4. Permutation representations.

If $G$ is a finite group and $S$ a finite $G$-set we can consider the associated permutation representation $V(S,F)$ of $S$ over the commutative ring $F$. The assignment $S \mapsto V(S,F)$ induces a ring homomorphism

$$h = h_F : A(G) \longrightarrow R(G;F)$$

of the Burnside ring into the representation ring. We shall describe some aspects of this homomorphism in particular when $F$ is a field or the ring of integers $\mathbb{Z}$. We describe the connection to the $J$-homomorphism of section 2 and to $\lambda$-rings.

4.1. p-adic completion.

Let $p$ be a prime number and let $G$ be a $p$-group. Let $A(G) = \lim_{\leftarrow n} A(G)/p^n A(G) \cong A(G) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ be the $p$-adic completion of $A(G)$.

If $|G| = p^n$ and $m = q(l,p)$ we have seen in exercise 1.9.4 that $m^{n+1} \subset p A(G) \subset m$. Hence

**Proposition 4.1.1.**

If $G$ is a $p$-group the $p$-adic and the $m$-adic topology on $A(G)$ coincide.

Let now $q$ be a prime different from $p$. Let $e : R(G,F_q) \rightarrow \mathbb{Z} : x \mapsto \dim x$ be the augmentation and $I(G,F_p) = \text{Kernel } e$ the augmentation ideal.

The ring $A(G) = \hat{A(G)}$ is a local ring with maximal ideal $m^\wedge$, the completion of $m$. 

We now consider the case $p \neq 2$. Since $A(G)^{-\frac{1}{p}} \subseteq A(G)_{\overline{p}}$ we obtain from 2.1 the $J$-homomorphism

$$J : R(G,F_q) \longrightarrow A(G)_{\overline{p}}^\wedge.$$ 

We notice that for an $F_{\overline{p}} G$-module $V$ we have $\text{dim} V = 1$. Hence

$$JI(G,F_q) \subseteq 1 + m^\wedge.$$ 

The set $1 + m^\wedge \subseteq A(G)_{\overline{p}}^\wedge$ is compact and a topological group with respect to multiplication. A fundamental system of neighbourhoods of $1$ is given by $(1+m^{\wedge^i})_{i \geq 1}$, or $(1+m^{\wedge^i}+p^{\wedge^j})$. Since

$$J(p^{\wedge^i}I(G,F_q)) \subseteq (1+m^{\wedge^i})^{\wedge^i} \subseteq 1+m^{\wedge^i+1}$$

we see that $J : I(G,F_q) \longrightarrow 1+m^\wedge$ is $p$-adically continuous and therefore induces a continuous map

$$J^\wedge : I(G,F_q)_{\overline{p}}^\wedge \longrightarrow 1+m^\wedge$$

homomorphimic from addition to multiplication.

4.2. Permutation representations over $F_q$.

We still assume that $p$ is odd and consider the permutation representation map and its $p$-adic completion

$$h : A(G) \longrightarrow R(G,F_q)$$

$$h^\wedge : A(G)_{\overline{p}}^\wedge \longrightarrow R(G,F_q)_{\overline{p}}^\wedge.$$
Since \( h(m) \subset p \cdot R(G,F_q) + I(G,F_q) \) and because the \( p \)-adic and \( I(G,F_q) \)-adic topology on \( R(G,F_q) \) coincide (see [6]) we obtain an induced continuous map between multiplicative topological groups

\[
(4.2.2) \quad h^\wedge: 1+m^\wedge \longrightarrow 1+I(G,F_q)^\wedge.
\]

**Definition 4.2.3.**

We call the prime \( q \) \( p \)-generic if it generates a dense subgroup of the \( p \)-adic units (i.e. if \( q \) generates \( \mathbb{Z}/p^2\mathbb{Z}^* \)).

**Theorem 4.2.4.**

Let \( q \) be a \( p \)-generic prime. Then the composition

\[
h^\wedge \circ J^\wedge: I(G,F_q)^\wedge \longrightarrow 1+I(G,F_q)^\wedge
\]

is an isomorphism.

In fact the proof will show that this is one of the isomorphisms which we had considered in the previous chapter on \( \lambda \)-rings, namely the map \( g_q \).

**Proof.**

In order to prove the equality \( h^\wedge \circ J^\wedge = g_q \) we need only consider cyclic \( \mathbb{Z}/p^n\mathbb{Z} \) because \( J^\wedge \), \( h^\wedge \) and \( g_q \) are compatible with restriction to subgroups and elements in \( R(G,F_q)^\wedge \) are detected by their restriction to cyclic subgroups.

We begin with the computation of \( g_q \) for \( G = \mathbb{Z}/p^n\mathbb{Z} \). The group algebra \( F_q G = F_q[x]/(x^a-1) \), \( a = p^n \), decomposes as \( \bigoplus_{1 \leq t \leq n} F_q[x]/\phi_t(x) \), where \( \phi_t(x) \) is the \( p^t \)-th cyclotomic polynomial. If \( q \) is \( p \)-generic then \( \phi_t(x) \) is irreducible. Hence the \( F_q[x]/\phi_t(x) =: V_t \) are the irreducible
\( F_q G\)-modules in our case. By 3.12.2 we have the identity

\[
\varphi_q(V_t - \dim V_t) \circ \varphi_q(\dim V_t) = \varphi_q(V_t).
\]

Over a splitting field \( F \) of \( G \) the module \( V_t \) splits \( V_t = \bigoplus_j V_t(j) \), where \( V_t(j) \) is one-dimensional and a generator of \( G \) acts as multiplication with \( u^j \), where \( u \) is a primitive \( p^t \)-th root of unity and \( j \in \mathbb{Z}/p^t\mathbb{Z}^\ast \). Since the \( \varphi_q \)-operations are compatible with field extension we obtain from 3.7.2

\[
\varphi_q(V_t) = \prod \varphi_q(V_t(j)) = \prod (1 + V_t(j) + \ldots + V_t(j)^{q-1}).
\]

It is enough, by naturality, to study this for \( t = n \). We claim that in \( R(G,F) \cong \mathbb{Z}[y]/(y^a-1) \) \( \varphi_q(V_n) = h(1+bG) \) where \( b \) satisfies \( 1+b p^n = q^a \).

This means we have to check

\[
\prod (1 + y^j + \ldots + y^j^{q-1}) = 1 + b(1 + y + \ldots + y^{a-1}).
\]

But this is true if we replace \( y \) by \( a \)-th roots of unity \( v \) and evaluation at such \( v \) determines elements of \( \mathbb{Z}[y]/(y^a-1) \). (This is essentially a computation with modular characters.) Now an easy checking of fixed point dimensions shows that \( J(V_n) = 1 + bG \). This shows \( hJ(V_t) = \varphi_q(V_t) \) and therefore \( h^J(V_t - \dim V_t) = \varphi_q(V_t - \dim V_t) \). The equality \( h^J = \varphi_q \) is now proved.

We now check that we are in a situation where 3.14.1 and 3.14.5 can be applied. To prove \( \psi^k V = V \) for \( (k,p) = 1 \) and \( F_q G\)-modules \( V \) we again need only consider cyclic \( G \) and then this follows from the determination of the irreducible \( F_q G\)-modules above.
Remark 4.2.5.

If $q$ is $p$-generic then the decomposition homomorphism

$$d : R(G,Q) \rightarrow R(G,F_q)$$

(Serre [147], 15.2) is an isomorphism.

4.3. Representations of 2-groups over $F_3$.

We now consider the analogue of 4.2 for 2-groups and restrict attention to representations over $F_3$. We first recall what the theory of oriented $\gamma$-rings tells us in this case.

In this section $G$ shall be a 2-group. We have the following objects

$$R(G,F_3) \triangleright RO(G,F_3) \triangleright RSO(G,F_3) \triangleright ISO(G,F_3).$$

Here $R(G,F_3)$ is the representation ring of $F_3G$-modules, $RO$ the subring of those modules possessing a $G$-invariant quadratic form, $RSO$ the subring of $F_3G$-modules on which each $g \in G$ acts with determinant one, and ISO is the augmentation ideal of zero-dimensional objects.

The ring $RSO(G,F_3)$ is an oriented $\Lambda$-ring (3.10.2) and ISO($G,F_3$) is an oriented $\gamma$-ring. Let a roof denote 2-adic completion. We have from 3.14.10

**Proposition 4.3.1.**

The map

$$\varphi_3^\text{or} : ISO(G,F_3)^\wedge \rightarrow 1 + ISO(G,F_3)^\wedge$$

is an isomorphism.
In order to relate this isomorphism to the J-homomorphism and to permutation representations we compute the map for cyclic groups \( G = \mathbb{Z}/2^n\mathbb{Z} \).

We start with the representation ring.

We have a decomposition of the group ring

\[
F_3^G \cong \bigoplus_{1 \leq t \leq n} F_3[x]/\phi_t(x)
\]

where \( \phi_t(x) \) is the \( 2^t \)-th cyclotomic polynomial. The \( \phi_t \) are no longer irreducible for \( t \geq 3 \). If \( K_t = F_3[u_t] \), where \( u_t \) is a primitive \( 2^t \)-th root of unity then \( [K_t : F_3] = 2^{t-2}, t \geq 3 \). Moreover \( \phi_2(x) = x^2 + 1 \) is irreducible and \( K_2 = F_3[u_t] = F_9 \).

First assume \( t \geq 3 \). Let \( V_t \) be the \( F_3 \)-module \( K_t \) where a fixed generator \( g \in G \) acts as multiplication with \( u_t \). Then the dual module \( V_t^* = \text{Hom}(V_t, F_3) \) is \( K_t \) and \( g \) acting as \( u_t^{-1} \). Moreover \( F_3[x]/\phi_t(x) \cong V_t \oplus V_t^* \) and \( V_t \) is not isomorphic to \( V_t^* \). The module \( V_t \) cannot carry a \( G \)-invariant quadratic form, because this would imply \( V_t \cong V_t^* \). But

\[
V_t \oplus V_t^* \longrightarrow F_3 : (x, y) \longmapsto Tr(xy)
\]

is a \( G \)-invariant, non-degenerate quadratic form (where \( Tr : K_t \longrightarrow F_3 \) is the trace map).

If \( t = 2 \) let \( V_t = F_3[u_2] = F_9 \) with \( g \) acting as multiplication with \( u_2 \).

Then the norm map \( N : F_9 \longrightarrow F_3 \) is a \( G \)-invariant quadratic form. The associated bilinear form is

\[
b : F_9 \times F_9 \longrightarrow F_3 : (x, y) \longmapsto \varphi(x)y + x \varphi(y)
\]

where \( \varphi \) is the Frobenius automorphism. The determinant of \( b \) is one.
Any G-invariant symmetric bilinear form must have determinant one in this case.

Finally there are two one dimensional representations, \( V_0 \) the trivial representation, and \( V_1 = \mathbb{F}_3 \) with \( g \) acting as multiplication with \(-1\). They both carry quadratic forms \( q : x \mapsto x^2 \) or \( q^- : x \mapsto -x^2 \).

We now enter the computation of \( \Phi^\text{or}_3 \) for the elements \( V_1 - \dim V_1, V_2 - \dim V_2, V_t + V_t^* - \dim(V_t + V_t^*) \). It is sufficient to compute \( \Phi^\text{or}_3 \) of the corresponding modules. Since character computations are easier, we compute for \( QG \)-module and then use the decomposition homomorphism. Let

\[
W_t = \mathbb{Q}[x]/\phi_t(x), \quad t \geq 1
\]

with \( g \) acting as multiplication with \( x \). Let \( S_t \) be the homogeneous \( G \)-set with \( 2^t \) elements and \( V(S_t) \) its permutation representation. Let \( a_t \) be the cardinality of \( K_t \). Then we have

**Proposition 4.3.1.**

For \( t \geq 3 \):

\[
\Phi^\text{or}_3(W_t) = V(S_1) - V(S_0) + 2^{t-1}(a_t-1)V(S_t).
\]

Moreover

\[
\Phi^\text{or}_3(W_2) = V(S_0) - V(S_1) + V(S_2)
\]

\[
\Phi^\text{or}_3(W_1 \oplus W_1) = V(S_0) - 2V(S_1).
\]
Proof.
Suppose $t \geq 3$. We compute the character of $\Phi_{3}^\text{Or}(W_{t})$. Over a splitting field $W_{t}$ decomposes as $W_{t} = \bigoplus_{j}(W_{t}(j) + W_{t}(-j))$ where $W_{t}(j)$ is one-dimensional with $g$ acting as multiplication with $(u_{t})^{j}$ and $1 \leq j \leq 2^{t-1}$. From 3.10.12 we obtain

$$\Phi_{3}^\text{Or}(W_{t}) = \prod_{j}(1 + W_{t}(j) \oplus W_{t}(-j))$$

with character value at $g$ equal to

$$\prod_{j}(1 + u^{j} + u^{-j}), \quad u = u_{t}.$$ 

This product is $-1$, as can be seen by using the identity

$$\prod_{j}(x + x^{-1} - (u^{j} + u^{-j})) = x^{-2^{t-2}} \phi_{t}(x)$$

and evaluating at $x$ a cubic root of unity. The character value of $\Phi_{3}^\text{Or}(W_{t})$ at non-generators $x \neq 1$ of $G$ is $1$. The character value at $1$ is $a_{t}$. It is an easy matter to check that the permutation representation of $S_{1} - S_{0} + 2^{-t}(a_{t}-1)S_{t}$ has the same character.

Finally $\Phi_{3}^\text{Or}(W_{2}) = 1 + W_{2}$, $\Phi_{3}^\text{Or}(W_{1} \oplus W_{1}) = 1 + W_{1} \oplus W_{1}$ and the assertion of the proposition is easily verified.

Connecting $\Phi_{3}^\text{Or}$ with the quadratic $J$-homomorphism and permutation representations presents the difficulty that permutation representations do not generally preserve the orientation. We deal therefore with this problem first.

Let $A_{O}(G) \subset A(G)$ be the subring generated by finite $G$-sets $S$ on which each $g \in G$ acts through even permutations.
If $S$ is any finite $G$-set we can assign to it a homomorphism

$$s(S) : G \rightarrow \mathbb{Z}^* : g \mapsto \text{signum}(l_g)$$

where $l_g : S \rightarrow S$ is left translation by $g$. The assignment $S \mapsto s(S)$ induces a homomorphism

$$s : A(G) \rightarrow \text{Hom}(G, \mathbb{Z}^*)$$

from the additive group of $A(G)$ into the multiplicative group $\text{Hom}(G, \mathbb{Z}^*)$. The kernel of $s$ is $A_0(G)$. Let

$$j : \text{Hom}(G, \mathbb{Z}^*) \rightarrow A(G)$$

be given by

$$j(f) = G/H_f - |G/H_f| + 1$$

where $H_f = \text{kernel } f$. Then $j$ maps into $A(G)^\wedge$. Since $2A(G) \in \text{kernel } s$ everything passes to the 2-adic completions. Let $\text{sign}$ be the composition

$$\text{(4.3.3)} \quad \text{sign} : A(G)^\wedge \xrightarrow{s} \text{Hom}(G/\mathbb{Z}^*) \xrightarrow{j} A(G) \subseteq A(G)^\wedge$$

Then $A(G)^\wedge \rightarrow A(G)^\wedge : x \mapsto x + \text{sign}(x) - 1$ has an image in $A_0(G)^\wedge$ and does not change the cardinality.

Let $QS(G, F_3)$ be the monoid of orientation preserving $F_3G$-modules with quadratic form under orthogonal sum. Denote $f : QS(G, F_3) \rightarrow \text{ISO}(G, F_3)$ the map $(M,q) \mapsto M - \text{dim } M$. 
We define a modified quadratic $J$-map

$$J' : QS(G,F_3) \longrightarrow A_0(G)^\wedge$$

by $J'(M,q) = (JQ(M,q) + \text{sign } JQ(M,q) - 1)_1$ where $(-)_1$ means that we divide the value in the bracket by its cardinality (which is a power of 3, hence invertible in $A_0(G)^\wedge$).

**Theorem 4.3.4.**

The following diagram is commutative

$$\begin{array}{ccc}
QS(G,F_3) & \xrightarrow{J'} & A_0(G)^\wedge \\
\downarrow f & & \downarrow h \\
ISO(G,F_3) & \xrightarrow{\phi} & RSO(G,F_3)^\wedge.
\end{array}$$

**Proof.**

It is sufficient to consider cyclic groups $G = \mathbb{Z}/2^n\mathbb{Z}$. In that case any $(M,q)$ is orthogonal sum of forms carried by one of the modules $V_t + V_t^*$, $t \geq 3$, $V_2$, $V_1 \oplus V_1$. In the case of $V_t + V_t^*$ the form must be hyperbolic. From 2.3.4 one obtains $JQ(V_t \oplus V_t^*, q) = 1 + 2^{-t}(a_t - 1)S_t$ (compare 4.3.2). Since $\text{sign } S_t = S_{t-1}$ we compute $J'(V_t \oplus V_t^*, q) = a_t^{-1}(S_{t-1} + 2^{-t}(a_t - 1)S_t)$ and with 4.3.2 we obtain the desired commutativity. The remaining cases give the following results:

- $JQ(V_2, q) = 1 - S_2$, $J'(V_2, q) = \frac{1}{3}(1 - S_1 + S_2)$
- $JQ(V_1 \oplus V_1, q \oplus q) = JQ(V_1 \oplus V_1, q^- \oplus q^-) = 1 - 2S_1$
- $J'(V_1 \oplus V_1, q \oplus q) = \frac{1}{3}(2S_1 - 1)$
- $JQ(V_1 \oplus V_1, q \oplus q^-) = 1 + S_1$, $J'(V_1 \oplus V_1, q \oplus q^-) = \frac{1}{3}(2S_1 - 1)$. 


Again with 4.3.2 we obtain the desired commutativity.

4.4. Permutation representations over $\mathbb{Q}$.

The previous investigations can be used to give a very round-about prove of

**Theorem 4.4.1.**

Let $G$ be a $p$-group. Then

$$h_\mathbb{Q} : A(G) \longrightarrow R(G, \mathbb{Q})$$

is surjective.

We make various remarks how this is related to the forgoing results. We have decomposition homomorphisms $d_q : R(G, \mathbb{Q}) \longrightarrow R(G, \mathbb{F}_q)$ and $d_3 : R(G, \mathbb{Q}) \longrightarrow R(G, \mathbb{F}_3)$. If $G$ is a $p$-group, $p \neq q$ and $q$ is $p$-generic then $d_q$ is an isomorphism. If $G$ is a 2-group then $d_3$ is an isomorphism. In order to show that $h_\mathbb{Q}$ is surjective one can therefore try to show the same for $h_\mathbb{F}_q$ or $h_\mathbb{F}_3$.

It is now easy to show that the cokernel of $h_\mathbb{Q}$ is annihilated by the order of the group $G$. This can be seen as follows. The characters in $R(G, \mathbb{Q})$ are constant on conjugacy classes and the set of generators of a cyclic group. If $H < G$ is cyclic then $h(G/H)(g)$ is non-zero if and only if $g$ is conjugate to an element in $H$ and $h(G/H)(g) = |G/H^g|$ is divisible by $|NH/H|$. Hence any class function which is constant on generator sets of cyclic groups is a $\mathbb{Z}$-linear combination of $|NH/H|^{-1} h(G/H), H < G$ cyclic. As a consequence $h_\mathbb{Q}$ is surjective for a $p$-group if the $p$-adic completion is surjective. For $p \neq 2$ this follows immediately from 4.2.4. For $p = 2$ one deduces from 4.3.4 that
$A^o(G) \longrightarrow RSO(G)$ is surjective. But if $V$ is any $Q[G]$-module let $D(V)$ be its determinant module. Then $D(V) \oplus 1$ is a permutation representation and $V \oplus D(V) \oplus 1$ is orientation preserving. Hence $V = V \oplus D(V) \oplus 1 - D(V) \oplus 1$ is in the image of $d^o_0$.

4.5. Comments.

The material in this section is taken from Segal [146]. The presentation in 4.3 is unsatisfactory; I hope some reader can elaborate on it. There are important connections between the Burnside ring and integral permutation representations, see Oliver [121], [122] and the references there to earlier work of Dress and Endo-Miyata. For 4.4.1 see also Ritter [133].