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Gamma functions for function fields and Drinfeld modules

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Introduction

The purpose of this paper is to study gamma functions in the context of the theory of function fields of one variable over a finite field. In particular, we explore, among other things, their connection with the theory of Drinfeld modules of rank one (which is function field cyclotomic theory), their interpolations, functional equations, the nature of their special values (transcendence, algebraicity, connection with "periods") and various analogies. Most of the facts about the classical gamma function are recalled when necessary. Readers may refer to [G-K] or Artin's book on the classical gamma function.

Now we describe more precisely the contents of this paper. Let $\mathbf{F}_q$ be a finite field of characteristic p. Section 0 covers the background material on Drinfeld modules. It also describes the gamma function Γ for $\mathbf{F}_q[T]$ (with domain $\mathbf{N}$ and with values in $\mathbf{F}_q[T]$) introduced by Carlitz and its interpolations $\bar{\Gamma}$, Γ_v (with domain $\mathbf{Z}_p$ and with values in the completions of $\mathbf{F}_q(T)$ at its infinite and finite places respectively) due to Goss. In Section 1, we relate this gamma function to the Carlitz module (which is a special Drinfeld $\mathbf{F}_q[T]$-module) by showing essentially that

$$\bar{\Gamma}(0) = \bar{\tilde{\pi}}, \qquad \bar{\Gamma}(1/2) = \sqrt{\bar{\tilde{\pi}}},$$

where $\tilde{\pi}$ is a period of the Carlitz module (the analogue of $2\pi i$), and putting a bar over a quantity which "removes the degree part". The second (first) of the equations stated above, is in analogy to (in contrast to) what we know about usual gamma functions. We also recall results of [T2], [T3] (relating special values of Γ_v to Gauss sums of [T2], [T3]), a weak corollary of which gives values of $\Gamma_v(0)$ and $\Gamma_v(1/2)$ for example. In the third and fourth sections we define gamma functions for function fields and interpolate them at all places by finding a suitable generalization of constructions of Carlitz and Goss and prove analogous results about the special values. Following a suggestion of Gekeler, we also include the degree part of the gamma function.

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The second section proves functional equations for these gamma functions, giving in particular, analogues of the reflection and the multiplication formula. The functional equations turn out to be equivalent to some relations between digit expansions of certain combinations of p-adic numbers. Material in Sections 0–4 formed a part of the author's thesis [T2].

The gamma function considered in this part has domain in characteristic zero and it reflects arithmetic of cyclotomic extensions which are constant field extensions. In Section 5, we introduce another gamma function with domain (and range) in characteristic p, now reflecting the cyclotomic theory of Carlitz-Hayes-Drinfeld and we interpolate it at all places. In Section 6, we prove functional equations for this gamma function (at all places). One interesting consequence is transcendence of values of gamma at all proper fractions and algebraicity of values of v-adic gamma at all fractions when $q = 2$. The seventh section analyses these results and tries to put the visible analogies studied so far in a better framework of brackets $\langle \ \rangle$ and partial zeta functions, following a suggestion by Anderson (see [A1] and references there). Section 8 studies similar questions for a two variable gamma function of Goss [Go3], of which the two types of gamma functions studied above are essentially specializations. The last section contains miscellaneous comments, analogies and partial results verifying special cases of Chowla-Selberg phenomena for one and two variable gamma functions.

It should be noted that we only have partial results on analogues of Chowla-Selberg and Gross-Koblitz formulae. Also, the analogue of Deligne's theorem we prove gives trivial Hecke characters. It seems that recent results of Greg Anderson (private communication) will provide complete results covering also the relations to the higher-dimensional generalizations of Drinfeld modules studied in [A2].

Warning on the notation. Symbols $\Pi, \Gamma, \Gamma_v, \tilde{\pi}, D_i, \langle \ \rangle$ etc. signify different objects according to the context. This has been done to avoid cumbersome notation and to make analogies more visible. The change of notation will be made explicit each time. We follow the standard convention where the empty sum evaluates to zero and the empty product evaluates to one. A prime over the summation or the product symbols means that the variable runs only through nonzero values.

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0. Background

This section describes the work of Carlitz, Drinfeld and Goss relevant to this paper. More precisely, we explain the basic set-up of Drinfeld modules and describe the gamma function for $\mathbf{F}_q[T]$ introduced by Carlitz and its interpolations due to Goss.

0.1. *Drinfeld modules* (for more details, see [Dr], [Ge]).

0.1.0. Let K be a function field of one variable with field of constants $k =: \mathbf{F}_q$ (where q is a power of a prime p); let ∞ be a place of K, δ be the residue class degree of ∞, A be the ring of integers outside ∞ of K, K_∞ be the completion of K at ∞, k_∞ be the field of constants $\mathbf{F}_{q^\delta}$ of K_∞ and Ω be the completion of an algebraic closure of K_∞. It will be useful to keep in mind analogies

$$K \leftrightarrow \mathbf{Q}, \quad A \leftrightarrow \mathbf{Z}, \quad K_\infty \leftrightarrow \mathbf{R}, \quad \Omega \leftrightarrow \mathbf{C}.$$

For $z \in K_\infty^\times$, let $\deg(z) =: -\delta \operatorname{ord}_\infty(z)$. For a subfield L of Ω, L^{sep} will denote the separable closure of L inside Ω.

0.1.1. Drinfeld introduced the key concept of an elliptic module (also called a Drinfeld module) in the theory of function fields. A Drinfeld module is an analogue of the multiplicative group, elliptic curve or of an elliptic curve with complex multiplication. To motivate the definition, we recall that classically an elliptic curve is just $\mathbf{C}/\Lambda$ where the lattice Λ is a rank two $\mathbf{Z}$-submodule of $\mathbf{C}$. Similarly, $\mathbf{C}^\times \equiv \mathbf{C}/\Lambda$ with Λ being a rank one $\mathbf{Z}$-submodule of $\mathbf{C}$. For a finite extension L, inside Ω, of K_∞, a lattice Λ over L will be, by definition, a finitely generated discrete A-submodule in L^{sep}, invariant with respect to $\operatorname{Gal}(L^{\text{sep}}/L)$. An important point, though irrelevant for this paper, is that, in contrast to the classical case, lattices of arbitrary rank exist.

0.1.2. Given such a lattice Λ, the corresponding exponential function (the analogue of exponential, sine or Weierstrass σ-functions)

$$e_\Lambda(z) =: z \prod_{\lambda \in \Lambda}{}' (1 - z/\lambda)$$

is an entire additive function and induces a group isomorphism $\Omega/\Lambda \equiv \Omega$. Since Λ is an A module, we obtain Ω as an A module. In fact, for $a \in A$,

$a \neq 0$, by comparing divisors, we see that

$$e_\Lambda(az) = c \prod_{b \in (1/a)\Lambda/\Lambda} (e_\Lambda(z) - e_\Lambda(b)) = \rho_a(e_\Lambda(z))$$

where ρ_a is a polynomial function and $c \in \Omega$.

From the analytic object Ω/Λ, we extracted a nice algebraic object ρ. It gives an A-module structure on the additive group scheme $\mathbf{G_a}$, i.e., a nontrivial embedding

$$\rho: A \to \operatorname{End}_L \mathbf{G_a} = L\{F\}$$

where $L\{F\}$ is a twisted polynomial ring in the Frobenius, i.e., the ring generated by elements of L and by F with the commutation relation $Fl = l^p F$. Also, the constant term of ρ_a is just a. This leads to:

Definition 0.1.3. Let i: $A \to L$ be an embedding of A in a field L. The elliptic (or Drinfeld) A-module ("of generic characteristic" according to the standard terminology; but we will drop this phrase, as we will have no occasion to use the more general notion) over L is a homomorphism ρ: $A \to L\{F\}$, $(a \to \rho_a)$ such that the constant term of ρ_a is just $i(a)$ and ρ is not just i: $A \to L \subset L\{F\}$ (the nontriviality condition).

0.1.4. This is analogous to "elliptic curve over L with complex multiplication by A". One should keep in mind the classical situations $\mathbf{Z} \to \operatorname{End} G_m$, $\mathbf{Z} \to \operatorname{End} E$, where E is an elliptic curve, and $\mathcal{O}_F \to \operatorname{End} E$ (where F is an imaginary quadratic field and E is a complex multiplication elliptic curve). (In the first case, $n \in \mathbf{Z}$ acts by the n-th power map, which is reflected in the functional equation $(e^z)^n = e^{nz}$. Compare with the functional equation for e_Λ in 0.1.2.)

Definition 0.1.5. Let ρ, ρ' be Drinfeld A-modules over L.

(1) An isomorphism (isogeny resp.) over L from ρ to ρ' is a nonzero element $\mu \in L^\times$ ($\mu \in L\{F\}$ resp.) such that $\mu\rho_a = \rho'_a\mu$ for $a \in A$.

(2) If the degree of ρ_a (viewed as polynomial in F) is $-r(\deg a)(\log_p q)$ for all $a \in A$, then we say that ρ has rank r.

It can be shown that, for any Drinfeld module ρ such an r exists and is a natural number. Drinfeld showed (see [Dr] for the definitions of the categories mentioned below):

PROPOSITION 0.1.6. *If L is a finite extension of K_∞, then the category of elliptic A-modules of rank r over L is isomorphic to the category of A-lattices of rank r over L. (So, in particular, there exist elliptic A-modules of arbitrary rank over K_∞^{sep}.)*

0.1.7. We have already described how to pass from Λ to $\rho = \rho^{\Lambda}$ via $e_{\Lambda} = e_{\rho}$. On the other hand, given ρ, we can essentially solve for e_{ρ} via $e_{\rho}(az) = \rho_a(e_{\rho}(z))$ (with any nonconstant a) and then recover Λ as Ker e_{ρ}.

0.2. *The special case.*

0.2.0. Some forty years before Drinfeld defined the general notion, by different considerations, Carlitz [C1] came across the $\mathbf{F}_q[T]$-module over $\mathbf{F}_q(T)$ given by $T \to \rho_T = T - F$ (T being the generator of $\mathbf{F}_q[T]$, ρ is well-defined as an $\mathbf{F}_q$-algebra homomorphism). Also F is now the "q-th power Frobenius" (abuse of notation).

Following Carlitz, define $[i]$, D_i (Carlitz used F_i in place of now-standard D_i) for nonnegative integers i by

$$[i] =: T^{q^i} - T, \quad D_0 =: 1, \quad D_i =: [i] D_{i-1}^q.$$

He showed that the exponential corresponding to ρ is given by

$$e_{\rho}(z) = \sum_{h=0}^{\infty} (-1)^h z^{q^h} \Big/ D_h$$

and the corresponding lattice is $\tilde{\pi}\mathbf{F}_q[T]$. (This defines $\tilde{\pi}$ up to multiplication by $\mathbf{F}_q^{\times}$) where

$$\tilde{\pi} = \lim [1]^{q^k/(q-1)} \Big/ [k] \cdots [1].$$

See [C1] for details. This period $\tilde{\pi}$ was shown to be transcendental over $\mathbf{F}_q[T]$ by Wade [W].

0.2.1. Let μ_n denote the set of n-th roots of unity. By adjoining μ_n to $\mathbf{Q}$ or K one obtains cyclotomic extensions with abelian Galois group. Over $\mathbf{Q}$, by the Kronecker-Weber theorem, any finite abelian extension of $\mathbf{Q}$ is contained in some cyclotomic extension. Over K, the corresponding statement is, of course, false. In fact, Carlitz [C3] showed that adjoining the roots in Ω of "the cyclotomic equation" $\rho_a(z) = 0$, $a \in A = \mathbf{F}_q[T] - \{0\}$ (i.e., the set of a-division points (let us denote it by Λ_a) of the action by ρ) to $K = \mathbf{F}_q(T)$, we get an abelian extension with Galois group $(A/aA)^{\times}$. (This should be compared with the cyclotomic extension of $\mathbf{Q}$ obtained by adjoining the division points of $\mathbf{Z} \to \operatorname{End} G_m$ and the situation in Lubin-Tate theory.) Carlitz developed an analogous cyclotomic theory for $\mathbf{F}_q[T]$. This aspect has been greatly generalized by the work of Drinfeld and Hayes (see [Dr], [H1], [H2], [H3]). In particular, see [H1] for an analogue of the Kronecker-Weber theorem for $\mathbf{F}_q(T)$. Roughly, the abelian extensions $K(\Lambda_a)$ are tamely ramified over the infinite place. To get the maximal abelian extension of K, one has to let the infinite place vary through all the places of K, choose sgn-normalized (see Section 3 for the definition) rank-1 Drinfeld A-modules, for corresponding A's and then adjoin all torsion points thus obtained.

0.3. *The zeta function for* $\mathbf{F}_q[T]$.

0.3.0. The Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s \in C, \qquad \mathrm{Re}(s) > 1.$$

By interpreting n as the cardinality of the residue class ring (i.e., the norm) of a nonzero ideal (n) in $\mathbf{Z}$ and carrying over the interpretation to function fields, Artin developed zeta functions of a complex variable for function fields and showed that they gave an excellent analogue.

Carlitz thought of n as a positive or monic (there are two choices ± 1 of signs for $\mathbf{Z}$ whereas there are $q-1$ choices of "signs" in $\mathbf{F}_q^{\times}$ for $\mathbf{F}_q[T]$) representative of a nonzero ideal (n). Hence he considered [C1] zeta values

$$Z(s) =: \sum_{n \text{ monic in } \mathbf{F}_q[T]} 1/n^s, \qquad s \in \mathbf{N}, \qquad Z(s) \in \mathbf{F}_q((1/T)).$$

In other words, instead of the norm which just depends on the degree of the polynomial, he used the whole polynomial, trading it for a smaller domain for s. More justification lies in the obvious Euler product expression and in the following proposition [C1], [C2]. (Notation is further explained in 0.3.3.)

Proposition 0.3.1.

$$Z((q-1)m) = \frac{B_{(q-1)m}}{\Pi((q-1)m)} \tilde{\pi}^{(q-1)m}$$

where m is a positive integer, $B_{(q-1)m} \in \mathbf{F}_q(T)$, $\Pi((q-1)m) \in \mathbf{F}_q[T]$ and $\tilde{\pi}$ is as above.

0.3.2. Example:

$$Z(q-1) = \frac{\tilde{\pi}^{q-1}}{T^q - T}.$$

0.3.3. This proposition should be compared with

$$\zeta(2m) = \frac{B_{2m}}{2(2m)!}(2\pi)^{2m}$$

where $B_{2m} \in \mathbf{Q}, (2m)! \in \mathbf{Z}, 2\pi$ is transcendental over $\mathbf{Q}$. Hence one should think of multiples of $(q-1)m$ as analogues of even integers, $B_{(q-1)m}$ as analogues of Bernoulli numbers B_{2m} (Carlitz [C2] defined these by a generating function analogous to the classical case and proved von-Staudt-type congruences for them to justify the terminology), $\Pi((q-1)m)$ as factorials (see 0.4.1 for the definition of Π) and $\tilde{\pi}$ as an analogue of $2\pi i$ (these analogies will be explained later). For interpolations and more on zeta functions, see [Go1] and other papers by Goss.

0.4. *The factorial function for* $\mathbf{F}_q[T]$.

The factorial function Π for $\mathbf{F}_q[T]$, which appeared in Proposition 0.3.1 was defined by Carlitz [C2] as follows:

0.4.1. For a nonnegative integer n, $n = \sum n_i q^i$, $0 \le n_i \le q-1$, define the factorial function by

$$\Pi(n) =: \prod D_i^{n_i} \in \mathbf{F}_q[T].$$

In addition to its analogous occurrence in the special values of the zeta function (see Proposition 0.3.1), its prime factorization

$$\Pi(n) = \prod_{P \text{ monic prime}} P^{n_P}; \qquad n_P = \sum_{e=1}^{\infty} [n/NP^e]$$

(where the norm NP is the cardinality of the residue class field) is an exact analogue of the classical formula

$$n! = \prod_{p \text{ a positive prime in } \mathbf{Z}} p^{n_p}, \qquad n_p = \sum [n/Np^e].$$

This was noticed by W. Sinnott and is easy to prove from the definitions since $[i]$ is the product of all monic primes whose degree divides i and D_i is the product of all monic polynomials of degree i (see [C1, p. 140]).

0.4.2. *Remark*. This factorization formula does not hold in general (i.e., for general A), for the definition of gamma functions we propose in Section 3, or for the definition proposed by Goss. Also, Goss in his papers called what we call the factorial function, the gamma function. We put $\Gamma(n) =: \Pi(n-1)$ in accordance with the classical convention, for all factorials occurring in Sections 0–4.

0.4.3. Carlitz [C2] also showed that $\Pi(a)\Pi(b)$ divides $\Pi(a+b)$ (integrality of binomial coefficients).

0.4.4. Goss [Go2, Appendix] made interpolations of the factorial at all places of $\mathbf{F}_q[T]$ as follows. Since $D_i = T^{iq^i} - T^{(i-1)q^i+q^{i-1}} +$ lower degree terms, the unit part

$$\overline{D_i} =: D_i/T^{\deg D_i} = 1 - 1/T^{(q-1)q^{i-1}} + \cdots$$

tends to 1 in $\mathbf{F}_q((1/T))$ as i tends to ∞. So the unit part of $\Pi(n)$ interpolates to a continuous function (called the ∞-adic factorial, this is not canonical: we are implicitly choosing uniformizer $1/T$ at ∞; see [Go2] and also Section 3) $\overline{\Pi}(n)$;

$$\overline{\Pi}: \mathbf{Z}_p \to \mathbf{F}_q((1/T)), \qquad \sum n_i q^i \to \prod \overline{D}_i^{\,n_i}.$$

Also, since D_i is the product of all monic elements of degree i, we have a Morita-style v-adic factorial Π_v: $\mathbf{Z}_p \to \mathbf{F}_q[T]_v$ for finite primes v of $\mathbf{F}_q[T]$ given by

$$\Pi_v(n) = \prod (-D_{i,v})^{n_i}$$

where $D_{i,v}$ is the product of all monic elements of degree i, which are relatively prime to v, and n_i are digits in the q-adic expansion of n. This makes sense since $-D_{i,v} \to 1$ v-adically as $i \to \infty$, as can be seen [Go2] by generalizing the standard group theoretical argument proving $(p-1)! \equiv -1$ modulo p.

1. The gamma function and the Carlitz module

In the last section, we introduced the gamma function and described many reasons why it is a good analogue. The aim of this section is to explain the analogy further by relating the gamma function to Drinfeld modules.

1.1. Let C be a rank-one $\mathbf{F}_q[T]$-module, given by $C_T =: T + F$. Then from 0.2.0 it is easy to see that the corresponding exponential is given by $e_C(z) = \Sigma z^{q^j}/D_j$. But $D_j = \Pi(q^j)$. This should be compared with $e^z = \Sigma z^n/\Pi(n)$. We can rephrase this in a more striking fashion as follows.

Let η denote the contour starting at $-\infty$ on the real axis, encircling the origin once in the positive sense and returning to the starting point. Then classically (closely related to Euler's famous integral representation of the gamma function) we have Hankel's formula:

$$\frac{1}{\Pi(n)} = \frac{1}{2\pi i}\int_\eta e^z z^{-n}\frac{dz}{z}.$$

Here, for a positive integer n, $1/\Pi(n)$ is a visible residue of $e^z z^{-n}\,dz/z$ (at $z = 0$) by the Taylor series expansion; and from that point of view, the same holds (with the classical exponential replaced by e_C and $2\pi i$ replaced by a period (which is just $\tilde{\pi}$ of 0.2.0 multiplied by the $q-1$-th root of -1; see 3.13 for a more detailed discussion) corresponding to C) in the function field case, but now only for $n = q^j$. (We get 0 otherwise.) The general value is then obtained multiplicatively.

1.2. Another interesting analogy is as follows. Carlitz [C1] considered a linear operator Δ given by $\Delta f(x) = f(Tx) - Tf(x)$ on linear functions. (Note that in characteristic p, one has a wide variety of linear functions.) It satisfies the derivation rule $\Delta(f \circ g) = f \circ (\Delta g) + (\Delta f) \circ g$, where $\circ$ denotes composition. Consider $d_F =: \Delta^{1/q}$ which is linear and satisfies the derivation rule for composition if one restricts to "linear" power series with coefficients in $\mathbf{F}_q$. Interestingly, the functional equation $e_C(Tx) = Te_C(x) + e_C(x)^q$ is equivalent to $d_F e_C(x) = e_C(x)$ just as $d/dx(e^x) = e^x$ (in contrast to $d/dx(e_C(x)) = 1$). In fact, this holds term by term, and one has $d_F(x^{q^n}/\Pi(q^n)) = x^{q^{n-1}}/\Pi(q^{n-1})$ and also $d_F(x^{q^n}/[n]) = d_F(x^{q^{n-1}})$ analogous to $d/dx(x^n/n!) = x^{n-1}/(n-1)!$ and $d/dx(x^n/n) = x^{n-1}$ resp. Also note that one has twisted recursions $\Pi(q^{n+1}) = [n+1]\Pi(q^n)^q$ and $[n+1] = [n]^q + [1] = [n] + [1]^{q^n}$ in place of the usual $(n+1)! = (n+1)n!$ and $(n+1) = n+1$ resp.

1.3. Now we relate the special values of $\bar{\Gamma}$ to the period $\tilde{\pi}$. Note that we are still dealing with Carlitz' ρ and $\tilde{\pi}$. Similar statements can be made about C and the corresponding periods; indeed all these will be a special case of results in Section 3, where we also take care of the degree part and the signs. In the rest of this section, for $0 \neq M \in \mathbf{F}_q((1/T))$, $M/T^{\deg M}$ will be denoted by $\overline{M}$. Recall the Carlitz formula for the period $\tilde{\pi}$ given in 0.2.0. Namely,

$$\tilde{\pi} = \lim [1]^{q^k/(q-1)}/[1] \cdots [k]$$

so that $\tilde{\pi}^{q-1} \in \mathbf{F}_q((1/T))$ and $\overline{\tilde{\pi}^{q-1}}$ makes sense. By $\bar{\tilde{\pi}}$ we will denote its unique $q-1$-th root which is a one-unit in $\mathbf{F}_q((1/T))$. A similar remark applies to $\tilde{\pi}^{1/(q-1)}$.

THEOREM 1.4. *For* $0 \leq a \leq q-1$,

$$\bar{\Gamma}\left(1 - \frac{a}{q-1}\right) = (\bar{\tilde{\pi}})^{a/(q-1)}.$$

In particular, if $q \neq 2^n$, *then*

$$\bar{\Gamma}(1/2) = \sqrt{\bar{\tilde{\pi}}}.$$

Proof. Since $-1 = \Sigma(q-1)q^i$, from the definitions of 0.4.4, we have

$$\bar{\Gamma}(0) = \bar{\Pi}(-1) = \lim \overline{(D_0 \cdots D_n)^{q-1}}.$$

It is easy to see from the definition of D_i that

$$(D_0 \cdots D_n)^{q-1} = D_{n+1}/[1] \cdots [n+1].$$

Hence

$$\overline{\Gamma}(0)^{q-1}/\overline{\tilde{\pi}^{q-1}} = \lim \overline{D_{n+1}^{q-1}}/\overline{[1]^{q^{n+1}}} = 1$$

since we have already seen that $\overline{D_i} \to 1$, and since $\overline{[1]}^{q^n} \to 1$, because any one-unit raised to the q^n-th power tends to 1 as $n \to \infty$.

Hence we have proved $\overline{\Gamma}(0) = \overline{\tilde{\pi}}$. Observing that $a/(1-q) = \Sigma a q^i$ for $0 \leq a \leq q-1$, we get the theorem.

Corollary 1.5. *$\overline{\Gamma}(i - a/(q-1)), \overline{\Gamma}(n)$ are transcendental, where a, i, n are integers, $0 < a < q-1$ and $n \leq 0$.*

Proof. The corollary follows from the fact that

$$\overline{\Gamma}(z+1)/\overline{\Gamma}(z) = \overline{[1] \cdots [n]}$$

if q^n divides z, (in the sense of valuations for p-adic numbers), and q^{n+1} does not (the empty product being defined always as one; this excludes only $z = 0$). This fact, in turn, is immediate from the definitions together with the observation made above that $(D_0 \cdots D_n)^{q-1} = D_{n+1}/([1] \cdots [n+1])$.

To investigate the nature of gamma values at all fractions, it is sufficient to look at all $\overline{\Pi}(q^j/(1-q^t))$ for $0 \leq j \leq t$. They can be related to the periods $\tilde{\pi}_t$'s of Carlitz modules for $\mathbf{F}_{q^t}[T]$. For example:

Theorem 1.6. $\overline{\tilde{\pi}_t} = \overline{\Pi}(q^{t-1}/(1-q^t))^q/\overline{\Pi}(1/(1-q^t))$.

Proof.

$$\begin{aligned}\frac{\overline{\Pi}(1/(1-q^t))}{\overline{\Pi}(q^{t-1}/(1-q^t))^q} &= \lim \overline{\frac{D_{tn} D_{t(n-1)} \cdots D_0}{D_{tn-1}^q \cdots D_{t-1}^q}} \\ &= \lim \overline{[tn][t(n-1)] \cdots [t]} \\ &= \left(\overline{\tilde{\pi}_t}\right)^{-1}.\end{aligned}$$

This is the Chowla-Selberg formula for constant field extensions, as will be explained in Section 7. Similarly, it can be shown, for example, that

$$\overline{\Pi}(1/(1-q^2))^{q^2-1} = \overline{\tilde{\pi}^q \tilde{\pi}_2^{-(q-1)}},$$

$$\overline{\Pi}(q/(1-q^2))^{q^2-1} = \overline{\tilde{\pi} \tilde{\pi}_2^{q-1}}.$$

Transcendence of these combinations of periods follows from [Y1].

1.7. In [T2] (see also [T3], Th. V, Cor. and Th. VI), we introduced Gauss sums for $\mathbf{F}_q[T]$ and related some special values of the v-adic gamma function to these. This also implied algebraicity of some special values. In Section 4, we provide a simpler proof of a weak corollary (for the $\mathbf{F}_q[T]$ case) of this result, to make a parallel statement to the theorem above. (Compare it to the analogous nature of $\Gamma_p(1/2)$, for Morita's p-adic gamma function Γ_p.)

2. Functional equations

2.0. Recall that the classical gamma function satisfies (1) the reflection formula: $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ and (2) the multiplication formula:

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right)\Big/\Gamma(nz) = (2\pi)^{(n-1)/2}n^{1/2-nz}.$$

We will prove analogues of these and also their p-adic counterparts in the function field case. The proof naturally falls into two parts. The first one, which is the subject of Sections 1, 3, 4, evaluates $\Gamma(0)$, $\Gamma_v(0)$ (Thms. 3.10 and 4.4 resp.) and the other part, which is the subject of this section, establishes relations between gamma values in the abstract setting below (Thms. 2.6 and 2.8).

2.1. Consider a function f defined on $\mathbf{Z}_p$ by

$$f\left(\sum a_j q^j\right) =: \prod A_j^{a_j}$$

for some A_j's which can be thought of as independent variables with the evident manipulation rules. Put $g(z) = f(z-1)$. The various factorial ("f") and gamma ("g") functions introduced in Sections 0–4 are all of this form.

We want to get formal relations satisfied by f. In particular, we would like to know when $\prod f(x_i)^{n_i} = 1$ formally, i.e., independently of A_i's. In other words, if x_i has q-adic expansion $\sum x_{ij}q^j$, then we want to know about the kernel of the map

$$\sum_{\text{formal}} n_i(x_i) \to \left(\sum_i n_i x_{ij}\right)_j.$$

If $\sum n_i(x_i)$ is in the kernel, then

$$\sum n_i x_i = \sum n_i \sum x_{ij} q^j = \sum_j \left(\sum_i n_i x_{ij}\right) q^j = 0.$$

Lemma 2.2. *For* $z \neq 0$, $g(z+1)/g(z)$ *depends only on* $\operatorname{ord}_q(z)$.

Proof. This is obvious from the definition.

Lemma 2.3. $g(z)g(1-z) = g(0)$.

Proof. This follows (replace z by $z+1$) since $-1 = \sum(q-1)q^j$ and if $z = \sum z_j q^j$ then $-1-z = \sum(q-1-z_j)q^j$. Another way to prove this is to notice that $g(1) = f(0) = 1$; so it is true for $z = 0$. And since $\operatorname{ord}_q(z) = \operatorname{ord}_q(-z)$, by Lemma 2.2, we have $g(z+1)/g(z) = g(1-z)/g(-z)$; hence by induction it is true for all integers z. Integers being dense in $\mathbf{Z}_p$, the lemma follows easily.

LEMMA 2.4. *For $z \in \mathbf{Z}_p$ and $(n, q) = 1$,*

$$g(z)g\left(z+\frac{1}{n}\right)\cdots g\left(z+\frac{n-1}{n}\right)/g(nz) = g(0)^{(n-1)/2}.$$

(Here, if n is even, so that q is odd, then by $g(0)^{1/2}$ the element $\prod_{j=0}^{\infty} A_j^{(q-1)/2}$ whose square is $g(0)$ is meant.)

Proof. Since $\operatorname{ord}_q(z) = \operatorname{ord}_q(nz)$, Lemma 2.2 implies that

$$\frac{g(z)g\left(z+\frac{1}{n}\right)\cdots g\left(z+\frac{n-1}{n}\right)}{g(nz)} \cdot \frac{g(nz+1)}{g\left(z+\frac{1}{n}\right)g\left(z+\frac{2}{n}\right)\cdots g(z+1)}$$

$$= \frac{g(z)}{g(z+1)} \cdot \frac{g(nz+1)}{g(nz)} = 1$$

and again, as before, it is enough to prove the claim for a single z, say $z = 1/n$. So we want to prove $g(1/n)\cdots g((n-1)/n) = g(0)^{(n-1)/2}$, which follows from the reflection formula (Lemma 2.3), by pairing $g(a/n)$ with $g((n-a)/n)$.

2.5. It is amusing to note that Lemma 2.4 also follows immediately from well-known results (e.g., Hardy and Wright, Chapter 9) on digit expansions; namely, if $(n, q) = 1$, $-1/n$ has a purely recurring expansion of r recurring digits where r is minimal such that n divides $q^r - 1$. Essentially, the recurring digits for $-a/n$ are obtained by permutations of those of $-1/n$, so that the sum of the i-th digits of all of them is constant. This constant is easily seen to be $(q-1)(n-1)/2$, as $-1/n + \cdots + -(n-1)/n = -(n-1)/2 = ((n-1)/2)\Sigma(q-1)q^j$.

Summarizing, we have proved the following:

THEOREM 2.6. *Let $z \in \mathbf{Z}_p$, $(n, q) = 1$. Then*
(1) $g(z)g(1-z) = g(0)$;
(2) $g(z)g(z+1/n)\cdots g(z+(n-1)/n)/g(nz) = g(0)^{(n-1)/2}$.

2.7. We now give a more general functional equation (Theorem 2.8). For the motivation and application of this result, see Sections 4 and 7. (In particular, 4.7 and 7.3 will explain the analogy with the classical case and how it generalizes

Theorem 2.6.) Let N be a positive integer prime to p. For $x \in \mathbf{Q}$, define $\langle x \rangle$ by $x \equiv \langle x \rangle$ modulo $\mathbf{Z}$, $0 \leq \langle x \rangle < 1$. If $\underline{a} = \Sigma m_i[a_i]$ ($m_i \in \mathbf{Z}$, $a_i \in (1/N\mathbf{Z} - \{0\})$) is an element of the free abelian group with basis $(1/N\mathbf{Z}) - \{0\}$, put $n(\underline{a}) =: \Sigma m_i \langle a_i \rangle$. Also, for $u \in (\mathbf{Z}/N\mathbf{Z})^{\times}$, let $\underline{a}^{(u)} =: \Sigma m_i[ua_i]$. By abuse of notation, we put

$$f(-\underline{a}) =: \prod f(-\langle a_i \rangle)^{m_i}.$$

Consider the hypothesis

$$(**) \qquad n\left(\underline{a}^{(q^j)}\right) \text{ is an integer independent of } j.$$

THEOREM 2.8. *If* $(**)$ *holds, then*

$$f(-\underline{a}) = f(-1)^{n(\underline{a})}.$$

Proof. Without loss of generality, we may assume that $N = q^r - 1$. Put $\underline{a} = \Sigma m_i a_i$ and

$$Na_i = b_i = b_{r-1,i} q^{r-1} + b_{r-2,i} q^{r-2} + \cdots + b_{0,i}.$$

For $1 \leq j \leq r - 1$, $(**)$ is equivalent to

$$\left(\sum m_i b_i\right)/N = \left(\sum m_i b_i q^j\right)/N - \sum m_i b_{r-1,i} q^{j-1} - \cdots - \sum m_i b_{r-j,i}.$$

So

$$\begin{aligned} f(-\underline{a}) &= \prod f\left(\frac{-q^j}{N}\right)^{\Sigma m_i b_{j,i}} \\ &= f\left(\frac{1}{1-q}\right)^{\Sigma m_i b_{r-1,i}} \prod f\left(\frac{-q^j}{N}\right)^{\Sigma m_i (b_{j,i} - b_{r-1,i})}. \end{aligned}$$

Claim. $\Sigma m_i(b_{j,i} - b_{r-1,i}) = 0$.

This is obvious if $j = r - 1$; suppose it is true for $j = r - 1, \ldots, r - t + 1$. Then $(**)$ for $j = t$ says that

$$\begin{aligned} \frac{\Sigma m_i b_i (q^t - 1)}{N} &= \left[\left(\sum m_i b_{r-1,i}\right)\left(q^{t-1} + q^{t-2} + \cdots + 1\right)\right] \\ &\quad - \left[\sum m_i b_{r-1,i} - \sum m_i b_{r-t,i}\right]. \end{aligned}$$

Now $(**)$ for $j = 1$ gives $(\Sigma m_i b_i (q-1))/N = \Sigma m_i b_{r-1,i}$ and hence $(\Sigma m_i b_i (q^t - 1))/N$ is equal to the first $[\cdots]$, and thus the second $[\cdots]$ is zero, proving the claim for $j = r - t$. Hence induction completes the proof.

3. General A

In this section we define the gamma function in general and relate its value at 0 to the period of the appropriate Drinfeld module.

3.1. *Definitions*. Recall the notation in 0.1.0. Note that for a in A, deg a is just the usual notion; i.e., $\deg a = \dim_{\mathbf{F}_q}(A/aA) =$ degree of the divisor of zeroes of a. The degree is always a multiple of δ. For an ideal $\mathscr{A}$ of A, put

$$\mathscr{A}_N =: \{a \in \mathscr{A}: \deg a \leq N\}.$$

Then the Riemann part of the Riemann-Roch theorem shows that $\#\mathscr{A}_{i\delta} = q^{i\delta+c}$ for $i \in \mathbf{N}$, $i \gg 0$, c some constant.

Choose a uniformizer t at ∞. This gives us splitting of $K_\infty^\times$ as $\mathbf{F}_{q^\delta}^\times \times U^1 \times t^{\mathbf{Z}}$ where U^1 is the group of one-units at ∞. In other words, z in $K_\infty^\times$ can be written uniquely as $z = \operatorname{sgn}(z) \times \bar{z} \times t^n$, with $\operatorname{sgn}(z) \in k_\infty^\times$, $\bar{z} \in U^1$, $n \in \mathbf{Z}$.

This gives a homomorphism sgn: $K_\infty^\times \to k_\infty^\times$ such that $\operatorname{sgn}(U^1) = 1$, $\operatorname{sgn}(t) = 1$, $\operatorname{sgn}(a) = a$ for $a \in k_\infty^\times$. There are $q^\delta - 1$ such sgn functions, depending on the choice of t. For $\sigma \in \operatorname{Gal}(k_\infty/\mathbf{F}_q)$, $\sigma \circ \operatorname{sgn}$ is called a twisted sign function and a Drinfeld module ρ is called sign-normalized (see [H3, p. 224]) if the leading coefficient of ρ is a twisting of sign.

By [H3, p. 224], every rank-one Drinfeld A-module over Ω is isomorphic to a sign-normalized ρ, for chosen sgn.

So fix a sign function (in 3.13–3.14 we discuss how the situation depends on the choice of sgn function) and let ρ be a corresponding sign-normalized rank-one Drinfeld A-module with corresponding rank-one lattice Λ and exponential (or sine) $e_\rho = e_\Lambda$. Since A is a Dedekind domain, Λ is isomorphic to an ideal, say $\mathscr{A}$, of A. Choose such an $\mathscr{A}$, and let $\tilde{\pi} \in \Omega$ be a corresponding "period" defined up to an element in $\mathbf{F}_q^\times$ by the equation $\Lambda = \tilde{\pi}\mathscr{A}$. Think, if you will, of this period $\tilde{\pi}$ of ρ as an analogue of period $2\pi i$ (up to ± 1) in the situation $\mathbf{Z} \hookrightarrow \operatorname{End} G_m$.

3.2. In this setting, E. U. Gekeler ([Ge, p. 36]) gave a nice formula, based on the formula ((*) below) of Hayes ([H3, p. 233]), for the period $\tilde{\pi}$. This formula will now be described (see also [Ge]) and reinterpreted as the identity $\Gamma(0) = \tilde{\pi}$ multiplied by a root of unity.

Let x be an element of A of degree > 0, say of degree d and with $\operatorname{sgn}(x) = 1$. Then, since ρ is sign-normalized of generic characteristic, we have

$$\rho_x(u) = u^{q^d} + \cdots + xu.$$

For each $a \in \mathscr{A} \bmod x\mathscr{A}$, $e_\rho(a\tilde{\pi}/x) = \tilde{\pi} e_{\mathscr{A}}(a/x)$ is an xA-torsion point of Ω

viewed as A-module via ρ. (Since

$$\rho_x^{\Lambda}\big(e_\rho(u)\big) = e_\rho(xu)$$

we have

$$\rho_x^{\Lambda}(\tilde{\pi} e_{\mathscr{A}}(a/x)) = \rho_x^{\Lambda}(e_{\tilde{\pi}\mathscr{A}}(\tilde{\pi} a/x)) = e_{\Lambda}(\tilde{\pi} a) = 0.)$$

In other words, we have the following commutative diagram whose rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \longrightarrow & \Omega & \xrightarrow{e_\Lambda} & \Omega & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \rho_a^{\Lambda}} & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \Omega & \xrightarrow{e_\Lambda} & \Omega & \longrightarrow & 0.
\end{array}$$

So these are distinct roots of $\rho_x(u)$. Since $(-1)^{q^d-1} = 1$ in characteristic p, comparison of the coefficients of t gives

$$x = \prod_{a\in\mathscr{A}/\mathscr{A}x}{}' \tilde{\pi} e_{\mathscr{A}}(a/x) = \tilde{\pi}^{q^d-1}\prod{}' e_{\mathscr{A}}(a/x).$$

Now

$$\tilde{\pi}^{1-q^d} = \frac{1}{x}\prod_{a\in\mathscr{A}/\mathscr{A}x}{}' e_{\mathscr{A}}(a/x).$$

This implies $\tilde{\pi}^{q^d-1} \in K_\infty$. We will see later that $\tilde{\pi}^{q^\delta-1} \in K_\infty$. Hence,

$$(*) \qquad \tilde{\pi}^{1-q^d} = 1/x \prod_{a\in\mathscr{A}/\mathscr{A}x} (a/x) \prod_{b\in\mathscr{A}} (1 - a/xb)$$

$$= x^{-q^d}\Big(\prod a \prod (xb - a)\big/xb\Big)$$

which is the limit of the same expression with $b \in \mathscr{A}$ replaced by $b \in \mathscr{A}_{N\delta}$ as N tends to infinity. (This follows, since the exponential itself is such a limit.) Now $(xb - 0)/xb = 1$, so that if N is large, then the numerator of the right-hand side of $(*)$, with $xb - 0$ allowed, is just the product of nonzero elements of $\mathscr{A}_{n\delta+d}$, whereas the denominator is $x^{(\#\mathscr{A}_{N\delta}-1)*q^d}$ times the q^d-th power of the product of nonzero elements of $\mathscr{A}_{N\delta}$ (as there are q^d a's).

Take the one-unit part of both sides and notice that

$$\bar{x}^{-q^d} * \bar{x}^{-(\#\mathscr{A}_{N\delta}-1)*q^d} = \bar{x}^{-\#\mathscr{A}_{N\delta}} = \bar{x}^{-q^{N\delta+c}} \to 1$$

since a one-unit raised to the q^r-th power tends to 1 as $r \to \infty$.

Hence,

$$\overline{\tilde{\pi}^{1-q^d}} = \overline{\left(\prod_{a\in\mathscr{A}_{N\delta+d}}' a\right)} \Big/ \overline{\left(\prod_{b\in\mathscr{A}_{N\delta}}' b\right)^{q^d}}.$$

Keeping in mind that we want to interpret this as

$$\overline{\Gamma}(0)^{1-q^d} = \overline{\Pi}(-1)^{1-q^d} = \left(\left(\overline{D_0 D_1} \cdots\right)^{q-1}\right)^{1-q^d}$$

suggests the following definitions for D_i, $\overline{D_i}$, $\overline{\Gamma}$:

3.3. First notice that, the field of constants being $\mathbf{F}_q$, if $x \in \mathscr{A}_i$ then $ax \in \mathscr{A}_i$, $a \in \mathbf{F}_q^\times$, so that the signs (elements of $\mathbf{F}_{q^\delta}^\times$) appear in $\mathbf{F}_q^\times$-equivalence classes. Choose representatives for $\mathbf{F}_{q^\delta}^\times/\mathbf{F}_q^\times$ and let D_i be the product of all elements a of $\mathscr{A}$ of degree $i\delta$ with $\operatorname{sgn}(a)$ being one of these representatives. Now $D_i \in \mathscr{A} \subset$ A. Also let d_i be the number of these elements. The one-unit part $\overline{D}_i$ is obviously independent of the choice of representatives. (Notice that for $\delta = 1$ and $1 \in \mathbf{F}_q^\times/\mathbf{F}_q^\times$ as the representative, D_i = product of all monic (i.e., with sign 1) elements in A of degree i.)

3.4. With these definitions, our equation becomes

$$\overline{\tilde{\pi}^{1-q^d}} = \lim_{N\to\infty} \left(\overline{D_0} \cdots \overline{D_{N+d/\delta}}\right)^{q-1} \Big/ \left(\left(\overline{D_0} \cdots \overline{D_N}\right)^{q-1}\right)^{1-q^d}.$$

So

$$\lim\left(\overline{D_{N+d/\delta}}^{q-1}\right) \Big/ \left(\overline{D_N}^{q-1}\right)^{q^d} = 1.$$

But in characteristic p, q^d power spreads out the power series expansion, so that, since $\overline{D_i}$ is a one-unit, we get $\overline{D_i}^{q-1} \to 1$, and $\overline{D_i} \to 1$.

Hence $\overline{\Gamma}\colon \mathbf{Z}_p \to K_\infty$, given by

$$\overline{\Gamma}\left(1 + \sum \alpha_i q^i\right) = \prod \overline{D_i}^{\alpha_i},$$

is well-defined and

$$\overline{\tilde{\pi}^{1-q^d}} = \overline{\Gamma}(0)^{1-q^d}.$$

In 3.8, we will prove the following:

LEMMA 3.5.

$$\gcd\{q^d - 1\colon d = \deg(x),\, x \in A,\, \operatorname{sgn}(x) = 1\} = q^\delta - 1.$$

Hence

$$\overline{\tilde{\pi}^{1-q^\delta}} = \overline{\Gamma}(0)^{1-q^\delta}.$$

If $\bar{\tilde{\pi}}$ is that $(1-q^\delta)$-th root of $\overline{\tilde{\pi}^{1-q^\delta}}$ which is a one-unit, then $\overline{\Gamma}(0) = \bar{\tilde{\pi}}$.

3.6. Gekeler showed how to interpret the degree as $\deg \Gamma$ (or $\log \Gamma$). Analysing the degrees on both sides of the equation, we get

$$\begin{aligned}\deg \tilde{\pi}^{q^d-1} &= \lim\left(q^d \sum_{b\in\mathscr{A}_{N\delta}}' (\deg b) - \sum_{a\in\mathscr{A}_{N\delta+d}}' (\deg a) + d\#\mathscr{A}_{N\delta}\right)\\ &= (q^d-1)\sum (q-1)i\delta d_i\end{aligned}$$

where $d_i = (q^\delta - 1)q^{i\delta+c}/(q-1) \to 0$ q-adically as i tends to infinity (by Riemann's theorem).

This implies that the map $\mathbf{N} \to \mathbf{Z}$ given by $z \to \deg \Pi(z)$ interpolates to a continuous function $\deg \Pi\colon \mathbf{Z}_p \to \mathbf{Z}_p$ given by $\Sigma z_i q^i \to \Sigma i\delta z_i d_i$. Since $-1 = \Sigma(q-1)q^i$, we get $\deg \Pi(-1) = \Sigma(q-1)i\delta d_i$. So $\deg \Pi(-1) = \deg \tilde{\pi}$ and hence if we change the range of the gamma function, so as to make sense out of p-adic integral powers of t, we would get $\Pi(-1) = \tilde{\pi}$ times a root of unity, if we define $\Pi(z) = \overline{\Pi}(z)t^{-\deg \Pi(z)/\delta}$. (Note that we use symbol Π again, as we have recovered the degree part also.) These would then be independent of the choice of the uniformizer, if it gives the same sign function.

3.7. The most natural way to do this is to complete $K_\infty^\times$ p-adically; i.e., define $\hat{K}_\infty^\times =: \varprojlim K_\infty^\times / K_\infty^{\times p^n}$. Since finite fields are perfect, signs in $K_\infty^\times$ project to 1 in $\hat{K}_\infty^\times$. We obtain a gamma function which is independent of the choice of sign.

If $\delta > 1$, as J. Tate pointed out to me, one can do a little better. Since d_i, for large i, is not only divisible by a large power of q, but also by $(q^\delta - 1)/(q-1)$, we can evidently put $\tilde{K}_\infty^\times =: \varprojlim K_\infty^\times / K_\infty^{\times(q^\delta-1)p^n/(q-1)}$ and take it as the range. The signs in $\mathbf{F}_{q^\delta}^\times$ survive now in $\mathbf{F}_{q^\delta}^\times/\mathbf{F}_q^\times$. Summarizing, one obtains $\Pi\colon \mathbf{Z}_p \to \tilde{K}_\infty^\times$ with $\Pi(z) = \overline{\Pi}(z)t^{-\deg \Pi(z)/\delta}$.

3.8. Before analysing the root of unity (i.e., the sign from $(*)$) and the question of the variation of the situation with respect to the choices we make, we first prove the lemma above.

Proof of Lemma 3.5. It is enough to prove that the gcd of d's is δ. First of all, by Riemann's theorem we know that the gcd of degrees of elements of A is δ. Next, since K is dense in K_∞, there is an element of degree 0 in K of any given sign. Multiplying by elements in $\mathscr{A}$ of high degree, clearing the denominators we see that $\mathscr{A}$ has, for some large i, elements of degree i of all signs.

Now, choose elements x_k's in A of degrees d_k such that the gcd of d_k's is δ. Then multiplying by powers of x_k, we get all signs in degree $i + nd_k$ for all positive n, so that the gcd in question divides the gcd of $i + nd_k$ which is δ, and the lemma is proved.

3.9. Now we analyse the signs in $(*)$. The sign of the right-hand side is the limit as $N \to \infty$ of the sign of the product of nonzero elements of $\mathscr{A}_{N\delta+d}$ divided by the product of nonzero elements of $\mathscr{A}_{N\delta}$ (as the q^d-th power is the identity on $\mathbf{F}_{q^\delta}$). When d varies, a simple gcd argument shows that $\tilde{\pi}^{1-q^\delta} = \varepsilon\Gamma(0)^{1-q^\delta}$ where the sign ε is the stationary limiting sign (we have shown that it exists; for a more direct proof see [Ge, p. 30]) of the product of elements of $\mathscr{A}$ of degree $N\delta$ as $N \to \infty$. If $\delta = 1$, as $\prod_{a \in \mathbf{F}_q^\times} a = -1$, and as $(-1)^{q^d} = -1$ it follows by straightforward counting using Riemann's theorem that $\varepsilon = -1$. Hence,

$$\Gamma(0)^{q-1} = -\tilde{\pi}^{q-1} \in \left(K_\infty^\times\right)_{\mathrm{sgn}=1} \subset \hat{K}_\infty^\times$$

as $(q-1)\deg\Gamma(0) \in \mathbf{Z}$ rather than just in $\mathbf{Z}_p$.

We summarize the discussion in the following:

THEOREM 3.10.

$$\Gamma(0) = \mu\tilde{\pi}$$

(*where* μ *is at most a* $(q^\delta - 1)^2$-th *root of unity. If* $\delta = 1$, *it is a* $(q-1)$-th *root of* -1), *in the sense that* $(q^\delta - 1)$-th *powers of both sides are the same, considered in* $K_\infty^\times$.

3.11. Jing Yu ([Y1, Th. 5.1]) has proved transcendence of $\tilde{\pi}$, which then implies transcendence of some special values of the gamma function as in Section 1.

3.12. Using this value of $\Gamma(0)$ in Theorem 2.6, we see that the powers of $\tilde{\pi}$ appearing are the same as in the classical formulae 2.0, but some interesting algebraic parts are missing.

3.13. It should be noted that in the case $A = \mathbf{F}_q[T]$ studied in Section 0, Γ and ρ did not correspond correctly from the point of view of this section. If uniformizer T corresponds to sign function sgn and for $a \in \mathbf{F}_q^\times$, T/a corresponds to sgn' say, then $C_T = T + F$, $\rho'_T = T = aF$ are sgn-normalized for sgn and sgn' respectively. So ρ is sgn-normalized for the uniformizer $-T$ rather than T. Now C and ρ' are isomorphic. Suppose the corresponding lattices are Λ and $\mu\Lambda$. Then $\mu^{-1}(T + aF)\,\mu = (T + F)$; i.e., $\mu^{q-1}a = 1$. In other words, in this case, change of sign function changes $\tilde{\pi}$ up to a $(q-1)$-th root of an element in $\mathbf{F}_q^{\times}$.

3.14. To summarize, given a rank-one Drinfeld module, we get an ideal class in A. If an ideal $\mathscr{A}$ is chosen representing the class, we get a gamma function and if in addition a sgn function is chosen $\tilde{\pi}$ is defined up to $\mathbf{F}_{q^\delta}^{\times}$ and $\Gamma(0) = \tilde{\pi}$ times a root of unity (at most $(q^\delta - 1)^2$-th in general and at most $2(q-1)$-th if $\delta = 1$). Observe that, if $q = 2$ and $\delta = 1$, the question of signs disappears. A change of sgn function may change $\tilde{\pi}^{q^\delta - 1}$ by an element in $\mathbf{F}_{q^\delta}^{\times}$, but leaves the formula intact. The Γ function is independent of choices of coset representatives of $\mathbf{F}_{q^\delta}^{\times}/\mathbf{F}_q^{\times}$, and depends on the uniformizer only through its sign. If t and t' are uniformizers with $t = at'$, $a \in \mathbf{F}_{q^\delta}^{\times}$, then $a^{\deg \Gamma/\delta}\Gamma = \Gamma'$; so there are $(q^\delta - 1)/(q-1)$ gamma functions (unique if $\delta = 1$) and the gamma function is independent of the sgn choice on a smaller disc of $\mathbf{Z}_p$. Notice that if you use the gamma function for A (i.e., choose $\mathscr{A} = A$) and use ρ (to get period $\tilde{\pi}$) which need not correspond to the lattice giving the principal ideal class, still the relation in the theorem is true up to an algebraic number, since all rank-one normalized Drinfeld A-modules are isogenous.

4. Interpolations at the finite places

In this section we show that the gamma function interpolates at all finite places v of A which are relatively prime to $\mathscr{A}$ and that $\Gamma_v(0) = (-1)^{\deg v - 1}$. We also establish an analogue of Deligne's theorem in this context.

4.1. Let v be a finite place of A relatively prime to $\mathscr{A}$, with residue class degree h. We form $\tilde{D}_i =: D_{i,v}$ as usual by removing the factors divisible by v.

Definition 4.1.1. Let $\tilde{D}_i$ be the product of elements a of degree $i\delta$ with $\operatorname{sgn}(a)$ one of the chosen representatives and $v(a) = 0$. Let S_i be the set of these elements.

4.2. We will prove $(-1)^\delta \tilde{D}_i \to 1$.

Definition 4.2.1.

$$\Pi_v\left(\sum z_i q^i\right) =: \prod \left((-1)^\delta \tilde{D}_i\right)^{z_i}$$

so that $\Pi_v \colon \mathbf{Z}_p \to K_v$. Even though D_i depends on a choice of representatives for $\mathbf{F}_{q^\delta}^{\times}/\mathbf{F}_q^{\times}$, $\tilde{D}_i$ for large i does not, because the number of elements in S_i of given sign is a multiple of $q^h - 1$ and $(q^h - 1)$-th power kills the choice. Similarly it is independent of the choice of sgn for large i. So Γ_v is again unique on a smaller disc of $\mathbf{Z}_p$. In any case, a value $\Gamma_v(z)$ is determined up to multiplication by an element in $\mathbf{F}_q^\times$. (So again, there is a unique function for $q = 2$.)

LEMMA 4.3. $(-1)^\delta \tilde{D}_i \to 1$.

Proof. It is enough to prove:

Claim. There is an integer l such that for $i \gg 0$, $(-1)^\delta \tilde{D}_i \equiv 1 \bmod v^w$ with $w = [i\delta/h] - l$.

Proof. Assume that i is sufficiently large. Then the product of elements of $(\mathscr{A}/\mathscr{A}v^w)^\times$ is $-1 \bmod v^w$, by a generalization of Wilson's theorem. (Notice that we need i (and hence w) large to control elements of order two; for example, if $A = \mathscr{A} = \mathbf{F}_q[T]$ then the relevant product is not -1 precisely when $q = 2$, v of degree 1 and $w = 2, 3$.) Now the elements of S_i are equidistributed among the cosets, since elements of S_i with fixed sign are. (This is because, for large i, all members of one coset can be transferred to any other by subtracting coset-representatives of degree less than $i\delta$, leaving the sign unchanged.) This takes care of the case $p = 2$; so now assume that the characteristic is odd. It is enough to prove that the number of elements of S_i in any coset is $\equiv \delta \bmod 2$. Now $\#\{a \in \mathscr{A}\colon \deg a \le k\delta\} = q^{k\delta + c}$, by Riemann's theorem, so that

$$\#\{a \in \mathscr{A}\colon \deg a = i\delta \text{ with } \operatorname{sgn}(a) \text{ one of the chosen}\}$$
$$= (q^\delta - 1)/(q-1)q^{(i-1)\delta + c}.$$

Now $\#S_i = ((q^\delta - 1)/(q-1))q^{(i-1)\delta - h + c}(q^h - 1)$, but the number of cosets is $(q^h - 1)q^{hw}$, so that the required number is

$$\{(q^\delta - 1)/(q-1)\}q^r \equiv (q^\delta - 1)/(q-1) \equiv \delta \bmod 2$$

as claimed. (Here l is chosen so that the equidistribution works and r is positive.) Hence the lemma is proved.

THEOREM 4.4. $\Gamma_v(0) = (-1)^{\deg v - 1}$ *for all* v *prime to* $\mathscr{A}$. *For* $0 \le a \le q - 1$, *the* $\Gamma_v(1 - a/(q-1))$ *are roots of unity and* $\Gamma_v(b/(q-1))$ *is algebraic for* $b \in \mathbf{Z}$.

Proof. The first statement of the theorem implies the rest and the first statement will follow, if we show that

$$(\Pi m)^{q-1} \equiv (-1)^{h-1}$$

mod v^{l_i} where m runs through monic polynomials prime to v and of degree not more than t_i and with $l_i, t_i \to \infty$ as $i \to \infty$. Given l_i, choose t_i so that $\{am\}$, as a runs through $\mathbf{F}_q^\times$, spans the reduced residue class system mod v^{l_i}. (For example, in the $\mathbf{F}_q[T]$ case, $t_i = hl_i - 1$ works.) Then it is easy to see that $\{am\}$ covers each reduced residue class an equal number (which is a power of q) of times. Hence we have

$$-1 \equiv (\Pi a)^{\#\{m\}}(\cdots)^{q-1}.$$

But $\Pi a = -1$, so we are done if $p = 2$. Assume p is not two: then we have to show that $\#\{m\} \equiv h \bmod 2$. But for some c we have

$$\#\{m\} = (q^{t_i - c} - q^{t_i - c - h})/(q-1) \equiv (q^h - 1)/(q-1) \equiv h \bmod 2.$$

This finishes the proof.

Using congruences to decide the root, one can pin down the roots of unity mentioned in the theorem.

4.5. For the rest of this section, we use the notation of 2.7. Let us recall some notation and results of [T3]. (For comparison with the classical case, see [De]. or [G-K].) Let $A = \mathbf{F}_q[T]$. Let N be a positive integer prime to p as in 2.7. Choose r such that N divides $q^r - 1$ and let $F = \mathbf{F}_{q^r}(T)$. For prime P of $\mathbf{F}_{q^r}[T]$, the "Gauss sum" $g(\underline{a}, P)$ is defined as in [T3, p. 111]. From the analogue of Weil's theorem on Jacobi sums as Hecke characters, proved in [T3, Thm. IX, p. 111] (also see forthcoming paper by D. Hayes proving it in full generality using different definitions and approach), we see that if $\underline{a}$ satisfies condition $(**)$ of 2.7, then $g(\underline{a}, P)/(\mathbf{N}P^{n(\underline{a})})$ is a Hecke character $\chi_{\underline{a}}(P)$ for F of finite order. More precisely $\chi_{\underline{a}}(P) = (-1)^{r \deg Pn(\underline{a})}$.

Now we prove the analogue of a theorem of Deligne ([De, p. 91]) in our situation. Note that we are using the Drinfeld module C of 1.1 to define the Gauss sums mentioned above and the corresponding period $\tilde{\pi}$ in Theorem 4.6 below.

If one takes a general A, with $\delta = 1$ and chooses a sgn-normalized Drinfeld module and a corresponding period $\tilde{\pi}$, Theorem 4.6 and its proof carry over word for word, when one takes $\chi_{\underline{a}}(P) = (-1)^{r \deg Pn(\underline{a})}$ as a definition of $\chi_{\underline{a}}$ (as a character of $F =: K(\mu_{q^r - 1})$ in this case). But in this general case, one loses the connection with the Gauss sums of [T3].

THEOREM 4.6 (analogue of Deligne's theorem). *If condition* $(**)$ *of* 2.7 *holds, and if* $\Omega_{\underline{a}} =: \Pi(-\underline{a})/\tilde{\pi}^{n(\underline{a})}$ *(this definition is to be understood in the sense explained in Theorem* 3.10), *then* $\Omega_{\underline{a}}^{\tau}/\Omega_{\underline{a}} = \chi_{\underline{a}}(\tau)$ *for any* $\tau \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$.

Proof. Theorem 3.10 shows that $M =: \Gamma(0)/\tilde{\pi} = (-1)^{1/(q-1)}$; if Frob_P denotes the Frobenius ($q^{r \deg P}$-th power), then $\mathrm{Frob}_P\, M/M = (-1)^{r \deg P}$.

It is sufficient to look at the action of $\tau = \mathrm{Frob}_P$. By Theorem 2.8 and by the analogue of Weil's theorem mentioned above, we see that both sides are $(-1)^{r \deg Pn(\underline{a})}$. Hence the theorem follows.

4.7. *Remark*. Note that $\text{Gal}(\mathbf{Q}(\mu_N)/\mathbf{Q})$ is $(\mathbf{Z}/N\mathbf{Z})^\times$ whereas $\text{Gal}(K(\mu_N)/K)$ in our case consists of Frobenius powers. Hence (see 7.2–7.3 for more details) the classical counterpart of condition $(**)$ is (condition 1.13 of [G-K]) "$n(\underline{a}^{(u)})$ is an integer independent of $u \in (\mathbf{Z}/N\mathbf{Z})^\times$". This being a much stronger condition, the gamma function in the function field case satisfying "more" relations, we can handle more $\underline{a}$'s and at the same time, we can prove the full result and not just up to sign (see [G-K], remark after Theorem 4.5). We do not need to use Kubert or Koblitz-Ogus results [K-O]. It should be noted that the multiplication and the reflection formulae follow easily from Theorem 2.8 (since they arise just as in the classical case (see, e.g., p. 577 of [G-K]) from $\underline{a}$'s satisfying the stronger condition mentioned above). Also, in those cases, we can use $\Gamma(\underline{a})$ instead of $\Pi(-\underline{a})$ in the definition of $\Omega_{\underline{a}}$.

Remark. Instead of using the analogue of Weil's theorem to get an expression for $\chi_{\underline{a}}$, we could have used the analogue of the Gross-Koblitz Theorem [T3, Thm. VI]. It immediately implies Theorem 4.8 below. (Since we will not use the rest of the section anywhere else, we use the notation of [T3] without further explanation.)

THEOREM 4.8 (for $\mathbf{F}_q[T]$ and $C_T = T + F$).

$$g(a, P) = (-1)^{a(q^{hf}-1)}(-1)^{f(h-1)}\lambda^{(q^h-1)\Sigma_{i=0}^{f-1}\langle q^{hi}a\rangle}\prod_{i=0}^{f-1}\Gamma_{\mathfrak{p}}(\langle q^{hi}a\rangle).$$

From this we get an expression

$$\chi_{\underline{a}}(P) = (-1)^{n(\underline{a})f}\Big/\left(\prod_{i=0}^{f-1}\Pi_{\mathfrak{p}}\left(-\underline{a}^{(q^{hi})}\right)\right)$$

for the Hecke character in terms of $\mathfrak{p}$-adic gamma functions and use of Theorem 2.8 gives the required expression.

For the rest of the paper, we restrict to the case $\delta = 1$ for simplicity.

5. Characteristic p gamma function

5.1. The gamma functions we have studied so far had domain in characteristic zero, even though the values were in characteristic p. This is connected to the fact that as we have seen, for example in the $A = \mathbf{F}_q[T]$ case, its arithmetic is linked up with cyclotomic extensions $\mathbf{F}_q(T)(\mu_n)$ of $\mathbf{F}_q(T)$ which are just constant field extensions. More precisely, the analogue of the Gross-Koblitz theorem in [T3, p. 110] and the Chowla-Selberg formula of Section 1 are related to Stickelberger elements of these constant field extensions. The fractions we handle there are p-integral, and so are of the form $m/(q^r - 1)$. The values of gamma function at these fractions are connected to extension $\mathbf{F}_q(T)(\mu_{q^r-1})$ which is a general finite constant field extension.

But we have another nice family (see 0.2.1) (among other cyclotomic families, given similarly, but with a different infinite place) of cyclotomic extensions $\mathbf{F}_q(T)(\Lambda_a)$ of Carlitz; so one would expect another gamma function with domain of characteristic p such that its special values at fractions with denominator a are related to the arithmetic of $\mathbf{F}_q(T)(\Lambda_a)$.

5.2. With all its nice analogies, the gamma function of the first part has one feature strikingly different from the classical gamma function. It has no poles. The usual gamma function has no zeroes and has simple poles exactly at 0 and the negative integers (negative of positive integers). In our non-archimedean case, the divisor determines function up to a multiplicative constant. (The simplest choice of constant we choose below also seems to be the best for the analògies we describe later.) As we have seen, monicity is an analogue of positivity (but note that positivity is closed under both addition and multiplication, but monicity only under multiplication).

5.3. Consider the meromorphic function defined as follows (note that we are changing the notation):

Definition 5.3.1.

$$\Gamma(x) =: \frac{1}{x} \prod_{n \text{ monic}} \left(1 + \frac{x}{n}\right)^{-1} \in \Omega \cup \{\infty\}, \qquad x \in \Omega.$$

(Note that for $p = 2$, positive is the same as negative and for $q = 2$ all integers are negative!) From the point of view of divisors, the factorial Π should be defined as follows:

Definition 5.3.2. $\Pi(x) =: x\Gamma(x)$.

Classically, $x\Gamma(x) = \Pi(x) = \Gamma(x + 1)$, whereas here the first equality is natural for the gamma and factorial defined here and the second equality holds for the ones considered previously in Sections 0–4. Consequently, gamma and factorial now differ by more than just a harmless change of variable. Also, in characteristic p, addition of p brings you back, so that giving the value at $x + 1$ in terms of that at x will not cover all integers by recursion anyway.

5.4. *Remark*. Definitions in 5.3 can be modified, as in 3.3, 4.1, so as to make good sense not only for A with $\delta = 1$, but for general A, by choice of sign representatives if $\delta > 1$ and use of the general ideal class $\mathscr{A}$ instead of A.

5.5. Taking logarithmic derivatives in the definition one sees that

$$\frac{\Gamma'}{\Gamma}(x) = -\sum_{n \text{ monic}} \frac{1}{x+n} - \frac{1}{x} = -Z(x,1), \qquad \frac{\Pi'}{\Pi}(0) = -Z(1)$$

where $Z(s)$ is as defined in 0.3.0 and $Z(x, s)$ is an analogue of Hurwitz's partial zeta function defined analogously. But classically, the value at zero of the logarithmic derivative of the factorial is $-\gamma$, where γ is Euler's constant. Hence in our case, the analogue, denoted by γ again, of Euler's constant gamma is (see [A-T] for details)

$$\gamma = Z(1) = -\frac{\Pi'}{\Pi}(0)$$

(and this is equal to log(1), where log is the inverse function (with log(0) = 0) to the exponential e_C of 1.1, if $A = \mathbf{F}_q[T]$, and also equal to $\tilde{\pi}/(T^2+T)$ if further, $q = 2$ (see [A-T]).

5.6. *Notation*. We will write "$n \in j$" (resp. $n \in j+$) for "n is integral (resp. monic integral) of degree j", A_+ (resp. A_-) for the set of "positive" (resp. "negative") elements of A.

5.7. If $j > \deg x$, $\prod_{n \in j+} n/(n+a) = 1$ and so if $a \in A - A_-$, then $\Gamma(a) \in K$. For example, when $A = \mathbf{F}_q[T]$, $1/\Gamma(T) = 2T(T+1)$ (so prime factorization does not depend only on the degree of the prime in contrast to the gamma function of Section 0).

5.8. *Remark*. In fact, interestingly enough, for $A = \mathbf{F}_q[T]$, if $a \in A$, $1/\Gamma(a) \in A$ (and even $2/\Gamma(a) \in A_+ \cup \{0\}$ when $a \in A_+$). To see this, it is enough to see that $\prod_{n \in k+} n$ divides $\prod_{n \in k+}(a+n)$. (Choosing $n_{\mathfrak{p},e}$ appropriately, of degree less than $e \deg \mathfrak{p}$, one sees that if $\mathfrak{p}^e$ divides n then it also divides $a + (n + n_{\mathfrak{p},e})$.) (This is not true for general A, because of the irregular behaviour in the Riemann-Roch for higher genus. For example, if $A = \mathbf{F}_q[x,y]/(y^2 = x^3 - x - 1)(q = 3)$, then neither $\Gamma(y)$ nor $1/\Gamma(y)$ is integral.) Another way to see this is: By Carlitz [C1, 2.17] one has (in the $\mathbf{F}_q[T]$ case)

$$\Pi(x)^{-1} = \prod_{j=0}^{\infty} \left(1 + \frac{\psi_j(x)}{D_j}\right)$$

where

$$\psi_k(x) =: x \prod_{j=0}^{k-1} \prod_{a \in j} (x - a)$$

and by [C4, p. 502], $\psi_j(x)/D_j$ is an integer when x is. For more on the

$\psi_j(x)/D_j$, which are analogues of binomial coefficients, see [Go4], [T4] and references there.

5.9. This rationality gives us hope for interpolation à la Morita. First we make the following definition (note that we are changing notation: The use will be clear from the context).

Definition 5.9.1. For $a \in A_v$, let $\bar{a} =: a$ (resp. 1) if $v(a) = 0$ (resp. $v(a) > 0$) and when $x \in A$, put

$$\Pi_v(x) =: \prod_{j=0}^{\infty} \left(\prod_{n \in j+} \frac{\bar{n}}{\overline{x+n}} \right).$$

Again terms are 1 for large j. Hence $\Pi_v(a) \in K$ for $a \in A$.

LEMMA 5.9.2. *Π_v interpolates to Π_v: $A_v \to A_v^*$ and is given by the same formula, as in the definition, even if $x \in A_v$. Similarly, $\Gamma_v(x) =: \Pi_v(x)/\bar{x}$ interpolates.*

Proof. In fact, it is easy to see that if $x \equiv y \bmod v^l$, then $\Pi_v(x) \equiv \Pi_v(y) \bmod v^l$.

5.10. *Example*. ($A = \mathbf{F}_q[T]$). $\Pi_T(T^2)$ is $(T + 1)/\{(T^2 + 1)(T - 1)(T^2 + T - 1)\}$ if $q = 3$ and is 1 if $q = 2$.

5.11. *Remark*. $1/\Pi_v(x) = \prod_{j=0}^{\infty} - \prod_{n \in j+} \overline{(x+n)}$ up to multiplication by a fixed root of unity by Theorem 4.4.

6. Functional equations

6.1. *The reflection formula*. For $q = 2$, all nonzero elements are monic, so that

$$\Gamma(x) = \frac{1}{e_A(x)} = \frac{\tilde{\pi}}{e(\tilde{\pi}x)},$$

(where e_A is the exponential corresponding to lattice A as in Section 1, e is the exponential corresponding to the sgn-normalized Drinfeld module with period lattice $\tilde{\pi}A$) so that for $x \in K - A$, $\Gamma(x)$ has algebraic ratio with $\tilde{\pi}$ and hence, in particular, $\Gamma(x)$ is transcendental (see 3.11). From the point of view of their divisors, $e(\tilde{\pi}x)$ being analogous to $\sin(\pi x)$ (i.e., both have simple zeros at integers and no poles), this observation suggests a relation between Γ and sine. We state these reflection relations (proof is immediate from the definitions) as follows (to make the analogy more visible).

The classical reflection formula can be stated as $\prod_{\theta \in Z^{\times}} \Pi(\theta x) = \pi x / \sin \pi x$ and here we have:

THEOREM 6.1.1.

$$\prod_{\theta \in A^{\times}} \Pi(\theta x) = \frac{\tilde{\pi} x}{e(\tilde{\pi} x)}.$$

Hence the classical name "half sine" for Γ should be replaced by "$1/(q-1)$-th exponential" in our case! Notice also that the interesting cyclotomic part missing from the reflection formula (see 3.12) is present now. One interesting difference is that for x a fraction, the ratio (denoted by $\Omega_{\underline{a}}$ again to stress analogy with the situation in Theorem 4.6) obtained by dividing the left-hand side by the first power of the period $2\pi i$ ($\tilde{\pi}$ resp.) lies in $Q(\mu_{2n})$($K(\Lambda_a)$ resp.) if the denominator of x is n (a resp.). Consequently, if τ is an element of the relevant Galois group fixing appropriate roots of unity (i.e., n-th in the classical case, a-th torsion in our case (and $(q^r - 1)$-th roots of unity in the case of Theorem 4.6), then from what we have just said, $\Omega_{\underline{a}}^{\tau}/\Omega_{\underline{a}}$ is 1 in our case. Hence the character $\chi_{\underline{a}}$ occurring as in the analogue of Deligne's theorem (take the formula in Theorem 4.6 as the definition of $\chi_{\underline{a}}$) is trivial in our case; whereas it can be nontrivial in the classical case (see [De, p. 91] or [G-K, p. 577]). On the other hand, if we look at $\Omega_{\underline{a}}^{\tau}/\Omega_{\underline{a}}$ for τ, not necessarily fixing the appropriate roots of unity, we get cyclotomic units in both cases.

6.2. *The multiplication formula.*

THEOREM 6.2.1. *Let $g \in A$ be monic of degree d and let α run through a full system of representatives modulo g. Then*

$$\prod_{\alpha} \Pi\left(\frac{x+\alpha}{g}\right) = \Pi(x)\tilde{\pi}^{(q^d-1)/(q-1)}\left((-1)^d g\right)^{q^d/(1-q)} R(x)$$

where

$$R(x) = \frac{\prod_{j=0}^{t} \prod_{\beta \in j+} \beta + x}{\prod_{\alpha} \prod_{j=0}^{t-d} \prod_{a \in j+} ga + \alpha + x},$$

where t is any integer larger than $\max(\deg \alpha, 2g) + d$.

Proof. To avoid confusion, here and for the rest of the paper, we use Π_0 (Γ_0 resp.) for the factorial (gamma resp.) of Section 3. Using Theorem 3.10 and the fact (see 3.4) that $\overline{D}_j \to 1$ as $j \to \infty$, we see that

$$\prod_{\alpha} \Pi\left(\frac{x+\alpha}{g}\right) = \prod_{j} \prod_{a \in j+} \prod_{\alpha} \frac{ga}{(ga+\alpha)+x}$$

$$= R(x) \lim_{j \to \infty} \left(\prod_{k=0}^{j+d} \prod_{a \in k+} \frac{(ga)^{q^d}}{a+x} \right) \left(\prod_{k=j+1}^{j+d} \prod_{a \in k+} \frac{1}{(ga)^{q^d}} \right)$$

$$= R(x)\Pi(x) \lim \frac{g^{q^d(q^{j-e}-1)/(q-1)} \prod_{k=0}^{j} \prod_{a \in k+} a^{q^d-1}}{D_{j+1} \cdots D_{j+d}}$$

$$= R(x)\Pi(x) g^{q^d/(1-q)} \Pi_0(-1)^{(q^d-1)/(q-1)}$$

$$= R(x)\Pi(x) \tilde{\pi}^{(q^d-1)/(q-1)} \left((-1)^d g\right)^{q^d/(1-q)}.$$

Hence the proof is complete.

6.2.2. *Remark*. Here $R(x)$ takes care of beginning irregularity in Riemann-Roch. For example, $R(x) = \prod_{\alpha \text{ monic}} (x+\alpha)$ in case $A = \mathbf{F}_q[T]$ and $\{\alpha\}$ is the set of all polynomials of degree not more than d.

6.2.3. *Remark*. $(*)$ of 3.2 is the multiplication and reflection formulae combined (for $x = 0$) and can be used to recover these formulae.

6.2.4. Analogy with the usual multiplication formula (2.0) is quite visible in that an analogous combination of factorials is an analogous power of the period (instead of $(n-1)/2$ here one has $(\mathbf{N}g - 1)/(q-1)$ but we have seen that 2 and $q-1$ represent choices of signs in respective situations whereas n and $\mathbf{N}g$ are the number of residue classes that are relevant) times an algebraic part. For $x \in K - A$, call the algebraic part $\Omega_{\underline{a}}$ again (see 4.6 and 6.1). Now $\underline{a}$ corresponds to the multiplication formula (see [G-K, p. 577]). Again it is interesting to note that $\Omega_{\underline{a}} \in K(\Lambda_g)$, which makes, just as in 6.1, the corresponding character $\chi_{\underline{a}}$ (see 4.6 and 6.1) trivial again, in contrast with the classical situation.

6.3. *The multiplication formula for* Π_v.

THEOREM 6.3.1. *Let* α, g *be as in Theorem* 6.2.1 *with* $(g, v) = 1$. *Let* $h =:$ deg v *and let* g_v *be the Teichmüller representative of g modulo v. Then*

$$\prod_{\alpha} \Pi_v\left(\frac{x+\alpha}{g}\right) = \Pi_v(x) g_v^{(q^h-1)/(q-1)} (-1)^{hd} g^m Q(x)$$

where

$$Q(x)=\frac{\prod_{j=0}^{t}\prod_{\beta\in j+}\overline{\beta+x}}{\prod_{\alpha}\prod_{j=0}^{t-d}\prod_{\alpha\in j+}\overline{ga+\alpha+x}},$$

t is any integer larger than $\max(\deg\alpha,2g)+d$ *and* $-m$ *is the number of* $n\in j+$, $j<d$ *with n not congruent to* $-x$ *modulo v.*

Proof. By manipulations similar to those in 6.2.1, we obtain

$$\prod_{\alpha}\Pi_v\left(\frac{x+\alpha}{g}\right)=\prod_{\alpha}\prod_{j}\prod_{a\in j+}\frac{\bar{a}}{\overline{\left(\dfrac{ga+\alpha+x}{g}\right)}}$$

$$=Q(x)\Pi_v(x)\lim_{j\to\infty}\frac{g^{m_j}\left(\tilde{D}_0\cdots\tilde{D}_j\right)^{q^d-1}}{\tilde{D}_{j+1}\cdots\tilde{D}_{j+d}}$$

where by Riemann-Roch-type counting m_j is seen to be

$$q^{j+c}(q^h-1)/(q-1)+m$$

for some integer c. Now (1) the q^k-th power of g/g_v, a one-unit at v, tends to one as k tends to ∞. By 4.2 and Theorem 4.4, (2) $-\tilde{D}_k\to 1$ and (3) $(\tilde{D}_0\cdots\tilde{D}_j)^{q-1}\to(-1)^{h-1}$. Hence the theorem follows easily.

6.3.2. *Remark*. Here $Q(x)$ takes care of beginning irregularity in Riemann-Roch, just as in 6.2.2. Also, $Q(x)=\Pi_{\alpha\text{ monic}}(x+\alpha)$ when A and $\{\alpha\}$ are as in 6.2.2. Note that when $x\in K$, $\Pi_\alpha\Pi_v((x+\alpha)/g)/\Pi_v(x)\in K^\times$.

6.4. *The reflection formula for* Π_v.

THEOREM 6.4.1. *Let* $x\in A_v$. *Then*

$$\prod_{\theta\in\mathbf{F}_q^\times}\Pi_v(\theta x)=\zeta_x\bar{x}$$

where $\zeta_x\in\mathbf{F}_q^\times$.

Proof. First, let $x=a/b$ with $(b,v)=1$. Let

$$R=:\prod_{\theta\in\mathbf{F}_q^\times}\Pi_v(\theta a/b).$$

Then

$$R=\prod_{\theta}\prod_{i}\prod_{n\in i+}\frac{(\theta b)^{w_i}\bar{n}}{\overline{\theta bn+a}}$$

where w_i is the number of $n \in i+$ with n not congruent to $-a/(\theta b)$ modulo v. For large i, by the equidistribution of $n \in i+$ among the residue classes modulo v, w_i is also the number of $n \in i+$ not congruent to 0 modulo v. Hence θ's and $\bar{n}$'s can be combined (i.e., the condition of monicity for n can be dropped; see below). On the other hand, Riemann-Roch gives $\sum_{i<j} w_i = e + (q^{j-c}-1)(q^h-1)/(q-1)$ where c and e are integers independent of j. Hence

$$R = \zeta \bar{a} \lim_{j \to \infty} b^{f+(q^{j-c}-1)(q^h-1)} \left(\prod_{\deg n \leq j,\, (n,v)=1} n \right) \Bigg/ \left(\prod_{\deg m \leq j+d,\, m \equiv a(b),\, (m,v)=1} m \right).$$

Here $\zeta \in \mathbf{F}_q^{\times}$ is a contribution of θ^{w_i}'s for low i's and f is an integer independent of j. Also ( )/( ) on the right-hand side is plus or minus one, since numerator and denominator both tend to plus or minus one by Theorem 4.4 and by proofs similar to those of 4.3 and 4.4.

(In more detail, we use the claim: There is an l such that $\{m \in A: (m,v)=1,\ m \equiv a(b),\ \deg m \leq j\}$ is equidistributed among residue classes mod v^{j-l} if j is large enough: By the Chinese remainder theorem, we may choose integers of degree $\leq j$ congruent to each given residue mod v^{j-l} and congruent to $0 \bmod b$; using these representatives to do the coset transfer, we see that the claim is true. Each residue is repeated some power of q number of times; hence $\prod m \equiv -1 \bmod v^{j-l}$.)

Combining this information, we see that $R = \zeta_x \bar{a} b^s$ for some integer s. But the answer should be the same if we replace a, b by ra, rb for $(r,v)=1$. Hence $R = \zeta_x \overline{a/b}$. The continuity of Π_v now enables us to finish the proof.

6.4.2. *Remark*. (1) It is easy to determine ζ_x using congruences. (2) The theorem implies that if $q=2$, $\Pi_v(a/b)$ is rational and in fact, $\Pi_v(x) = \bar{x}$ and so $\Gamma_v(x) = 1$. In general, for $x \in K$, $\prod \Pi_v(\theta x) \in K^{\times}$.

7. Framework of brackets

In this section, we will try to explain the visible analogies studied so far in the framework of brackets $\langle\ \rangle$ and partial zeta functions. For comparison with the classical case, readers should refer to [De], [G-K], [K-O] or [A1].

7.1. Unless clear from the context, Γ will stand for either the classical gamma or the p-adic gamma function of Morita or gamma functions and their interpolations in previous sections. Γ_∞ will stand for gamma at the infinite place (note that it is just Euler's gamma in the classical case) and Γ_v for gamma functions at a finite place (p-adic or v-adic). We use shorthand "$a \sim b$" for "a/b is a nonzero algebraic number" (over $\mathbf{Q}$ or K according to the context).

The idea of the framework of brackets is: Given a certain type of gamma function, to identify the relevant definition for $\langle\ \rangle$, so that simple recipes using $\langle\ \rangle$'s (see, e.g., 7.3, 7.4, 7.12) will yield results on algebraicity and relations to periods of values of the gamma function at fractions. For certain types of gamma functions, these recipes are proved in full and in other cases they are to be considered as conjectures, for which some evidence is then provided.

7.2. If a is a proper (i.e., nonintegral) fraction in the relevant domain, then up to $\sim$ (in fact, up to "rational"), $\Gamma(a)$ depends only on a modulo the integers. Classically, $\langle\ \rangle: \mathbf{Q} - \mathbf{Z} \to (0,1) \cap \mathbf{Q}$ is defined by $0 < \langle a \rangle < 1$, $a \equiv \langle a \rangle \bmod \mathbf{Z}$.

We consider $\underline{a}$ etc. as in 2.7. Without loss of generality, one can take $N = q^r - 1$. Identifying $\mathrm{Gal}(K(\mu_N)/K)$ with $q^{\mathbf{Z}/r\mathbf{Z}}$, one defines $\underline{a}^{\sigma}$ for σ in the Galois group, as in 2.7. (In other words, we think of σ as some $q^j \in (\mathbf{Z}/N\mathbf{Z})^{\times}$.)

Definition 7.2.1. $m(\underline{a}) =: \Sigma m_i \langle -a_i \rangle$.

Notice that $\langle a \rangle + \langle -a \rangle = 1$ for any proper fraction a and that for $\underline{a}$'s corresponding to multiplication or reflection formulae, one has $\Sigma m_i = 2n(\underline{a})$. Hence Theorem 2.8 together with Theorems 3.10 and 4.4 can be reformulated, in general, as (proved for Γ of Sections 3, 4):

7.3. *Recipe/conjecture*. If $m(\underline{a}^{(\sigma)})$ is independent of $\sigma \in \mathrm{Gal}(K(\mu_N)/K)$ then $\Gamma_\infty(\underline{a}) \sim \tilde{\pi}^{m(\underline{a})}$ and $\Gamma_v(\underline{a}) \sim 1$.

The classical counterpart of this can be found in [K-O], [De], [G-K]. Also, in both ours and the classical case, $\sim$ can be replaced by the more precise "up to multiplication of a rational number and a root of unity", if $m(\underline{a})$ is an integer.

The proof of Theorem 2.8 shows that the hypothesis of 7.3 can be rephrased as "$\underline{a}$" is made up of Galois orbits" (i.e., if one expresses $\underline{a}$ in terms of basic $a_i =: q^i/(q^r - 1)$ (using digit expansion), then all a_i's occur with the same multiplicity). This suggests:

7.4. *Recipe/conjecture*. $\Gamma_v(\underline{a}) \sim 1$ if $\underline{a}$ is made up of Frob_v-orbits (observe that this is not equivalent to "$n(\underline{a}^{(\mathbf{N}v^j)})$ is independent of j").

7.4 is true classically by [G-K, Cor. 1.11], and it is true for Γ functions of Section 4, if $A = \mathbf{F}_q[T]$, by Theorem 4.8 (e.g. $\Pi_v(q^j/(1 - \mathbf{N}v))$ is algebraic, because $q^j/(1-\mathbf{N}v)$ is a Frob_v-orbit; i.e., one has $\langle q^j \mathbf{N}v^k/(1 - \mathbf{N}v) \rangle = \langle q^j/(1 - \mathbf{N}v) \rangle$). Some evidence for 7.4 in other cases is presented in Section 9.

7.5. One wonders whether 7.3 and 7.4 are best possible in full generality (i.e., for all the different gamma functions under consideration in this paper), as far as values at proper fractions are concerned.

If one wants to extend this framework to Γ and Γ_v in Sections 5 and 6, and to use the other cyclotomic family $K(\Lambda_a)/K$, then one has to deal with $\langle\ \rangle$ defined for proper fractions in K as follows. (I would like to thank Greg Anderson for explaining this to me.)

7.6. Greg Anderson defined $\langle\ \rangle$ as follows. (See also [A1] where he deals with the classical case.)

Definition 7.6.1. We define $\langle\ \rangle: K - A \to \{0, 1\}$ by putting $\langle -f/g \rangle =: 1$ if f is monic and zero otherwise, where we first normalize, using translation by elements in A, by making g monic and $\deg f < \deg g$.

7.7. Motivation for this is as follows. Classically, one has the Hurwitz formula $\zeta(x, s) = \langle -x \rangle - 1/2 + (\log \Gamma(x))s + o(s^2)$ around $s = 0$. Hence one expects connection between (e.g., between distribution relations satisfied by them) partial zeta values at $s = 0$, $\langle\ \rangle$'s and Γ's.

Now we do not have such a relation (yet), but we can consider (for $A = \mathbf{F}_q[T]$)

$$\zeta\left(\frac{f}{g}, s\right)\Bigg|_{s=0} = \sum_{n \text{ monic} \equiv f \bmod g} q^{-s \deg n}\Big|_{s=0}$$

which is $1 - 1/(q-1)$ if f is monic and $-1/(q-1)$ otherwise. The analogy $1/2 \leftrightarrow 1/(q-1)$ now explains the definition of $\langle\ \rangle$ in 7.6.

7.8. The important thing is 7.3 holds (even with the refinement mentioned there) with this definition of $\langle\ \rangle$, with μ_N replaced by Λ_a and $\underline{a}$ having fractions with denominator $a \in A$ in place of $N \in \mathbf{Z}$ and $n(\underline{a})$ and $m(\underline{a})$ defined similarly, of course. (First note the case $q = 2$, when the $\langle\ \rangle$ is 1 and correspondingly, $\Gamma(a) \sim \tilde{\pi}$ (see 6.1) and $\Gamma_v(a) \sim 1$ (see 6.4).) The reason is that the reflection and multiplication formulae in Section 6 are special cases of 7.3, and they generate (K-linearly) all of the relations of 7.3, as can be seen by straightforward modification of the Koblitz-Ogus proof [K-O], by the analogies mentioned. (More precisely, signs ± 1 are now to be replaced by $\mathbf{F}_q^\times$ and even and odd to be interpreted in that context. The ζ and $\langle\ \rangle$ connection there works by definition and $L(0, \chi_d) \neq 0$ has a similar expression.)

7.9. As noted in 4.7, for Γ of Sections 2, 3, 4, Theorem 2.8 gives many more relations than those generated by reflection and multiplication, since the condition in this case (q^j-powers) is weaker than that in [K-O] (power prime to denominator).

7.10. For $A = \mathbf{F}_q[T]$, if $0 \le r < k$, the basic partial zeta value $\sum_{n \in j+,\, j \equiv r \bmod k} q^{-s \deg n}$ is $q^r/(1-q^k)$ at $s = 0$. These are (negatives of) basic $\langle\ \rangle$'s for Γ of Sections 2, 3, 4.

7.11. Now we turn to an "explanation" of Theorem 1.6 as the Chowla-Selberg formula. Recall that if E is an elliptic curve over $\overline{\mathbf{Q}}$ with complex multiplication by $(\mathbf{Q}(\sqrt{-d}))$, then the classical Chowla-Selberg formula expresses period π_E in terms of Γ values at fractions with denominators d. "Discriminant" d can also be viewed via $\mathbf{Q}(\sqrt{-d}) \subset \mathbf{Q}(\mu_d)$ (or $\mathbf{Q}(\mu_{2d})$). Hence considering the Carlitz module for $\mathbf{F}_{q^t}[T] \subset \mathbf{F}_q[T](\mu_{q^t-1})$ as an analogue of E, we see that Theorem 1.6 is a Chowla-Selberg formula. (Unit parts in the statement can be removed as in Section 3.) But, as explained to me by Greg Anderson (implicit in [A1] and in references given there), the particular combination of Γ values occurring there is no accident, but can be explained via $\langle\ \rangle$'s as follows. (For more details see [A1] and forthcoming work by Anderson.)

7.12. *Recipe/conjecture*. The general set-up in our case is roughly as follows. We have the Drinfeld module E over K^{sep} with complex multiplication via integral closure of A in an abelian extension L over base K. (This forces L to have only one place above the infinite place of K; one must use higher-dimensional generalizations of Drinfeld modules to remove this restriction.)

Define $h(a)\colon \mathrm{Gal}(K^{\text{sep}}/K) \to \mathbf{Q}$, by $h(a)(\sigma) =: \langle -b \rangle$ where $\exp(2\pi i a)^\sigma = \exp(2\pi i b)$. (Here the exp is the classical exponential, since we are dealing with constant field extensions and $\langle\ \rangle$'s which have values in characteristic zero. In dealing with Sections 5 and 6 one needs $e(\tilde{\pi} a)$, of course, in place of $\exp(2\pi i a)$.) Let $\chi_{L/K}$ be the characteristic function of $\mathrm{Gal}(L/K)$; i.e., $\chi_{L/K}\colon \mathrm{Gal}(K^{\text{sep}}/K) \to \mathbf{Z}$ such that $\chi_{L/K}(\sigma) =: 1$ if $\sigma|_L = 1$ and 0 otherwise. Then if $\chi_{L/K} = \sum m_a h(a)$ then the recipe for expressing the period π_E is:

$$\pi_E \sim \prod \Gamma_\infty(a)^{m_a}.$$

7.13. In our example, $L = \mathbf{F}_{q^t}(T)$, the $K = \mathbf{F}_q(T)$, $\chi_{L/K}(F^n) = 1$ if $n \equiv 0 \bmod t$ and 0 otherwise, where F is the q-power Frobenius. Now

$$\left[q h\left(\frac{q^{t-1}}{1-q^t}\right) - h\left(\frac{1}{1-q^t}\right)\right](F^n)$$
$$= q\left\langle \frac{q^{t+n-1}}{q^t - 1}\right\rangle - \left\langle \frac{q^n}{q^t-1}\right\rangle = \chi_{L/K}(F^n)$$

explains Theorem 1.6.

7.14. By the Chowla-Selberg phenomenon, we mean the expression, in a complex multiplication situation, for the period in terms of a certain combination (predicted by simple calculation on $\langle\ \rangle$'s as in 7.12) of gamma values at appropriate fractions. We will give more examples of this phenomenon in Section 9.

8. Two variable gamma functions of Goss

In this section, we make a similar study of a two variable gamma function introduced by Goss [Go3].

8.1. Goss introduced a two-variable gamma function $\Gamma: \Omega \times \mathbf{Z}_p \to \Omega$ which, up to small factors and change of variables, can be expressed as $\Gamma(x, y) =: (1/x \prod_{j=0}^{\infty} \prod_{n \in j+} (1 + x/n)^{-y_j}) \bar{\Gamma}(y)$, where as usual $y = \sum y_j q^j$ is a q-adic expression and $\bar{\Gamma}$ is the gamma function of Sections 0 and 1. $\Gamma(y)$ can be replaced by $\Gamma_0(y)$ (recall the notation from 6.2) if one makes adjustments similar to those in Section 3. We ignore such technical difficulties below and compare natural variations:

Definition 8.1.1. $\Pi_3(x, y) =: (\prod_{j=0}^{\infty} \prod_{n \in j+} (1 + x/n)^{-y_j})$, $\Pi_2(x, y) =: \Pi_3(x, y)/\Pi_0(y)$ and $\Pi_1(x, y) =: \Pi_3(x, y)\Pi_0(y)$. For $i = 1, 2, 3$, $\Gamma_i(x, y) =: \Pi_i(x, y - 1)/x$.

The definition of Γ_i seems to be a natural definition in view of previous sections.

8.2. Note that $\Pi_1(0, y)$ is the factorial of Section 3 (now denoted by $\Pi_0(y)$ to distinguish it from others) and $\Pi_1(x, 1/(1-q))$ is up to constant $(= \Pi_0(1/(1-q)))$ the factorial of Section 5. Hence two gamma functions studied so far are essentially specializations of Π_1. Similar statements can be made for Π_2. But, though $\Pi_3(x, 1/(1-q))$ gives the factorial of Section 5, one cannot recover the factorial of Section 3 from it; on the other hand Π_3 is more simply defined, in that technical problems of $\Pi_0(y)$ are absent and, as we will see below, $\langle\ \rangle$ for Π_3 turns out to be simple and integral. See also [Go4], where Mahler coefficients of Π_3 are identified with analogues of binomial coefficients (see 5.8). At the moment, it seems to be a matter of taste, whether one should use Π_1 or Π_3 and Π_0 etc. We will ignore questions of v-adic interpolations, integrality (these can be treated as in Section 5) and turn to the subject of functional equations.

8.3. Evaluation of the partial zeta function $\sum q^{-s \deg n}$ (where n is monic congruent to f modulo g and the degree of n is d modulo t) at $s = 0$ suggests the following definition of bracket $\langle\ \rangle$ corresponding to Π_3 (brackets for Π_1 and Π_2 are obtained just by adding or subtracting $\langle\ \rangle$ for Π_0).

Definition 8.3.1. $\langle\, ,\rangle: (K - A) \times (\mathbf{Q} \cap \mathbf{Z}_p - \mathbf{Z}) \to \mathbf{Z}$ is defined by:

(1) $\langle x, y\rangle$ depends only on $x \bmod A$ and $y \bmod \mathbf{Z}$.

(2) If $f \in A$, $g \in A_+$, $\deg f < \deg g$, $0 \le s < t$ then $\langle -f/g, q^s/(q^t - 1)\rangle =: 1$ if f is monic and $s \equiv \deg f - \deg g \bmod t$ and 0 otherwise.

(3) Given a proper fraction y, one can write $\sum_{j=0}^{t-1} n_j q^j/(q^t - 1)$; then $\langle -f/g, y\rangle =: \sum n_j \langle -f/g, q^j/(q^t - 1)\rangle$.

8.4. Let us state functional equations in [Go3] adapted to Π_3. First of all, Π_3 is of the form "f" in 2.1, so that Theorem 2.8 applies. On the other hand, for $y = 1/(1 - q)$, Π_3 becomes the factorial of Section 5. (Notice that $\langle\ \rangle$ reduces to one considered before.) In other words, straight modification of 7.3 works, if restricted to the first or second variable. From Theorem 4.5.4 of [Go3], it is easy to see the following:

THEOREM 8.4.1.

$$\prod_{a \bmod g} \Pi_3\left(\frac{x + a}{g}, y\right) \sim \Pi_3(x, q^d y)\Pi_0(y)^{q^d}/\Pi_0(q^d y).$$

We do not yet know whether analogues of 7.3 and 7.4 are true for Π_3 and whether they give all relations. Also, we have not yet tried to work out, generalizing 6.3 and 6.4, functional equations for v-adic interpolations of Π_3.

More evidence that the two-variable gamma function deals with both cyclotomic families (i.e., constant field extensions and Carlitz' family) is presented in Section 9.

9. Miscellaneous results

This section contains some partial miscellaneous results: cases of the Chowla-Selberg phenomenon (see 7.14), special cases of 7.3 and 7.4. We also discuss some open questions.

9.1. We now give some examples of the Chowla-Selberg phenomenon for gamma functions of Sections 5–8. For 9.1–9.3, let the base be $A = \mathbf{F}_q[T]$.

9.1.1. If one wants a rank-one Drinfeld B-module (complex multiplication) where B is the integral closure of A in the full cyclotomic field $K(\Lambda_a)$, then the condition: "one infinite place" forces $\deg a \le 1$. So without loss of generality consider $B =: \mathbf{F}_q[T](\Lambda_T) = \mathbf{F}_q[y]$ where $y^{q-1} = -T$.

We will show that if π_B is a period of the Carlitz module of B, then $\pi_B \sim \Gamma(1/T)$. (Notice that for $q = 2$, $\pi_B = \tilde{\pi} \sim \Gamma(a/b)$ for any proper fraction a/b as noted before. This is consistent with the fact that $\langle\ \rangle = 1$, when $q = 2$.)

Let $a \in A$, $b \in A_+$, $\deg a < \deg b = h$ say, and let $D_{r,a,b}$ be the product of monic elements of A of degree r, which are congruent to a modulo b. We will show how to evaluate some of these in 9.3; in particular, we will show

$$D_{j+1,1,T} = D_j T^{q^j}\left((-T)^{(q^{j+1}-1)/(q-1)} - 1\right)/(-T)^{(q^{j+1}-1)/(q-1)}.$$

Hence, we have

$$\Gamma\left(\frac{1}{T}\right) \sim \prod_{j=0}^{\infty} \prod_{n \in j+} \left(1 + \frac{1}{nT}\right)^{-1} \sim \prod_{j=0}^{\infty} \frac{D_j T^{q^j}}{D_{j+1,1,T}}$$

$$\sim \prod_{j=0}^{\infty} \frac{(-T)^{(q^{j+1}-1)/(q-1)}}{(-T)^{(q^{j+1}-1)/(q-1)} - 1}.$$

On the other hand, since $y^{q^n} - y = y((-T)^{(q^n-1)/(q-1)} - 1)$, by 0.2.0, we have

$$\pi_B \sim \lim \frac{y^{q^n/(q-1)}(-T^{q^n} - 1)^{1/(q-1)}}{y^n\left((-T)^{(q^n-1)/(q-1)} - 1\right) \cdots (-T - 1)}.$$

By straightforward algebraic manipulation, as in Section 1, one now sees that $\pi_B \sim \Gamma(1/T)$ as claimed.

9.1.2. It is easy to see that if one takes, for example, an intermediate case $B' =: \mathbf{F}_q[T^{2/(q-1)}]$ between A and B, then $\pi_{B'} \sim \Gamma(1/T)\Gamma(-1/T)$, by a similar proof. (Note consistency with $\langle\ \rangle$ philosophy and case $q = 3$, when $A = B'$ and $\pi_{B'} = \tilde{\pi} \sim \Gamma(1/T)\Gamma(-1/T)$ by the reflection formula.)

9.1.3. Now we turn to the two-variable case. First notice the following special case of the multiplication formula: $\prod_{\theta \in \mathbf{F}_q} \Pi_3(\theta/T, y) = \Pi_0(y)^q / \Pi_0(qy)$, which with $y = (q^{t-1})/(1 - q^t)$ gives $\tilde{\pi}_t$ of 1.6, consistent with $\langle\ \rangle$ for subextension $\mathbf{F}_{q^t}(T)$ of $\mathbf{F}_{q^t}(\Lambda_T)$.

9.1.4. In fact, if π_B denotes the period of Carlitz module for $\mathbf{F}_{q^t}[\Lambda_T]$ then $\pi_B \sim \Pi_3(1/T, q^{t-1}/(1 - q^t))$ can be easily seen from the proof of one variable case (i.e., Theorem 1.6).

9.2. We now give some examples of 7.4 for the gamma function of Section 5. We already know, by 6.4, that $\Gamma_T(1/(T + 1)) \sim 1$, for $q = 2$. Using the formula for $D_{j,1,T}$ in 9.1.1, we now generalize this to $p = 2$. Put $W_j =: \{n\colon\ n \in j+,\ (n, T) = 1,\ n \equiv 1 (\operatorname{mod} T + 1)\}$ and $T_j =: \{n\colon\ n \in j+,\ n \equiv$

1(mod $T+1$)}. Then,

$$\Gamma_T\left(\frac{1}{T+1}\right) \sim \prod_{j=0}^{\infty} \prod_{n\in j+,(n,T)=1} \frac{(T+1)n}{(T+1)n+1} \sim \lim_{r\to\infty} \prod_{j=0}^{r} \frac{\tilde{D}_j(1+T)^{q^j-q^{j-1}}}{\prod_{n\in W_j} n}$$

$$\sim \lim\left(\prod_{j=0}^{r} \prod_{n\in W_j} n\right)^{-1}$$

since the numerator $\tilde{D}_0 \cdots \tilde{D}_r(1+T)^{q^\gamma}$ of ( ) tends to 1, by Theorem 4.4 and the fact that $1+T$ is a one-unit at T. Now,

$$\prod_{n\in W_j} n = \left(\prod_{n\in T_j} n\right)\Big/\left(\prod_{n\in T_{j-1}} n\right) T^\beta$$

where β is such that the quantity is a unit at T. (Observe that this is the step which expresses the product as a telescoping product, where we use $T \equiv 1(\text{mod } T+1)$ (as $p=2$); i.e., one has Frob_T-orbit.)

Combining this with the formula for $\prod_{n\in T_r} n$, which is just $D_{j,1,T}$ obtained above with T replaced by $T+1$, we are reduced to showing algebraicity of the T-adic limit

$$\lim_{r\to\infty}\left(1+\frac{1}{(1+T)^{(q^r-1)/(q-1)}}\right) D_{r-1}(1+T)^{q^{r-1}}/T^{\beta'}$$

where, again, β' is such that the quantity is a unit at T. Now since $1+T$ is a one-unit at T, its q^{r-1}-th power tends to 1; also by [T3, p. 109], $D_r/T^{\text{val}_T(D_r)}$ tends to 1. Now, $(1+T)^{(q^r-1)/(q-1)} = 1+T+\cdots$; hence the limit above is

$$\lim\left(1+\frac{1}{(1+T)^{(q^r-1)/(q-1)}}\right)\Big/T = \left(1+\frac{1}{(1+T)^{1/(1-q)}}\right)\Big/T.$$

The claim is established.

The same method shows algebraicity of Γ_v on Frobenius orbits of fractions, with denominators of degree not more than one.

9.3. We now show how to "evaluate" $D_{r,a,b}$, if a/b has a denominator which factors into distinct linear factors over $\mathbf{F}_q$.

Proposition 9.3.1 (Moore [M]). *If G runs through $G = a_0x_0 + \cdots + a_rx_r$, $a_i \in \mathbf{F}_q$ and $a_i = 1$ for the lowest i for which a_i is nonzero, then* $\prod G = \det(x_i^{q^{r-j}})$.

9.3.2. We use shorthand $[a,b]_r$ (or just $[a,b]$ if there is no chance of ambiguity) for the determinant of the square matrix of order $r+1$ whose

(i, j)-th entry is $x_i^{q^{r-j}}$, where $x_r =: a$ and $x_k =: bT^{r-k}$ for $0 \le k < r$. For $\theta \in \mathbf{F}_q$, we also write $[T + \theta]_r$ (or just $[T + \theta]$) for scalar $(T + \theta)^{(q^{r+1}-1)/(q-1)}$.

9.3.3. For example, by 0.4.1, we have $[1, 1]_r = D_r \cdots D_0$.

9.3.4. Observe that, for $\theta \in \mathbf{F}_q^\times$, $[\theta a, b]_r = \theta[a, b]_r$ and $[a, b]_r + [c, b]_r = [a + c, b]_r$.

9.3.5. First let us compute $D_{r,1,T}$. Taking $x_0 = T^{r+1} + 1$ and $x_i = T^{r-i+1}$ for $i > 0$ in 9.3.1, we see that

$$\begin{aligned}\prod G &= D_{r+1,1,T}\left(T^{q^{r-1}}D_{r-1}\right) \cdots (TD_0) \\ &= D_{r+1,1,T}D_{r-1} \cdots D_0[T]_{r-1}.\end{aligned}$$

On the other hand, by 9.3.1 and elementary row operations, one sees that

$$\prod G = \left([T]_r + (-1)^r\right)[1, 1]_r.$$

This, together with 9.3.3, gives the formula in 9.1.1.

9.3.6. Let $\{a_i\}$ be an arbitrary set of distinct elements of $\mathbf{F}_q^\times$. We "know" [1, 1] by 9.3.3.

$$[T + a_1][1, 1] + [-a_1, T + a_1] = [T, T + a_1] = [T][1, 1].$$

(After one "takes out" $[T]$, the last equality follows from addition of suitable multiples of the next row to the given row from the bottom up.) Hence, by 9.3.4, one "knows" $[1, T + a_1]$. By induction, similarly, the equalities

$$\begin{aligned}&[T + a_{n+1}][1, (T + a_1) \cdots (T + a_n)] + [a_1 - a_{n+1}, (T + a_1) \cdots (T + a_{n+1})] \\ &\quad = [T + a_1][1, (T + a_2) \cdots (T + a_{n+1})],\end{aligned}$$

together with 9.3.4, show how to evaluate $D_{r,a,b}$, if a/b has a denominator which factors into distinct linear factors over $\mathbf{F}_q$.

Even though the expression thus obtained for $D_{r,a,b}$ is complicated, it seems that using it one can generalize algebraicity results in 9.2, by the same method. We have also proved a case of the Chowla-Selberg formula for a higher-dimensional generalization [A2] of Drinfeld modules, combining 9.3 with techniques of 9.1.1. Since this is just a special case and since it seems that recent results of Anderson mentioned in the introduction will prove much more in a better way, we do not go into details.

9.4. Functional equations and all examples of the Chowla-Selberg phenomenon so far, can be thought of as giving, for some $\underline{a}$'s (note that $n(\underline{a})$ is an integer), a motive (in the sense of [A2]) $M(\underline{a})$ with complex multiplication (with the infinity type corresponding to $\underline{a}$), whose period is (up to an algebraic quantity) $\Gamma(\underline{a})$ and for which eigenvalues of Frob_v are Gauss (or rather Jacobi) sums $g(\underline{a}, v)$ with prime factorization $v^{\Sigma n(\underline{a}^{(\sigma)})\sigma^{-j}}$. In our examples, $M(\underline{a})$ is just a Carlitz module for $\mathbf{F}_q[T]$, $\mathbf{F}_{q^t}[T]$, $\mathbf{F}_{q^t}[\Lambda_T]$, etc. Since the cyclotomic equation $C_v(u) = 0$ is Eisenstein at v by [C3], the eigenvalue of Frob_v is v. This fits with our situations: $n(\underline{a}) = 1$, $n(\underline{a}^{\sigma}) = 0$ for σ not the identity (see also the analogue of Weil's theorem on Jacobi sums as Hecke characters [T3] for Gauss sums for $\mathbf{F}_q[T]$ introduced in [T3]). (Using the tensor products of [A2], one can create more examples of $M(\underline{a})$'s.) For v in the base, the v-adic periods of $M(\underline{a})$ (in the sense of A. Ogus: essentially eigenvalues of Frobenius at v) should be $\Gamma_v(\underline{a})$ (up to an algebraic quantity). In our cases, when v is "ordinary", our results show that $\Gamma_v(\underline{a})$ is algebraic. This seems to fit with the statement above.

9.5. In [H3], Hayes introduced "Gauss sums" (in the sense of providing "correct Stickelberger elements") in general, explicitly. (Their existence was shown by Tate and Deligne (see [H3]).) The interesting feature of his construction is that the Gauss sum for a prime v occurs as a torsion point for some rank-one Drinfeld A-module, where the infinite place for A lies above v. But since the infinite place is arbitrary, by Theorem 3.10 and reflection formula 6.1, we get a v-adic expression for the Gauss sum for prime v, in terms of values of gamma functions (for v) at appropriate fractions. The difference from the Gross-Koblitz phenomenon is that this gamma function at v, though defined uniformly at all places, does not arise as interpolated from a fixed gamma function at infinity.

9.6. Some results of this paper are mentioned or proved only for $\mathbf{F}_q[T]$ and not for general A. We now want to separate out what does not hold for general A from what is not proved, but may be generalized: 0.4.1, 0.4.3 do not hold (for example, for $A =: \mathbf{F}_2[x, y]/(y^2 + y = x^3 + x + 1)$, $\Pi(8)/\Pi(4)$ is not an integer); 1.1 and 1.2 do not hold; 1.6 and 1.7 should generalize, but 1.7 does not, if one defines "Gauss sums" as in [T3]; but 1.7 may generalize with the definition as in [H3] mentioned above. More on this subject will appear in a separate paper. Equality $Z(1) = \log(1)$ in 5.5 does not hold in general.

9.7. We finish this paper by mentioning briefly some of the open problems apart from the generalizations:

(a) How are the gamma functions related to the zeta functions of [Go1]?

(b) Why do the characters coming from the analogue of Deligne's theorem (4.6, 6.1, 6.2) turn out to be trivial?

(c) Why does the $\langle\ \rangle$, which was obtained from the partial zeta function for $\mathbf{F}_q[T]$, seem to work in general?

(d) Can gamma functions be defined in a better fashion treating sign and class number problems better?

(e) Is there a v-adic Chowla-Selberg phenomenon? (I.e., can one decompose a combination, similar to that occurring in the Chowla-Selberg formula, of Γ_v values into natural transcendental ("v-adic periods) and algebraic parts?.) A. Ogus has recently described (unpublished) the p-adic Chowla-Selberg formula using Morita's p-adic gamma function. We just note here that the p-adic periods he uses are different from the Fontaine-Messing periods and that there is some indication (see, e.g., 9.4) that the analogue of his formula holds in the function field case.

(f) In view of the analogies of 0.1.1, can one define reasonable, useful gamma functions and Gauss sums in a situation where there is an elliptic curve with complex multiplication by a quadratic imaginary field?

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