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Automata and Transcendence

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Interesting quantities of number theory or geometry are often defined by analytic processes such as integration or infinite series or product expansions. Central question which transcendence theory addresses is whether they can also be described by a simpler ('finite') algebraic methods, in particular, whether they turn out to be algebraic. For example, the most common analytic description of a number is by its infinite decimal or p-adic expansion and of a function is by power series expansion. It is well-known that eventually periodic expansion represents a rational number (or a function). On the other hand, rational numbers and rational functions over a finite field give eventually periodic expansions.

More generally, we can ask for a similar, simple characterization of digit patterns for algebraic numbers or functions. We will begin by describing the 'automata' criterion of the discrete mathematics for this and will prove two transcendence results in number theory as applications.

First major advance in answering this general question was made by Furstenberg [F67]:

For $r = \sum r_{n_1,\cdots,n_k} x_1^{n_1} \cdots x_k^{n_k}$, put 'diagonal' $Dr = \sum r_{n,\cdots,n} x^n$.

Theorem 1. *For $k = \mathbb{C}$ or $\mathbb{F}_q$, the set of algebraic power series $f(x)$ over $k(x)$ is the same as the set of diagonals Dr of two variable rational functions $r(x_1, x_2)$. The diagonal of a rational function of many variables over $\mathbb{F}_q$ (but not over $\mathbb{C}$) is algebraic.*

Sketch of the proof. : Over $\mathbb{C}$, for small ϵ and $|x|$, we have

$$Dr(x) = \frac{1}{2\pi i} \int_{|z|=\epsilon} r(z, \frac{x}{z}) \frac{dz}{z}$$

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Evaluating by residues gives algebraic function. On the other hand, suppose f is algebraic, satisfying a polynomial equation $P(x, f(x)) = 0$. Also assume $f(0) = 0$ and that 0 is an isolated root of $P(0, w) = 0$, then expressing

$$f(x) = \int_\gamma w \frac{\partial P}{\partial w}(x, w)/P(x, w) dw$$

as integral above, we see that f is a diagonal of a rational function. Also, if we have more than two variables, the resulting contour integration of algebraic function of two variables is in general transcendental. Though this proof does not work over $\mathbb{F}_q$, the resulting formulas can be directly checked. □

Deligne [D84] generalized the last statement of the Theorem to algebraic functions of many variables.

From now on, we will restrict to the base field $k = \mathbb{F}_q$.

Corollary 1. *(1) Algebraic power series are closed under Hadamard (term-by-term) product.*

(2) $\sum a_n x^n$ is algebraic if and only if $\sum_{a_n = a} x^n$ is algebraic for each a.

Sketch of the proof. : If $\sum a_n x^n = Dr_1(x_1, x_2)$, $\sum b_n x^n = Dr_2(x_1, x_2)$, then $\sum a_n b_n x^n = D(r_1(x_1, x_2) r_2(x_3, x_4))$. This implies (1). Then (2) follows from (1), by expanding out $\sum_{a_n = a} x^n = \sum (1 - (a_n - a)^{q-1}) x^n$. □

By the Corollary, we can focus on characteristic sequences on subsets of natural numbers, i.e., on the series $\sum x^{n_i}$, if we want. The main theorem giving the automata criterion for algebraicity is the following theorem due to Christol [C79], [CKMR80]. In fact, the equivalence of the last two conditions (as well as another description in language of substitutions) of the Theorem is due to Cobham [C072]. We sketch the proof later.

Theorem 2. *(i) $\sum f_n x^n$ is algebraic over $\mathbb{F}_q(x)$ if and only if (ii) $f_n \in \mathbb{F}_q$ is produced by a finite-state-q-automata if and only if (iii) there are only finitely many subsequences of the form $f_{q^k n + r}$ with $r < q^k$.*

Here, m-automata (we will usually use $m = q$) consists of a finite set S of states, a table of how D_m, the digits (base m) operate on S and a map τ from S to $\mathbb{F}_q$ (or some alphabet in general). For given input n, fed in digit by digit from left, each digit changing the state by the rule provided by the table, the output is $\tau(n\alpha)$ where α is some chosen initial state. For more details, see the expository article [A87].

Example: The following table together with $\tau(\alpha) = \tau(\gamma) = 0$ and $\tau(\beta) = 1$ defines a p-automata whose output f_n is the characteristic sequence of $\{p^m\}$, i.e., the numbers of the form $1000\cdots$ base p.

	α	β	γ
0	α	β	γ
1	β	γ	γ
i	γ	γ	γ

It is easy to see directly that $f = \sum x^{p^n}$ satisfies $f^p - f + x = 0$, in accordance with the Theorem 2. In fact, this f is Mahler's famous counterexample to analogue of Roth's theorem in characteristic p: The partial sums to f approximate f with Liouville bounds.

As a warm-up to the proof in the second application, we now prove that f is transcendental in finite characteristic $\ell \neq p$: Clearly, there are infinitely many k's such that $0 < p^m - \ell^k \mu < \ell^k$, for some m and some $0 < \mu < \ell$. So the subsequence $f_{\ell^k n + (p^m - \ell^k)}$ assumes value 1, for $n < \ell$. The next n for which it is 1 corresponds to $\ell^k n + (p^m - \ell^k \mu) = p^{m+w}$. So ℓ^k divides $p^w - 1$ and hence $n = \mu + p^m(p^w - 1)/\ell^k > p^m \to \infty$ as k (and hence m) tends to ∞, there are infinitely many such subsequences and (iii) of Theorem 2 proves the claim.

In fact, this is a special case of a very general result of Cobham [C069]:

Theorem 3. *Non-periodic sequences produced by m-automata can not be produced by n-automata, if m and n are multiplicatively independent.*

We do not sketch the proof. To quote Eilenberg [E74], "The proof is correct, long and hard. It is a challenge to find a more reasonable proof of this fine theorem". Combining with the Theorem 2, we immediately get

Corollary 2. *If $\sum x^{n_i}$ is irrational and algebraic in one finite characteristic, then it is transcendental in all other finite characteristics.*

It should be noted here that the proof of often quoted result of Loxton and van der Poorten that under the hypothesis of the Corollary, the real number $\sum 10^{-n_i}$ is transcendental, has a gap, as van der Poorten has mentioned to the author. Thus, this statement and similar p-adic statement remain as challenging open questions.

We now present an example due to a Richie [Ri63], which we will use and which does not look as special as the first example.

Theorem 4. *Characteristic sequence of squares is not produced by any 2-automata.*

Sketch of the proof. : Consider the 2-automata given by the following table and $\tau(f) = 1$, $\tau(\text{rest}) = 0$.

	α	s_1	s_2	s_3	s_4	f	n
0	n	s_3	s_4	s_4	s_3	n	n
1	s_1	s_2	s_1	n	f	n	n

A straight entry chase in the table shows that this produces the characteristic function χ_A of the set $A = \{1^n0^m1 : n, m > 0, n+m \text{ odd}\}$. It is easy to see that the intersection of A with the set of squares is $B = \{1^n0^{n+1}1 : n > 0\}$ and also that in general, the intersection corresponds to the direct product of Automata or the Hadamard product of series.

But χ_B can not be produced by 2-automata: As there are only a finite number of states, $1^\ell\alpha = 1^{\ell+m}\alpha$, for some ℓ, m. But $1^{\ell+m}0^{\ell+1}1$ is not in B whereas $1^\ell 0^{\ell+1}1 \in B$. □

We finish this introduction by giving a very brief sketch of ideas involved in the proof of the Theorem 2. For more details, see [CKMR80] or [A87].

(ii) implies (iii): There are only a finitely many possible maps $\beta : S \to S$ and any f_{q^kn+r} is of the form $\tau(\beta(n\alpha))$.

(iii) implies (i): Let V be the vector space over $\mathbb{F}_q(x)$ generated by monomials in $\sum f_{q^kn+r}x^n$. Then V is finite dimensional with $fV \subset V$, so f satisfies its characteristic polynomial.

(i) implies (iii): For $0 \leq r < q$, define C_r (twisted Cartier operators, as Anderson pointed out to me) by $C_r(\sum f_nx^n) = \sum f_{qn+r}x^n$. Considering $\mathbb{F}_q$ vector space generated by the roots of the polynomial satisfied by f, we can assume that $\sum_{i=0}^k a_if^{q^i} = 0$, with $a_0 \neq 0$. Using $g = \sum_{r=0}^{q-1} x^r(C_r(g))^q$ and $C_r(g^qh) = gC_r(h)$, we see that

$$\{h \in \mathbb{F}_q((x)) : h = \sum_{i=0}^k h_i(f/a_0)^{q^i}, h_i \in \mathbb{F}_q[x], \deg h_i \leq \max(\deg a_0, \deg a_ia_0^{q^{i-2}})\}$$

is a finite set containing f and stable under C_r's.

(iii) implies (ii): If there are m subsequences $f_n^{(i)}$ with $f_n^{(1)} = f_n$ say, put $S := \{\alpha := \alpha_1, \cdots, \alpha_m\}$. Define digit action : $r\alpha_i := \alpha_k$ if $f_{qn+r}^{(i)} = f_n^{(k)}$. Define $\tau(\alpha_i) := f_n$, if $n^-\alpha_1 = \alpha_i$ with n^- being base q expansion of n written in the reverse order. □

Now we will present two applications of the Theorem 2 to Number theory, where the quantities are not naturally presented as power series, but are convertible to manageable power series.

1. Application I

Let k be an algebraic closure of $\mathbb{F}_p$, q be a variable and let

$$a_4 := a_4(q) := \sum_{n\geq 1} \frac{-5n^3q^n}{1-q^n}, \quad a_6 := a_6(q) := \sum_{n\geq 1} \frac{-(7n^5+5n^3)q^n}{12(1-q^n)}$$

THEOREM 5. *The period q of the Tate elliptic curve $y^2 + xy = x^3 + a_4x + a_6$ over $K := k(a_4, a_6)$ is transcendental over K.*

This Theorem was proved by Voloch [V94] using Igusa theory and we provided another proof [T94]. The Theorem can be considered as an analogue of the result of Siegel and Schneider on the transcendence of the period of elliptic curve (over a number field) over its field of definition. If, more appropriately, we consider the q as the multiplicative version of the period, then it can be considered as analogues of conjectures of Mahler (in p-adic setting) and of Manin (in the complex setting), as pointed out to us by Waldschmidt. These conjectures themselves were settled since then by Barre-Sirieix, Diaz, Gramain, Philibert [BDGP96]. In particular, they show that the '$\log_p q$' appearing in the p-adic Birch-Swinnerton-Dyer conjectures of Mazur, Tate and Teitelbaum (Theorem of Stevens/Greenberg) does not vanish, so that the order of vanishing is exactly as predicted in the conjectures.

Sketch of the proof. : First, let $p = 2$. Then

$$a_4 = a_6 = \sum_{n\ odd \geq 1} q^n/(1-q^n) = \sum_{n\ odd\geq 1} \sum_{k=0}^{\infty} q^{kn} = \sum_{m=1}^{\infty} d_o(m) q^m,$$

where $d_o(m)$ = number of odd positive divisors of m. Hence, if $m = 2^k \prod p_i^{m_i}$, then $d_o(m) = \prod(m_i + 1)$. So $d_o(m)$ is odd if and only if $m = n^2$ or $2n^2$. Hence, with $f := \sum q^{n^2}$ (essentially theta), we have

$$a_4 = \sum_{n=1}^{\infty} (q^{n^2} + q^{2n^2}) = f + f^2,$$

Now the Theorems 2 and 4 imply that f and so $a_4 = a_6 = f + f^2$ is transcendental over $k(q)$ i.e., q is transcendental over $K = k(a_4)$, finishing the proof when $p = 2$.

Now consider the case of general p: We will show:

(1) f is transcendental over $k(q)$: This follows from the generalization [E74] of the Theorem 4 to any base.

(2) a_4 and a_6 are algebraically dependent over k.

(3) f is algebraic over $k(\overline{a}_4, \overline{a}_6)$, where $\overline{a}_4 := a_4(q^2)$, $\overline{a}_6 := a_6(q^2)$.

It is easy to see that (1), (2) and (3) imply the Theorem.

Proof of (2): Elementary congruences show that $a_4 = a_6$ if $p = 2$, $a_4 = 0$ if $p = 5$ and $a_4 = 5a_6$ if $p = 7$.

Swinnerton-Dyer [S-D73] noticed that expressing the fact that the Hasse-invariant of the Tate elliptic curve is one in terms of a_4 and a_6 gives (2) for $p > 3$.

For $p = 3$, the Hasse- invariant is identically one and we do not get a relation this way. Nonetheless, we claim that $a_6 + a_4 + 2a_4^2 = 0$: In fact, $a_4 + a_6$ is

$$\sum_{n\equiv 2,4(9)} \frac{q^n}{1-q^n} + 2 \sum_{n\equiv 5,7(9)} \frac{q^n}{1-q^n} = \sum_{3\nmid n} \frac{n + (-1)^{n-3\lfloor n/3 \rfloor}}{3} \frac{q^n}{1-q^n} = a_4^2$$

First two equalities follow by analyzing n modulo 9 and the last equality is a rearrangement of Ramanujan's identity (19) of [R16].

Proof of (3): Note that $\overline{a_4}$, $\overline{a_6}$, f are connected to the well-known Eisenstein series and theta function: $\theta = 1 + 2f$, $e_4 = 1 - 48\overline{a}_4$ and $e_6 = 1 - 72\overline{a}_4 + 864\overline{a}_6$.

We have following explicit algebraic dependency relation between the three modular forms:

$$4e_6^2 - 4e_4^3 + 27e_4^2\theta^8 - 54e_4\theta^{16} + 27\theta^{24} = 0$$

A straight translation to the original variables implies (3). □

2. Application II

Here we give an application to the transcendence of the gamma monomials (in the function field case) evaluated at fractions.

The most well-known classical result is : $\Gamma(1/2) = (-1/2)! = \sqrt{\pi}$. Chudnovsky [Chu84] showed that Γ values at proper fractions with denominator 4 or 6 are transcendental. This is basically all that is known about the transcendence of the individual gamma values at proper fractions.

We will consider $A = \mathbb{F}_q[T]$, $K = \mathbb{F}_q(T)$, $K_\infty = \mathbb{F}_q((t))$ (with $t = 1/T$) and $\Omega =$ the completion of an algebraic closure of K_∞. These are analogues of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively, in the theory of function fields.

Carlitz factorial $\Pi(n) \in A$ for $n \in \mathbb{Z}_{\geq 0}$ is defined by

$$[n] := T^{q^n} - T, \ D_n := \prod_{k=1}^{n} [k]^{q^{n-k}}$$

$$\Pi(n) := \prod D_i^{n_i}, \text{ for } n = \sum n_i q^i, \ 0 \leq n_i < q.$$

David Goss noticed that

$$\overline{D_i} := \frac{D_i}{T^{\deg D_i}} = 1 - t^{q^i - q^{i-1}} + \text{higher degree terms} \to 1 \text{ as } i \to \infty$$

gives an interpolation: $\overline{\Pi}(n) \in K_\infty$ for $n \in \mathbb{Z}_p$ by

$$n! := \overline{\Pi}(n) := \prod \overline{D}_i^{n_i} \text{ for } n = \sum n_i q^i, \ 0 \leq n_i < q.$$

Why is this gamma function good? We mention some reasons by catch-words: analogous prime factorization, divisibility properties, analogous occurrence in the Taylor coefficients in the relevant exponential (of Carlitz-Drinfeld module), right functional equations, interpolations at finite primes and connection with Gauss sums (Gross-Koblitz type formula), connection with periods (Chowla-Selberg type formula). See [T92] for details and references.

For our factorial $(-1/2)! = \sqrt{\tilde{\pi}}$, when $p \neq 2$, where $\tilde{\pi} \in \Omega$ is the period of the Carlitz-Drinfeld exponential (analogue of $2\pi i \in \mathbb{C}$). This is known [W41] to be transcendental. See also [A87] for Automata style proof.

In fact, the correct analogue for classical $(-1/2)!$ is $(1/(1-q))!$, because 2 and $q-1$ are the number of roots of unity in $\mathbb{Z}$ and $\mathbb{F}_q[T]$, respectively. And analogue of Chudnovsky's $4, 6$ (which are numbers of roots of unity in quadratic imaginary fields) is $q^2 - 1$ (number of roots of unity in $\mathbb{F}_{q^2}[T]$).

We have analogue of the Chowla-Selberg formula for constant field extensions, expressing periods of Drinfeld modules with complex multiplication in terms of gamma values at some particular fractions. Combining with their results on transcendence of periods, Jing Yu [Y92] and Thiery [Thi92] (by different techniques) proved that $(1/(1-q^2))!$ is transcendental. So far, the results are parallel in the classical and the function field case.

Using Christol's criterion, we proved [T95] the transcendence of gamma values with any denominator, but with some restrictions on numerators. Using logarithmic derivatives on the product formula, which makes exponents relevant only modulo p, instead (trick also used by L. Denis), Allouche [A95] then proved the transcendence for all values at fractions. Finally, we have shown how Allouche's technique, in fact, settles completely the question of which monomials in gamma values at fractions are algebraic and which are transcendental. Let us describe the result in detail and sketch the simplest case. For the full details and the history, we refer to [A95].

For a proper fraction f, let $\langle f\rangle$ denote its fractional part. For a finite formal sum $\underline{f} = \sum m_i[f_i]$, with $m_i \in \mathbb{Z}$ and $f_i \in \mathbb{Q}$, put $m(f) = \sum m_i\langle -f_i\rangle$ and $\Gamma(\underline{f}) = \prod \Gamma(f_i)^{m_i}$. Also, for $\sigma \in \mathbb{Z}$, put $\underline{f}^{(\sigma)} = \sum m_i[f_i\sigma]$

Let $\underline{f}$ be given, with all f_i's having a common denominator say N.

Theorem 6. *(Usual Gamma) If $m(\underline{f}^{(\sigma)}) = 0$ for all σ relatively prime to N, then $\Gamma(\underline{f})$ is algebraic.*

The way we have presented it, this was conjectured (together with Galois action) by Deligne [D79] (proved in [D82]). But, using the ideas of Lang and Kubert on distributions, it was shown in the appendix by Koblitz and Ogus to [D79], that the algebraicity also follows by taking right combinations of multiplication and reflection formula. The converse is not known, but is conjectured, because it follows from the general belief that the functional equations force all the relations and also from conjectures [D82] in algebraic geometry.

Theorem 7. *(Our case) $m(\underline{f}^{(\sigma)}) = 0$ for all $\sigma = q^j$ if and only if $\Gamma(\underline{f})$ is algebraic.*

The conditions of the two theorems are analogous, since the Galois group in the relevant cyclotomic theory is $(\mathbb{Z}/N\mathbb{Z})^* = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ in the classical case and $\mathrm{Gal}(\mathbb{F}_q(T)(\zeta_k)/\mathbb{F}_q(T))$ (generated by q-power Frobenius) in our case. In our case, the monomials are not combinations of naive analogues of multiplication and reflection formula, but the only if part was proved directly [T91] by showing that in the multiplicative basis of factorials of $1/(1-q^i)$ our monomial turns out to be a trivial monomial. Automata theory takes care of the converse as mentioned above. We explain the simplest case below.

Sketch of the proof. : Simplest case: $(1/(1-q^\mu))!$ is transcendental:

After some simple manipulations, we get

$$P := (\frac{1}{1-q^\mu})!^{1-q^\mu} = \prod_{i=1}^{\infty}\prod_{j=0}^{\mu-1}(1-t^{q^{\mu i}-q^j})$$

Write $P = \sum a_n t^n$ and use Christol's criterion (Theorem 2).
Consider the representations

$$n = \sum (q^{\mu i} - q^j), \quad \text{all terms distinct}, \ 0 \leq j < \mu.$$

If such a representation is impossible, then $a_n = 0$, whereas if such a representation is unique (not always the case), then $a_n = \pm 1$.

Claim: There are infinitely many subsequences of the form $b_n := a_{q^k n + (q^k - k)} = a_{q^k(n+1)-k}$. (Hence P is transcendental).

We show that for sufficiently large k, b_n is 0 for at least the first $k/q^{\mu-1} - 1$ values of n. Since $k/q^{\mu-1} - 1 \to \infty$ as $k \to \infty$, to show that there are infinitely many distinct subsequences (b_n), it is enough to show that infinitely many of these subsequences are not identically zero.

Let $m := k/q^{\mu-1} - 1$ and $n := q^\mu + q^{2\mu} + \cdots + q^{m\mu}$. Then

$$q^k(n+1) - k = (q^k - q^{\mu-1}) + (q^{k+\mu} - q^{\mu-1}) + \cdots + (q^{k+m\mu} - q^{\mu-1})$$

is the unique representation of the form required and so $b_n = \pm 1 \neq 0$. $\square$

Since then another proof of this case has been given by Hellegouarch [H95], using De Mathan's criterion instead.

In fact, there is another gamma function in function field theory and breakthrough in establishing the transcendence in its values at proper fractions has been recently achieved by S. Sinha [S95], using Anderson's theory of t- motives and solitons, in particular. The proof expresses the gamma values as periods on analogues of Fermat Jacobians in t- motives setting and uses Jing Yu's transcendence results for this. In a sense the reason why we can do better than the classical case by similar (philosophically!) methods is that in the setting of Drinfeld modules and t- motives, we can have arbitrary fractions (and not just half integers) as 'weights'.

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