Equivariant Cohomology Theories

Definition
A $G$-complex $X$ is a CW-complex with an action of $G$ so that $X^H$ for any $H \leq G$ is a subcomplex.

We would like to give the cohomology of a $G$-complex so that information regarding the action of $G$ is incorporated.

Definition
An equivariant cohomology theory is a sequence of contravariant functors $H^n_G : G$-complexes $\rightarrow Ab$
Equivariant Cohomology Theories

Definition
The orbit category $O_G$ is the category consisting of objects $G/H$ for $H \leq G$ and morphisms $G/H \to G/K$ whenever $g^{-1}Hg \subset K$ for some $g \in G$.

Since $G$-complexes are built from the orbits $G/H$ using equivariant maps $G/H \to G/K$, any ECT should include groups $H^n(G/H)$ and homomorphisms $H^n(G/K) \to H^n(G/H)$. 
Coefficient Systems

Definition
A coefficient system $\underline{M}$ is a contravariant functor from $O_G$, the orbit category, to $Ab$.
The collection of coefficient systems forms a category $\mathcal{C}_G$.

In equivariant ordinary cohomology:

$$H^*(G/H; \underline{M}) = H^0(G/H; \underline{M}) = \underline{M}(G/H)$$

for any $\underline{M} \in \mathcal{C}_G$
A Mackey functor $M$ is a pair of functors $M^*: O^\text{op}_G \to \text{Ab}$ and $M_*: O_G \to \text{Ab}$ such that $M^*(X) = M_*(X) = M(X)$ and which send disjoint unions to direct sums and satisfy certain commutativity relations.

Notation: For $f: G/H \to G/K$ we call $M^*(f)$ a restriction and $M_*(f)$ a transfer.
Bredon Cohomology

If $X$ is a $G$-complex, define the chains on $X$ by:

$$C_*(X)(G/H) = C_*(X^H)$$

Then for $M \in C_G$ define the cochains by:

$$C^n_G(X; M) = \text{Hom}_{C_G}(C_n(X), M)$$

and so we define Bredon cohomology to be

$$H^n_G(X; M) = H^n(C_G^*(X; M))$$
An alternative definition for Bredon cohomology can be given since it is, in fact, representable. To give this, we must have definitions for equivariant homotopy groups.

**Definition**

Let $X$ be a $G$-space. For each $H \leq G$ the equivariant homotopy groups of $X$ are given by

$$\pi^H_n(X) = [S^n \wedge G/H_+, X]_G$$
Stable Equivariant Homotopy Groups

Definition
A G-spectrum $X$ is a collection of G-spaces $X_k$ together with equivariant maps $\Sigma X_k \to X_{k+1}$ (or equivalently $X_k \to \Omega X_{k+1}$)

Definition
The equivariant homotopy groups of the G-spectrum $X$ are given by

$$\pi^H_n(X) = \left[ \Sigma^\infty S^n \wedge (G/H)_+, X \right]_G$$

Note: These homotopy groups form a Mackey functor:

$$\pi_n(X) = \left[ \Sigma^\infty S^n \wedge (G/H)_+, X \right]_G$$

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Mackey Functors in Eq. Homotopy and Cohomology Theory
Stable Equivariant Homotopy Groups

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Definition
The equivariant homotopy groups of the G-spectrum $X$ are given by

$$\pi_n^H(X) = [\Sigma^\infty S^n \wedge (G/H)_+, X]_G$$

or equivalently

$$\pi_n^H(X) = \colim_k \pi_n^H(X_k)$$

Note: These homotopy groups form a Mackey functor:

$$\pi_n(X)(G/H) = [\Sigma^\infty S^n \wedge G/H_+, X]_G = \pi_n^H(X)$$
Equivariant Homotopy Group Mackey Functor

\[
\pi_k(X)(G/H) = [G/H_+ \wedge S^k, X]_G \\
= [G_+ \wedge_H S^k, X]_G \\
= [S^k, X]_H \\
= \pi_k(X^H)
\]

Restriction Map \( \pi_k(X^K) \rightarrow \pi_k(X^H) \)
Induced from inclusion of fixed points \( X^K \rightarrow X^H \)

Transfer Map \( \pi_k(X^H) \rightarrow \pi_k(X^K) \)
Induced from \( X^H \rightarrow X^K \)

\[
x \rightarrow \sum_{gH \in K/H} g \cdot x
\]
Cohomology Theories from G-Spectra

Let $X$ be a $G$-space and $Y$ be a $G$-spectrum.

The groups $[\Sigma^{k-n}X, Y_k]_G$ form a direct system:

$$[\Sigma^{k-n}X, Y_k]_G \to [\Sigma^{k-n+1}X, \Sigma Y_k]_G \to [\Sigma^{k-n+1}X, Y_{k+1}]_G$$

So we can define cohomology:

$$\tilde{Y}_G^n(X) = \colim_k [\Sigma^{k-n}X, Y_k]_G$$

$$= \colim_k \pi_{k-n}(F(X, Y^G)_k)$$

$$= \pi_{-n}(F(X, Y)^G)$$

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Equivalence of Categories

From which $G$-spectrum do we obtain Bredon cohomology?

**Proposition**

There is an equivalence of categories between the category of Mackey functors and the homotopy category consisting of $G$-spectra $X$ such that $\pi_i(X) = 0$ for $i \neq 0$.

In particular, for any Mackey functor $M$, we have an associated Eilenberg-MacLane spectrum $HM$ satisfying:

$$\pi_k(HM) = \begin{cases} M & k=0 \\ 0 & \text{otherwise} \end{cases}$$
Bredon Cohomology

Now for any Mackey functor $M$, we may obtain Bredon Cohomology from $HM$ as follows:

$$\tilde{H}^n_G(X; M) = \colim_k [\Sigma^{k-n}X, (HM)_k]_G$$

$$= [\Sigma^{-n}X, HM]_G$$

$$= \pi_{-n}(F(X, HM))^G$$

$$= \pi_{-n}(F(X, HM))(G/G)$$

Note: For a group $G$, Bredon Cohomology is the image of $G/G$ under a Mackey functor.
RO(G)-grading

In working with equivariant theories, we want to consider spheres with nontrivial $G$-action. In particular, we will look at linear spheres arising from representations of $G$.

**Definition**
For a group $G$ and a vector space $V$, we will say a representation of $G$ is a homomorphism $\rho : G \rightarrow O(V)$.

**Definition**
For a representation space $V$ we will write $S^V$ to denote the one-point compactification of $V$. 
RO(G)-graded Homotopy Groups

If $V \in RO(G)$ then it is also an $H$-representation for any $H \leq G$ so we have RO(G)-graded homotopy groups:

$$\pi^H_V(X) = [S^V, X]_H = [G_+ \wedge_H S^V, X]_G$$

Note: Our original $\mathbb{Z}$-graded homotopy groups $\pi^H_n(X)$ are the homotopy groups associated to the trivial representation $n \in RO(G)$ where $n$ stands for $\mathbb{R}^n$. 

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Mackey Functors in Eq. Homotopy and Cohomology Theory
RO(G)-graded Cohomology

In addition to usual $\mathbb{Z}$-suspensions we have:

$$\Sigma^V X = X \land S^V$$

for any $V \in RO(G)$

So extending the usual suspension axiom

$$\sigma_n : H^n(X) \to H^{n+1}(\Sigma X)$$

we obtain RO(G)-graded cohomology groups:

$$H^\alpha_G(X) \cong H^{\alpha+V}_G(\Sigma^V X)$$

for $\alpha, V \in RO(G)$
Why is the RO(G)-grading important?

A few examples:

- (Lewis) Let $X$ be a $\mathbb{Z}/p$-complex constructed from even dimensional unit disks of real $G$-representations. The $H^*_G(X)$ is a free $RO(G)$-graded module over the equivariant ordinary cohomology of a point.

- (Lewis) Let $V$ be a complex $G$-representation and $P(V)$ the associated complex projective space. Then all generators of $H^*_G(P(V))$ live in dimensions corresponding to nontrivial representations of $G$.

- (tom Dieck) RO(G)-graded cohomology theories admit important splitting theorems.
When can we extend?

In the $RO(G)$-graded setting we have transfer maps

$$\tau(G/H) : S^V \to (G/H)_+ \wedge S^V$$

These induce transfer homomorphisms

$$\tilde{H}^n_H(X; M) \cong \tilde{H}^{V+n}_G(\Sigma^V (G/H_+ \wedge X); M)$$

If $n = 0$ and $X = S^0$ we get a transfer map

$$\underline{M}(G/H) \to \underline{M}(G/G)$$
If this argument is elaborated a bit we get that the coefficient system $M$ must actually be a Mackey functor.

Additionally it can be shown that this necessary condition is also sufficient:

**Theorem** (May, Waner) The ordinary $\mathbb{Z}$-graded cohomology theory $\tilde{H}^*_G(-; M)$ extends to an RO($G$)-graded theory if and only if $M$ extends to a Mackey functor.
We may additionally think of our Equivariant Cohomology Theory as being Mackey functor valued:

\[ H^\alpha_G(X; M) = \pi_{-k}(X)(G/G) \]

and

\[ H^\alpha_H(X; M) = \pi_{-k}(X)(G/H) \]

In general we have

\[ H^\alpha_G(X; M)(G/H) = H^\alpha_G(G/H_+ \wedge X; M) \]
An Example

\[
\begin{align*}
H_{C_p}^\alpha(X; M)(C_p/C_p) &= H_{C_p}^\alpha(X; M) \\
(H_C^\alpha(X; M)(C_p/e) &= H_C^\alpha(C_p \times X; M)
\end{align*}
\]

\(\pi^*\) is induced from the projection \(\pi : C_p \times X \to X\)

\(\pi!\) is the transfer map arising from regarding the projection \(\pi\) as a covering space.

Note: \(H_G^\alpha(G \times X; M) \cong H_{\lvert \alpha \rvert}(X; M(G/e))\)