\longrightarrow\; L_m B \;\longrightarrow\; A(\mathbf{m}) \cup_{L_m A} L_m B \;\xrightarrow{\nu_m i}\; B(\mathbf{m})$$

and the composite of $\nu_m i$ with the canonical morphism $A(\mathbf{m}) \longrightarrow A(\mathbf{m}) \cup_{L_m A} L_m B$ is the m-th level of the morphism i.

At the end of the day we have indeed factored $f = qi$ in the category $\mathcal{D}^*$ such that q is a level equivalence and level fibration. Moreover, the m-th latching morphism comes out to be the morphism $\nu_m i : A(\mathbf{m}) \cup_{L_m A} L_m B \longrightarrow B(\mathbf{m})$ in the factorization (3.28), which is a cofibration in the model structure $\mathcal{C}(m)$. So the morphism i is indeed a cofibration.

The second factorization $f = pj$ as a cofibration and level equivalence j followed by a level fibration p is similar, but instead of the factorization (3.28) we use a factorization, in the model category $\mathcal{C}(m)$, as an acyclic cofibration followed by a fibration. Then the resulting morphism p is a level fibration and the morphism j has the property that all its latching morphisms $\nu_m j$ are acyclic cofibrations. So j is a cofibration (by definition) and a level equivalence (by the part of (ii) established above).

It remains to show the lifting axioms. In each of the model structures $\mathcal{C}(m)$ the cofibrations have the left lifting property with respect to the acyclic fibrations; so by Proposition 3.25 the cofibrations in $\mathcal{D}^*$ have the left lifting property with respect to level equivalences which are also level fibrations.

We postpone the proof of the other lifting property and prove the remaining direction of (ii) next. We let $i : A \longrightarrow B$ be a cofibration and a level equivalence. The second factorization axiom proved above provides a factorization $i = pj$ where $j : A \longrightarrow D$ is a level equivalence such that each latching morphism $\nu_m j$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$, and $p : D \longrightarrow B$ is a fibration in $\mathcal{C}(m)$. Since i and j are level equivalences, so is p. So the cofibration i has the left lifting property with respect to the level equivalence and level fibration p by the previous paragraph. In particular, a lift $\lambda : B \longrightarrow D$ in the square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & D \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dotted, "\lambda"] & B \end{tikzcd}$$

shows that the morphism i is a retract of the morphism j. So the latching morphism $\nu_n i$ is a retract of the latching morphism $\nu_n j$, hence also an acyclic cofibration in the model structure $\mathcal{C}(m)$. This proves (ii).

Now we prove the remaining half of the lifting properties. We let $i : A \longrightarrow B$ be a cofibration that is also a level equivalence. By (ii), which has just been shown, each latching morphism $\nu_m i$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$. So i has the left lifting property with respect to all level fibrations by Proposition 3.25.

Property (iii) is a straightforward consequence of the fact that the functor G_m is left adjoint to evaluation at $\mathbf{m}$. $\square$

Bibliography

[1] J. F. Adams, J.-P. Haeberly, S. Jackowski, J. P. May, *A generalization of the Atiyah-Segal completion theorem.* Topology **27** (1988), no. 1, 1–6.

[2] J. F. Adams, *Prerequisites (on equivariant stable homotopy) for Carlsson's lecture.* Algebraic topology, Aarhus 1982, 483–532, Lecture Notes in Math. 1051, 1984.

[3] M. A. Aguilar, C. Prieto, *Quasifibrations and Bott periodicity.* II Iberoamerican Conference on Topology and its Applications (Morelia, 1997). Topology Appl. **98** (1999), no. 1-3, 3–17.

[4] J. C. Alexander, *The bordism ring of manifolds with involution.* Proc. Amer. Math. Soc. **31** (1972) 536–542.

[5] L. Alonso Tarrío, A. Jeremías López, M. J. Souto Salorio, *Construction of t-structures and equivalences of derived categories.* Trans. Amer. Math. Soc. 355 (2003), no. 6, 2523–2543.

[6] M. Aschbacher, R. Kessar, B. Oliver, *Fusion systems in algebra and topology.* London Mathematical Society Lecture Note Series, 391. Cambridge University Press, Cambridge, 2011. vi+320 pp.

[7] M. F. Atiyah, *Bott periodicity and the index of elliptic operators.* Quart. J. Math. Oxford Ser. (2) **19** (1968), 113–140.

[8] M. F. Atiyah, R. Bott, A. Shapiro, *Clifford modules.* Topology **3** (1964) suppl. 1, 3–38.

[9] M. F. Atiyah, G. B. Segal, *Equivariant K-theory and completion.* J. Differential Geometry **3** (1969), 1–18.

[10] H. Bass, *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam 1968 xx+762 pp.

[11] J. C. Becker, D. H. Gottlieb, *The transfer map and fiber bundles.* Topology **14** (1975), 1–12.

[12] R. P. Beem, *Generators for G bordism.* Proc. Amer. Math. Soc. **67** (1977), no. 2, 335–343.

[13] R. P. Beem, *The structure of $Z/4$ bordism.* Indiana Univ. Math. J. **27** (1978), no. 6, 1039–1047.

[14] R. B. Beem, R. J. Rowlett, *The fixed point homomorphism for maps of period 2^k.* Indiana Univ. Math. J. **30** (1981), no. 4, 489–500.

[15] M. J. Behrens, *A new proof of the Bott periodicity theorem.* Topology Appl. **119** (2002), no. 2, 167–183.

[16] A. A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers.* Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[17] A. L. Besse, *Einstein manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **10**. Springer-Verlag, Berlin, 1987. xii+510 pp.

[18] J. M. Boardman, *On Stable Homotopy Theory and Some Applications.* PhD thesis, University of Cambridge (1964)

[19] J. M. Boardman, R. M. Vogt, *Homotopy-everything H-spaces.* Bull. Amer. Math. Soc. **74** (1968), 1117–1122.

[20] J. M. Boardman, R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973. x+257 pp.

[21] A. M. Bohmann, *Topics in equivariant stable homotopy theory.* PhD thesis, University of Chicago, 2011.

[22] A. M. Bohmann, *Global orthogonal spectra.* Homology, Homotopy and Applications **16** (2014), no. 1, 313–332.

[23] R. Boltje, *A canonical Brauer induction formula.* Astérisque No. 181-182 (1990) **5**, 31–59.

[24] R. Boltje, V. Snaith, P. Symonds, *Algebraicisation of explicit Brauer induction.* J. Algebra **148** (1992), no. 2, 504–527.

[25] F. Borceux, *Handbook of categorical algebra. 2.* Categories and structures. Encyclopedia of Mathematics and its Applications, 51. Cambridge University Press, Cambridge, 1994. xviii+443 pp

[26] R. Brauer, *On Artin's L-series with general group characters.* Ann. of Math. (2) **48** (1947), 502–514.

[27] G. E. Bredon, *Introduction to compact transformation groups.* Pure and Applied Mathematics, Vol. 46. Academic Press, New York-London, 1972. xiii+459 pp.

[28] T. Bröcker, E. C. Hook, *Stable equivariant bordism.* Math. Z. **129** (1972), 269–277.

[29] T. Bröcker, T. tom Dieck, *Representations of compact Lie groups.* Graduate Texts in Math., Vol. 98, Springer-Verlag, New York, 1985.

[30] K. S. Brown, *Abstract homotopy theory and generalized sheaf cohomology.* Trans. Amer. Math. Soc. **186** (1974), 419–458.

[31] M. Brun, *Witt vectors and equivariant ring spectra applied to cobordism.* Proc. London Math. Soc. (3) **94** (2007), 351–385.

[32] M. Brun, B. I. Dundas, M. Stolz, *Equivariant structure on smash powers.* `arXiv:1604.05939`

[33] R. Bruner, J. Greenlees, *The connective K-theory of finite groups.* Mem. Amer. Math. Soc. **165** (2003), no. 785, viii+127 pp.

[34] R. Bruner, J. P. May, J. McClure, M. Steinberger, *H_∞ ring spectra and their applications.* Lecture Notes in Mathematics, 1176. Springer-Verlag, Berlin, 1986. viii+388 pp.

[35] G. Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture.* Ann. of Math. (2) **120** (1984), no. 2, 189–224.

[36] J. D. Christensen, M. Hovey, *Quillen model structures for relative homological algebra.* Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 2, 261–293.

[37] M. Cole, *Mixing model structures.* Topology Appl. **153** (2006), no. 7, 1016–1032.

[38] P. E. Conner, E. E. Floyd, *Differentiable periodic maps.* Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33 Academic Press Inc., Publishers, New York; Springer-Verlag, 1964 vii+148 pp.

[39] S. R. Costenoble, *Equivariant cobordism and K-theory.* PhD thesis, University of Chicago, 1985.

[40] B. Day, *On closed categories of functors.* 1970 Reports of the Midwest Category Seminar, IV pp. 1–38. Lecture Notes in Mathematics, Vol. 137. Springer, Berlin

[41] I. Dell'Ambrogio, H. Emerson, T. Kandelaki, R. Meyer, *A functorial equivariant K-theory spectrum and an equivariant Lefschetz formula.* `arXiv:1104.3441`

[42] A. Dold, *Lectures on algebraic topology.* Second edition. Grundlehren der Mathematischen Wissenschaften, 200. Springer-Verlag, Berlin-New York, 1980. xi+377 pp.

[43] A. Dress, *Notes on the theory of representations of finite groups.* Duplicated notes, Bielefeld, 1971.

[44] E. Dror, W. G. Dwyer, D. M. Kan, *An arithmetic square for virtually nilpotent spaces.* Illinois J. Math.**21** (1977), no. 2, 242–254.

[45] D. Dugger, *An Atiyah-Hirzebruch spectral sequence for KR-theory.* K-theory **35** (2005), 213–256.

[46] D. Dugger, D. C. Isaksen, *Topological hypercovers and $\mathbb{A}^1$-realizations.* Math. Z. **246** (2004), no. 4, 667–689.

[47] W. G. Dwyer, J. Spalinski, *Homotopy theories and model categories.* In: Handbook of algebraic topology, ed. I. M. James, Elsevier (1995), 73–126.

[48] A. D. Elmendorf, M. A. Mandell, *Rings, modules, and algebras in infinite loop space theory.* Adv. Math. **205** (2006), no. 1, 163–228.

[49] A. D. Elmendorf, I. Kriz, M. A. Mandell, J. P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole*, Mathematical Surveys and Monographs, **47**, American Mathematical Society, Providence, RI, 1997, xii+249 pp.

[50] H. Fausk, *Equivariant homotopy theory for pro-spectra.* Geometry & Topology **12** (2008), 103–176.

[51] M. Feshbach, *The transfer and compact Lie groups.* Trans. Amer. Math. Soc. **251** (1979), 139–169.

[52] M. Firsching, *Real equivariant bordism for elementary abelian 2-groups.* Homology Homotopy Appl. **15** (2013), no. 1, 235–251.

[53] B. Fresse, *Modules over operads and functors.* Lecture Notes in Mathematics, 1967. Springer-Verlag, Berlin, 2009. x+308 pp.

[54] E. M. Friedlander, B. Mazur, *Filtrations on the homology of algebraic varieties.* With an appendix by Daniel Quillen. Mem. Amer. Math. Soc. **110** (1994), no. 529, x+110 pp.

[55] R. Fritsch, R. Piccinini, *Cellular structures in topology.* Cambridge Studies in Advanced Mathematics, 19. Cambridge University Press, Cambridge, 1990. xii+326 pp.

[56] S. Galatius, U. Tillmann, I. Madsen, M. Weiss, *The homotopy type of the cobordism category.* Acta Math. **202** (2009), no. 2, 195–239.

[57] N. Ganter, *Global Mackey functors with operations and n-special lambda rings.* `arXiv:1301.4616`

[58] D. Gepner, A. Henriques, *Homotopy theory of orbispaces.* `arXiv:math.AT/0701916`

[59] P. Goerss, M. Hopkins, *Moduli spaces of commutative ring spectra.* Structured ring spectra, 151–200, London Math. Soc. Lecture Note Ser., 315, Cambridge Univ. Press, Cambridge, 2004.

[60] S. Gorchinskiy, V. Guletskiĭ, *Symmetric powers in abstract homotopy categories.* Adv. Math. **292** (2016), 707–754.

[61] B. Gray, *Homotopy theory. An introduction to algebraic topology.* Pure and Applied Mathematics, Vol. 64. Academic Press, 1975. xiii+368 pp.

[62] J. A. Green, *Axiomatic representation theory for finite groups.* J. Pure Appl. Algebra **1** (1971), 41–77.

[63] J. P. C. Greenlees, *Equivariant connective K-theory for compact Lie groups.* J. Pure Appl. Algebra **187** (2004) 129–152.

[64] J. P. C. Greenlees, J. P. May, *Localization and completion theorems for MU-module spectra.* Ann. of Math. (2) **146** (1997), 509–544.

[65] J. P. C. Greenlees, J. P. May, *Generalized Tate cohomology.* Mem. Amer. Math. Soc. **113** (1995), no. 543, viii+178 pp.

[66] E. Guentner, N. Higson, J. Trout, *Equivariant E-theory for C^*-algebras.* Mem. Amer. Math. Soc. **148** (2000), no. 703, viii+86 pp.

[67] A. Haar, *Der Massbegriff in der Theorie der kontinuierlichen Gruppen.* Ann. of Math. (2) **34** (1933), no. 1, 147–169.

[68] A. Hatcher, *Algebraic topology.* Cambridge University Press, Cambridge, 2002. xii+544 pp.

[69] A. Hatcher, *Vector bundles and K-theory.* Book, available from the authors web page.

[70] H. Hauschild, *Bordismentheorie stabil gerahmter G-Mannigfaltigkeiten.* Math. Z. **139** (1974), 165–171.

[71] M. Hausmann, *Global equivariant homotopy theory of symmetric spectra.* Master thesis, Universität Bonn, 2013.

[72] M. Hausmann, *Symmetric spectra model global homotopy theory of finite groups.* `arXiv:1509.09270`

[73] M. Hausmann, D. Ostermayr, *Filtrations of global equivariant K-theory.* `arXiv:1510.04011`

[74] N. Higson, G. Kasparov, J. Trout, *A Bott periodicity theorem for infinite-dimensional Euclidean space.* Adv. Math. **135** (1998), no. 1, 1–40.

[75] N. Higson, E. Guentner, *Group C^*-algebras and K-theory.* Noncommutative geometry, 137–251, Lecture Notes in Math., 1831, Springer, Berlin, 2004.

[76] M. Hill, M. Hopkins, D. Ravenel, *On the nonexistence of elements of Kervaire invariant one.* Ann. of Math. **184** (2016), 1–262.

[77] P. S. Hirschhorn, *Model categories and their localizations.* Mathematical Surveys and Monographs, 99. American Mathematical Society, Providence, RI, 2003. xvi+457 pp.

[78] H. Hohnhold, S. Stolz, P. Teichner, *From minimal geodesics to supersymmetric field theories.* A celebration of the mathematical legacy of Raoul Bott, 207–274, CRM Proc. Lecture Notes 50, Amer. Math. Soc., Providence, RI, 2010.

[79] M. Hovey, *Model categories.* Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999, xii+209 pp.

[80] M. Hovey, B. Shipley, J. Smith, *Symmetric spectra.* J. Amer. Math. Soc. **13** (2000), 149–208.

[81] W. Hurewicz, *Beiträge zur Topologie der Deformationen. IV. Asphärische Räume.* Proc. Kon. Akad. Wet. Amsterdam **39** (1936), 215–224.

[82] S. Illman, *The equivariant triangulation theorem for actions of compact Lie groups.* Math. Ann. **262** (1983), 487–501.

[83] S. Illman, *Restricting the transformation group in equivariant CW complexes.* Osaka J. Math. 27 (1990), no. 1, 191–206.

[84] M. Joachim, *K-homology of C^*-categories and symmetric spectra representing K-homology.* Math. Ann. **327** (2003), no. 4, 641–670.

[85] M. Joachim, *Higher coherences for equivariant K-theory.* Structured ring spectra, 87–114, London Math. Soc. Lecture Note Ser., 315, Cambridge Univ. Press, Cambridge, 2004.

[86] M. Karoubi, *Sur la K-théorie équivariante.* 1970 Séminaire Heidelberg-Saarbrücken-Strasbourg sur la K-théorie (1967/68) 187–253. Lecture Notes in Mathematics, Vol. 136 Springer, Berlin

[87] J. L. Kelley, *General topology.* D. Van Nostrand Company, Inc., Toronto-New York-London, 1955. xiv+298 pp.

[88] G. M. Kelly, *Basic concepts of enriched category theory.* Reprint of the 1982 original. Repr. Theory Appl. Categ. No. 10 (2005), vi+137 pp.

[89] C. Kosniowski, *A note on $RO(G)$ graded G bordism theory.* Quart. J. Math. Oxford Ser. (2) **26** (1975), no. 104, 411–419.

[90] H. Krause, *A Brown representability theorem via coherent functors.* Topology **41** (2002), no. 4, 853–861.

[91] I. Kriz, *The Z/p-equivariant complex cobordism ring.* Homotopy invariant algebraic structures (Baltimore, MD, 1998), 217–223, Contemp. Math., 239, Amer. Math. Soc., Providence, RI, 1999.

[92] R. K. Lashof, *Equivariant bundles.* Illinois J. Math. **26** (1982), no. 2, 257–271.

[93] R. K. Lashof, J. P. May, G. B. Segal, *Equivariant bundles with abelian structural group.* Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982), 167–176, Contemp. Math., 19, Amer. Math. Soc., Providence, RI, 1983.

[94] L. G. Lewis, Jr., *The stable category and generalized Thom spectra.* Ph.D. thesis, University of Chicago, 1978

[95] L. G. Lewis, Jr., *When projective does not imply flat, and other homological anomalies.* Theory Appl. Categ. **5** (1999), no. 9, 202–250.

[96] L. G. Lewis, Jr., J. P. May, J. E. McClure, *Classifying G-spaces and the Segal conjecture.* Current trends in algebraic topology, Part 2 (London, Ont., 1981), pp. 165–179, CMS Conf. Proc., 2, Amer. Math. Soc., Providence, R.I., 1982.

[97] L. G. Lewis, Jr., J. P. May, M. Steinberger, *Equivariant stable homotopy theory.* Lecture Notes in Mathematics, **1213**, Springer-Verlag, 1986.

[98] J. A. Lind, *Diagram spaces, diagram spectra and spectra of units.* Algeb. Geom. Topol. **13** (2013), 1857–1935.

[99] H. Lindner, *A remark on Mackey-functors.* Manuscripta Math. **18** (1976), no. 3, 273–278.

[100] P. Löffler, *Equivariant unitary cobordism and classifying spaces.* Proceedings of the International Symposium on Topology and its Applications (Budva, 1972), pp. 158–160. Savez Društava Mat. Fiz. i Astronom., Belgrade, 1973.

[101] W. Lück, B. Uribe, *Equivariant principal bundles and their classifying spaces.* Algebr. Geom. Topol. 14 (2014), no. 4, 1925–1995.

[102] D. MacDuff, *Configuration spaces.* K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pp. 88–95. Lecture Notes in Math., Vol. 575, Springer, Berlin, 1977.

[103] S. Mac Lane, *Categories for the working mathematician.* Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp.

[104] M. A. Mandell, J. P. May, S. Schwede, B. Shipley, *Model categories of diagram spectra.* Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512.

[105] M. A. Mandell, J. P. May, *Equivariant orthogonal spectra and S-modules.* Mem. Amer. Math. Soc. 159 (2002), no. 755, x+108 pp.

[106] M. A. Mandell, *Algebraization of E_∞ ring spectra.* Preprint, 1998.

[107] J. P. May, *E_∞ ring spaces and E_∞ ring spectra.* With contributions by F. Quinn, N. Ray, and J. Tornehave. Lecture Notes in Mathematics, Vol. 77. Springer-Verlag, Berlin-New York, 1977. 268 pp.

[108] J. P. May, *The geometry of iterated loop spaces.* Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972. viii+175 pp.

[109] J. P. May, *Equivariant homotopy and cohomology theory.* With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. CBMS Regional Conference Series in Mathematics, 91. American Mathematical Society, Providence, RI, 1996. xiv+366 pp.

[110] J. P. May, *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. x+243 pp.

[111] J. McClure, R. Schwänzl, R. Vogt, *$THH(R) \cong R \otimes S^1$ for E_∞ ring spectra.* J. Pure Appl. Algebra **121** (1997), no. 2, 137–159.

[112] M. C. McCord, *Classifying spaces and infinite symmetric products.* Trans. Amer. Math. Soc. **146** (1969), 273–298.

[113] D. Montgomery, L. Zippin, *A theorem on Lie groups.* Bull. Amer. Math. Soc. **48**, (1942), 448–452.

[114] F. Morel, V. Voevodsky, *$\mathbb{A}^1$-homotopy theory of schemes.* Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143.

[115] G. D. Mostow, *Equivariant embeddings in Euclidean space.* Ann. of Math. (2) **65** (1957), 432–446.

[116] M. Murayama, *On G-ANRs and their G-homotopy types.* Osaka J. Math. **20** (1983), no. 3, 479–512.

[117] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability.* J. Amer. Math. Soc. **9** (1996), 205–236.

[118] A. Neeman, *Triangulated categories.* Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, 2001. viii+449 pp.

[119] J. von Neumann, *Zum Haarschen Maß in topologischen Gruppen.* Compositio Math. **1** (1935), 106–114.

[120] R. Oliver, *The representation ring of a compact Lie group revisited.* Comment. Math. Helv. **73** (1998), no. 3, 353–378.

[121] D. Ostermayr, *Equivariant Γ-spaces.* Homology Homotopy Appl. **18** (2016), no. 1, 295–324.

[122] R. S. Palais, *Imbedding of compact, differentiable transformation groups in orthogonal representations.* J. Math. Mech. **6** (1957), 673–678.

[123] R. S. Palais, *The classification of G-spaces.* Mem. Amer. Math. Soc. . **36** (1960), iv+72 pp.

[124] W. Pulikowski, *$RO(G)$-graded G-bordism theory.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 991–995.

[125] D. Puppe, *Bemerkungen über die Erweiterung von Homotopien.* Arch. Math. (Basel) **18** (1967) 81–88.

[126] D. Quillen, *Homotopical algebra.* Lecture Notes in Mathematics, **43**, Springer-Verlag, 1967.

[127] D. Quillen, *Higher algebraic K-theory. I.* Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[128] C. Rezk, *Spaces of algebra structures and cohomology of operads.* PhD thesis, Massachusetts Institute of Technology, 1996.

[129] . J. J. Rotman, *An introduction to homological algebra.* Second edition. Universitext. Springer, New York, 2009. xiv+709 pp.

[130] S. Sagave, C. Schlichtkrull, *Diagram spaces and symmetric spectra.* Adv. Math. **231** (2012), no. 3-4, 2116–2193.

[131] P. dos Santos, *A note on the equivariant Dold-Thom theorem.* J. Pure Appl. Algebra **183** (2003), 299–312.

[132] S. Schwede, *Morita theory in abelian, derived and stable model categories.* Structured ring spectra, 33–86, London Math. Soc. Lecture Note Ser., 315, Cambridge Univ. Press, Cambridge, 2004.

[133] S. Schwede, B. Shipley, *Algebras and modules in monoidal model categories.* Proc. London Math. Soc. **80** (2000), 491–511.

[134] S. Schwede, B. Shipley, *Stable model categories are categories of modules.* Topology **42** (2003), no. 1, 103–153.

[135] S. Schwede, B. Shipley, *A uniqueness theorem for stable homotopy theory.* Mathematische Zeitschrift **239** (2002), 803–828.

[136] S. Schwede, *The p-order of topological triangulated categories.* J. Topol. **6** (2013), no. 4, 868–914.

[137] S. Schwede, *Equivariant properties of symmetric products.* `arXiv:1403.1355`

[138] S. Schwede, *Orbispaces, orthogonal spaces, and the universal compact Lie group.* Preprint (2016), available from the author's homepage.

[139] G. Segal, *Equivariant stable homotopy theory.* Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 59–63, 1971.
[140] G. Segal, *Categories and cohomology theories.* Topology **13** (1974), 293–312.
[141] G. Segal, *Equivariant K-theory.* Inst. Hautes Études Sci. Publ. Math. No. **34** (1968), 129–151.
[142] G. Segal, *K-homology theory and algebraic K-theory.* K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pp. 113–127. Lecture Notes in Math., Vol. 575, Springer, Berlin, 1977.
[143] G. Segal, *Some results in equivariant homotopy theory.* Preprint, 1978.
[144] G. Segal, *The representation ring of a compact Lie group.* Inst. Hautes Études Sci. Publ. Math. No. 34 (1968), 113–128.
[145] K. Shimakawa, *Infinite loop G-spaces associated to monoidal G-graded categories.* Publ. Res. Inst. Math. Sci. **25** (1989), 239–262.
[146] J. Singer, *Äquivariante λ-Ringe und kommutative Multiplikationen auf Moore-Spektren.* PhD thesis, Universität Bonn, 2007. `http://hss.ulb.uni-bonn.de/2008/1400/1400.pdf`
[147] V. P. Snaith, *Explicit Brauer induction.* Invent. Math. **94** (1988), no. 3, 455–478.
[148] E. H. Spanier, *Algebraic topology.* Corrected reprint. Springer-Verlag, New York-Berlin, 1981. xvi+528 pp.
[149] N. E. Steenrod, *A convenient category of topological spaces.* Michigan Math. J. **14** (1967), 133–152.
[150] M. Stolz, *Equivariant structure on smash powers of commutative ring spectra.* PhD thesis, University of Bergen, 2011. `http://folk.uib.no/hus001/data/thesismartinstolz.pdf`
[151] R. E. Stong, *Unoriented bordism and actions of finite groups.* Memoirs of the American Mathematical Society, No. 103 Amer. Math. Society, Providence, R.I. (1970), 80 pp.
[152] N. P. Strickland, *Complex cobordism of involutions.* Geom. Topol. **5** (2001), 335–345.
[153] N. P. Strickland, *Realising formal groups.* Algebr. Geom. Topol. **3** (2003), 187–205.
[154] N. P. Strickland, *The category of CGWH spaces.* Preprint, available from the author's homepage.
[155] D. P. Sullivan, *Localization, Periodicity, and Galois Symmetry.* Geometric topology I Notes. M.I.T. 1970, revised 1971.
[156] D. P. Sullivan, Geometric topology: localization, periodicity and Galois symmetry. The 1970 MIT notes. Edited and with a preface by Andrew Ranicki. K-Monographs in Mathematics, 8. Springer, Dordrecht, 2005. xiv+283 pp.
[157] P. Symonds, *A splitting principle for group representations.* Comment. Math. Helv. **66** (1991), no. 2, 169–184.
[158] D. Tambara, *On multiplicative transfer.* Comm. Algebra **21** (1993), no. 4, 1393–1420.
[159] J. Thévenaz, *Some remarks on G-functors and the Brauer morphism.* J. Reine Angew. Math. **384** (1988), 24–56.
[160] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.
[161] T. tom Dieck, *Faserbündel mit Gruppenoperation.* Arch. Math. (Basel) **20** (1969), 136–143.
[162] T. tom Dieck, *Bordism of G-manifolds and integrality theorems.* Topology **9** (1970), 345–358.
[163] T. tom Dieck, *Orbittypen und äquivariante Homologie. I.* Arch. Math. (Basel) **23** (1972), no. 1, 307–317.
[164] T. tom Dieck, *Orbittypen und äquivariante Homologie. II.* Arch. Math. (Basel) **26** (1975), no. 6, 650–662.
[165] T. tom Dieck, *Transformation groups and representation theory.* Lecture Notes in Mathematics, 766. Springer, Berlin, 1979. viii+309 pp.
[166] T. tom Dieck, *Transformation Groups.* De Gruyter Studies in Mathematics, 8. Walter de Gruyter & Co., Berlin, 1987. x+312 pp.
[167] T. tom Dieck, *Algebraic topology.* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. xii+567 pp.
[168] J. Ullman, *Tambara functors and commutative ring spectra.* `arXiv:1304.4912`
[169] A. Verona, *Triangulation of stratified fibre bundles.* Manuscripta Math. **30** (1979/80), no. 4, 425–445.
[170] S. Waner, *Equivariant $RO(G)$-graded bordism theories.* Topology Appl. 17 (1984), no. 1, 1–26.
[171] S. Waner, *Equivariant classifying spaces and fibrations.* Trans. Amer. Math. Soc. **258** (1980), 385–405.
[172] A. G. Wasserman, *Equivariant differential topology.* Topology **8** (1969), 127–150.
[173] P. Webb, *Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces.* J. Pure Appl. Algebra **88** (1993), no. 1–3, 265–304.
[174] C. A. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp
[175] D. White, *Model structures on commutative monoids in general model categories.* `arXiv:1403.6759`
[176] K. Wirthmüller, *Equivariant homology and duality.* Manuscripta Math. **11** (1974), 373–390.
[177] Ch.-T. Yang, *On a problem of Montgomery.* Proc. Amer. Math. Soc. **8** (1957), 255–257.

Index