On Jacobi Sums, Multinomial Coefficients, and $p$-adic Hypergeometric Functions

Paul Thomas Young

Department of Mathematics, University of Charleston, Charleston, South Carolina 29424

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We extend the methods of our previous article to express certain special values of $p$-adic hypergeometric functions in terms of the $p$-adic gamma function and Jacobi sums over general finite fields. These results are obtained via $p$-adic congruences for Jacobi sums in terms of multinomial coefficients, and allow one to more fully exploit classical hypergeometric identities to obtain $p$-adic unit root formulae.

1. Introduction

In [15] we gave some explicit formulae relating Jacobi sums over the prime field $\mathbb{F}_p$ to values of $p$-adic hypergeometric functions. These formulae were obtained from combinatorial identities and the methods of Dwork ([4, 5]) and Koblitz [7], and may be viewed as $p$-adic analogues of classical results. The primary focus of this article is the generalization of these results to include Jacobi sums defined over finite extensions of $\mathbb{F}_p$.

We begin in Section 2 by giving congruence results for general Jacobi sums over finite fields of characteristic $p > 2$ in terms of multinomial coefficients. The main tools are the Gross–Koblitz formula and the properties of the $p$-adic gamma function. For Jacobi sums which are not $p$-adic units, the congruences we give are stronger than those typically predicted by the theory of formal group laws. We then apply these results to hypergeometric functions in Section 3 to give $p$-adic analogues of classical formulae. Equation (3.17) below is perhaps the best example (particularly in the case $n = 2m$), and the cohomological interpretation given in [15] remains valid relative to the Frobenius map $(x, y) \mapsto (x^p, y^q)$. The results (3.24), (3.27), (3.28), and (3.44) of [15] may also be extended by the methods found in this paper.
In Section 4 we consider the elliptic curve with affine equation $y^2 = x^3 - x$ which has supersingular reduction modulo $p$ when $p \equiv -1 \pmod{4}$, and show that the roots of its zeta function over $\mathcal{F}_p$ may be obtained from a limit of $p$-adic hypergeometric functions, although this is not the specialization of a uniform limit. As a further application, we also express the formal-group congruences associated to an Apéry sequence in terms of Jacobi sums.

2. JACOBI SUMS AND MULTINOMIAL COEFFICIENTS

Throughout this paper $p$ will denote an odd prime, $\mathbb{F}_q$ the finite field of $q = p^f$ elements, $\mathbb{Z}_p$ the ring of $p$-adic integers, $\mathbb{Q}_p$ the field of $p$-adic numbers, $K$ the unramified extension of $\mathbb{Q}_p$ of degree $f$, $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$, "ord" the valuation on $\mathbb{C}_p$ normalized so that $\text{ord}(p) = 1$, and $\mathcal{O}$ the ring of integers of $\mathbb{C}_p$. We let $\pi \in \mathcal{O}$ be a fixed solution to $\pi^{p-1} = -p$ and let $\zeta$ be the unique $p$th root of unity in $\mathcal{O}$ such that $\zeta \equiv 1 + \pi \pmod{\pi^2 \mathcal{O}}$.

We define a map $x \mapsto x'$ on $\mathcal{O} \cap \mathbb{Z}_p$ by requiring that $px' - x = \mu_x \in \{0, 1, 2, \ldots, p - 1\}$. We write $x^{(0)} = x$, and $x^{(i)} = (x^{(i-1)})'$ for $i > 0$; we also will write $\mu_x^{(i)}$ for $\mu_{x^{(i)}}$. It follows that the $\mu_x^{(i)}$ are the digits in the $p$-adic expansion of $-x$, that is, $-x = \sum_{i=0}^{\infty} \mu_x^{(i)} p^i$. It is easy to verify that this map is well-defined and continuous; that $x^{(i)} = 0$ for some $i$ if and only if $x$ is zero or a negative integer; and that $x^{(i)} = x$ if and only if $x$ is a rational number in $[0, 1]$ with denominator dividing $q - 1$.

The $p$-adic gamma function $\Gamma_p$ is defined for positive integers $n$ by

$$\Gamma_p(n) = (-1)^n \prod_{0 < j < n} \frac{1}{j},$$

and has an extension to a continuous function $\Gamma_p: \mathbb{Z}_p \to \mathbb{Z}_p^*$, which is Lipschitz with constant 1, and satisfies the functional equations of translation and reflection

$$\Gamma_p(x + 1) = \begin{cases} -x \Gamma_p(x), & x \in \mathbb{Z}_p^*, \\ -\Gamma_p(x), & x \in p \mathbb{Z}_p; \end{cases}$$

$$\Gamma_p(x) \Gamma_p(1 - x) = (-1)^{n_x}, \quad x \in \mathbb{Z}_p.$$  \hfill (2.2)

Let $\psi: \mathbb{F}_p \to \mathbb{Q}_p(\zeta)$ be the additive character on $\mathbb{F}_p$ defined by $\psi(t) = \zeta^t$, and let $\psi: \mathbb{F}_q \to \mathbb{Q}_p(\zeta)$ denote the additive character on $\mathbb{F}_q$ defined by $\psi(t) = \psi(\text{Tr}(t))$, where $\text{Tr}: \mathbb{F}_q \to \mathbb{F}_p$ is the trace map. The Teichmüller character $\omega_j: \mathbb{F}_q \to K$ is the unique multiplicative character on $\mathbb{F}_q$ such that,
for all $t \in \mathbb{F}_q$, the reduction of $\omega_f(t) \mod p$ is $t$. (We extend all multiplicative characters $\chi$ using the convention $\chi(0) = 0$.)

For $x = a/(q-1)$ with $a \in \mathbb{Z}$, the Gauss sum $g(\omega_f^{-x})$ over $\mathbb{F}_q$ associated to the characters $\psi_f$ and $\omega_f^{-x}$ is defined by

$$g(\omega_f^{-x}) = -\sum_{t \in \mathbb{F}_q} \psi_f(t) \omega_f^{-x}(t).$$  \hfill (2.4)

Write $a = t(q-1) + c$ with $t, c \in \mathbb{Z}$ and $0 \leq c \leq q-1$, and put $\gamma = c/(q-1)$; then from the Gross–Koblitz formula [6] we have

$$g(\omega_f^{-x}) = g(\omega_f^{-\gamma}) = \frac{\prod_{i=0}^{\frac{S(c)}{p-1}} \zeta_{p}(y^{(i)})}{G_1},$$  \hfill (2.5)

where

$$G_1 = \begin{cases} 1, & \text{if } c < q-1, \\ q, & \text{if } c = q-1, \end{cases}$$  \hfill (2.6)

and where $S(c)$ denotes the sum of the digits in the base $p$ expansion of $c$. (Allowing both $c = 0$ and $c = q-1$ will be useful later.)

If $s \geq 2$ and $\chi_1, \ldots, \chi_s : \mathbb{F}_q \to K$ are multiplicative characters, the Jacobi sum $J(\chi_1, \ldots, \chi_s)$ is defined by

$$J(\chi_1, \ldots, \chi_s) = -\sum_{t_1 + \cdots + t_s = 1} \chi_1(t_1) \cdots \chi_s(t_s).$$  \hfill (2.7)

A modification of ([13, Lemma 6.2]), using the results of [14] or ([9, Theorem 1.1]) shows that

$$J(\chi_1, \ldots, \chi_s) = \frac{(-1)^s}{G_2} \frac{g(\chi_1) \cdots g(\chi_s)}{g(\chi_1 \cdots \chi_s)},$$  \hfill (2.8)

where

$$G_2 = \begin{cases} 1, & \text{if } \chi_1 \cdots \chi_s \text{ is nontrivial}, \\ q, & \text{if } \chi_1 \cdots \chi_s \text{ is trivial but each } \chi_j \text{ is nontrivial}. \end{cases}$$  \hfill (2.9)

The following lemma will be used to relate Jacobi sums to multinomial coefficients via (2.5) and (2.8).

**Lemma 2.1.** Suppose $m_1, \ldots, m_s$ are nonnegative integers and write $m_j = k_j p + l_j$ with each $l_j \in \{0, 1, \ldots, p-1\}$; set $m = m_1 + \cdots + m_s$. 

\[ k = k_1 + \cdots + k_s, \text{ and } l = l_1 + \cdots + l_s. \] Let \( \varepsilon \) be a nonnegative integer and set 
\[ \delta = \left\lfloor \frac{(l + \varepsilon)}{p} \right\rfloor. \] Then

\[
\frac{(m + \varepsilon)! \; k_1! \cdots k_s!}{(k + \delta)! \; m_1! \cdots m_s!} = (-p)^\delta \frac{I_p^{\left(-m_1\right)} \cdots I_p^{\left(-m_s\right)}}{I_p^{\left(-m - \varepsilon\right)}}.
\]

**Proof.** We note that \((-m_j) = -k_j\) for each \( j \) and \((-m - \varepsilon) = -k - \delta\). From the definition of \( I_p^\rho \) we have

\[ -I_p^{\left(1 + m_j\right)} = (-1)^m \; p^{-k_j} \frac{m_j!}{k_j!}, \] (2.10)

and a similar expression for \(-I_p^{\left(1 + m + \varepsilon\right)}\). Therefore we have

\[
\frac{(m + \varepsilon)! \; k_1! \cdots k_s!}{(k + \delta)! \; m_1! \cdots m_s!} = (-1)^{1 + \varepsilon} \; p^\delta \frac{I_p^{\left(1 + m + \varepsilon\right)}}{I_p^{\left(1 + m_1\right)} \cdots I_p^{\left(1 + m_s\right)}}.
\] (2.11)

The lemma then follows by applying the reflection formula (2.3), noting that each \( \mu_{-m_0} = l_0 \) and \( \mu_{-m - \varepsilon} = l + \varepsilon - \rho \delta \).

We now give our principal congruence result for general Jacobi sums.

**Theorem 2.2.** Let \( x_1, \ldots, x_s \in \mathbb{Z}_p \cap [0, 1) \) satisfy \( x_j = a_j / (q - 1) \) with each \( a_j \in \mathbb{Z} \), and set \( x = x_1 + \cdots + x_s \). We assume that \( x > 0 \), and if \( x \in \mathbb{Z} \) we also assume each \( x_j > 0 \). For \( r > 0 \) define the nonnegative integers \( n_j, r \) as \( (q - 1) x, n_r = (q - 1) x \). Let \( t \) be the greatest integer strictly less than \( x \), and suppose \( t < p \). Let \( \varepsilon \) be the \( p \)-adic ordinal of the Jacobi sum \( J(\omega_{-a_0}^{q-1}, \ldots, \omega_{-a_s}^{q-1}) \). Then for each \( r > 0 \) we have the congruence

\[
\frac{n_r + t}{n_{r-1} + t} \quad \equiv \quad (-1)^r J(\omega_{-a_1}^{q-1}, \ldots, \omega_{-a_s}^{q-1}) \quad \pmod{p^{1 + r q'} \mathbb{Z}_p}.
\]

**Proof.** For \( j = 1, \ldots, s \) we may write \( n_{j,r} = \sum_{i=0}^{r-1} \mu_{x_j}^{i} \; p^{i} \). For \( 0 \leq i \leq f - 1 \) we will apply Lemma 2.1 with \( m_j = (-n_{j,r})^{i+1} \) and \( l_j = \mu_{x_j}^{i} \). For each \( i \) we choose the nonnegative integer \( \varepsilon_i = \varepsilon_i \) so as to satisfy \((-n_{r-1} + t)^{i+1} = -n_{r-1}^{i} \cdots - n_{r-1}^{0} - \varepsilon_i \); this implies that \( \delta = \varepsilon_i + 1 \) in the notation of the lemma. Thus \( e_{i+1} = \left\lfloor \left( \sum_{i=0}^{r-1} \mu_{x_j}^{i} + \varepsilon_i \right) / p \right\rfloor \); i.e., \( e_{i+1} \) is the number of carries from the \((i+1)\)st to the \((i+2)\)nd digit in the base \( p \) addition of \( a_1 + \cdots + a_r + t \). Writing \( x = a / (q - 1) \) with \( a = (q - 1) t + c \) and \( 0 < c \leq q - 1 \), and setting \( \gamma = c / (q - 1) \), it follows that \( n_r + t = q't + (q' - 1) \gamma \) for each \( r \geq 0 \). From this we see that \((-n_{r-1} - t)^{i+1} = -n_{r-1} - t \) for each \( r > 0 \), implying \( \varepsilon_f = \varepsilon_0 = t \).
We take the product on both sides of these equalities from Lemma 2.1, as \( i \) runs from 0 to \( f - 1 \). On the left, the product telescopes, yielding

\[
\frac{(n_r + t)! n_{1,r-1}! \cdots n_{s,r-1}!}{(n_{r-1} + t)! n_{1,r}! \cdots n_{s,r}!} = (-p)^e \prod_{i=0}^{f-1} \frac{\Gamma_p((-n_{1,r})^{(i)}) \cdots \Gamma_p((-n_{s,r})^{(i)})}{\Gamma_p((-n_{r-1} - t)^{(i)})},
\]

(2.12)

where \( e = e_1 + \cdots + e_f \) is the number of carries in the base \( p \) addition \( a_1 + \cdots + a_s + t \), since we assume \( e_f = t < p \). Since \( \Gamma_p \) is unit-valued and Lipschitz with constant 1 we have the congruence

\[
\prod_{i=0}^{f-1} \frac{\Gamma_p((-n_{1,r})^{(i)}) \cdots \Gamma_p((-n_{s,r})^{(i)})}{\Gamma_p((-n_{r-1} - t)^{(i)})} \equiv \prod_{i=0}^{f-1} \frac{\Gamma_p(x_1^{(i)}) \cdots \Gamma_p(x_s^{(i)})}{\Gamma_p(y_1^{(i)})} \pmod{pq^{e-1}Z_p},
\]

(2.13)

and therefore

\[
\frac{(n_r + t)! n_{1,r-1}! \cdots n_{s,r-1}!}{(n_{r-1} + t)! n_{1,r}! \cdots n_{s,r}!} \equiv (-p)^e \prod_{i=0}^{f-1} \frac{\Gamma_p(x_1^{(i)}) \cdots \Gamma_p(x_s^{(i)})}{\Gamma_p(y_1^{(i)})} \pmod{p^{e-1}q^{e-1}Z_p}.
\]

(2.14)

We claim that the right member of the congruence (2.14) is precisely \((-1)^e J(\omega_f^{-a_1}, \ldots, \omega_f^{-a_s})\). From (2.5) and (2.8), we see that

\[
(-1)^e J(\omega_f^{-a_1}, \ldots, \omega_f^{-a_s}) = \pi^{a_1} \prod_{i=0}^{f-1} \frac{\Gamma_p(x_1^{(i)}) \cdots \Gamma_p(x_s^{(i)})}{\Gamma_p(y_1^{(i)})},
\]

(2.15)

where \( g = S(a_1) + \cdots + S(a_s) - S(c) \); note that, since \( 0 < c \leq q - 1 \), we have \( c = q - 1 \) if and only if \( w_f^{-a} \) is trivial, so that the factors \( G_1 \) and \( G_2 \) from (2.6) and (2.9) always cancel. Thus we need only show that \( e = g/(p - 1) \). Since \( a + t = t q + c \), we have \( S(a + t) = S(t q + c) = S(t) + S(c) \), and therefore \( g = S(a_1) + \cdots + S(a_s) + S(t) - S(a + t) \). As it is well-known that \( \text{ord}(n!) = (n - S(n))/(p - 1) \), we see that \( g/(p - 1) \) is the ordinal of the multinomial coefficient \( \binom{a_1 + \cdots + a_s}{t} \), which is precisely the number \( e \) of carries in the base \( p \) addition \( a_1 + \cdots + a_s + t \). The proof is now complete.

When \( e > 0 \) these congruences become stronger than those generally obtained from formal group laws. Considering the simplest case, suppose that \( f = 1 \) (so \( p = q \)); then \( e = e_1 = t \). The result for \( s = 2 \), \( e = 0 \) has been
given previously ([15, Corollary 2.2]). The congruences of Theorem 2.2 hold modulo \( p^{e+r} \mathbb{Z}_p \), and therefore one has the result
\[
\left( \begin{array}{c} n_r + t \\ n_{1,r}, \ldots, n_{s,r}, t \end{array} \right) \equiv (-1)^s J(\omega_1^{-a_1}, \ldots, \omega_1^{-a_s}) \times \left( \begin{array}{c} n_{r-1} + t \\ n_{1,r-1}, \ldots, n_{s,r-1}, t \end{array} \right) \pmod{p^{1+\epsilon+r} \mathbb{Z}_p},
\]
(2.16)
since the multinomial coefficient on the right side of (2.16) has \( p \)-adic ordinal \((r-1)t\). For \( t \geq 1 \) such congruences have been called supercongruences.

A second interesting case is obtained by taking an integer \( d > 2 \), an odd prime \( p \) such that \( p \equiv -1 \pmod{d} \), \( s = 2 \), and \( \alpha_1 = \alpha_2 = 1/d \). Taking \( f = 2 \), \( q = p^2 \), one then has \( e = 1, t = 0 \), and the congruences read
\[
\frac{\binom{2(q'-1)/d}{(q'-1)/d}}{\binom{2(q'-1)/d}{(q'-1)/d}} + p \equiv 0 \pmod{q' \mathbb{Z}},
\]
(2.17)
as it is easily verified from (2.8), (2.5), and (2.3) that \( J(\omega_2^{-a_1}, \omega_2^{-a_2}) = -p \).

The \( r = 1 \) case of these congruences has been given in ([8, Proposition 3.1]); in the cases \( d = 3, 4 \) they arise from formal groups associated to certain supersingular elliptic curves and are related to elliptic cohomology. In Section 4 below we examine the supersingular elliptic curve with \( j = 12^3 \) which corresponds to the \( d = 4 \) case.

As the occurrence of the integer \( t \) in the multinomial coefficients of Theorem 2.2 is rather artificial, it is natural to remove it, which we now do.

**Corollary 2.3.** Under the hypotheses of Theorem 2.2, for each \( r > 0 \) we have the congruence
\[
\left( \begin{array}{c} n_r \\ n_{1,r}, \ldots, n_{s,r} \end{array} \right) \equiv (-1)^s J(\omega_f^{-a_1}, \ldots, \omega_f^{-a_s}) \pmod{p^{b+d+r}q'^{-1} \mathbb{Z}_p},
\]
where \( d = 0 \) if \( \mu_z + t < p \) and \( d = -1 - \text{ord}(\alpha' - 1) \) if \( \mu_z + t \geq p \), and \( b = 1 - \text{sgn}(t) \). Furthermore, in all cases we have \( b + d + e \geq 1 \), so the congruence always holds modulo \( pq'^{-1} \mathbb{Z}_p \).
Proof. As before we write \( \alpha = a/(q-1) \) with \( a = t(q-1) + c \) and \( 0 < c \leq q-1 \). If \( t = 0 \) the results are immediate, so assume \( 1 \leq u \leq t < p \); then for all \( r > 0 \) we have

\[
\frac{n_{r-1} + u}{n_r + u} = 1 + \frac{(q-1) - q'}{1 + q'(u - \alpha)}.
\]

(2.18)

If \( \text{ord}(\alpha - u) > 0 \), then \( \mu_s = p - u \), whence \( \mu_s + t \geq p \); in this case set \( -d = \text{ord}(\alpha - u) \), otherwise set \( d = 0 \). In the former case, we note that \( \alpha - u = \alpha + \mu_s - p = p(\alpha' - 1) \), showing that the two definitions of \( d \) coincide. By writing

\[
\alpha - u = \frac{-qu + (a + u)}{q - 1} = \frac{(t - u)q + (c - t + u)}{q - 1}
\]

(2.19)

we see that \( -d \leq f \), and that \( -d = f \) if and only if \( c + u = t \), in which case \( a + u = tq \) with \( t \geq 2 \) and \( \alpha - u = cq/(q-1) \). It follows that the denominator of the right side of (2.18) is a \( p \)-adic unit for all \( r > 0 \), and therefore

\[
\frac{(n_{r-1} + 1) \cdots (n_r + t)}{(n_r + 1) \cdots (n_r + t)} \equiv 1 \pmod{p^d q^{r-1} \mathbb{Z}_p},
\]

(2.20)

from which it follows that

\[
\left( \begin{array}{c} n_r \\ n_1, \ldots, n_{r-1} \end{array} \right) \equiv \left( \begin{array}{c} n_r + t \\ n_1, \ldots, n_{r-1} + t \end{array} \right) \pmod{p^{f+e} q^{r-1} \mathbb{Z}_p},
\]

(2.21)

since the right member of this congruence has \( p \)-adic ordinal \( e \). Since \( d \leq 1 \), comparison with Theorem 2.2 proves the first assertion of the corollary.

We now recall that \( e = \varepsilon_1 + \cdots + \varepsilon_f \) is the number of carries in the base \( p \) addition \( a_i + \cdots + a_s + t \). From (2.19) we see that \( \text{ord}(\alpha - u) = \text{ord}(a + u) = -d \). Since the first \( -d \) digits of \( a + u \) are zero and \( a < a + u \leq a + t \), we have \( \varepsilon_i > 0 \) for \( 0 \leq i < -d \). If \( -d < f \) then we have \( e \geq -d + \varepsilon_f \geq -d + 1 \) since we assume \( t = \varepsilon_f \). If \( -d = f \) then \( t \geq 2 \) by the remarks following (2.19); thus all \( \varepsilon_i > 0 \) and \( \varepsilon_f \geq 2 \), whence \( e \geq f + 1 \). Thus \( d + e \geq 1 \) in all cases, completing the proof.

3. Jacobi Sums and \( p \)-adic Hypergeometric Functions

We recall from our previous article [15] certain results and definitions concerning the \( p \)-adic theory of hypergeometric functions. If \( \alpha_1, \ldots, \alpha_k, \)
\( \gamma_1, \ldots, \gamma_{k-1} \in \mathbb{Q} \cap \mathbb{Z}_p \) and none of the \( \gamma_j \) are zero or negative integers, we denote for \( i \geq 0 \) the hypergeometric series

\[
F^{(i)}(X) = \sum_{n=0}^{\infty} A^{(i)}(n) X^n
\]

and for \( i, s \geq 0 \) set \( F^{(i)}_s(X) = \sum_{n=0}^{s-1} A^{(i)}(n) X^n \). Suppose that the parameters satisfy the conditions

(C1) \( |\gamma_j^{(i)}| = 1 \) for all \( i \geq 0, j = 1, \ldots, k-1 \);

(C2) For each fixed \( i \geq 0 \), supposing the indices are rearranged so that \( \mu_1^{(i)} \leq \cdots \leq \mu_{s_i}^{(i)} \) and \( \nu_1^{(i)} \leq \cdots \leq \nu_{m_i}^{(i)} \), where \( \gamma_j \neq 1 \) for \( 1 \leq j \leq m \) and \( \gamma_j = 1 \) for \( m < j \leq k-1 \), we have \( \mu_j^{(i)} > \mu_{m_j}^{(i)} \) for \( j = 1, \ldots, m \).

Then for all \( i \geq 0 \) we have \( F^{(i)}(X) \in \mathbb{Z}_p[[X]] \), and for \( r \geq s \geq 0 \) there are formal congruences

\[
F^{(i)}_{r+s}(X) \equiv F^{(i)}_{r+s}(X^p) F^{(i)}_{s+1}(X) \pmod{p^{r+1} \mathbb{Z}_p[[X]]}.
\]  

These congruences imply that the ratio \( F^{(0)}(X)/F^{(1)}(X^p) \) is the restriction to the disk \( \{ x \in \mathbb{C}_p : |x| < 1 \} \) of the analytic element (i.e., uniform limit of rational functions)

\[
k \bar{\Theta}_{k-1} \left( \frac{\alpha_1, \ldots, \alpha_k}{\gamma_1, \ldots, \gamma_{k-1}} ; X \right) = \lim_{r \to \infty} F^{(0)}_{r+s}(X)/F^{(1)}_{s+1}(X^p),
\]

which is supported on the Hasse domain

\[
\mathcal{D} = \{ x \in \mathbb{C}_p : |F^{(i)}_1(x)| = 1 \text{ for all } i \geq 0 \}.
\]

For series satisfying \( F^{(i)} = F^{(0)} \), we will obtain products of Jacobi sums over \( \mathbb{F}_q \) as values of the analytic element of support \( \mathcal{D} \) defined by

\[
k \bar{\Theta}^{(i)}_{k-1} \left( \frac{\alpha_1, \ldots, \alpha_k}{\gamma_1, \ldots, \gamma_{k-1}} ; X \right) = \lim_{r \to \infty} F^{(0)}_{r+s}(X)/F^{(i)}_{s+1}(X^q)
\]

(cf. [4, p. 42]) whose existence as a uniform limit for \( x \in \mathcal{D} \) follows from the the formal congruences (3.2) and the observation that

\[
F^{(0)}_{r+s}(X)/F^{(i)}_{s+1}(X^q) = \prod_{i=0}^{f-1} F^{(i)}_{r+s-i}(X^p)/F^{(i+1)}_{r+s-i+1}(X^{p^{i+1}}).
\]

We note that the limits in (3.3), (3.5) may exist in \( \mathbb{Q}_p \) for certain values of \( x \in \mathcal{D} \) not lying in \( \mathcal{D} \), or when the hypotheses (C1), (C2) are not satisfied;
but in this case they need not be specializations of uniform limits. Here we will evaluate the limits (3.3), (3.5) at \( x = \pm 1 \in \mathcal{D} \) for functions satisfying (C1), (C2). Our method will be to \( p \)-adically approximate the given series by terminating series which can be evaluated by combinatorial results. For fixed \( x_0 \in \mathcal{D} \) the truncated series \( F^{(i)}_1(x_0) \) are rational functions of the parameters \( x^{(i)}, \gamma^{(i)} \), and therefore one may appeal to their continuity with respect to the parameters (cf. [7]), although this need not be the case for the nonterminating \( F^{(i)}_1(x_0) \).

We generalize our previous result ([15, Theorem 3.1]) giving a \( p \)-adic analogue of Kummer's theorem, which gives the value of a well-poised \( _2F_1(-1) \).

**Theorem 3.1.** Let \( T \) denote the set of all \((x, \beta) \in \mathbb{Z}_p^2\) such that \(-1 \in \mathcal{D} \) and both (C1), (C2) are satisfied for \( F(X) = _2F_1(2x, \beta; 1 + 2x - \beta; X) \). Then \((x, \beta) \in T \) if and only if

(i) \( 2\mu_x^{(i)} \leqslant \mu_{\beta}^{(i)} \) for all \( i \geqslant 0 \);

(ii) If \( \beta \neq 2x \) then \( \mu_{\beta}^{(i)} - \mu_x^{(i)} < (p - 1)/2 \) for all \( i \geqslant 0 \).

Furthermore, if \((x, \beta) \in T \) then

\[
2\gamma_1 \left( \begin{array}{c} 2x, \beta \\ 1 + 2x - \beta \\ -1 \end{array} \right) = (-1)^{\mu_1} \frac{\Gamma_p(x) \Gamma_p(\beta - x)}{\Gamma_p(2x) \Gamma_p(\beta - 2x)},
\]

and if in addition \((q - 1)x, (q - 1)\beta) = (a, b) \in \{0, 1, \ldots, q - 1\}^2 \) then

\[
2\gamma_1^{(i)} \left( \begin{array}{c} 2x, \beta \\ 1 + 2x - \beta \\ -1 \end{array} \right) = (-1)^{\mu_1} \frac{J(\omega_p^{-a}, \omega_p^{-b})}{J(\omega_p^{-2a}, \omega_p^{-2b})}.
\]

**Proof.** Suppose \((x, \beta) \in T \), and set \( \gamma = 1 + 2x - \beta \). If \( \gamma = 1 \) then \( \beta = 2x \), so \( \mu_\beta^{(i)} = \mu_x^{(i)} \) for all \( i \). Since \( (\beta - 2x) + \gamma = 1 \) we have \( \mu_\beta^{(i)} - 2x + \mu_x^{(i)} = p - 1 \) for all \( i \), so if \( \gamma \neq 1 \) then by (C2) we have \( \mu_x^{(i)} > \mu_x^{(i)} \), \( \mu_{\beta}^{(i)} \), which implies that for all \( i \),

\[
\mu_\beta^{(i)} + \mu_x^{(i)} < p - 1.
\]

(3.7)

\[
\mu_{\beta}^{(i)} + \mu_x^{(i)} - 2x < p - 1.
\]

(3.8)

From (3.7) we see that in fact

\[
\mu_\beta^{(i)} + \mu_x^{(i)} - 2x = \mu_\beta^{(i)} < p - 1.
\]

(3.9)

so in any event we have \( 0 \leqslant \mu_x^{(i)} \leqslant \mu_{\beta}^{(i)} \) for all \( i \).

Since we assume that \( F(X) \) satisfies (C1), (C2), we note that

\[
F^{(i)}_1(-1) \equiv _2F_1 \left( \begin{array}{c} -M, -n \\ 1 + n - M \\ -1 \end{array} \right) \pmod{p\mathbb{Z}_p},
\]

(3.10)
where $M = \mu_{2s}^{(i)}$ and $n = \mu_{\gamma}^{(i)}$. By equating coefficients of $T^M$ in the expansions of $(1 - T)^n (1 + T)^n (1 - T^2)^n$ one obtains, for $0 \leq M \leq n$,

$$
\sum_{k=0}^{M} (-1)^k \binom{n}{k} \binom{M-k}{n} = \begin{cases} (-1)^m \binom{n}{m} & \text{if } M = 2m, \\ 0 & \text{if } M \text{ is odd.} \end{cases} \tag{3.11}
$$

Applying the identity

$$
\binom{n}{M-k} = \binom{n}{M} \binom{M}{k} \binom{n-M+k}{k}^{-1} \tag{3.12}
$$

in (3.11) shows that

$$
\sum_{k=0}^{M} (-1)^k \binom{M}{k} \binom{n}{k} \binom{n-M+k}{k} = (-1)^m \binom{n}{m} \binom{n}{2m} \tag{3.13}
$$

if $M = 2m \leq n$, whereas the sum in (3.13) is zero if $M$ is odd. But since this sum is precisely $\binom{n}{m}$, we see from (3.10) that $-1 \notin \mathcal{D}$ for our $F(X)$ unless $M = \mu_{2s}^{(i)}$ is an even integer for all $i \geq 0$.

Since $\mu_{2s}^{(i)}$ must be even for all $i$, we have $\mu_{2s}^{(i)} = 2\mu_{s}^{(i)}$ for all $i$. From (3.9) we see that $2\mu_{s}^{(i)} \leq \mu_{\gamma}^{(i)}$ for all $i$, giving (i). Then substituting the equality in (3.9) into (3.8) yields $2\mu_{s}^{(i)} - 2\mu_{s}^{(i)} < p - 1$, giving (ii).

Now suppose (i) and (ii) hold. From (i) we see that $\mu_{2s}^{(i)} = 2\mu_{s}^{(i)}$, $\mu_{s}^{(i)} + \mu_{\gamma - s}^{(i)} = \mu_{\gamma}^{(i)}$, and $\mu_{2s}^{(i)} + \mu_{\gamma - 2s}^{(i)} = \mu_{\gamma}^{(i)}$ for all $i$. If $y \neq 1$, then from (ii) we have

$$
\mu_{s}^{(i)} = (p-1) - \mu_{\gamma - 2s}^{(i)} = (p-1) + \mu_{2s}^{(i)} - \mu_{s}^{(i)} = (p-1) + 2(\mu_{s}^{(i)} - \mu_{s}^{(i)}) + \mu_{\gamma}^{(i)} > \mu_{\gamma}^{(i)} \geq 2\mu_{s}^{(i)} \geq 0, \tag{3.14}
$$

and therefore the hypotheses (C1), (C2) are satisfied for $F(X)$. By comparison with Theorem 2.2, we see that for $m = \mu_{s}^{(i)}$, $n = \mu_{\gamma}^{(i)}$, $M = 2m$, the binomial coefficients on the right side of (3.13) are $p$-adic units, showing via (3.10) that $-1 \notin \mathcal{D}$. Thus $(x, \beta) \in T$, proving the first statement of the theorem.

The basic idea of Koblitz [7] shows that $(x, \beta) \mapsto \binom{2x}{\beta, 1+2x-\beta; -1}$ is continuous on the set $T$. For $r > 0$ and $(x, \beta) \in T$ set $m_r = \sum_{i=0}^{r-1} \mu_{s}^{(i)} p^i$ and $n_r = \sum_{i=0}^{r-1} \mu_{\gamma}^{(i)} p^i$. Since

$$
(\mu_{2m_r}^{(i)}, \mu_{m_r}^{(i)}, \mu_{s}^{(i)} + \mu_{r-s}^{(i)}) = \begin{cases} (\mu_{2s}^{(i)}, \mu_{\gamma}^{(i)}, \mu_{\gamma}^{(i)}), & \text{if } 0 \leq i \leq r, \\ (0, 0, 1), & \text{if } i \geq r, \end{cases} \tag{3.15}
$$
we see that \((-2m_r, -n_r) \in T\) as well, because the pair \((0, 0)\) clearly satisfies conditions (i), (ii) above. We note that \((1 + 2x - \beta) = 1 + 2x' - \beta'\) and \((1 + n + 2m_r \gamma') = 1 + n' + 2m_r'\) for all \(r\). Therefore if \(s > 0\), it follows from (3.2) that there exists \(R \geq s + 1\) such that

\[
\tilde{N}_1\left(\frac{2x, \beta}{\gamma}; -1\right) \equiv \frac{F_{s+1}\left(\frac{2x, \beta}{\gamma}; -1\right)}{F_{s}\left(\frac{2x', \beta'}{\gamma'}; (-1)^p\right)}
\]

\[
\equiv \frac{F_{s+1}\left(-2m_r, -n_r; -1\right)}{F_{s}\left(-2m_r', -n_r'; (-1)^p\right)}
\equiv \frac{F\left(-2m_r, -n_r; -1\right)}{F\left(-2m_r', -n_r'; (-1)^p\right)} \quad (\text{mod } p^{s+1}\mathbb{Z}_p) \quad (3.16)
\]

for all \(r \geq R\), the second congruence holding because the \(F_s((-1)\) are rational functions of their parameters. We then use (3.13) with \(M = 2m_r\), \(n = n_r\), and with \(M = 2m_r'\), \(n = n_r'\), and Lemma 2.1, to obtain

\[
\tilde{N}_1\left(\frac{2x, \beta}{\gamma}; -1\right) = \lim_{r \to \infty} \frac{\tilde{F}_1\left(-2m_r, -n_r; -1\right)}{\tilde{F}_1\left(-2m_r', -n_r'; (-1)^p\right)}
\]

\[
= \lim_{r \to \infty} \frac{(-1)^{m_r-n_r}}{\binom{n_r}{m_r}} \frac{\binom{n_r'}{m_r'}}{\binom{n_r}{2m_r}}
\]

\[
= \lim_{r \to \infty} (-1)^{m_r-n_r} \frac{\Gamma_p(-m_r) \Gamma_p(m_r, -n_r)}{\Gamma_p(-2m_r) \Gamma_p(2m_r, -n_r)}
\]

\[
= (-1)^{m_r} \frac{\Gamma_p(\beta) \Gamma_p(\beta - 2x)}{\Gamma_p(2x) \Gamma_p(\beta - 2x)}, \quad (3.17)
\]

as desired.
The Jacobi-sum values of associated $\tilde{\mathfrak{S}}_1^{(f)}$ for $\alpha, \beta$ lying also in $(1/(q-1)) \mathbb{Z} \cap [0, 1]$ may be obtained from (3.17) with the aid of (3.6), (2.5), and (2.8), or directly as follows: Noting that $m_{\alpha} = (q' - 1) \alpha$, $n_{\beta} = (q' - 1) \beta$, $(-2m_{\alpha})^{(f)} = -2m_{(\alpha-1)f}$, $(-n_{\beta})^{(f)} = -n_{(\beta-1)f}$, and $(1 + n_{\beta} - 2m_{\alpha})^{(f)} = 1 + n_{(\beta-1)f} - 2m_{(\alpha-1)f}$, we substitute (3.16) in (3.6), yielding

$$
\tilde{\mathfrak{S}}_1^{(f)} \left( 2\alpha, \beta; -1 \right) = \lim_{r \to \infty} \frac{\tilde{\mathfrak{S}}_1 \left( -2m_{\alpha}, -n_{\beta}; -1 \right)}{\tilde{\mathfrak{S}}_1 \left( 1 + n_{(\beta-1)f} - 2m_{(\alpha-1)f}; -1 \right)}
$$

$$
= \lim_{r \to \infty} \left( -1 \right)^{n_{\beta} - m_{\alpha}} \frac{\binom{n_{\beta}}{m_{\alpha}} \binom{n_{\beta}}{2m_{\alpha-1f}}}{\binom{n_{\beta}}{m_{\alpha-1f}} \binom{n_{\beta}}{2m_{\alpha}}}
$$

$$
= \left( -1 \right)^{\alpha} \frac{\text{J}(\omega_f, \omega_f - 2\alpha; \omega_f - \beta)}{\text{J}(\omega_f, \omega_f - 2\alpha)}
$$

via (3.13) with $M = 2m_{\alpha}$, $n = n_{\beta}$, and with $M = 2m_{(\beta-1)f}$, $n = n_{(\beta-1)f}$, and Corollary 2.3. This completes the proof.

We remark that, by (3.13) and Corollary 2.3, the limit of hypergeometric functions given in (3.18) is indeed correct for any $\alpha, \beta \in (1/(q-1)) \mathbb{Z} \cap [0, 1]$ such that $2\alpha \leq \beta$. However when $(\alpha, \beta) \notin T$ the symbol $\tilde{\mathfrak{S}}_1^{(f)}$ is not justified for this limit, as it need not be the specialization to $-1$ of a uniform limit on that residue class.

We now give a generalization of Dixon's theorem, which gives the value of a well-poised $\tilde{\mathfrak{S}}_2(1)$. We give a proof along somewhat different lines than the one given in [15] for elements of $(1/(p-1)) \mathbb{Z}$.

**Theorem 3.2.** Let $T$ denote the set of all $(\alpha, \beta, \gamma) \in \mathbb{Z}_+^3$ such that $1 \in T$ and both (C1), (C2) are satisfied for the series $F(X) = \chi_2(\alpha, \beta, \gamma; 1 + 2\alpha - \beta, 1 + 2\alpha - \gamma, X)$. Then $(\alpha, \beta, \gamma) \in T$ if and only if

(i) $2\mu_{\beta}(i) \leq \mu_{\alpha}(i)$, $\mu_{\gamma}(i)$ and $\mu_{\beta}(i) + \mu_{\gamma}(i) - \mu_{\alpha}(i) \leq p - 1$ for all $i \geq 0$;

(ii) If $2\alpha, \beta, \gamma$ are not all equal then $\mu_{\beta}(i) + \mu_{\gamma}(i) - 2\mu_{\alpha}(i) < p - 1$ for all $i \geq 0$.

Furthermore, if $(\alpha, \beta, \gamma) \in T$ then

$$
\tilde{\mathfrak{S}}_2 \left( 2\alpha, \beta, \gamma; 1 + 2\alpha - \beta, 1 + 2\alpha - \gamma \right) = \left( -1 \right)^{m_{\alpha}} \frac{\Gamma_p(\alpha) \Gamma_p(\beta - \alpha) \Gamma_p(\gamma - \alpha) \Gamma_p(\beta + \gamma - 2\alpha)}{\Gamma_p(2\alpha) \Gamma_p(\beta - 2\alpha) \Gamma_p(\gamma - 2\alpha) \Gamma_p(\beta + \gamma - 2\alpha)}
$$
and if in addition \((q - 1) \alpha, (q - 1) \beta, (q - 1) \gamma = (a, b, c) \in \{0, 1, \ldots, q - 1\}\) \(^3\), then

\[
\psi_{2s}^{(i)} \left( \frac{2x, \beta, \gamma}{1 + 2x - \beta, 1 + 2x - \gamma} ; 1 \right) = (-1)^{a} J(\omega_{\gamma-a}^{\alpha}, \omega_{\gamma-a}^{\beta}, \omega_{\gamma-a}^{\gamma}) J(\omega_{\beta-a}^{\alpha}, \omega_{\beta-a}^{\gamma}) J(\omega_{\alpha-a}^{\beta}, \omega_{\alpha-a}^{\gamma}).
\]

**Proof.** Suppose \((\alpha, \beta, \gamma) \in T\). Set \(s = 1 + 2x - \beta\) and \(t = 1 + 2x - \gamma\). If \(s = t = 1\) then \(2x = \beta = \gamma\), so \(\mu_{\alpha-2s}^{(i)} = \mu_{\beta-2s}^{(i)} = \mu_{\gamma-2s}^{(i)}\) for all \(i\). Note that \(\mu_{\beta-2s}^{(i)} = p - 1\) and \(\mu_{\gamma-2s}^{(i)} = p - 1\) for all \(i\). Therefore if \(s, t\) are not both equal to 1, (C2) implies that for all \(i \geq 0\), either \(\mu_{\alpha-2s}^{(i)} > \mu_{\beta-2s}^{(i)}\) or \(\mu_{\gamma-2s}^{(i)} > \mu_{\beta-2s}^{(i)}\), so that for all \(i\),

\[
\mu_{\alpha-2s}^{(i)} + \mu_{\beta-2s}^{(i)} < p - 1 \quad \text{or} \quad \mu_{\alpha-2s}^{(i)} + \mu_{\gamma-2s}^{(i)} < p - 1. \tag{3.19}
\]

Now if for some \(i\), say the former half of (3.19) holds, then in fact

\[
\mu_{\alpha-2s}^{(i)} + \mu_{\beta-2s}^{(i)} = \mu_{\beta-2s}^{(i)} + \mu_{\gamma-2s}^{(i)} + 1 = p - 1, \tag{3.20}
\]

so in particular \(\mu_{\alpha-2s}^{(i)} \leq \mu_{\beta-2s}^{(i)}\) for such \(i\). But then (C2) requires that both \(\mu_{\alpha-2s}^{(i)}, \mu_{\gamma-2s}^{(i)} > \mu_{\beta-2s}^{(i)}\), so both inequalities in (3.19) must hold; thus in fact for any \(i\) we have

\[
\mu_{\alpha-2s}^{(i)} + \mu_{\beta-2s}^{(i)} = \mu_{\beta-2s}^{(i)}, \quad \mu_{\beta-2s}^{(i)} + \mu_{\gamma-2s}^{(i)} = \mu_{\gamma-2s}^{(i)}. \tag{3.21}
\]

so in any event we have \(\mu_{\alpha-2s}^{(i)} \leq \mu_{\beta-2s}^{(i)}, \mu_{\gamma-2s}^{(i)}\) for all \(i\).

Since we assume that \(F(X)\) satisfies (C1), (C2), we note that

\[
F_{1}^{(i)}(1) \equiv \psi_{2s}^{(i)} \left( \frac{-S, -m, -n}{1 + m - S, 1 + n - S} ; 1 \right) \pmod{pZ, \rho}. \tag{3.22}
\]

where \(S = \mu_{\alpha-2s}, m = \mu_{\beta-2s}\) and \(n = \mu_{\gamma-2s}\). A terminating from of the classical Dixon's theorem (cf. [10, eq. (III.9)]) states that for \(n, m \in \mathbb{Z}^+\), we have

\[
\psi_{2s}^{(i)} \left( \frac{-2s, -m, -n}{1 + m - 2s, 1 + n - 2s} ; 1 \right) = \frac{(1 - 2s)_n (1 + m - s)_n (1 + m - 2s)_n}{(1 - s)_n (1 + m - 2s)_n} \tag{3.23}
\]

if \(2s\) is not a positive integer. We invoke the identity \((1 + x)_n = \Gamma(1 + x + n)/\Gamma(1 + x)\) to express each factor on the right side of (3.23) in terms of the classical gamma function. Using the classical functional
equations $\Gamma(1 + z) = z\Gamma(z)$ and $\Gamma(z)\Gamma(1 - z) = \pi\csc\pi z$ (here $\pi$ has its more usual meaning) we obtain

\[
\frac{\Gamma(1 - s)}{\Gamma(1 - 2s)} = \frac{\Gamma(1 + 2s)}{\Gamma(1 + s)} \cos \pi s, \tag{3.24}
\]

which shows that the value of the $\zeta F_2(1)$ in (3.23) is equal to

\[
\cos \pi s \frac{\Gamma(1 + 2s)\Gamma(1 + n - 2s)\Gamma(1 + m + n - s)\Gamma(1 + m - 2s)}{\Gamma(1 + s)\Gamma(1 + n - s)\Gamma(1 + m - s)\Gamma(1 + n + m - 2s)}, \tag{3.25}
\]

As functions of $s$, both the $\zeta F_2(1)$ in (3.23) and the expression (3.25) are continuous at $s$ when $2s$ is a positive integer such that $2s \leqslant m, n$, so their equality holds also in this case. Note that if $2s$ is a positive odd integer and $2s \leqslant m, n$ then $\cos \pi s = 0$ and the expression (3.25) is therefore zero. Since $\cos \pi N = (-1)^N$ and $\Gamma(1 + N) = N!$ for $N \in \mathbb{Z}^+$, we obtain the identity

\[
\zeta F_2\left(\frac{-2s, -m, n}{1 + m - 2s, 1 + n - 2s}; 1\right) = (-1)^s \frac{(m + n - s)_s}{(n - s)_s} \frac{(m)_s}{(2s)_s}, \tag{3.26}
\]

valid for integers positive integers $s, n, m$ such that $2s \leqslant m, n$, whereas this value is zero if $2s$ is a positive odd integer and $2s \leqslant m, n$. Comparing this result with (3.22), we see that $1 \notin \mathcal{D}$ for our $F(X)$ unless $S = \mu_{\mu}^{(i)}$ is an even integer for all $i \geq 0$. Therefore $\mu_{\mu}^{(i)} = 2\mu_{\mu}^{(i)}$ for all $i$, giving the first half of (i).

Since $2\mu_{\mu}^{(i)} \leq \mu_{\mu}^{(i)}$, $\mu_{\mu}^{(i)}$ for all $i$, we see that the binomial coefficients $\binom{m}{s}$, $\binom{n}{s}$, and $\binom{n - s}{s}$ all lie in $\mathbb{Z}_p^*$, where $s = \mu_{\mu}^{(i)}$, $m = \mu_{\mu}^{(i)}$ and $n = \mu_{\mu}^{(i)}$. Comparing (3.22) and (3.26) (with $S = 2s$) shows that the ensure $1 \in \mathcal{D}$ we need $(m + n - s) \in \mathbb{Z}_p^*$, which requires $m + n - s \leq p - 1$, giving the second half of condition (i).

Finally, if $2x, \beta, \gamma$ are not all equal, then from (C2) we know that for all $i$, either $\mu_{\mu}^{(i)} > \mu_{\mu}^{(i)}$ or $\mu_{\mu}^{(i)} > \mu_{\mu}^{(i)}$. Thus for all $i$,

\[
\mu_{\mu}^{(i)} - \mu_{\mu}^{(i)} \leq 2x \leq p - 1 \quad \text{or} \quad \mu_{\mu}^{(i)} + \mu_{\mu}^{(i)} \leq 2x < p - 1, \tag{3.27}
\]

but in either case (3.21) yields $\mu_{\mu}^{(i)} + \mu_{\mu}^{(i)} - 2\mu_{\mu}^{(i)} < p - 1$, which is condition (ii).

Now suppose (i) and (ii) hold. It follows easily that $\mu_{\mu}^{(i)} = 2\mu_{\mu}^{(i)}$ and (3.21) holds for all $i$. Furthermore, if $2x, \beta, \gamma$ are not all equal, then $\mu_{\mu}^{(i)} + 2x - \beta > \mu_{\mu}^{(i)} + 2x - \gamma > \mu_{\mu}^{(i)} + 2x - \beta$, so $F(X)$ satisfies (C1) and (C2). From (i) we see that for $s = \mu_{\mu}^{(i)}$, $m = \mu_{\mu}^{(i)}$ and $n = \mu_{\mu}^{(i)}$, each binomial coefficient on the right side of (3.26) is a $p$-adic unit. Then (3.22) shows that $1 \in \mathcal{D}$, proving the first statement of the theorem.
Now let \((\alpha, \beta, \gamma) \in T\), and for \(r > 0\) set \(s = \sum_{i=0}^{r-1} \mu^{(i)} \rho^i\), \(m_r = \sum_{i=0}^{r-1} \mu_r^{(i)} \rho^i\), and \(n_r = \sum_{i=0}^{r-1} \mu_r^{(i)} \rho^i\). Since

\[
(\mu^{(i)}_{-2s_r}, \mu^{(i)}_{-m_r}, \mu^{(i)}_{-n_r}, \mu^{(i)}_{1+m_r-2s_r}, \mu^{(i)}_{1+n_r-2s_r}) = \begin{cases} 
(\mu^{(i)}_{2s_r}, \mu^{(i)}_{m_r}, \mu^{(i)}_{n_r}, \mu^{(i)}_{1-2s_r}, \mu^{(i)}_{1+2s_r-2r}) & \text{if } 0 \leq i < rf, \\
(0, 0, 0, 1, 1) & \text{if } i = rf,
\end{cases}
\]

we see that \((-2s_r, -m_r, -n_r) \in T\) as well. Therefore, for all \(s > 0\) there exists \(R \geq s + 1\) such that

\[
\tilde{\mu}_2\left(\frac{2\alpha, \beta, \gamma}{1 + 2\alpha - \beta, 1 + 2\alpha - \gamma}; 1\right) \equiv \frac{F_{s+1}\left(\frac{2\alpha, \beta, \gamma}{1 + 2\alpha - \beta, 1 + 2\alpha - \gamma}; 1\right)}{F_r\left(\frac{1 + 2\alpha' - \beta', 1 + 2\alpha' - \gamma'}{1 + 2\alpha' - \beta', 1 + 2\alpha' - \gamma'}; 1\right)}
\]

\[
= \frac{F_{s+1}\left(\frac{-2s_r, -m_r, -n_r}{1 + m_r - 2s_r, 1 + n_r - 2s_r}; 1\right)}{F_r\left(\frac{-2s_r', -m_r', -n_r'}{1 + m_r' - 2s_r', 1 + n_r' - 2s_r'}; 1\right)}
\]

\[
= \frac{F\left(\frac{-2s_r, -m_r, -n_r}{1 + m_r - 2s_r, 1 + n_r - 2s_r}; 1\right)}{F\left(\frac{-2s_r', -m_r', -n_r'}{1 + m_r' - 2s_r', 1 + n_r' - 2s_r'}; 1\right)}
\]

(mod \(\rho^{s+1}\mathbb{Z}^r\))

for all \(r \geq R\). Then from (3.26) and Lemma 2.1 we have

\[
\tilde{\mu}_2\left(\frac{2\alpha, \beta, \gamma}{\delta, \epsilon}; 1\right) = \lim_{r \to \infty} \frac{3F_2\left(\frac{-2s_r, -m_r, -n_r}{1 + m_r - 2s_r, 1 + n_r - 2s_r}; 1\right)}{3F_2\left(\frac{-2s_r', -m_r', -n_r'}{1 + m_r' - 2s_r', 1 + n_r' - 2s_r'}; 1\right)}
\]

\[
= \lim_{r \to \infty} \left(\frac{m_r + n_r - s_r}{m_r'} \frac{n_r' - s_r'}{s_r'} \frac{m_r'}{2s_r'} \frac{s_r'}{s_r'} \frac{2s_r'}{2s_r}\right)
\]

\[
= (-1)^{n'} \frac{\Gamma_r(\alpha) \Gamma_r(\gamma - \alpha) \Gamma_r(\beta - \alpha) \Gamma_r(\beta + \gamma - 2\alpha)}{\Gamma_r(2\alpha) \Gamma_r(\gamma - 2\alpha) \Gamma_r(\beta - 2\alpha) \Gamma_r(\beta + \gamma - \alpha)}
\]

(3.30)

giving the second statement of the theorem.
As in (3.18), the Jacobi-sum values for \( \mathcal{K}_{2}^{(f)}(\alpha, \beta, \gamma; \delta, \epsilon, 1) \) for \( \alpha, \beta, \gamma \) lying also in \((1/(q - 1)) \mathbb{Z} \cap [0, 1]\) may be obtained by using (3.29), (3.26), (3.6), and Corollary 2.3 to evaluate

\[
\lim_{r \to \infty} \frac{\binom{-2s_{r}, 1 + m_{r} - 2s_{r} - n_{r}, -n_{r} - 2s_{r} 3F_{2}}{1}}{\binom{1 + m_{r} - 2s_{r} - n_{r} - 2s_{r}, 1 + m_{r} - 2s_{r} - n_{r} - 2s_{r} 3F_{2}}{1}} = \lim_{r \to \infty} \frac{(-1)^{s_{r} - s_{r - 1} f}}{
\binom{m_{r} + n_{r} - s_{r}}{s_{r}} \binom{n_{r - 1} f - s_{r - 1} f}{s_{r - 1} f} \binom{m_{r - 1} f}{2s_{r - 1} f} \binom{m_{r - 1} f}{2s_{r - 1} f} \binom{n_{r} - s_{r}}{s_{r}} \binom{m_{r}}{2s_{r}}}{\binom{m_{r - 1} f + n_{r - 1} f - s_{r - 1} f}{s_{r - 1} f} \binom{n_{r - 1} f - s_{r - 1} f}{s_{r - 1} f} \binom{m_{r - 1} f}{2s_{r - 1} f} \binom{m_{r}}{2s_{r}}}

= (-1)^{s_{r}} \frac{J(\omega_{-a}, \omega_{2a - b - c})}{J(\omega_{-a}, \omega_{2a - c})} \frac{J(\omega_{-a}, \omega_{2b - c})}{J(\omega_{-a}, \omega_{2b - c})} \times \frac{J(\omega_{-a}, \omega_{2a - c})}{J(\omega_{-a}, \omega_{2b - c})} \times \frac{J(\omega_{-a}, \omega_{2a - c})}{J(\omega_{-a}, \omega_{2b - c})} \times \frac{J(\omega_{-a}, \omega_{2a - c})}{J(\omega_{-a}, \omega_{2b - c})}
\tag{3.31}
\]

Again, the limit of hypergeometric functions given in (3.31) is correct for any \( \alpha, \beta, \gamma \in (1/(q - 1)) \mathbb{Z} \cap [0, 1] \) such that \( 2 \alpha \leq \beta, \gamma \) but the symbol \( \mathcal{K}_{2}^{(f)} \) is not justified for this limit unless \( (\alpha, \beta, \gamma) \in T \).

4. Applications

In ([4, eq.(6.29)]) Dwork showed that for the Legendre family of elliptic curves

\[ E_{\lambda}: y^{2} = x(x - 1)(x - \lambda) \quad (\lambda \neq 0, 1), \tag{4.1} \]

if \( \lambda^{q} = \lambda \) and the reduction of \( E_{\lambda} \) to \( \mathbb{F}_{q} \) is non-supersingular then the reciprocal unit root \( \alpha(\lambda) \) of the zeta-function of the reduced curve \( E_{\lambda}/\mathbb{F}_{q} \) is given by

\[ \alpha(\lambda) = (-1)^{s_{r}} \binom{3^{1/2}}{1} \mathcal{K}_{2}^{(f)}(\frac{3^{1/2}}{1}, \lambda). \tag{4.2} \]

In [15] we noted that if \( p \equiv 1 \pmod{4} \) and \( q = p \) then \( -1 \in \mathcal{D}, \) and the value

\[ \alpha(1) = (-1)^{p - 1/4} J(\omega_{1}^{1 - p/4}, \omega_{1}^{1 - p/4}) \tag{4.3} \]
follows from (4.2) and our $p$-adic analogue of Kummer's theorem. However, if $p \equiv -1 \pmod{4}$ then the curve $E_{-1}$ has supersingular reduction mod $p$. Here we show that in this case the roots of the zeta function of $E_{-1}$ over $\mathbb{F}_p$ are also given by a limit of hypergeometric functions as in (3.18). The symbol $\mathcal{R}^{2}\left(\frac{1}{2}, \frac{1}{2} ; 1, -1\right)$ is not justified for this limit, however, because it is not the specialization to $-1$ of a uniform limit on that residue class.

This result is obtained from Corollary 2.3 and a theorem of Stienstra [12] as follows: Taking the double cover $U^2 = T_0 T_1^2 - T_1 T_0 = T_0 T_1 (T_0 - T_1) (T_0 + T_1)$ of $\mathbb{P}^1$ as a model of $E_{-1}$ and applying ([12, Theorem 0.1], one obtains the congruences

$$\beta_{m'q'} + a \beta_{m'q'-1} + q \beta_{m'q'-2} \equiv 0 \pmod{pq^{-1}Z}, \quad (4.4)$$

where

$$\beta_n = \begin{cases} (-1)^n \binom{2m}{m} \sum_{k=0}^{2m} \binom{2m}{k} (-1)^k, & \text{if } n = 2m + 1, \\ 0, & \text{otherwise}, \end{cases} \quad (4.5)$$

and where $P_1(T) = 1 + aT + qT^2$ is the numerator of the zeta function of $E_{-1}$ over $\mathbb{F}_q$. When $p \equiv -1 \pmod{4}$ and $q = p^2$, using the first expression $\beta_{4m+1} = (-1)^m \binom{2m}{m}$ and Corollary 2.3 yields

$$\frac{\beta_{q'}}{\beta_{q'-1}} \equiv J(\omega_2^{1-q/4}, \omega_2^{1-q/4}) \pmod{q'Z_p}, \quad (4.6)$$

since this Jacobi sum has $p$-adic ordinal $e = 1$. Therefore the ratios $\beta_{q'}/\beta_{q'-1}$ converge in $Z_p$, and we also find by induction in (4.6) that $\text{ord } \beta_{q'} = r$. Setting $m = 1$ and dividing the congruence (4.4) by $\beta_{q'}$ then yields

$$1 + a \frac{\beta_{q'-1}}{\beta_{q'}} + q \frac{\beta_{q'-2}}{\beta_{q'-1}} \frac{\beta_{q'-1}}{\beta_{q'}} \equiv 0 \pmod{p^{r-1}Z_p}, \quad (4.7)$$

and then letting $r \to \infty$ in (4.7) shows that the ratios $\beta_{q'-1}/\beta_{q'}$ in fact converge to a root of $P_1(T)$. We note that in general congruences such as (4.4) do not imply convergence of these ratios in the supersingular case (ord $a > 0$); in particular when $p \equiv -1 \pmod{4}$ and $q = p$ this is evident from (4.5). However, having some other means (such as (4.6)) of establishing convergence, it is then easy to see that the limit is a root of the associated polynomial.

One may also check directly via character sums that the Jacobi sum in (4.6) is in fact the reciprocal root of the zeta function of $y^2 = x^3 - x$ over $\mathbb{F}_p$; that is, $P(T) = (1 - \alpha T)^2$ with $\alpha = J(\omega_2^{1-q/4}, \omega_2^{1-q/4})$. Indeed, one easily obtains $J(\omega_2^{1-q/4}, \omega_2^{1-q/4}) = -p$ as in (2.17), since the reciprocal
roots of the zeta function over $\mathbb{F}_p$ are $\pm \sqrt{-p}$, the reciprocal roots over $\mathbb{F}_q$ are both $-p$. Using the second expression $\beta_{4m+1} = \sum_{k=0}^{2m} (2m)^2 (-1)^k$ in (4.5) we see that the limit of hypergeometric functions in (3.18) exists when $\alpha = \frac{1}{4}$, $\beta = \frac{1}{4}$, $\gamma = 1$, and $f = 2$; and (4.6), (4.7) show that the limit is indeed a reciprocal root of the zeta function of $E_{1,1}$, although the root is not a unit root.

One should not expect to obtain the supersingular roots over $\mathbb{F}_p$ in this manner, because they have $p$-adic ordinal $\frac{1}{3}$ and thus do not lie in $\mathbb{E}_p$. As noted in ([15, pp. 239, 245]), one may view the Jacobi-sum expression for the limit of hypergeometric functions as arising from the complex multiplication $(x, y) \mapsto (-x, \sqrt{-1} y)$ on $E_{1,1}$ by the fourth roots of unity; this map commutes with the Frobenius $(x, y) \mapsto (x^q, y^q)$ if and only if $q \equiv 1 \pmod{4}$, showing why $f = 2$ is necessary to make this argument when $p \equiv -1 \pmod{4}$; but in view of ([4, eq. (6.28)]) does not explain why the value is a root of the zeta function, because the limit is not the specialization of a uniform limit.

We conclude with an application to the study of the Apéry numbers. In ([15, Corollary 4.2(iii)]) we proved that

$$\beta_{2, p} = (-1)^{(p-1)/2} \frac{3}{\mathbb{F}_2} \left(\begin{array}{ccc} 2, 2, \frac{1}{2} \\ 1, 1 \\ -1 \end{array} \right),$$

where $\beta_{2, p}$ is the reciprocal of the $p$-adic unit root of the polynomial $P_{2, p}(T) = 1 - (4a^2 - 2p) T + p^2 T^2$, whenever $p = a^2 + 2b^2$ with $a, b \in \mathbb{Z}$; this polynomial is the $p$th Hecke polynomial associated to a certain cusp from the weight 3 and level 8. The proof was obtained from formal-group congruences associated to the Apéry sequence

$$d(n) = \sum_{k=0}^{n} \binom{n}{k} \left(\begin{array}{cccc} 3 \\ n-n-n-n \\ 1, 1 \end{array} \right)$$

which were discovered by Stienstra and Beukers [11], and which exhibit $\beta_{2, p}$ as the reciprocal of the $p$-adic unit root of the zeta function of a certain $K3$-surface. Here we express $\beta_{2, p}$ in terms of Jacobi sums over $\mathbb{F}_p$ and over $\mathbb{F}_p^2$, using a classical hypergeometric identity and Theorem 3.1.

The value of the hypergeometric function in (4.8) may be obtained by applying the well-known formula

$$\left. _3F_2 \left( \begin{array}{ccc} 2a, a+b, 2b \\ a+b+\frac{1}{2}, 2a+2b \end{array} ; x \right) = \left. _3F_1 \left( \begin{array}{ccc} a, b \\ a+b+\frac{1}{2} ; x \end{array} \right)^2 \right|$$

of Clausen ([1, p. 185]) with $a = \beta = \frac{1}{4}$. When $p \equiv 1 \pmod{8}$ we have $\frac{1}{4} \equiv 1$, and so $f = 1$ suffices, while if $p \equiv 3 \pmod{8}$ we have $\frac{1}{4} = \frac{1}{4}$ and $\frac{3}{4} = \frac{1}{4}$, so we take $f = 2$ in that case. Since $\frac{1}{4} \equiv 1$ in either case, we have
\[ 3\tilde{\varphi}_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; -1\right) = 3\tilde{\varphi}_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; -1\right) \]
\[ = \tilde{\varphi}_1\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}; -1\right)^2 \]
\[ = J(\omega_1^{1-p/8}, \omega_1^{1-p^2/8})^2. \quad (4.11) \]

When \( p \equiv 1 \pmod{8} \) this yields
\[ \beta_{2, p} = J(\omega_1^{1-p/8}, \omega_1^{1-p^2/8})^2, \quad (4.12) \]
while for \( p \equiv 3 \pmod{8} \) we get
\[ \beta_{2, p} = \varepsilon \cdot J(\omega_2^{1-p/8}, \omega_2^{1-p^2/8}) \quad (4.13) \]
where \( \varepsilon = \pm 1 \).

The value in (4.12) is readily seen to be consistent with the result of Berndt and Evans ([2, Corollary 3.13]). In like fashion, the value \( \varepsilon = -1 \) in (4.13) may be determined by comparison with ([3, Theorem 4.6]).

One may also determine a Jacobi sum formula for the \( p \)-adic integer \( \beta_{3, p} \) appearing in ([15, Corollary 4.2(iv)]) indirectly via (4.10) and the theory of elliptic curves. However, this question appears to remain open for the \( p \)-adic integer \( \alpha_p \) in ([15, Corollary 4.2(vi)]).

REFERENCES


