5. The Burnside-Ring of a Compact Lie Group.

5.1. Euler Characteristic.
We collect the properties of the Euler-Characteristic that we shall need in the sequel and indicate proofs when appropriate references cannot be given.

Let \( R \) be a commutative ring and let \( A \) be an associative \( R \)-algebra with identity (e.g. \( A = R \); \( A = R[G] \), \( G \) a finite group). In general, an Euler-Poincaré map is a map from a certain category of \( A \)-modules to an abelian group which is additive on certain exact sequences. We consider the following sufficiently general situation:

Let \( \text{Gr}_R(A) \) be the abelian group (Grothendieck group) with generators \([M]\) where \( M \) is a left \( A \)-module which is finitely generated and projective as an \( R \)-module, with relations \([M] = [M'] + [M'']\) for each exact sequence \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) of such modules. Let \( \text{Gr}(A) \) be the Grothendieck group of finitely generated left \( A \)-modules and the analogous relations for exact sequences. A ring \( R \) is called regular if it is noetherian and every finitely generated \( R \)-module has a finite resolution by finitely generated projective \( R \)-modules.

Proposition 5.1.1.
Let \( R \) be a regular ring and \( A \) an \( R \)-algebra which is finitely generated and projective as an \( R \)-module. Then the forgetful map \( \text{Gr}_R(A) \rightarrow \text{Gr}(A) \) is an isomorphism.

Proof.
Swan-Evans [158], p. 2. (The symbol \( G_0 \) is used in [158] where we use \( \text{Gr} \). Since we do not need \( G_1 \) and use \( G \) to denote groups we have chosen this non-standard notation.)
Remark.
In the case of the group ring $A = S[\pi]$, $S$ a commutative ring, we denote $\text{Gr}^S(A)$ by $R(\pi, S)$. Tensor product over $S$ induces a multiplication and $R(\pi, S)$ becomes a commutative ring the representation ring of $\pi$ over $S$.

We call the assignment $M \mapsto [M] \in \text{Gr}^R(A)$ a universal Euler-Characterisitic for the modules under consideration, because any map $M \mapsto e(M)$, $e(M) \in B$, $B$ an abelian group, such that $e(M) = e(M') + e(M'')$ whenever $0 \to M' \to M \to M'' \to 0$, is induced from a unique homomorphism $h : \text{Gr}^R(A) \to B$, $e(M) = h[M]$. (Similar definition for $\text{Gr}(A)$.) If $R = A$ is a field then $M \mapsto \text{dim}_R M \in \mathbb{Z}$ is such a universal map, establishing $\text{Gr}(R) \cong \mathbb{Z}$. If $R = A = \mathbb{Z}$ then $M \mapsto \text{rank}(M) = \text{dim}_Q(M \otimes Q) \in \mathbb{Z}$ is a universal Euler-Characteristic. (by 5.1.1 $\text{Gr}^R(\mathbb{Z}) = \text{Gr}(\mathbb{Z})$).

If $M : 0 \to M_0 \to M_1 \to \ldots \to M_n \to 0$ is a complex of $A$-modules which are finitely generated and projective as $R$-modules then we define

\begin{equation}
\chi(M) = \sum_{i=0}^{n} (-1)^i [M_i] \in \text{Gr}^R(A)
\end{equation}

to be the Euler-Characteristic of the complex. We use the same terminology in case of $\text{Gr}(A)$. If submodules of finitely $A$-modules are again finitely generated then for the homology groups $H_i(M)$ of a complex

$$
\chi(H_i(M)) := \chi(M) \in \text{Gr}(A).
$$

If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of complexes then

\begin{equation}
\chi(M) = \chi(M') + \chi(M'')
\end{equation}

when everything is defined. If one works with $\text{Gr}^R(A)$ then one has to
use hereditary rings, i.e. submodules of projective modules are pro-
jective (see Cartan-Eilenberg [45], p. 14 for this notion). Examples
are Dedekind rings R, i.e. integral domains in which all ideals are
projective (see Swan-Evans [153], p. 212 for various characterisations
of Dedekind rings).

We now consider the special cases that are relevant for topology.
Let \( (Y,A) \) be a pair of spaces such that the (singular) homology groups
with integral coefficients \( H_i(Y,A) \) are finitely generated and zero for
large \( i \). Then, by abuse of language, we define the Euler-Characteristic
\( \chi(Y,A) \) of the pair \( (Y,A) \) to be the integer

\[
\chi(Y,A) = \sum_{i \geq 0} (-1)^i \text{rank} H_i(Y,A)
\]

with the usual convention \( \chi(Y) = \chi(Y,\emptyset) \). Standard properties are
(see Dold [75], p. 105):

**Proposition 5.1.6.**

(i) If two of the numbers \( \chi(Y), \chi(A) \) and \( \chi(Y,A) \) are defined then
so is the third, and

\[
\chi(Y) = \chi(A) + \chi(Y,A).
\]

(ii) If \( (Y;Y_1,Y_2) \) is an excisive triad and if two of the numbers
\( \chi(Y_1 \cup Y_2), \chi(Y_1 \cap Y_2), \chi(Y_1) + \chi(Y_2) \) are defined then so is
the third, and

\[
\chi(Y_1) + \chi(Y_2) = \chi(Y_1 \cup Y_2) + \chi(Y_1 \cap Y_2).
\]

(iii) If \( (Y,A) \) is a relative CW-complex with \( Y-A \) containing many cells
then \( \chi(Y,A) \) is defined and

\[
\chi(Y,A) = \sum_{i \geq 0} (-1)^i n_i
\]
where \( n_i \) is the number of \( i \)-cells in \( Y - A \).

If \( F \) is a field we can consider

\[
\chi(Y, A; F) = \sum_{i \geq 0} (-1)^i \dim_F H_i(Y, A; F),
\]

if this number is defined. Then 5.1.6 also holds with this type of Euler-Characteristic.

**Proposition 5.1.8.**

(i) If \( F \) has characteristic zero then \( \chi(Y, A) \) is defined if and only if \( \chi(Y, A; F) \) is defined and \( \chi(Y, A) = \chi(Y, A; F) \).

(ii) If \( \chi(Y, A) \) is defined and \( (Y, A) \) has finitely generated integral homology then \( \chi(Y, A; F) \) is defined for any field and \( \chi(Y, A) = \chi(Y, A; F) \).

**Proof.** This is a simple application of the universal coefficient formula. (See Dold [75], p. 156).

One can also define the Euler-Characteristic using various types of cohomology (singular-, Alexander-Spanier-, sheaf-, etc.) and use the universal coefficient formulas to see that homology and cohomology gives the same result under suitable finiteness conditions.

**Proposition 5.1.9.**

Let \( p : E \rightarrow B \) be a Serre-fibration with typical fibre \( F \). If \( \chi(B) \) and \( \chi(F) \) are defined and the local coefficient system \( (H_*(p^{-1}b; \mathbb{Q})) \) is trivial then \( \chi(E) \) is defined and

\[
\chi(E) = \chi(F) \chi(B).
\]
Proof.
Use the existence of the Serre spectral sequence; apply the Künneth-formula to the $E_2$-term; use 5.1.3 (see Spanier [152], p. 481).

We actually need a more general result where fibrations are replaced by relative fibrations and the coefficient system may be non-trivial. This will be done in the next section when a suitable class of spaces with Euler-Characteristic (the Euclidean neighbourhoods retracts) has been described. A really general and satisfactory treatment of the Euler-Characteristic (and its generalization: the Lefschetz number) does not seem to exist.

5.2. Euclidean neighbourhood retracts.
We single out a convenient class of $G$-spaces $X$ such that for all fixed point sets and other related spaces the Euler-Characteristic is defined.

Let $G$ be compact Lie group. We define a $G$-ENR (Euclidean Neighbourhood Retract) to be a $G$-space $X$ which is ($G$-homeomorphic to) a $G$-retract of some open $G$-subset in a $G$-module $V$.

Proposition 5.2.1.
If $X$ is a $G$-ENR and $i : X \rightarrow W$ a $G$-embedding into a $G$-module $W$ then $iX$ is a $G$-retract of a neighbourhood.

Proof. As in Dold [75], p. 81, using the Tietze-Gleason extension theorem (Bredon [37], p. 36; Palais [124], p. 19).

Proposition 5.2.2.
A differentiable $G$-manifold with a finite number of orbit types is a $G$-ENR.
Proof.
Embed the manifold differentiably into a $G$-module (Wasserman [165]) where it is a retract of a $G$-invariant neighbourhood.

If we have no group $G$ acting we simply talk about ENR's. The following basic result of Borsuk shows that being an ENR is a local property. Recall that a space $X$ is called locally contractible if every neighbourhood $V$ of every point $x \in X$ contains a neighbourhood $W$ of $x$ such that $W \subset V$ is nullhomotopic fixing $x$. It is easy to see that an ENR is locally contractible (Dold [75], p. 81). A space is locally $n$-connected if every neighbourhood $V$ of every point $x$ contains a neighbourhood $W$ such that any map $S^j \to W$, $j \leq n$, is nullhomotopic in $V$.

Proposition 5.2.3.
If $X \subset \mathbb{R}^n$ is locally $(n-1)$-connected and locally compact then $X$ is an ENR.

Proof.
Dold [75], IV 8.12, and 8.13 exercise 4.

Remarks 5.2.4.
A basic theorem of point set topology says that a separable metric space of (covering) dimension $\leq n$ can be embedded in $\mathbb{R}^{2n+1}$; see Hurewicz-Wallman [38] for the notion of dimension and this theorem. Hence a space is an ENR if and only if it is locally compact, separable metric, finite-dimensional and locally contractible. Using a local Hurewicz-theorem (Raußen [131]) one can express the local contractibility in terms of homology conditions.
Proposition 5.2.5.
Let \( X \) be a G-ENR. Then the orbit space \( X/G \) is an ENR.

Proof.
Let \( X \rightarrow U \rightarrow X \) be a presentation of \( X \) as a neighbourhood retract (i.e. \( U \) open G-subset in a G-module, \( ri = id \)). We pass to orbit spaces. A retract of an ENR is an ENR. Hence we have to prove the Proposition for \( X \) a differentiable G-manifold (and then apply it to the manifold \( U \)). Let \( p : X \rightarrow X/G \) be the quotient map. Given \( x \in V \subset X/G, V \text{ open}, p^{-1}V \) contains a G-invariant tubular neighbourhood \( W \) of the orbit \( p^{-1}x \). Hence \( pW \) is contractible. Therefore \( X/G \) is locally contractible. Moreover \( X/G \) is locally compact (Bredon \([37]\), p. 38), separable metric (Palais \([124]\), 1.1.12) and \( \dim X/G \leq \dim X \) (use Hurewicz-Wallman \([38]\)). Now apply 5.2.3, and 5.2.4.

Using 5.2.3 and the following result of Jaworowski we see that being a G-ENR is a local property too.

Proposition 5.2.6.
Let \( X \) be a G-space which is separable metric and finite-dimensional. Then \( X \) is a G-ENR if and only if \( X \) is locally compact, has a finite number of orbit types, and for every isotropy group \( H < G \) the fixed point set \( X^H \) is an ENR.

Proof.
Jaworowski \([102]\).

Corollary 5.2.7.
If \( X \) is a G-ENR then \( X(H) \) is a G-ENR for every \( H < G \).
Proposition 5.2.8.
If $X$ is a compact ENR then the Euler-Characteristic $\chi(X)$ is defined.

Proof.
$X$ is a retract of a space $K$ which may be given as a finite union of cubes in a Euclidean space. Hence $H_iX$ is a direct summand in $H_iK$, which is finitely generated and zero for large $i$.

Proposition 5.2.9.
Let $E \rightarrow B$ be a fibre bundle with typical fibre $F$. If $F$ and $B$ are ENR's then $E$ is an ENR.

Proof.
Apply 5.2.3.

We now come to the generalization of 5.1.9.

Proposition 5.2.10.
Let $F : (X,A) \rightarrow (Y,B)$ be a continuous map between compact ENR's such that $F(X \setminus A) = Y \setminus B$. Suppose the induced map $f : X \setminus A \rightarrow Y \setminus B$ is a fibration with typical fibre $Z$ a compact ENR. Then

$$\chi(X,A) = \chi(Z) \chi(Y,B).$$

The Euler-Characteristic $\chi_c(X \setminus A)$ of $X \setminus A$ computed with Alexander-Spanier cohomology with compact support and coefficients in a field exists and $\chi(X,A) = \chi_c(X \setminus A)$.

Proof.
Since the integral homology groups are finitely generated, we can compute the Euler-Characteristic using any field of coefficients and
homology or cohomology. We use cohomology with \( \mathbb{Z}/2 \)-coefficients. Since ENR's are locally contractible, 5.2.3, we can use singular or Alexander-Spanier cohomology (Spanier \[152\], 6.9.6.). Using Alexander-Spanier cohomology with compact support we have by Spanier \[152\], 6.6.11, that

\[
H_c^i(X, A) = H_c^i(X \setminus A)
\]

and similarly for \((Y, B)\). The fibration \( f : X \setminus A \to Y \setminus B \) gives us a Leray spectral sequence with \( E_2 \)-term

\[
E_2^{p, q} = H_c^p(Y \setminus B; H_c^q(Z))
\]

where the coefficients are \( H_c^q(Z) \) considered as a local coefficient system on \( Y \setminus B \) (Borel \[25\], XVI. 4.3; \[27\]). If this local coefficient system is trivial then our assertion follows as in 5.1.9. If it is non-trivial then the following ad hoc argument of Becker and Gottlieb reduces it to the case of a trivial coefficient system: Since \( H_c^q(Z) \) is a finite group (\( \mathbb{Z}/2 \)-coefficients!) a finite covering of \( Y \setminus B \) will make the coefficient system trivial. The relation

\[
\chi(U') = N \chi(U)
\]

for a finite covering \( U' \to U \) of degree \( N \) (which will be proved in 5.3) and the result for trivial coefficients implies

\[
\chi_c(X \setminus A) = \chi(Z) \chi_c(Y \setminus B).
\]

**Problem 5.2.11.**

Give a satisfactory and general (not just for ENR's) proof for 5.2.10 and its generalization to Lefschetz numbers (compare Dold \[77\]).

**Proposition 5.2.11.**

Finite \( G \)-CW-complexes are \( G \)-ENR's.
Proof.
See Illman [100] for the notion of G-CW-complexes.
Use 5.2.3, 5.2.6.

5.3. Equivariant Euler-Characteristic.
If $G$ is a compact Lie group and $X$ is a $G$-space then the $G$-action on $X$ induces a $G$-action on the cohomology groups $H^i(X;M)$ where $M$ is an $R$-module. If $G_0$ is the component of the identity of $G$ then $G_0$ acts trivially on $H^i(X;M)$ so that $H^i(X;M)$ becomes an $R \left[ G/G_0 \right]$-module. If $H^*(X;M) = (H^i(X;M))_{i \geq 0}$ is $R$-finite, i.e. zero for large $i$ and finitely generated as $R$-module, then we define the equivariant Euler-Characteristic of the $G$-space $X$ to be the element

$$\chi_G(X;R) = \sum_{i \geq 0} (-1)^i H^i(X;R) \in \text{Gr}(R[G/G_0]).$$

If $R = \mathbb{C}$, the complex numbers, then $\chi_G(X;\mathbb{C}) \in R(G)$, where $R(G)$ denotes the complex representation ring. We use similar definitions for pairs of $G$-spaces and homology. Actually for general spaces one has to specify the cohomology theory. For simplicity we make the following

Assumption 5.3.2:
$X$ is a $G$-ENR. Cohomology is Alexander-Spanier cohomology with compact support (in this case isomorphic to sheaf- or presheaf cohomology with compact support; see Spanier [152], Chapter 6; Bredon [35], Chapter III).

Our task in this section is the computation of (5.3.1) in case $R$ is the field of rational numbers. The computation will be in terms of non-equivariant Euler-Characteristics. The reader should convince himself that most of the results to follow are obvious if a finite group acts simplicially on a finite complex. In this case one can compute on
the chain level.

**Proposition 5.3.3.**

Let \( G \) be a p-group acting freely on \( X \). Suppose \( H^*(X;F_p) \) is \( F_p \)-finite. Then \( \chi(X/G;F_p) \) is defined and

\[
\chi(X;F_p) = |G| \chi(X/G;F_p).
\]

(Recall 5.3.2 and that \( \chi \) is defined using cohomology with compact support.)

**Proof.**

If \( H \triangleleft G \) then \( G/H \) acts freely on the \( G/H \)-ENR (by 5.2.5, 5.2.6) \( X/H \). Hence using induction on the order of \( G \) it is sufficient to prove the proposition for \( G = \mathbb{Z}/p \). We use the following fact:

\[
(5.3.4) \quad H^i(X;F_p) \cong H^i(X/G;A)
\]

where \( A \) is the local coefficient system ( = locally constant sheaf, Spanier [152], p. 360) with stalks \( H^0(\pi^{-1}(x);F_p) \cong F_p[G] \), \( \pi: X \to X/G \) the quotient map. In our case the group action on \( H^i(X;F_p) \) corresponds via 5.3.4 to the group action on the coefficient system, which is a system of \( F_p[G] \)-modules (for a verification see Floyd [83], III. 1). Since an \( F_p[G] \)-module always contains non-trivial \( G \)-fixed submodules if \( G \) is a p-group (e.g. by 1.3.) we can find a filtration

\[
A = A_1 \supset A_2 \supset \ldots \supset A_k = 0
\]

of the coefficient system such that \( A_i/A_{i+1} \) is the constant system. The Cartan spectral sequence of a covering (Bredon [35], p. 154) shows \( H^i(X/G;F_p) \) to be finite dimensional. From the additivity of the Euler-Characteristic \( \chi(X/G;A_i) = \chi(X/G;A_{i+1}) + \chi(X/G;A_i/A_{i+1}) \) we obtain the result.
Proposition 5.3.4.

Let the finite group $G$ act freely on $X$. Suppose $H^*(X;\mathbb{Z})$ is $\mathbb{Z}$-finite. Then $\chi(X/G;\mathbb{Q})$ is defined and

$$\chi_G(X;\mathbb{Q}) = \chi(X/G;\mathbb{Q}) \cdot [G] \in R(G;\mathbb{Q}).$$

(Here $Q[G]$ denotes the regular representation of $G$ over $Q$.)

Proof.

Two elements of $R(G;\mathbb{Q})$ are equal if their characters are equal. Thus the assertion of the Proposition is equivalent to:

(5.3.5) \[ \chi(X) = |G| \chi(X/G), \]

(5.3.6) \[ \chi_G(X)(g) = 0 \quad \text{for } g \neq 1. \]

(Note that $\chi_G(X)(g)$ is the Lefschetz-number)

$$L(g,X) = \sum_{i \geq 0} (-1)^i \text{Trace}(g,H^i(X;\mathbb{Q}))$$

for the action of $g$; and under reasonable circumstances the Lefschetz-number of a map without fixed points should be zero.)

We first prove 5.3.5 and 5.3.6 for cyclic groups. Since $H^*(X;\mathbb{Z})$ is finite the universal coefficient formula for cohomology with compact support (Spanier [152], p. 338) shows

(5.3.7) \[ \chi(X;\mathbb{Q}) = \chi(X;\mathbb{F}_p). \]

The Cartan spectral sequence of a covering shows that $H^*(X/G;\mathbb{Z})$ is $\mathbb{Z}$-finite. Hence we obtain from 5.3.3 and 5.3.7, using induction on $|G|$,
that 5.3.5 is true for cyclic $G$.

The existence of the transfer for finite groups implies the isomorphism (Bredon \[37\] , III 7.2)

\[(5.3.8) \quad H^i(X,\mathbb{Q})^G \cong H^i(X/G;\mathbb{Q}).\]

Since for any character $\psi$ of $G \dim \psi^G = |G|^{-1} \sum \psi(g)$ we obtain from 5.3.5 and 5.3.8

\[(5.3.9) \quad \sum_{g \neq 1} \chi_G^G(X)(g) = 0.\]

Using this we prove 5.3.6 for cyclic groups by induction over the group order: We start with

\[H^i(X,\mathbb{Q}) \cong H^i(X/G;A)\]

where $A$ again is the local coefficient system with typical stalk $\mathbb{C}[G]$.
Let $g$ be a generator of $G$. We decompose the coefficient system $A$ according to the irreducible $\mathbb{C}[G]$-modules

\[A = \bigoplus A_j, \quad 0 \leq j < m = |G|\]

where $g$ acts on $A_j$ through multiplication with $\zeta^j = \exp(2\pi ij/m)$. The equalities

\[\text{Tr}(g^k, H^i(X;\mathbb{Q})) = \sum_j \text{Tr}(g^k, H^i(X/G;A_j)) = \sum_j \zeta^{jk} \dim H^i(X/G;A_j)\]

yield for the Lefschetz-number
But \(L(g^k, X) \in \mathbb{Z}\) is obtained from \(L(g, X)\) for \((k, m) = 1\) by applying a Galois automorphism of \(\mathbb{Q}(\xi)\) over \(\mathbb{Q}\). Therefore \(L(g^k, X) = L(g, X)\) for \((k, m) = 1\). From 5.3.9 we obtain

\[
(5.3.10) \quad 0 = \sum_{(k,m)=1} L(g^k, X) + \sum_{(k,m)\neq 1} L(g^k, X)
\]

By the inductive assumption the second sum in 5.3.10 is zero, and since the summands of the first sum are all equal we see that \(L(g, X) = 0\). This proves 5.3.6 in general. Again using 5.3.8 and 5.3.9 we obtain 5.3.5 for general \(G\).

We have actually proved in 5.3.4 a special case of the Lefschetz fixed point theorem.

Proposition 5.3.11.

Let \(X\) be a compact \(G\)-ENR where \(G\) is a cyclic group with generator \(g\). Then the Lefschetz number

\[
L(g, X) = \sum_{i \geq 0} (-1)^i \text{Trace } (g, H^i(X; \mathbb{Q}))
\]

is equal to the Euler-Characteristic \(\chi(X)^g\).

Proof.

Let \(X_1 = X^g, X_2, \ldots, X_r\) be the orbit bundles of \(X\). Then \(H^*(X, \mathbb{Z})\) (cohomology with compact support) is \(\mathbb{Z}\)-finite and \(L(g, X_j) = 0\) for \(j > 1\) by 5.3.4. Hence

\[
L(g, X) = \sum_j L(g, X_j) = L(g, X_1)
\]
and clearly $L(g, x^1) = \chi(x^g)$.

**Corollary 5.3.12.**

Let $G$ be a finite group and let $X$ be a compact $G$-ENR. Then

$$\chi(x^G) = \frac{1}{|G|} \sum_{g \in G} \chi(x^g).$$

**Proof.**

From $H^i(X/G) \cong H^i(X)^G$ and $\dim H^i(X)^G = \frac{1}{|G|} \sum_{g \in G} \text{Trace}(g, H^i(X))$ the result follows, using 5.3.11.

We can now compute the equivariant Euler-Characteristic $\chi_G(X)$.

**Theorem 5.3.13.**

Let $G$ be a compact Lie group and $X$ be a compact $G$-ENR. Then

$$\chi_G(X) = \sum (H) \chi(X^{(H)}/G) \chi_G(G/H)$$

where the sum is taken over those isotropy types $(H)$ of $X$ such that $NH/H$ is finite.

**Proof.**

By additivity of the Euler-Characteristic

$$\chi_G(X) = \sum (H) \chi_G(X^{(H)}).$$

Thus we have to show: $\chi_G(X^{(H)}) = 0$ if $NH/H$ is infinite and

$$L(g, x^{(H)}) = \chi(x^{(H)}/G) L(g, G/H)$$

(5.3.14) otherwise ($g \in G$). Let $C$ be the closed subgroup of $G$ generated by $g$. 

Since \( L(g, Y) \) only depends on the image of \( g \) in the group of components of \( C \) we can find an element \( h \in C \) of finite order such that \( L(g, Y) = L(h, Y) \) for all \( Y \). We fix \( h \) with this property. Since \( X \) is a compact \( G \)-ENR we can find compact \( G \)-ENR's \( Y \supseteq Z \) in \( X \) such that \( Y \setminus Z = X_{(H)} \). The proof of 5.3.11 shows

\[
L(h, X_{(H)}) = \chi(X_{(H)}^h).
\]

Using the fibre bundle

\[
G/H \longrightarrow X_{(H)} \longrightarrow X_{(H)}/G
\]

and 5.2.10 we obtain

\[
\chi(X_{(H)}^h) = \chi(X_{(H)}/G) \chi(G/H^h).
\]

Again by 5.3.11 \( \chi(G/H^h) = L(g, G/H) \), so we see that 5.3.14 is true in general. But \( \chi(G/H^h) = 0 \) if \( NH/H \) is infinite because \( NH/H \) acts freely on \( G/H^h \).

Remark 5.3.15.

If \( G \) is finite then \( \chi_G(G/H) \) is just the permutation representation associated to the \( G \)-set \( G/H \). In general \( \chi_G(G/H) \in \mathbb{R}(G/G_0; \mathbb{Q}) \) where \( G_0 \) is the component of the identity of \( G \). We would like to see that this is actually a permutation representation.

Problem 5.3.16.

What are the most general assumptions on the spaces which imply the decomposition formula 5.3.13? A similar formula holds for the equivariant Lefschetz number of a \( G \)-map \( f : X \rightarrow X \) between compact \( G \)-ENR's. Also this should be generalized to more general spaces.
5.4. Universal Euler-Characteristic for G-spaces.

The classical computation of the Euler-Characteristic from a cell decomposition of a space indicates that suitable axioms (like 5.16 (i), (ii)) determine the Euler-Characteristic uniquely. This is carried out in Watts [166]. We present a similar argument for G-spaces without insisting on a minimal set of axioms.

An Euler Characteristic for finite G-CW-complexes consists of an abelian group \(A\) and map \(b\) which associates to each finite CW-complex \(X\) an element \(b(X) \in A\) such that:

(i) If \(X\) and \(Y\) are G-homotopy-equivalent then \(b(X) = b(Y)\).

(ii) If \(X\) and \(Y\) are subcomplexes of \(Z\) then

\[ b(X) + b(Y) = b(X \cup Y) + b(X \cap Y). \]

Given such an Euler-Characteristic \(b\) we show

Proposition 5.4.1.

Let \(X\) be a finite G-CW-complex. Then

\[ b(X) = \sum_{(H)} n_H b(G/H) \]

where

\[ n_H = \sum_{i \geq 0} (-1)^i n(H,i) \]

\(n(H,i)\) the number of \(i\)-cells of type \((H)\), and the sum is taken over conjugacy classes of subgroups of \(G\).

Proof.

Induction on the number of cells and dimension. Let \(Z = X \cup (G/H \times e^n)\) be obtained from \(X\) by attaching an \(n\)-cell of type \((H)\). Let \(Y = G/H \times D^n(1/2)\)
be the closed cell in $G/H \times e^n$ of radius 1/2. If we remove $Y$ from $Z$ then the resulting space is $G$-homotopy-equivalent to $X$. Therefore

$$b(Z) = b(X) + b(G/H \times D^n) - b(G/H \times S^{n-1}).$$

One shows by induction

$$b(G/H \times S^n) = 1 + (-1)^{n+1} b(G/H);$$

namely if $D_+$ and $D_-$ are the upper and lower hemisphere of $S^n$ respectively then

$$b(G/H \times S^n) = b(G/H \times D_+) + b(G/H \times D_-) - b(G/H \times S^{n-1})$$

$$= 2b(G/H) - (1+(-1)^n) b(G/H)$$

$$= 1 + (-1)^{n+1} b(G/H).$$

Put together we obtain

$$b(Z) = b(X) + (-1)^n b(G/H),$$

the induction step.

An Euler-Characteristic $(U(G),u)$ for finite $G$-CW-complexes is called universal, if every Euler-Characteristic $(A,b)$ as above is obtained from $(U(G),u)$ by composing with a unique homomorphism $U(G) \rightarrow A$. As usual for universal objects uniqueness up to isomorphism follows.

From 5.4.1 we obtain existence:
(5.4.2) $U(G)$ free abelian group with basis

$$[G/H], \ (H) \in \mathcal{C}(G).$$

$$u(x) = \sum_{(H)} n_{(H)} [G/H].$$

Instead of $u(x)$ we also write $[x]$, in accordance with the notation $[G/H]$ for the basis elements. We now aim at another characterisation of $U(G)$ which is not based on CW-complex and which shows that $b(x)$ in 5.4.1 is independent of the cell decomposition.

**Proposition 5.4.3.**

We have $[x] = [y]$ in $U(G)$ if and only if for all $H \leq G$

$$\chi(x^H/N_H) = \chi(y^H/N_H).$$

**Proof.**

Suppose $[x] = [y]$. We consider the mapping

$$b_H : \mathbb{Z} \rightarrow \chi(z^H/N_H)$$

from finite $G$-CW-complexes into $\mathbb{Z}$. This mapping satisfies (i) and (ii) in the definition of an Euler-Characteristic for finite $G$-CW-complexes. From the universal property of $U(G)$ we obtain $b_H(x) = b_H(y)$. For the converse be have to show that the totality of maps $b_H : U(G) \rightarrow \mathbb{Z}$ defines an injective map $U(G) \rightarrow \mathfrak{f}(H)\mathbb{Z}$. Let $0 \neq x = \sum a_H [G/H] \in U(G)$. Let $H$ be maximal such that $a_H \neq 0$. Then

$$b_H(x) = a_H \chi((G/H^H)/N_H) = a_H \neq 0.$$
Definition and Proposition 5.4.4.

On the set of compact G-ENR introduce the equivalence relation: 
\[ X \sim Y \iff \text{for all } H < G \text{ the equality } x(x^H / NH) = x(y^H / NH) \text{ holds.} \]
Let \( U(G) \) be the set of equivalence classes and let \([X] \in U(G)\) be the class of \( X \). Disjoint union induces on \( U(G) \) the structure of an abelian group. This group is free abelian with basis \([G/H], H \in C(G)\). We have

\[
(5.4.5) \quad [X] = \sum_{(H)} x_c (X(H)/G) [G/H].
\]

Proof.

We have to show that inverses exist for addition. Let \( K \) be a compact ENR with trivial G-action and \( x(K) = -1 \). Then \([X] + [K \times H] = 0\) in \( U(G) \) because \( x(x^H) + x((K \times H)^H) = 0\) for all \( H < G \). As in the proof of 5.4.3 one shows that the \([G/H]\) are linearly independent. We show that the \([G/H]\) span \( U(G) \) by proving 5.4.5. By additivity of the Euler-Characteristic we have

\[
x(x^K / NK) = \sum_{(H)} x_c (X(H)^H / NK).
\]

Now \( X(H) \longrightarrow X(H)/G \) is a fibre bundle with fibre \( G/H \) and as G-space \( X(H) \) has the form \( G/H \times_{NH} X_H \) (see Bredon , p. 88). Hence \( X^K(H) / NK \longrightarrow X(H)/G \) is a fibre bundle with fibre \( G/H^K / NK \). From we obtain

\[
x_c (X(H)^K / NK) = x((G/H^K) / NK) x_c (X(H)/G).
\]

This shows that both sides of 5.4.5 describe the same element in \( U(G) \).

Definition and Proposition 5.4.6.

Cartesian product of representatives induces a multiplication on \( U(G) \). Addition and multiplication make \( U(G) \) into a commutative ring with identity. This ring is called the Euler-ring of the compact Lie group \( G \).
Proof.

We need only show that multiplication is well-defined, i.e. we have to show that the numbers $\chi((X \times Y)^K/NK)$ can be computed from the $\chi(X^H/NH)$, $\chi(Y^H/NH)$ or, equivalently, from the $\chi_c(X^H/NH)$, $\chi_c(Y^H/NH)$. We begin with

$$\chi((X \times Y)^K/NK) = \sum (H) \chi_c((X \times Y)^K/(NH)).$$

The map

$$(X \times Y)^K/(NH) \to (Y^H/G)$$

is a fibre bundle with fibre

$$(X^K \times G/H^K)/NK.$$ 

Now we use the fact that $G/H^K$ consists of a finite number of $NK$-orbits (Bredon [37], p. 87), say

$$G/H^K = \sum U^{NK/U}$$

as $NK$-space. Using this information and 5.2.10 we obtain

$$\chi_c((X \times Y)^K/(NH)) = \sum U \chi_c(Y^H/G) \chi(X^K/U).$$

Finally, using

$$\chi(X^K/U) = \sum (H) \chi_c(X^H/G) \chi((G/H^K)/U),$$

we see that $\chi((X \times Y)^K/NK)$ can be computed from the $\chi_c(X^H/NH)$, $\chi_c(Y^H/NH)$. 
We show in the next section that for finite $G$ $U(G)$ is the Burnside ring of $G$. For non-finite $G$ $U(G)$ contains nilpotent elements. In order to obtain the product structure one has to compute $[G/H] [G/K]$.

**Proposition 5.4.7.**

Suppose $NH/H$ is not finite. Then $[G/H] \in U(G)$ is nilpotent.

**Proof.**

By the descending chain condition for subgroups of $G$ the spaces $G/H^k$, $k \geq 1$, altogether only contain a finite number of isotropy groups. If $[G/H]^K = \sum_{(K)} a_K [G/K]$ with $a_K \neq 0$ and $(K)$ maximal with this property then $[G/H]^{k+1}$ does not contain $[G/K]$ with a non-zero coefficients:

Expanding $[G/H]^{k+1}$ then $G/K$ could only occur from the expansion of $a_K [G/H] [G/K]$. But $(G/H \times G/K)_K = G/H^K \times NK/K$ and therefore

$\chi_c((G/H \times G/K)_K/NK) = \chi(G/H^K) = 0$ because $NH/H$ acts freely on $G/H^K$ and $\chi(NH/H)$ is zero if $NH/H$ is not finite (e.g. because a circle group acts freely on $NH/H$).

**5.5. The Burnside ring of a compact Lie group.**

Let $G$ be a compact Lie group. On the set of compact $G$-ENR's consider the equivalence relation: $X \sim Y$ $\iff$ for all $H < G$ the Euler-Characteristics $\chi(X^H)$ and $\chi(Y^H)$ are equal. Let $A(G)$ be the set of equivalence classes and let $[X] \in A(G)$ be the class of $X$. Disjoint union and cartesian product induce composition laws addition and multiplication, respectively, on $A(G)$. It is easy to see that $A(G)$ with these composition laws is a commutative ring with identity. We call $A(G)$ the Burnside ring of $G$. We will show in a moment that this definition is consistent with the earlier one of section 1. (for finite $G$).

Let $\Phi(G)$ be the set of conjugacy classes $(H)$ such that $NH/H$ is finite.
Proposition 5.5.1.

Additively, $A(G)$ is the free abelian group on $[G/H]$, $(H) \in \Phi(G)$. For a compact $G$-ENR $X$ we have the relation

$$[x] = \sum_{(H) \in \Phi(G)} \chi(x_{(H)/G}) [G/H].$$

The assignment $X \mapsto \chi(x^H)$ induces a ring homomorphism $\psi_H : A(G) \to \mathbb{Z}$.

Proof. (Compare 5.4.4).

The last assertion is obvious from the definition. The $[G/H]$, $(H) \in \Phi(G)$, are linearly independent: Given $x = \sum a_H [G/H] \in A(G)$. Choose $(H)$ maximal such that $a_H \neq 0$. Then

$$\psi_H x = a_H \chi(G/H^H) = a_H \mid NH/H \mid \neq 0$$

and therefore $x \neq 0$.

Given a compact $G$-ENR $X$. Then

$$\chi(x^K) = \sum_{(H)} \chi(x_{(H)/G}^H) = \sum \chi(G/H^K) \chi(x_{(H)/G}).$$

The summands with $NH/H$ not finite vanish, because $NH/H$ then acts freely on $G/H^K$ so that $\chi(G/H^K) = 0$. This computation shows that $[X]$ and $\sum \chi(x_{(H)/G}) [G/H]$ have the same image under $\psi_K$, $(K) \in \Phi(G)$, hence are equal in $A(G)$.

The map

$$v : U(G) \longrightarrow A(G) : [X] \longrightarrow [X]$$

is a well-defined ring homomorphism. By 5.5.1 and 5.4.4 it is surjective, and bijective for finite $G$. In particular we have
Proposition 5.5.2.
For finite $G$ the rings $U(G)$, $A(G)$, and the Burnside ring of finite $G$-sets are canonically isomorphic.

Proposition 5.5.3.
The kernel of $v : U(G) \to A(G)$ is the nilradical (= set of nilpotent elements) of $U(G)$.

Proof.
Since the $\mathcal{V}_H : A(G) \to \mathbb{Z}$ detect the elements of $A(G)$ the ring $A(G)$ cannot have nilpotent elements (different from zero). Now use 5.4.4, 5.4.7, and 5.5.1.

Remark 5.5.4.
The previous Proposition implies in particular that $U(G)$ and $A(G)$ have the same prime ideal spectrum.

Remark 5.5.5.
In contrast to the situation in section 1 with our new definition of $A(G)$ also the negatives of all elements are represented by geometric objects.

We now give some immediate applications of the geometric definition of $A(G)$.

Recall that we have in 5.3 associated with every compact $G$-ENR $X$ the equivariant Euler-Characteristic

$$\chi_G(X) = \sum_{i \geq 0} (-1)^i H^i(X;\mathbb{Q}) \in R(G;\mathbb{Q}).$$
Proposition 5.5.6.

The assignment $X \mapsto \chi_G(X)$ induces a ring homomorphism

$$\chi_G : A(G) \longrightarrow R(G;\mathbb{Q}).$$

Proof. In order to show $\chi_G$ is well-defined we have to show that the character $\chi_G(X)$ can be computed from Euler-Characteristics of fixed point sets. But this is the content of 5.3.11, and the same Proposition shows that $\chi_G$ respects addition and multiplication.

Remark 5.5.7.

The homomorphism $\chi_G$ generalizes the permutation representation of finite $G$-sets.

We have mentioned in 1.5 the construction of units of $A(G)$ using representations. We can now make this precise.

The homomorphisms $\varphi_H : A(G) \longrightarrow Z$ combine to an injective (by definition $A(G)$) ring homomorphism

$$\varphi : A(G) \longrightarrow \prod_{(H)} Z \quad (5.5.8)$$

where the product is taken over the set $C(G)$ of conjugacy classes of closed subgroups of $G$. We use $\varphi$ to identify elements of $A(G)$ with functions $C(G) \longrightarrow Z$ (see 5.6 for an elaboration of this point of view).

Proposition 5.5.9.

Let $V$ be a real representation of $G$. Then $u(V) : (H) \mapsto (-1)^{\dim V_H}$ is a unit of $A(G)$. The assignment $V \mapsto u(V)$ induces a homomorphism

$$u : R(G;\mathbb{R}) \longrightarrow A(G)^*.$$
Proof.
Let $S(V)$ be the unit sphere in $V$. Then

$$\chi(S(V)^H) = 1 - (-1)^{\dim V^H}.$$

Hence $1 - [SV] \in \Lambda(G)$ represents the function $u$.

Proposition 5.5.10.
The multiplication table of the $[G/H] \in \Lambda(G)$ has non-negative coefficient, i.e. if

$$[G/H] [G/K] = \sum_{L} n_L [G/L]$$

then $n_L \geq 0$.

Proof.
We have $n_L = \chi((G/H \times G/K)_{(L)}/G)$.
Moreover

$$(G/H \times G/K)_{(L)}/G \cong (G/H \times G/K)_{L}/NL \subset (G/H \times G/K)^{L}/NL.$$

But by Bredon [37], II. 5.7, the space $G/H^L$ consists of finitely many $NL/L$-orbits. Since $NL/L$ is finite the set $(G/H \times G/K)_{(L)}/G$ is finite and its Euler-Characteristic therefore non-negative.

5.6. The space of subgroups.
We recall some notions from point set topology. Let $E$ be a metric space with bounded metric $d$. Let $F(E)$ be the set of non-empty subsets of $E$. 

(Here $R(G;\mathbb{R})$ is the real representation ring of $G$, also denoted $RO(G)$.)
On $F(E)$ one has the Hausdorff metric $h$ defined by

$$h(A,B) = \max(r(A,B), r(B,A))$$

with $$r(A,B) = \sup(d(x,B) \mid x \in A).$$

If $E$ is complete then $F(E)$ is complete. If $E$ is compact then $F(E)$ is compact.

The convergence of a sequence $X_i$ to the limit $X$ can be expressed as follows: For any $\varepsilon > 0$ there exists $n_0$ such that for $n > n_0$:

(a) for $x \in X_n$ there exists $y \in X$ with $d(x,y) < \varepsilon$.  
(b) for $x \in X$ there exists $y \in X_n$ with $d(x,y) < \varepsilon$.

If $Y_n$ is the closure of $\bigcup_{p \geq n} X_p$ then $X$ is the intersection of the $Y_n$.

We want to use this metric on the set $S(G)$ of closed subgroups of the compact Lie group $G$.

**Proposition 5.6.1.**

(i) $S(G)$ is a closed (hence compact) subset of $F(G)$.

(ii) The action $G \times S(G) \to S(G) : (g,H) \mapsto gHg^{-1}$ is continuous. The quotient space $C(G)$ is a countable, hence a totally disconnected, compact Hausdorff space.

(iii) $\Phi(G) \subset C(G)$ is a closed subspace.

**Proof.**

(i) We start with a bi-invariant metric $d$ on $G$. Let $X = \lim H_i$, $H_i \in S(G)$. Given $x,y \in X$, $\varepsilon > 0$, choose $n_0$ such that for $n > n_0$ there exists $x_n, y_n \in H_n$ with $d(x,x_n) < \varepsilon/2$, $d(y,y_n) < \varepsilon/2$. Then $d(xy^{-1}, x_ny_n^{-1}) < \varepsilon$. If $xy^{-1} \notin X$ then $X \cup \{xy^{-1}\}$ would satisfy conditions
(a), (b) above, a contradiction.

(ii) Let \( \lim g_i = g \) in \( G \) and \( \lim H_i = H \) in \( S(G) \). Using
\[
d(g_n x_n g_n^{-1}, g x g^{-1}) \leq 2d(g, g_n) + d(x, x_n),
\]
which follows from the triangle inequality and bi-invariance, one shows that \( gHg^{-1} \) is precisely the set of points satisfying (a) and (b) above for the sequence \( g_n H g_n^{-1} \). The space \( C(G) \) is countable: see Palais [124], 1.7.27.

(iii) We show that \( S_0(G) = \{ H \mid NH/H \text{ finite} \} \) is closed in \( S(G) \). Let \( H = \lim H_i, H_i \in S(G) \). By a theorem of Montgomery and Zippin (Bredon [37], p. 87) there exists an \( \varepsilon > 0 \) such that any subgroup in the \( \varepsilon \)-neighbourhood of \( H \) is conjugate to a subgroup of \( H \). Hence the \( H_i \) are eventually conjugate to subgroup of \( H \). But if \( K \in S_0(G) \) and \( K < H \) then \( H \in S_0(G) \); this follows e.g. from Bredon [37], II. 5.7, because \( G/H \) consists of finitely many \( NK/K \)-orbits hence is a finite set with free \( NH/H \)-action.

We now show that convergence in \( S(G) \) and \( C(G) \) is equivalent in the following sense.

Proposition 5.6.2.

Let \( (H) = \lim (H_i) \) in \( C(G) \). There exists an \( n_0 \) and \( K_n \in S(G), n > n_0 \), such that \( (K_n) = (H_n), K_n < H, \lim K_n = H \).

Proof.

By the theorem of Montgomery and Zippin (Bredon [37], II. 5.6) we can find for each \( \varepsilon > 0 \) an integer \( n_0(\varepsilon) \) such that for \( n > n_0(\varepsilon) \) there exists an \( u_n \) with \( d(u_n, 1) < \varepsilon \) and \( u_n H u_n^{-1} < H \). Therefore we can find a sequence \( g_n \in G \) converging to \( 1 \) such that for almost all \( n \)
\[
g_n H g_n^{-1} < H.
\]

In view of the preceding Proposition it is interesting to know which compact Lie groups \( G \) are limits of a sequence of proper subgroups.
Proposition 5.6.3.

G is a limit of proper subgroups if and only if G is not semi-simple.

Proof.

Suppose \( G = \lim H_n \), \( H_n \not\supseteq G \). Let \( G^0 \) be the component of 1 of G and put \( K_n = G^0 \cap H_n \). Then \( \lim K_n = G^0 \) so that without loss of generality we can assume that G is connected. By passing to a subsequence we can assume that the components \( H_n^0 \) converge to \( H \) and therefore must have eventually the same dimension as \( H \). But then the \( H_n^0 \) are conjugate to \( H \) and by conjugating the whole sequence we arrive at the situation:

\[
G = \lim L_n, \quad L_n^0 =: L \text{ for all } n, \quad L \not\supseteq G.
\]

Since \( L \not\supseteq L_n \) we must have \( L \not\supseteq G \) and \( G/L \) is the limit of finite subgroups \( L_n/L \). We now invoke the theorem of Jordan (Wolf [169]) which says that there exists an integer \( j \) such that any finite subgroup of \( G/L \) has a normal abelian subgroup of index less than \( j \). Choose such a large abelian normal subgroup \( A_n \) in \( L_n/L \). The limit \( A \) of the \( A_n \) is then an abelian normal subgroup of index less than \( j \) in \( G/L \). Since \( G/L \) is connected we must have \( G/L = A \) a torus and therefore \( G \) is not semi-simple.

Conversely if \( G \) is not semi-simple we can find a normal subgroup \( L \) of \( G^0 \) such that \( G^0/L \) is a non-trivial torus (Hochschild [37], XIII Theorem 1.3). By Lie algebra considerations (e.g. Helgason [36], II. Proposition 6.6) the group \( L \) is a characteristic subgroup of \( G^0 \) and therefore a normal subgroup of \( G \). Therefore \( G/L =: P \) is a finite extension of a torus

\[
1 \rightarrow T \rightarrow P \rightarrow F \rightarrow 1,
\]

\( T \) a torus, \( F \) finite. If we show that \( P \) is a limit of proper subgroups then \( G \) is a limit of proper subgroups. We shall show in section 5.10 what the finite subgroups of \( P \) are, in particular we shall see that \( P \) is a
limit of finite subgroups.

**Proposition 5.6.4.**

If $X$ is a compact $G$-ENR then the mapping $C(G) \rightarrow Z : (H) \mapsto \chi(x^H)$ is continuous ($Z$ carries the discrete topology).

**Proof.**

Let $(H) = \lim (H(i))$. By 5.6.2 we can assume $H(i) < H$ and $H = \lim H(i)$. We can and do assume $H = G$ (otherwise consider the $H$-space $X$). We choose a bi-invariant metric on $X$. Put $\varepsilon = \min h(K,G)$ where $(K)$ runs through the finite set of orbit-types of $X$ unequal to $G$. Since $(L) < (K)$ implies $h(L,G) > h(K,G)$ we have: $h(L,G) < \varepsilon$ implies $(L) \not< (K)$ for all isotropy types of $X$ except possibly $(G)$. Thus if $h(H(i),G) < \varepsilon$ then $X^H(i) = \bigcup X^H(i) \cdot H(i) = X^G(K) = X^G$.

**5.7. The prime ideal spectrum of $A(G)$.**

Recall the ring homomorphisms $\Phi_H : A(G) \rightarrow Z$ (see 5.5). If $(p) \subseteq Z$ is a prime ideal then

$$q(H,p) := \Phi_H^{-1}(p) \subseteq A(G)$$

is a prime ideal of $A(G)$. We show that all prime ideals of $A(G)$ arise in this way.

**Proposition 5.7.1.**

Given $H \triangleleft K \triangleleft G$. Assume that $K/H$ is an extension of a torus by a finite $p$-group ($K/H$ a torus if $p = 0$). Then $q(H,p) = q(K,P)$.

**Proof.**

For a certain $L$ we have $H \triangleleft L \triangleleft K$, $L/H$ is a torus, and $K/L$ a finite $p$-group. Let $X$ be a compact $G$-ENR. The group $K/L$ acts on $M^L$ with fixed
point set \( M^K \). Hence \( \chi(M^K) \equiv \chi(M^L) \mod p \) and \( \chi(M^L) = \chi(M^H) \) by an easy application of Theorem 5.3.

**Theorem 5.7.2.**

Every prime ideal \( q \) of \( A(G) \) has the form \( q(H,p) \) for a suitable \( (H) \in \phi(G) \). Given \( q \) there exists a unique \( (K) \in \phi(G) \) with \( q = q(K,p) \) and \( \varphi_K(G/K) \neq 0(p) \) where \( p \) is the characteristic of \( A(G)/q \).

**Proof.**

We closely follow Dress [73]. Let

\[
T(q) = \{ (H) \in \phi(G) \mid [G/H] \notin q \}.
\]

Then \( T(q) \) is not empty because \( (G) \in \phi(G) \) and \( [G/G] = 1 \notin q \). Let \( (H) \) be minimal in \( T(q) \); this exists because compact Lie groups satisfy the descending condition. We claim that for any \( x \in A(G) \) we have a relation of the type

\[
(5.7.3) \quad [G/H] x = \varphi_H(x) [G/H] + \sum a_K [G/K]
\]

where the sum is over \( (K) < (H) \), \( (K) \notin (H) \). To see this we take \( x = [X] \) look at the orbits of \( G/H \times X \) and see from 5.5.1 that a relation must hold as claimed with some constant \( c \) instead of \( \varphi_H(x) \). We then determine \( c \) if we apply \( \varphi_H \) to both sides of this equation. (This uses \( \varphi_H(G/H) \neq 0 \), i.e. \( (H) \in \phi(G) \).) But 5.7.3 implies \( [G/H] x \equiv \varphi_H(x)[G/H] \mod q \) (by minimality of \( G/H \)) and dividing by \( [G/H] \notin q \) we get

\[
x \equiv \varphi_H(x) \mod q \text{ or } q = q(H,p) \text{ with } p \text{ the characteristic of } A(G)/q.
\]

If \( K \) is any subgroup of \( G \) with \( q = q(K,p) \) and \( \varphi_K(G/K) \neq 0 \mod p \) for \( p = \text{char } A(G)/q \) then for an \( (H) \) as in the beginning of the proof

\[
\varphi_K(G/K) \equiv \varphi_H(G/K) \neq 0 \mod p.
\]

In particular \( G/H^K \) is not empty; and
similarly $G/K^H$ is not empty. This can only happen if $(H) = (K)$.

**Proposition 5.7.4.**

Every homomorphism $f : A(G) \rightarrow R$ into an integral domain $R$ has the form $f(x) = \psi_K(x)^1$ for a suitable $K < G$.

**Proof.**

The kernel of $f$ is a prime ideal $q(K,p)$. Therefore

$$f : A(G) \rightarrow A(G)/q(K,p) \rightarrow R$$

must be the map $x \mapsto \psi_K(x)^1$, because there is a unique isomorphism $A(G)/q(K,p) \cong \mathbb{Z}/(p)$.

**Proposition 5.7.5.**

(i) If $q(K,o) = q(L,o)$ and $(K) \notin \phi$ then (up to conjugation) $L \triangleleft K$ and $K/L$ is a torus.

(ii) Given $L < G$ there exists $K \in \phi$ such that $L \triangleleft K$ and $K/L$ is a torus. Moreover we have in this case $\psi_L = \psi_K$.

**Proof.**

(i) Since $q(K,o) = q(L,o)$ by 5.7.2 $\psi_K = \psi_L$. From

$$\chi(G/K^L) = \psi_L(G/K) = \psi_K(G/K) = |NK/K| \neq 0,$

we see that $G/K^L$ is non-empty and hence $(L) < (K)$. We take $L < K$. Let $T$ be a maximal torus in $NL/L$ and let $P$ be its inverse image in $NL$. By 5.7.1 $q(P,o) = q(L,o)$. We show $P \in \phi$; then by 5.7.2 $(P) = (K)$. Assume $(P) \notin \phi$. Then $NP/P$ contains a non-trivial maximal torus $S$. We let $Q$ be its inverse image in $NP$. We claim that $L$ is still normal in $Q$. Let $q \in Q$ induce the conjugation automorphism $c_q$ on $P$. Since $Q/P$ is a torus, $c_q$ is homotopic to an inner automorphism, hence (e.g. by Conner-Floyd [47], 38.1) an inner automorphism itself and preserves the normal subgroup $L$. From the exact sequence

$$0 \rightarrow P/L \rightarrow Q/L \rightarrow S \rightarrow 0$$
and $P/L = T$ we conclude that $Q/L$ is a torus and hence $T$ is not a maximal torus.

(ii) Use the proof of (i) and 5.7.1.

As a corollary of 5.6.4 and 5.7.5 we obtain

**Corollary 5.7.6.**

Let $C(\phi(G),\mathbb{Z})$ be the ring of continuous (= locally constant, in this case) functions. Then

\[
\psi : A(G) \longrightarrow C(\phi(G),\mathbb{Z})
\]

\[
\psi(x) : (H) \longmapsto \psi_H(x), \text{ is defined and an injective ring homomorphism.}
\]

The possible equalities $q(H,p) = q(K,p)$ are not so easy to describe. We show that in a certain sense 5.7.1 is the only reason for such equalities. Given $K \triangleleft G$. If $NK/K$ is not finite or $|NK/K| \equiv 0 \mod p$ we find a subgroup $K \triangleleft P$ with $q(K,p) = q(P,p)$ as follows: Either by the procedure in the proof of 5.7.5 we let $P$ be the inverse image in $NK$ of a maximal torus in $NK/K$ or we let $P$ be the inverse image in $NK$ of a Sylow $p$-group of $NK/K$. Then $(P) \triangleleft \phi$ but it may happen that $|NP/P| \equiv 0 \mod p$. In this case we can iterate the procedure. Either we arrive after a finite number of steps at a group $Q$ with $|NQ/Q| \not\equiv 0 \mod p$, or we get a sequence

\[
P_0 = K \triangleleft P_1 \triangleleft P_2 \triangleleft P_3 \triangleleft \ldots
\]

of groups with $q(P_i,p) = q(P_{i-1},p)$ and $|NP_i/P_i| \equiv 0 \mod p$ for $i \geq 1$.

Let in this case $Q$ be the closure in $G$ of $\bigcup P_i$ (this is the limit in the space of subgroups, see 5.6). By continuity 5.6.4 we still have
\[ q(Q, p) = q(K, p). \] Now again we can apply our construction to Q if \[ |NQ/Q| \equiv 0 \mod p. \] Sooner or later we arrive at the defining group L of the prime ideal with \[ |NL/L| \not\equiv 0 \mod p. \]

That an infinite chain as above can actually occur is shown by the group \( G = O(2) \). The groups in \( \Phi \) are \( O(2) \), \( SO(2) \) and the dihedral groups \( D_m \). We have \( ND_m = D_{2m} \). Hence

\[ q(D_m, 2) = q(D_n, 2) \quad \text{if} \quad n = 2^j m. \]

For finite \( G \) the situation is more tractable.

**Proposition 5.7.8.**

Suppose \( q(H, p) = q(K, p), H \not\cong K, IK/K \not\equiv 0 \mod p, \) where \( K_0 \) is the component of the identity in \( K \), and \( p \not\equiv 0 \). Then up to conjugation \( K \triangleleft H \) and \( H/K \) is a finite \( p \)-group.

**Proof.**

Choose \( P \) such that \( NK > P > K \) and \( P/K \) a Sylow \( p \)-group of \( NK/K \). We claim that \( NP < NK \). Take \( a \in NP \) and let \( K^a \) be the \( a \)-conjugate of \( K \). Then \( K/(K \cap K^a) < P/K^a \), hence \( K/(K \cap K^a) \) is a finite \( p \)-group. On the other hand \( K, K^a \), and \( P \) have the same component \( K_0 \) of the identity, hence \( K/(K \cap K^a) \) is a quotient of \( K/K_0 \) which has order prime to \( p \) by assumption. Therefore \( K = K \cap K^a = K^a \) and \( a \in NK \). But then \( |NP/P| \not\equiv 0 \mod p \), because \( P/K \) was a Sylow \( p \)-group of \( NK/K \). Now 5.7.1 and 5.7.2 imply \( (P) = (H) \) and hence the assertion.

In particular if \( G \) is finite and \( |NH/H| \not\equiv 0 \mod p \) then there exists a unique smallest normal subgroup \( H_p \) of \( H \) such that \( H/H_p \) is a \( p \)-group and we have (with these notations)
Proposition 5.7.9.

\( q(H,p) = q(K,p) \) if and only if \( (H_p) \leq (K) \leq (H) \).

We shall see later that the cokernel of 5.7.7 is a torsion group of bounded exponent. We now make some remarks on the topology of \( \text{Spec } A(G) \), the prime ideal spectrum of \( A(G) \) with the Zariski topology.

Proposition 5.7.10.

The map

\[
q : \phi(G) \times \text{Spec } \mathbb{Z} \longrightarrow \text{Spec } A(G)
\]

\( (H), (p) \longmapsto q(H,p) \)

is continuous, closed and surjective.

Proof.

An element \( x \in C(\phi(G), \mathbb{Z}) =: C \), being a locally constant function, is an integral linear combination of idempotent functions. Therefore this ring is integral over any subring. By an elementary result of commutative algebra (Atiyah-Mac Donald [11], p. 67, Exercise 1) the mapping

\[
\text{Spec } \varphi : \text{Spec } C \longrightarrow \text{Spec } A(G)
\]

is closed (and surjective by 5.7.2). Hence the Proposition follows from the next Lemma.

Lemma 5.7.11.

Let \( X \) be a compact, totally disconnected space. Then

\[
(x,(p)) \longmapsto \{ f \mid f(x) \in (p) \}
\]
defines a homeomorphism

\[ F : X \times \text{Spec } Z \longrightarrow \text{Spec } C(X, Z). \]

Proof.

We ask the reader to recall the topology on Spec (Bourbaki [33], Ch. II). Certainly \( \{ f \mid f(x) \in (p) \} \) is a prime ideal in \( C(X, Z) \) for any \( x \) and \( (p) \), so that \( F \) is defined. To define an inverse, let \( k : Z \to C(X, Z) \) take \( n \) to the constant function \( k_n : x \mapsto n \). This induces a continuous map \( k^\ast : \text{Spec } C(X, Z) \to \text{Spec } Z \). Given \( b \in \text{Spec } C(X, Z) \), let \( p \) be the element generating \( k^\ast(0) \). Then we claim that \( P = \bigcap f \in b f^{-1}(p) \) consists of a single element of \( X \). For if \( p \neq 0 \) and \( P \) is empty, then for each \( x \in X \) there is a function \( g_x \in b \) with \( g_x(x) \notin (p) \). Since \( k_p \in b \), for each \( x \in X \) there is an \( f_x \in b \) with \( f_x(x) = 1 \), i.e. the sets \( f_x^{-1}(1) \) form a closed-open cover of \( X \). Choose a finite subcover

\[ U_i = f_x^{-1}(1), \quad 1 \leq i \leq n. \]

Then one shows by induction on \( i \) that the characteristic function \( K(V_i) \) of \( V_i = U_i \cup \ldots \cup U_i \) is in \( b \) and in particular \( k_1 \in b \), a contradiction. For \( p = 0 \), the same type of argument shows that \( k_m \) with \( m = \text{l.c.m. } (g_{x_i}(x_i)) \) is in \( b \), contradicting \( k^\ast b = (0) \). But if \( x, y \in P \), choose \( f \in b \) with \( f(x) \in (p) \), and choose a closed-open \( U \) with \( x \in U, y \in U \). Then setting

\[ f_1 = f K(U) + (1-K(U)) \]
\[ f_2 = f(1-k(U)) + K(U) \]

we have \( f_1 f_2 = f \in b \). Since \( f_2(x) = 1 \), \( f_2 \notin b \), hence \( f_1 \notin b \), but \( f_1(y) = 1 \), hence \( y \notin P \). Now we have a map \( d : \text{Spec } C(X, Z) \longrightarrow X \) taking \( b \) to the unique element \( P \), and the maps \( F \) and \( d \times k^\ast \) are clearly...
For the continuity of \( d \times x^* \) we need only show \( d \) continuous. But for a closed-open \( V \subset X \), \( d^{-1}(V) = \{ b \mid x(V) \not\in b \} \), which is open, while such \( V \) form a base of the topology of \( X \).

It remains to be seen that \( F \) itself is continuous. But if \( U = \{ b \mid f \not\in b \} \) is a basic open set for some \( f \in C(X,Z) \), and \( q \subset U \), then writing \( q \) as \( F(x,(p)) \) we have \( f(x) = m \not\in (p) \), and \( V = f^{-1}(m) \) is closed-open in \( X \) containing \( x \). Thus \( q \subset F(V \times \{(p) \mid m \not\in (p)\}) \subset U \).

5.8. Relations between Euler-Characteristics.
We have described the Burnside ring of finite \( G \)-sets using congruences among fixed point sets (see 1.3). We generalize this description to compact Lie groups. The geometric interpretation of the Burnside ring then shows that we obtain a complete set of congruences that hold among the Euler characteristics of fixed point sets. We have already used the classical relations:

\[
(5.8.1) \quad \chi(X) \equiv \chi(X^P) \mod p, \quad P \text{ a } p\text{-group}
\]

\[
(5.8.2) \quad \chi(X) = \chi(X^T), \quad T \text{ a torus.}
\]

Using 5.8.2 we have shown in 5.7 that is suffices to consider subgroups \( H \) with finite index in their normalizer. Therefore we pose the problem: Describe the image of

\[
\varphi : A(G) \longrightarrow C(\phi(G),Z) =: C
\]

The next Proposition shows that this can be done by using congruences.
Proposition 5.8.3.

C is a free abelian group with basis \( x_H = |NH/H|^{-1} \varphi(G/H), (H) \in \Phi(G) \).

Proof.

A priori the \( x_H \) are only contained in \( C \otimes \mathbb{Q} \). But since \( NH/H \) acts freely on every fixed point set \( G/H^K \), \( (K) \in \Phi(G) \), we see that the numbers \( \chi(G/H^K) \) are divisible by \( |NH/H| \), and therefore \( x_H \in C \). The elements \( x_H \) are linearly independent over \( \mathbb{Z} \) because the \( G/H \) are. We have to show that each \( x \in C \) is an integral linear combination of the \( x_H \). Since \( x \) is continuous it attains only a finite number of values. Let \( (H_1), \ldots, (H_k) \) be the maximal elements of \( \Phi(G) \) such that \( x(H_i) \neq 0 \). Consider \( x - \sum_{1 \leq i \leq k} x(H_i)x_{H_i} =: y \in C \). If \( y(K) \neq 0 \) then \( (K) \) is strictly smaller than one of the \( (H_i) \). Induction, using the descending chain condition for subgroups, gives the result.

Now let \( X \) be a compact \( G \)-ENR. For \( (H) \in \Phi(G) \) we consider the \( NH/H \)-space \( X^H \). Since \( NH/H \) is a finite group we obtain as in 1.3

\[
\sum_{n \in NH/H} \chi_{NH/H}(X^H)(n) \equiv 0 \mod |NH/H|
\]

and this congruence can be rewritten in the form, using 5.3.,

\[
(5.8.4) \quad \sum_{(K)} n(H,K) \chi(X^K) \equiv 0 \mod |NH/H|
\]

where the sum is taken over conjugacy classes \( (K) \) of \( K \lhd G \) such that \( K \triangleright H \) and \( K/H \) is cyclic; the \( n(H,K) \) are integers such that \( n(H,H) = 1 \).

Proposition 5.8.5.

The congruences 5.8.4 are a complete set of congruences for the image of \( \varphi : A(G) \rightarrow C \), i.e. \( z \in C \) is contained in \( \varphi A(G) \) if and only if for all \( (H) \in \Phi(G) \).
\[
\sum_{(K)} n(H,K) z(H) \equiv 0 \mod |NH/H|,
\]

with the summation convention as for 5.8.4.

**Proof.**

Write \( z \) according to 5.8.3 as integral linear combination 
\[
z = \sum n_K x_K
\]
and suppose that \( z \) satisfies the congruences. If we can show that \( n_K \)
is divisible by \( |NK/K| \) then \( z \in \mathcal{A}(G) \). Choose \( (H) \) maximal with
\( n_H \neq 0 \). Consider the congruence belonging to \( H \). The only term which
is non-zero is \( n(H,H) z(H) = n_H \) which has to be zero mod \( |NH/H| \). There-
fore \( n_H x_H \in \mathcal{A}(G) \). Apply the same argument to \( z - n_H x_H \) etc.
Induction on the "length" of \( z \) in terms of the \( x_K \) gives that \( z \in \mathcal{A}(G) \).

Proposition 5.8.5 tells you which congruences hold among the Euler-
Characteristics of fixed point sets \( X^H \) if \( X \) is a compact \( G \)-ENR. One
would like to know the most general class of spaces for which such
congruences hold. We must ensure that the results of 5.3 are applicable:
The equivariant Euler-Characteristics \( \chi_{NH/H}(X^H) \) should be defined
and the decomposition formula 5.3 should hold.

**Remark 5.8.6.**

A different proof for 5.8.5 in the more general context of certain
modules over \( A(G) \) was given in tom Dieck - Petrie [69].

**Remark 5.8.7.**

As in 1.2.4 one shows that \( \varphi : A(G) \rightarrow C \) can be recovered from the
ring structure of \( A(G) \): namely \( \varphi \) is the inclusion of \( A(G) \) into the
integral closure in its total quotient ring.
5.9. Finiteness theorems.

We collect some finiteness theorems for compact Lie groups.

Proposition 5.9.1.

Let $M$ be a compact differentiable $G$-manifold. Then $M$ has only a finite number of orbit-types.

Proof.

Induct over $\dim M$. An equivariant tubular neighbourhood $U$ of an orbit $X \subset M$ is a $G$-vector bundle hence has only isotropy groups appearing on $X$ or on the unit sphere of a fibre. By induction $U$ has finite orbit type. (See Palais [124], 1.7.25 for more details.)

Proposition 5.9.2.

Let $G$ be a compact Lie group. There are only a finite number of conjugacy classes of subgroups which are normalizers of connected subgroups.

Proof.

(Bredon [26], VII Lemma 3.2) Let $L$ be the Lie algebra of $G$, $E$ its exterior algebra, and $P(E)$ the projective space of $E$. If $h$ is a linear subspace of $L$ with basis $h_1, \ldots, h_k$ then $h_1 \wedge \ldots \wedge h_k$ determines a point $ph$ of $P(E)$ which is independent of the choice of the basis. The adjoint action of $G$ on $L$ induces an action of $G$ on $P(E)$. A subgroup $N$ of $G$ leaves $h$ invariant if and only if $ph$ is fixed under $N$. If $H$ is a subgroup with Lie algebra $h$ then:

$$gHg^{-1} = H \iff \text{ad}(g)h = h \iff g(ph) = ph.$$  

Thus $NH = G_{ph}$. Now apply 5.9.1.
Proposition 5.9.3.
A compact Lie group G contains only a finite number of conjugacy classes (K) where K is the centralizer of a closed subgroup.

Proof.
Let G act on M = G via conjugation. If H < G then M^H is the centralizer ZH. Apply 5.9.1.

We now come to a classical theorem of Jordan. Let σ(G) be the set of finite subgroups of G.

Theorem 5.9.4.
There exists an integer j, depending only on the dimension and the number of components of G, with the following properties: For each \( H \in \sigma(G) \) there exists an abelian normal subgroup \( A_H \) of \( H \) such that \( |H/A_H| < j \). Moreover the \( A_H \) can be chosen such that \( H < K \) implies \( A_H < A_K \).

Proof.
(Boothby and Wang [24], Wolf [169].) Given integers k and d there are only a finite number of groups G with \( |G/G_0| = k \) and \( \text{dim } G = d \), up to isomorphism (see 5.9.5). These groups can therefore be embedded into a fixed \( O(n) \). Hence it suffices to prove the theorem for \( G = O(n) \). A simply proof may be found for instance in Wolf [169], p. 100 - 103.

Theorem 5.9.5.
There exist only a finite number of non-isomorphic compact Lie groups of a given dimension and number of components.

Proof.
This depends on various classical results. We only describe the ingredients.
We begin with connected groups $G$. Then $G$ is of the form

$$G = (T \times H)/D$$

where $T$ is a torus, $H$ is compact semi-simple, $D$ is a finite central subgroup of $T \times H$ such that $D \cap T$ and $D \cap H$ are trivial (Hochschild [97] XIII Theorem 1.3). Therefore the projection of $D$ into $H$ is injective with image contained in the center $ZH$ of $H$. This center $ZH$ is finite by a theorem of Weyl (Helgason [96], II. 6.9.). Hence given $T$ and $H$ there only a finite number of $G$'s. By the classification theorem for semi-simple groups there are only a finite number of $H$'s (Bourbaki [34]). This establishes the theorem for connected groups.

For the general case one has to study finite extensions

$$1 \longrightarrow G_o \longrightarrow G \longrightarrow E \longrightarrow 1$$

where $G_o$ is connected and $E$ is finite. By the general theory of group extensions and the finiteness of the cohomology of finite groups (Mac Lane [112], IV) one sees that the following has to be proved: There are only a finite number of conjugacy classes of homomorphisms $E \longrightarrow \text{Aut}(G_o)/\text{In}(G_o)$ into the group of automorphisms modulo inner automorphisms. In case $G_o$ is a torus the required finiteness follows from the Jordan-Zassenhaus theorem (Curtis-Reiner [48], §79) and the general case is easily reduced to this case.

Theorem 5.9.6.

Let $G$ be a compact connected Lie group. Then there exist only finitely many conjugacy classes of connected subgroups of maximal rank.
Proof.
Borel - de Siebenthal \[23\].

We now consider solvable groups. A compact Lie group is called **solvable** if it is an extension of a torus by a finite solvable group. The **derived group** $G^{(1)}$ of $G$ is the closure of the subgroup generated by commutators. We put inductively $G^{(n)} = (G^{(n-1)})^{(1)}$. A group $H$ is called **perfect** if $H = H^{(1)}$. If $1 \to A \to B \to C \to 1$ is an exact sequence of compact Lie groups, then $B$ is solvable if and only if $A$ and $C'$ are solvable. A compact Lie group $G$ is solvable if and only if there exists an integer $n$ such that $G^{(n)} = \{1\}$. We list the following elementary facts.

**Proposition 5.9.7.**

a) Any subgroup $H$ of $G$ has a unique minimal normal subgroup $H_{a}$ such that $H/H_{a}$ is solvable.

b) For each $H$ there exists an integer $n$ such that $H^{(n)} = H_{a}$.

c) $H_{a}$ is a perfect characteristic subgroup of $H$.

d) $H = H_{a}$ if and only if $H$ is perfect.

e) $(H) = (K) \Rightarrow (H_{a}) = (K_{a})$.

f) $K \trianglelefteq H$, $H/K$ solvable $\Rightarrow K_{a} = H_{a}$.

**Proof.**

a): If $K \trianglelefteq H$, $L \trianglelefteq H$ and $H/K$, $H/L$ are solvable then $K \cap L \trianglelefteq H$ and $H/(K \cap L)$ is solvable. By the descending chain condition for subgroups there is a minimal group as stated. b), c) and d): Since $H/H^{(1)}$ is abelian, by induction $H/H^{(k)}$ is solvable, hence $H^{(k)} \supset H_{a}$ for all $k$, and $H^{(k)}/H_{a}$ is solvable. If $H^{(k)} \neq H_{a}$ then $H^{(k)}$ has a non-trivial abelian quotient, hence $H^{(k)} \neq H^{(k+1)}$. By the descending chain condition there is an $n$ such that $H^{(n)} = H^{(n+1)}$ and for this $n$ necessarily $H^{(n)} = H_{a}$ and $H^{(n)}$ is perfect. The $H^{(n)}$ are characteristic subgroups.
e) and f) are obvious.

Theorem 5.9.8.
Let \( G \) be a compact Lie group. There exists an integer \( n \) such that for all \( H < G \) we have \( H^{(n)} = H_a \).

Proof.
Note that \( H^{(n)} = H_a \) if and only if \( (H/H_a)^{(n)} \) is the trivial group. Therefore we consider pairs \( H, K \) such that \( H < K < G \) and \( K/H \) is solvable. We show that there is an integer \( n \) such that for all such pairs \( (K/H)^{(n)} \) is the trivial group. Let us call the smallest integer \( k \) such that \( L^{(k)} = 1 \) for a solvable group \( L \) the length \( l(L) \) of \( L \).

Take a pair \( K, H \) as above. Since \( K/H \) is solvable we have an exact sequence

\[
1 \longrightarrow T \longrightarrow K/H \longrightarrow F \longrightarrow 1
\]

where \( T \) is a torus and \( F \) is finite solvable. We have

\[
l(K/H) \leq l(T) + l(T) = 1 + l(F).
\]

So we need only show that the length of finite solvable subquotients is bounded. Let generally \( K_0 \) denote the component of 1 of \( K \). Then \( K/H \longrightarrow F \) induces a surjection \( p : K/K_0 \longrightarrow F \). We show in a moment that there exists an integer \( b(G) \) such that for any \( K < G \) there exists an abelian normal subgroup \( A_K \) of \( K/K_0 \) such that \( |K/K_0 : A_K| \leq b(G) \). Let be \( F_0 \) be a \( pA_K \). Then \( F/F_0 \) has order less than \( b(G) \). But \( l(F) \leq l(F_0) + l(F/F_0) = 1 + l(F/F_0) \) because \( F_0 \) is abelian. But \( l(F/F_0) \) is bounded because only a finite number of groups occur.
The existence of the integer \( b(G) \) is proved by induction over \( \dim G \) and \( |G : G_0| \). Given \( G \), the bound exists for the finite subgroups of \( G \) by Theorem 5.9.4. Let \( K \) be a subgroup of positive dimension. Consider

\[
K_0 \leq K < N_K < N_K_0.
\]

Then \( K/K_0 \) is a finite subgroup of \( N_K_0/K_0 =: U \), and \( \dim U < \dim G \). By 5.9.2. only a finite number of \( U \) occur up to isomorphism. This gives by induction the required finiteness.

We put \( WH = NH/H \).

**Theorem 5.9.9.**

There exists an integer \( b \) such that for each closed subgroup \( H \) of \( G \) the index \( |WH : (WH)_0| \) is less than \( b \).

**Proof.**

The proof proceeds in three steps: We first reduce to the case that \( WH \) is finite; then we reduce to the case that \( H \) is finite; and finally we show that for finite \( H \) with finite \( WH \) the order of \( WH \) is uniformly bounded.

The group \( \text{Aut } H/\text{In } H \) of automorphism modulo inner automorphisms is discrete. Conjugation induces an injective homomorphism

\[
NH/ZH \cdot H \longrightarrow \text{Aut } H/\text{In } H
\]

where \( ZH \) is the centralizer of \( H \). Hence \( NH/ZH \cdot H \) being compact and discrete is finite. Hence
Lemma 5.9.10.

WH is finite if and only if ZH/ZH \cap H is finite.

Lemma 5.9.11.

For any H < G the group ZH \cdot H has finite index in its normalizer.

This follows from the previous Lemma and the relations Z(ZH \cdot H) < ZH < ZH \cdot H.

If n \in G normalizes H then also ZH and hence ZH \cdot H. We therefore have

$$\frac{NH}{ZH \cdot H} < \frac{N(ZH \cdot H)}{ZH \cdot H}.$$ 

Using Lemma 5.9.11 and the existence of an upper bound for the set

$$F(G) := \{ |WH| \mid H < G, WH \text{ finite} \}$$

we obtain

Lemma 5.9.12.

There exists an integer c such that for all H < G we have |NH/ZH \cdot H| < c.

Now we obtain the first reduction of our problem. From the exact sequence

$$1 \longrightarrow ZH/ZH \cap H \longrightarrow WH \longrightarrow NH/ZH \cdot H \longrightarrow 1$$

we see that WH/(WH) \circ \rightarrow NH/ZH \cdot H has the kernel which is a quotient of ZH/(ZH) \circ. Now Proposition 5.9.3 and Lemma 5.9.12 show that

$$\{ |WH/(WH)\circ| \mid H < G \}$$

is bounded.
We show by induction over $|G/G_0|$ and $\dim G$ that $F(G)$ has an upper bound $a = a(G/G_0, \dim G)$. For finite $G$ we can take $a = |G|$. Suppose that an upper bound $a(K/K_0, \dim K)$ is given for all $K$ with $\dim K < \dim G$. Let $T(G) = \{H < G \mid WH \text{ finite}\}$. Suppose $H \in T(G)$ is not finite. We consider the projection

$$p : NH_0 \longrightarrow NH_0/H_0 =: U.$$ 

Let $V$ be the normalizer of $H/H_0$ in $U$. Then $WH = V/(H/H_0)$ and therefore $H/H_0 \in T(U)$. Since $\dim U < \dim G$ we obtain by induction hypothesis $|WH| \leq a(U/U_0, \dim U)$. We show that the possible values for $|U/U_0|$ are finite: This follows from 5.9.2. Hence for a given $G$ the possible $|U/U_0|$ are bounded, say $|U/U_0| \leq m(G)$. We have

$$|U/U_0| \leq |G/G_0| m(G_0).$$

By the classification theory of compact connected Lie groups there are only a finite number in each dimension. Hence there exists a bound for $|U/U_0|$ depending only on $|G/G_0|$ and $\dim G$. This proves the induction step as far as the non-finite $H$ in $T(G)$ are concerned.

For the remaining case we use 5.9.4. and 5.9.6.

If $H \in T(G)$ is finite then also $K = NH$ is finite and by Lemma 5.9.11 $K \in T(G)$. We choose $j = j(|G/G_0|, \dim G)$ and $A_H, A_K$ according to 5.9.4. We have

$$|K/H| \leq |K/A_K| \cdot |A_K/H \cap A_K| \leq j|A_K/H \cap A_K|.$$ 

Hence it suffices to find a bound for the $|A_K/H \cap A_K|$. Consider the exact sequence $1 \longrightarrow A_H \longrightarrow H \longrightarrow S \longrightarrow 1$. The conjugation $c(a)$ with
a \in \Lambda_K \text{ is trivial on } \Lambda_H, \text{ because } \Lambda_K > \Lambda_H, \text{ and hence } c(a) \text{ induces an automorphism of } S. \text{ Since } |S| \leq j \text{ this automorphism has order at most } J = j!, \text{ i.e. } c(a^r) \text{ is the identity on } S \text{ and } \Lambda_H \text{ for a suitable } r \leq J. \text{ The group of such automorphisms modulo the subgroups of inner automorphisms by elements of } \Lambda_H \text{ is isomorphic to } H^1(S; \Lambda_H), \text{ with } S \text{ acting on } \Lambda_H \text{ by conjugation. Since this group is annihilated by } |S| \text{ we see that } c(a^s) \text{ is an inner automorphism by an element of } \Lambda_H \text{ for a suitable } s \leq J|S| \leq jJ. \text{ In other words: } a^s h^{-1} \in \Lambda_H. \text{ Hence it is sufficient to find a bound for the order of }

\Lambda_K \cap \Lambda_H \cap \Lambda_K \cap \Lambda_H.

Let U_1 = \Lambda_K \cap \Lambda_H. \text{ By Borel-Serre } [28], \text{ Théorème 1, } U_1 \text{ is contained in the normalizer } NT \text{ of a maximal torus of } G. \text{ Put } U = U_1 \cap T. \text{ Then } |U_1/U| \leq |G/G_o| |wG_o| \text{ where } wG_o \text{ denotes the Weyl group of } G_o. \text{ We estimate the order of } U. \text{ Since } U \text{ is abelian we have } U < ZU. \text{ Moreover } H < ZU \text{ by definition of } ZH. \text{ Since } U \text{ is contained in the center } C = CZU \text{ of } ZU. \text{ The inclusion } H < ZU \text{ implies } C < NH. \text{ Hence } C \text{ is finite.}

We proceed to show that for the order of a finite center } C(G) \text{ of } G \text{ there exists a bound depending only on } |G/G_o| \text{ and dim } G. \text{ We let } G/G_o \text{ act by conjugation on } C(G_o). \text{ Then } C(G) \cap G_o \text{ is the fixed point set of this action. We have } C(G_o) = A \times T_1, \text{ where } A \text{ is a finite abelian group and } T_1 \text{ is a torus. The group } A \text{ is the center of a semisimple group and therefore } |A| \text{ is bounded by a constant } c \text{ depending only on dim } G. \text{ The exact cohomology sequence associated to the universal covering}

\[
0 \longrightarrow \Phi(T_1) \longrightarrow V \longrightarrow T_1 \longrightarrow 0
\]

shows, that the fixed point set of the action of } G/G_o \text{ on } T_1 = C(G_o)_o \text{ is isomorphic to } H^1(C(G_o), \Phi(T_1)), \text{ hence its order is bounded by a}
constant $d$ depending only on $|G/G_o|$ and the rank of $T_1$. Hence

$$|C(G)| \leq |G/G_o| cd .$$

Finally we show that for the possible groups $ZU$ the order $|ZU/(ZU)_o|$ is bounded. $U$ is contained in a maximal torus of $G$. Therefore $ZU$ is a subgroup of maximal rank and $(ZU)_o$ a connected subgroup of maximal rank. By Theorem 5.9.6 there exist only finitely many conjugacy classes of connected subgroups of maximal rank. We have

$$|ZU/(ZU)_o| \leq |G/G_o| \ |N_o(ZU)_o/(ZU)_o| .$$

There are only finitely many possibilities for normalizers $N_o(ZU)_o$ in $G_o$ of $(ZU)_o$.

This finishes the proof of Theorem 5.9.9.

The last Theorem together with Proposition 5.8.3 gives the following result.

**Proposition 5.9.13.**

Let $n$ be the least common multiple of the numbers $|NH/H|$ where $(H) \in \Phi(G)$. Then the cokernel of $A(G) \rightarrow C(\Phi(G),\mathbb{Z})$ is annihilated by $n$. 

5.10. Finite extensions of the torus.

We have seen earlier that the appearance of infinitely many elements in \( \phi(G) \) is connected with subgroups of \( G \) which are not semi-simple. The typical situation is given, when \( G \) itself is an extension of a torus \( T \) by a finite group \( F \)

\[
(5.10.1) \quad a \rightarrow T \rightarrow G \rightarrow F \rightarrow 1.
\]

In particular if we are given a homomorphism \( h : F \rightarrow \text{Aut}(T) = \text{GL}(n, \mathbb{Z}) \), \( n = \dim T \), we can form the semi-direct product of \( T \) with \( F \) and \( h \) as twisting, call this \( G_h \). Note that \( h \) is an integral representation of \( F \). It would be interesting to know what the Burnside ring \( A(G_h) \) can say about the integral representation (or vice versa). We are going to make a few elementary remarks concerning the Burnside ring \( A(G) \) for groups \( G \) as in 5.10.1.

Given \( G \) as in 5.10.1 let \( h : F \rightarrow \text{Aut}(T) \) be the homomorphism induced by conjugation. We call a pair \((F', T')\) with \( F' \leq F \), \( T' \leq T \) and \( T' \) invariant under \( F' \) admissible, and call \( H \leq G \) an \((F', T')\)-subgroup if \( \rho(H) = F' \) and \( H \cap T = T' \).

Let \( \sigma \in H^2(F, T) \) be the class given by 5.10.1. We have maps

\[
k_* : H^2(F', T) \rightarrow H^2(F', T/T') \\
i_* : H^2(F, T) \rightarrow H^2(F', T).
\]

Elementary diagram chasing then tells us

**Proposition 5.10.2.**

An \((F', T')\)-subgroup exists in \( G \) if and only if \( \sigma \in \text{Ker}(k_* i^*) \).
Now choose any section \( s : F \rightarrow G \) to \( p \) and parametrize \( G \) as \( F \times T \) via \( g \mapsto (pg, spg^{-1} \cdot g) \). The multiplication in \( G \) takes the form

\[
(f, t)(f', t') = (ff', (t)f' + t' + \mu(f, f'))
\]

where \( (t)f = g^{-1}tg \) for \( g \epsilon p^{-1}(f) \) and

\[
\mu(f, f') = s(ff')^{-1} s(f)s(f').
\]

We always assume \( s(1) = 1 \) from now on.

**Proposition 5.10.3.**

If \( H \) is an \((F', T')\)-subgroup of \( G \), and \( s \) is a section with \( s(F') \subset H \), then a 1-1 correspondence between the \((F', T')\)-subgroups of \( G \) and the crossed homomorphisms \( \alpha : F' \rightarrow T/T' \) is established by associating to \( H' \) the crossed homomorphism

\[
\alpha(f') = k(s(f')^{-1}h(f'))
\]

for \( h(f') \) any element of \( H' \cap p^{-1}(f') \).

We leave the proof as an exercise. We denote the group described in 5.10.3 by \((F', T', \alpha)\). If \( G \) is a semi-direct product then \( \sigma = 0 \) and \((F', T')\)-subgroups always exist; in this case it is advisable to choose \( s \) as a homomorphism.

We now describe the effect of conjugation. For conjugation by elements of \( T \), note that in our parametrization

\[
(1, t)(f', t')(1, t)^{-1} = (f', t' + (t)f' - t).
\]

Thus denoting by \( \delta_t : F' \rightarrow T \) the principal crossed homomorphism
\( dt(f') = (t)f' - t \), the result of conjugating \((F', T', \alpha)\) by \((1, t)\) is \((F', T', \alpha')\) with \( \alpha'(f') = \alpha(f') + k(dt(f')) \).

Proposition 5.10.4.
Given a choice of \( H \) and \( s \) as in 5.10.3. There is a 1-1 correspondence between classes of \((F', T')\)-subgroups under conjugation by elements of \( T \) and the elements of \( H^1(F', T/T') \).

Proposition 5.10.5.
If \( H < G \) is an \((F', T')\)-subgroup then

\[ NH \cap T/H \cap T = \text{Fix}(F', T/T') \]

Proofs are again left as exercises.

Proposition 5.10.6.
If \( H \) is an \((F', T')\)-subgroup then the following are equivalent:

i) \( H \in \phi(G) \).

ii) \( \text{Fix}(F', T/T') \) is finite

iii) \( T' \) contains the zero-component of \( \text{Fix}(F', T) \).

Proof.
The equivalence i) \( \iff \) ii) follows immediately from 5.10.5. The equivalence i) \( \iff \) iii) is elementary Lie group theory and will be left to the reader.

From 5.10.2. and 5.10.6 one obtains

Proposition 5.10.7.
\( \phi(G) \) is infinite if and only if the action of \( F \) on \( T \) is non-trivial.
This can be used to give an analogous result for an arbitrary compact Lie group.

**Proposition 5.10.8.**

Let $G$ be a compact Lie group. Then $\Phi(G)$ is finite if and only if the action of the Weyl group on the maximal torus is trivial.

**Proof.**

If the action is trivial then $G_\circ$ can have no semi-simple component. Hence $G$ is of type 5.10.1 and 5.10.6 says that $\Phi(G)$ is finite.

Now assume that in

$$
0 \rightarrow T \rightarrow NT \rightarrow WT \rightarrow 1,
$$

$T$ a maximal torus, the action of the Weyl group $WT$ on $T$ is non-trivial.

By 5.10.7 $\Phi(NT)$ is infinite. We show that an infinite number of elements of $\Phi(NT)$ are contained in $\Phi(G)$. We know that $NT = \lim H_i$, $H_i \neq NT$.

By continuity our assertion follows with the help of the next Lemma.

**Lemma 5.10.9.**

Let $H \lhd K \lhd G$. Then $(H) \in \Phi(G)$ if and only if $(H) \in \Phi(K)$ and $G/K^H$ is finite.

**Proof.**

If $(H) \in \Phi(G)$ then, of course, $(H) \in \Phi(K)$ and $G/K^H$ is finite because it consists of a finite number of $NH/H$-orbits. For the other direction, note that $H \lhd N_KH \lhd N_GH$ yields a fibre bundle $N_KH/H \rightarrow N_GH/H \rightarrow N_GH/N_KH$.

But the inclusion $N_GH \rightarrow G$ induces an injective map

$$
N_GH/N_KH = N_GH/N_GH \cap K \rightarrow G/K^H.
$$

Thus if $(H) \in \Phi(K)$ and $G/K^H$ is finite, both base and fibre are finite.
We now report briefly about cyclic extensions of a torus (see Gordon [87]).

Proposition 5.10.10.

If \( G \) is an extension of \( T \) by \( F \) and \( F' \triangleleft F \) is cyclic, then any two \((F',T')\) subgroups of \( G \) are conjugate under an element of \( T \).

Proof.

If \( F' \) is cyclic and \( \text{Fix}(F',T/T') \) is finite, then \( H^1(F',T/T') = 0 \). Now use 5.10.4.

If \( f \) is cyclic of order \( n \) with generator \( f \) and \( M \) is any \( F \)-module then

\[
H^2(F,M) \cong \text{Fix}(F,M)/N^*M
\]

where \( N^*M = \sum_{i=0}^{n-1} (m)f^i \). Since for an \( r \)-torus \( M = T^r \) we have \( H^2(F,T^r) \cong H^3(F,Z^r) \) this group is finite. Thus \( N^*_T^r \) contains the zero-component of \( \text{Fix}(F,T^r) \). On the other hand, if \( \varphi: I \to T^r \) is any path from \( 0 \) to \( t \), then \( \sum_{i=0}^{n-1} (\varphi(t))f^i \) is a path in \( N^*_T^r \) from \( 0 \) to \( \sum_{i=0}^{n-1} (t)f^i \), so that \( N^*_T^r \) is connected. Hence for any torus \( T^r \), \( N^*_T^r \) is precisely \( \text{Fix}(F,T^r)_0 \).

The isomorphism \( H^2(F,T) \cong \text{Fix}(F,T)/N^*T \) means that the extension \( G \) is characterized by a component of \( \text{Fix}(F,T) \). Now note that it is no essential restriction to assume \( N^*_T = 0 \). For if \( L \) is any compact Lie group and \((ZL)_0\) the zero-component of its center, then \( L \to L/(ZL)_0 \) induces an isomorphism of rings

\[
A(L/(ZL)_0) \cong A(L).
\]

Now choose any element \( s(f) \in p^{-1}(f) \) and construct a section \( s \) by
putting $s(f_i^1) = s(f_i^1)$, $0 \leq i < n$. Then since $s(f)^{-1}s(f)^ns(f) = s(f)^n$, $\tau = s(f)^n \in \text{Fix}(F,T)$, and $\tau$ is the image of $[G] \in H^2(F,T)$ in $\text{Fix}(F,T) = \text{Fix}(F,T)/N^+T$. If now $(F',T')$ is an admissible pair then there exists an $(F',T')$ subgroup $H$ with $NH/H$ finite if and only if both the zero-component and the $\tau$-component of $\text{Fix}(F',T')$ are in $T'$.

Suppose $\tau \in \text{Fix}(F,T)$, latter being discrete, and let $T'$ be the (finite) subgroup generated by $\tau$. Then $T/T'$ inherits an $F$-operation. With these notations one has

**Theorem 5.10.11.**

If $G$ is the extension of $T$ by $F$ defined by $\tau$, and $G'$ is the semi-direct product of $T/T'$ and $F$ in the action above, then $A(G) \cong A(G')$.

**Proof.**

There exists a map $t : G \rightarrow G'$ making the following diagram commutative

\[
\begin{array}{ccc}
T & \longrightarrow & G & \longrightarrow & F \\
\downarrow k & & \downarrow t & & \downarrow \text{id} \\
T/T' & \longrightarrow & G' & \longrightarrow & F
\end{array}
\]

By the analysis of $(F',T')$-subgroups of $G$ given above it is seen that $t$ induces the required isomorphism.
5.11. Idempotent elements.

In section 1.4 we have described the idempotents of $A(G)$ for finite $G$. We generalize this to compact Lie group, using results of 5.9. and 5.6.

Let $S = S(G)$ be the space of closed subgroups of $G$ and $cS$ the quotient space under the conjugation action (see 5.6). Let $H^{(1)}$ be the commutator subgroup of $H$ and $H_a$ the smallest normal subgroup of $H$ such that $H/H_a$ is solvable (see 5.9.8). Let $P$ be the space of perfect subgroups in $S$.

**Proposition 5.11.1.**

The maps $H \mapsto H^{(1)}$ and $H \mapsto H_a$ are continuous maps $S \to S$. The space $P$ is closed in $S$.

**Proof.**

In view of the compactness of $S$ and 5.9.8 we need only show that $H \mapsto H^{(1)}$ is continuous. Let $H_1, H_2, \ldots$ be a sequence of subgroups converging to $H$. Without loss of generality we can assume that the $H_i$ are conjugate to subgroups of $H$. Moreover by 5.6.2 we can find a sequence $g_i \in G$ converging to 1 such that $K_i = g_i H g_i^{-1}$ is contained in $H$. We show that $\lim K_i^{(1)}$ exists and is equal to $H^{(1)}$. Fix $\varepsilon > 0$ and choose $n$ such that in the Hausdorff metric $d(K_i, H) < \varepsilon$ for $i \geq n$. Let $c^K$ be the closed subspace of a group $K$ consisting of elements which are product of at most $k$ commutators. Then $d(K_i, H) < \varepsilon$ implies $d(c^K_{K_i}, c^K_{H}) < 4k\varepsilon$.

Choose $k$ such that $d(c^K_{H}, H^{(1)}) < \varepsilon$. Then for $i \geq n$ we have $d(c^K_{K_i}, H^{(1)}) < (4k+1)\varepsilon$ and a fortiori $d(K_i^{(1)}, H^{(1)}) < (4k+1)\varepsilon$.

As a corollary we obtain
Proposition 5.11.2.

Given a perfect subgroup \( H \) of \( G \). Then \( \{ K \mid K_a = H \} \) and \( \{ K \mid K_a \sim H \} \) are closed subsets of \( S \).

In 5.7 we obtained the closed quotient map

\[
q : S \times \text{Spec } Z \longrightarrow \text{Spec } A(G) : (H), (p) \longmapsto q(H, p).
\]

Let \( r \) be the composition

\[
\begin{array}{cccc}
S \times \text{Spec } Z & \longrightarrow & S & \longrightarrow & P & \longrightarrow & \text{cP} \\
pr & & a & & c
\end{array}
\]

where \( pr \) is the projection, \( a \) the map \( aH = H_a \), and \( c \) the map \( cH = (H) \) into the space \( \text{cP} \) of conjugacy classes of perfect subgroups. Then \( r \) is continuous by 5.11.1.

Proposition 5.11.3.

The map \( r \) factors over \( q \) inducing a continuous surjective map

\[
s : \text{Spec } A(G) \longrightarrow \text{cP}.
\]

Proof.

Suppose \( q(H, p_1) = q(K, p_2) \). Since \( p \) is the residue characteristic of \( q(H, p) \) we must have \( p_1 = p_2 \). Put \( p = p_1 \). Let \( (H^\ast) \) be the unique conjugacy class such that \( q(H, p) = q(H^\ast, p) \) and \( NH^\ast/H^\ast \) is finite (see 5.7.2). By 5.7 we can find a countable transfinite sequence \( H \triangleleft H_1 \triangleleft H_2 \ldots H_\lambda \sim H^\ast \) such that \( H_{i+1}/H_i \) is solvable and \( H_j \) is the limit of the preceding subgroups if \( j \) is a limit ordinal. It follows from Proposition 5.11.1 that \( H_a = (H^\lambda)_a \).

The space \( \text{cP} \) being a countable compact metric space is totally dis-
connected. Hence we get a unique continuous map $e$ which makes the following diagram commutative

\[
\begin{array}{ccc}
\text{Spec } A(G) & \xrightarrow{\pi} & \text{Spec } A(G) \\
\text{cP} & \xrightarrow{e} & \text{Spec } A(G)
\end{array}
\]

Here $\pi$ is the projection onto the space of components.

**Theorem 5.11.4.**

The map $e$ is a homomorphism.

**Proof.**

$\pi \text{Spec } A(G)$ is a quotient of a quasi compact space hence quasi compact. The space $cP$ is a Hausdorff space. We therefore need only show that $e$ is bijective. We already know that $e$ is surjective.

Given two components $B$ and $C$ of $\text{Spec } A(G)$. Choose elements $q(H,p) \in B$, $q(K,l) \in C$. Assume that $e(B) = e(C)$, hence

\[(H_a) = sq(H,p) = sq(K,l) = (K_a).\]

Since $H/H_a$ is solvable we can find a finite chain of subgroups

\[H = H_1 \triangleright H_2 \triangleright \ldots \triangleright H_k = H_a\]

such that $H_i/H_{i+1}$ is a torus or finite cyclic of prime order. By 5.7.1 $q(H_1, p_1) = q(H_{i+1}, p_1)$ for a suitable prime. If $\overline{q}(H,p)$ denotes the closure of the point $q(H,p)$ we have
\[ q(H,p) \in \overline{q}(H_i, o), \quad \overline{q}(H_i, o) \cap \overline{q}(H_{i+1}, o) \neq \emptyset, \]

and therefore \( q(H_a, o) \in B \). Similarly \( q(K_a, o) \in C \) and therefore \( B = C \).

We now show how Theorem 5.11.4 leads to a description of idempotent elements.

Let \( U \) be an open and closed subset of \( \text{Spec } A(G) \). Then \( U \) is a union of components and projects into an open and closed subset of \( cP \) called \( s(U) \). Let \( e(U) \) be the idempotent element of \( A(G) \) which corresponds to \( U \) (Bourbaki [33], II. 4.3, Proposition 15). Let \( S(U) = \{ H \in G | \varphi^o_H e = 1 \} \)

**Proposition 5.11.5.**

\[ H \in S(U) \iff (H_a) \in S(U). \]

**Proof.**

Since \( e(U) \) is idempotent, \( \varphi^o_H(e(U)) \) is 0 or 1. We have to recall how to pass from \( U \) to \( e(U) \). Let \( Z \) be the complement of \( U \) in \( \text{Spec } A(G) \).

Then

\[ Z = V(A(G)e(U)) = \{ q \in \text{Spec } A(G) | q \supset A(G)e(U) \} . \]

Moreover \( e(Z) = 1 - e(U) \). Suppose \( \varphi^o_H e(U) = 1 \), then \( \varphi^o_H(e(Z)) = 0 \), so \( \varphi^o_H A(G)e(Z) = (0) \), which means \( q(H, o) \supset A(G)e(Z) \), \( q(H, o) \in V(A(G)e(Z)) = U \) and therefore \( (H_a) \in s(U) \).

Conversely, if \( (H_a) \in s(U) \), then \( q(H, o) \in U \), \( \varphi^o_H A(G)e(Z) = (0) \), \( \varphi^o_H e(U) = 1 \).

The idempotent is indecomposable if and only if \( U \) is a component. If the perfect subgroup \( H \) of \( G \) is not a limit of perfect subgroups then
\{q(K,p) \mid (K_a) = (H) \} := U(H) is a component and H yields an indecomposable idempotent \(e_H := e(U(H))\).

We are now going to show that the topological considerations above are necessary in that usually an infinite number of conjugacy classes of perfect subgroups exists. Let \(1 \to T \to G \to F \to 1\) be an exact sequence where \(T\) is a torus and \(F\) a finite group. Conjugation in \(G\) induces a homomorphism \(\varphi : F \to \text{Aut}(T)\) which we also interpret as action of \(F\) on \(T\) (compare section 5.10.) Let \(F_U\) be the kernel of \(\varphi\).

**Proposition 5.11.6.**

*Let \(G\) be a finite extension of a torus as above. Then the number of conjugacy classes of perfect subgroups of \(G\) is finite if and only if \(F/F_U\) is solvable. If \(F/F_U\) is solvable then the set of perfect subgroups is finite.*

**Proof.**

A quotient of a perfect group is perfect. Let \(F/F_U\) be solvable. Let \(H < G\) be perfect. Then the image under \(G \to F \to F/F_U\) is perfect hence trivial. Therefore \(H\) is an extension \(1 \to H \cap T \to H \to P \to 1\) with \(P < F_U\) perfect and trivial action of \(P\) on \(H \cap T\) and \(T\). Let \(K\) be the pre-image of \(P\) under \(p : G \to F\). Then \(H \cap K\) since \(T\) is contained in the center of \(K\). The group \(K/H = T/H \cap T\) is solvable. Hence \(H = K_a\). There a perfect group comes via the map \(K \to K_a\) from a finite set of subgroups.

Now let us assume that \(F/F_U\) is not solvable. Let \(P < F/F_U\) be a non-trivial perfect subgroup. Let \(H\) be the pre-image of \(P\) under \(G \to F/F_U\) and \(Q < F\) be its group of components. Let \(T_o\) be the component of \(1\) in the fixed point set of the \(Q\) action on \(T\). Since \(Q > F_U\), \(Q \neq F_U\) we have \(T_o \neq T\). The group \(T_o\) is contained in the center of \(H\) and \(H \to H/T_o\) induces an injective ring homomorphism \(A(H/T_o) \to A(H)\). If \(A(H/T_o)\) has
an infinite number of idempotents then $A(H)$ has an infinite number of idempotents, hence an infinite number of conjugacy classes of perfect subgroups. The action of $Q$ on $H/T_\circ$ has zero-dimensional fixed point set. Hence we have reduced the problem to the case $T_\circ = \{1\}$. But then a subgroup $L$ of $H$ which projects onto $P$ under $H \rightarrow Q \rightarrow P$ has finite index in its normalizer. Let $L$ be such a group and consider its derived group $L^{(1)}$. Then $L^{(1)}$ also projects onto $P$ because $P$ is perfect. Therefore $NL^{(1)}/L^{(1)}$ is finite and $L/L^{(1)} < NL^{(1)}/L^{(1)}$. But we have shown in 5.9.4 that there exists a number $b$ such that for any $L < H$ with finite index in its normalizer $|NL/L| < b$. Together with 5.9.8 we see that there is an integer $n$ such that $L/L_a$ is finite of order less than $b^n$. Hence if there exists an infinite set of subgroups of $H$ which projects onto $P$ and which contains groups of arbitrary large order then the set of conjugacy classes of perfect subgroups is infinite. But infinite sequence of subgroups of the required sort is easily constructed, using the techniques of 5.10.
5.12. Functorial properties.

If $X$ is a $G$-space and $H \triangleleft G$ then $X$ can be considered as $H$-space. This induces the forgetful functor $r_H^G : \text{G-Top} \rightarrow \text{H-Top}$ from the category of $G$-spaces to the category of $H$-spaces. This functor has a left adjoint, called extension from $H$-spaces to $G$-spaces. On objects it is defined by

$$e_H^G(X) = Gx_HX$$

for an $H$-space $X$. The adjointness means that for $H$-spaces $X$ and $G$-spaces $Y$ we have a natural bijection

$$\text{Map}_G(Gx_HX, Y) \cong \text{Map}_H(X, r_H^GY) ,$$

where $\text{Map}_G$ is the set of $G$-maps. If $f : X \rightarrow Y$ is an $H$-map then $f' : Gx_HX \rightarrow Y : (g,x) \mapsto gf(x)$ is the adjoint $G$-map.

**Proposition 5.12.1.**

The assignment $X \mapsto Gx_HX$ induces an additive homomorphism

$$e_H^G : \mathcal{A}(H) \rightarrow \mathcal{A}(G).$$

($X$ a compact $H$-ENR.)

**Proof.**

Given $K \triangleleft G$, then $Gx_HX^K \neq \emptyset \Rightarrow G/H^K \neq \emptyset \Rightarrow (K) \triangleleft (H)$. Assume $K \triangleleft H$. We have to show that $\chi((Gx_HX)^K)$ can be computed from Euler-Characteristics of fixed point sets $X^L$. The set $G/H^K$ is finite (if $K \subseteq \phi(G)$). The fibre of $Gx_HX^K \rightarrow G/H^K$ over $gH$ is homeomorphic to $x^{gKg^{-1}\cap H}$. Hence

$$\chi((Gx_HX)^K) = \sum_{gH \in G/H^K} \chi(x^{gKg^{-1}\cap H}) .$$
If $f : H \longrightarrow K$ is a continuous homomorphism between compact Lie groups then a $K$-space $X$ can be considered via $f$ as an $H$-space. This induces a ring homomorphism

$$A(f) = f^* : A(K) \longrightarrow A(H)$$

and $A(-)$ becomes a contravariant functor from compact Lie groups to commutative rings. If $f : H \subset K$ then $f$ is called restriction, also denoted $r^K_H$.

We want to investigate the various interrelations between the $e^G_H$ and $r^G_H$. We need a slightly more general map than the $e^G_H$. This is done best by redefining $e^G_H$ and $r^G_H$ using a more general concept than the Burnside ring.

Let $S$ be a closed differentiable $G$-manifold and let $a(S)$ be the set of differentiable $G$-maps $M \longrightarrow S$ which are proper submersions. On $a(S)$ we induce the following equivalence relation: $p : M \longrightarrow S$ equivalent to $q : N \longrightarrow S$ if and only if for all $s \in S$ and all $H < G$ the equality

$$\chi(p^{-1}(s)^H) = \chi(q^{-1}(s)^H)$$

holds. Disjoint union (addition) and fibre product over $S$ (multiplication) makes the set of equivalence classes into a commutative ring with identity, denotes $A[S]$. If $S$ is a point this is the Burnside ring; hence we call $A[S]$ the Burnside ring of $G$-manifolds over $S$. We are going to describe the functorial properties of this ring.

Let $f : T \longrightarrow S$ be a differentiable $G$-map. Let $p : M \longrightarrow S$ be a submersion as above. Then in the pull-back diagram
the map \( q \) is a proper submersion and defines an element in \( A[T] \). The assignment \( p \mapsto q \) induces a ring homomorphism \( f^*: A[S] \to A[T] \).

We also have covariant maps. Let \( f: T \to S \) be a submersion. Then composition with \( f \) induces an additive (but not multiplicative) map \( f_*: A[T] \to A[S] \). These maps have the following properties.

**Proposition 5.12.2.**

i) \( f^* \) is a homomorphism of rings. We have \((id)^* = id \text{ and } (fg)^* = g^* f^* \).

ii) For any submersion \( f: T \to S \) the map \( f_* \) is well-defined and additive. We have \((id)_* = id \text{ and } (fg)_* = f_* g_* \).

iii) For \( a \in A[S] \) and \( b \in A[T] \) we have

\[
af_*(b) = f^*(a) b.
\]

iv) Let

\[
\begin{array}{ccc}
T' & \longrightarrow & S' \\
\downarrow F & & \downarrow P \\
T & \longrightarrow & S \\
\downarrow f & & \downarrow \quad \\
\end{array}
\]

be a pull-back diagram with \( f \) and hence \( F \) a submersion. Then \( p^* f^* = F^* p^* \).

v) If \( f_0^*, f_1^*: T \to S \) are \( G \)-homotopic then \( f_0^* = f_1^* \).
The proofs are straightforward and left to the reader. The connection with material at the beginning of this section if obtained using a canonical isomorphism $A[G/H] \cong A(H) : p : M \rightarrow G/H \mapsto p^{-1}(H/H)$.

Proposition 5.12.2 iv) generalizes the main property of Mackey functors in the sense of Dress [80] to compact Lie groups. But in the case of non-finite Lie groups there exists a double coset formula which is a less formal generalization of the Mackey axiom and is more accessible to computation. We are going to describe this formula.

We consider a pull-back diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\overline{h}} & G/L \\
\downarrow{\overline{k}} & & \downarrow{k} \\
G/K & \xrightarrow{h} & G/P
\end{array}
\]

The problem is to compute $k^* h_*$. We use a decomposition of $S$ into homogeneous spaces but slightly more refined than the decomposition in the Burnside ring. As in Section 5.5 we have the decomposition $S = \bigcup S_{(H)}$ into the subspaces of a given orbit type. We let $S_{(H),b}$ be the inverse image in $S_{(H)}$ of the connected components of $S_{(H)}/G$. So the index $b$ distinguishes the components. Then we still have a decomposition

\[
S = \sum n_{(H),b} [M_{(H),b}]
\]

in $A(G)$ with $n_{(H),b} := \chi_{c}(S_{(H),b}/G) \in \mathbb{Z}$ and $M_{(H),b}$ an orbit in $S_{(H),b}$. We let $k_{(H),b} : M_{(H),b} \rightarrow G/K$ and $h_{(H),b} : G/L$ be the maps which are compositions of the inclusion $M_{(H),b} \subset S$ with the maps $\overline{k}$ and $\overline{h}$ respectively. Then we claim
Theorem 5.12.3.

We have the equality of maps

$$k^* h_x = \sum_{(H), b} n_{(H), b}(h_{(H), b})^* (k_{(H), b})^*.$$ 

Proof.

Given an element $x$ in $A[G/K]$ represented by $f : M \rightarrow G/K$. Then $k^* h x$ is represented by $\bar{h} F$ in the pull-back diagram below (where the squares and hence also the rectangle are pull-backs).

$$\begin{array}{ccc}
\bar{M} & \xrightarrow{F} & S \\
\downarrow & & \downarrow h \\
\bar{k} & & k \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & G/K \\
\downarrow & & \downarrow h \\
G/P & & G/L
\end{array}$$

Since pull-backs are transitive the pull-back of $f : M \rightarrow G/K$ along $k_{(H), b}$ is the fibre of $F : \bar{M} \rightarrow S$ over $M_{(H), b}$, say $F_{(H), b} : \bar{M}_{(H), b} \rightarrow M_{(H), b}$ and this represents $k_{(H), b}^* x$. Hence $(h_{(H), b})^* (k_{(H), b})^* x$ is represented by the composition

$$h_{(H), b} F_{(H), b} : \bar{M}_{(H), b} \rightarrow M_{(H), b} \rightarrow G/L.$$

So we have to show that the following two elements are equal in $A[G/L]$, namely $[\bar{h} F]$ and $\sum n_{(H), b} [h_{(H), b} F_{(H), b}]$. This means by definition of $A[G/L]$ that we have to show: For each $U \subset L$ the $U$-fixed points of the fibres over the coset $L/L$ of $G/L$ have the same Euler-characteristic.

The fibre of $\bar{h} F$ is the fibre of $h f$ over $k(L/L)$, considered as
L-manifold. Since we are now dealing with G-spaces over G/L the whole situation can be reconstructed from the fibres over L/L, which we denote by an index zero, using canonical G-diffeomorphisms like $G \times_{L} \overline{M}^{O} = \overline{M}$. We have for $V < L$

$$M(H), b = G \times_{L} M^{O}(H), b', S(V) = G \times_{L} S^{O}(V), S(V), b = G \times_{L} S^{O}(V), b$$

using the identification $S(V)/G = S^{O}(V)/L$.

Let $F : \overline{M}^{O} \rightarrow S^{O}$ be the restriction of the map $F : \overline{M} \rightarrow S$. As in Section 5.5 we have

$$(5.12.4) \chi((\overline{M}^{O})^{U}) = \sum_{V,b} \chi_{c}(F^{-1}(S^{O}(V), b)^{U}) .$$

The map

$$F^{-1}(S^{O}(V), b) \rightarrow S^{O}(V), b \rightarrow S^{O}(V), b/L$$

is a fibre bundle with the fibre $F^{-1}(S^{O}(V), b)$ such that the $U$-fixed points again yield a fibration with typical fibre $F^{-1}(S^{O}(V), b)^{U}$. Then the $((V), b)$-summand in (5.12.4) is by Proposition equal to

$$\chi_{c}(F^{-1}(M^{O}(V), b)^{U}) \chi_{c}(S^{O}(V), b/L)$$

$$= \chi_{c}(F^{-1}(S^{O}(V), b)^{U}) \chi_{c}(S^{O}(V), b/G)$$

$$= \chi_{c}(F^{-1}(S^{O}(V), b)^{U})_{S^{O}(V), b}$$

and this was to be shown.
5.13. Multiplicative induction and symmetric powers.

Let $K$ be a subgroup of finite index in $G$. Let $\text{Hom}_K(G,X)$, for a $K$-space $X$, be the space of $K$-maps $G \rightarrow X$ with $G$-action induced by right translation on $G'$. The functor $X \mapsto \text{Hom}_K(G,X)$ from $K$-spaces to $G$-spaces is right adjoint to the restriction functor and preserves in particular products. Explicitly, we have a natural bijection

$$\text{Map}_G(Y,\text{Hom}_K(G,X)) \cong \text{Map}_K(Y,X),$$

where $Y$ is any $G$-space. Given $f : Y \rightarrow \text{Hom}_K(G,X)$ in the set on the left we compose with the $K$-map $\text{Hom}_K(G,X) \rightarrow X : f \mapsto f(1)$ to obtain the corresponding element in the set on the right side. We have chosen $K$ to be of finite index in $G$ in order to avoid some technical problems: In our case $\text{Hom}_K(G,X)$ as a topological space is simply the product

$$\prod_{y \in G/K} X$$

Proposition 5.13.1.

The assignment $X \mapsto \text{Hom}_K(G,X)$ induces a map $A(K) \rightarrow A(G)$ which, in general, is not additive but preserves products ($X$ a compact $G$-ENR).

Proof.

Given $H \leq G$ we have to compute $X(\text{Hom}_K(G,X)^H)$. Since $K$ has finite index in $G$ the space $G/H$ is $K$-homeomorphic to a finite disjoint union $\bigcup_i K/K(i)$ of homogeneous spaces. The equalities

$$\text{Hom}_K(G,X)^H = \text{Hom}_G(G/H, \text{Hom}_K(G,X)) = \text{Hom}_K(G/H,X) = \text{Hom}_K(\bigcup_i K/K(i),X) = \prod_i \text{Hom}_K(K/K(i),X) = \prod_i X^{K(i)}$$
show that the Euler-Characteristics in question can be computed from Euler-Characteristics of fixed point sets $X^L$, $L \leq K$.

We call $X \mapsto \text{Hom}_K(G, X)$ and the map induced on the Burnside ring multiplicative induction.

**Proposition 5.13.2.**

Let $L$ be a finite normal subgroup of $G$. The assignment $X \mapsto X/L$ induces a map $A(G) \rightarrow A(G/L)$. ($X$ a compact $G$-ENR.)

**Proof.**

Given $H \leq G/L$ we have to show that $\chi(X/L^H)$ is determined by Euler-Characteristics of fixed point sets of $X$. Let $P$ be the inverse image of $H$ in $G$. Let $B = p^{-1}(X/L^H)$ where $p : X \rightarrow X/L$ is the quotient map. We consider $X$ and $B$ as $P$-spaces. An Orbit of $X$ isomorphic to $P/U$ is contained in $B$ if and only if $P \leq LU$. Hence $B$ is a union of orbit bundles. From Proposition $[B] = [B']$ in $A(P)$ where $B' \subset X'$ has a similar meaning as $B$. Now

$$
\chi(X/L^H) = \chi(B/L) = |L|^{-1} \sum_{g \in L} \chi(B^g).
$$

Here we have used 5.3.12. Hence $\chi(X/L^H)$ can be computed from Euler-Characteristics as we wanted. We still have to show that $X/L$ is a $G/L$-ENR. By 5.2.6 if suffices to see that all $M/L^H = B/L$ are ENR. But $B$ is an ENR by 5.2.6 and hence $B/L$ an ENR by 5.2.5.

We now discuss symmetric powers. Let $S_r$ be the symmetric group on $r$ symbols. If $X$ is a $G$-space then the diagonal action of $G$ on $X^r$ and the permutation action of $S_r$ commute, so we can view $X^r$ as $(S_r \times G)$-space. If $M$ is an $S_r$-space with trivial $G$-action then $M \times X^r$ is an $(S_r \times G)$-space. Dividing out the $S_r$-action yields the $G$-space $(M \times X^r)/S_r$. 

Proposition 5.13.3.

The assignment $(M,X) \mapsto (M \times X^r)/S_r$ induces a map

$$A(S_r) \times A(G) \longrightarrow A(G).$$

$(M,X \text{ compact } G\text{-ENR's.})$

Proof.

We begin by showing that $X \mapsto X^r$ induces a map $w : A(G) \longrightarrow A(S_r \times G)$. The standard embedding $S_{r-1} < S_r$ gives $S_{r-1} \times G$ as a subgroup of finite index in $S_r \times G$. Viewing $X$ as an $(S_{r-1} \times G)$-space via the projection $S_{r-1} \times G \longrightarrow G$ then the $(S_r \times G)$-space $X^r$ is obtained from $X$ using the multiplicative induction corresponding to $S_{r-1} \times G < S_r \times G$. Therefore $w$ is well-defined by 5.13.1. Now consider the following composition of maps

$$A(S_r) \times A(G) \xrightarrow{p \times w} A(S_r \times G) \times A(S_r \times G) \xrightarrow{m \otimes q} A(G).$$

where $w$ is as above, $p$ is induced by the projection $S_r \times G \longrightarrow S_r$, $m$ is ring-multiplication, and $q$ is the quotient map of 5.13.2. We check that on representatives the above composition is $(M,X) \mapsto (M \times X^r)/S_r$.

Let $\pi S_r$ be a subgroup. Then $X^r / \pi \pi S_r$ is the $\pi$-symmetric power, a $G$-space if $X$ is a $G$-space. Note that $(S_r / \pi \times X^r)/S_r = X^r / \pi$. Hence we have

Corollary 5.13.4.

$X \mapsto X^r / \pi$ induces a map $A(G) \longrightarrow A(G)$. 
We are going to analyse the formal properties of the map 5.13.3. We write this map

\[(5.13.5) \quad A(S_r) \times A(G) \rightarrow A(G) : (x, y) \mapsto x \cdot y .\]

We recall some constructions with the symmetric group. Let \(X, Y\) be \(S_r\)-, \(S_t\)-spaces, respectively. We write

\[(5.13.6) \quad X \cdot Y = S_{r+t} \times S_r \times S_t (X \times Y)\]

using the standard embedding \(S_r \times S_t \subset S_{r+t}\).

Let \(S_r \subset G\) be the wreath-product of \(G\) with \(S_r\). This is the set \(S_r \times G^r\) with group-law

\[(s; g_1, \ldots, g_r)(t; h_1, \ldots, h_r) = (st; g_1 h_1^{-1}, \ldots, g_r h_r^{-1}) .\]

If \(M\) is a \(G\)-space then \(M^r\) becomes an \(S_r \subset G\)-space with action

\[(s; g_1, \ldots, g_r)(m_1, \ldots, m_r) = (g_1 m_1^{-1}, \ldots, g_r m_r^{-1}) .\]

We consider \(S_r \subset S_t\) as a subgroup of \(S_{rt}\): If \(M = M^r\) as \(S_t\)-space then \(S_r \subset S_t\) acts as a group of permutations on \(M^r\); now identify \(M^r\) in a sensible way with \(\{1, 2, \ldots, rt\}\). (The conjugacy class of \(S_r \subset S_t\) in \(S_{rt}\) is then uniquely determined.) Let \(X, Y\) be \(S_r\)-, \(S_t\)-spaces respectively. We write

\[(5.13.7) \quad X \star Y = S_{rt} \times (S_r \wr S_t) (X \times Y^r) .\]

**Proposition 5.13.8.**

The constructions \((X, Y) \mapsto X \cdot Y\) and \((X, Y) \mapsto X \star Y\) induce maps
respectively. The graded additive group

\[ A = \bigoplus_{r \geq 0} A(S_r) \]

becomes a graded ring with multiplication \( \star \). Moreover one has

\[
\begin{align*}
(a+b) \star c &= a \star c + b \star c \\
(a\star b) \star c &= (a \star c) \star (b \star c) \\
(a \star b) \star c &= a \star (b \star c) \\
b \star 1 &= b.
\end{align*}
\]

Here \( 1 \in B(S_0) = \mathbb{Z} \).

**Proof.** The formal algebraic properties of these constructions follow by considering representatives once we have shown that there are well-defined induced maps \( \star \) and \( \star \).

We factorise the required map \( \star \) as

\[
\begin{array}{ccc}
A(S_r) \times A(S_t) & \longrightarrow & A(S_r \times S_t) \times A(S_r \times S_t) \\
\downarrow p_1^\star \times p_2^\star & & \downarrow \text{extension homomorphism} \\
& & A(S_{r+t})
\end{array}
\]

where \( p_1, p_2 \) are the projections, the second map is the multiplication in the ring \( A(S_r \times S_t) \) and the third map is the extension homomorphism 5.12.1.
Similarly we factorise the map $\ast$ as

$$A(S_r) \times A(S_t) \xrightarrow{p_S \times w} A(S_r \cup S_t) \times A(S_r \setminus S_t) \xrightarrow{\text{id} \times \text{id}} A(S_r \cup S_t)$$

where $p$ is the projection $S_r \cup S_t \to S_r$ and where $w$ is induced by $Y \leftrightarrow Y^r$ (this well-defined!); the second map is again multiplication and the third extension.

We return to the map 5.13.5 which, obviously, is additive in the first variable, so that we obtain an action $A \times A(G) \to A(G)$. Moreover the constructions of 5.13.8 have the following properties.

**Proposition 5.13.9.**
For $a_1, a_2 \in A$ and $b \in A(G)$

$$(a_1 \circ a_2) \cdot b = (a_1 \circ b)(a_2 \circ b)$$

$$(a_1 \ast a_2) \cdot b = a_1 \cdot (a_2 \cdot b)$$

The interpretation of these formulas is this: $a \in A$ induces an operation $b \mapsto a \cdot b$ on $A(G)$. Addition and multiplication in $A$ corresponds to pointwise addition and multiplication of operation. Finally $\ast$ is composition of operations. Hence $A$ is a ring of operations. The operations have some obvious naturality properties which we do not write down. The proof of the identities is given by looking at representatives.
5.14. An example: The group $\text{SO}(3)$.

Using Wolf [163], 2.6., one can see that $\text{SO}(3)$ has the following conjugacy classes of subgroups:

- $\text{SO}(3)$
- $S^1 \cong \text{SO}(1)$ maximal torus
- $S \cong \text{NS}^1 \cong \text{O}(1)$ normalizer of $S^1$
- $I = A_5$ icosahedral group
- $O = S_4$ octahedral group
- $T = A_4$ tetrahedral group
- $D_n$, $n \geq 2$ dihedral group of order $2n$
- $Z/n$, $n \geq 1$ cyclic group.

One has $\text{ND}_n = D_{2n}$, $n = 2$; $\text{ND}_2 = S_4$, $\text{NA}_4 = S_4$, $\text{NS}_4 = S_4$, $\text{NA}_5 = A_5$, $\text{NO}(1) = O(1)$. The cyclic groups do not have finite index in their normalizer.

The ring $\Lambda(\text{SO}(3))$ is the set of functions $z \in C(\phi, \mathbb{Z})$ such that

i) $z(H)$ arbitrary for $H = \text{SO}(3)$, $A_5$, $S_4$, $\text{NT}$.

ii) $z(D_n) \equiv z(D_{2n}) \mod 2$, $n \neq 2$

iii) $z(A_4) \equiv z(S_4) \mod 2$

iv) $z(S) \equiv z(S^1) \mod 2$

v) $z(D_2) + 2z(A_4) + 3z(D_4) \equiv 0 \mod 6$.

The continuity of $z$ means $\lim_{j} z(D_{2j}^n) = z(S)$.

If $H$ is a subgroup of $\text{SO}(3)$ we denote for simplicity with the same symbol the element $[G/H]$ of $\Lambda(\text{SO}(3))$. We give the multiplication table of the elements $H$. We put $(k,n)$ for the greatest common divisor and let $d(k,n) = 1$ if $(k,n) = k$ and $d(k,n) = 0$ otherwise.
\[(S)^2 = S + D_2 \quad \text{and} \quad S_1 S^1 = S^1\]

\[S \cdot D_k = D_k + 2d(2,k)D_2 \quad \text{and} \quad SI = D_5 + D_3 + D_2\]

\[S \cdot T = D_2 \quad \text{and} \quad SO = D_4 + D_3 + D_2\]

\[(S^1)^2 = 2S^1 \quad \text{and} \quad S_1 H = 0 \text{ for } H \neq S, S^1.\]

\[(D_k)^2 = 2D_k + 4d(2,k)D_2\]

\[D_k D_n = 2D(k,n) + 4d(2,(k,n))D_2\]

\[D_k O = 2d(4,k)D_4 + 2d(3,k)D_3 + 2d(2,k)(2-d(4,k))D_2\]

\[D_k T = 2d(2,k)D_2\]

\[I^2 = I + T + D_5 + D_3\]

\[IT = 2T\]

\[IO = T + 2D_3 + D_2\]

\[O^2 = O + D_4 + D_3 + D_2\]

\[OT = T + D_2\]

\[T^2 = 2T\]

\[D_k I = 2d(5,k)D_5 + 2d(3,k)D_3 + 2d(2,k)D_2\]

The ring \(A(SO(3))\) contains the following idempotent elements

\[x = I - T - D_5 - D_3\]

\[y = S + O - D_4 - D_3\]

\[x+y, 1-x, 1-y, 1-x-y.\]

5.15. Comments.

The general theory of the Burnside ring of a compact Lie group is based on the authors' papers \([64]\), \([65]\), \([66]\). As far as the equivariant Euler characteristic is concerned there has been a parallel development in the cohomology of groups, see K. Brown \([39]\), \([40]\), \([41]\). We have been guided in 5.3 by Brown \([39]\). For 5.3.3 see Floyd \([33]\, III \S 3). For 5.3.4 see \([39]\).
It would be desirable to give a unified treatment of the Burnside ring and results in Brown [33]. Also Bass [16] is relevant. The universal ring for Euler-Characteristic in 5.4 has been introduced by Oliver [118] and has also been used by Becker-Gottlieb 5.5.10 and 5.14 is due to Schwänzl [140], 5.7 is an extension of work of Dress [79]. For general compact groups see Gordon [86]. It would be interesting to find a more general class of G-spaces which satisfy the relations between Euler-Characteristics 5.8.5; suitable finiteness conditions for cohomology should suffice. For 5.9.8 see Zassenhaus [171] and Raghunathan [130]. The results of 5.10 are based on the thesis of Gordon [86]; see also Gordon [87]. The reader can see that a purely algebraic definition of the Burnside ring for finite torus extensions can be given. This algebraic definition is then also applicable to other arithmetic situations, e.g. representations over p-adic integers. If G acts on a disk D such that all $D^H$ are either empty or contractible then D represents an idempotent in $A(G)$. Oliver [118] has shown that essentially all idempotents of $A(G)$ arise in this way. For 5.13 I could make use of an unpublished manuscript of Rymer [137]. For operations in the Burnside ring see also Siebeneicher [143].

5.16. Exercises.

1. Compute the ring $U(G)$ of 5.4 for $G = SO(3)$.

2. Given a natural number $n \geq 2$. Can $U(G)$ contain elements $x$ such that $x^{n-1} \neq 0$, but $x^n = 0$?

3. Show that $A(SO(3))$ has three indecomposable idempotent elements.

4. Compute the units of $A(SO(3))$ and compare with the units obtainable from 1.5.3.

5. If G is cyclic then permutation representations given an isomorphism $A(G) \cong R(G;Q)$.
6. Use 5.13 to define a $\Lambda$-ring structure on $A(G)$ by symmetric powers and show that the isomorphism of exercise 5 is compatible with $\Lambda$-operations (but don't take exterior powers!).

7. Let $S$ be any subring of the rationals. Determine the idempotent elements of $A(G) \otimes_{\mathbb{Z}} S$, in particular for $S = \mathbb{Z}(p)$.

8. Let $S_G$ be the homotopy category of pointed $G$-CW-complexes. Consider the Grothendieck group $K(S_G)$ of this category: The universal abelian group $S_G \rightarrow K(S_G) : X \mapsto [X]$, where each cofibration sequence $A \rightarrow X \rightarrow X/A$ gives rise to a relation $[X] = [A] + [X/A]$. Show that smashed-product $(X,Y) \mapsto X \wedge Y$ makes $K(S_G)$ into a commutative ring. Show that $K(S_G) \cong U(G)$. 