Automata Methods in Transcendence

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Abstract. The purpose of this expository article is to explain diverse new tools that automata theory provides to tackle transcendence problems in function field arithmetic. We collect and explain various useful results scattered in computer science, formal languages, logic literature and explain how they can be fruitfully used in number theory, dealing with transcendence, refined transcendence and classification problems.

2010 Mathematics Subject Classification. Primary 11J93, 68Q45; Secondary 11B85, 11G09.

Keywords. Automata, periods, Drinfeld modules, special values.

1. Introduction

Naturally occurring interesting numbers (say real, complex, p-adic or their function field counterparts) in number theory or algebraic geometry, such as periods, special values of Γ, ζ, L or other special functions, are usually defined by analytic processes such as infinite sums, products, limits or integrals. In transcendence theory, we are interested in knowing whether they are linked algebraically or not, i.e., whether they are transcendental, algebraically independent etc.

Though in science, the usual way to exhibit a number given by a limiting process is by its decimal (base p, p-adic, Laurent series etc.) expansion, such a description had been usually useless for transcendence purposes, for naturally occurring numbers. For usual numbers such as e and π we do not know good description of expansion, and carry over makes it hard to manipulate the expansions.

Work of Christol [C79, CKMR80] (also see Furstenberg [F67]) showed that the series $\sum f_n t^n \in \mathbb{F}_q[[t]]$ for an algebraic function over $\mathbb{F}_q(t)$ ('numbers' in function field arithmetic) can be generated by a finite q-automaton (weakest machine model with no memory and which accepts q different inputs only) taking the expansion digits of n base q one by one as input and producing f_n as the output at the end. Not only that, but conversely such machines produce algebraic series.

This translates the finite algebraic description of the polynomial that function satisfies over $\mathbb{F}_q(t)$ to finite computer science description of patterns of digits. There are similar nice equivalent descriptions given by formal language theory, logic etc. These subjects have been developed a lot and generated various equivalent viewpoints and many more implications, which are enough for transcendence applications and often directly applicable as we will see.

*The author is supported by NSA grant H98230-08-1-0049.

Our previous accounts [T94, T96, T98, T04] explained applications in detail while quoting automata results used. Here we explain automata tools in more detail (at least with definitions and full statements and sketches of proofs of things we use) and only sketch and quote applications. We describe a few more things than we use in the hope that they might be of future use.

For more extensive treatments of automata theory, we refer to books [HU79, S85, AS03], and especially the excellent bibliography and bibliographical notes of [AS03].

2. Automata: Implications, equivalences: Definitions and statements

2.1. Automata: Definition. Let q be a positive integer. *A q-automaton* is a 'machine' which can be in one of finitely many states and takes q possible inputs which act on these states, and finally each state has an output associated with it. Mathematically, we can consider it as a quadruple (S, s_0, T, O) where S is a finite set (think of set of 'states'), $s_0 \in S$ (think of 'initial state'), $T : \{0, 1, \cdots q-1\} \times S \to S$ is a function (called transition function describing how possible inputs marked by digits from 0 to $q-1$ act on states) and $O : S \to F$ is a function (called output function describing the output in F when the machine is in 'final state').

Of course, one is interested in infinitely many inputs represented by all integers $n \geq 0$. This is achieved as follows. The input n is expended into its base q expansion as $\sum n_i q^i$, $0 \leq n_i < q$, and is fed digit by digit, say from left to right, and output on the final state ns_0 is read when one is finished.

Without loss of generality, F is a finite set, which will in fact be a finite field in our applications. Often the question can be reduced to only two output values: 0 (rejection), and 1 (acceptance or recognition). Hence we call a subset M of $\mathbb{Z}_+$ *q-automatic subset*, if for some q-automaton we have $O(ns_0) = 1$ if and only if $n \in M$. Our main interest is of course infinite subsets, and thus we call an infinite increasing sequence of positive integers listing all elements of such a set *a q-automatic sequence.*

2.2. Visualization and representations. Those familiar with Turing machine format can visualize a machine whose control head moves, depending on input and transition function, on a tape representing (outputs on) states and once it stops, one just reads the output at that position. Those familiar with neural network models can visualize active neuron (current state s) firing on input i and making another neuron $T(i, s)$ active.

If N is the number of states, one can describe the automaton by a transition function by giving $q \times N$ table and output function; or one can describe it by labeled graph, with vertices representing the states with initial and accepting states specially marked and edges from a state to another labeled by digits giving those transitions. Then accepted words are just ordered lists of labels of edges of all paths from initial to accepted states.

For automaton with output $0, 1$ and with N states, we have a matrix representation $h : \{0, 1, \cdots, q-1\} \to Mat_{N\times N}(\{0, 1\})$ with $h(d)_{ij}$ being 1, if the digit d takes the state i to state j, and being 0 otherwise. This is extended by taking concatenation of digits to multiplication of corresponding matrices. For N-vectors v_1, v_2 with $0, 1$ entries, corresponding to initial and final states, $v_1 h(w) v_2 > 0$, if and only if the word w is accepted.

2.3. Variants on automata models. The notion of q-automata (also called finite automata in the literature) is quite robust in the sense that many a priori different variants like (i) non-deterministic automata (think of a parallel computer), where the transition function is multi-valued, so that acceptance is through some possible transitions path, (ii) incomplete automata, where transition function is partial i.e., not always defined, (iii) non-deterministic automata with ϵ-moves, that is state can change without input, (iv) two-way automata, (v) with one marker, where the control head can return as memory etc.

We refer to [HU79, S85, AS03] for definitions and proofs of equivalences. It is easy to see that at the expense of exponential blow up of the number of states, we can use usual automata simulating non-deterministic one, by using its new states to be all subsets of set of states of the non-deterministic one and keeping track of possibilities at each stage. The non-deterministic ones are quite useful in proofs because of their flexibility. For example, to make a machine accepting n's accepted by two machines, we just feed the new start state into start states of the two machines on any input, possible in non-deterministic realm.

2.4. Variants on digit models. Also, whether (a) we allow leading zeros in base q expansions of n, whether (b) we read from left to right or right to left, or whether (c) we use base q or q^k, does not change the final outcome.

To see that there is no difference on (a), we just introduce a new start state, which stays the same on input zero and on other input goes to the start state of the new machine.

For (b), we just reverse arrows and interchange initial and final states, which is possible in non-deterministic realm (if we insist on one start state, introduce one and feed into (old) terminal states on any input).

For (c), given a q-automaton, corresponding q^k-automaton would have same states, start and final states with transition on a digit base q^k being the transition on corresponding base q word, and conversely given a q^k-automaton, from each state make k-length paths with new states according to q-digits of the base q word corresponding to q^k base digit.

2.5. Christol's theorem. Automata method is based on the following theorem of Christol [C79, CKMR80].

Theorem 1. *The series* $\sum f_n t^n \in \mathbb{F}_q[[t]]$ *is algebraic over* $\mathbb{F}_q(t)$*, if and only if there is a* q*-automaton which gives output* f_n *on input* n*.*

2.6. Remark. Note that for algebraicity questions, general (finite tail) Laurent series immediately reduce to corresponding power series part, and also whether a series in t or in $1/t$ does not make a difference.

2.7. Some simple consequences. The power of the method derives from the non-obvious equivalence, as well as various viewpoints and tools from which automata has been studied over the years by computer scientists, logicians, formal language theory experts. We will describe these and the applications. But first let us see, how such unusual equivalence gives new perspective on algebraic relationships.

Theorem 2. *(1) If $\sum f_n t^n, \sum f'_n t^n \in \mathbb{F}_q[[t]]$ are algebraic, then so is $\sum f_n f'_n t^n$.*

(2) The series $\sum f_n t^n \in \mathbb{F}_q[[t]]$ is algebraic, if and only if each $\sum_{f_n=f} t^n$ is algebraic for each $f \in \mathbb{F}_q$.

Proof. To see (1), just imagine two corresponding automata running in parallel (take direct product) and at the end combine their outputs by multiplying, which can be achieved by a table. (Hence, the algebraicity of this Hadamard or term-by-term product is obvious from the automata viewpoint, but hard from the definition of algebraicity, whereas for the algebraicity of usual product of two algebraic series, the situation is reversed!)

The 'if' direction of (2) is clear, by taking linear combinations. The hard converse direction is clear from automata viewpoint: Change the output function by sending f to 1 and the other elements to 0. □

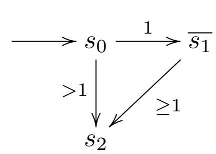
Here is another direct nice consequence (pointed out to me by Allouche): The series $\sum a_i t^{p^i} \in \mathbb{F}_p[[t]]$ is algebraic over $\mathbb{F}_p(t)$ if and only if $\sum a_i t^i \in \mathbb{F}_p(t)$. This is since both the statements are equivalent to the fact that a_i is an eventually periodic sequence.

2.8. Examples. Now we give three examples, all with outputs only $0, 1$, so that the corresponding series is of form $s = \sum_{m \in M} t^n$.

Example I: We describe the transition function by the following table, where $2 \leq i < q$, and the output function separately by $O(s_0) = O(s_2) = 0$ and $O(s_1) = 1$

	s_0	s_1	s_2
0	s_0	s_1	s_2
1	s_1	s_2	s_2
i	s_2	s_2	s_2

or both by graph, with accepted states overlined and only movements shown:



To get the final state s_1 to end up with output 1, digit expansion of m has to be 1 followed by all zeros, i.e., m is a power of q, so that the corresponding series is $s = \sum t^{q^n}$, which is algebraic, as you can see by telescoping and noticing that q-th power is linear: $s^q - s = -t$.

Example II: Let $q = 2$. Consider the output $O(s_0) = O(s_1) = 1$ and $O(s_2) = O(s_3) = 0$, and the transition function

	s_0	s_1	s_2	s_3
0	s_0	s_2	s_1	s_3
1	s_1	s_3	s_1	s_3

Now $m \in M$ precisely when ms_0 is either s_0 or s_1. Note $0, 1 \in M$ and we claim that $m \in M$ precisely when $4m, 4m+1 \in M$. This is immediately seen by diagram-chase of automata theory: Note that base 2 expansions of $4m, 4m+1$ are obtained by appending 00, 01 respectively to the expansion of m and the diagram chase shows that if you are in (out respectively) $\{s_0, s_1\}$ you stay in (out respectively) by the action of 00 or 01. We can reformulate this in terms of the generating function s as $s = \sum t^m = (1+t)\sum t^{4m} = (1+t)s^4$, so that we have algebraic series $s = (1+t)^{-1/3}$.

Example III: Let $q = 3$. Consider two state machine whose initial state s_0 is the only accepting state and machine stays in the initial state on input 0 or 2, but goes to error (i.e., the other non-accepting) state s_1 on input 1, and once it is there, stays there on any input.

Here it is immediate that $m \in M$ precisely when its base 3 expansion does not contain digit 1 (Cantor type description!), so that $s = s^3 + t^2 s^3$.

Example IV: We leave it as a fun exercise to the reader to build a 2-automaton representing Thue-Morse sequence containing m's whose base 2 expansion has even number of ones (by keeping track of parity of the number) and get algebraic equation for the corresponding series. The reader should also work out details of other equivalences mentioned below on this example. We also mention a simple non-example: $\{1^n 0^n\}$ cannot be recognized by automata.

2.9. Languages and grammars. We saw how digit pattern of algebraic power series is described by machines. Other common way of describing patterns is to give generating or accepting rule for the language exponents or describe it by logical sentences. In fact, all these a priori different and independent descriptions of classes have converged to the same robust classes. We will exploit this fact later.

In this language perspective, the digits are now possible words so that $\mathbb{Z}_{\geq 0}$ consists of all possible tries to make sentences, whereas a subset M of the set of non-negative integers can be considered as a language of particular class of 'grammatical' sentences.

Grammar teaches us, for example, how a sentence can break up into a noun phrase and a verb phrase; a noun phrase can break up into an adjective and another noun phrase; a noun phrase can be a noun like the word 'man', and adjective can be a word like 'tall'.

In a formal language theory, this is abstracted by defining *a grammar* to be a tuple $(V, T, \{P \to Q\}, N)$. Here V is a finite set (think of variables or non-terminals or syntactic classes), T is a disjoint finite set (think of terminals or words), $P \to Q$ are finitely many 'production rules' (think of 'sentence' goes to 'noun phrase', 'verb phrase' etc.) where P is a string on $V \cup T$ containing at least one element of V and Q is a string on $V \cup T$, and $N \in V$ (think of a special start symbol). Finally *the language* generated by this grammar is collections of all possible grammatical sentences, i.e., strings on words (terminals) arising via the allowed productions starting from the start symbol (i.e., $N \to \cdots \to$ string on terminals).

For our applications to q-automata, $T = D_q := \{0, 1, \cdots, q-1\}$, so the digits are possible words, non-negative integers are sentences (do not have to worry about leading zeros as we saw) and a language will thus correspond to a set M of non-negative integers or equivalently a power series $\sum_{m \in M} t^m$. In other words, we are talking about the language of 'exponents'.

The class of a language is defined by the production types allowed.

(i) The language is *regular language* if each production rule is of form $X \to YP$ or $X \to P$, where $X, Y \in V$ and P a string on T.

(ii) The language is *context-free language* if each production rule is of form $X \to Q$, where $X \in V$ and Q a string on $V \cup T$.

(iii) The language is *context-sensitive language* if each production rule is of form $Q_1 X Q_2 \to Q_1 P Q_2$, where Q_i are strings over $V \cup T$ and X and P as above.

Warning: Though the definition of context-sensitive allows replacement only within some context as you would imagine, the context-free and context-sensitive are not opposite notions. In fact, regular is context-free and context-free is context-sensitive.

Our examples I, II, III correspond to regular languages given by production rules (I) $N \to 1$, $N \to N0$; (II) $N \to 0$, $N \to 1$, $N \to N00$, $N \to N01$; (III) $N \to 0$, $N \to 2$, $N \to N0$, $N \to N2$, where in the last two we have allowed leading zeros for simplicity. If we do not want to do that, we use, e.g., for (III), $N \to 0$, $N \to N'$, $N' \to 2$, $N' \to N'0$, $N' \to N'2$.

2.10. Regular expressions. Regular languages on words T can also be described by *regular expressions* defined recursively by strings of symbols in T and operations of concatenation (denoted by juxtaposition), or (denoted by $+$), and repetition (denoted by *).

Rather than giving a formal definition [HU79], we note that our examples I, II, III correspond to regular expressions (I) 10^* on $T = D_q$, (II) $0 + 1(00 + 01)^*$ on $T = D_2$, (III) $0 + 2(0 + 2)^*$ (or simply $(0 + 2)^*$, if we allow leading zeros) on $T = D_3$.

2.11. Fixed point of a q-substitution. A *q-substitution* over S is just a function $f : S \to S^q$. If $q > 1$, writing $f(s) = sw$ and extending by juxtaposition, we see that $f^{(\infty)}(s) = swf(w)f^{(2)}(w) \cdots = s_1 s_2 \cdots$ is a fixed point. If $I : S \to F$ is some function, then we say that $f_n = I(s_n)$ is a sequence which is image of fixed point (starting at s) of q-substitution. Again, we also call the subset of integers where the sequence is 1 by the same name.

The Example I above corresponds to $S = \{a, b, c\}$, $f(a) = abc\cdots c$, $f(b) = bc\cdots c$, $f(c) = c\cdots c$ and $I(a) = I(c) = 0$, $I(b) = 1$, with starting point a, the n-th position of $I(f^\infty(a))$ being one, exactly when n is a power of q.

The Example II above corresponds to $S = \{a, b, c, d\}$. $f(a) = ab$, $f(b) = cd$, $f(c) = bb$, $f(d) = dd$, and $I(a) = I(b) = 1$, $I(c) = I(d) = 0$, with starting point a. We give a little more details to help understand the notation. The fixed point is $abcdbbddcdcddddddbbddbbdddd\cdots$, its image is $11001100000000001100110000\cdots$, with 1's at (base 10) positions $0, 1, 4, 5, 16, 17, 20, 21, \cdots$.

The Example III above corresponds to $S = \{a, b\}$, $f(a) = aba$, $f(b) = bbb$ and $I(a) = 1$, $I(b) = 0$, with starting point a.

2.12. *q*-definability. Let $v_q(x)$ denote the largest power of q dividing x, for a non-zero x, and $v_q(0) = 1$.

Again, rather than giving a formal definition [BHMV94], we describe *q-definable set* informally as a subset of $\mathbb{Z}_{\geq 0}$ which can be defined by a first order formula over the structure $(\mathbb{Z}_{\geq 0}, +, v_q)$. Roughly, this means that the subset is defined by a statement involving sentential logical operations and quantifiers running over non-negative integers (but not, for example, over subsets or functions), and involving only $+$, v_q (but not, say multiplication, or individual non-negative integers,unless they are defined by such properties). The following examples should make this clearer.

The Example I above corresponds to $v_q(x) = x$, which isolates powers of q. Let us see, following [BHMV94], how to build a formula for example III. Note that '0' can be defined as x with 'for all y, $x+y = y$' and '$x \geq y, x > y$' etc. can be defined by 'there is z such that $x = y+z$' and so on. Then the formula we need is 'there is no y such that y is a power of 3 and 1 is a coefficient of y in the base 3 expansion of x', where the first part is done in Example I above and second part is taken care of by 'there exist w, z such that $x = z + 1 * y + w$ and $w < y$ and ($v_3(z) > y$ or $z = 0$)'. We leave (II) to interested readers by saying that the accepted words can be described as those having digit zero at any even numbered position from right.

We refer to [BHMV94] for detailed description and references to literature with this viewpoint and for example, work of Büchi in 1960 giving 'second order arithmetic' description of automata.

2.13. $q = 1$. There is a way to define things, so that for $q = 1$, the notions give periodic sequences. We ignore this, except for pointing out this terminology.

2.14. Equivalent notions to automata.

Theorem 3. *Let $q > 1$ be an integer, and F be a finite set. Consider a F-valued sequence f_n and for each $f \in F$, put $M_f := \{m : f_m = f\}$. Then the following are equivalent.*

(i) The sequence f_n is q-automatic.

(ii) There are only finitely many distinct subsequences of the form $n \to f_{q^k n+r}$, where k varies through positive integers and r through $0 \leq r < q^k$.

(iii) The sequence f_n is q-definable.

(iv) For each $f \in F$, M_f is a regular language.

(v) For each $f \in F$, M_f is given by a regular expression.

(vi) For each $f \in F$, M_f is an image of a fixed point of a q-substitution.

Further, if q is a power of a prime p, $F = \mathbb{F}_q$ then these are equivalent to the following.

(vii) The series $s = \sum f_n t^n$ is algebraic over $\mathbb{F}_q(t)$.

(viii) The series s is (a diagonal) $\sum f_{n\cdots n} t^n$, where $\sum f_{n_1\cdots n_k} x_1^{n_1} \cdots x_k^{n_k}$ is a rational function in $\mathbb{F}_q(x_1, \cdots, x_k)$.

We have already given several equivalences in our examples. For the Example I, the series is a diagonal of the rational function $x_1/(1 - x_1^{q-1} - x_2)$, and the only subsequences of the type we consider are clearly f_n itself and identically 0 sequence. We leave the other examples to the reader.

2.15. Some properties of automatic sequences. Instead of equivalent notions, we now look at implications of automata which are simple to check and thus good tools to prove transcendence results.

Language theoretic viewpoint immediately suggests that long enough grammatical sentences should contain some parts, which can be pumped any number of times retaining grammatical property. For example, from 'he is a friend of mine' we can have 'he is a friend of a friend of mine' etc. Indeed, there are such pumping lemmas for many languages, as we will see and use below.

Theorem 4. *(Pumping lemma for finite automata/regular languages) Let $\mathcal{S}$ be an q-automatic set. Then there is N such that any word in $\mathcal{S}$ of length at least N can be written as juxtaposition xyz, with y being a non-empty word and length of xy not more than N and so that for all i, the juxtaposition xy^iz is also in $\mathcal{S}$.*

Proof. The number N is just (greater than) the number of states in the corresponding automaton. As you keep inputing the digits from left, we will get a repeat of states: the portion between the repeat is y, and can be clearly pumped. □

Here are some results [Co72, E74, A87, AS03] about restrictions on densities, gaps and asymptotic behavior of automatic sequences:

Theorem 5. *Let $\mathcal{S} = \{n_i\}$ be a q-automatic set, with n_i an increasing sequence. Define its maximum growth rate to be $\limsup\ n_{i+1}/n_i$, its natural density to be $\lim s_n/n$, where $s_x := |\{n_i \leq x\}|$, and its logarithmic density to be $\lim(\sum 1/n_i)/\log(n)$, where the sum is over $n_i \leq n$. Then*

(1) The maximum growth rate is finite.

(2) Either the maximum growth rate is more than one, or limsup $(n_{i+1} - n_i) < \infty$, and these are mutually exclusive.

(3) The logarithmic density exists and if the natural density exists, then both are the same and rational. If the natural density is zero, then either there is an integer $d \geq 1$ and a real number s with $0 < s < 1$ such that

$$0 < \liminf\ s_x/(x^s \log^{d-1} x) < \limsup\ s_x/(x^s \log^{d-1} x) < \infty,$$

or there are integers $d \geq 1$, $m \geq 2$ and rational $c > 0$ such that s_x is asymptotic to $c(\log_m x)^{d-1}$.

(4) Consider the characteristic sequence χ of $\mathcal{S}$, namely the sequence $\chi(n)$ which takes values one or zero according as n is in $\mathcal{S}$ or not. There is c such that the number $\beta(n)$ of distinct blocks of given length n occurring in this sequence of zero-ones is at most cn. The number $\beta(n)$ is at least n, for $\chi(n)$ not ultimately periodic.

(5) There is a subsequence n'_i of n_i such that $n'_{i+1}/n'_i \to q^d$. More precisely, there are non-negative integers a, $b > 0$, c, $d > 0$ such that for all positive integers n, $aq^{nd} + b(q^{nd} - 1)/(q^d - 1) + c \in \mathcal{S}$.

(6) The series $\sum N_k t^k \in \mathbb{Q}[[t]]$, where N_k is number of elements in $\mathcal{S}$ of k digits, is a rational function in $\mathbb{Q}(t)$.

(7) If $\limsup \ s_n/\log(n)$ is infinite, then $\liminf \ (n_{i+1} - n_i) < \infty$ (i.e., small gaps infinitely often). If the natural density is zero, then maximum growth rate is more than one (i.e., large gaps infinitely often).

3. Sketches of proofs

We now sketch proofs of some parts of Theorems 3 (which implies Theorem 1), 5, as well as of some other facts mentioned above. For full proofs or parts we do not cover, we refer to several available treatments such as [HU79, S85, AS03, CKMR80, Co69, Co72, E74, F67, A87, BHMV94].

3.1. Ideas connecting parts of Theorem 3. (i) implies (ii): If we note that the base q expansion of $q^k n + r$ is obtained from that of n by just appending the expansion of r (with leading zeros, if necessary, to make it of size k), this is immediate, since there are only finitely many possible maps $\beta : S \to S$ and any $f_{q^k n+r}$ is of the form $O(\beta(ns_0))$. (Equivalence of regular languages with (ii) is known as Myhill-Nerode theorem in language theory perspective.)

(ii) implies (vii): Let V be the vector space over $\mathbb{F}_q(t)$ generated by monomials in $\sum f_{q^k n+r} t^n$. Then V is finite dimensional with $sV \subset V$, so s satisfies its characteristic polynomial.

(vii) implies (ii): For $0 \leq r < q$, define C_r (twisted Cartier operators) by $C_r(\sum f_n x^n) = \sum f_{qn+r} x^n$. Considering the vector space over $\mathbb{F}_q$ generated by the roots of the polynomial satisfied by s, we can assume that $\sum_{i=0}^{k} a_i s^{q^i} = 0$, with $a_0 \neq 0$. Using $g = \sum_{r=0}^{q-1} x^r (C_r(g))^q$ and $C_r(g^q h) = gC_r(h)$, we see that

$$\{h \in \mathbb{F}_q((x)) : h = \sum_{i=0}^{k} h_i (s/a_0)^{q^i}, h_i \in \mathbb{F}_q[x], \deg h_i \leq \max(\deg a_0, \deg a_i a_0^{q^{i-2}})\}$$

is a finite set containing s and stable under C_r's.

(ii) implies (i): If there are m subsequences $f_n^{(i)}$ with $f_n^{(1)} = f_n$ say, put $S := \{s_0 := \alpha_1, \cdots, \alpha_m\}$. Define a digit action by, $r\alpha_i := \alpha_k$ if $f_{qn+r}^{(i)} = f_n^{(k)}$. Define

$O(\alpha_i) := f_n$, if $n^- \alpha_1 = \alpha_i$ with n^- being the base q expansion of n written in the reverse order.

Equivalence of (i) and (vi): Given a q-automaton (S, s_0, T, O), construct q-substitution with same start state and image map and with substitution function $f(s) = T(0, s)T(1, s) \cdots T(q-1, s)$. Conversely, given a q-substitution, we define the corresponding automaton similarly with the transition function $T(j, s)$ being the $(j+1)$-th letter in the word $f(s)$. It is easy to see that the image of the fixed point starting with s_0 represents the sequence $O(ns_0)$.

Equivalence of (i) and (v): We just mention that it is done constructively, by induction on the number of operations $+, ^*$ and concatenation, by constructing an automaton, and in the other direction, by induction on the number of edges in the non-deterministic automaton, by building the regular expression from regular expressions obtained (by induction) by four non-deterministic automata obtained by erasing an edge from p to q say, and by using initial s_0, final state set A replaced by (s_0, A), $(s_0, \{p\})$, $(q, \{p\})$, (q, A) in the four cases.

For equivalence of (vii) and (viii), due to Furstenberg, see [F67] and also [T04, 11.1]. This was one of the first result, and was used in [C79] to prove equivalence of (i) and (vii), whereas [CKMR80] gave a direct proof.

For (iii), see [BHMV94]. For (iv), see [HU79, 9.1].

3.2. Remarks. A given algebraic series can be generated by several different automata. Even if you restrict to 'smallest' automaton, there is no simple way to connect basic numerical data on both sides, for example genus, degree, height of a series versus number of states, or invariants of corresponding labeled graph. Following the proofs above, rough bounds on the degree and height in terms of the size of the automaton, and rough bounds on the size in terms of the degree and height (together) can be given.

3.3. Parts of theorem 5. Part (4): We use the substitution equivalence. Let $q^{m-1} \le n < q^m$, and $jq^m \le i < (j+1)q^m$. Then $I(s_i \cdots s_{i+n-1})$ is a sub-block of $I(f^m(s_j s_{j+1}))$, and depends only on $s_j s_{j+1}$ (at most N^2 possibilities where N is the cardinality of S) and $i - jq^m$ (at most $q^m < qn$ possibilities). This proves the first part, with $c = qN^2$. Now β is non-decreasing function, thus for the least k such that $\beta(k) < k$, we have $\beta(k) = \beta(k-1) = k-1$. Thus, for $j \ge k-1$, $I(s_j)$ is uniquely determined by the $(k-1)$-block immediately preceding it. This implies eventual periodicity.

Part (5) is nothing but the pumping lemma above read in base q. It also implies (1) showing, in fact, that the maximum growth rate $\le q^d$, in the notation of (5). Alternately, note that if s is algebraic of degree d, then truncation approximation with Liouville theorem shows that growth rate cannot be bigger than d.

Now we prove the first part of (7): Let u be smallest such that every symbol occurring at least once in $s_{q^u} s_{q^u+1} \cdots$ occurs infinitely often there. If $q^u \le i < q^{u+1}$, and $k \ge 1$, such that 1 occurs at least twice in $I(f^k(s_i))$, then since s_i occurs infinitely often, this block occurs infinitely often, and hence gaps of size at most q^k occur infinitely often. Otherwise, each q^{u+k+1}- image block can contain 1 at most

$q^k - (k+1)(q^{u+1} - q^u)$ times and thus $s_n/\log(n) \leq (q-1)q^k/\log(q) < \infty$.

Now we explain ideas and techniques coming in the proof of (3). These density and asymptotics results come through study of powers and eigenvalues of incidence matrix: Let N be the cardinality of S as usual, and consider square matrix (m_{ij}) of size N, with m_{ij} being the number of occurrences of the j-th entry of S in f applied to the i-th entry, so that corresponding entry in the k-th power of the matrix denotes the same number with f replaced by f^k. Now the matrix M obtained by multiplying the above matrix by $1/q$, giving proportions is stochastic matrix with all row sums being one and all entries non-negative. Powers and eigenvalues of such matrices are described by Perron-Frobenius theory. In particular, there is h such that M^h has 1 as the largest eigenvalue and it is simple. Then M^{hk} tends to a limit, say M_∞, which can be described by rational operations and density (if the limits exist) can also be expressed in terms of entries rationally, because of interpretation given above. Hence the density, if it exists, is rational. The other asymptotics results mentioned depend on how entries of powers grow and hence on finer analysis of eigenvalues and Jordan blocks, and we just refer to [Co72] or [AS03, Cha. 8].

For Part (6) Consider automaton with N states and output 1 on subset S_f. If $M = (m_{ss'})$ is $N \times N$ matrix with $m_{ss'}$ being the number of digits taking s to s'. Then $(I - tM)^{-1} = \sum t^n M^n$ has entries rational functions in t and ss'-th entry being number of words of length n taking s to s'. Let X be a row vector with 1 at the place s_0 and 0 otherwise, Y be a column vector with 1 at places corresponding to S_f and 0 otherwise. Then the series in (6) is just $X(I - tM)^{-1}Y$, and thus rational.

Part (6) is due to Chomsky and Schützenberger, who also gave a general version with series in noncommuting variables (words giving monomials in the alphabet and summing over all the words in the language), in which context rationality is equivalent to regular languages. Note that when we specialize to all variables being equal, we get the series in (6). They also showed that unambiguous (only one way to derive from grammar) context-free languages give algebraic series (in different perspective than we are considering). See [SS78] for details.

We see part (2) as follows. If for each n, ns_0 is a state from which a final (accepting) state can be reached, then since maximum lengths of such paths is bounded, say by k, for any n, there is $0 \leq r < q^k$ such that $nq^k + r$ is accepted, and thus limsup $(n_{i+1} - n_i) \leq 2q^k$. Otherwise, let m be such that ms_0 is a state from which accepting state cannot be reached. Then there is no n_i between mq^n and $(m+1)q^n$, for any n, so that the maximum growth rate is at least $(m+1)/m > 1$. It is clear that the two statements are mutually exclusive.

For (7), we refer to [Co72].

4. Applications to function field arithmetic

We will focus on how and which automata techniques were used in these applications, leaving full detailed definitions of the objects, statements of the theorems and detailed proofs to the references provided.

4.1. Special values of gamma function. The gamma function $\Gamma : \mathbb{Z}_p \to \mathbb{F}_q((1/t))$ associated with $\mathbb{F}_q[t]$ by Carlitz and Goss is defined by

$$\Gamma(n+1) = \Pi(n) = \prod (D_i/t^{\deg D_i})^{n_i}, \quad n = \sum n_i q^i, \ 0 \leq n_i < q,$$

where

$$D_i := \prod_{k=0}^{i-1} (t^{q^i} - t^{q^k}) \in \mathbb{F}_q[t]$$

is the product of all monic polynomials of degree i. The author had proved functional equations and related some special values to periods of Drinfeld modules, given an analog of Chowla-Selberg formula. Combined with Thiery and Jing Yu's transcendence results [Thi92, Y92] on periods, the known transcendence results (for fractions with only a few denominators) were exactly parallel [T04, pa. 334] in this case and in the number field case. In [T96], the automata method, namely (ii) of Theorem 4, was used to show that Γ value at any proper fraction in $\mathbb{Z}_p$, with some restrictions on the numerator is transcendental. Using logarithmic derivatives (which reduce modulo p the exponents in the monomials occurring in the products and causing troubles and restrictions on numerators with their size, and thus removing size problem), Allouche [A96] proved the transcendence of all values at proper fractions. Author [A96, T98, T04] then used the same trick combined with his earlier work to show that all the monomials in gamma values at fractions that were not earlier shown by him to be algebraic were, in fact, transcendental. Soon afterwards, it was shown [Mf-Y97] that any $\Pi(n)$ is transcendental, for $n \in \mathbb{Z}_p - \mathbb{Z}_{\geq 0}$.

I will now give a quick sketch of this very nice proof in [Mf-Y97]. Note that if s is algebraic non-zero, $s' := ds/dt$ (and thus s'/s) is also algebraic, as one can see directly, or by applying (1) of Theorem 2. Thus, as mentioned before, using logarithmic derivatives, products are turned into sums and it is enough to prove that if the sequence $n_j \in \mathbb{F}_q$ is not ultimately zero, then $\sum n_j/(t^{q^j} - t)$ is transcendental over $\mathbb{F}_q(t)$. This is then achieved by writing the series as $\sum c(m)t^{-m-1}$ and using (ii) of Theorem 4 to conclude by showing that there are infinitely many distinct subsequences $c_r(m) = c(q^r m + 1)$. In fact, one shows by direct manipulation using elementary number theory of divisibility arguments that $c_a \neq c_b$ if $a > b$ and n_a and n_b are non-zero by showing that these sequences differ at their $(q^h - 2)$-th term, where h is the least positive integer s dividing a, but not dividing b such that $n_s \neq 0$.

See [T04, Cha.4, 11] and the references quoted above for more details and motivations and comparison with classical case.

4.2. Tate multiplicative period for elliptic curves. Mahler-Manin conjecture asserted transcendence (over the field of definition) of the fundamental multiplicative period (traditionally denoted by q not to be confused with our usage in this paper) of Tate elliptic curve $G_m/q^{\mathbb{Z}})$ over (complex or p-adic) local field.

Voloch proved [V94] analog over finite characteristic local field of Laurent series using Igusa towers and cohomology. When he lectured on this at University of Arizona, the author could give automata style proof [T94] as follows. By transcendence degree considerations (and algebraic identities between coefficients) the question of transcendence of q over the field of coefficients, which are essentially 'q-expansions' of certain Eisenstein series, is equivalent to whether these expansions represent transcendental series over field of rational functions over q. By algebraic identities between modular forms this in turn can be deduced from the transcendence of theta series $\sum q^{n^2}$ modular form. This is equivalent to the set of squares being not automatic, well-known fact to computer scientists, in fact, one of the first application of automata due to Richie, Büchi etc. For us it may be the easiest to deduce this from (2) or (7) of Theorem 5, since ratio of successive squares tends to one, but the gap between them tends to infinity. (See another proof in the next subsection.) The original conjecture was soon proved [BDGP96] by Mahler method. It had a nice application to BSD conjecture case studied by Mazur, Tate and Teitelbaum. See the papers quoted or [T04, Cha. 11] for more details on this.

4.3. *q*-expansions of Eisenstein and fake Eisenstein series. In [AT99], more direct (more automatic!) proof of the transcendence of q was given, by noticing that coefficients of the expansions of eisenstein series are given by arithmetic functions called higher divisor functions $\sigma_u(n)$, and modulo p distribution of their values has been well-studied by number theorists. In particular, Rankin proved some asymptotics which does not fit in classification of automatic sequence asymptotics studied by Cobham, more precisely with (3) of Theorem 5, so that these series are transcendental.

Another application [AT99] was showing that if $\zeta(p)/\zeta(p-1)$ is an irrational real number (here ζ is the Riemann zeta function, so this is expected but not known), and if $p-1$ divides u, then $S_u = \sum \sigma_u(n) q^n \in \mathbb{F}_p[[q]]$ is transcendental over $\mathbb{F}_p(q)$. First note that when, as in this case, u is even, the expansions are no longer connected to Eisenstein series (which have even weight and odd u), thus we call them fake Eisenstein series. Again, Rankin's result gives in this case the natural density as rational multiple of this zeta values ratio, and thus proof just consists of quoting that together with (3) of Theorem 5. The result is amusing concluding transcendence of a finite characteristic Laurent series representing (fake) modular form from irrationality of a real number representing (ratio of) zeta value!

Soon afterwards, Yazdani [Yaz01], using stronger automata criterion (2) of Theorem 3 showed the transcendence unconditionally and dropping the condition on u.

4.4. Special values of *v*-adic gamma. For a monic irreducible polynomial v of $\mathbb{F}_q[t]$, Goss defined v-adic gamma function $\Gamma_v : \mathbb{Z}_p \to \mathbb{F}_q[t]_v$ by

$$\Gamma_v(n+1) = \Pi_v(n) = \prod (-D_{i,v})^{n_i}, \quad n = \sum n_i q^i, \ 0 \le n_i < q,$$

where

$$D_{i,v} := (D_i/D_{i-\deg v})v^{-ord_v(D_i/D_{i-\deg v})} \in \mathbb{F}_q[t]$$

is the product of monic polynomials not divisible by v and of degree i. The author had proved functional equations and analog of Gross-Koblitz formula and conjectured which monomials in values at fractions would be transcendental.

It was proved in [T98] in the special case of fractions and in [WY02] in the general case, with a simpler proof based on results of [Mf-Y97], that for v a prime of degree one, for $n \in \mathbb{Z}_p$, $\Pi_v(n)$ is transcendental, if and only if the digit sequence n_j is not ultimately constant. (The only if part was proved earlier by the author.)

Using translation automorphisms, it is enough to prove the claim for $v = t$. Again one turns products into sums by using logarithmic derivatives and is led to prove transcendence of the series $\sum(n_j - n_{j+1})/((1/t)^{q^j} - (1/t))$ and one applies the result of [Mf-Y97] quoted above.

Similar transcendence results are not known for higher degree v. See [T04, Cha.4, 11] and the references quoted above for more details and motivations and comparison with classical case.

4.5. Algebraicity criterion for hypergeometric functions. We now quote a generalization (due independently to Sharif and Woodcock, Harase [SW88, Ha88, A89]) of Christol's theorem, or rather of equivalence (ii) of Theorem 4, to function fields over any field (not necessarily finite) of characteristic p and just mention its recent application [TWYZ11, TWYZ09] to characterization of all parameters for which the hypergeometric function of [T04, 6.5(a)] is an algebraic function.

Theorem 6. *Let F be a field of characteristic p. The series $\sum f_n t^n \in F[[t]]$ is algebraic over $F(t)$ if and only if the $\overline{F}$-vector space generated by the sequences $n \to f_{n+k}^{1/p^k}$, as k runs through positive integers, is of finite dimension.*

4.6. Carlitz periods, logarithm and zeta values. We have only looked at applications which are (or were when introduced) new results. For other automata applications in Drinfeld module context for Carlitz period, Carlitz zeta values and logarithm values, as well as many others outside this context, we refer to papers by Allouche, Berthé and Yao [A87, A90, AS03, B92, B93, B94, B95, Ya97] and references there.

5. Comparison with other tools

When one tries to prove transcendence results for naturally occurring quantities in function field arithmetic, say from the theory of Drinfeld modules, t-motives or varieties, there are several tools available. When automata method applies, which is surprisingly often, it leads to quick direct proofs. But when the other methods such as functional equations, Mahler method, *period methods* using algebraic groups and motives etc. apply, they usually give more general and structurally more satisfying results.

For example, the transcendence result for appropriate gamma monomials at fractions mentioned above has been now generalized [CPTY10] to complete algebraic independence results between appropriate gamma as well as Carlitz zeta values, by using transcendence techniques of Anderson, Brownawell, Papanikolas covered in Brownawell's and other lectures of this Banff workshop. Using author's generalizations for gamma functions to other function fields and his results connecting them to periods of Drinfeld modules together with Jing Yu's results about transcendence of such periods prove transcendence of some special gamma values but for any function field.

As for the *method of Diophantine approximation* is considered, it is interesting to note that very often, in function field case, it is much easier to obtain faster approximations than for the number field counterparts of the quantities, but at the same time, because of the failure of Roth theorem analog, it is harder in function field case to conclude transcendence from such fast approximations!

As we have seen above, automata leads to equations of type handled in *Mahler's method*, such as $\sum a_i(x)f(x^{p^i}) = 0$, which becomes algebraic equation in charactersitic p, as we can take the exponent out to get powers of f. We refer to Pellarin's lectures at this workshop for more on Mahler method.

The applicability of the automata method is somewhat limited so that we do not know how to generalize it to get similar results for quantities occurring in the context of higher genus function fields, or to get strong algebraic independence results. For example, period methods have finally, not only caught up with automata methods for transcendence of gamma values at fractions, but have also provided much stronger full independence results. On the other hand, some results such as Mendès France-Yao result on transcendence of gamma values at p-adic integers which are non-fractions and v-adic gamma results, as well as refined transcendence results are still only obtained by automata method and not by other methods. Same can be said for transcendence results for quantities not strongly related to geometry, such as 'wrong weight' Eisenstein series discussed above.

Many of these function field results were proved by different methods by Denis, Hellagoauarch, de Mathan, Yao etc. References and comparison of these methods can be found in [FKdM00, Ya09].

In the next section, we discuss another big strength of automata methods.

We end this section with a challenge: It is well-known that the power series for the Artin-Hasse exponential

$$\exp(\sum_0^\infty \frac{x^{p^n}}{p^n}) = \prod_{n \not\equiv 0 \mod p} (1 - x^n)^{-\mu(n)/n}$$

has coefficients in $\mathbb{Z}_p$. I wonder which method will settle the open question about transcendence over $\mathbb{F}_p(x)$ of its reduction modulo p.

6. Refined transcendence classification based on strength of computers

6.1. Computational classification with algebraic properties. We now briefly explain computational classification [BT98] with good algebraic properties and give an illustration of refined transcendence classification of some important Laurent series. We saw above how various ways of thinking of automata have helped giving transcendence proofs by completely different methods when usual methods do not apply. Now usual numbers/Laurent series coming up in number theory and geometry are computable (already a small countable subclass) and like automata, computability has various incarnations studied by various viewpoints, such as Turing machines, languages generated by unrestricted grammar, recursive function theory, Post systems, Church's lambda calculus etc. Computer scientists, logicians, linguists have also studied intermediate strength classes (e.g., Chomsky hierarchy) and often converged to the same notions. So we examined [BT98] these robust classes from computational, series perspective as we have been studying above and found that many of these have good algebraic properties, such as forming a field, a field algebraically closed in Laurent series etc.; in addition to closure, logical properties, such as closure under union, concatenation, complementation etc., explored before. The algebraic properties allow you to move by algebraic operations the problem about one series to another series which might be more convenient to deal by these generalized automata tools. We will give an example below.

For example, the context-sensitive languages form a field, and are equivalent to Turing machine which uses work space at most linear in input size. Context-free languages are equivalent to 'pushdown automata', but have very weak algebraic properties. On the other hand, Turing machines, Turing machines which take polynomial space to operate etc. form fields algebraically closed in Laurent series. For proofs, precise definitions, various equivalences and algebraic and closure properties, we refer to [BT98] and only sketch here some applications.

6.2. Refined transcendence of π by language tools. Let $\tilde{\pi}$ be the fundamental period of Carlitz module for $\mathbb{F}_q[t]$, so that $\pi := t^{-q/(q-1)}\tilde{\pi}$ is a Laurent series. By result of Wade in 1941, it is known to be transcendental, thus non-automatic. (For a direct automata proof, see [A90]). We show that it (or rather its reciprocal) is not context-free (which gives, in particular, a language theoretic proof of its transcendence), but is context-sensitive. The tools here are language theoretic closure properties, moving to convenient series by algebraic properties and getting contradiction by pumping lemma for context-free languages.

Here is a part of the argument illustrating this dealing with series with coefficients $f_n = \sum_{q^j-1|n} 1 \in \mathbb{F}_q$. Just dividing n by $q^j - 1$'s, one at a time, in a linear space, and reusing it, we see the series to be context-sensitive, but if it were context-free, intersection with regular language $\{q^u - 1\} \leftrightarrow (q-1)\cdots(q-1)$ is also context-free. Now $c_{q^u-1} = d(u) = \prod(u_i + 1)$ for $u = \prod p_i^{u_i}$. Hence, the subset of these $q^n - 1$'s where $d(n) = 2$ (odd respectively), is also context-free. By pumping, the only digits being $q - 1$, we get same value (modulo p) of d on some arithmetic progression and are led to a contradiction by elementary number theoretic arguments.

6.3. Refined transcendence of e, θ. Using computational and language tools, we [BT98] show that Carlitz analog of e, (known to be non-automatic by Wade 1941) is context-sensitive and theta series or set of squares (explained to be non-automatic above) is context-sensitive (even in logarithmic space under GRH), but (for $q = 2$) not context-free.

6.4. Algebraic dependence. We also give a computational criterion for algebraic independence, but have not found any application to natural quantities. Algebraic properties mentioned above and theta or Eisenstein calculations above show that modular forms expansions are in polynomial space class. It was suggested in [T04, pa. 369] that construction of non-periods can probably be done by computational methods, because periods are probably in polynomial or exponential space class. Mahler put all complex numbers in four classes of A, S, T, U numbers, with property that algebraically dependent transcendental numbers belong to the same class. We have divided countable class of computable numbers into infinitely many finer classes with this property, but do not have a good diophantine approximation theoretic description yet. For some applications of automata tools in diophantine approximation questions, we refer to [T03].

6.5. Remarks. In these equivalences, the role of generalization in substitution viewpoint is not yet clear. For example, what do context-free and context-sensitive correspond to on substitution side, and what do non-uniform substitutions (different letters can go to strings with different lengths) correspond to in other viewpoints? For some other interesting open questions, see [BT98, pa. 816].

7. Beyond function field real numbers

Automata method immediately applies to Puiseux series, which are Laurent series in $t^{1/k}$ (for some k), which is an algebraic quantity. But unlike the characteristic zero case, the Puiseux series field no longer gives the algebraic closure of the Laurent series field, for example, as Chevalley pointed out, in characteristic p, the polynomial $x^p - x - t^{-1}$ has no root in Puiseux series field. But as Abhyankar noticed, it has a root $\sum t^{-1/p^i}$ in the (algebraically closed) field of 'generalized fractional power series' (considered first by Hahn and studied in detail in this context by Huang) of the form $\sum_{i \in S} f_i t^i$, where $S \subset \mathbb{Q}$ is a well-ordered subset such that for some m the elements of mS have denominators powers of p. Kedlaya [K01] described the algebraic closure of $\mathbb{F}_q((t))$ as subfield of this field and described [K06] the algebraic closure of $\mathbb{F}_q(t)$ in it by adapting Christol's algebraicity criterion to these type of series by considering automata with radix point, a special symbol s_k, so that one considers a string of symbols $s_1 \cdots s_{k-1} s_k s_{k+1} \cdots s_n$, with $s_1 \cdots s_{k-1}$ is integral part and $s_{k+1} \cdots s_n$ is the fractional part. We refer to [K01, K06] for the detailed definitions, statements and proofs.

I do not know yet any naturally occurring such generalized series, which is not a Puiseux series, to have a nice application of this generalization.

8. Strong characteristic dependence for algebraicity and real numbers

8.1. Finite characteristic. No carry over of power series expansions, finite possibilities of coefficients and $\mathbb{F}_q$-linearity of q-power map makes function field case amenable to combinatorial description of automata. But in fact, even in function field case, this algebraicity description is very strongly dependent on characteristic as the following theorem by Cobham (whose combinatorial proof is a 'challenge to algebraists' according to Eilenberg [E74]) shows.

Theorem 7. *Non-periodic sequence cannot be m-automatic and n-automatic, if $m, n > 1$ are multiplicatively independent (e.g., if m, n are powers of distinct primes).*

Corollary 8. *If $s = \sum_{m \in M} t^m$ is irrational algebraic in one finite characteristic, then it is transcendental in all other finite characteristics.*

8.2. Characteristic zero. Natural question is what happens for similar expansions in characteristic zero. Corresponding power series is transcendental over $\mathbb{Q}(t)$ as reduction shows. But how about related p-adic, real numbers? Despite earlier claimed proofs in the literature, the following theorem was only recently proved [ABL04] by Adamczewski, Bugeaud, Luca, making very nice use of Schmidt's subspace theorem and automata equivalences. (Note that naive analogs of this theorem and of its consequence Roth's theorem, fail in finite characteristic.) We sketch an important special case.

For words W in alphabet, say S or F, we denote by $w = |W|$ its length, and for $x > 0$, we put $W^x := W^{\lfloor x \rfloor} W'$, where W' prefix of W of length $\lceil (x - \lfloor x \rfloor) w \rceil$.

Theorem 9. *Let p be a prime. If f_n is p-automatic non-periodic sequence taking values in $F = \{0, 1, \cdots, p-1\}$ (so that $\sum f_n t^n$ is algebraic over $\mathbb{F}_p(t)$), then $\mu := \sum f_n / p^n \in \mathbb{R}$ and p-adic number $\nu := \sum f_n p^n$ are transcendental over $\mathbb{Q}$.*

Proof. In the notation of 2.11, f_n corresponds to p-substitution (f, I), with S of cardinality N, and a fixed point $u = u_1 u_2 \cdots$ whose image is the sequence f_n. Write a prefix of length $N+1$ of u as $W_1 a W_2 a W_3$, with $a \in S$ and W_i words (possibly empty) over S. Put $U_n = I(f^n(W_1))$, $V_n = I(f^n(aW_2))$. Put $r = 1 + 1/N$. Then (1) $U_n V_n^r$ is a prefix of non-eventually periodic sequence f_n, (2) $u_n / v_n \leq c := N - 1$, (3) v_n is increasing sequence.

Put $f_k^{(n)} := f_k$, if $k \leq u_n + r v_n$ and $:= f_{k+v_n}^{(n)}$, if $k \geq u_n + v_n$. For every n, it gives ultimately periodic sequence of preperiod U_n and period V_n. Put $\mu_n := \sum f_k^{(n)} / p^k$, so there is p_n such that $\mu_n = p_n / (p^{u_n}(p^{v_n} - 1))$ and $|\mu - \mu_n| < p^{-(u_n + r v_n)}$, by (1).

Consider six linear forms in x, y, z with algebraic coefficients:

$$L_1 = \mu x + \mu y + z, \ L_2 = y, \ L_3 = z, \ \ L_1' = x, \ L_2' = y, \ L_3' = z.$$

The forms L_i (L'_i respectively) are linearly independent over $\mathbb{Q}$. Put $X_n = (p^{u_n+v_n}, -p^{u_n}, -p_n)$. Evaluated on X_n, we have

$$|L_1|_\infty < \frac{1}{p^{(r-1)v_n}}, |L_2|_\infty = p^{u_n}, |L_3|_\infty = p_n, |L'_1|_p = \frac{1}{p^{u_n+v_n}}, |L'_2|_p = \frac{1}{p^{u_n}}, |L'_3|_p \leq 1.$$

Then, by (2), there is $\delta > 0$ such that

$$\prod |L_i(X_n)|_\infty |L'_i(X_n)|_p << \frac{1}{p^{(r-1)v_n}} < \frac{1}{p^{(u_n+v_n)\delta}} \leq ||X_n||^{-\delta},$$

where $||X_n||$ is the max norm. Hence by Schmidt's subspace theorem, X_n's lie in union of proper subspaces of $\mathbb{Q}^3$, so that there are rationals x_0, y_0, z_0, not all zero such that $x_0 - y_0(p^{u_n}/p^{u_n+v_n}) - z_0(p_n/p^{u_n+v_n}) = 0$. By (3), taking limit, $x_0 = z_0\mu$. But μ is irrational, so that $z_0 = 0$, so $x_0 = y_0 = 0$ giving a contradiction and thus proving the first claim.

For the p-adic case, we put $\nu_n = \sum f_k^{(n)} p^k$. Then $|\nu - \nu_n|_p \leq p^{-(u_n+rv_n)}$ and $\nu_n = b_n/(p^{v_n} - 1)$, with $b_b \leq p^{u_n+v_n}$. Now choose L_i to be x, y, z and L'_i to be $x, y, \nu x + \nu y + z$, and $X_n = (p^{v_n}, -1, -b_n)$. Again, the straight estimates work, and Schmidt's subspace theorem applied to the same product (with new notation) gives similar contradiction. □

8.3. Errata. Finally, we take this opportunity to correct some misprints in the chapter on automata of [T04].

(i) Pa. 344, Example 11.1.5: (1) In the automata table, entries in the column headed by s_2 should be s_3, s_4. In (2) line 2, s_2 and s_3 should be switched.

(ii) Pa. 348, Thm. 11.2.2 (5) n'_{i+1}/n_i should be n'_{i+1}/n'_i, and 0 is missing from $d > 0$. It should have been mentioned that n_i is increasing sequence and thus S is infinite.

(iii) Pa. 349, last para. It should be added to 'turns out to be a trivial monomial' that 'after a preliminary reduction as in Thm. 4.6.4, which changes it by a rational function'.

(iv) Pa. 353, first paragraph of 11.4 is misplaced and should be dropped.

Acknowledgments This article is based on the mini-course the author gave at ETH, Zurich in 2006 and the lecture at BIRS, Banff in September 2009. I would like to thank both the institutions, Richard Pink and the organisers of this conference. I would also like to thank my collaborators Jean-Paul Allouche, and Robert Beals, from whom I learned many things. I am grateful to the referee for a very careful reading and suggestions.

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