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Patterns of Continued Fractions for the Analogues of *e* and Related Numbers in the Function Field Case*

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Patterns for simple continued fractions of the analogues of $(xe^{2/f}+y)/(ze^{2/f}+w)$ in the $\mathbf{F}_q[t]$ case are described. In contrast to the classical case where they consist of arithmetic progressions, in this case they involve an interesting inductive scheme of block repetition and reversals, especially for $q=2$.

This is a sequel to [T2]. For perspective, motivation, and background material on continued fractions and the Carlitz exponential, the reader may profit from looking at [BS, T1, T2], but this paper is mostly self-contained. We start with a brief introduction:

0.0. Classical results of Euler and Hurwitz ([H1], [P]) show that the sequence of partial quotients for the (simple) continued fraction for $(xe^{2/f}+y)/(ze^{2/f}+w)$, with x, y, z, w, f integers, $f \neq 0$, $xw-yz \neq 0$ (the so-called Hurwitz numbers), eventually consist of a fixed number of arithmetic progressions.

EXAMPLES.

$$e=[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2n, 1, \ldots]=[2, \overline{1, 2i, 1}]_{i=1}^{\infty}$$

$$\frac{e+1}{3}=[1, 4, \overline{5, 4i-3, 1, 1, 36i-16, 1, 1, 4i-2, 1, 1,}\ \overline{36i-4, 1, 1, 4i-1, 1, 5, 4i, 1}]_{i=1}^{\infty}.$$

There is no "formula" giving the sequence of partial quotients: the reduction process giving a procedure to work out the sequence is quite involved. See [P, R, MW, vP] for more information.

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0.1. Let q be a power of a prime p and let $\mathbf{F}_q$ denote a finite field of q elements. From now on, we will be interested in the case where integers are replaced by polynomials (though we sometimes refer to them as integers) in $\mathbf{F}_q[t]$ and the exponential function is replaced by the Carlitz–Drinfeld exponential. See [C1], [Ha] for background and motivation for 0.1.1 and 0.1.2. We will not need any knowledge of the theory for this paper.

0.1.1. The Carlitz–Drinfeld exponential $e(z)$ is defined as follows: Let $[i] := t^{q^i} - t$, $d_0 := 1$ and $d_i := [i]\, d_{i-1}^q$, for $i > 0$. Put

$$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}.$$

0.1.2. By Hurwitz numbers we will mean $(xe(\theta/f)+y)/(ze(\theta/f)+w)$, with $x, y, z, w, f \in \mathbf{F}_q[t]$, $f \neq 0$, $xw - yz \neq 0$ and $\theta \in \mathbf{F}_q^*$. For $q=2$, we also allow $\theta = t$, $t+1$, t^2+t. The motivation for this is that these are exactly the values whose Carlitz torsion is "rational", i.e., the roots of Carlitz cyclotomic polynomial (see [Ha] for the definition and properties) $C_\theta(u)$ are in $\mathbf{F}_q(t)$. This is analogous to the fact that the only roots of unity which are rational are the first and the second roots. So analogue of 2, in this context, is $[1] = t^2 + t$.

0.1.3. We do have "patterns" for analogues, but of completely different kind. We have complete results [T2] for $q > 2$. They can be roughly described (see below for the precise description) as "block reversals at each stage." The case $q = 2$ leads to the most interesting and subtle patterns, involving "an inductive scheme of block reversal and repetition at each stage." We study and settle this case completely for $\theta = 1$. For $\theta = t$, $t+1$, t^2+t, we will be content with partial results and conjectures based on experimental evidence.

0.1.4. As the simplest example of "patterns" we will encounter, we have [T1, 2], for $q = 2$,

$$e := e(1) = [1, \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4],$$

$$\underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], \ldots].$$

0.1.5. A number and its integral linear fractional transformation of determinant ± 1 have continued fractions with the same "tail" and they can be easily computed from one another. Hurwitz numbers can be transformed [T2] by such a transformation to numbers of the form $\alpha = (a/b)\, e(\theta/f) + (c/d)$, with θ and f as before, a, b, c, d integral and $a, b, d \neq 0$. Without loss of generality, we will assume that a/b and c/d are in reduced form. We will show/observe that these numbers have "pure patterns." In other words, the patterns start immediately after the first partial quotient, rather than after some number of partial quotients.

0.1.6. Let $\alpha_n := (a/b) \sum_{i=0}^{n} (\theta/f)^{q^i}/d_i + (c/d)$. For large n, the α_n give convergents to α (See Theorem 1 and Propositions 1 and 4 below). We write their continued fraction expansion as $[\lfloor \alpha \rfloor, M_n]$, where "the integral (polynomial) part" $\lfloor x \rfloor$ is defined to be $\sum_{i=0}^{m} x_i t^i$, when $x = \sum_{i=-\infty}^{m} x_i t^i$. We show how M_n is obtained by "an inductive scheme of block reversals and repetitions up to one well defined sequence of partial quotients".

0.1.7. *Basic notation*: A capital letter such as X denotes a finite tuple (a vector) of partial quotients and X^- denotes the tuple with the same entries in the reverse order. We sometimes write X^+ for X. Also, $-X$ denotes the tuple obtained from X by putting a negative sign in the front of each entry. We also write $M_{n+1} = (M_n, t_{n+1}, N_{n+1})$.

Hence to describe the continued fraction we describe the sequence of partial quotients t_n and new vectors N_n at each stage.

1. THE CASE $q > 2$

1.0. The exponential is $\mathbf{F}_q$ linear, so without loss of generality, $\theta = 1$ in this case. The completely satisfactory answer [T2] to our problem is:

THEOREM 1. *We have* $N_{n+1} = -M_n^-$ *and* $t_n = \pm a(f^{n-1} d_{n-1})^{q-2} [n]/b$ (*with an explicit sign*) *for* $n-1$ *larger than the maximum of the degrees of* a, b, d.

Hence all such α's have continued fractions made up of block reversals:

$$[a_0, X, x_1, -X^-, x_2, X, -x_1, -X^-, x_3, X, x_1,$$
$$-X^-, -x_2, X, -x_1, -X^-, x_4, \ldots],$$

with a_0, X, x_i explicitly known.

2. THE CASE $q = 2$, $\theta = 1$

2.0. We will describe the main results in this section. The proofs will be given later.

PROPOSITION 1. *If* $n > C := 1 + \max(\deg d, \deg a, \deg b)$, *then* α_n *is a convergent to* α *and* $t_{n+1} = \lfloor a[n+1]/b \rfloor$.

2.1. Since we have described t_n's it remains to describe N_n.

2.2. Let $l := l(\alpha) := l(b)$ be the least common multiple of the degrees of the primes dividing b.

2.2.0 Patterns for all the α's with the same l will be described in a uniform fashion. In fact, for each positive integer z, we will give a pattern which works for all the α's with l a divisor of z. But as we will see, the complexity of the patterns grows with z, so for practical purposes, the reader would do well to concentrate on the case $z = l$.

2.3. *Below, n will be assumed to satisfy* $n > C + z$.

2.3.0. For such n, the patterns will be given inductively, so that the continued fraction pattern for all the α's will be described effectively. Note that $\deg c$ does not come into the picture, as it can be assumed to be less than $\deg d$ by a harmless shift of an integer (polynomial).

We will start with the simple cases, which will be needed anyway for the proof of the main theorem.

THEOREM 2. *For* $n > C + z$, *we have*

(P1) *If* l *divides* $z = 1$, *then* $N_n = M_{n-1}$.

(P2) *If* l *divides* $z = 2$, *then* $N_n = (N_{n-1}, t_{n-1}, M_{n-2})$.

2.4. We will show (the reader is also urged to look at example given in 5.0) in the course of the proof of the main theorem that N_n has a well-defined decomposition into 2^j smaller blocks (for any n sufficiently large compared to any j in fact, but we will only use this for n and j appearing in the main theorem, namely for n as in 2.3 and for $j \leqslant z-2$) as follows:

2.4.1. We will write i_n^j (for $1 \leqslant i \leqslant 2^j$) for the vector of partial quotients giving the "ith part" in the "2^jth fold decomposition" of N_n. These are related inductively as follows:

$$N_n = 1_n^0, \qquad i_n^j = ((2i-1)_n^{j+1}, \qquad t_{n-j-1}, (2i)_n^{j+1}).$$

2.4.2. For example,

$$N_n = 1_n^0 = (1_n^1, t_{n-1}, 2_n^1) = (\underbrace{1_n^2, t_{n-2}, 2_n^2}, t_{n-1}, \underbrace{3_n^2, t_{n-2}, 4_n^2})$$

$$= (\underbrace{\underbrace{1_n^3, t_{n-3}, 2_n^3}, t_{n-2}, \underbrace{3_n^3, t_{n-3}, 4_n^3}}, t_{n-1}, \underbrace{\underbrace{5_n^3, t_{n-3}, 6_n^3}, t_{n-2}, \underbrace{7_n^3, t_{n-3}, 8_n^3}}).$$

2.4.3. We also define $s_k = s_{k,j}$, for $0 \leqslant k \leqslant j$, by $s_k := \sum_{u=1}^{k} 2^{j-u}$. Note that $s_0 = 0$ and $s_j = 2^j - 1$.

We are now ready to state our main result:

THEOREM 3. *For $n > C + z$, we have*:

(P3) *If l divides $z = 3$, then $1_n^1 = (2_{n-1}^1, t_{n-2}, 1_{n-1}^1)$ and $2_n^1 = (N_{n-2}, t_{n-2}, M_{n-3})$.*

(P4) *If l divides $z = 4$, then*

$$1_n^2 = (4_{n-1}^2, t_{n-3}, 3_{n-1}^2), \qquad 2_n^2 = (2_{n-1}^2, t_{n-3}, 1_{n-1}^2),$$

$$3_n^2 = (2_{n-2}^1, t_{n-3}, 1_{n-2}^1), \qquad 4_n^2 = (N_{n-3}, t_{n-3}, M_{n-4}).$$

(Pz) *More generally, for $z > 1$, if l divides z, then with $j = z - 2$, for $1 \leqslant i < 2^j$, i_n^j is given by*

$$((2^{j-k} - 2(i - s_k - 1)))_{n-k-1}^{j-k}, t_{n-j-1}, (2^{j-k} - 2(i - s_k - 1) - 1)_{n-k-1}^{j-k}),$$

$$\textit{if} \quad s_k < i \leqslant s_{k+1}$$

and

$$(2^j)_n^j = (N_{n-j-1}, t_{n-j-1}, M_{n-j-2}).$$

2.5. To help visualize these patterns without a complicated scheme of subscripts and superscripts, we reformulate the patterns as follows: The pattern in (P3) is equivalent to

$$N_n = (A_n, t_{n-1}, B_n) \qquad \text{with} \quad A_n = (B_{n-1}, t_{n-2}, A_{n-1})$$

and

$$B_n = (N_{n-2}, t_{n-2}, M_{n-3}).$$

At this point, the reader should look at the example in 5.0 *and check that the pattern follows our description.*

Pattern in (P4) is equivalent to

$$N_n = (X_n, t_{n-2}, Y_n, t_{n-1}, Z_n, t_{n-2}, W_n),$$

$$X_n = (W_{n-1}, t_{n-3}, Z_{n-1}), \qquad Y_n = (Y_{n-1}, t_{n-3}, X_{n-1}),$$

$$Z_n = (\underbrace{Z_{n-2}, t_{n-4}, W_{n-2}}, t_{n-3}, \underbrace{X_{n-2}, t_{n-4}, Y_{n-2}}),$$

$$W_n = (N_{n-3}, t_{n-3}, M_{n-4}).$$

Finally, (Pz) can be described in rough words as: N_n is formed by first writing the 2^{z-2}-fold decomposition into blocks of N_{n-1} in the reverse order, followed by the 2^{z-3}-fold decomposition in reverse for N_{n-2} and so on, until you write $N_{n-(z-1)}$ (1-fold) followed by M_{n-z}, all punctuated by the appropriate t_i's. A simpler description (which fails to illuminate the richer interconnections though) is that for larger n, $N_n = (X_k, x_k, X_{k-1}, x_{k-1}, ..., X_0)$ where $M_{n-1} = (X_0, x_1, X_1, ..., x_k, X_k)$ is the unique decomposition where the x_i are entries of degree $\geqslant 2^{n-l(b)+1} + \deg a - \deg b$ and the X_i are vectors which do not contain such entries.

2.6. Since $M_n = (M_{n-1}, t_n, N_n)$, we can rephrase (Pz) by giving an induction scheme purely in terms of parts of N_i's, an aesthetic advantage being that one has to remeber only the last few new vectors to write the next new vector.

3. ADDITIONAL/SIMPLER PATTERNS

3.0. These continued fractions exhibit many more additional and simpler symmetries in special cases. (In fact, this made the general pattern in the theorem as well as the role of $l(b)$ difficult to discover).

3.1. For example, (see [T2], or Corollary to Proposition 2 below) if b divides [1], then $N_n = M_{n-1}^-$ for large n and if b is square-free, the same formula holds for n any multiple of $l(b)$. So in the first case, we have $M_n = M_n^-$ for all n, whereas in the second case, we have complicated patterns much more simply expressed for many n's. Similarly [T2], for $\alpha = e/t^m$, we have $N_{n+1} = M_n$ and for $\alpha = e/(t^2 + t + 1)$, if n is odd (more than 1), we have $N_{n+1} = M_n^-$ and if n is even (more than 2), we have $N_{n+1} = (N_n, t_n, N_n^-)$.

3.2. Note that (Pz) implies that after some initial (explicit) vector, the only new partial quotients appearing are the t_n's. Moreover, every single partial quotient (after a_0) occurs infinitely many times. In particular, we know the exact set of partial quotients.

3.3 The simplest case of block reversal: $N_{n+1} = M_n^-$ can equally well be presented as: $N_n = (N_{n-1}^-, t_{n-1}, N_{n-1})$.

3.4. For $e/(t^2+t+1)$, the scheme can also be presented [T2] as: $N_{2n} = (N_{2n-1}^-, t_{2n-1}, N_{2n-2}^-, t_{2n-2}, N_{2n-2})$ and $N_{2n+1} = (N_{2n}, t_{2n}, N_{2n}^-)$, instead of using M's as above. This scheme also works for α having b square-free, with $l = 2$.

3.5. For b square-free, with $l(b) = 3$ and large n, the scheme can also be presented as: Let us put $X_n = (M_{3n-2}^-, t_{3n-1}, N_{3n-1}^-)$. Then $N_{3n} = M_{3n-1}^-$ (by 3.1), $N_{3n+1} = (X_n, t_{3n}, X_n^-)$ and $N_{3n+2} = (X_n^-, t_{3n}, X_n, t_{3n+1}, M_{3n-1}^-, t_{3n}, M_{3n-1})$. It can also be presented without involving M's, by just noting that 3.1 implies $M_{3n-2}^- = (N_{3n-2}^-, t_{3n-2}, N_{3n-3}^-, t_{3n-3}, N_{3n-3})$ and $M_{3n-1} = (M_{3n-2}, t_{3n-1}, N_{3n-1})$. We leave the proof, which is a straightforward modification of the proof of the main Theorem below, to the interested reader. See also 4.2.3.

3.6. For $e/(t^4+t+1)$ or for $e/(t(t^4+t+1))$, we have $N_7 = (N_5, t_5, M_4, t_6, N_6^-)$, $N_8 = M_7^- = (N_6, t_6, M_5, t_7, M_6^-)$ and $N_9 = (N_7, t_7, M_6, t_8, M_6^-, t_7, N_7^-)$; whereas for $e/(t^4+t^3+1)$, N_7 has new digits, N_8 is given by the first but not the second equality above and $N_9 = (M_5^-, t_6, N_6^-, t_7, A_7^-, t_6, B_7^-, t_8, B_7, t_6, A_7, t_7, N_6, t_6, M_5)$, where $N_7 = (A_7, t_6, B_7)$. Here we note that t^4+t+1 is invariant for the automorphism $t \to t+1$, but t^4+t^3+1 is not.

3.7. In fact, most of the time, the patterns we have described seem to have sub-patterns inside the initial part itself.

4. PROOFS

4.0. We can assume $\lfloor \alpha \rfloor$ to be zero, by subtracting it from α, if necessary.

4.0.1. We will use frequently the recursion relation 0.1.1 for d_i, as well as the fact [C1] that d_i is the product of the polynomials of degree i.

4.0.2. We use the standard notation (see [HW, T2]) for continued fractions. Our first lemma (see [BS]) is an analogue of the standard approximation criterion for the convergents of the usual continued fractions.

LEMMA 1. *Let* $f \in \mathbf{F}_2((1/t))$. *If* $\deg(f - p/q) + 2\deg q = -d < 0$, *with* $p, q \in \mathbf{F}_2[t]$ *having no common factor, then* $q = q_k$, $p = p_k$ *and* $\deg a_{k+1} = d$ *for some* k.

Proof of Proposition 1. By 4.0.1, if we write $q = bf^{2^n} d_n/a$, we have $\alpha_n = p/q$ for some p prime to q, since $n > C$. (This is proved in the proof of Theorem 2 [T2] and also follows from the integrality of Q proved in the proof of Proposition 2 below. Also, to show that α_n is a convergent, we just need the obvious fact that the denominator of α_n divides q, because the upper bound for $\deg q$ thus obtained is sufficient). Now,

$$\alpha - \alpha_n = (a/b) \sum_{i=n+1}^{\infty} 1/(f^{2^i} d_i), \qquad \deg(\alpha - \alpha_n) = \deg(a/(bf^{2^{n+1}} d_{n+1})).$$

Hence by a straightforward calculation, using 0.1.1, we have

$$\deg(\alpha - \alpha_n) + 2\deg q = \deg(b/(a[n+1]) = \deg b - \deg a - 2^{n+1} < 0.$$

By Lemma 1, α_n is a convergent to α.

4.1.1. We write $\alpha_n = p_{k_n}/q_{k_n}$, in the standard notation. To avoid double subscripts, we write k for k_n below. The reader should not confuse between $k - 1 = k_n - 1$ and k_{n-1}.

Let $u_n = \lfloor a[n]/b \rfloor$. We want to prove that $u_{n+1} = t_{n+1}$. It is enough to show that $\mu := (u_{n+1} p_k + p_{k-1})/(u_{n+1} q_k + q_{k-1})$ is a convergent to α. By Lemma 1, it is enough to show that $\deg(\alpha - \mu) + 2\deg(u_{n+1} q_k) < 0$.

Now, $\alpha - \alpha_n$ is $(a[n+1]\, q_k^2/b)^{-1}$ plus terms of degree less than $-2\deg(u_{n+1} q_k)$, as is apparent from a straightforward calculation using the series representation above. On the other hand, $\alpha_n - \mu = (u_{n+1} q_k^2 + q_k q_{k-1})^{-1}$. Adding the two quantities we have,

$$\deg(\alpha - \mu) + 2\deg(u_{n+1} q_k)$$
$$\leqslant \deg \frac{(a[n+1]/b + u_{n+1})\, q_k^2 + q_k q_{k-1}}{(a[n+1]\, q_k^2/b)(u_{n+1} q_k^2 + q_k q_{k-1})} + 2\deg(u_{n+1} q_k) < 0$$

since $\deg(a[n+1]/b + u_{n+1}) < 0$. This finishes the proof. ∎

4.2.0. We have

$$q_{k_n} = \frac{bf^{2^n} d_n}{a}, \qquad p_{k_n} = q_{k_n}\left((a/b) \sum_{i=0}^{n} 1/(f^{2^i} d_i) + (c/d)\right).$$

4.2.1. Let us write $t_n = \lfloor a[n]/b \rfloor$, $\overline{t_n} = a[n]/b$, $r_n = t_n + \overline{t_n}$.

PROPOSITION 2. *We have*

(1) $[0, N_n] = r_n + (q_{k_{n-1}-1}/q_{k_{n-1}})$,

(2) $q_{k_n-1} = q_{k_n} r_n + p_{k_n}$ *i.e.*, $r_n = (p_k/q_k) + (q_{k-1}/q_k)$.

Proof. We have

$$\frac{p_{k_n}}{q_{k_n}} = [0, M_n] = [0, M_{n-1}, t_n, N_n], \qquad \frac{p_{k_{n-1}}}{q_{k_{n-1}}} = [0, M_{n-1}],$$

so that

$$\frac{p_{k_n}}{q_{k_n}} - \frac{p_{k_{n-1}}}{q_{k_{n-1}}} = \frac{a}{bf^{2^n} d_n} = \frac{1}{q_{k_{n-1}}^2 (t_n + [0, N_n] + q_{k_{n-1}-1}/q_{k_{n-1}})},$$

which simplifies to (1).

To prove (2), we first note that $Q := q_k r_n + p_k$ is clearly an integer (i.e., a polynomial) and claim that $P := (1 + p_k Q)/q_k = (1 + p_k^2 + p_k q_k r_n)/q_k$ is also an integer.

To prove the claim, note that

$$p_k = q_k \left(\frac{a}{b} \sum_{i=0}^{n} \frac{1}{f^{2^i} d_i} + \frac{c}{d} \right) = 1 + q_k \left(\frac{a}{b} \sum_{i=0}^{n-1} \frac{1}{f^{2^i} d_i} + \frac{c}{d} \right),$$

so that

$$\begin{aligned} P &= q_k \left(\frac{a^2}{b^2} \sum_{i=0}^{n-1} \frac{1}{f^{2^{i+1}} d_i^2} + \frac{c^2}{d^2} \right) + q_k \left(\frac{a}{b} \sum_{i=0}^{n} \frac{1}{f^{2^i} d_i} + \frac{c}{d} \right) r_n \\ &= \sum_{i=1}^{n} f^{2^n - 2^i} \left(\frac{a d_n}{b d_{i-1}^2} + \frac{d_n r_n}{d_i} \right) + \left\{ f^{2^n - 1} d_n r_n + q_k \left(\frac{c^2}{d^2} + \frac{c r_n}{d} \right) \right\}. \end{aligned}$$

By 2.3, d^2 and bd divide q_k and the quantity in the curly brackets is an integer. Hence it is enough to show the integrality of $ad_n/(bd_{i-1}^2) + d_n r_n/d_i = (d_n/d_i)(\overline{t_i} + r_n)$. But b divides d_n/d_i, if $i < n$ and if $i = n$ the integrality follows, since t_n is an integer by definition. Hence the claim is justified. Finally, $p_k Q + q_k P = 1$ implies $Q = q_{k-1}$ (eg. as in [T2]), since $\deg(p_{k-1} - P) < \deg p_k$ (This is equivalent to $\deg(1 + p_k^2 + p_k q_k r_n) < \deg p_k q_k$ and this in turn follows from $\deg p_k < \deg q_k$ and $\deg r_n < 0$). This finishes the proof of the proposition. ∎

COROLLARY 1. *We have* $[0, N_n] = r_n + [0, M_{n-1}^-] = r_n + r_{n-1} + [0, M_{n-1}]$.

Proof. This follows by combining (1) and (2) of the Proposition 2 and by using the following standard formula (see [HW]) to reverse a continued fraction:

4.2.2. If $[0, X] = p_i/q_i$, then $[0, X^-] = p_i^-/q_i^-$, with

$$p_i^- = q_{i-1}, \qquad q_i^- = q_i, \qquad p_{i-1}^- = p_{i-1}, \qquad q_{i-1}^- = p_i.$$

4.2.3. The Corollary explains some of the simpler additional patterns, when b is square-free: then, $r_{il(b)} = 0$ for large i.

4.3.0. We note another key fact about the divisibility, which will be used frequently: The condition that $l(b)$ divides z is equivalent to the condition that b divides some power of $[z]$, eg. the $\deg b$th power, which is equivalent to $r_n + r_{n-z} = 0$ (since $2^{n-1} \geqslant n \geqslant \deg b$). Hence, the Corollary proves part (P1) of the Theorem 2.

4.3.1. Now, we have $q_{k-1} = q_k r_n + p_k$ and $p_{k-1} = (1 + p_k q_{k-1})/q_k$, by the basic determinant relation.

4.3.2. We write $[0, N_n] = \tilde{p}_s/\tilde{q}_s$, with $s = s_n$.

PROPOSITION 3. *We have* $\tilde{p}_s = r_n q_{k_{n-1}} + q_{k_{n-1}-1}$, $\tilde{q}_s = q_{k_{n-1}}$,

$$\tilde{q}_{s-1} = q_{k_{n-1}}(r_n + p_k/q_k) + 1/(\overline{t_n} q_{k_{n-1}}), \qquad \tilde{p}_{s-1} = (1 + \tilde{p}_s \tilde{q}_{s-1})/\tilde{q}_s.$$

Proof. The first two formulas follow easily from the Proposition 2 and the last is just the determinant identity, so it remains to prove the third formula. We have, by the reversal formula quoted above,

$$\frac{q_{k-1}}{q_k} = [0, M_n^-] = [0, N_n^-, t_n, M_{n-1}^-]$$

$$= \frac{a\tilde{p}_s^- + \tilde{p}_{s-1}^-}{a\tilde{q}_s^- + \tilde{q}_{s-1}^-} = \frac{a\tilde{q}_{s-1} + \tilde{p}_{s-1}}{a\tilde{q}_s + \tilde{p}_s},$$

with $a = t_n + p_{k_{n-1}}^-/q_{k_{n-1}}^- = t_n + q_{k_{n-1}-1}/q_{k_{n-1}}$. Plugging in the value for $\tilde{p}_{s-1}$ above, we see that

$$\tilde{q}_{s-1} = (q_{k-1}\tilde{q}_s(a + \tilde{p}_s/\tilde{q}_s)/q_k + 1/\tilde{q}_s)/(a + \tilde{p}_s/\tilde{q}_s).$$

By the first two formulas, we have $a + \tilde{p}_s/\tilde{q}_s = t_n + r_n = \overline{t_n}$. Using (2) of Proposition 2 finishes the proof. ∎

4.4.0. So we have given formulas, in general, for the $p_m, q_m, p_{m-1}, q_{m-1}$ (in terms of p_{k_n}'s, q_{k_n}'s, r_n's and t_n's) for each of the fractions $[0, M_n^{\pm}]$ and $[0, N_n^{\pm}]$, so that given any claimed induction scheme involving only M's or N's for a particular example, it should be mechanical (though laborious) to prove it, modulo simple divisibility arguments. We illustrate this philosophy now, by proving part (P2) of Theorem 2.

Write $w = t_n + p_{k_{n-1}}/q_{k_{n-1}}$, then

$$[0, N_n, t_n, M_{n-1}] = (w\tilde{p}_s + \tilde{p}_{s-1})/(w\tilde{q}_s + \tilde{q}_{s-1}).$$

By Propositions 2 and 3, the denominator is

$$wq_{k_{n-1}} + q_{k_{n-1}}(r_n + p_k/q_k + 1/q_k) = \overline{t_n}\, q_{k_{n-1}}$$

and the numerator is

$$\begin{aligned} &q_{k_{n-1}}(w(r_n + r_{n-1} + p_{k_{n-1}}/q_{k_{n-1}}) + (r_n + p_k/q_k + 1/q_k) \\ &\quad \times (r_n + r_{n-1} + p_{k_{n-1}}/q_{k_{n-1}}) + 1/q_{k_{n-1}}^2) \\ &= q_{k_{n-1}}(\overline{t_n}(r_n + r_{n-1} + p_{k_{n-1}}/q_{k_{n-1}}) + 1/q_{k_{n-1}}^2. \end{aligned}$$

4.4.1. Hence the ratio is $r_n + r_{n-1} + \alpha_{n-1} + 1/q_k$, which can be written, by Proposition 2, as

$$[0, N_n, t_n, M_{n-1}] = [0, N_n] + 1/q_k = r_{n+1} + r_{n-1} + [0, N_{n+1}].$$

By 4.3.0, since $l(b)$ divides 2, we have $r_{n+1} + r_{n-1} = 0$. Hence, we have proved Theorem 2. ▮

4.5. Now we start the proof of the main theorem.

First we prove two general lemmas, which extract all we need from the theory of continued fractions and the particular form of the α's. After exploiting them, the proof will be reduced to induction book-keeping and an application of 4.3.0.

LEMMA 2. *Let*

$$[0, X] = s + \alpha_{n-i}, \qquad [0, X^-] = u + \alpha_{n-i}, \qquad [0, Y] = v + \alpha_{n-i},$$

with $s + \alpha_{n-i}$ (*and hence* $u + \alpha_{n-i}$ *also*) *having denominator exactly* $q_{k_{n-i}}$, *and with the denominator of* v *dividing* $q_{k_{n-i}}$. *Then we have*

$$[0, X, t_{n-i+k}, Y] = s + \alpha_{n-i+1}, \qquad \text{if} \quad u + v = \overline{t_{n-i+1}} + t_{n-i+k}.$$

Proof. Recursion formulae (see [HW] or [T2]) for the numerators and the denominators for the continued fractions imply that if (in standard notion for continued fractions [HW]) $[a_0, a_1, ..., a_n] = p_n/q_n$, then

$$[a_0, a_1, ..., a_{n+1}] = \frac{a_{n+1}(p_n/q_n) + (p_{n-1}/q_n)}{a_{n+1} + (q_{n-1}/q_n)},$$

and $p_{n-1}/q_n = (\pm 1 + p_n q_{n-1})/q_n^2 = \pm 1/q_n^2 + (p_n/q_n)(q_{n-1}/q_n)$.

Hence $[0, X, t_{n-i+k}, Y]$ is

$$\frac{(t_{n-i+k} + v + \alpha_{n-i})(s + \alpha_{n-i}) + 1/q_{k_{n-i}}^2 + (s + \alpha_{n-i})(u + \alpha_{n-i})}{(t_{n-i+k} + v + \alpha_{n-i}) + (u + \alpha_{n-i})}$$

$$= \frac{(s + \alpha_{n-i})\, \overline{t_{n-i+1}} + 1/q_{k_{n-i}}^2}{\overline{t_{n-i+1}}} = s + \alpha_{n-i} + 1/q_{k_{n-i+1}} = s + \alpha_{n-i+1},$$

by 0.1.1, 4.2.0, and 4.2.1. This finishes the proof. ∎

LEMMA 3. *Let $i < z$ and $[0, X] = g + \alpha_{n+i}$ with denominator of g a divisor of b. Then the denominator of $g + \alpha_{n-i}$ is exactly $q_{k_{n-i}}$ and $[0, X^-] = g + r_{n-i} + \alpha_{n-i}$.*

Proof. Let us write $q = q_{k_{n-i}}$ and $p = p_{k_{n-i}}$ temporarily for the proof. If the denominator is not exactly q, then some prime (i.e., an irreducible polynomial) divides q but does not divide q/b. This is impossible, because b^2 divides q by 4.0.1, 2.3, and 4.2.0.

The rest of the proof is very similar to the proof of (2) of Proposition 2. In fact, $Q := (g + r_{n-i})\, q + p$ is clearly an integer and we claim that

$$P := ((gq + p)\, Q + 1)/q = g(g + r_{n-i})\, q + (1 + p^2 + pqr_{n-i})/q$$

is an integer (i.e., a polynomial).

Now the first part is an integer, since b^2 divides q and the fact that the second part is an integer is proved in the proof of Proposition 2. (One has only to replace n there by $n - i$, which is valid, because of the bound on i and 2.3). Finally, a straightforward degree estimate, as in the proof of Proposition 2, shows that P and Q are the penultimate convergents to $[0, X]$ and 4.2.2 finishes the proof. ∎

To help better understand the argument by induction on z, we give the arguments for part (P3) in detail together with the general arguments.

4.5.1. By Lemma 1 and Corollary to Proposition 2, a straightforward calculation (exactly as in the proof of Proposition 1) shows that $[0, 1_n^1] := r_n + r_{n-1} + \alpha_{n-2}$ is a convergent to $[0, N_n] = [0, 1_n^0] = r_n + r_{n-1} + \alpha_{n-1}$

with the next partial quotient t_{n-1}. So we write $[0, N_n]=[0, 1^1_n, t_{n-1}, 2^1_n]$. By Lemma 3, $[0, 1^{1-}_n]=r_n+r_{n-1}+r_{n-2}+\alpha_{n-2}$. Lemma 2 now implies that $[0, 2^1_n]=r_n+r_{n-2}+\alpha_{n-2}$. One more application of Lemma 3 gives $[0, 2^{1-}_n]=r_n+\alpha_{n-2}$. (Notice that by Proposition 3, $[0, N^-_n]=r_n+\alpha_{n-1}$ which is consistent with its convergent $[0, 2^{1-}_n]=r_n+\alpha_{n-2}$, which we would have started with, if we had argued with the reverse vector first).

4.5.2. Exactly in the same fashion, we see that, $i^j_n=((2i-1)^{j+1}_n, t_{n-j-1}, (2i)^{j+1}_n)$, as claimed in 2.4.1, together with

$$[0, (2i-1)^{j+1}_n]=[0, i^j_n]+\alpha_{n-j-2}+\alpha_{n-j-1}.$$

4.5.3. By Lemma 3, we have $[0, i^j_n]=[0, i^{j-}_n]+r_{n-j-1}$. Applying Lemma 2 to this, together with 4.5.2, we get

$$[0, (2i)^j_n]=[0, (2i-1)^j_n]+r_{n-j}+r_{n-j-1}.$$

Also, by 4.5.2, we have

$$[0, (2i-1)^{j+1}_n]=[0, i^j_n]+\alpha_{n-j-1}+\alpha_{n-j-2}.$$

By induction, this gives the values of all i^j_n's and i^{j-}_n's.

Here is a table for $j=0, 1, 2$.

$[0, 1^0_n]=r_n+r_{n-1}+\alpha_{n-1}$	$[0, 1^{0-}_n]=r_n+\alpha_{n-1}$
$[0, 2^1_n]=r_n+r_{n-2}+\alpha_{n-2}$	$[0, 2^{1-}_n]=r_n+\alpha_{n-2}$
$[0, 1^1_n]=r_n+r_{n-1}+\alpha_{n-2}$	$[0, 1^{1-}_n]=r_n+r_{n-1}+r_{n-2}+\alpha_{n-2}$
$[0, 4^2_n]=r_n+r_{n-3}+\alpha_{n-3}$	$[0, 4^{2-}_n]=r_n+\alpha_{n-3}$
$[0, 3^2_n]=r_n+r_{n-2}+\alpha_{n-3}$	$[0, 3^{2-}_n]=r_n+r_{n-2}+r_{n-3}+\alpha_{n-3}$
$[0, 2^2_n]=r_n+r_{n-1}+r_{n-2}+r_{n-3}+\alpha_{n-3}$	$[0, 2^{2-}_n]=r_n+r_{n-1}+r_{n-2}+\alpha_{n-3}$
$[0, 1^2_n]=r_n+r_{n-1}+\alpha_{n-3}$	$[0, 1^{2-}_n]=r_n+r_{n-1}+r_{n-3}+\alpha_{n-3}$

Now we explain how Lemma 2 together with the table for $j=0$ and 1 finish the proof of (P3). (We urge the reader to verify (P4) from the table above for $j\leqslant 2$ in exactly similar fashion to help better understand the general arguments that follow.):

By the $j=1$ entries of the table (for n replaced by $n-1$) and Lemma 2, $[0, 2^1_{n-1}, t_{n-2}, 1^1_{n-1}]=r_{n-1}+r_{n-3}+\alpha_{n-2}$, but by 4.3.0, this is the same as $r_{n-1}+r_n+\alpha_{n-2}$ which is $[0, 1^1_n]$. This proves the first claim of (P3). The second claim follows by comparing the 2^1_n entry with the value given 4.4.1 and Corollary 1, using 4.3.0.

4.5.4. We now claim that in general, for $1 \leqslant i < 2^j$,

$$[0, i_n^j] + r_n + r_{n-j-2} = [0, (2^{j-k} - 2(i - s_k - 1))_{n-k-1}^{j-k}, \\ t_{n-j-1}, (2^{j-k} - 2(i - s_k - 1) - 1)_{n-k-1}^{j-k}]$$

if $s_k < i \leqslant s_{k+1}$.

First we notice that by Lemma 2, the right hand side is $[0, (2^{j-k} - 2(i - s_k - 1))_{n-k-1}^{j-k}] + \alpha_{n-j-1} + \alpha_{n-j-2}$. (In more detail, by the first formula of 4.5.3 (in the notation of Lemma 2), we have $u = s + r_{n-j-2}$ and by the second formula of 4.5.3, we have $v = s + r_{n-j-1} + r_{n-j-2}$, so that $u + v = r_{n-j-1} = t_{n-j-1} + \overline{t_{n-j-1}}$ as required.)

So we are reduced to proving that

$$[0, i_n^j] = [0, (2^{j-k} - 2(i - s_k - 1))_{n-k-1}^{j-k}] + \alpha_{n-j-1} + \alpha_{n-j-2} + r_n + r_{n-j-2}.$$

The proof is by induction on j:

We will work out the case when i is odd and leave the case when i is even, which is similar, to the reader. First note that $s_{k,j} = 2s_{k,j-1}$, so that if $s_{k,j} < i \leqslant s_{k+1,j}$, then $s_{k,j-1} < (i+1)/2 \leqslant s_{k+1,j-1}$. We use this below in the induction step. We have

$$\begin{aligned}[0, i_n^j] &= [0, ((i+1)/2)_n^{j-1}] + \alpha_{n-j} + \alpha_{n-j-1} \\ &= [0, (2^{j-1-k} - 2((i+1)/2 - s_{k,j-1} - 1))_{n-k-1}^{j-1-k} + r_n + r_{n-j-1},\end{aligned}$$

where the first equality is by the third formula in 4.5.3 and the second is by the induction hypothesis. On the other hand, we have

$$\begin{aligned}&[0, (2^{j-k} - 2(i - s_k - 1))_{n-k-1}^{j-k}] \\ &\quad = [0, (2^{j-k} - 2(i - s_k - 1) - 1)_{n-k-1}^{j-k}] + r_{n-1-j} + r_{n-j-2} \\ &\quad = [0, (2^{j-1-k} - (i - s_k - 1))_{n-k-1}^{j-1-k} + r_{n-1-j} + r_{n-j-2} + \alpha_{n-j-1} + \alpha_{n-j-2},\end{aligned}$$

where the first (resp. second) equality is by the second (resp. third) formula of 4.5.3. Comparison proves the reduction and hence the claim.

4.5.5. Now we proceed to the proof of (Pz). So let $j = z - 2$. The fact 4.3.0 shows that under the hypothesis of the theorem, $r_n + r_{n-z} = 0$ and by 4.5.4, we get the proof of (Pz) for $i < 2^j$. Finally, the claim for the case $i = 2^j$ follows by comparing the value obtained by 4.5.2 and 4.5.3 with the one obtained in 4.4.1 and Corollary 1, using 4.3.0. ∎

5. AN EXAMPLE

5.0. For the benefit of the reader, we provide the C program output of an example giving the partial quotients for M_8 for $\alpha = e/(t^3+t+1)$. We have put in some empty spaces and new lines to facilitate the pattern recognition. Each entry in a squared bracket is a partial quotient, [2, 3, 4] standing for $t^2+t^3+t^4$ for example. This temporary notion, used for this example only, should not be confused with square brackets denoting continued fractions or with the square brackets defind in 0.1.1. The larger size entries are t_i's of degree 2^i-3. The reader should check the general pattern (P3), which works for N_6, N_7, N_8 and in fact the reader can check it for N_9 also, as $N_9 = M_8^-$. The reader should also check the special patterns mentioned above: for example, the pattern in 3.5 and $N_6 = M_5^-$.

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6. THE REMAINING CASES $\theta = t,\ t+1,\ t^2+t$

6.0. In these cases, we will be content with giving guesses deduced from examples worked out. Note that we have proved ([T2] Theorem 3) the pure pattern for α, when θ divides b and divides t^2+t. We now prove a basic proposition in the general case.

PROPOSITION 4. *Let θ divide $[1]$. If n is large enough, then α_n is a convergent to α.*

Proof. By 0.1.1, $[1]$ divides $[i]$, for every i, so that θ^{2^i-1} divides d_i. Hence, if q is the denominator of α_n, then q divides $b\,df^{2^i}\,d_i/\theta^{2^i-1}$. So

$$\deg(\alpha-\alpha_n)+2\deg(q)\leqslant \deg(ab\,d^2\theta^2/[n+1])<0,$$

for large n and hence by Lemma 1, α_n is a convergent to α. ∎

6.0.1. We get a glimpse of why $t^2+t=t(t+1)$ serves as an analogue of 2 in Hurwitz numbers, at the crude level of divisibility analysis as in the proof above. Classically, $\mathrm{ord}_p(n!)/n\to 1$ as $n\to\infty$, if and only if $p=2$ and in fact, $\mathrm{ord}_2(2^k!)=2^k-1$. Our d_i is an analogue of the factorial of q^i. (See [T1] for more analogies). We have $\mathrm{ord}_{\wp}(d_k)/q^k\to 1$ as $k\to\infty$ for a prime $\wp$, if and only if $q=2$ and $\wp=t$ or $t+1$ (since the norm of t or $t+1$ is 2, in the $\mathbf{F}_2[t]$ case) and also $\mathrm{ord}_{\wp}(d_k)=2^k-1$ exactly in this case, as is seen in the proof above.

6.0.2. We also see that possibilities of common factors between a, b, θ, d, f make the analysis of the exact denominators of α_n more complicated, when $\theta\notin\mathbf{F}_q^*$.

6.1. *First we consider $\theta=t$.*

We do not consider $\theta=t+1$ separately, as $t\to t+1$ is an automorphism of the whole situation.

Qualitatively, we expect the same kinds of inductive schemes, but it seems that the sequence of new partial quotients need not be from the same sequence as t_n's and the recipe for t_n also gets more complicated.

Let k be the valuation at t of a/b. Write $a_n=\lfloor [n]\,a/b\rfloor$ and $b_n=\lfloor [n]\,t^2a/b\rfloor$.

If $b=t$, $N_n=M_{n-1}^-$ with $t_n=a_n=[n]\,a/t$.

If b divides t^2+t, then $N_n=(M_{n-2}^-, a_{n-1}, M_{n-2})$, with $t_n=\lfloor [n]\,a/(t^{2k+2}b)\rfloor$. Note that $a_{n-1}=[n-1]\,a/b$ now.

If $l(b)=1$, then we have

$$\begin{aligned}N_n&=(N_{n-2}, t_{n-2}^*, A_{n-1}, a_{n-1}, B_{n-1}, t_{n-2}, N_{n-2})\\&=(N_{n-2}, t_{n-2}^*, A_{n-1}, a_{n-1}, M_{n-2})\end{aligned}$$

where $N_n = (A_n, a_{n-1}, B_n)$ and with t_n given by the formula above if $k \geqslant -1$, but with $t_{2n} = b_{2n}$ and $t_{2n+1} = a_{2n+1}$, if $k < -1$; and with $t_n^* = t_n$ if $k \geqslant -1$, but with $t_{2n}^* = a_{2n}$ and $t_{2n+1}^* = b_{2n+1}$, if $k < -1$.

6.2. *Finally we consider* $\theta = t^2 + t$.

For $l(b) = 1$, N_n is given by exactly the same formula (the special cases are also similar, but we will not spell them out here) as above, with t_n^* having exactly the same relation with t_n as before, but with t_n given as follows:

Write $a/b = t^k (t+1)^m f/g$, with f and g relatively prime to $t^2 + t$. If $k, m \geqslant -1$, we have $t_n = \lfloor [n] f/(t^{k+2}(t+1)^{m+2} g) \rfloor$.

Otherwise, if $k, m \leqslant -1$ then $t_{2n+1} = \lfloor [2n+1] a/b \rfloor$. If not, and if $m \geqslant -1$, then $t_{2n+1} = \lfloor [2n+1] a/((t+1)^{2m+2} b) \rfloor$. The case $k \geqslant -1$ follows by the automorphism sending t to $t+1$ i.e., in that case we replace $(t+1)^{2m+2}$ by t^{2k+2}.

On the other hand, $t_{2n} = \lfloor [2n] f t^u (t+1)^v / g \rfloor$, with $u = 0$, if $k = -2$; $u = k+4$, if $k < -2$; and $v = -m-2$, if $k \leqslant -2$ and $m \geqslant -1$. Automorphism then determines all the cases: i.e., $v = 0$, if $m = -2$; $v = m+4$, if $m < -2$; and $u = -k-2$, if $m \leqslant -2$ and $k \geqslant -1$.

The author will greatly appreciate hearing about any progress on proofs (or counter-examples), or more general guesses.

7.0. Some important open questions are (1) What is the deeper reason underlying the existence of strong patterns (of totally different kind) for Hurwitz numbers in both the classical and function field cases? (2) What is the deeper reason behind the appearance of the rationality question of torsion? (3) Is there a similar easy universal scheme for describing the patterns in the classical case? Which Hurwitz numbers give the pure patterns? (4) What is the general theory behind the special patterns we have illustrated?

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