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Tensor powers of the Carlitz module and zeta values

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Notation

$A =: \mathbf{F}_q[T]$ (T: a variable, q: a power of a prime number p).

$A_+ =:$ the set of monic elements of A.

$v =:$ a monic prime element of A.

$K =:$ the fraction field of A.

$K_\infty =: \mathbf{F}_q\left(\left(\frac{1}{T}\right)\right)$ (the completion of K at the place $T \to \infty$).

$\overline{K}_\infty =:$ an algebraic closure of K_∞.

$K^{\mathrm{sep}} =:$ the separable algebraic closure of K in $\overline{K}_\infty$.

$\mathbf{G}_a =:$ the additive group over A.

Introduction

The power sum

$$\zeta(n) =: \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty \qquad (n\text{: positive integer})$$

was first considered by Carlitz [C1]. In many respects, e.g., the existence of an Euler product representation, $\zeta(n)$ is analogous to the corresponding value of the Riemann zeta function. A review of pertinent arithmetic properties of $\zeta(n)$ and its v-adic analogue can be found in Section 3 below.

Our aim is to relate $\zeta(n)$ to a certain A-module-valued functor of A-algebras, namely the n^{th} *tensor power* $C^{\otimes n}$ of the *Carlitz module*. We shall also obtain for each prime v a corresponding result concerning the v-adic analogue of $\zeta(n)$. Using our results, J. Yu [Y2] has proved the transcendence of $\zeta(n)$ over

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K; see Section 3 for further discussion. We focus in this introduction on the functors $C^{\otimes n}$.

The functor $C = C^{\otimes 1}$, the *Carlitz module* ([C1], [H1]), assigns to each A-algebra R the A-module obtained by equipping the additive group of R with the unique $\mathbf{F}_q$-linear action of A such that T acts by the (nonlinear) rule

$$r \mapsto Tr + r^q \colon R \to R.$$

The group underlying C is the additive group $\mathbf{G}_a$. Given any $f \in A_+$, one can show that the f-torsion submodule $C_f(K^{\mathrm{sep}})$ of $C(K^{\mathrm{sep}})$ is a rank one free A/f-module; let

$$\chi_f \colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to (A/f)^{\times}$$

be the associated character. Given any monic prime v such that $(v, f) = 1$, one can show that the character χ_f is unramified at v and for any arithmetic Frobenius element $\sigma_v \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$,

$$\chi_f(\sigma_v) \equiv v \qquad \mathrm{mod}\, f.$$

The *exponential map*

$$\exp\colon \mathrm{Lie}(C)(\overline{K}_\infty) \to C(\overline{K}_\infty)$$

associated to the Carlitz module C is by definition the unique analytic additive A-equivariant map tangent to the identity map of $\mathrm{Lie}(C)(\overline{K}_\infty)$. (Given an A-module valued functor G of A-algebras, $\mathrm{Lie}(G)$ denotes the A-module-valued functor of A-algebras given by the rule

$$\mathrm{Lie}(G)(R) =: \ker\big(G(R[\varepsilon]/(\varepsilon^2)) \to G(R)\big).)$$

Identifying the domain and range of exp with $\overline{K}_\infty$ in the evident fashion, one has

$$\exp(x) = x \prod_{0 \neq a \in A} \left(1 - \frac{x}{\bar{\pi} a}\right) = \sum_{i=0}^{\infty} \frac{x^{q^i}}{(T^{q^i} - T^{q^{i-1}}) \cdots (T^{q^i} - T)},$$

where $\bar{\pi} \in K_\infty^{\mathrm{sep}}$ (well-defined up to a factor in $\mathbf{F}_q^{\times}$) is the fundamental period of the Carlitz module; $\bar{\pi}$ is an analogue of $2\pi i$, the fundamental period of the multiplicative group. Note that for each $f \in A_+$, $\exp(\bar{\pi}/f)$ generates $C_f(K^{\mathrm{sep}})$. These observations are key points of the explicit class field theory for K developed by Hayes [H1], who built upon earlier ideas of Carlitz [C3]. All in all, it is reasonable to regard C as an A-analogue of the Tate motive $\mathbf{Z}(1)$.

The functor $C^{\otimes n}$ would appear to be an analogue of the n^{th} tensor power $\mathbf{Z}(n)$ of the Tate motive; it was this analogy, together with Deligne's results [De] concerning the fundamental group of the projective line minus three points, which suggested to the authors the possibility of a connection between $C^{\otimes n}$ and

$\zeta(n)$. We note key properties of the functor $C^{\otimes n}$ here. See Section 1 below for the definition.

(0.1) The group-valued functor underlying $C^{\otimes n}$ is isomorphic to $\mathbf{G}_a^n$.

The functor $C^{\otimes n}$ is a higher-dimensional generalization of a Drinfeld module. A class of functors including $C^{\otimes n}$ (over base rings that are perfect fields) was studied in [A], building upon the foundation laid by Drinfeld [Dr].

(0.2) The group $C_f^{\otimes n}(K^{\text{sep}})$ of K^{sep}-valued f-torsion points of $C^{\otimes n}$ is a free A/f-module of rank one and the associated character $\text{Gal}(K^{\text{sep}}/K) \to (A/f)^{\times}$ is χ_f^n.

Property (0.2) provides some justification for regarding $C^{\otimes n}$ as the n^{th} tensor power of C.

For the functor $C^{\otimes n}$ one has an analytic theory generalizing that for the Carlitz module:

(0.3) There exists a unique A-equivariant additive analytic map $\exp_n$: $\text{Lie}(C^{\otimes n})(\overline{K}_\infty) \to C^{\otimes n}(\overline{K}_\infty)$ tangent to the identity map of $\text{Lie}(C^{\otimes n})(\overline{K}_\infty)$.

(0.4) $\Lambda_n =: \ker(\exp_n) \subseteq \text{Lie}(C^{\otimes n})(\overline{K}_\infty)$ is a discrete A-submodule free of rank one.

See Section 2 below for details.

An important point is that for $n > 1$, the derivative action of T on $\text{Lie}(C^{\otimes n})$ *differs from* multiplication by T by a nonzero nilpotent endomorphism consisting of a single Jordan block. Hence the largest quotient of $\text{Lie}(C^{\otimes n})$ on which the derivative and multiplication actions of A coincide is isomorphic to $\mathbf{G}_a$; let ℓ_n: $\text{Lie}(C^{\otimes n}) \to \mathbf{G}_a$ induce an isomorphism of this largest quotient with $\mathbf{G}_a$. The coordinate ℓ_n is unique up to a factor in $\mathbf{F}_q^{\times}$ and has a special role to play; e.g., we show (Cor. 2.5.8) that

(0.5) $$\ell_n(\Lambda_n) = \bar{\pi}^n A.$$

Property (0.5) is another reason for regarding $C^{\otimes n}$ as the n^{th} tensor power of C.

At last we can state our main result (Theorem 3.8.3):

(0.6) There exist $Z_n \in C^{\otimes n}(A)$ and $z_n \in \text{Lie}(C^{\otimes n})(K_\infty)$ such that

$$\exp_n(z_n) = Z_n, \qquad \ell_n(z_n) = \Gamma_n \zeta(n).$$

Here $\Gamma_n \in A_+$ is a certain polynomial defined by Carlitz [C2] analogous to $(n-1)! = \Gamma(n)$. We give explicit constructions for the points Z_n and z_n. See

Section 3 for details. Concerning the explicit point Z_n we prove (Cor. 3.8.4):

(0.7) For n not divisible by $q-1$, Z_n is not an A-torsion point of $C^{\otimes n}$.

We prove as well a v-adic analogue of (0.6), relating the "v-adic logarithm" of the "v-twist" of Z_n to the v-adic analogue of $\zeta(n)$.

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1. Definitions and first properties

Notation (continued)

$\# =:$ "cardinality of".

$f =:$ an element of A_+.

$\deg(a) =:$ the degree in T of $a \in A$.

$\tau =: (x \mapsto x^q) \in \mathrm{End}(\mathbf{G}_a)$ (the q^{th} power endomorphism).

1.1. The *Carlitz action*

$$a \mapsto [a] \colon A \to \mathrm{End}(\mathbf{G}_a)$$

of A upon $\mathbf{G}_a$ is defined to be the unique $\mathbf{F}_q$-linear ring homomorphism such that

$$[T] =: (x \mapsto Tx + x^q) \in \mathrm{End}(\mathbf{G}_a).$$

Then, e.g.,

$$[T^2 + T] = \left(x \mapsto (T^2+T)x + (T^q + T + 1)x^q + x^{q^2}\right) \in \mathrm{End}(\mathbf{G}_a).$$

The subring of $\mathrm{End}(\mathbf{G}_a)$ consisting of the $\mathbf{F}_q$-linear endomorphisms may be identified with the ring of polynomials in τ with coefficients in A equipped with the "twisted" multiplication law

$$\left(\sum_i a_i \tau^i\right)\left(\sum_j b_j \tau^j\right) = \sum_{i,j} a_i b_j^{q^i} \tau^{i+j}.$$

In particular,

$$[T] = T + \tau.$$

The *Carlitz module* C is a copy of $\mathbf{G}_a$ equipped with the Carlitz action of A. Given any A-algebra R, let $C(R)$ denote the ring R equipped with A-module structure via the Carlitz action. We shall briefly review some of the arithmetic properties which motivate the study of the Carlitz module before defining "higher" Carlitz modules.

1.2. The key property of the Carlitz module is:

PROPOSITION 1.2.1 [C3, p. 178]. *The endomorphisms*

$$[v],\ \tau^{\deg(v)} \in \mathrm{End}(\mathbf{G}_a)$$

are congruent modulo v.

Proof. Set $d =: \deg(v)$. Since $\tau = [T] - T$, it follows by induction that

$$\tau^d = \tau^{d-1}([T] - T) = \tau^{d-1}[T] - T^{q^{d-1}}\tau^{d-1} = \sum_{i=0}^{d} Q_{di}[T^i]$$

where the coefficients $Q_{di} \in A$ are defined by

$$\sum_{i=0}^{d-1} (t - T^{q^i}) = \sum_{i=0}^{d} Q_{di}t^i \qquad (t\text{: a variable independent of } T).$$

Call the left-hand side above $Q_d(t)$, a polynomial in t with coefficients in A. Then clearly

(1.2.2) $$Q_d(t) \equiv v(t) \mod vA[t],$$

where $v(t)$ denotes the polynomial in t obtained by substituting t for T in v. Note that we made use of the assumption that v is monic to deduce the congruence above. The desired congruence now follows. q.e.d.

Remark. The preceding proposition is the analogue of the fact that the endomorphism $x \mapsto x^p\colon \mathbf{G}_m \to \mathbf{G}_m$ of the multiplicative group reduces modulo p to the Frobenius endomorphism of $\mathbf{G}_m/\mathbf{F}_p$.

1.3. Put

$$C_f =: \ker([f]\colon \mathbf{G}_a \to \mathbf{G}_a),$$

the f-torsion submodule of the Carlitz module C. Since

$$[f] = \tau^{\deg(f)} + \text{terms of lower degree in } \tau,$$

C_f is a finite flat group scheme over A. Since

$$[f] = f + \text{terms of higher degree in } \tau,$$

C_f is étale over $A[f^{-1}]$. In particular

(1.3.1) $$\#C_f(K^{\mathrm{sep}}) = q^{\deg(f)}.$$

Since (1.3.1) holds for all monic divisors of f, $C_f(K^{\text{sep}})$ must be an A-module isomorphic to A/f.

PROPOSITION 1.3.2 [H1, p. 83]. *Let $\sigma_v \in \text{Gal}(K^{\text{sep}}/K)$ be an arithmetic Frobenius element at v and assume that v does not divide f. Then for all $c \in C_f(K^{\text{sep}})$,*

$$\sigma_v c = [v]c.$$

Consequently the representation

$$\text{Gal}(K^{\text{sep}}/K) \to \text{Aut}_A\big(C_f(K^{\text{sep}})\big) = (A/f)^\times$$

is surjective.

Proof. This follows directly from Proposition 1.2.1. q.e.d.

Remark. Therefore the torsion points of the Carlitz module C in K^{sep} generate an abelian extension of K, just as the torsion points of $\mathbf{G}_m$ in $\overline{\mathbf{Q}}$, the roots of unity, generate an abelian extension of $\mathbf{Q}$. (Of course, in the latter case, the extension in question is the maximal abelian extension, whereas this is not true in the former case.) It was Carlitz's insight [C3] that C could be the basis of explicit class field theory for K. This explicit class field theory was subsequently developed by Hayes [H1], and later vastly generalized by Hayes to the case of positive genus in [H2].

1.4. Let n be a positive integer. Given any ring R, R^n denotes the module consisting of all *column vectors* of length n with entries in R; we denote by $\mathbf{G}_a^n$ the functor of A-algebras $R \mapsto R^n$. Given a matrix M (e.g., a column vector of length n) and nonnegative integer k, let $M^{(k)}$ denote the matrix obtained by raising all the entries of M to the $(q^k)^{\text{th}}$ power. The n^{th} *higher Carlitz action* of A upon $\mathbf{G}_a^n$

$$(a \mapsto [a]_n): A \to \text{End}(\mathbf{G}_a^n)$$

is defined to be the unique $\mathbf{F}_q$-linear ring homomorphism from A to the ring of endomorphisms of $\mathbf{G}_a^n$ such that

$$[T]_n =: \big(x \mapsto Tx + Nx + Ex^{(1)}\big): \mathbf{G}_a^n \to \mathbf{G}_a^n,$$

where

$$N = N_n =: \begin{pmatrix} 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 1 \\ 0 & 0 & \cdot & 0 \end{pmatrix}$$

is the nilpotent n-by-n matrix with 1's along the superdiagonal and 0's elsewhere, and

$$E = E_n =: \begin{pmatrix} 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & 0 \end{pmatrix}$$

is the n-by-n elementary matrix with 1 in the lower left-hand corner and 0's elsewhere. For example

$$[T]_3 = \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} Tx_1 + x_2 \\ Tx_2 + x_3 \\ Tx_3 + x_1^q \end{pmatrix} \right) \in \operatorname{End}(\mathbf{G}_a^3).$$

The n^{th} *higher Carlitz module* $C^{\otimes n}$ is a copy of $\mathbf{G}_a^n$ equipped with the n^{th} higher Carlitz action of A. Given any A-algebra R, $C^{\otimes n}(R)$ denotes the group R^n upon which A operates by the n^{th} higher Carlitz action. Let τ denote the endomorphism $x \mapsto x^{(1)}$ of $\mathbf{G}_a^n$. In evident fashion, we may identify $\operatorname{End}(\mathbf{G}_a^n)$ with the ring of polynomials in τ with coefficients in the ring of n-by-n matrices with entries in A, equipped with the "twisted" multiplication law

$$\left(\sum_i P_i \tau^i\right)\left(\sum_j Q_j \tau^j\right) = \sum_{i,j} P_i Q_j^{(i)} \tau^{i+j}. \tag{1.4.1}$$

Under this identification, we have simply

$$[T]_n = T + N + E\tau.$$

1.5. The A-module-valued functor $C^{\otimes n}$ of A-algebras admits another presentation which is convenient for our purposes. Given any A-algebra R, set

$$W_n(R) =: \left\{ w \in R((t^{-1}))/R[t] \,\middle|\, w^{(1)} \equiv (t-T)^n w \bmod R[t] \right\},$$

where here and elsewhere throughout the paper we denote by $R((t^{-1}))$ the ring of Laurent series

$$\sum_{i \in \mathbf{Z}} r_i t^i \qquad (r_i \in R,\ r_i = 0 \text{ for } i \gg 0)$$

and write

$$\left(\sum r_i t^i\right)^{(k)} =: \sum r_i^{q^k} t^i \qquad (k = 0, 1, 2, \ldots).$$

Let e_i: $\mathbf{G}_a^n \to \mathbf{G}_a$ ($i = 1, \ldots, n$) denote the i^{th} coordinate projection.

PROPOSITION 1.5.1. *The map*

$$(*)_n =: c \mapsto -\sum_{i=1}^{\infty}\left(e_1[T^{i-1}]_n c\right)t^{-i}\colon C^{\otimes n}(R) \to R((t^{-1}))/R[t]$$

is injective. The image of $(*)_n$ *is* $W_n(R)$.

Proof. We have

$$e_{i+1} = e_i[T]_n - Te_i \qquad (i = 1,\dots,n-1),$$
$$\tau e_1 = e_n[T]_n - Te_n.$$

Therefore, by an evident induction,

$$(1.5.2) \qquad e_k = \sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}T^i e_1[T^{k-1-i}]_n \qquad (k = 1,\dots,n),$$

$$(1.5.3) \qquad \tau e_1 = \sum_{i=0}^{n}(-1)^i\binom{n}{i}T^i e_1[T^{n-i}]_n.$$

In particular, if $w \in W_n(R)$ is the image of $c \in C^{\otimes n}(R)$ under $(*)_n$, then

$$(1.5.4) \qquad e_k c = \operatorname{Res}_{t=\infty}\left((t-T)^{k-1}w\,dt\right) \qquad (k = 1,\dots,n),$$

where here and throughout the paper, given a Laurent series

$$\sum r_i t^i \in R((t^{-1})) \qquad (r_i \in R),$$

we set

$$\operatorname{Res}_{t=\infty}\left(\sum r_i t^i\,dt\right) =: -r_{-1}.$$

Therefore $(*)_n$ is injective and moreover, by (1.5.3), takes values in $W_n(R)$. Now fix $w \in W_n(R)$ arbitrarily, let $c \in C^{\otimes n}(R)$ be defined by the equations (1.5.4), and let $w \in W_n(R)$ be the image of c under $(*)_n$. Set $w'' =: w - w'$. Then by (1.5.2) and (1.5.4),

$$\operatorname{Res}_{t=\infty}(t^i w''\,dt) = 0 \qquad (i = 0,\dots,n-1).$$

Since w'' belongs to $W_n(R)$, it follows by an evident induction that $\operatorname{Res}_{t=\infty}(t^i w''\,dt)$ vanishes for all $i \geq 0$. Therefore $w'' \equiv 0 \bmod R[t]$. Hence w is the image of c under $(*)_n$. q.e.d.

As an immediate consequence of the definition of $(*)_n$,

$$(1.5.5) \qquad (*)_n([a]_n c) = a(t)((*)_n(c)) \qquad (a \in A,\ c \in C^{\otimes n}(R)),$$

where here and elsewhere throughout the paper, given $a \in A$, $a(t)$ denotes the polynomial obtained by replacing T by t.

1.6. The generalization for $C^{\otimes n}$ of Proposition 1.2.1 is:

PROPOSITION 1.6.1. *The endomorphisms*

$$[v^n]_n,\ \tau^{\deg(v)} \in \mathrm{End}(\mathbf{G}_a^n)$$

are congruent modulo v.

Proof. Let R be any A-algebra such that $vR = 0$, let $w \in W_n(R)$ be given. Set $d =: \deg(v)$. It will suffice by (1.5.5) to prove that

$$w^{(d)} \equiv v(t)^n w \qquad \mathrm{mod}\ R[t]. \tag{1.6.2}$$

In any case,

$$w^{(d)} \equiv Q_d(t)^n w \qquad \mathrm{mod}\ R[t],$$

where $Q_d(t) \in A[t]$ is just as defined in the proof of Proposition 1.2.1. Relation (1.6.2) follows now by the congruence (1.2.2). q.e.d.

1.7. Let R be an A-algebra, $c \in C^{\otimes n}(R)$ a point. We begin the study of "Kummer theory" for higher Carlitz modules by studying the functor of R-algebras

$$R' \mapsto \{e \in C^{\otimes n}(R') | [f]_n e = c\}. \tag{1.7.1}$$

Let $w \in t^{-1}R[[t^{-1}]]$ be the unique power series in t^{-1} without constant term congruent modulo $R[t]$ to the image of c under $(*)_n$. Set

$$c_+ =: w^{(1)} - (t-T)^n w \in R[t],$$

a polynomial in t with coefficients in R of degree strictly less than n. One can check that

$$c_+ = \sum_{i=1}^{n} (e_i c)(t-T)^{n-i}, \tag{1.7.2}$$

by using formula (1.5.4). Now under the map induced by $(*)_n$, the functor (1.7.1) of R-algebras is isomorphic to

$$R' \mapsto \{y \in t^{-1}R'[[t^{-1}]] | f(t)y \equiv w \text{ and } y^{(1)} \equiv (t-T)^n y \bmod R'[t]\}. \tag{1.7.3}$$

The functor above is, in turn, isomorphic to the functor

$$R' \mapsto \{z \in R'[t]/f(t)R'[t] | z^{(1)} - (t-T)^n z \equiv c_+ \bmod f(t)R'[t]\} \tag{1.7.4}$$

under the map

$$y \mapsto (f(t)y)_{\geq 0},$$

where, given any Laurent series

$$\sum r_i t^i \in R((t^{-1})) \qquad (r_i \in R),$$

we set

$$\left(\sum r_i t^i\right)_{\geq 0} =: \sum_{i \geq 0} r_i t^i \in R[t].$$

1.8. We consider multiplication by f in $C^{\otimes n}$.

PROPOSITION 1.8.1. *The morphism*

$$[f]_n \colon \mathbf{G}_a^n \to \mathbf{G}_a^n$$

is faithfully flat, finite and of rank $q^{\deg(f)}$. *Moreover, after the base change* $\mathrm{Spec}(A[f^{-1}]) \to \mathrm{Spec}(A)$, *it becomes étale.*

Proof. Set $d =: \deg(f)$. Since

$$[f]_n = f + \text{nilpotent matrix} + \text{terms in positive powers of } \tau,$$

the second statement follows by the Jacobian criterion. We turn to the proof of the first statement. We shall employ the observations of the preceding paragraph. Let R be any A-algebra, $c \in C^{\otimes n}(R)$ any point and let R_1 be the R-algebra representing the functor (1.7.4) of R-algebras; it will suffice to prove that R_1 is a free R-module of rank q^d. Let $x_1, \dots, x_d$ be independent variables. In any case, it is clear that there exist polynomials $F_j \in R[x_1, \dots, x_d]$ $(j = 1, \dots, d)$ such that

$$F_j = x_j^q + \text{linear terms and constant term},$$

and such that R_1 is isomorphic as an R-algebra to $R[x_1, \dots, x_d]/(F_1, \dots, F_d)$. By the lemma to be proved below, R_1 is free of rank q^d over R. q.e.d.

1.9. In this paragraph we prove a lemma of commutative algebra needed to complete the proof of Proposition 1.8.1. We are grateful to O. Gabber for the proof. Let R be a ring, $x_1, \dots, x_d$ variables, m a positive integer and $F_1, \dots, F_d \in S =: R[x_1, \dots, x_d]$ polynomials such that

$$F_j = x_j^m + \text{terms of total degree} < m \qquad (j = 1, \dots, d).$$

Let I be the ideal of S generated by $F_1, \dots, F_d$.

LEMMA 1.9.1. *S/I is a free R-module of rank m^d.*

Proof. Consider the filtered complex

$$K.(F_1,\dots,F_d) =: K.(F_1) \underset{S}{\otimes} \cdots \underset{S}{\otimes} K.(F_d),$$

where for any $F \in S$ of total degree not exceeding m, $K.(F)$ denotes the filtered complex

$$\cdots \leftarrow 0 \leftarrow S \overset{F}{\leftarrow} S \leftarrow 0 \cdots$$

concentrated in degrees 0 and 1, filtered by the rule

$$\mathrm{Fil}_i K_j(F) =: \begin{cases} \text{polynomials of total degree} \le i - m & (j = 1), \\ \text{polynomials of total degree} \le i & (j = 0), \\ 0 & (j \ne 0, 1). \end{cases}$$

Now $K.(F_1,\dots,F_d)$ is the Koszul complex of the sequence $F_1,\dots,F_d$ in S and, in particular,

$$H_0(K.(F_1,\dots,F_d)) = S/I.$$

Further, there exist isomorphisms of complexes

$$\mathrm{Gr}_p K.(F_1,\dots,F_d) \xrightarrow{\sim} \mathrm{Gr}_p K.(x_1^m,\dots,x_d^m),$$

where $\mathrm{Gr}_p =: \mathrm{Fil}_p/\mathrm{Fil}_{p-1}$. Now since $x_1^m,\dots,x_d^m$ is a regular sequence, $H_i(\mathrm{Gr}_p K.(x_1^m,\dots,x_d^m)) = 0$ for $i > 0$ and by direct calculation one can show that $H_0(\mathrm{Gr}_p K.(x_1^m,\dots,x_d^m))$ is a free R-module of finite rank vanishing for $|p| \gg 0$. Therefore the sequence

$$0 \to H_0(\mathrm{Fil}_{p-1} K.(F_1,\dots,F_d)) \to H_0(\mathrm{Fil}_p K.(F_1,\dots,F_d)) \\ \to H_0(\mathrm{Gr}_p K.(F_1,\dots,F_d)) \to 0$$

is exact for all indices p and, by induction on p, $H_0(\mathrm{Fil}_p(F_1,\dots,F_d))$ is a free R-module independent of p for $p \gg 0$. Since homology commutes with direct limits, $H_0(\mathrm{Fil}_p K.(F_1,\dots,F_d)) = S/I$ for all $p \gg 0$. Therefore S/I is a free R-module. Clearly $\mathrm{rank}_R(S/I) = \mathrm{rank}_R(S/(x_1^m,\dots,x_d^m)) = m^d$. q.e.d.

1.10. Set

$$C_f^{\otimes n} =: \ker([f]_n\colon \mathbf{G}_a^n \to \mathbf{G}_a^n).$$

Then by Proposition 1.8.1, $C_f^{\otimes n}$ is a finite flat group scheme over A of rank $q^{\deg(f)}$, étale over $A[f^{-1}]$.

PROPOSITION 1.10.1. *Let L be any separably algebraically closed field equipped with the structure of an A-algebra and suppose that $f \neq 0$ in L. Then $C_f^{\otimes n}(L)$ is isomorphic to A/f as an A-module.*

Proof. In any case, since $C_f^{\otimes n}$ is étale over $A[f^{-1}]$,

$$\#C_f^{\otimes n}(L) = q^{\deg(f)}. \tag{1.10.2}$$

Because (1.10.2) holds also for all divisors of f and $C_f^{\otimes n}(L)$ is annihilated by f, $C_f^{\otimes n}(L)$ must in fact be isomorphic to A/f. q.e.d.

Set

$$\mathbf{F}_v =: A/v,$$

$$\mathrm{Frob}_v =: \text{the arithmetic Frobenius automorphism of } \overline{\mathbf{F}}_v/\mathbf{F}_v.$$

PROPOSITION 1.10.3. Frob_v *and* $[v^n]_n$ *act in the same way upon* $C^{\otimes n}(\overline{\mathbf{F}}_v)$ *and, in particular,* $C^{\otimes n}(\mathbf{F}_v)$ *is isomorphic as an A-module to* $A/(v^n - 1)$.

Proof. The first part follows from Proposition 1.6.1. Since the fixed points of Frob_v in $C^{\otimes n}(\overline{\mathbf{F}}_v)$ are precisely the points killed by $[v^n - 1]_n$, the second part follows from Proposition 1.10.1. q.e.d.

1.11. The generalization of Proposition 1.3.2 for $C^{\otimes n}$ is:

PROPOSITION 1.11.1. *Let $\sigma_v \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be an arithmetic Frobenius at v. Then for all $c \in C_f^{\otimes n}(K^{\mathrm{sep}})$, provided that v does not divide f,*

$$\sigma_v c = [v^n]_n c.$$

Consequently the image of the Galois representation

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}_A\big(C_f^{\otimes n}(K^{\mathrm{sep}})\big) = (A/f)^{\times}$$

consists of the n^{th} powers in the group $(A/f)^{\times}$.

Proof. This follows easily from Proposition 1.10.1. q.e.d.

Remark. We might rephrase Proposition 1.11.1 as the assertion that $C_f^{\otimes n}(K^{\mathrm{sep}})$ is A-linearly and $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariantly isomorphic to the n-fold tensor power over A of $C_f(K^{\mathrm{sep}})$. It is this fact which suggests that $C^{\otimes n}$ ought to be regarded as the n^{th} tensor power of C. See Anderson [A] for further discussion of higher-dimensional Drinfeld modules and tensor products.

PROPOSITION 1.11.2. $C_f^{\otimes n}(K) = \{0\}$ *unless $q - 1$ divides n.*

Proof. We may assume that f is irreducible, i.e., that $f = v$; suppose that $C_v^{\otimes n}(K^{\mathrm{sep}}) = C_v^{\otimes n}(K) \neq \{0\}$. Then, by the preceding proposition, raising to the

n^{th} power must annihilate the cyclic group $\mathbf{F}_v^\times$ which is of order divisible by $q-1$. q.e.d.

2. The logarithm and the exponential

2.1. A positive integer n is fixed throughout this section. We define the *formal logarithm* $\log_n$ attached to the n^{th} higher Carlitz module $C^{\otimes n}$ to be the unique series

$$\log_n = \sum_{i=0}^{\infty} P_i \tau^i \qquad (P_i\text{: an } n\text{-by-}n \text{ matrix with entries in } K)$$

determined by the conditions

$$P_0 =: 1, \tag{2.1.1}$$

$$\log_n(T + N + E\tau) = (T+N)\log_n, \tag{2.1.2}$$

where N and E are the n-by-n matrices intervening in the definition of the higher Carlitz module $C^{\otimes n}$. The formal series under consideration here are to be added and multiplied according to the rule (1.4.1) which makes sense for arbitrary sequences of coefficient matrices, and not just for those sequences almost all of whose terms vanish. In order to show that $\log_n$ is well defined, we make explicit the recursion relations forced upon the coefficients P_i by (2.1.2). We have

$$(T+N)P_{i+1} = P_{i+1}(T^{q^{i+1}} + N) + P_i E.$$

After a little rearrangement we obtain

$$P_{i+1} - \frac{[N, P_{i+1}]}{(T^{q^{i+1}} - T)} = -\frac{P_i E}{(T^{q^{i+1}} - T)}.$$

(Here $[X, Y] =: XY - YX$.) Since N is nilpotent, the relation above can be solved "by geometric series" and we obtain

$$P_{i+1} = -\sum_{j=0}^{2n-2} \frac{\operatorname{ad}(N)^j (P_i E)}{(T^{q^{i+1}} - T)^{j+1}} \qquad (i = 0, 1, 2, \ldots). \tag{2.1.3}$$

(Here $\operatorname{ad}(X)^0(Y) =: Y$, $\operatorname{ad}(X)^{j+1}(Y) =: [X, \operatorname{ad}(X)^j(Y)]$.) Therefore $\log_n$ is well defined. For example we have

$$P_i = L_i^{-1} \qquad (\text{when } n = 1), \tag{2.1.4}$$

where (cf. [C1], up to a sign)

$$L_i =: \prod_{j=1}^{i} (T - T^{q^j}) \qquad (i = 1, 2, \dots),$$
$$L_0 =: 1.$$

More generally, we have the following:

PROPOSITION 2.1.5. *The entry in the lower right corner of P_i is L_i^{-n}.*

Proof. It will be enough to show that

$$N^{n-1}P_{i+1}E = \frac{N^{n-1}P_iE}{(T - T^{q^{i+1}})^n} \qquad (i = 0, 1, 2, \dots).$$

Note the following:

(2.1.6)
$$\begin{aligned} EN^iE &= E && \text{if } i = n-1, \\ &= 0 && \text{otherwise.} \\ N^i &= 0 && \text{if } i > n-1. \end{aligned}$$

If we multiply the right-hand side of (2.1.3) on the left by N^{n-1} and on the right by E, the only surviving term is the one we want. q.e.d.

Remark. Therefore one has formally

$$\log_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} \vdots \\ \sum_{k=0}^{\infty} \frac{x^{q^k}}{L_k^n} \end{pmatrix}.$$

Recalling that, classically, the n^{th} multilogarithm function

$$\sum_{k=0}^{\infty} \frac{z^k}{k^n}$$

is the n-fold Hadamard convolution of

$$-\log(1 - z) = \sum_{k=0}^{\infty} \frac{z^k}{k},$$

and given that

$$\sum_{k=0}^{\infty} \frac{x^{q^k}}{L_k}$$

is the logarithm of C, it is reasonable to think of the n-fold Hadamard convolution .

$$\sum_{k=0}^{\infty} \frac{x^{q^k}}{L_k^n}$$

as the "n^{th} multilogarithm" associated to the Carlitz module. Therefore one might hope for a relationship with zeta values as in the classical situation; such a relationship will be established in Section 3 below.

2.2. We define the *formal exponential* $\exp_n$ associated to the higher Carlitz module $C^{\otimes n}$ to be the unique series

$$\exp_n = \sum_{i=0}^{\infty} Q_i \tau^i \qquad (Q_i\text{: an } n\text{-by-}n \text{ matrix with coefficients in } K)$$

such that

$$(2.2.1) \qquad Q_0 =: 1,$$

$$(2.2.2) \qquad (T + N + E\tau)\exp_n = \exp_n(T + N).$$

In order to prove that $\exp_n$ is well-defined, we exhibit a recursion relation for the coefficients Q_i similar to (2.1.3). By (2.2.2),

$$(T + N)Q_{i+1} + EQ_i^{(1)} = Q_{i+1}(T^{q^{i+1}} + N),$$

which after rearrangement yields

$$Q_{i+1} - \frac{[N, Q_{i+1}]}{(T^{q^{i+1}} - T)} = \frac{EQ_i^{(1)}}{(T^{q^{i+1}} - T)}.$$

Solving for Q_{i+1} we obtain

$$(2.2.3) \qquad Q_{i+1} = \sum_{j=0}^{2n-2} \frac{\operatorname{ad}(N)^j\left(EQ_i^{(1)}\right)}{\left(T^{q^{i+1}} - T\right)^{j+1}} \qquad (i = 0, 1, 2, \ldots).$$

For example, we have

$$(2.2.4) \qquad Q_i = D_i^{-1} \qquad (\text{when } n = 1),$$

where (cf. [C1]; however Carlitz writes F_i in place of D_i),

$$D_i =: \prod_{j=0}^{i-1} (T^{q^i} - T^{q^j}) \qquad (i = 1, 2, \ldots),$$

$$D_0 =: 1.$$

In the general case, we have:

PROPOSITION 2.2.5. *The entry in the upper left corner of* Q_i *is* D_i^{-n}.

Proof. It will be enough to show that

$$EQ_{i+1}N^{n-1} = \frac{EQ_i^{(1)}N^{n-1}}{\left(T^{q^{i+1}} - T\right)^n} \qquad (i = 0, 1, 2, \ldots).$$

If we multiply the right-hand side of (2.2.3) on the left by E and on the right by N^{n-1}, the only surviving term is the one we want. q.e.d.

Remark. Therefore one has formally

$$\exp_n \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{\infty} \frac{x^{q^i}}{D_i^n} \\ \vdots \end{pmatrix}.$$

For $n = 2$, the "multi-exponential" appearing above is (up to signs) the Bessel function analogue studied by Carlitz [C4].

2.3. It is easy to see that

$$\log_n \exp_n = 1, \tag{2.3.1}$$

$$\exp_n \log_n = 1, \tag{2.3.2}$$

i.e., that $\log_n$ and $\exp_n$ are formally inverse one to the other. Now let us rewrite the characteristic functional equations (2.1.2) and (2.2.2) in a more suggestive manner, as follows.

Identify $\mathrm{End}(\mathbf{G}_a^n)$ with a subring of the ring of formal series to which $\log_n$ and $\exp_n$ belong in the manner of subsection 1.4 above. Given any $a \in A$, set

$$d[a]_n =: \text{the coefficient of } \tau^0 \text{ in the expansion of } [a]_n \text{ in powers of } \tau.$$

When we identify $\mathrm{Lie}(\mathbf{G}_a^n)$ with $\mathbf{G}_a^n$ in the evident fashion, $d[a]_n$ is clearly the matrix representing the endomorphism of $\mathrm{Lie}(\mathbf{G}_a^n)$ induced by $[a]_n$ and

$$\log_n [a]_n = d[a]_n \log_n, \tag{2.3.3}$$

$$\exp_n d[a]_n = [a]_n \exp_n. \tag{2.3.4}$$

Therefore $\log_n$ and $\exp_n$ are the analogues for the higher Carlitz module $C^{\otimes n}$ of the logarithm and exponential, respectively, attached to an algebraic group.

Remark. For all $a \in A$, $d[a]_n$ is an upper triangular matrix with each diagonal entry equal to a. In particular, one has

$$d[a]_n \begin{pmatrix} * \\ \vdots \\ * \\ x \end{pmatrix} = \begin{pmatrix} * \\ \vdots \\ * \\ ax \end{pmatrix}.$$

Therefore the distinguished coordinate ℓ_n: $\mathrm{Lie}(C^{\otimes n}) \to \mathbf{G}_a$ mentioned in the introduction is none other than the last coordinate e_n: $\mathbf{G}_a^n \to \mathbf{G}_a$.

2.4. Let K_v denote the v-adic completion of K, let $\overline{K}_v$ denote an algebraic closure of K_v and let $|\ |_v$ be the unique extension to $\overline{K}_v$ of an ultrametric absolute value of K_v that defines the v-adic topology of K_v. Given any matrix M with entries in $\overline{K}_v$, set

$$|M|_v =: \sup_{i,j} |M_{ij}|_v.$$

The unique place of K at which T has a pole is denoted ∞. Let $|\ |_\infty$ be the unique extension to $\overline{K}_\infty$ of an ultrametric absolute value of $K_\infty = \mathbf{F}_q((1/T))$ inducing the ∞-adic topology of K_∞. We turn to the investigation of the v-adic and ∞-adic convergence properties of $\log_n$ and $\exp_n$.

PROPOSITION / DEFINITION 2.4.1. *For any $x \in \overline{K}_v^n$ such that $|x|_v < 1$, the series*

$$\sum_{i=0}^{\infty} P_i x^{(i)}$$

converges v-adically to a value in $\overline{K}_v^n$, denoted $\log_{n,v}(x)$.

Proof. By (2.1.3),

$$|P_{i+1}|_v \le |P_i|_v |T^{q^i} - T|_v^{1-2n} \qquad (i = 0, 1, 2, \dots)$$

and one has the crude estimate

$$|T^{q^i} - T|_v \ge |v|_v,$$

whence by induction

$$|P_i|_v \le |v|_v^{-i(2n-1)} |P_0|_v.$$

This last guarantees v-adic convergence. q.e.d.

PROPOSITION / DEFINITION 2.4.2. *For any $x \in \overline{K}_\infty^n$ the series*

$$\sum_{i=0}^{\infty} Q_i x^{(i)}$$

converges ∞*-adically to a value denoted by* $\exp_{n,\infty}(x)$. *Consequently*

$$\exp_{n,\infty}(d[a]_n x) = [a]_n \exp_{n,\infty}(x) \qquad \left(x \in \overline{K}_\infty^n, a \in A\right).$$

Proof. By (2.2.3),

$$|Q_{i+1}|_\infty \leq |Q_i|_\infty^q |T|_\infty^{-q^{i+1}},$$

whence by induction

$$|Q_i|_\infty \leq |T|_\infty^{-iq^i} |Q_0|_\infty.$$

This last estimate guarantees convergence. q.e.d.

PROPOSITION / DEFINITION 2.4.3. *For any* $x \in \overline{K}_\infty^n$, *the components* x_i $(i = 1, \ldots, n)$ *of which satisfy*

$$|x_i|_\infty < |T|_\infty^{i-n+(nq/(q-1))},$$

the series

$$\sum_{i=0}^{\infty} P_i x^{(i)}$$

converges ∞*-adically to a value denoted by* $\log_{n,\infty}(x)$. *Consequently, for all such* x,

$$\exp_{n,\infty}(\log_{n,\infty}(x)) = x.$$

Proof. It will be enough to show that

$$|P_k N^{n-i} E|_\infty \leq |T^{-n((q^{k+1}-q)/(q-1))+(n-i)q^k}|_\infty \qquad (2.4.4)$$

$$(1 \leq i \leq n, 0 \leq k < \infty)$$

because, since $N^{n-i}E$ is the matrix with entry 1 in the i^{th} row and first column and zeroes elsewhere, the left-hand side above is the absolute value of the largest entry in the i^{th} column of P_k. We proceed by induction on k. The case $k = 0$ is trivial. We have

$$P_{k+1} N^{n-i} E = \sum_{m=0}^{2n-2} \sum_{j=0}^{m} -(-1)^j \binom{m}{j} \frac{N^{m-j} P_k E N^{n+j-i} E}{\left(T^{q^{k+1}} - T\right)^{m+1}}.$$

Now by (2.1.6), the summand above vanishes unless $j = i - 1$ and $n + i - 2 \geq m \geq i - 1$. We are left with

$$P_{k+1} N^{n-i} E = \sum_{m=i-1}^{n+i-2} -(-1)^j \binom{m}{j} \frac{N^{m-i+1} P_k E}{\left(T^{q^{k+1}} - T\right)^{m+1}}.$$

By the induction hypothesis,

$$|P_{k+1}N^{n-i}E|_\infty \le |T^{-n((q^{k+1}-q)/(q-1))-iq^{k+1}}|_\infty = |T^{-n((q^{k+2}-q)/(q-1))+(n-i)q^{k+1}}|_\infty .$$

q.e.d.

2.5. We shall determine the structure of

$$\Lambda_n =: \ker(\exp_{n,\infty}: \bar{K}_\infty^n \to \bar{K}_\infty^n)$$

by adapting techniques of [A]. We first consider the set Ω_n of power series

$$h(t) = \sum_{i=0}^{\infty} a_i t^i \qquad (a_i \in \bar{K}_\infty)$$

with the following properties:

(2.5.1) $\{a_i\}$ generates a finite algebraic extension of K_∞.

(2.5.2) $h(t)$ converges for $|t|_\infty \le 1$.

(2.5.3) $h^{(1)}(t) = (t-T)^n h(t)$,

where

$$h^{(1)}(t) =: \sum_{i=0}^{\infty} a_i^q t^i.$$

The set Ω_n has the structure of a module over $\mathbf{F}_q[t]$.

LEMMA 2.5.4. *Ω_n is a free module of rank one over* $\mathbf{F}_q[t]$.

Proof. Consider the power series

$$\omega_1(t) =: \sqrt[q-1]{-T} \prod_{i=0}^{\infty} \left(1 - \frac{t}{T^{q^i}}\right)^{-1}$$

and set

$$\omega_n(t) =: \omega_1(t)^n,$$

where $\sqrt[q-1]{-T}$ denotes a choice of the $(q-1)^{\text{st}}$ root of $-T$ in $\bar{K}_\infty$ fixed hereafter. Then $\omega_n(t)$ belongs to Ω_n and has no zeroes in the disc $|t|_\infty \le 1$. Given now any $h \in \Omega_n$, set $g =: h/\omega_n$. Then g satisfies (2.5.1), (2.5.2) and $g^{(1)} = g$. The power series g can have these properties only if g is a polynomial in t with coefficients in $\mathbf{F}_q$. Consequently h has a factorization $h = g\omega_n$ with $g \in \mathbf{F}_q[t]$ which is unique by the Weierstrass preparation theorem. Therefore Ω_n is free of rank one over $\mathbf{F}_q[t]$ on the basis ω_n. q.e.d.

Therefore each $h \in \Omega_n$ is a meromorphic function of t with poles of order no greater than n and, in particular, possesses a unique Laurent expansion

$$h(t) = \sum_{i=-n}^{\infty} a_i (t-T)^i \qquad (a_i \in \overline{K}_\infty)$$

convergent for t near T; we write

$$\mathrm{RES}_n(h(t)) =: \begin{pmatrix} a_{-1} \\ \vdots \\ a_{-n} \end{pmatrix} \in \overline{K}_\infty^n.$$

PROPOSITION 2.5.5. *The map* $\mathrm{RES}_n\colon \Omega_n \to \overline{K}_\infty^n$ *is injective, and the image of* RES_n *is* Λ_n.

Proof. Let $h(t) \in \Omega_n$ be arbitrary. Let $\tilde{h}(t)$ be the column vector of length n with i^{th} entry $(t-T)^{i-1}h(t)$. Set $c_0 =: 0 \in \overline{K}_\infty^n$ and define coefficients $c_i = c_i(h) \in \overline{K}_\infty^n$ $(i = 1, 2, \dots)$ by the rule

$$\tilde{h}(t) =: \sum_{i=0}^{\infty} c_{i+1} t^i.$$

Necessarily, $\lim_{i\to\infty} |c_i|_\infty = 0$ because $\tilde{h}(t)$ is a column vector each of whose entries converges on the closed unit disc $|t|_\infty \leq 1$. Also,

$$[T]_n c_{i+1} = (T+N)c_{i+1} + Ec_{i+1}^{(1)} = c_i \qquad (i = 0, 1, 2, \dots)$$

as can be checked by direct calculation; i.e., the coefficients c_i form a T-division sequence. Consequently

$$(T+N)^i \log_{n,\infty}(c_i)$$

is defined for all $i \gg 0$ and is independent of i; call it λ. Necessarily $\lambda \in \Lambda_n$ and

$$c_i = \exp_{n,\infty}\big((T+N)^{-i}\lambda\big) \qquad (i = 0, 1, 2, \dots).$$

Equivalently,

$$\tilde{h}(t) = \sum_{i=0}^{\infty} \exp_{n,\infty}\big((T+N)^{-i-1}\lambda\big)t^i. \tag{2.5.6}$$

The polar part of $\tilde{h}(t)$ at $t = T$ is clearly the same as that of

$$\sum_{i=0}^{\infty} (T+N)^{-i-1}\lambda t^i = -\lambda(t-T)^{-1} - \sum_{j=1}^{n-1} N^j \lambda (t-T)^{-j-1},$$

whence it follows that

$$\mathrm{RES}_n(h(t)) = -\lambda. \tag{2.5.7}$$

Therefore RES_n takes values in Λ_n and, by (2.5.6), RES_n is injective. Now let $\lambda \in \Lambda_n$ be given arbitrarily and let $F(t)$ be the column vector of power series defined by the right-hand side of (2.5.6). Let $h(t)$ be the first entry of $F(t)$. Then one check that $h(t) \in \Omega_n$ and that $\tilde{h}(t) = F(t)$. By (2.5.7) the image of $h(t)$ under RES_n is $-\lambda$. Therefore RES_n is surjective. q.e.d

Put

$$\bar{\pi} =: \mathrm{RES}_1(\omega_1(t)).$$

COROLLARY 2.5.8. *There exists a vector* $\lambda \in \bar{K}_\infty^n$ *with last entry* $\bar{\pi}^n$ *such that* $\Lambda_n =: \{d[a]_n \lambda | a \in A\}$.

Proof. The image of $\omega_1(t)^n = \omega_n(t)$ under RES_n, by Lemma 2.5.4 and Proposition 2.5.5, is the desired vector λ. q.e.d.

Remark. Note that the entries of the period vector λ of $C^{\otimes n}$ are the Laurent coefficients of the expansion in powers of $(t - T)$ of $\omega_1(t)^n$. In particular, if n is a power of the characteristic p, all entries except the last are zero.

COROLLARY 2.5.9. $\bar{\pi}^n$ *belongs to* K_∞ *if and only if* $q - 1$ *divides* n.

Proof.

$$\bar{\pi} = \mathrm{RES}_1(\omega_1(t)) = \left(\sqrt[q-1]{-T}\right)^q \prod_{i=1}^{\infty} \left(1 - \frac{T}{T^{q^i}}\right)^{-1}. \qquad \text{q.e.d.}$$

3. Zeta values

3.1. Following Carlitz ([C1], [C2]) we define

$$\zeta(n) =: \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty \qquad (n = 1, 2, \ldots).$$

If one regards monicity as a sign condition analogous to positivity, then one might regard $\zeta(n)$ as an analogue of the corresponding value of the Riemann zeta function. The existence of a representation

$$\zeta(n) = \prod_{\substack{v \in A_+ \\ v:\ \text{prime}}} \left(1 - \frac{1}{v^n}\right)^{-1}$$

as a convergent *Euler product* is more evidence for the existence of an analogy and, of course, guarantees that $\zeta(n) \neq 0$. (The nonvanishing of $\zeta(n)$ is in any case a triviality since clearly $|\zeta(n)|_\infty = 1$.) Note, however, that in contrast to the classical case of power sums of positive integers, the products and the sums under consideration here converge for $n = 1$; one might reasonably regard $\zeta(1)$ as the analogue of the Euler-Mascheroni constant. Note also the curious identity

$$\zeta(np^m) = \zeta(n)^{p^m}.$$

Nonetheless Carlitz showed that for "even" positive integers n, i.e., those positive integers divisible by $q - 1$, $\zeta(n)$ is quite analogous to the corresponding value of the Riemann zeta function.

THEOREM 3.1.1 (Carlitz [C2, p. 503]). *If n is a positive integer divisible by $q - 1$, then $\zeta(n)/\bar{\pi}^n$ belongs to K.*

Carlitz went on to show that the ratio in question can naturally be written as the quotient of a "Bernoulli number" by a "factorial". We have also:

THEOREM 3.1.2 (Wade [W1]). *$\bar{\pi}$ is transcendental over K.*

COROLLARY 3.1.3. *If n is a positive integer divisible by $q - 1$, then $\zeta(n)$ is transcendental and $\zeta(n)/\bar{\pi}^n$ is rational over K.*

This is analogous to what is known about the value of the Riemann zeta function at even positive integers. (But note that in the case $q = 2$, all positive integers are "even".)

One knows nothing about the transcendence of the values of the Riemann zeta function at odd positive integers; even about irrationality, beyond Apéry's celebrated result, nothing is known. But recently Jing Yu has obtained results about $\zeta(n)$ for n "odd" which go far beyond what is known for the special values of the Riemann zeta function at odd positive integers.

THEOREM 3.1.4 (Yu [Y2]). *If n is a positive integer not divisible by $q - 1$, then $\zeta(n)$and $\zeta(n)/\bar{\pi}^n$ are transcendental over K.*

In the paper [Y1], Yu obtained quite general results concerning the transcendence properties of a class of functions including $\exp_n$ and $\log_n$ that, in combination with the results of subsection 3.8 of this paper, are used to prove the cited result. In [T1] and [T2], the transcendence of $\zeta(n)$ for some "odd" values of n was deduced by applying methods of [W1] and evidence was presented for the transcendence of $\zeta(n)/\bar{\pi}^n$ for some "odd" values of n, namely, the existence of "fast approximation by rationals". It should be noted that Roth's inequalities do not imply transcendence in positive characteristic, in contrast to the characteristic zero case.

It seems that Carlitz and Wade had enough machinery to prove Theorem 3.1.4 for $n = 1$, and so also for n equal to a power of p. More precisely, by [C1, p. 160], $\zeta(1) = \log_1(1)$. (Carlitz used $T - \tau$ rather than $T + \tau$ for the T-division polynomial, so that, strictly speaking, with his definition of log, $\zeta(1) = \log(1)$ holds only in characteristic 2. However, $\zeta(1) = \log(u)/u$, where u is any $(q-1)^{\text{st}}$ root of -1 holds in general.) On the other hand, Wade [W1, p. 720] showed that the log of a nonzero number in $\overline{K}$ is transcendental, so that $\zeta(1)$ is transcendental. In [W2] he also proved an analogue of the Siegel-Schneider criterion, namely that, for nonzero α and irrational β, at least one of α, β, $\exp_1(\beta \log_1(\alpha))$ is transcendental. In particular, taking $\alpha = 1$ and $\beta = \bar{\pi}/\zeta(1)$, one recovers the transcendence of $\zeta(1)/\bar{\pi}$ when $q \neq 2$. But this was not noticed then.

3.2. It can be shown ([C1, p. 162], [G, p. 110]) that, with a fixed positive integer n, the sum

$$\sum_{\substack{a \in A_+ \\ \deg(a) = k}} a^n \tag{3.2.1}$$

vanishes for all sufficiently large k. By grouping terms according to degree, Goss [G] has given an "analytic continuation" to the power sums of Carlitz defining an analogue of the Riemann zeta function. We refer to [G] for a complete exposition. We simply note here that one can define $\zeta(-n)$ for any nonnegative integer n by the formula

$$\zeta(-n) =: \sum_{k=0}^{\infty} \left(\sum_{\substack{a \in A_+ \\ \deg(a) = k}} a^n \right),$$

because the sum over k is in fact finite.

Remark. Even though an analogue of Riemann's functional equation is not known for the Goss zeta function, Goss has shown that the special values at negative integers have interesting properties. For example, he has shown that $\zeta(n)$ vanishes for all negative "even" integers, is nonvanishing for all negative "odd" integers and belongs to A, i.e., is integral. Note that while the first two properties are analogous to those of the corresponding values of the Riemann zeta function, the third sharply distinguishes the characteristic p and characteristic zero situations. It should be noted here that vanishing of ζ at negative even integers and integrality at negative integers are also consequences of Lemma 7.1 of Lee [L].

3.3. Having defined $\zeta(n)$ for nonpositive n, we naturally try to push the analogy with the Riemann zeta function further by seeking an interpolation of the values of ζ at nonpositive integers for each prime v of A. This has been carried out by Goss; we refer the reader to [G] for the complete story. The following ad hoc approach will suffice for us:

If $a \in A$ is prime to v, the function a^n interpolates to a continuous function with range

$$A_v =: \text{the completion of } A \text{ at } v$$

and domain

$$S_v =: \lim_{\leftarrow} \mathbf{Z}/\left(q^{\deg(v)} - 1\right)p^n.$$

From this fact it follows (see [G] for the details) that

$$\zeta_v(-n) = (1 - v^n)\zeta(-n) \qquad (n\text{: nonnegative integer})$$

interpolates to a continuous A_v-valued function on S_v.

The interpolation procedure yields as a by-product a representation of $\zeta_v(n)$ for any integer n (positive or not) as a v-adically convergent infinite sum

$$\zeta_v(n) = \sum_{k=0}^{\infty} \left(\sum_{\substack{a \in A_+ \\ (v,a)=1 \\ \deg(a)=k}} \frac{1}{a^n} \right), \tag{3.3.1}$$

where, of course, it is necessary that the interior summation be carried out first. This sum will play an important role later. It follows that, in particular, $\zeta_v(n)$ for n positive may be recovered as the v-adic limit of values of ζ_v for negative integers by the formula

$$\zeta_v(n) = \lim_{j \to \infty} \zeta\left(n - \left(q^{\deg(v)} - 1\right)p^j\right).$$

Remark. Observe that the classical p-adic Kummer congruences may be interpreted as the assertion that the values at negative integers of the Riemann zeta function multiplied by the p^{th} Euler factor interpolate so as to yield a $\mathbf{Q}_p$-valued function on the space

$$\lim_{\leftarrow} \mathbf{Z}/(p-1)p^i$$

that is continuous apart from a "simple pole" at 1.

3.4. In this section we review some ideas of [C1], especially pp. 160–162. Note, however, that we use different notation. Consider the polynomials

$$\Psi_k(x) \in K[x] \qquad (x\text{: a variable, } k = 0,1,2,\dots)$$

defined by

$$(\exp_1)x(\log_1) = \sum_{k=0}^{\infty} \Psi_k(x)\tau^k,$$

where the right-hand side is obtained by multiplying out the left-hand side according to the rule (1.4.1). Roughly speaking, the polynomials $\Psi_k(x)$ are to the Carlitz module as the polynomials

$$\binom{x}{n} \in \mathbf{Q}[x] \qquad (n = 0,1,2,\dots)$$

defined by the power series identity

$$(1+t)^x = \exp(x\log(1+t)) = \sum_{n=0}^{\infty} \binom{x}{n} t^n$$

are to the multiplicative group. The following properties of $\Psi_k(x)$ are easy to check:

(3.4.1) $\Psi_k(x)$ is a polynomial of degree q^k.

(3.4.2) $\Psi_k(x)$ is a K-linear combination of terms of the form x^{q^i} $(0 \le i \le k)$.

Let $a \in A$. Consider the equality of τ-expansions

$$(\exp_1)a(\log_1) = (\exp_1)(\log_1)[a] = [a] = \sum_{k=0}^{\infty} \Psi_k(a)\tau^k.$$

Now the coefficient of τ^k in the τ-expansion of $[a]$ vanishes for $k > \deg(a)$ and equals 1 if a is monic of degree k. Therefore

(3.4.3) $\Psi_k(a) = 0$ for all $a \in A$ such that $\deg(a) < k$.

(3.4.4) $\Psi_k(T^k) = 1$.

Having determined all the zeroes of $1 - \Psi_k(x)$ and their multiplicities, by taking the logarithmic derivative, we have

(3.4.5) $$\sum_{\substack{a \in A_+ \\ \deg(a) = k}} \frac{1}{(x-a)} = -\frac{\Psi_k'(0)}{(1-\Psi_k(x))}.$$

It follows that

$$(3.4.6)\qquad \sum_{n=0}^{\infty}\left(\sum_{\substack{a\in A_+\\ \deg(a)=k}}\frac{1}{a^{n+1}}\right)x^n$$

$$= \text{Laurent expansion of } \frac{\Psi_k'(0)}{(1-\Psi_k(x))} \qquad \text{at } x=0.$$

$$(3.4.7)\qquad \sum_{n=0}^{\infty}\left(\sum_{\substack{a\in A_+\\ \deg(a)=k}} a^n\right)\frac{1}{x^{n-1}}$$

$$= \text{Laurent expansion of } -\frac{\Psi_k'(0)}{(1-\Psi_k(x))} \qquad \text{at } x=\infty.$$

Note that the vanishing of the sum (3.2.1) for fixed n and all sufficiently large k follows from (3.4.1) and (3.4.7).

Another consequence of the definition is

$$\log_1 = \sum_{k=0}^{\infty} \Psi_k'(0)\tau^k;$$

hence by (2.1.4),

$$(3.4.8)\qquad \Psi_k'(0) = L_k^{-1}.$$

By comparing the coefficients of the τ-expansions

$$(\exp_1)Tx(\log_1) - T(\exp_1)x(\log_1) = \tau(\exp_1)x(\log_1),$$

one finds that

$$(3.4.9)\qquad \Psi_{k+1}(Tx) - T\Psi_{k+1}(x) = \Psi_k(x)^q.$$

3.5. For any nonnegative integer n, write (uniquely)

$$n = \sum_{i=0}^{\infty} n_i q^i \qquad (0 \le n_i < q,\ n_i = 0 \text{ for } i \gg 0),$$

and define

$$\Gamma_{n+1} =: \prod_{i=0}^{\infty} D_i^{n_i}.$$

Then Γ_{n+1} is the factorial of n (gamma of $n + 1$) introduced by Carlitz [C2]. Observe that

$$(3.5.1)\qquad \frac{\Gamma_{m+n+1}}{\Gamma_{m+1}\Gamma_{n+1}} \in A.$$

3.6. We define polynomials

$$G_n(y) \in A[y] \qquad (n = 0, 1, 2, \dots)\ (y\text{: a variable})$$

by the formula

$$G_n(y) = \prod_{i=1}^{n} \left(T^{q^n} - y^{q^i}\right).$$

Note that

$$(3.6.1)\qquad G_0(y) =: 1.$$

$$(3.6.2)\qquad G_{n+1}(y^q) =: (T - y^q)^{q^{n+1}} G_n(y)^q.$$

$$(3.6.3)\qquad \text{degree in } y \text{ of } G_i(y) = \frac{q^{i+1} - q}{q - 1} \le q^i \frac{q}{q - 1}.$$

We claim that

$$(3.6.4)\qquad \Psi_k(x) = \sum_{i=0}^{k} \frac{G_i(T^{q^k})}{D_i} \left(\frac{x}{L_k}\right)^{q^i}.$$

In order to prove this, let the right-hand side above be denoted by $F_k(x)$. Then by definition, $F_0(x) = \Psi_0(x)$ and $F_k'(0) = \Psi_k'(0)$. In order to prove that $F_k(x) = \Psi_k(x)$ in general, it will be enough to check that $F_k(x)$ satisfies the functional equation (3.4.9). Equivalently, we must check that for all i, $k \ge 0$,

$$(3.6.5)\qquad \left(\frac{G_{i+1}(T^{q^{k+1}})}{D_{i+1} L_{k+1}^{q^{i+1}}}\right)\left(T^{q^{i+1}} - T\right) = \left(\frac{G_i(T^{q^k})}{D_i L_k^{q^i}}\right)^q.$$

But this follows from the definitions of $G_i(y)$, L_k and D_i. The claim is proved.

3.7. Define a sequence of polynomials

$$H_n(y) \in A[y] \qquad (n = 0, 1, \dots)$$

by means of the generating function identity

$$\sum_{n=0}^{\infty} \frac{H_n(y)}{\Gamma_{n+1}} x^n =: \left(1 - \sum_{i=0}^{\infty} \frac{G_i(y)}{D_i} x^{q^i}\right)^{-1}. \tag{3.7.1}$$

A priori the polynomials $H_n(y)$ merely belong to $K[y]$, but in fact belong to $A[y]$ by (3.5.1). Note that

$$H_n(y) = 1 \qquad (0 \le n \le q-1). \tag{3.7.2}$$

From the estimate (3.6.3), we obtain

$$\deg(H_n(y)) \le n\frac{q}{q-1} < (n+1)\frac{q}{q-1}. \tag{3.7.3}$$

Substituting T^{q^k} for y and x/L_k for x in (3.7.1), we get by (3.6.4) and the definitions,

$$\sum_{n=0}^{\infty} \frac{H_n(T^{q^k})x^n}{\Gamma_{n+1}L_k^n} = \frac{1}{(1-\Psi_k(x))}.$$

By (3.4.6) and (3.4.8),

$$\frac{H_n(T^{q^k})}{L_k^{n+1}} = \Gamma_{n+1} \sum_{\substack{a\in A_+ \\ \deg(a)=k}} \frac{1}{a^{n+1}}. \tag{3.7.4}$$

3.8. Fix a positive integer n. We define coefficients $h_{ni} \in A$ $(i = 0, 1, 2, \dots)$ thus:

$$\sum_{i=0}^{\infty} h_{ni} y^i =: H_{n-1}(y).$$

(Note the index shift.) Of course, by (3.7.3)

$$h_{ni} = 0 \qquad \text{for } i \ge \frac{nq}{q-1}. \tag{3.8.1}$$

Set

Y_{ni} =: the column vector of length n with last entry T^i and all other entries vanishing.

Now define an A-valued point of $C^{\otimes n}$ by the formula

$$Z_n =: \sum_i [h_{ni}]_n Y_{ni}. \tag{3.8.2}$$

We call Z_n the *special point* of $C^{\otimes n}$. Here are some examples for small values of n.

$$Z_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad (n \leq q),$$

$$Z_n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + [T^q - T]_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad (n = q + 1),$$

$$Z_n = \left[(T^q - T)^{q-1}\right]_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad (n = q^2).$$

For each monic irreducible element v of A, set

$$Z_{n,v} =: [v^n - 1]_n Z_n \in C^{\otimes n}(A) = A^n.$$

We call $Z_{n,v}$ the *v-twist* of the special point Z_n. The v-twist operation corresponds to the operation of "deleting the Euler factor at v". For example, as we shall show presently,

$$\left(1 - \frac{1}{T}\right)\zeta(1) = \left(1 - \frac{1}{T}\right)\log_1(Z_1) = \left(1 - \frac{1}{T}\right)\log_1(1) = \frac{\log_1(T)}{T} \in K_\infty,$$

$$\zeta_T(1) = \frac{\log_{1,T}(Z_{1,T})}{T} = \frac{\log_{1,T}(T)}{T} \in K_T.$$

Note that by Proposition 1.6.1,

$$Z_{n,v} \equiv 0 \bmod v;$$

hence $\log_{n,v}(Z_{n,v})$ is defined.

THEOREM 3.8.3. (I) *There exists* $z_n \in K_\infty^n$ (*constructed explicitly in the proof*) *such that* $\exp_{n,\infty}(z_n) = Z_n$ *and*

$$z_n = \begin{pmatrix} \vdots \\ \Gamma_n \zeta(n) \end{pmatrix}.$$

(II) *For each monic irreducible element v of A,*

$$\log_{n,v} Z_{n,v} = \begin{pmatrix} \vdots \\ v^n \Gamma_n \zeta_v(n) \end{pmatrix}.$$

COROLLARY 3.8.4. *If $q - 1$ does not divide n, then Z_n is not a torsion point of $C^{\otimes n}$.*

Proof of corollary. Suppose that Z_n is a torsion point. Then by (I) of the theorem and Corollary 2.5.8, the last coordinate of z_n would be a K-multiple of $\bar{\pi}^n$ and, a fortiori, a K_∞-multiple. But this possibility is ruled out by Corollary 2.5.9. q.e.d.

Remark. Using Theorem 3.1.1, together with transcendence theory, Yu [Y2] has proved the converse to Corollary 3.8.4. For example, 1 is $T^2 + T$-torsion if $q = 2$, but non-torsion otherwise.

3.9. *Proof of Theorem* 3.8.3(I). For $0 \le i < n(q/q-1)$, the column vector Y_{ni} is in the region of convergence of $\log_{n,\infty}$ by Proposition 2.4.3. Therefore

$$y_{ni} =: \log_{n,\infty} Y_{ni} \in K_\infty^n$$

is defined and moreover,

$$Y_{ni} = \exp_{n,\infty}(y_{ni}).$$

Set

$$z_n =: \sum_i d[h_{ni}]_n y_{ni} \in K_\infty^n.$$

Then the sum above breaks off after finitely many terms and, in view of the functional equation satisfied by $\exp_{n,\infty}$,

$$\exp_{n,\infty}(z_n) = Z_n.$$

It remains to examine the last coordinate of z_n. By Proposition 2.1.5 and the remark made at the end of subsection 2.3,

$$z_n = \begin{pmatrix} \vdots \\ \displaystyle\sum_{i,k} h_{ni} \frac{T^{iq^k}}{L_k^n} \end{pmatrix}.$$

By (3.7.4) and the definition of the coefficients h_{ni}, the last coordinate of the right-hand side above is indeed $\Gamma_n \zeta(n)$. q.e.d.

3.10. In this paragraph we prepare for the proof of Theorem 3.8.3(II). Let X be a variable and let $K[[X]]$ be the ring of power series in X with coefficients in K. We define a map $\log_{n,X}$: $XK[[X]]^n \to XK[[X]]^n$ by the rule

$$\log_{n,X} g = \sum_{i=0}^{\infty} P_i g^{(i)}.$$

Here the coefficients P_i (see subsection 2.1 above) are the coefficients of the formal logarithm $\log_n$ associated to $C^{\otimes n}$, and $g^{(i)}$ denotes the column vector whose entries are the (q^i)th powers of the entries of g. Then $\log_{n,X}$ satisfies the expected functional equation:

$$\log_{n,X}([a]_n g) = d[a]_n \log_{n,X}(g) \qquad (a \in A,\ g \in XK[[X]]^n).$$

Let $K_v\{X\}$ denote the ring of power series with coefficients in K_v convergent in the "closed" v-adic unit disc $|X|_v \le 1$. Then, in particular, we may assign to each $g(X) \in K_v\{X\}^n$ of length n a value $g(1) \in K_v^n$ at $X = 1$ by summing v-adically. Now let $b \in A^n$ be a column vector with entries in A and let i be a nonnegative integer. Then, by Proposition 2.4.1, $\log_{n,X}(vX^i b)$ belongs to $K_v\{X\}^n$; clearly the result of evaluation at $X = 1$ is $\log_{n,v}(vb)$. Therefore, by the linearity of evaluation at $X = 1$, of $\log_{n,X}$ and of $\log_{n,v}$, the diagram below commutes:

(3.10.1)
$$\begin{array}{ccc} vXA[X]^n & \xrightarrow{g(X) \mapsto g(1)} & vA^n \\ {\scriptstyle \log_{n,X}}\downarrow & & \downarrow{\scriptstyle \log_{n,v}} \\ K_v\{X\}^n & \xrightarrow[g(X) \mapsto g(1)]{} & K_v^n \end{array}$$

3.11. *Proof of Theorem* 3.8.3(II). Set

$$z_n(X) = \sum_i d[h_{ni}]_n \log_{n,X}(XY_{ni}) \in XK[[X]]^n.$$

Then by the remark at the end of subsection 2.3 and formula (3.7.4),

$$z_n(X) = \begin{pmatrix} \vdots \\ \Gamma_n \sum_{k=0}^{\infty} \left(\sum_{\substack{a \in A_+ \\ \deg(a)=k}} \frac{1}{a^n} \right) X^{q^k} \end{pmatrix}.$$

Set

$$z_{n,v}(X) =: d[v^n]_n z_n(X) - z_n(X^{q^{\deg(v)}}).$$

Then

(3.11.1)
$$z_{n,v}(X) = \begin{pmatrix} \vdots \\ v^n \Gamma_n \sum_{k=0}^{\infty} \left(\sum_{\substack{a \in A_+ \\ (v,A)=1 \\ \deg(a)=k}} \frac{1}{a^n} \right) X^{q^k} \end{pmatrix}.$$

Set

$$Z_n(X) =: \sum_i [h_{ni}]_n (XY_{ni}) \in XA[X]^n.$$

Then $Z_n(X)$ is an $A[X]$-valued point of $C^{\otimes n}$ from which, upon evaluating at $X = 1$, one recovers the special point Z_n. Set

$$Z_{n,v}(X) =: [v^n]_n Z_n(X) - Z_n(X^{q^{\deg(v)}}) \in XA[X]^n.$$

Note that, upon evaluating $Z_{n,v}(X)$ at $X = 1$, we recover the v-twist $Z_{n,v}$ of the special point Z_n and that,

(3.11.2) $$z_{n,v}(X) = \log Z_{n,v}(X).$$

Now by Proposition 1.6.1,

$$Z_{n,v}(X) \in vA[X]^n,$$

and hence, by Proposition 2.4.1, the entries of the column vector $z_{n,v}(X)$ are power series convergent in the v-adic unit disk $|X|_v \leq 1$. By the commutativity of diagram (3.10.1),

$$z_{n,v}(1) = \log_{n,v} Z_{n,v}.$$

By (3.3.1) and (3.11.1), the last entry of $z_{n,v}(1)$ is $\zeta_v(n)$. q.e.d.

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Added in proof. Jing Yu has also proved the transcendence of the v-adic zeta values for n not divisible by $q - 1$.

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