TAF 2/18/11 Scott Bailey

The dual abelian variety

$\mathcal{L}$ - invertible sheaf, a locally free rank 1 $\mathcal{O}_A$-module.

$\mathrm{Pic}(A)$ - isomorphism classes of invertible sheaves on A.

$[\mathcal{L}]\cdot[\mathcal{L}'] = [\mathcal{L}\otimes\mathcal{L}']$. The "unit" is $\mathcal{O}_A$.

$\mathcal{L}^\vee = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_A)$

We have a natural map $\mathcal{L}\otimes\mathcal{L}^\vee \to \mathcal{O}_A$, isomorphism, so
$(x, f) \mapsto f(x)$
$[\mathcal{L}]\cdot[\mathcal{L}^\vee] = [\mathcal{O}_A]$

Thus $\mathcal{L}^{-1} = \mathcal{L}^\vee$.

Theorem of the Square (TOTS) For all invertible sheaves over A, $a, b \in A(k)$, we have $t_{a+b}^*\mathcal{L}\otimes\mathcal{L} = t_a^*\mathcal{L}\otimes t_b^*\mathcal{L}$.

Tensor $\mathcal{L}^{-2} = \mathcal{L}^{-1}\otimes\mathcal{L}^{-1}$

$$t_{a+b}^*\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}^{-2} = t_a^*\mathcal{L}\otimes t_b^*\mathcal{L}\otimes\mathcal{L}^{-2}$$

So $t_{a+b}^*\mathcal{L}\otimes\mathcal{L}^{-1} = (t_a^*\mathcal{L}\otimes\mathcal{L}^{-1})\otimes(t_b^*\mathcal{L}\otimes\mathcal{L}^{-1})$

This implies the map:

$$\lambda_{\mathcal{L}} : A(k) \to \mathrm{Pic}(A)$$
$$a \mapsto t_a^*\mathcal{L}\otimes\mathcal{L}^{-1}$$ is an isomorphism.

(Compare to the curve case: D divisor, $\lambda_D : C(k) \to \mathrm{Pic}(C)$
$a \mapsto D_a - D$.)

$\lambda_D = (\deg D)^{1 \text{ or } 2}\lambda_{D_0}$, and $\deg D = 0 \iff \lambda_D = 0$.

We want analogous definition of $\mathrm{Pic}^0(A)$.

Let $\mu_A : A\times A \to A$
$(a, b) \mapsto a+b$

$$p_1 : A \times A \to A$$
$$(a, b) \mapsto a$$

Consider the sheaf over $A \times A$:

$$\mu_A^* \mathcal{L} \otimes p_1^* \mathcal{L}$$

$$\mu_A \circ (x \mapsto (x, a)) = x + a = t_a$$

$$p_1 \circ (x \mapsto (x, a)) = x = Id_A = I$$

$$\mu_A^* \mathcal{L} \otimes p_1^* \mathcal{L} \big|_{A \times \{a\}} = t_a^* \mathcal{L} \otimes \mathcal{L}^{-1} = \lambda_L(a) \text{ for } a \in A(k).$$

Let $K(\mathcal{L}) = \{ a \in A : \mu_a^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} |_{A \times \{a\}}$ is trivial $\}$.

$Pic^0(A) = \{$ iso. classes of sheaves st. $K(\mathcal{L}) = A \}$.

__Proposition:__ $\mathcal{L}$ invertible sheaf st. $\Gamma(A, \mathcal{L}) \neq 0$, then $\mathcal{L}$ is ample $\iff K(\mathcal{L})$ has dim 0.

($\mathcal{L}$ is __ample__ if for every coherent sheaf $\mathcal{F}$, $\exists\, n$ st. $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ generated by global sections.)

(compare to: D effective divisor, $L(D)$ linear system, $\mathcal{L} = \mathcal{L}(D)$ Cartier divisor, $\Gamma(A, \mathcal{L}(D)) = L(D)$.)

(as a scheme / $\bar{k}$)

__Prop:__ D ample divisor $\iff \lambda_D : A(k) \to Pic(A_{\bar{k}})$ has finite kernel.

$$a \mapsto D_a - D$$

The elements of $A^\vee$ should parametrize $Pic^0(A)$

Universal property of $A^\vee$:

$(A^\vee, \mathcal{P}_A)$: $A^\vee$ abelian variety

$\mathcal{P}_A$ invertible sheaf over $A \times A^\vee$.

Assume (a) $\mathcal{P}|_{A \times \{b\}} \in \mathrm{Pic}^0(A_b)$ $\forall b \in A^\vee$

(b) $\mathcal{P}|_{\{0\} \times A^\vee}$ trivial

If for any pair $(T, \mathcal{L})$ T ab var, $\mathcal{L}$ inv. sheaf over $A \times T$, we have:

(a') $\mathcal{L}|_{A \times \{t\}} \in \mathrm{Pic}^0(A)$

(b') $\mathcal{L}|_{\{0\} \times T}$ trivial

then $\exists!$ regular map $\alpha : T \to A^\vee$

$$A \times T \xrightarrow[1 \times \alpha]{} A \times A^\vee$$

$\mathcal{L} = (1 \times \alpha)^* P$ $\qquad$ $A^\vee$ dual ab. var.

P poincaré sheaf

<u>Note:</u> $\mathrm{Hom}(T, A^\vee) = \{$ iso. classes of inv. sheaf over $A \times T$ satisfying (a'), (b') $\}$

<u>Note:</u> $\beta : A \to B$ homomorphism of abelian varieties

Using $A \times B^\vee \xrightarrow{\beta \times 1} B \times B^\vee$ we get $(\beta \times 1)^* P$ on $A \times B^\vee$ inv. sheaf

so $\exists!$ regular $\alpha : B^\vee \to A^\vee$ s.t. $(1 \times \alpha)^* P_A = (\beta \times 1)^* P_B$.

Call $\beta^\vee = \alpha : B^\vee \to A^\vee$

$$\mathrm{Pic}^0(B) \to \mathrm{Pic}^0(A).$$

<u>Note:</u> iso classes satisfying (a'), (b'), is the same as $\alpha : T \to A^\vee$
$\alpha(0_T) = 0_{A^\vee}$.

Prop: $\mathrm{Hom}(T, A^\vee) = \{$ iso. classes of inv. sheaves on $A \times T$ trivial on $\{0\} \times T$ and $A \times \{0\}$. $\}$

A homomorphism $\lambda : A \to A^\vee$ is symmetric if $\lambda^\vee : \underset{\substack{\| \\ A}}{(A^\vee)^\vee} \to A^\vee$ satisfies $\lambda^\vee = \lambda$.

Example: $\mathcal{L}$ invertible sheaf on A, $p_1, p_2, m_A : A \times A \to A$ as usual.

$m_A^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}$ inv. sheaf on $A \times A$, degree 0, trivial on $A \times \{0\}$ and $\{0\} \times A$.

So $\exists\ \lambda_{\mathcal{L}} \in \mathrm{Hom}(A, A^\vee)$ st. $\lambda_{\mathcal{L}}$ is symmetric.

Converse: Given symmetric $\alpha : A \to A^\vee$ there exists an $\mathcal{L}$ over A st. $\alpha = \lambda_{\mathcal{L}}$ and $\mathcal{L}^2 = \Delta^* \mathcal{L}_\alpha$.

Definition: Let A be an abelian variety. A polarization of A is a symmetric isogeny $\lambda : A \to A^\vee$ associated to an ample sheaf $\mathcal{L}$.

Example: Bilinear forms on A:

$$\mathcal{L} \longleftrightarrow \lambda_{\mathcal{L}} : A \to A^\vee \in \mathrm{Hom}_k(A, k)$$

$$\begin{array}{c} m_A^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1} \\ \updownarrow \\ Q(x+y) - Q(x) - Q(y) \end{array} \qquad \lambda_{\mathcal{L}}(x) = B(x, \cdot)$$

This is a symmetric bilinear form $B(x,y)$ on A.

$\mathcal{L}^2 = \Delta^* \mathcal{L}$. $\qquad 2Q(x) = B(x,x)$

$$Q(x) = \tfrac{1}{2} B(x,x).$$

If $\lambda = \lambda_{\mathcal{L}}$ is a polarization then $n\lambda = \lambda_{\mathcal{L}^n}$ is a polarization. If $\exists\, m, n \in \mathbb{Z}_+$ s.t. $\lambda = m\lambda'$ then λ, λ' are
$\lambda_{\mathcal{L}^n} = \lambda'_{\mathcal{L}^n}$
equivalent. A <u>weak polarization</u> is an equivalence class of this type.

Given a scheme $A \to S$
$$\lambda : A \to A^\vee$$
restricting to polarization on geometric fibers.

<u>Weil pairings:</u> There is a pairing
$$e_m : A_m(\bar{k}) \times A^\vee_m(\bar{k}) \to \mu_m(\bar{k})$$ nondegenerate and commutes with action of $Gal(\bar{k}/k)$.

<u>Construction:</u> $a \in A_m(\bar{k})$, $a' \in A^\vee_m(\bar{k}) \subset Pic^0(A)$, $\exists$ inv. sheaf $\mathcal{L}$ on A.
$$m_A = \underbrace{1_A + \cdots + 1_A}_{m \text{ times}} \qquad m_A^* \mathcal{L} = \mathcal{L}^m.$$

Let a' represented by Weil divisor D.

$m_A^* D = mD = 0$, $\exists$ rational fns f, g,

$mD = (f) \quad m_A^* D = (g)$.

$div(f \circ m_A) = m_A^*(div(f)) = m_A^*(mD) = m(m_A^*(D)) = div(g^m)$

$\Rightarrow \dfrac{g^m}{f \circ m_A}$ rational function with no zeros or poles.

Thus $\exists$ constant c with $g^m = c(f \circ m_A)$.

$$g(x+a)^m = c(f\circ m_A)(x+a)$$
$$= c f(mx + ma)$$
$$= c f(mx)$$
$$= g(x)^m.$$

Thus, $1 = \left(\frac{g}{g\circ t_a}\right)^m$

<u>Define</u>: $\boxed{e_m(a,a') = \frac{g}{g\circ t_a}}$,

If $\lambda : A \to A^\vee$ is a polarization, then $\lambda(-,-) : A_m(\bar{k}) \times A_m(\bar{k}) \to \mu_m(\bar{k})$ by $\lambda(a,b) = e_m(a, \lambda b)$.