Conjecture. (Greenlees) For any compact Lie group $G$ there is an abelian category $\mathcal{A}(G)$ such that

$$\mathbb{Q} \text{- } G\text{-spectra } \cong_{\mathbb{Q} \text{ d. g.}} \mathcal{A}(G)$$

where $\mathcal{A}(G)$ has injective dimension equal to the rank of $G$.

Verified for finite groups, SO(2), O(2), SO(3) (G.-May, G., S., Barnes)
Theorem 1. (G.-S., ’09) For $G$ connected compact Lie, 
$\mathbb{Q}$ free $G$-spectra $\simeq_{\mathbb{Q}} \text{tor-} H^*BG \text{-Mod}$

Theorem 2. (preprint in progress) The conjecture holds for $G$ any torus.

The rest of this talk will outline the five steps of the proof of Theorem 2 for $G = S^1$.

We will concentrate on step one.
Step 1 Variation on fixed point diagram.

Definitions. Let $\mathcal{F} = \{F\}$ be the family of finite subgroups of $G$. Define

$$(E\mathcal{F})^H = \begin{cases} \text{pt} & H \text{ finite} \\ \emptyset & H \text{ not finite} \end{cases}$$

Define $\tilde{E}\mathcal{F}$ as the cofiber of the map $E\mathcal{F}_+ \to S^0$.

Define $DE\mathcal{F}_+ = \text{Hom}(E\mathcal{F}_+, S^0)$.

Proposition. For $G = SO(2)$ there is a homotopy pullback of $G$-equivariant commutative ring spectra.

$$
\begin{tikzcd}
S^0 \ar[r] \ar[d] & \tilde{E}\mathcal{F} \\
DE\mathcal{F}_+ \ar[r] \ar[d] & DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}
\end{tikzcd}
$$
Analogues

**Proposition.** There is a pullback square

\[
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathbb{Q} \\
\downarrow & & \downarrow \\
\prod_p \mathbb{Z}_p & \longrightarrow & \prod_p \mathbb{Z}_p \otimes \mathbb{Q}
\end{array}
\]

**General Case:** Assume given a homotopy pullback of rings (ring spectra or DGAs):

\[
\begin{array}{ccc}
R & \longrightarrow & R_1 \\
\downarrow & & \downarrow \\
R_2 & \longrightarrow & R_3
\end{array}
\]

Let \( R' \) denote the diagram of rings above with \( R \) deleted.

\[
\begin{array}{ccc}
R_1 & \longrightarrow & R_3 \\
\downarrow & & \\
R_2 & \longrightarrow & R_3
\end{array}
\]
**Definition.** $R^\downarrow$-modules is the category of modules over the ring with three objects with $\text{Hom}(1, 3) = R_3$ and $\text{Hom}(2, 3) = R_3$.

Such a module is a collection $\{M_i\}_{i=1,2,3}$ of $\{R_i\}$-modules with structure maps $R_3 \otimes_{R_1} M_1 \to M_3$ and $R_3 \otimes_{R_2} M_2 \to M_3$. (The adjoints of these structure maps are an $R_1$-morphism $M_1 \to M_3$ and an $R_2$-morphism $M_2 \to M_3$.)

Note $R^\downarrow$ determines such a module.

$R^\downarrow$-Mod has three generators $R_\cdot$-Mod has only one.
**Proposition.** The derived category of $R$-modules is equivalent to the localizing subcategory of $R^\downarrow$-modules generated by $R^\downarrow$. This equivalence is induced by a Quillen equivalence of model categories.

\[ R\text{-Mod} \simeq_Q \text{cell}_{\{R^\downarrow\}} - R^\downarrow\text{-Mod} \]

**Proof.** Consider the adjoint functors on the generators.

\[ M \to R^\downarrow \otimes_R M \]

\[ \text{pullback}(\{M_i\}) \leftarrow \{M_i\} \]
Step 1:
Rational $G$-spectra are $S^0$-modules; apply above proposition with above square with $R^\downarrow = \tilde{E}\mathcal{F}$

\[\begin{array}{c}
\tilde{E}\mathcal{F} \\
\downarrow \\
DEF_+ \longrightarrow DEF_+ \wedge \tilde{E}\mathcal{F}.
\end{array}\]

Here cellularize with respect to $\{G/H_+ \wedge R^\downarrow\}_H$.

Conclude:

\[\mathbb{Q} - G\text{-spectra} = S^0\text{-Mod} \simeq \text{cell}_{\{G/H_+ \wedge R^\downarrow\} - R^\downarrow\text{-Mod}}\]
**Step 2:** Move from $G$-spectra to spectra.

\[ A \text{-Mod} \langle G \text{- spectra} \rangle \leftrightarrow A^G \text{-Mod} \langle \text{spectra.} \rangle \]

This induces an equivalence on each of the cells \( \{ G/H \wedge R^\Delta \}^H \) for each of the relevant rings.

\[ S^0 \text{-Mod}_G \simeq_1 \text{cell-} R^\Delta \text{-Mod}_G \simeq_2 \text{cell-} (R^\Delta)^G \text{-Mod} \]

**Step 3:** Make algebraic:

rational commutative ring spectra are modeled by rational commutative DGA’s

\[ \simeq_2 \text{cell-} (R^\Delta)^G \text{-Mod} \simeq_3 \text{cell-d.g.-} (R^\Delta)^G_{DGA} \text{-Mod} \]
Step 4: Rigidity

\((R^\downarrow)^G_{DGA}\) is intrinsically formal.

1. \(\pi_*(\tilde{E}\mathcal{F})^G \cong \pi_*S^0 \cong \mathbb{Q}[0]\).

2. Note \(E\mathcal{F}_+\) rationally splits as \(\vee E\langle F\rangle\). Since \(E\langle 1 \rangle = EG\), then

\[ (DEG_+)^G = \text{Hom}(EG_+, S^0)^G \cong \text{Hom}(BG_+, S^0). \]

So \(\pi_*(DE\mathcal{F}_+)^G \cong \prod_F H^*(BG/F) =: \vartheta_{\mathcal{F}}. \)

3. \(\pi_*(DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F})^G \cong E^{-1}_G \vartheta_{\mathcal{F}}. \)

Thus \((R^\downarrow)^G_{DGA}\) is quasi-isomorphic to \(R^\downarrow_{alg} = H_*(R^\downarrow)^G_{DGA}. \)

\[
\begin{array}{c}
\mathbb{Q} \\
\vartheta_{\mathcal{F}} \rightarrow \mathcal{E}_G^{-1} \vartheta_{\mathcal{F}}
\end{array}
\]

Summary:

\[S^0\text{-Mod}_G \cong_1 \text{cell-}R^\downarrow\text{-Mod}_G \cong_2 \text{cell-} (R^\downarrow)^G\text{-Mod} \]

\[\cong_3 \text{cell-d.g.-} (R^\downarrow)^G_{DGA}\text{-Mod} \cong_4 \text{cell-d.g.-} R^\downarrow_{alg}\text{-Mod} \]
**Step 5:** Small algebraic model.

For $G = SO(2)$, $\mathcal{A}(G)$ is the category of modules $N \to M \leftarrow V$ over

$$
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\vartheta_{\mathcal{F}} \\
\mathcal{E}_G^{-1} \vartheta_{\mathcal{F}}
\end{array}
$$

such that both structure maps are isomorphisms.

1. Quasi-coherence: $\mathcal{E}_G^{-1} \vartheta_{\mathcal{F}} \otimes_{\vartheta_{\mathcal{F}}} N \cong \mathcal{E}_G^{-1} N \xrightarrow{\cong} M$.
2. Extended: $\mathcal{E}_G^{-1} \vartheta_{\mathcal{F}} \otimes_{\mathbb{Q}} V \cong M$.

$$
\simeq_4 \text{cell-d.g.-} R^j_{alg} \text{-Mod} \simeq_5 \text{d.g.A}(G)
$$

**Theorem 2.** For $G = SO(2)$, the homotopy theory of rational $G$-spectra is modeled by differential graded $\mathcal{A}(G)$-modules. Here $\mathcal{A}(G)$ has injective dimension one.
General outline is the same for all tori, just have larger diagrams. For $G$ a 2-torus, the diagram shape is:

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \rightarrow \bullet \\
\downarrow \downarrow \\
\bullet \rightarrow \bullet \rightarrow \bullet \\
\end{array}
\]

For an $n$-torus there are $n$ layers.

Can restrict to families of fixed points.

For example, free $G$-spectra with $G = SO(2)$: have a module $N$ over $H^*(BG)$, with $V = 0, M = 0$. The quasi-coherence condition says $\mathcal{E}_{-1}N \cong M = 0$; that is, $N$ is torsion.

**Theorem 1.** The homotopy theory of free rational $G$-spectra is modeled by torsion modules over $H^*BG$. 