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Continued Fraction for the Exponential for $\mathbf{F}_q[T]$

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We show that the exponential $e(z)$ for $\mathbf{F}_q[T]$, whose definition and properties are recalled in Section 0, has a continued fraction expansion with an interesting pattern. © 1992 Academic Press, Inc.

0. ANALOGUE OF THE EXPONENTIAL

Let $\mathbf{F}_q$ be a finite field of cardinality q. Let $A := \mathbf{F}_q[T]$, $K := \mathbf{F}_q(T)$, $K_\infty := \mathbf{F}_q((1/T))$ and let Ω be the completion of an algebraic closure of K_∞. Then A, K, K_∞, Ω are well-known analogues of $\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$, respectively. Carlitz [C1] introduced an entire function $e: \Omega \to \Omega$ as an analogue of the exponential function. We will now give its power series expansion and describe some analogies with the classical exponential function. (Our notations and normalizations of signs are different from [C1]. In particular, Carlitz uses the symbol ψ for the exponential. It is the same as our e for characteristic 2, but in odd characteristic ψ and e differ by a simple change of variable. Also, we will ignore historical motivation. For a wider perspective, see [G, T] and references there.)

Let $[i] := T^{q^i} - T$. This is just the product of monic irreducible elements of A of degree dividing i. Note $[i+1] = [i]^q + [1] = [i] + [1]^{q^i}$. Let $D_0 := 1$, $D_i := [i]\, D_{i-1}^q$, $i > 0$. (D_i is the same as Carlitz' F_i.) This is the product of monic elements of A of degree i. Let

$$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{D_i}.$$

Some analogies:

(1) $e(z)$ is an entire function (in the sense that the power series defining it converges for all $z \in \Omega$), but it is additive unlike the classical

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exponential, which is multiplicative. For $a \in A$, $e(az) = C_a(e(z))$, where $C_a(u)$ is a polynomial in u (e.g., $C_T(u) = Tu + u^q$), somewhat analogous to the classical case $e^{nz} = (e^z)^n$. (For nonzero a, the degree of C_a is Norm $a := q^{\text{degree } a}$, which is just the number of residue classes modulo a. Note that n is the number of residue classes modulo n.) In fact, associating this polynomial to a gives an embedding of A as a ring in the endomorphism ring of the additive group, similar to the embedding of $\mathbf{Z}$ (by sending n to the nth power map) in the endomorphism ring of the multiplicative group. Also as an analogue of $e^z = \lim(1 + z/n)^n$, one has $e(z) = \lim C_a(z/a)$, where now the limit is taken as the degree of a tends to infinity. (See [H, T].)

(2) $e(z)$ has analogous transcendence properties, for example, analogues of the Siegel–Schneider and Hermite–Lindemann theorems hold. In particular, $e := e(1)$ is transcendental. (The irrationality of e can be proved easily by imitating the classical proof (see, e.g., [HW, Theorem 47]). The kernel of $e(z)$ is of the form $\tilde{\pi}A$, for some $\tilde{\pi} \in \Omega$. Compare this to $e^z = 1$ iff $z \in 2\pi i\mathbf{Z}$. In fact, $\tilde{\pi}$ is an analogue of $2\pi i$: It is known to be transcendental and it occurs in analogous fashion in the special values of analogues of zeta and gamma functions. (See [C2, W1, W2, T].)

(3) Adjoining $e(\tilde{\pi}a/b)$, for some $a, b \in A$, to K, one gets analogues of cyclotomic extensions, with similar Galois actions, ramification, and prime splitting properties. (See [H].)

(4) Compare $e^z = \sum(z^n/n!)$ with the power series expansion above. In fact, D_i gives a good analogue of the factorial of q^i: It has analogous prime factorization, divisibility properties, growth rate, interpolations at all places with analogous functional equations, and special values. (See [T].)

(5) Analogues of the Bernoulli numbers can be obtained from the generating function $z/e(z)$; classically one uses $z/(e^z - 1)$. These satisfy von-Staudt–Clausen type congruences and appear analogously in the special values of the zeta function and in connection with the class groups of the cyclotomic fields. (See [C2, Ge].)

(6) Classically the derivative of e^z with respect to z is e^z itself. Here the usual derivative of $e(z)$ is 1, but for an analogue, see [T].

1. Continued Fractions

We recall some standard facts and notation (see, e.g., [HW]): $[a_0, a_1, a_2, \ldots]$ (there should be no confusion with $[i]$ of Section 0, since it has only one entry), denotes the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

a'_n denotes $[a_n, a_{n+1}, ...]$.

(A) Let

$$p_0 = a_0, \qquad q_0 = 1, \qquad p_1 = a_1 a_0 + 1, \qquad q_1 = a_1$$
$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2}$$

then

(B) $p_n/q_n = [a_0, ..., a_n]$,
(C) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$,
(D) $[a_0, a_1, ...] = (a'_n p_{n-1} + p_{n-2})/(a'_n q_{n-1} + q_{n-2})$, and
(E) $q_n/q_{n-1} = [a_n, ..., a_1]$.

In the classical case, Euler proved $e = [2, 1, 2, 1, 1, 4, 1, ..., 1, 2n, 1, ...]$. It is also well known that

$$e^z = 1 + \cfrac{z}{1 - \cfrac{z}{2 + z - \cfrac{2z}{3 + z - \cfrac{3z}{4 + z - \cdots}}}} = \cfrac{1}{1 - \cfrac{z}{1 + \cfrac{z}{2 - \cfrac{z}{3 + \cfrac{z}{2 - \cfrac{z}{5 + \cfrac{z}{2 - \cdots}}}}}}}.$$

Our main result is

THEOREM 1. *Define a sequence x_n by setting $x_1 := [0, z^{-q}[1]]$ and if $x_n = [a_0, a_1, ..., a_{2^n-1}]$, then setting*

$$x_{n+1} := [a_0, ..., a_{2^n-1}, -z^{-q^n(q-2)} D_{n+1}/D_n^2, -a_{2^n-1}, ..., -a_1],$$

then

$$x_n = \sum_{i=1}^{n} \frac{z^{q^i}}{D_i}.$$

In particular, $e(z) = z + \lim_{n\to\infty} x_n$ and the continued fraction for e is obtained by patting $z = 1$. In particular, for $q = 2$,

$$e=[1, \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4], \underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], \ldots]$$

$$e(z)=z+\cfrac{z^2}{[1]+\cfrac{z^2}{[2]+\cfrac{z^2}{[1]+\cdots}}}.$$

Proof. Let $e_n:=\sum_{i=1}^{n} z^{q^i}/D_i$ and $\bar{e}_n:=e_n-2z^{q^n}/D_n$. Also, for x_n as in the statement of the theorem, we let $\bar{x}_n:=[0, -a_{2^n-1}, \ldots, -a_1]$. Let $p_i, q_i, \bar{p}_i, \bar{q}_i$ have the obvious meanings, corresponding to the continued fractions $x_n, \bar{x}_n$'s. (Note that the continued fraction for x_n is obtained by truncating that of x_{n+1}, but the analogous statement is not true for $\bar{x}_n$ and $\bar{x}_{n+1}$, so $\bar{p}_i, \bar{q}_i$ depend on the particular $\bar{x}_n$ we are considering). Let the induction hypothesis H_n be

$$q_{2^n-1}=-\bar{q}_{2^n-1}=z^{-q^n}D_n, \qquad p_{2^n-1}=e_n q_{2^n-1}, \qquad \bar{p}_{2^n-1}=\bar{e}_n\bar{q}_{2^n-1}.$$

Then H_1 clearly holds. Assume H_n. Applying (E) to the definitions of $x_n, \bar{x}_n$, we see that q_{2^n-2} is the same as $\bar{p}_{2^n-1}$, which is the same as $-p_{2^n-1}+2$, by the induction hypothesis. Application of (C) then shows that $p_{2^n-2}=-(p_{2^n-1}-1)^2/q_{2^n-1}$. On the other hand, (D) together with H_n implies that

$$x_{n+1}=\frac{(-z^{q^n(q-2)}D_{n+1}/D_n^2+\bar{x}_n)\,p_{2^n-1}+p_{2^n-2}}{(-z^{q^n(q-2)}D_{n+1}/D_n^2+\bar{x}_n)\,q_{2^n-1}+q_{2^n-2}}.$$

Using the formulae we have obtained, a simple manipulation shows that $-D_n/z^{q^n}$ times the denominator (the numerator resp.) of this expression is $z^{-q^{n+1}}D_{n+1}(z^{-q^{n+1}}D_{n+1}e_{n+1}$ resp.). From this evaluation of x_{n+1}, we see that $p_{2^{n+1}-1}, q_{2^{n+1}-1}$ are as stated in H_{n+1}, by an easy count, using (A), of their sign and degrees in T and z. Similarly, one sees using (D) and H_n, that

$$\bar{x}_{n+1}=\frac{(z^{q^n(q-2)}D_{n+1}/D_n^2+\bar{x}_n)\,p_{2^n-1}+p_{2^n-2}}{(z^{q^n(q-2)}D_{n+1}/D_n^2+\bar{x}_n)\,q_{2^n-1}+q_{2^n-2}}.$$

Similar manipulation then shows that $\bar{p}_{2^{n+1}-1}, \bar{q}_{2^{n+1}-1}$ are also as stated in H_{n+1}, thus proving H_{n+1}. This completes the proof by induction.

Remarks. (1) In fact, the proof shows that the partial sums of any series of the form $1/r_1 - 1/(r_1^2 r_2) - 1/(r_1^4 r_2^2 r_3) + \cdots$ have continued fractions with similar "negative reverse repetition" with the terms $r_1, r_2, r_3, \ldots$.

(2) For $q=2$, use of the auxiliary continued fraction $\bar{x}_n$ can be avoided and a simpler proof can be given by just using (E).

(3) In general, it is much easier to express the given quantity as a generalized continued fraction, since there is no uniqueness. For example, we have a term by term equality

$$\frac{1}{a} - \frac{1}{ab} + \frac{1}{abc} - \cdots = \cfrac{1}{a + \cfrac{a}{b - 1 + \cfrac{b}{c + 1 + \cfrac{c}{\cdots}}}}.$$

This gives generalized continued fractions, with interesting looking partial quotients in terms of the $[i]$'s, for $e(z)$, its inverse $\log(z)$, $\zeta(1), \ldots, \zeta(q)$ (where $\zeta(s) := \sum_{n \text{ monic} \in A} 1/n^s$, $s \in \mathbf{N}$, is an analogue of the Riemann zeta function), the analogue $\gamma := \zeta(1)$ of Euler's gamma constant, $\tilde{\pi}^{q-1}/[1]$. For more information about these quantities and series expansions for them, see [T] and references there. We just give a simple example. For $q=2$,

$$\frac{\tilde{\pi}}{[1]} = \zeta(1) = \log(1) = \gamma = \cfrac{1}{1 + \cfrac{1}{[1] + 1 + \cfrac{[1]}{[2] + 1 + \cfrac{[2]}{[3] + 1 + \cfrac{[3]}{\cdots}}}}}.$$

(4) After circulating a preliminary version of this article, I was informed by J. Shallit that in [S, PS], continued fractions of some real numbers were shown to have similar "reverse repetition" patterns. He also suggested a nice alternate description of a_n in Theorem 1, when $q=2$, $z=1$, $n>0$: $a_n = [\mathrm{ord}_2(2n)]$, where $\mathrm{ord}_2(2n)$ is the largest integer k such that 2^k divides $2n$.

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