ARITHMETIC OF GAMMA, ZETA AND MULTIZETA VALUES FOR FUNCTION FIELDS

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Abstract. We explain work on the arithmetic of Gamma and Zeta values for function fields. We will explore analogs of the gamma and zeta functions, their properties, functional equations, interpolations, their special values, their connections with periods of Drinfeld modules and t-motives, algebraic relations they satisfy and various methods showing that there are no more relations between them. We also briefly describe work on multizeta and many open problems in the area.

In the advanced course given at Centre de Recerca Matemática (CRM), Barcelona consisting of twelve hour lectures during 22 February-5 March 2010, we described the results and discussed some open problems regarding the gamma and zeta functions in the function field context. The first four sections, dealing with gamma, roughly correspond to the first four lectures of one and half hour each, and the last three sections, dealing with zeta, cover the last three two hour lectures. Typically, in each part, we first discuss elementary techniques, then easier motivating examples with Drinfeld modules in detail, and then outline general results with higher dimensional t-motives. The section 4 is independent of section 3, whereas the last part (last three sections) is mostly independent of the first part, except that the last two sections depend on section 3. At the end, we include a guide to the relevant literature.

We will assume that the reader has basic familiarity with the language of function fields, cyclotomic fields and Drinfeld modules and t-motives, though we will give quick reviews at the appropriate points. We will usually just sketch the main points of the proofs, leaving the details to references.

We will use the following setting and notation, sometimes it will be specialized.

$\mathbb{F}_q$: a finite field of characteristic p having q elements
X: a smooth, complete, geometrically irreducible curve over $\mathbb{F}_q$
K: the function field of X
∞: a closed point of X, i.e., a place of K
d_∞: the degree of the point ∞
A: the ring of elements of K with no pole outside ∞
K_∞: the completion of K at ∞
C_∞: the completion of an algebraic ('separable', equivalently) closure of K_∞
$\overline{K}$, $\overline{K_\infty}$: the algebraic closures of K, K_∞ in C_∞
$\mathbb{F}_\infty$: the residue field at ∞
A_v: the completion of A at a place $v \neq \infty$
g: the genus of X
h: the class number of K

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h_A: the class number of A $(= hd_\infty)$

We will consider ∞ to be the distinguished place at infinity and call any other place v a finite place. It can be given by a (non-zero) prime ideal $\wp$ of A. As we have not fixed ∞ above, we have already defined K_v, $K_\wp$, $\mathbb{F}_\wp$, C_v, d_v etc.

Since $\mathbb{F}_\infty^*$ has $q^{d_\infty}-1$ elements, when $d_\infty > 1$, in defining zeta and gamma, sign conditions giving analogs of positivity have to be handled more carefully by choosing sign representatives in $\mathbb{F}_\infty^*/\mathbb{F}_q^*$ rather than just saying monic. Having said that, for simplicity, **we will assume** $d_\infty = 1$ **throughout**, leaving the technicalities of the more general case to the references.

Basic analogs are

$$K \leftrightarrow \mathbb{Q}, \quad A \leftrightarrow \mathbb{Z}, \quad K_\infty \leftrightarrow \mathbb{R}, \quad C_\infty \leftrightarrow \mathbb{C}.$$

Instead of $\mathbb{Q}$ we can also have an imaginary quadratic field with its unique infinite place and the corresponding data.

The Dedekind domain A sits discretely in K_∞ with compact quotient, in analogy with $\mathbb{Z}$ inside $\mathbb{R}$ or the ring of integers of an imaginary quadratic field inside $\mathbb{C}$.

We will see that analogies are even stronger when X is the projective line over $\mathbb{F}_q$ and ∞ is of degree 1, so that $K = \mathbb{F}_q(t)$, $A = \mathbb{F}_q[t]$, $K_\infty = \mathbb{F}_q((1/t))$.

Comparing sizes of A^* and $\mathbb{Z}^*$, which are $q-1$ and 2 respectively, in our situation, we call multiples of $q-1$ 'even' and other integers in $\mathbb{Z}$ 'odd'.

It should be kept in mind that, unlike in number field theory, there is neither a canonical base like the prime field $\mathbb{Q}$, nor a canonical (archimedean) place at infinity. Unlike $\mathbb{Q}$, K has many automorphisms. Another crucial difference, a big plus point as we will see, is that we can combine (even isomorphic copies of our) function fields by using independent variables.

In many respects, in classical function field theory one works over $\overline{\mathbb{F}_q}$, which is the maximal 'cyclotomic' extension of $\mathbb{F}_q$, and uses Frobenius to descend. Classical study of function fields, their zeta functions, geometric class field theory goes smoothly (once we understand genus well with Riemann-Roch) for all function fields, any q, and treats all places similarly, while we do distinguish a place at infinity. Some aspects, such as zeta zero distributions, show much more complexity in our case, when q is not a prime.

We denote by H the Hilbert class field for A, i.e., the maximal abelian, unramified extension of K in which ∞ splits completely. There are h sign-normalized, non-isomorphic, rank one Drinfeld A-modules, they are Galois conjugates and are defined over H, which is a degree h extension of K. We use e and ℓ for the exponential and the logarithm for a Drinfeld module ρ. Recall that $e(az) = \rho_a(e(z))$, $\ell(\rho_a(z)) = a\ell(z)$. We write Λ_a for the corresponding a-torsion. We denote by Λ the lattice corresponding to ρ, namely the kernel of the exponential of ρ. Recall that $e(z) = z\prod'(1-z/\lambda)$, where the λ runs over the non-zero elements of Λ. If ρ corresponds to the principal ideal class (eg. for the Carlitz module for $A = \mathbb{F}_q[t]$ or for class number one A's), then we write $\Lambda = \tilde{\pi}A$ and think of $\tilde{\pi}$ as analog of $2\pi i$. The Carlitz module C is given by $C_t(z) = tz + z^q$.

Warning on the notation: We use the same notation for the same concepts, for example, for various gamma in function fields, or the complex gamma. It will be always made clear, sometimes at the start of section or the subsection, which concept we would talk about.

1. Gamma: Definitions, properties and functional equations

Once we fix a place at infinity, we have two kinds of families of cyclotomic extensions. The first family, the one mentioned above, is the family of constant field extensions. These are cyclotomic abelian (everywhere unramified) extensions obtained by solving $x^n - 1 = 0$, for (non-zero) integers n. (The classical Iwasawa theory exploits these analogies). The second cyclotomic family is obtained by adjoining a-torsion of appropriate rank one Drinfeld A-module (rather than n-torsion of the multiplicative group as above) for (non-zero) integers $a \in A$.

We will see two kinds of gamma functions closely connected to these two theories. The second one is often called geometric as (or when) there is no constant field extension involved. The first one is then called arithmetic.

1.1. **Arithmetic Gamma for $\mathbb{F}_q[t]$: Definitions and analogies.** The easiest way to introduce the Carlitz factorial Π associated to $A = \mathbb{F}_q[t]$ is to define it, for $n \in \mathbb{Z}_{\geq 0}$, by

$$\Pi(n) := n! := \prod_{\wp \text{ monic prime}} \wp^{n_\wp} \in \mathbb{F}_q[t], \quad n_\wp := \sum_{e \geq 1} \lfloor \frac{n}{\text{Norm}\wp^e} \rfloor$$

in analogy with the well-known prime factorization of the usual factorial. We define $\Gamma(n) = \Pi(n-1)$ as usual.

The following formula, which is the key to everything that follows, also gives a much faster way to compute it.

$$\Pi(n) = \prod D_i^{n_i}, \quad \text{for } n = \sum n_i q^i, \text{ with } \ 0 \leq n_i < q,$$

where

$$D_i = (t^{q^i} - t)(t^{q^i} - t^q) \cdots (t^{q^i} - t^{q^{i-1}}).$$

Here is a quick sketch of how you see the equivalence. The case $n = q^i$ immediately follows from the

claim: D_i is the product of monic polynomials of degree i.

The τ-version (i.e., q-linearized version) of the more familiar Vandermonde determinant $|x_j^{i-1}| = \prod_{i>j}(x_i - x_j)$ is the Moore determinant (a very useful tool)

$$M(x_i) := |\tau^{i-1}(x_j)| = |x_j^{q^{i-1}}| = \prod_i \prod_{f_j \in \mathbb{F}_q} (x_i + f_{i-1}x_{i-1} + \cdots + f_1 x_1).$$

(The proof of the last equality is similar to that of the Vandermonde identity.) Now $M(1, t, \cdots, t^d)$ is determinant of Moore as well as Vandermonde, so that the two evaluations give us the claim (after taking the ratio of terms for d and $d-1$).

The general case follows from $\sum \lfloor n/\text{Norm}(\wp)^e \rfloor = \sum n_i \lfloor q^i/\text{Norm}(\wp)^e \rfloor$, where $n = \sum n_i q^i$ is the base q expansion of n, i.e., $0 \leq n_i < q$.

Examples 1.1. For $q = 3$, for $i = 0, 1, 2$ we have $\Pi(i) = 1$, $\Pi(3+i) = t^3 - t$, $\Pi(6+i) = (t^3 - t)^2$ and $\Pi(9+i) = (t^9 - t)(t^9 - t^3)$.

Note that $D_i = (q^i)!$ fits in with the analogy:

$$e^z = \sum z^n/n!, \quad e(z) = \sum z^{q^n}/D_n,$$

where we compare the usual exponential e^z with the Carlitz exponential $e(z)$. This can be seen by substituting $e(z) = \sum e_i z^{q^i}$ in $e(tz) = te(z) + e(z)^q$ coming from functional equation from Carlitz module action and solving for $e_i = 1/D_i$, with initial value $e_0 = 1$ to get $D_i = (t^{q^i} - t)D_{i-1}^q$.

The general factorial is obtained then by these basic building blocks by digit expansion, a phenomenon which we will see again and again in various contexts. It can be motivated in this case by the desire to have integral binomial coefficients: For example, $q^{n+1}!/q^n! = D_{n+1}/D_n = (t^{q^{n+1}} - t)D_n^{q-1} = (t^{q^{n+1}} - t)q^n!^{q-1}$, which suggests that the factorial of $q^{n+1} - q^n = (q-1)q^n$ should be (as it is for the Carlitz factorial) D_n^{q-1}.

Here are some naive analogies.

The defining polynomial $[n] = t^{q^n} - t$ of $\mathbb{F}_{q^n}$, which can also be described as the product of monic irreducible polynomials of degree dividing n, does sometimes play some role analogous to the usual n, or rather q^n.

We have $q^n! = D_n = ([n]-[0])([n]-[1])\cdots([n]-[n-1])$ in this context looking like a factorial of $[n]$. One has twisted recursions

$$\Pi(q^{n+1}) = [n+1]\Pi(q^n)^q, \quad [n+1] = [n]^q + [1]$$

in place of the usual $(n+1)! = (n+1)n!$ and $(n+1) = n+1$ respectively.

In this vein, note that $[n][n-1]\cdots[1] =: L_n$, and $([k+1]-[k])([k+2]-[k])\cdots([n]-[k]) = L_{n-k}^{q^k}$ also play fundamental roles, as we will see.

These vague analogies are made much more precise by a definition, due to Manjul Bhargava [Bha97, Bha00], of a factorial in a very general context.

Let X be an arbitrary nonempty subset of a Dedekind ring (i.e., noetherian, locally principal and with all non-zero primes maximal) R. Special cases would be Dedekind domains $\mathcal{O}_S$ coming from global fields or their quotients. Bhargava associates to a natural number k, an ideal $k!_X := \prod \wp^{v_k(X,\wp)}$ of R, with the exponents v_k of the primes $\wp$ of R defined as follows: Let a_0 be any element of X. Choose a_k to be an element of X which minimizes the exponent of the highest power of a prime $\wp$ dividing $(a_k - a_0)(a_k - a_1)\cdots(a_k - a_{k-1})$ and $v_k(X,\wp)$ be this exponent. It can be proved that it is well-defined, independent of the choices involved.

The sequence a_i is called a $\wp$-ordering. If a_i is $\wp$-ordering for all $\wp$, then the ideal is the principal ideal generated by $k! = (a_k - a_0)\cdots(a_k - a_{k-1})$.

Examples 1.2. (1) The sequence $0, 1, 2, \cdots$ in $X = R = \mathbb{Z}$ gives simultaneous p-ordering, for all p and leads to the usual factorial, once we choose the positive generator of the corresponding factorial ideal. For X consisting of q-powers for integer $q > 1$ in $R = \mathbb{Z}$, we have $k!_X = (q^k - 1)\cdots(q^k - q^{k-1})$. For the set of $(q^j - 1)/(q-1)$'s, we get the q-factorial.

(2) For $X = R = \mathbb{F}_q[t]$, we have the following simultaneous $\wp$-ordering: Let $0 = a_0, a_1, \cdots, a_{q-1}$ be the elements in $\mathbb{F}_q$ and put $a_n = \sum a_{n_i} t^i$ where $n = \sum n_i q^i$ is the base q expansion of n. Hence the monic generator of the factorial ideal is $n! = (a_n - a_0)\cdots(a_n - a_{n-1}) = \prod [i]^{n_i + n_{i+1}q + \cdots + n_h q^{h-i}}$, which is the Carlitz factorial of n.

If q is a prime, then as a nice mnemonic we can think of associating to base q expansion $n = n(q) = \sum n_i q^i$ a polynomial $a_n = n(t) = \sum n_i t^i$ and with this ordering the Carlitz factorial can be described by the usual formula $n! = (n-0)(n-1)\cdots(n-(n-1))$. (But keep in mind that addition of n's is like integers, with carry-overs and not like polynomials!) The same works for general q, except we have to identify n_i between 0 and $q-1$ with elements of $\mathbb{F}_q$ by force then.

(3) Let $R = \mathbb{F}_q[t]$. We saw that if $X = R$, then $D_i = (q^i)!_X$. We also have $D_i = i!_X$, for $X = \{t^{q^j} : j \geq 0\}$ or for $X = \{[j] : j \geq 0\}$ justifying the naive analogies mentioned above.

Bhargava shows that the generalized factorial, though its values are ideals which may not be principal even for $R = A$, retains the most important divisibility properties of the usual factorial, such as the integrality of binomial coefficients.

On the other hand, the Carlitz factorial (but not its generalizations below for general A) even satisfies the following analog of the well-known theorem of Lucas.

Theorem 1.1. *Let $A = \mathbb{F}_q[t]$, let $\binom{m}{n}$ denote the binomial coefficient for the Carlitz factorial, and let $\wp$ be a prime of A of degree d. Then we have*

$$\binom{m}{n} \equiv \prod \binom{m_i(d)}{n_i(d)} \mod \wp,$$

where $m = \sum m_i(d)q^{di}$ and $n = \sum n_i(d)q^{di}$ are the base q^d-expansions of m and n respectively, so that $0 \leq m_i(d), n_i(d) < q^d$.

In particular, if $m > n$, the left side is zero modulo $\wp$ if and only if there is a carry over of q^d-digits in the sum $n + (m - n)$.

Proof. First observe that if there is no carry over of base q-digits, then all the binomial coefficients above are equal to one, because of the digit expansion definition of Carlitz factorial. Now suppose there is a carry over at (base q) exponents $i, i+1, \cdots, j-1$, but not at $i-1$ or j. Let $\sum m_k q^k$, $\sum n_k q^k$ and $\sum \ell_k q^k$ be the base q expansions of $m, n, m-n$ respectively. Then $n_k + \ell_k$ is $m_i + q$, $m_k + q - 1$ or $m_j - 1$ according as whether k is i, $i+1 \leq k \leq j-1$ or $k = j$. Thus the contribution of this block of digits to the binomial coefficient expression using the digit expansion is

$$\frac{D_j}{D_{j-1}^{q-1} \cdots D_{i+1}^{q-1} D_i^q} = [j] \cdots [i].$$

On the other hand, the congruence class of $[k]$ modulo $\wp$ depends on the congruence class of k modulo d, and both are zero if d divides k. □

1.2. **Arithmetic Gamma for $\mathbb{F}_q[t]$: Interpolations.** Goss made interpolations of the factorial at all places of $\mathbb{F}_q[t]$ as follows:

Since $D_i = t^{iq^i} - t^{(i-1)q^i + q^{i-1}} +$ lower degree terms, the unit part

$$\overline{D_i} := D_i / t^{\deg D_i} = 1 - 1/t^{(q-1)q^{i-1}} + \cdots$$

tends to 1 in $\mathbb{F}_q((1/t))$ as i tends to ∞. So the unit part of $\Pi(n)$ interpolates to a continuous function called ∞-adic factorial, $\overline{\Pi}(n)$:

$$\overline{\Pi} : \mathbb{Z}_p \to \mathbb{F}_q((1/t)), \ \sum n_i q^i \to \prod \overline{D_i}^{n_i}.$$

Let v (sometimes we use symbol $\wp$) be a prime of A of degree d. Since D_i is the product of all monic elements of degree i, we have a Morita-style v-adic factorial $\Pi_v : \mathbb{Z}_p \to \mathbb{F}_q[t]_v$ for finite primes v of $\mathbb{F}_q[t]$ given by

$$\Pi_v(n) = \prod (-D_{i,v})^{n_i}$$

where $D_{i,v}$ is the product of all monic elements of degree i, which are relatively prime to v, and n_i are the digits in the q-adic expansion of n. This makes sense since $-D_{i,v} \to 1$, v-adically, as $i \to \infty$. This is because, if $m = \lfloor i/d \rfloor - \ell$, for sufficiently large fixed ℓ, then $D_{i,v}$ is a q-power power of the product of all elements in $(A/v^mA)^*$ and hence is -1 mod v^m, for large i, by analog of the usual group-theoretic proof of Wilson's theorem $(p-1)! \equiv -1 \mod p$.

Here is a direct proof, in the case where v is of degree one. Using the automorphism sending t to $t+\alpha$, $\alpha \in \mathbb{F}_q$, we can assume without loss of generality that $v = t$. Now, for a general monic prime v of degree d, we have $D_{i,v} = D_i/v^w D_{i-d}$, where w is such that $D_{i,v}$ is a unit at v. So in our case,

$$-D_{n,t} = (\prod_{i=0}^{n-1}(1-t^{q^n-q^i}))/(\prod_{i=0}^{n-2}(1-t^{q^{n-1}-q^i})) \to 1, \text{ as } n \to \infty.$$

In fact, it is easy to evaluate the t-adic gamma value:

$$\Pi_t(\frac{1}{1-q}) = \lim \prod_{i=0}^{N} -D_{i,t} = -\lim \prod_{j=0}^{N-1}(1-t^{q^N-q^j}) = -1$$

because the product telescopes after the first term $-D_{0,t} = -1$

1.3. **Arithmetic Gamma for general A: Definitions and interpolations.** The different analogies that we have discussed for $\mathbb{F}_q[t]$ diverge for general A, giving different possible generalizations and we have to choose the ones with best properties. It turns out that the factorial coming from the prime factorization analogy, which is the same as the Bhargava factorial, though excellent for divisibility and combinatorial properties, is local in nature and in general there is no simultaneous p-ordering. No good interpolation for this ideal valued factorial is known. To get good global properties connecting with Drinfeld modules and cyclotomic theory, we proceed as follows. (We will deal with the exponential analogy in section 4).

The arithmetic of gamma is closely connected with cyclotomic theory, thus with rank one Drinfeld A-modules, thus with A-lattices, or projective rank one A-modules, and thus with ideals.

Let $\mathcal{A}$ be an ideal of A, and D_i be the product of all monic elements a of $\mathcal{A}$ of degree i. (Note that even for $h_A = 1$ cases, now D_i need not divide D_{i+1}, unlike the $\mathbb{F}_q[t]$ case.) So $D_i \in \mathcal{A} \subset A$. Also let d_i be the number of these elements. We choose a uniformizer $u = u_\infty$ at ∞. The one-unit part $\overline{D}_i$ with respect to u satisfies $\overline{D_i} \to 1$ as $i \to \infty$. We then define $\overline{\Pi}$ and $\overline{\Gamma}$ similarly.

By Riemann-Roch theorem, $d_i = q^{i+c}$, which tends to zero, q-adically as i tends to infinity. Thus (following a suggestion by Gekeler), we can recover the degree of the Gamma as follows:

The map $\mathbb{N} \to \mathbb{Z}$ given by $z \to \deg \Pi(z)$ interpolates to a continuous function $\deg \Pi : \mathbb{Z}_p \to \mathbb{Z}_p$ given by $\sum z_i q^i \to \sum i z_i d_i$.

Hence we p-adically complete $K_\infty^\times$, i.e. define $\hat{K}_\infty^\times := \varprojlim K_\infty^\times / K_\infty^{\times^{p^n}}$. Since finite fields are perfect, signs in $K_\infty^\times$ project to 1 in $\hat{K}_\infty^\times$.

Then we define the ∞-adic interpolation $\Pi = \Pi_\infty : \mathbb{Z}_p \to \hat{K}_\infty^\times$ with

$$\Pi(z) = \overline{\Pi}(z) u^{-\deg \Pi(z)}.$$

We use the symbol Π again, as we have recovered the degree part.

Let v be a finite place of A relatively prime to $\mathcal{A}$, and of degree d. We form $\tilde{D}_i = D_{i,v}$ as usual by removing the factors divisible by v.

Definition 1.1. *Let $\tilde{D}_i$ be the product of monic elements a of degree i and $v(a) = 0$.*

Again, generalized Wilson theorem type argument, which we omit, shows that $-\tilde{D}_i \to 1$, so we put

Definition 1.2.

$$\Pi_v(\sum z_i q^i) := \prod (-\tilde{D}_i)^{z_i}$$

so that $\Pi_v : \mathbb{Z}_p \to K_v$.

1.4. **Functional equations for arithmetic gamma.** We will now see how the structure of the functional equations for the factorial functions, for all places and all A's follows just from manipulation of p-adic digits of the arguments. So the proofs of functional equations reduce to this plus calculation of one single value: the value of gamma at 0, which we take up later.

After the more familiar reflection and multiplication formula, we will prove a general functional equation directly. We will see later how the cyclotomy and the Galois groups play a role in this structure.

Recall that the classical gamma function Γ satisfies (1) Reflection formula: $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ and (2) Multiplication formula:

$$\Gamma(z)\Gamma(z+\frac{1}{n})\cdots\Gamma(z+\frac{n-1}{n})/\Gamma(nz) = (2\pi)^{(n-1)/2} n^{1/2-nz}.$$

We will prove analogs of these and also their p-adic counterparts in the function field case, by first proving relations in the abstract setting below.

Consider a function f defined on $\mathbb{Z}_p$ via base q expansions by

$$f(\sum n_j q^j) := \prod f_j^{n_j}$$

for some f_j's. You can think of f_j's as independent variables with the evident manipulation rules. Put $g(z) = f(z-1)$. The various factorial functions ('f') and gamma functions ('g') introduced above, and below in Section 4, are all of this form.

We want to get formal relations satisfied by f. In particular, we would like to know when $\prod f(x_i)^{n_i} = 1$ formally, i.e., independently of f_i's.

First we have a reflection formula:

Theorem 1.2. $g(z)g(1-z) = g(0)$ *or equivalently (after a suitable change of variables)* $f(z)f(-1-z) = f(-1)$.

Proof. Let the digit expansion of z be $z = \sum z_j q^j$. Since $-1 = \sum (q-1)q^j$, and $0 \le q-1-z_j < q$, $-1-z = \sum (q-1-z_j)q^j$ is a digit expansion. Hence the relation with f's follows. □

Next we have multiplication formula:

Theorem 1.3. *For* $z \in \mathbb{Z}_p$ *and* $(n,q) = 1$,

$$g(z)g(z+\frac{1}{n})\cdots g(z+\frac{n-1}{n})/g(nz) = g(0)^{(n-1)/2}$$

(here, if n is even, so that q is odd, then we mean by $g(0)^{1/2}$ the element $\prod_{j=0}^{\infty} f_j^{(q-1)/2}$ whose square is $g(0)$).

Proof. If $(n,q) = 1$, $-1/n$ has a purely recurring q base expansion of r recurring digits where r is minimal such that n divides $q^r - 1$. The recurring digits for $-a/n$'s are related in such a way that the sum of the i-th digits of all of them is constant independent of i. (This can be seen by considering orbits under multiplication by

q, but we omit the details.) This constant is easily seen to be $(q-1)(n-1)/2$, as $-1/n + \cdots + -(n-1)/n = -(n-1)/2 = ((n-1)/2)\sum (q-1)q^j$. □

We now give a more general functional equation, explained in an uniform framework in the next section.

Let N be a positive integer prime to p. For $x \in \mathbb{Q}$, define $\langle x \rangle$ by $x \equiv \langle x \rangle$ modulo $\mathbb{Z}$, $0 \le \langle x \rangle < 1$. If $\underline{a} = \sum m_i [a_i]$ ($m_i \in \mathbb{Z}, a_i \in \frac{1}{N}\mathbb{Z} - \{0\}$) is an element of the free abelian group with basis $\frac{1}{N}\mathbb{Z} - \{0\}$, put $n(\underline{a}) := \sum m_i \langle a_i \rangle$. Also, for $u \in (\mathbb{Z}/N\mathbb{Z})^\times$, let $\underline{a}^{(u)} := \sum m_i [u a_i]$.

Theorem 1.4. *If*

$$n(\underline{a}^{(q^j)}) \text{ is an integer independent of } j, \tag{$**$}$$

then

$$\prod f(-\langle a_i \rangle)^{m_i} = f(-1)^{n(\underline{a})}.$$

We skip the proof involving digit manipulations [T04, Sec. 4.6] or [T91a]. We will discuss the interesting value $f(-1) = g(0)$ in the next section, by relating it to periods.

Remarks 1.1. (1) By integer translation any proper fraction in $\mathbb{Z}_p$ can be brought strictly between -1 and 0 and thus can be written as $\sum_{i=0}^{n-1} a_i q^i/(1-q^n)$, with $0 \le a_i < q$. The proof of the above theorem reduces ultimately, after these reduction steps and some combinatorics, to

$$(\frac{\sum a_i q^i}{1-q^n})! = \prod (\frac{q^i}{1-q^n})!^{a_i}.$$

This in turn is a simple consequence of $(z_1 + z_2)! = z_1! z_2!$, if there is no carry over base p of digits of z_1 and z_2. There are uncountably many pairs satisfying these conditions, which would have implied a limit point and thus functional identity in the real or complex case, but does not imply in our case because of the differences in the function theory.

(2) We will see (discussion after Theorem 6.4) that the 'basis elements' $(q^i/(1-q^n))!$, with $0 \le i < n$, are algebraically independent, for a fixed n.

1.5. **Geometric Gamma: Definitions and interpolations.** The Gamma function we studied so far has domain in characteristic zero, even though the values are in characteristic p. With all its nice analogies, it has one feature strikingly different than classical gamma function: It has no poles. The usual gamma function has no zeros and has simple poles exactly at 0 and negative integers, which we interpret as negative of the positive integers and replace 'positive' by 'monic'. (The monicity is an analog of positivity, but the positivity is closed under both addition and multiplication, while the monicity only under multiplication. Also, for $p = 2$, positive is the same as negative and for $q = 2$ all integers are negative!) Having thus decided upon the location of the poles, note that in our non-archimedean case, the divisor determines the function up to a multiplicative constant. (This follows easily from Weierstrass preparation theorem associating a distinguished polynomial to a power series, or from the Newton polygon method). The simplest constant that we choose below also seems to be the best for the analogies we describe later.

We denote by $A+$ (A_d+ respectively) the set of monic elements (monic of degree d respectively) in A. Similarly, we define $A_{<d}$ etc.

Hence we define geometric gamma function as a meromorphic function on C_∞ by

Definition 1.3.

$$\Gamma(x) := \frac{1}{x}\prod_{a\in A+}(1+\frac{x}{a})^{-1} \in C_\infty \cup \{\infty\}, \;\; x \in C_\infty.$$

From this point of view of divisors, the factorial Π should be defined as $\Pi(x) := x\Gamma(x)$.

Remarks 1.2. (1) Classically, we have $x\Gamma(x) = \Pi(x) = \Gamma(x+1)$, whereas in our situation, the first equality is natural for the gamma and factorial defined here and the second equality is natural for the ones considered previously. Consequently, the geometric gamma and factorial now differ by more than just a harmless change of variable. Also, in characteristic p, addition of p brings you back, so giving value at $x+1$ in terms of that at x will not cover all integers by recursion anyway.

(2) Unlike the arithmetic gamma case, where we started with the values at positive integers and interpolated them, in the geometric gamma case, the values at integers do not even exist for $q = 2$. For general q, in the $A = \mathbb{F}_q[t]$ case, the reciprocals of the values at integers are integral. In general, even this property is lost! At least, the values at integers (when they exist) are rational, as the gamma product then is finite, and the terms for a given degree contribute 1 to the product, when the degree is large enough.

This rationality of values at integers gives hope for interpolation a la Morita.

Definition 1.4. *For $a \in A_v$, let $\overline{a} := a$ or 1 according as whether $v(a) = 0$ or $v(a) > 0$ respectively, and when $x \in A$, put*

$$\Pi_v(x) := \prod_{j=0}^{\infty}(\prod_{n\in A_j+}\frac{\overline{n}}{\overline{x+n}}).$$

Note that the terms are 1 for large j. Hence $\Pi_v(a) \in K$ for $a \in A$.

Lemma 1.1. *Π_v interpolates to $\Pi_v : A_v \to A_v^*$ and is given by the same formula, as in the definition, even if $x \in A_v$. Similarly, $\Gamma_v(x) := \Pi_v(x)/\overline{x}$ interpolates to a function on A_v.*

Proof. It is easy to see that if $x \equiv y \mod v^l$, then $\Pi_v(x) \equiv \Pi_v(y) \mod v^l$. □

1.6. **Functional equations for geometric gamma. Reflection formula**:

For $q = 2$, all nonzero elements are monic, so

$$\Gamma(x) = \frac{1}{e_A(x)} = \frac{\tilde{\pi}}{e(\tilde{\pi}x)}$$

where e_A is the exponential corresponding to the lattice A and e is the exponential corresponding to the sgn-normalized Drinfeld module with the period lattice $\tilde{\pi}A$. Hence, for $x \in K - A$, $\Gamma(x)$ has algebraic (even 'cyclotomic') ratio with $\tilde{\pi}$ and so, as we will see, $\Gamma(x)$ is transcendental.

From the point of view of their divisors, $e(\tilde{\pi}x)$ being analogous to $\sin(\pi x)$ (i.e. both have simple zeros at integers and no poles), this observation suggests relation between Γ and sine. We reformulate the reflection relation as follows to make the analogy more visible.

The classical reflection formula can be stated as $\prod_{\theta\in\mathbb{Z}^\times}\Pi(\theta x) = \frac{\pi x}{\sin(\pi x)}$, and here, for general q, we clearly have

Theorem 1.5.
$$\prod_{\theta\in A^\times}\Pi(\theta x) = \frac{\tilde{\pi}x}{e(\tilde{\pi}x)}.$$

Multiplication formula for geometric Π:

Theorem 1.6. *Let $g\in A$ be monic of degree d and let α run through a full system of representatives modulo g. Then*
$$\prod_\alpha \Pi(\frac{x+\alpha}{g}) = \Pi(x)\tilde{\pi}^{(q^d-1)/(q-1)}((-1)^d g)^{q^d/(1-q)}R(x)$$
where
$$R(x) = \frac{\prod_{\beta\in A_{\leq m}+}\beta + x}{\prod_\alpha\prod_{a\in A_{\leq m+d}+} ga+\alpha+x},$$
with m being any integer larger than $\max(\deg\ \alpha, 2g_K)+d$.

Here $R(x)$ takes care of irregularity in Riemann-Roch at the low degrees. For example, $R(x) = \prod_{\alpha \text{ monic}}(x+\alpha)$, when $A=\mathbb{F}_q[t]$ and $\{\alpha\}$ is the set of all polynomials of degree not more than d.

Multiplication and reflection formula for Π_v:

Theorem 1.7. *(1) Let α, g be as in the previous Theorem and with $(g,v)=1$. Then*
$$\prod_\alpha \Pi_v(\frac{x+\alpha}{g})/\Pi_v(x) \in K(x)^\times.$$
(2) For $a\in A_v$,
$$\prod_{\theta\in\mathbb{F}_q^\times}\Pi_v(\theta a)/\overline{a}\in\mathbb{F}_q^*$$
and can be prescribed by congruence conditions. For example, it is $-sgn(a_v)^{-1}$ or 1 respectively, according as whether a_v, the mod v representative of a of degree less than $\deg(v)$ *(if it exists, as it always does when $A=\mathbb{F}_q[t]$), is zero or not.*

We have omitted proofs, which follow by manipulating cancellations in the product expansions.

2. Special Γ-values, relations with Drinfeld modules and uniform framework

2.1. **Arithmetic gamma: $\mathbb{F}_q[t]$ case.** First we deal with the case $A=\mathbb{F}_q[t]$ and relate the special values of $\overline{\Gamma}$ to the period $\tilde{\pi}$ of the Carlitz module. Later, we will describe how to derive more general results in a different fashion.

For $0 \neq f \in \mathbb{F}_q((1/t))$, $f/t^{\deg f}$ will be denoted by $\overline{f}$.

The most well-known gamma value at a fraction is $\Gamma(1/2)=\sqrt{\pi}$. In our case, when $p \neq 2$, so that $1/2 \in \mathbb{Z}_p$, we have $\Gamma(1/2)=\Pi(-1/2)=\Pi(-1)^{1/2}$, where the last equality follows directly by the digit expansion consideration. We will now prove an analog of this fundamental evaluation, which will also complete our functional equation in this case of the arithmetic gamma for $\mathbb{F}_q[t]$.

Carlitz [Car35] (see also [T04, Sec. 2.5] proved that

$$\tilde{\pi}=(-1)^{1/(q-1)}\lim [1]^{q^k/(q-1)}/[1]\cdots[k]$$

so $\tilde{\pi}^{q-1}\in\mathbb{F}_q((1/t))$ and $\overline{\tilde{\pi}^{q-1}}$ makes sense. By $\overline{\tilde{\pi}}$ we will denote its unique $(q-1)$-th root which is a one unit in $\mathbb{F}_q((1/t))$.

Theorem 2.1. *Let $A=\mathbb{F}_q[t]$. For $0\le a\le q-1$, we have*

$$\overline{\Gamma}(1-\tfrac{a}{q-1})=(\overline{\tilde{\pi}})^{a/(q-1)}.$$

In particular, we have $\overline{\Gamma}(0)=\overline{\tilde{\pi}}$, and if $q\neq 2^n$, then

$$\overline{\Gamma}(1/2)=\sqrt{\overline{\tilde{\pi}}}.$$

Proof. Since $-1=\sum (q-1)q^i$, we have

$$\overline{\Gamma}(0)=\overline{\Pi}(-1)=\lim \overline{(D_0\cdots D_n)^{q-1}}.$$

Now

$$(D_0\cdots D_n)^{q-1}=D_{n+1}/[1]\cdots[n+1].$$

Hence

$$\overline{\Gamma}(0)^{q-1}/\overline{\tilde{\pi}^{q-1}}=\lim \overline{D_{n+1}^{q-1}}/\overline{[1]^{q^{n+1}}}=1$$

since we have already seen that $\overline{D_i}\to 1$, and since $\overline{[1]}^{q^n}\to 1$, because any one unit raised to the q^n-th power tends to 1 as $n\to\infty$. Hence $\overline{\Gamma}(0)=\overline{\tilde{\pi}}$.

(We will not prove here the Carlitz formula above, but will see in 3.3 another formula, which also leads to the same calculation. Also, our proof below, for general A, gives another approach. The reason for giving this incomplete proof here is that the same idea generalizes in the next Chowla-Selberg analog).

Since $a/(1-q)=\sum aq^i$ for $0\le a\le q-1$, we get the theorem. □

Corollary 2.1. *For $A=\mathbb{F}_q[t]$, we have $\Gamma(0)^{q-1}=-\tilde{\pi}^{q-1}$.*

Proof. Both sides have degree $q/(q-1)$ and $\tilde{\pi}^{q-1}\in\mathbb{F}_q((1/t))$ has sign -1 as we see from any of the formulas for it, whereas any $\Gamma(z)$ has sign one by construction. □

To investigate the nature of gamma values at all fractions (with denominator not divisible by p), it is sufficient to look at all $\overline{\Pi}(q^j/(1-q^k))$ for $0\le j<k$, since a general value is (up to a harmless translation of the argument by an integer resulting in rational modification in the value) a monomial in these basic ones. They can be related to the periods $\tilde{\pi}_k$ of the Carlitz module for $\mathbb{F}_{q^k}[t]$, a rank k A-module with complex multiplication by this cyclotomic ring. For example,

Theorem 2.2. *Let $A = \mathbb{F}_q[t]$. We have* $\overline{\tilde{\pi}_k} = \frac{\overline{\Pi}(q^{k-1}/(1-q^k))^q}{\overline{\Pi}(1/(1-q^k))}$.

Proof. We have

$$\begin{aligned}\frac{\overline{\Pi}(1/(1-q^k))}{\overline{\Pi}(q^{k-1}/(1-q^k))^q} &= \lim \overline{\frac{D_{kn}D_{k(n-1)}\cdots D_0}{D_{kn-1}{}^q \cdots D_{k-1}{}^q}} \\ &= \lim \overline{[kn][k(n-1)]\cdots[k]} \\ &= (\overline{\tilde{\pi}_k})^{-1}.\end{aligned}$$

□

This is the Chowla-Selberg formula for constant field extensions, as will be explained in 2.5. Similarly, it can be shown, for example, that

$$\overline{\Pi}(1/(1-q^2))^{q^2-1} = \overline{\tilde{\pi}^q \tilde{\pi}_2{}^{-(q-1)}}$$

$$\overline{\Pi}(q/(1-q^2))^{q^2-1} = \overline{\tilde{\pi}\tilde{\pi}_2{}^{q-1}}.$$

2.2. **Arithmetic gamma: General A case.** Next, we relate for general A, $\Gamma(0)$ with a period $\tilde{\pi}$, defined up to multiplication from $\mathbb{F}_q^*$, of sign-normalized rank one Drinfeld A-module ρ, with corresponding rank one lattice $\Lambda = \tilde{\pi}\mathcal{A}$, with $\mathcal{A}$ an ideal of A, and exponential $e_\rho = e_\Lambda$.

Let x be an element of A of degree > 0, say of degree d and with $\mathrm{sgn}(x) = 1$. The coefficient of the linear term of ρ_x is x, and ρ is sgn-normalized. Hence x is the product of the nonzero roots of the polynomial ρ_x:

$$x = \prod_{a \in \mathcal{A}/\mathcal{A}x}{}' \tilde{\pi} e_{\mathcal{A}}(a/x) = \tilde{\pi}^{q^d-1} \prod{}' e_{\mathcal{A}}(a/x).$$

So

$$\tilde{\pi}^{1-q^d} = \frac{1}{x} \prod_{a \in \mathcal{A}/Ax}{}' e_{\mathcal{A}}(a/x).$$

By using the product expansion of the exponential, careful grouping and manipulation of signs, degrees, limits etc. one can prove the following theorem (though we omit the details).

Theorem 2.3.

$$\Gamma(0) = \mu\tilde{\pi},$$

where μ is $(q-1)$-th root of -1.

We note that all rank 1 normalized Drinfeld A-modules are isogenous, so periods for different choices of $\mathcal{A}$ are algebraic multiples of each other.

2.3. **Special values of arithmetic Γ_v.** We first prove strong results for the $\mathbb{F}_q[t]$ case, but only weak results for general A. We will show, in Section 6, how comparable strong results follow after developing more machinery.

The Gross-Koblitz formula, based on crucial earlier work by Honda, Dwork and Katz, expresses Gauss sums lying above a rational prime p in terms of values of Morita's p-adic gamma function at appropriate fractions.

Honda conjectured and Katz proved a formula for Gauss sums made up from p-th roots of unity in terms of p-adic limits involving factorials, combining two different calculations of Frobenius eigenvalues on p-adic cohomology (Crystalline or Washnitzer-Monsky) of Fermat and Artin-Schreier curves. Gross and Koblitz interpreted this as a special value of Morita's then recently developed p-adic interpolation of the classical factorial.

For a prime $\wp \in A$ of degree d, and $0 \leq j < d$ an analog g_j of Gauss sums was defined in author's thesis (see section 4 for details) as 'a character sum' with multiplicative character coming from theory of cyclotomic extensions of constant field extension type and analog of additive character coming from the Carlitz-Drinfeld cyclotomic theory. It was shown that for a prime $\wp$ of degree d the corresponding Gauss sum satisfies $g_j \in K(\Lambda_\wp)(\zeta_{q^d-1})$ and the corresponding Jacobi sum $g_j^{q^d-1} \in K(\zeta_{q^d-1})$ (so that, in particular, the cyclotomic extension $K(\Lambda_\wp)$ is a Kummer extension of $K(\zeta_{q^d-1})$ given by a root of the Gauss sum), and also analog of Stickelberger factorization and congruence was proved for these Gauss sums.

Theorem 2.4. *('Analog of the Gross-Koblitz formula'): Let $A = \mathbb{F}_q[t]$ and $\wp$ be a monic prime of A of degree d. Then for $0 \leq j < d$, we have*

$$g_j = -\lambda^{q^j}/\Pi_\wp(\frac{q^j}{1-q^d}),$$

where λ is a $q^d - 1$-th root of $-\wp$ (fixed by a congruence condition we omit). In particular, these values are algebraic (cyclotomic, in fact).

Proof. We have $\tilde{D}_a = D_a/D_{a-d}\wp^l$, where l is such that $\tilde{D}_a$ is a unit at $\wp$. Hence, using the base q expansion $q^j/(1-q^d) = \sum q^{j+id}$ we get,

$$\Pi_\wp(\frac{q^j}{1-q^d}) = \lim(-1)^{m+1}\tilde{D}_j \cdots \tilde{D}_{j+md} = \lim(-1)^{m+1}D_{j+md}/\wp^{w_m}$$

where $w_m = \mathrm{ord}_\wp D_{j+md}$. Moreover, the recursion formula for D_i gives

$$D_{j+md} = [j+md][j-1+md]^q \cdots [j+1+(m-1)d]^{q^{d-1}} D_{j+(m-1)d}^{q^d}.$$

Without loss of generality, we can assume that $\wp \neq t$. Thus t is a unit in $K_\wp$ and we can write in form $t = au$, as the product of its 'Teichmüler representative' $a = \lim t^{q^{md}}$ and its one unit part u. As $a^{q^{md}} = a$ and $u^{q^n} \to 1$ as $n \to \infty$, we have, as $m \to \infty$, $[l+md] = ((au)^{q^{md+l}} - t) \to (a^{q^l} - t)$, which is just $-\wp_{1-l}$, the negative of one of the monic primes $\wp_j$'s above $\wp$. Using this in the limit above and counting powers of $\wp$, using the description of $[i]$ given above, we see that

$$\Pi_\wp(\frac{q^j}{1-q^d})^{1-q^d} = (-\wp_{1-j})(-\wp_{2-j})^q \cdots (-\wp_{-j})^{q^{d-1}}/\wp^{q^j}.$$

Comparing with the Stickelberger factorization (note naive analogy with $\sum a\sigma_a$ where $a \in (\mathbb{Z}/n\mathbb{Z})^*$, when the cyclotomic Galois group $(\mathbb{Z}/n\mathbb{Z})^*$ is replaced by $q^{\mathbb{Z}/n\mathbb{Z}}$) we see that factorizations are the same and we fix the root of unity by comparing the congruences. We omit the details. ☐

This proof is quite direct and does not need a lot of machinery, unlike the proof in the classical case.

For general A, we evaluate below only a few simple values.

Theorem 2.5. $\Gamma_v(0) = (-1)^{\deg v - 1}$ *for all v prime to $\mathcal{A}$. For $0 \leq a \leq q-1$ $\Gamma_v(1 - \frac{a}{q-1})$ are roots of unity and $\Gamma_v(\frac{b}{q-1})$ is algebraic for $b \in \mathbb{Z}$.*

Proof. The first statement of the theorem implies the first part of the second statement just using the definitions in terms of relevant digit expansions. This implies the last claim immediately from definitions, by considering the effect of integral translations of the arguments.

Let $d = \deg v$. The first statement will follow, if we show that

$$(\prod m)^{q-1} \equiv (-1)^{d-1} \qquad \bmod \ v^{l_i}$$

where m runs through monic polynomials prime to v and of degree not more than t_i and with $l_i, t_i \to \infty$ as $i \to \infty$. Given l_i, choose t_i so that $\{am : a \in \mathbb{F}_q^\times\}$ spans the reduced residue class system mod v^{l_i} (for example, in $\mathbb{F}_q[t]$ case, $t_i = dl_i - 1$ works). Then it is easy to see that $\{am\}$ covers each reduced residue class equal number (which is a power of q) of times. Hence, again by the usual Wilson theorem argument, we have

$$-1 \equiv (\prod a)^{\#\{m\}}(\prod m)^{q-1} \qquad \bmod \ v^{l_i}.$$

But $\prod a = -1$, so we are done if $p = 2$. Assume p is not two, then we have to show that $\#\{m\} \equiv d \bmod 2$. But for some c we have

$$\#\{m\} = (q^{t_i - c} - q^{t_i - c - d})/(q-1) \equiv (q^d - 1)/(q-1) \equiv d \bmod 2.$$

□

2.4. **Geometric gamma values.** Functional equations (see 1.6) express many monomials at fractions as period times cyclotomic numbers. In particular, for $q = 2$, all geometric gamma values at proper fraction are algebraic multiples of $\tilde{\pi}$ and all v-adic gamma values are algebraic.

Now we look at special value results in the simplest case, that of $A = \mathbb{F}_q[t]$, q any prime power, and v a prime of degree one, to give a flavor of what can be done. It can be shown by Moore determinant calculations as before that

Lemma 2.1. *Let $D_{r,\eta,t}$ denote the product of monic polynomials of degree r, which are congruent to $\eta \in \mathbb{F}_q^*$ modulo t. Then*

$$D_{r+1,\eta,t} = D_r t^{q^r}(1 - \eta/(-t)^{(q^{r+1}-1)/(q-1)}).$$

Now notice that we are dealing with denominator t, and t-th torsion of Carlitz module is $\lambda_t = (-t)^{1/q-1}$. So the formula in the lemma can be rewritten as

$$\prod_{n \in A_d+} (1 + \eta/nt) = 1 - \eta\lambda_t/\lambda_t^{q^{d+1}}.$$

Consider the Carlitz module over $B = \mathbb{F}_q[\lambda_t]$, the integral closure of $A = \mathbb{F}_q[t]$ in the t-th cyclotomic field. This is a Drinfeld module of rank $q-1$ over A and of rank one over B. Hence this can be considered as t-th Fermat motive, having complex multiplications by t-th cyclotomic field. Denote its period by π_B.

Theorem 2.6. *With this notation, we have $\pi_B = \lambda_t^{q/(q-1)}\Pi(1/t)$.*

Proof. The displayed formula after the lemma shows that

$$\Pi(1/t) = \prod_{d=1}^{\infty}(1 - \lambda_t/\lambda_t^{q^d})^{-1}.$$

On the other hand using the Carlitz period formula (see 2.1) in the case of base B, we see that

$$\pi_B/\Pi(1/t) = \lim(\lambda_t^q - \lambda_t)^{q^n/(q-1)}/(\lambda_t^q)^{(q^n-1)/(q-1)} = \lambda_t^{q/(q-1)},$$

as $\lim(1-\lambda_t^{1-q})^{q^n/(q-1)} = 1$. □

We remark that $\Pi(-1/t)$ turns out to be a 'quasi-period'.

Now let us look at the v-adic values. First we set up some preliminary notation. The Galois group of $K(\lambda_t)$ over K can be identified with $\mathbb{F}_q^*$, with $\eta \in \mathbb{F}_q^*$ acting as $\sigma_\eta(\lambda_t) = \eta\lambda_t$. If v is a prime of degree d, then the Galois group of $K\mathbb{F}_v$ over K can be identified with $\mathbb{Z}/d\mathbb{Z}$, with $j \in \mathbb{Z}/d\mathbb{Z}$ acting as $\tau_j(\zeta_{q^d-1}) = \zeta_{q^d-1}^{q^j}$. Let $\overline{\lambda}$ be the Teichmüller representative of λ_t, so that $\lambda_t^{q^{di}} \to \overline{\lambda} \in \mathbb{F}_v \subset K_v$ as $i \to \infty$. Write $\wp = \lambda_t - \overline{\lambda}$. Then $\wp^\eta$'s are the primes of $K(\lambda_t)\mathbb{F}_v$ above v.

Theorem 2.7. *Let $A = \mathbb{F}_q[t]$, v be a monic prime of degree d of A which is congruent to 1 modulo t and $\eta \in \mathbb{F}_q^*$. Then*

$$\Pi_v(\eta/t) = (-\overline{\lambda})^{(q^d-1)/(q-1)}(\wp^{\sigma_\eta})^{-\sum \tau_j}, \ \textit{for } \eta \neq 1, \ \Pi_v(1/t) = \frac{v}{t}(\wp)^{-\sum \tau_j}.$$

In particular, $\Pi_v(a/t)$ is algebraic, for any $a \in A$.

Proof. Write $w_j := \prod_{n \in A_{j+}} \overline{n/n+\eta/t}$ and $x_j := \prod_{n\in A_{j+}} n/(n+\eta/t)$. Since v is monic congruent to 1 modulo t, for $j \geq d$, we have $w_j = x_j/x_{j-d}v^{r_j}$, where r_j is such that the right side is a unit at v.

For $0 \leq j < d$, $w_j = x_j$, unless $j = d-1$ and $\eta = 1$, in which case, they differ in $n = (v-1)/t$ term giving $w_{d-1}t/v = x_{d-1}$. Hence, for $\eta \neq 1$, the product telescopes and we get

$$\Pi_v(\eta/t) = \prod w_j = \lim \prod_{i=0}^{d-1}(1-\eta\lambda_t/\lambda_t^{q^{nd+i}})^{-1} = \prod_{i=0}^{d-1}(1-\eta\lambda_t/\overline{\lambda}^{q^i}).$$

Similarly, we calculate $\eta = 1$ case, where the answer is a unit at v. □

2.5. **Uniform framework.** Now we compare and explain analogies by giving unified treatment for the gamma functions in the three cases: classical, arithmetic and geometric.

Classical	Arithmetic	Geometric
$\Gamma(\mathbb{Z}_+) \subset \mathbb{Z}_+$	$\Gamma(\mathbb{Z}_+) \subset A_{>0}$	$\Gamma(A - A_{\leq 0}) \subset K^*$
$\Gamma : \mathbb{C} - \mathbb{Z}_{\leq 0} \to \mathbb{C}^*$	$\Gamma : \mathbb{Z}_p \to \hat{K}_\infty^*$	$\Gamma : C_\infty - A_{\leq 0} \to C_\infty^*$
$\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^*$	$\Gamma_v : \mathbb{Z}_p \to K_v^*$	$\Gamma_v : A_v \to A_v^*$

We look at the questions of algebraicity, transcendence, relations to the periods of the special values of gamma functions at fractional arguments. First note how special value combinations occurring in the reflection and multiplication formulas, for z a fraction, introduce cyclotomic ('sin $\pi a/b$') and Kummer ('$n^{1/2-nz}$') extensions and keep in mind the vague connection $\Gamma(1/a) \leftrightarrow e(\tilde{\pi}/a) \leftrightarrow a$-torsion of ρ.

A unified treatment requires some unified notation and identification of similar objects:

	Classical	Arithmetic	Geometric
I: integers in domain	$\mathbb{Z}$	$\mathbb{Z}$	A
F: fractions in domain	$\mathbb{Q}$	$\{a/b \in \mathbb{Q} : (p,b)=1\} = \{a/(q^m-1)\}$	K
$\overline{\mathcal{A}}$: algebraic nos. in range	$\overline{\mathbb{Q}}$	$\overline{K}$	$\overline{K}$
underlying objects	$\mathbb{G}_m$, CM elliptic curves	$\mathbb{G}_m$, CM ρ's	CM ρ's
exponential e	classical e	classical for domain, e_ρ for range	e_ρ
period θ	$2\pi i$	$2\pi i$ for domain, $\tilde{\pi}$ for range	$\tilde{\pi}$
B: base field	$\mathbb{Q}$	K	K
$\mathcal{O}$: base ring	$\mathbb{Z}$	A	A
extension $B(e(\theta f))$	usual cyclotomic	constant field	Drinfeld cyclotomic
Galois group G identified with	$\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^*$	$\mathrm{Gal}(K(\mu_N)/K) = q^{(\mathbb{Z}/r\mathbb{Z})}$	$\mathrm{Gal}(K(\Lambda_N)/K) = (A/NA)^*$

In this table, $f = a/N$, with $a, N \in I$ and r was defined by $\mathbb{F}_{q^r} = \mathbb{F}_q(\mu_N)$ and in the geometric case we restricted to $h = d_\infty = 1$ or even to $A = \mathbb{F}_q[t]$ first for simplicity. See below (and [T04, 4.12.5]) for comments on more general situation.

We restrict our attention to gamma values at proper fractions. In all the three cases, if $f \in F - I$, $i \in I$, then $\Gamma(f+i)/\Gamma(f) \in B \subset \overline{\mathcal{A}}$. So for our question of algebraicity, it is enough to look at $f \in (F-I)/I$. To extract this information, we define for $f \in F - I$, a rational number $\langle f \rangle \in \mathbb{Q}$ such that $\langle f \rangle$ depends only on f mod I:

Definition 2.1. *(1) For classical and arithmetic gamma: For unique $n \in \mathbb{Z}$, $0 < f - n < 1$. Put $\langle f \rangle := f - n$.*

(2) For geometric gamma: Let $A = \mathbb{F}_q[t]$. For unique $a \in A$, $f - a = -a_1/a_2$, where $a_1, a_2 \in A$, $\deg a_1 < \deg a_2$, a_2 monic. Put $\langle f \rangle := 1$ or 0, according as a_1 is monic or not.

Definition 2.2. *Consider a finite formal sum $\underline{f} = \oplus m_i[f_i]$, $m_i \in \mathbb{Z}$, $f_i \in F - I$. This is nothing but an integral linear combination of symbols $[f_i]$, i.e. a divisor. Let N be a common denominator for the f_i's. Put $\Gamma(\underline{f}) := \prod \Gamma(f_i)^{m_i}$ and make similar definitions for Π, Γ_v, Π_v etc. Also put $m(\underline{f}) := \sum m_i \langle -f_i \rangle \in \mathbb{Q}$.*

For $\sigma \in G$, let $\underline{f}^{(\sigma)} := \oplus m_i[f_i^{(\sigma)}]$, where $f_i^{(\sigma)}$ is just multiplication of f_i by σ as an element of the identification of the Galois group given in the table.

For example, in the case of arithmetic gamma, if σ corresponds to q^j, then $f_i^{(\sigma)} = q^j f_i$.

For a finite place v, let $\mathrm{Frob}_v \in G$ be the Frobenius, so that it makes sense to talk of Frob_v-orbits of f_i or of $\underline{f}$.

Consider the hypothesis/recipe/conjecture:

(H1): If $m(\underline{f}^{(\sigma)})$ is independent of $\sigma \in G$, then $\Gamma(\underline{f})/\theta^{m(\underline{f})}$ and $\Gamma_v(\underline{f})$ belong to $\overline{\mathcal{A}}$.

(H2): If $\underline{f}$ is a linear combination of $Frob_v$-orbits, then $\Gamma_v(\underline{f}) \in \overline{\mathcal{A}}$ (or rather algebraic in appropriate v-adic context).

Examples 2.1. With $\underline{f} = \langle f \rangle + \langle 1-f \rangle$, $m(\underline{f}) = 1$ and with $\underline{f} = \oplus_{j=0}^{n-1} \langle z + j/n \rangle - \langle nz \rangle$, $m(\underline{f}) = (n-1)/2$ the hypothesis of H1 is satisfied in the first two cases. These correspond to the reflection and multiplication formulas. In case II, the theorems above show that $\Gamma(-\underline{f})/\tilde{\pi}^{n(\underline{f})} \in \overline{\mathcal{A}}$. This implies that $\Gamma(\underline{f})/\tilde{\pi}^{\sum m_i - n(\underline{f})} \in \overline{\mathcal{A}}$, but in case of reflection and multiplication, $\sum m_i = 2n(\underline{f})$. Hence $\Gamma(\underline{f})/\tilde{\pi}^{\sum m_i \langle -f_i \rangle}$ works in both cases under the independence hypothesis.

What is known about H1 and H2 and are they best possible?

H1 is true in all the three cases, but still we have written it as a hypothesis, because we will consider more situations later, which fit in this framework: For gamma of the fourth section, it is true with $\theta = \Gamma(0)$, but we do not understand yet the nature, or connection with the period, of this value.

In the classical case, $\underline{f}$ satisfying the condition in H1 is a linear combination, with rational coefficients, of the examples above (i.e. the conclusion follows from the known reflection and the multiplication formulas) by a result of Koblitz and Ogus. With a proof analogous to that of Koblitz and Ogus, the same can be proved for the geometric case. (We do need rational coefficients, so that we get some non-trivial χ_a's for a's which are not integral linear combinations of our functional equations. See [Sin97a] for analysis of rational versus integral linear combinations of functional equations and Anderson's 'ϵ-generalization' of Kronecker-Weber theorem [A02] in the classical case, inspired by this analysis.) But, since the independence condition in arithmetic case is much weaker than the one in the classical case, the general functional equations above which prove H1 in this case directly are more general than those generated by multiplication and reflection formulas.

H2 follows from the Gross-Koblitz theorem in classical case, and its analog above for the arithmetic case with $A = \mathbb{F}_q[t]$. We will see a full proof for geometric case, when $A = \mathbb{F}_q[t]$. There is some evidence in other cases.

In the complex multiplication situation, we also have a Chowla-Selberg type formula (up to multiplication by element of $\overline{\mathcal{A}}$) for the period, in terms of the gamma values at appropriate fractions. This formula is predicted via a simple calculation involving the 'brackets' introduced above. It works as follows:

Let E be a Drinfeld A-module (or elliptic curve for the case I) over B^{sep} with complex multiplication by the integral closure of $\mathcal{O}$ in the appropriate abelian extension L of B as in the table, eg., a constant field extension for arithmetic and Drinfeld cyclotomic extension for geometric case etc. (In fact, we can get much more flexibility using higher dimensional A-motives of Anderson, eg., the solitons give rise to higher dimensional A-motives with Drinfeld cyclotomic CM whose periods are values of $\Gamma(z)$, for $z \in K$, in the geometric case with $A = \mathbb{F}_q[t]$. But here we will be content with this simple case.)

For $f \in F - I$, let $h(f) : \mathrm{Gal}(B^{sep}/B) \to \mathbb{Q}$ be defined by $h(f)(\sigma) := \langle -f' \rangle$, where $e(\theta f)^\sigma = e(\theta f')$. (Note that in arithmetic case, the exponential used here is the classical exponential according to the table.)

Let $\chi_{L/B}$ be the characteristic function of $\mathrm{Gal}(L/B)$. In other words, $\chi_{L/B}$: $\mathrm{Gal}(B^{sep}/B) \to \mathbb{Z}$ is such that $\chi_{L/B}(\sigma)$ is 1, if σ is identity on L and is 0 otherwise. We then have the hypothesis/recipe/conjecture:

(H3): If $\chi_{L/B} = \sum m_f h(f)$, $m_f \in \mathbb{Z}$, then $\mathrm{Period}_E / \prod \Gamma(f)^{m_f} \in \overline{\mathcal{A}}$.

This is known in classical case. In function fields, there is only some evidence.

Examples 2.2. $\mathcal{O} = A = \mathbb{F}_q[t]$, $K = B = \mathbb{F}_q(t)$, $L = \mathbb{F}_{q^k}(t)$. Then the rank k Drinfeld A-module: $u \mapsto tu + u^{q^k}$ is the rank one Carlitz module over $\mathbb{F}_{q^k}[t]$. Hence we want to express its period $\tilde{\pi}_k$ in terms of the gamma values at fractions with the gamma function built up from the base. If τ is the q-power Frobenius, then $\chi_{L/B}(\tau^n)$ is 1, if k divides n and is 0 otherwise. The same is true for

$$[qh(\frac{q^{k-1}}{1-q^k}) - h(\frac{1}{1-q^k})](\tau^n) = q\langle \frac{q^{k+n-1}}{q^k-1}\rangle - \langle \frac{q^n}{q^k-1}\rangle.$$

Hence H3 predicts that $\Gamma(q^{k-1}/(1-q^k))^q/\Gamma(1/(1-q^k))$ is an algebraic multiple of $\tilde{\pi}_k$, as we have seen before.

Remarks 2.1. (1) In case I, the Γ function is closely connected with ζ and L functions, it is in some sense a factor at infinity for the ζ function and hence appears in functional equations. Such connections are missing in the other cases. We only have $\Pi'(x)/\Pi(x) = \zeta(x,1)$ in the third case.

(2) The Hurwitz formula $\zeta(x,s) = \langle -x\rangle - 1/2 + s \log \Gamma(x) + o(s^2)$ around $s = 0$ connects partial zeta, 'brackets' and logarithm of gamma. For case III, the partial zeta value for $x = a_1/a_2$ (with $\deg a_1 < \deg a_2$ and $a_2 \in A_+$) is obtained by summing $q^{-s \deg a}$ over monic a congruent to a_1 modulo a_2. Hence at $s = 0$, it equals $1 - 1/(q-1)$ or $-1/(q-1)$ according as a_1 is monic or not. This is the motivation for the definition, due to Anderson, of the brackets as above in the geometric case

(3) See [T04, Cha. 4] for the connection of this set-up with function field gauss and jacobi sums we mentioned before. Chris Hall made the following interesting remark about the connection of the condition in (H1) with the classical gauss sums and their occurrence in the work related to Ulmer's parallel lecture series on ranks of elliptic curves over function fields. The classical Gauss and Jacobi sums arise, e.g., in Weil's work, as (reciprocal) zeros of zeta functions of Fermat varieties $X/\mathbb{F}_q$. When such a sum arises as a (reciprocal) eigenvalue of Frobenius acting on the even-index cohomology of X and when the n-th power of the sum is a power of q, then the Tate conjectures predict that there are corresponding algebraic cycles on $X/\mathbb{F}_{q^n}$. While finding such algebraic cycles is usually very difficult, the seemingly simpler question of determining whether or not some power of a Gauss or Jacobi sum is a power of q also appears quite difficult. (In the literature (cf. papers of R. Evans and N. Aoki), such sums are called 'pure'). One can use Stickelberger's theorem to reformulate the condition for a particular sum to be pure in terms of a fractional sum (cf. section 2.3 of [Ulmer, Math. Res. Lett. 14 (2007), no. 3, 453–467.]), and the resulting condition resembles hypothesis (**) of theorem 3, Sec. 1.4 or (H1).

We will see, in section 4, that H1 is best possible in arithmetic case by automata method. In both the function field cases, by motivic method, which was developed a little later, we can even prove (see sections 3 and 6) stronger algebraic independence results. In the classical case, much less (see sec. 4.1) is known.

3. Solitons, t-motives and complete gamma relations for $\mathbb{F}_q[t]$

In this section, we restrict to $A = \mathbb{F}_q[t]$.

Our examples in the last section can be considered as simple cases of Fermat motives: Drinfeld modules with complex multiplication by ring of integers of 'cyclotomic fields'. In these cases, we saw connections between periods and gamma values and Frobenius eigenvalues at finite primes, the gauss sums connecting to values of interpolated gammas at that prime. Now we will outline how complete generalization using t-motives, which are higher dimensional generalization of Drinfeld modules, works.

So far, we handled gamma values by connecting them to Drinfeld modules with complex multiplications by (integral closure of A in) cyclotomic fields, such as constant field extensions $K(\mu_n)$ of K or Carlitz-Drinfeld cyclotomic extensions $K(\Lambda_\wp)$. Drinfeld modules can handle only one point at infinity. For the geometric gamma function, this restricted us to $\Gamma(1/\wp)$ (and its v-adic counterparts), with degree one prime $\wp$. To handle $\Gamma(1/f)$, for any $f(t) \in A$, we need objects with multiplication from $K(\Lambda_f)$. Thus we need 'higher dimensional t-motives' which can handle many infinite places. We will focus on the geometric gamma in this section and return to the general case in section 6.

3.1. **Anderson's solitons: General overview.** If you look at the proof of theorems 2.6, 2.7, the crucial point was to produce special functions (coined 'solitons' by Anderson, the reason will be explained below) on $X_f \times X_f$ (where X_f is the f-th cyclotomic cover of the projective line, namely the curve corresponding to $K(\Lambda_f)$), with the property that they specialize on graph of d-th power of Frobenius to the d-th degree term in the product defining the gamma value.

Examples 3.1. For $f(t) = t$, as we saw before by Moore determinants, this corresponded to a 'compact formula' (for the left side which has exponentially growing size)

$$(\prod_{\substack{a\in A_{i+} \\ a\equiv 1 \mod t}} a)/(t^{q^{i-1}} D_{i-1}) = 1 - \frac{\zeta_t}{\zeta_t^{q^i}} = 1 - (-t)^{-(q^i-1)/(q-1)},$$

with corresponding 'soliton' $\phi = (t/T)^{1/q-1} = \zeta_t/\zeta_T$, if you use two independent variables t and T for the two copies of the line. The additivity of the solitons (as functions of a/f) was established using Moore determinants and this formula was used (essentially the method of partial fractions) to get formulas for (what is now understood as) the solitons when f was a product of distinct linear factors.

In general, we get 'compact' formulas for the product of the monic polynomials in $\mathbb{F}_q[t]$ of degree i in a given congruence class (essentially the same as asking for a formula for the term in the definition of the geometric gamma function at a fraction). Here is another example

$$(\prod_{\substack{a\in A_{i+} \\ a\equiv 1 \mod t^2}} a)/((t^2)^{q^{i-2}} D_{i-2}) = 1 - \frac{\zeta_{t^2}(-t)^{q^{i-1}/(q-1)} - \zeta_{t^2}^{q^{i-1}}(-t)^{1/(q-1)}}{(-t)^{2q^{i-1}/(q-1)}}.$$

Here $f(t) = t^2$, and

$$\phi = (\zeta_{t^2} C_T(\zeta_{T^2}) - \zeta_{T^2} C_t(\zeta_{t^2}))/C_T(\zeta_{T^2})^2.$$

Remarks 3.1. For $A = \mathbb{F}_q[t]$, we have good analogs of binomial coefficient polynomials $\{{x \atop n}\}$ obtained by digit expansions, just as in factorial case, from the basic ones for $n = q^d$, defined via analogy $e(x\ell(z)) = \sum \{{x \atop q^d}\} z^{q^d}$. We have the exponential $e(z) = \sum z^{q^i}/d_i$ and the logarithm $\ell(z) = \sum z^{q^i}/\ell_i$, in general, so that $D_i = d_i$ and $L_i = (-1)^i \ell_i$, in our case of Carlitz module for $A = \mathbb{F}_q[t]$.

It can be easily shown that

$$\{{x \atop q^k}\} = \sum x^{q^i}/(d_i \ell_{k-i}^{q^i}) = e_k(x)/d_k =: \binom{x}{q^k},$$

with $e_k(x) = \prod(x - a)$ where the product runs over all $a \in A$ of degree less than k. Same definitions easily generalize to all A's, but the curly and round brackets binomials differ in general and we will return to this issue. For several analogies, we refer to [T04, Sec. 4.14].

It is easy to verify that the degree d term in the geometric gamma product definition for $\Gamma(a/f)$ is nothing but $1 + \binom{a/f}{q^d}$. So solitons are 'interpolations' of these binomial coefficients, for $x = a/f$, as d varies.

Some such functions constructed by the author were constructed using Moore determinants. Anderson had a great insight that such determinant techniques are analogs of techniques found in the so called tau-function theory or soliton theory. Applying deformation techniques from soliton theory, which occur in Krichever theory of solving, using theta functions, certain integrable system partial differential equations, such as KdV, to the arithmetic case of Drinfeld dictionary, Anderson obtained [A92, A94, ABP04], for any f, the required functions, which he coined solitons. Anderson used tau- functions, theta functions, and explicit constructions (using exponential, torsion and adjoints) respectively in the three papers mentioned. We refer to these papers and [T01] for motivations and analogies.

We will only describe the last construction. See [T04, 8.5] or [T99] for alternate approach of the author giving Frobenius semilinear difference equations analogous to partial differential equations.

Interpolating the partial gamma products, by such algebraic functions on the cyclotomic curve times itself, Anderson constructed t-motives and showed that their periods are essentially the Gamma values, and that they also give ideal class annihilators. This vastly generalizes very simple examples we looked at. The applications to special values and transcendence go way beyond the classical counterparts this time, because the t-motives occurring can have arbitrary fractions as weights in contrast to the classical motives

We saw that the gamma product for $\Gamma(a/f)$, restricted to degree d, is essentially the binomial coefficient $\{a/f, q^d\}$, which is the coefficient of t^{q^d} in $e(\ell(t)a/f)$, which can be thought of as deformation of torsion $e(\ell(0)a/f)$, since $\tilde{\pi}$ is a value of $\ell(0)$. In function fields, we have the luxury of introducing more copies of variables (tensor products, products of curves, generic base change) with no direct analog for number fields. (In other words, we do not know how to push polynomials versus numbers analogy to multi-variable polynomials.)

Before describing constructions of solitons, we sketch how they are used to construct t-motives which have periods and quasi-periods the corresponding gamma values.

3.2. **t-modules, t-motives and dual t-motives.** Let F be an A-field, $\iota : A \to F$ being the structure map (not needed in the number field case where we have the canonical base $\mathbb{Z}$). We assume F is perfect. This is needed for technical reasons, but for our transcendence applications, we can even assume it to be algebraically closed. The kernel of ι is called the characteristic. We will usually stay in 'generic characteristic' situation of zero kernel.

Recall that a Drinfeld A-module ρ over F is roughly a non-trivial embedding of (commutative ring) A in the (non-commutative ring of) $\mathbb{F}_q$-linear endomorphisms of additive group over F. It gives, for a given $a \in A$, a polynomial $\rho_a \in F\{\tau\}$ (where τ is the q-th power map) with constant term $\iota(a)$. Now ρ_t, for a single non-constant $t \in A$, determines ρ_a, for any $a \in A$, by commutation relations. So without loss of generality, we can just look at t-modules, which one can think of as special case $A = \mathbb{F}_q[t]$, or the general case can be thought of as having extra multiplications. In other words, a t-module G is an algebraic group isomorphic to the additive group together with a non-trivial endomorphism denoted by t, such that at the Lie algebra level t acts by scalar $\theta = \iota(t) \in F$.

(**Warning**: Since we have stated everything with t-variable at the start, we will do so for the end results also, but in proofs we will distinguish t and θ, just as we distinguish the role of coefficient and multiplication field, even when they are the same. Our answers will thus often be in terms of θ, which we then replace by t! As we progress, we will see the advantage of having two isomorphic copies of $\mathbb{F}_q(x)$ together, something which we can not do in the number field case).

We can generalize to d-dimensions by replacing the additive group by its d-th power, and Anderson realized that rather than requiring t acting by a scalar matrix θ at the Lie algebra level, one should relax the condition to t having all eigenvalues θ or equivalently $t - \theta$ should be a nilpotent matrix. We will see the need for the relaxation clearly when we calculate $C^{\otimes n}$ in section 6.

Anderson called $M(G) := \mathrm{Hom}(G, \mathbb{G}_a)$ the corresponding t-motive. By composing with t-action on left and τ-action on right, this is a module for (non-commutative ring) $F[t, \tau]$. This dual notion is called motive, because it is a nice concrete linear/τ-semilinear object from which cohomology realizations of Drinfeld modules and t-modules can be obtained by simple linear algebra operations.

We will not go into details (see [G94] or [T04, 7.5]) here, but just note that by efforts of Drinfeld, Deligne, Anderson, Gekeler, Yu, cohomologies of Betti, v-adic, DeRham, Christalline types, comparison isomorphisms etc. are developed and that the recent works of Pink, Böckle, Hartl, Vincent Lafforgue, Genestier etc. have developed Hodge theoretic and Christalline aspects much further into a mature theory.

The dimension d and rank r of M are, by definition, the ranks of M (freely generated) over $F[\tau]$ and $F[t]$ respectively. Choosing a basis, a t-motive can thus be described by a size r matrix M_τ with entries in $F[t]$. Unlike the one dimensional case of Drinfeld modules, in general, the corresponding exponential function need not be surjective. The surjectivity or uniformizability criterion can be expressed as existence of solution of matrix equation (see below) in the rigid analytic realm. (The uniformizability is equivalent to the rank of lattice being r). The period matrix is then obtained by residue (at $t = \theta$) operation from this solution.

Anderson later used the 'adjoint' duality [T04, 2.10] developed by Ore, Poonen, Elkies, between τ objects and $\sigma := \tau^{-1}$-objects and considered 'dual t-motives' which are $F[t,\sigma]$-modules. For t-module G, one associates such $M^*(G) := \mathrm{Hom}(\mathbb{G}_a, G)$. When F is perfect, eg. algebraically closed (as is sufficient for transcendence purposes), and when we do not bother about finer issues of field of definition etc., the q-th root operation is fine too.

The technical advantages are quick, clean description [ABP04] as $M^*/(\sigma-1)M^*$ for points of G and $M^*/\sigma M^*$ for its Lie algebra in terms of M^*, and being able to replace residue operation by the simpler evaluation operation.

In [ABP04], this was put to great use and transcendence theory papers now use this language and call the dual t-motives, the t-motives or Anderson t-motives. Just be warned that this basic term means slightly different (but related) concepts in different papers. In transcendence literature, the rigid analytically trivial (equivalently uniformizable) dual t-motive is often called t-motive.

We just remark that in the first part of these notes, dealing with gamma values, we use 'pure' complex multiplication (rank one over higher base) t-motives (coming from solitons), while dealing with zeta and multizeta values in the second part, we use mixed motives of higher rank and the language of Tannakian theory in terms of motivic galois group, which we could have used, to simplify once developed, in the first part also.

3.3. **Period recipe and examples.** Let Φ be size r square matrix with entries in $F[t]$ giving the σ action $\sigma m = m^{(-1)}$ on the (dual!) t-motive M. Consider the Frobenius-difference equation $\Psi^{(-1)} = \Phi\Psi$. If it has a solution Ψ with entries in the fraction field of power series convergent in closed unit disc (fraction field of the Tate algebra), then Anderson proved that M is uniformizable with period matrix being the inverse of Ψ evaluated at $t = \theta$. (Here n-th twist ${}^{(n)}$ represents entry wise q^n-th power on elements of F and identity on t).

For example, for the Carlitz module, we have $\Phi = t - \theta$. Then we have

$$\Omega := \Psi = (-\theta)^{-q/(q-1)} \prod_{i=1}^{\infty} (1 - t/\theta^{q^i})$$

and $\tilde{\pi} = 1/\Psi(\theta)$. We will denote this particular Ψ by Ω, as is the common notation.

Remarks 3.2. We remark here that though C_∞ being of infinite degree over K_∞ is quite different than $[\mathbb{C} : \mathbb{R}] = 2$, the formulas for $\tilde{\pi}$ show that for many purposes the role of C_∞ is played by the degree $q-1$ extension

$$K_\infty(\tilde{\pi}) = K_\infty(\zeta_t) = K_\infty((-t)^{1/(q-1)}) = K_\infty((-[1])^{1/(q-1)}) \leftrightarrow \mathbb{C} = \mathbb{R}(i) = \mathbb{R}(2\pi i).$$

The algebraic elements of $K_\infty(\tilde{\pi})$ are separable over K.

Note that ignoring simple manipulations needed for the convergence in the right place, we always have a formal solution $\Psi = \prod_{i=1}^{\infty} \Phi^{(i)}$, so that in the rank one case, the (reciprocal of the) period is given as product of Φ evaluated at graphs of powers of Frobenius, as we mentioned above.

So given a soliton Φ, one gets a motive with period the corresponding gamma value at the proper fraction, up to a simple factor.

3.4. **Explicit construction.** We follow the notations of [ABP04], the T (t respectively) below corresponds to our t (θ respectively).

Let $\mathbf{e}(z) = e(\tilde{\pi} z)$. Now put $\Omega^{(-1)}(T) = \sum a_i T^i$, $a_i \in K_\infty(\tilde{t})$. Let $\mathrm{Res} : K_\infty \to \mathbb{F}_q$ be the unique $\mathbb{F}_q$-linear functional with kernel $\mathbb{F}_q[t] + 1/t^2 \mathbb{F}_q[[t]]$ and with $\mathrm{Res}(1/t) = 1$, i.e. the usual residue function for parameter t. For $x \in K_\infty$, put

$$\mathbf{e}_*(x) := \sum_{i=0}^{\infty} \mathrm{Res}(t^i x) a_i.$$

We have

$$\Omega^{(-1)}(T) = \sum_{i=0}^{\infty} \mathbf{e}_*(\frac{1}{t^{i+1}}) T^i, \qquad \frac{1}{\Omega^{(-1)}(T)} = \sum_{i=0}^{\infty} \mathbf{e}(\frac{1}{t^{i+1}}) T^i$$

So $\mathbf{e}_*(1/t) = 1/\tilde{t}$. From the functional equation above, we get recursion on a_i's which implies that

$$t\mathbf{e}_*(x)^q + \mathbf{e}_*(x) = \mathbf{e}_*(tx)^q.$$

Comparison with adjoint $C_t^* = t + \tau^{-1}$ of Carlitz module shows that $\mathbf{e}_*(a/f)$ are q-th roots of f-torsion points of this adjoint.

Definition 3.1. *For $f \in A_+$, we say that families $\{a_i\}$, $\{b_j\}$ ($i, j = 1$ to $\deg f$) of elements in A are f-dual if Res$(a_i b_j / f) = \delta_{ij}$.*

Theorem 3.1. *Fix $f \in A_+$ and f-dual families $\{a_i\}$, $\{b_j\}$ as in the definition above. Fix $a \in A$ with $\deg a < \deg f$. Then*

$$\sum_{i=1}^{\deg f} \mathbf{e}_*(a_i/f)^{q^{N+1}} \mathbf{e}(b_i a/f) = -\binom{a/f}{q^N}$$

for all $N \geq 0$. Also, if $a \in A_+$,

$$\sum_{i=1}^{\deg f} \mathbf{e}_*(a_i/f) \mathbf{e}(b_i a/f)^{q^{\deg f - \deg a - 1}} = 1.$$

Thus we define soliton by

Definition 3.2. *For $x = a_0/f \in f^{-1}A - A$, put*

$$g_x := 1 - \sum_{i=1}^{\deg f} \mathbf{e}_*(a_i/f)(C_{a_0 b_i}(z)|_{t=\theta}).$$

If y is 'fractional part' of ax,

$$g_x^{(N+1)}(\xi_a) = 1 + \binom{y}{q^N} = \prod_{n \in A_N+} (1 + y/n),$$

where $\xi_a := (t, \mathbf{e}(a/f))$. Thus on graph of $N+1$-th Frobenius power it gives N-th term of gamma product.

For $x \in K_\infty = \mathbb{F}_q((1/t))$ and integers $N \geq 0$, let us define $\langle x \rangle_N$ to be 1, if the fractional part power series of x starts with $(1/t)^{n+1}$, and 0 otherwise. Then for bracket defined in uniform framework, we have $\langle x \rangle = \sum_{N=0}^{\infty} \langle x \rangle_N$.

Theorem 3.2. *For $x \in f^{-1}A - A$, we have equality of divisors of f-th cyclotomic cover X_f of the projective line*

$$(g_x) = -\frac{1}{q-1}\infty_{X_f} + \sum_{a \in A_{<\deg f}, (a,f)=1} \sum_{N=0}^{\infty} \langle ax \rangle_N \xi_a^{(N)}.$$

The divisor D on the right (note the sum is finite) being the Stickelberger divisor of the cyclotomic theory, you get through the partial zeta function corresponding to a/f, we get the soliton specialization as the Stickelberger element, generalizing Coleman's constructions [A92] corresponding to few denominators f.

Specializing the solitons at appropriate geometric points, Anderson has proved [A92] two dimensional version of Stickelberger's theorem. For connection with the arithmetic of zeta values and theta functions, see [A94, T92b]. The reason for the name soliton is that the way it arises in the theory of Drinfeld modules or Shtukas, when dealing with the projective line with some points (in support of f) identified, is analogous to the way the soliton solutions occur in Krichever's theory of algebro-geometric solutions of differential equations, as explained in [Mum78, pa. 130, 145] and [A92, A94, T01].

3.5. **Analog of Gross-Koblitz for Geometric Gamma: $\mathbb{F}_q[t]$ case.** Using Anderson's analog of Stickelberger, the proof of Gross-Koblitz analog generalizes, from the simplest case considered in 2.3, to the general $\mathbb{F}_q[t]$ case by a proof exactly as in previous cases:

Theorem 3.3. *Let $A = \mathbb{F}_q[t]$. Let v be a monic prime of degree d such that f (of positive degree) divides $v-1$. Let a be an element of degree less than that of f. Then $\Gamma_v(a/f)$ is (explicit) rational multiple of the a/f-Stickelberger element applied to v. In particular, it is algebraic. (So If $\underline{a}$ consists of $Frob_v$-orbits, up to translation by elements in A, then $\Gamma_v(\underline{a})$ is algebraic.)*

3.6. **Fermat t-motive.** With soliton function $g = g_x$, consider $M_f :=$ as $\overline{K}[t,\tau]$-module by left multiplication action of $\overline{K}[t]$ and by τ action given by g. The underlying space here is rank one over 'cyclotomic base'. (See [ABP04] for details).

Let $f \in A_+$, $I := \{a \in A : (a,f) = 1, \deg(a) < \deg(f)\}$, and $I_+ = I \cup A_+$. Sinha, in his University of Minnesota thesis, proved

Theorem 3.4. *With the notation as above, M_f is a uniformizable abelian t-motive over $\overline{K}$ of dimension $\phi(f)/(q-1)$ and rank $\phi(f) := |(A/fA)^*|$. In fact, the corresponding t-module is HBD module with multiplications by $A[\zeta_f]_+$ and is of CM type with complex multiplication by $A[\zeta_f]$. Its period lattice is free rank one over $A[\zeta_f]$, with the a-th coordinate (for $a \in I_+$) of any non-zero period being (with appropriate explicitly given $C_\infty\{\tau\}$-basis of M to give co-ordinates) the (explicit non-zero algebraic multiple of) value $\Gamma(a/f)$.*

Brownawell and Papanikolas [BP02] proved

Theorem 3.5. *The coordinates of periods and quasi-periods of M_f (in the same coordinates as above) are exactly the (explicit non-zero algebraic multiples of) values $\Gamma(a/f)$, with $a \in I$.*

We refer to [Sin97b, BP02, ABP04] for very nice and clean treatment of these issues considered in 3.4, 3.6-3.8.

3.7. **ABP criterion: Period relations are motivic.** Here we use $F = K$, C_∞ with variable θ and t an independent variable.

Let $C_\infty\{t\}$ be the ring of power series over C_∞ convergent in closed unit disc.

Theorem 3.6. *[ABP] Consider $\Phi = \Phi(t) \in Mat_{r\times r}(\overline{K}[t])$ such that det Φ is a polynomial in t vanishing (if at all) only at $t = \theta$ and $\psi = \psi(t) \in Mat_{r\times 1}(C_\infty\{t\})$ satisfying $\psi^{(-1)} = \Phi\psi$.*

If $\rho\psi(\theta) = 0$ for $\rho \in Mat_{1\times r}(\overline{K})$, then there is $P = P(t) \in Mat_{1\times r}(\overline{K}[t])$ such that $P(\theta) = \rho$ and $P\psi = 0$.

Thus $\overline{K}$-linear relations between the periods are explained by $\overline{K}[T]$-level linear relations (which in our set-up are the motivic relations and thus 'algebraic relations between periods are motivic', as analog of Grothendieck's conjecture for motives he defined). In terms of special functions of our interest, this makes the vague hope that 'there are no accidental relations and the relations between special values come from known functional equations' precise and proves it.

Motives are thus simple, concrete linear algebra objects and have tensor products via which algebraic relations between periods i.e., linear relations between powers and monomials in them reduce to linear relations between periods (of some other motives). In this sense, the new [ABP] criterion below is similar to Wüstholz type sub-t-module theorem proved by Jing Yu, as remarked in 3.1.4 of [ABP]. The great novelty is, of course, the direct simple proof as well as its perfect adaptation to the motivic set-up here. Because our motives are concrete linear algebra objects, it is 'easy' (compared to the cycle-theoretic difficulties one encounters in the classical theory) to show that their appropriate category (up to isogeny) is neutral Tannakian category over $\mathbb{F}_q(t)$, with fiber functor to vector spaces over $\mathbb{F}_q(t)$ category given by lattice Betti realization, because the formalism of such categories is based on linear algebra motivation anyway. (Soon afterwards [Be06], Beukers proved similar criterion for dependence of values of E-functions, but it does not have such strong applications to relations between periods of classical motives, because of differences in period connections in this case). Each motive M also generates such (sub-) category and thus equivalent to a category of finite dimensional representations over $\mathbb{F}_q(t)$ of an affine group scheme Γ_M over $\mathbb{F}_q(t)$, called the motivic Galois group of M, which can be described as the group of tensor automorphisms of the fiber functor.

After developing [P08] this machinery, the following very useful theorem [P08] (which is analog of Grothendieck period conjecture for our abelian t-modules) follows [P08, pa. 166-167] easily from the ABP criterion and goes one step further in the quantitative direction:

Theorem 3.7. *If M is uniformizable t-motive over $\overline{K}$, then transcendence degree of the field extension of $\overline{K}$ generated by its periods is the dimension of the motivic Galois group of M (i.e., the group corresponding to Tannakian category generated by M).*

In [P08], we have further a description of the motivic Galois group as 'difference equations Galois group' for the 'Frobenius semilinear difference equation' $\Psi^{(-1)} = \Phi\Psi$. This allows the calculation of the dimensions and proofs of theorems described in section 6.

Here is the proof from [ABP04] of the Theorem for the simplest $r = 1$ case. Without loss of generality we can assume $\rho \neq 0$, so that we have to conclude ψ

vanishes identically from $\psi(\theta) = 0$. For $v \geq 0$, we have

$$(\psi(\theta^{q^{-v}}))^{q^{-1}} = \psi^{(-1)}(\theta^{q^{-v-1}}) = \Phi(\theta^{q^{-v-1}})\psi(\theta^{q^{-v-1}}).$$

Since $\Phi(\theta^{q^{-v-1}}) \neq 0$, ψ has thus infinitely many zeros $\theta^{q^{-v}}$ in the disc $|t| \leq |\theta|$ and so it vanishes identically.

The general case makes similar beautiful use of functional equation of the hypothesis by manipulating suitable auxiliary function to vanish identically (so as to recover P). This is done by applying the standard transcendence theory tools such as Siegel lemma (to solve system of linear equations thus arising) and Schwarz-Jensen, Liouville inequalities (to estimate bounds needed). We refer the reader to the clean treatment in [ABP].

3.8. **Complete determination of geometric gamma relations for $\mathbb{F}_q[t]$.** Brownawell and Papanikolas [BP02] developed complex multiplication theory and theory of quasi-periods for general t-motives, and analyzing the connection between the CM types of soliton t-motives showed, that only $\overline{K}$- linear relations among the gamma values at fractions are those explained by the bracket criterion in our uniform framework, as they mirror relations coming via CM type relations.

Anderson, Brownawell and Papanikolas [ABP04], using tensor powers to realize all monomials as periods (do not need quasi-periods then) and ABP criterion, proved

Theorem 3.8. *Let Γ stand for geometric gamma function for $A = \mathbb{F}_q[T]$. By a Γ-monomial, we mean an element of the subgroup of C_∞^* generated by $\tilde{\pi}$ and Γ-values at proper fractions in K.*

*Then a set of Γ-monomials is $\overline{K}$-linearly dependent exactly when some pair of Γ-monomials is, and pairwise $\overline{K}$-linear dependence is entirely decided by bracket criterion (**H1**).*

In particular, for any $f \in A_+$ of positive degree, the extension of $\overline{K}$ generated by $\tilde{\pi}$ and $\Gamma(x)$ with x ranging through proper fractions with denominator (not necessarily reduced) f, is of transcendence degree $1 + (q-2)|(A/f)^*|/(q-1)$ over $\overline{K}$.

We will talk about important applications of Papanikolas theorem to understanding relations between gamma (arithmetic or geometric) and zeta values in section 6.

4. Automata method, General A, v-adic situation and another gamma mystery

This section is independent of the last section.

4.1. **Automata and transcendence for arithmetic gamma.** In this subsection, we will restrict to $A = \mathbb{F}_q[t]$ and look at the arithmetic gamma. We will show that

Theorem 4.1. *(Arithmetic gamma) In the case* $A = \mathbb{F}_q[t]$, $m(\underline{f}^{(\sigma)}) = 0$ *for all* $\sigma = q^j$ *if and only if* $\Gamma(\underline{f})$ *is algebraic.*

The 'only if' part is (H1) in this case and was already explained, so new result that theorem implies is that (H1) is best possible for arithmetic gamma.

This was proved by the 'automata method'. Before that the methods based on the transcendence of periods and the Chowla-Selberg formula in the section two gave transcendence results (due to Thiery [Thi] and Jing Yu [Y92] independently) parallel to those known in the number field case. In the number fields case, the transcendence is known only for gamma values at fractions with denominators dividing 4, 6, while in our case, it was known only for denominators dividing $q^2 - 1$, the analogy being that $4, 6$ are numbers of roots of unity in imaginary quadratic fields, whereas $q^2 - 1$ is the number of roots of unity in quadratic extension field $\mathbb{F}_{q^2}(t)$ and the reason being the Chowla-Selberg formula and complex multiplication from these fields that we saw. The automata method settled completely [T96, All96] the question of which monomials in gamma values at fractions are algebraic and which are transcendental. (In section 6, we will mention recent [CPTY] results proving much stronger result than the theorem above). It is based on the following theorem [Chr79, CKMR80] by Christol and others.

Theorem 4.2. *(i)* $\sum f_n x^n$ *is algebraic over* $\mathbb{F}_q(x)$ *if and only if (ii)* $f_n \in \mathbb{F}_q$ *is produced by a q-automaton if and only if (iii) there are only finitely many subsequences of the form* $f_{q^k n + r}$ *with* $0 \leq r < q^k$.

For our proof, we will only need (and prove) (i) implies (iii). But (ii) is the reason for the name of the method and the reason for various other techniques from this viewpoint and results, for which we refer to the surveys [All87, AS03, T98, T04]. Thus we also say a few words about the concept of automata and how (ii) quickly implies (iii), though it is logically not necessary. The concept of automata will not be used in these notes after the next paragraph, except for the references to the method.

Here, an m-automaton (we shall usually use $m = q$, a prime power in the applications) consists of a finite set S of states, a table of how the digits base m operate on S, and a map Out from S to $\mathbb{F}_q$ (or some alphabet in general). For a given input n, fed in digit by digit from the left, each digit changing the state by the rule provided by the table, the output is $\mathrm{Out}(n\alpha)$ where α is some chosen initial state. So instead of our ideal Turing machine which has infinite tape and no restriction on input size (and still is a good approximation to computers because of enormous memories available these days) the finite automata has a restricted memory, so integer has to be fed in digit by digit, with the machine retaining no memory of previous digits fed except through its changed states.

Proof. (ii) implies (iii): There are only finitely many possible maps $\beta : S \to S$ and any $f_{q^k n+r}$ is of the form $\mathrm{Out}(\beta(n\alpha))$.

(i) implies (iii): For $0 \leq r < q$, define C_r (twisted Cartier operators) by $C_r(\sum f_n x^n) = \sum f_{qn+r}x^n$. Considering the vector space over $\mathbb{F}_q$ generated by the roots of the polynomial satisfied by f, we can assume that $\sum_{i=0}^k a_i f^{q^i} = 0$, with $a_0 \neq 0$. Using $g = \sum_{r=0}^{q-1} x^r (C_r(g))^q$ and $C_r(g^q h) = gC_r(h)$, we see that

$$\{h \in \mathbb{F}_q((x)) : h = \sum_{i=0}^k h_i (f/a_0)^{q^i}, h_i \in \mathbb{F}_q[x], \deg h_i \leq \max(\deg a_0, \deg a_i a_0^{q^{i-2}})\}$$

is a finite set containing f and stable under C_r's. $\square$

The particular case of theorem 4.1 is the transcendence at any proper fraction, proved by Allouche generalizing the author's result for any denominator, but with restrictions on the numerator. Mendès-France and Yao [MFY97] generalized to gamma values at p-adic integers from the values at fractions, and simplified further. We present below an account based on their method.

Lemma 4.1. *For positive integers a, b, c; $q^c - 1$ divides $q^a(q^b - 2) + 1$ if and only if c divides (a, b), the greatest common divisor of a and b.*

The proof, which is short, straightforward and elementary, is omitted.

Theorem 4.3. *If the sequence $n_j \in \mathbb{F}_q$ is not ultimately zero, then $\sum_{j=1}^\infty n_j/(t^{q^j} - t) \in K_\infty$ is transcendental over K.*

Proof. We have $\sum t\, n_j/(t^{q^j} - t) = \sum c(m)t^{-m}$, where

$$c(m) = \sum_{(q^j-1)|m} n_j.$$

Consider the subsequences $c_t(m) := c(q^t m + 1)$, as there are infinitely many non-zero n_j's, it is enough to show, by Christol's theorem, that $c_a \neq c_b$, for any $a > b$ such that n_a and n_b are non-zero.

Let h be the least positive integer s dividing a, but not dividing b and with $n_s \neq 0$. (Note $s = a$ satisfies the three conditions, so h exists.) By the lemma,

$$\begin{aligned} c_a(q^h - 2) - c_b(q^h - 2) &= \sum_{(q^l-1)|(q^a(q^h-2)+1)} n_l - \sum_{(q^l-1)|(q^b(q^h-2)+1)} n_l \\ &= \sum_{l|\gcd(a,h)} n_l - \sum_{l|\gcd(b,h)} n_l \\ &= n_h \neq 0. \end{aligned}$$

Hence $c_a \neq c_b$ and the theorem follows. $\square$

Theorem 4.4. *Let $A = \mathbb{F}_q[t]$. If $n \in \mathbb{Z}_p$ is not a non-negative integer, then $n! = \Gamma(n+1)$ is transcendental over K. In particular, the values of the gamma function at the proper fractions and at non-positive integers are transcendental.*

Proof. If a power series f is algebraic, so is its derivative f', and hence also the logarithmic derivative f'/f. In other words, transcendence of the logarithmic derivative implies the transcendence. This is a nice tool to turn products into sums, sometimes simplifying the job further because now exponents matter modulo p only.

(This nice trick due to Allouche got rid of the size restrictions on the numerator that author had before).

Write the base q expansion $n = \sum n_j q^j$ as usual, so that $n! = \prod D_j^{n_j}$, and hence

$$\frac{n!'}{n!} = \sum n_j \frac{D_j'}{D_j} = -\sum \frac{n_j}{t^{q^j} - t}.$$

Now if all sufficiently large digits n_j are divisible by p^k, then modifying the first few digits (which does not affect transcendence) we can arrange that all are divisible by p^k and then take p^k-th root (which does not affect transcendence). So without loss of generality, we can assume that the sequence n_j is not ultimately zero (modulo p). Hence the previous theorem applies. □

Now we describe the proof of Theorem 4.1 at the beginning of the section.

Proof. By taking a common divisor and using Fermat's little theorem, any monomial of gamma values at proper fractions can be expressed as a rational function times monomial in $(q^j/(1-q^d))!$'s, where d is fixed and $0 \leq j < d$. It was shown in the proof [T04, pa.107] of Theorem 1.4 that if the hypothesis of H1 is not met, this monomial is non-trivial. So again taking p-powers out as necessary, as in the proof of the previous theorem, we can assume that the exponents are not all divisible by p. But the exponents matter only modulo p, when we take the logarithmic derivative. So this logarithmic derivative is logarithmic derivative of some gamma value also, and hence is transcendental by the previous theorem. □

Remarks 4.1. We cannot expect to have analogous result for the classical gamma function, because its domain and range are archimedean, and continuity is quite a strong condition in the classical case. In other words, a non-constant continuous real valued function on an interval cannot fail to take on algebraic values.

Morita's p-adic gamma function has domain and range $\mathbb{Z}_p$, which being non-archimedean is closer to our situation. Let us now look at interpolation of $\Pi(n)$ at a finite prime v of $A = \mathbb{F}_q[t]$:

We have proved, as a corollary to analog of Gross-Koblitz theorem that if d is the degree of v, then $\Pi_v(q^j/(1-q^d))$ $(0 \leq j < d)$ is algebraic. The straight manipulation with digits then shows that $\Pi_v(n)$ is algebraic, if the digits n_j are ultimately periodic of period d. The converse, in a case n is a fraction, is a question raised earlier whether (H2) is best possible. In analogy with the Theorem above, Yao has conjectured the converse for $n \in \mathbb{Z}_p$.

Things become quite simple [T98] when v is of degree one, so that without loss of generality we can assume that $v = t$. Yao used [MFY97] result to simplify and generalize again.

Theorem 4.5. *Let $A = \mathbb{F}_q[t]$. If v is a prime of degree 1, then $\Pi_v(n)$ is transcendental if and only if the digits n_j of n are not ultimately constant.*

Proof. Using the automorphism $t \to t + \theta$ for $\theta \in \mathbb{F}_q$ of A, we can assume without loss of generality that $v = t$. Then $\Pi_v(n) = \prod(-D_{j,v})^{n_j}$, for $n = \sum n_j q^j$. Since $q = 0$ in characteristic p, when we take logarithmic derivative, it greatly simplifies to give

$$t\frac{\Pi_v(n)'}{\Pi_v(n)} = \sum n_j \left(\frac{t^{q^j}}{1-t^{q^j-1}} - \frac{t^{q^{j-1}}}{1-t^{q^{j-1}-1}}\right) = \sum \frac{n_j - n_{j+1}}{(1/t)^{q^j} - (1/t)}.$$

This power series in $\mathbb{F}_q((t))$ is transcendental over $\mathbb{F}_q(t) = \mathbb{F}_q(1/t)$ by previous theorem, by just replacing t by $1/t$, because the hypothesis implies that $n_j - n_{j+1}$ is not ultimately zero. Then the theorem follows as before. □

Remarks 4.2. What should be the implications for the Morita's p-adic gamma function? The close connection to cyclotomy leads us to think that the situation for values at proper fractions should be parallel. But then this implies that the algebraic values in the image not taken at fraction should be taken at irrational p-adic integers. Thus we do not expect a Mendès France-Yao type result for Morita's p-adic gamma function, but it may be possible to have such a result for Π_v's, for any v. This breakdown of analogies seems to be due to an important difference: in the function field situation, the range is a 'huge' finite characteristic field of Laurent series over a finite field, and the resulting big difference in the function theory prevents analogies being as strong for non-fractions.

We will see in section 6 that by powerful techniques of [ABP04, P08] there is now much stronger and complete independence result [CPTY] for values at fractions for arithmetic gamma, whereas the non-fraction result (Theorem 4.4) or v-adic result (Theorem 4.5) above is still not provable by other methods.

4.2. **General** A**.** What happens for general A? In short, it seems that several analogies that coincide for $A = \mathbb{F}_q[t]$ now diverge, but still generalize to different concepts with some theorems generalizing well. So we can think that this divergence helps us to focus on core relations in the concepts by throwing out accidents.

In 4.3-4.9 we discuss a new gamma and return in 4.10 to gammas of section 1, for general A. We start with the summary in 4.3, followed by some details in 4.4-4.9.

4.3. **Another gamma function coming from the exponential analogy: Summary.** Consider the exponential $e_\rho(z) = \sum z^{q^i}/d_i$ for sign-normalized rank one Drinfeld A-module ρ. Now $d_i \in K_\infty$ lie in the Hilbert class field H for A. It turns out that still we can define a factorial by digit expansion and interpolate its unit parts to get gamma functions at ∞ and v, by generalizing our earlier approach for the arithmetic gamma in one particular way. The fundamental two variable function $t - \theta$ specializing to $[n]$'s on the graphs of Frobenius power, in the $\mathbb{F}_q[t]$ case, now generalizes to Shtuka function corresponding to ρ, by Drinfeld-Krichever correspondence. Our proof of Gross-Koblitz formula generalizes very nicely connecting the values of v-adic interpolation at appropriate fractions to gauss sums that the author had defined, even though these gauss sums in general no longer have Stickelberger factorization, but have quite strange factorizations, which now get explained by the geometry of theta divisor! This fundamental correspondence with shtuka and these v-adic results suggest that there should be very interesting special value theory at the infinite place too.

To me, the most interesting mystery in the gamma function theory is that we do not even understand the nature of the basic value $\Gamma(0)$ for this gamma at the infinite place. Is it related to the period? The question is tied to the switching symmetry (or quantifying the lack of it) of two variables of the shtuka function.

4.4. **Some details: Drinfeld correspondence in the simplest case.** We describe the Drinfeld's geometric approach for the simplest case when the rank and the dimension is one, namely for Drinfeld modules of rank one.

Let F be an algebraically closed field containing $\mathbb{F}_q$ and of infinite transcendence degree. Let $\overline{X}$ denote the fiber product of X with $\mathrm{Spec}(F)$ over $\mathbb{F}_q$. We identify closed points of $\overline{X}$ with F-valued points of X in the obvious way. For $\xi \in X(F)$, let $\xi^{(i)}$ denote the point obtained by raising the coordinates of ξ to the q^i-th power. We extend the notation to the divisors on $\overline{X}$ in the obvious fashion. For a meromorphic function f on $\overline{X}$, let $f|_\xi$ denote the value (possibly infinite) of f at ξ and let $f^{(i)}$ denote the pull-back of f under the map $id_X \times \mathrm{Spec}(\tau^i) : \overline{X} \to \overline{X}$, where $\tau := x \to x^q : F \to F$. (Note that our earlier mention of two copies of the curve is taken care of $\overline{X}$, as F can contain even several copies of (extensions of) function field of X). For a meromorphic differential $\omega = f dg$, let $\omega^{(i)} := f^{(i)} d(g^{(i)})$.

Fix a local parameter $t^{-1} \in \mathcal{O}_{\overline{X},\infty}$ at ∞. For a nonzero $x \in \mathcal{O}_{\overline{X},\infty}$, define $\deg(x) \in \mathbb{Z}$ and $\mathrm{sgn}(x) \in F$ to be the exponent in the highest power of t and the coefficient of the highest power respectively, in the expansion of x as Laurent series in t^{-1}, with coefficients in F.

Drinfeld proved, by nice argument analyzing cohomology jumps provided by Frobenius twists and multiplications by f's of sections and using Riemann-Roch that

Theorem 4.6. *Let $\xi \in X(F)$, a divisor V of $\overline{X}$ of degree g and a meromorphic function f on $\overline{X}$ be given such that*

$$V^{(1)} - V + (\xi) - (\infty) = (f).$$

If $\xi \neq \infty$, then

$$H^i(\overline{X}, \mathcal{O}_{\overline{X}}(V - (\infty))) = 0 \quad (i = 0, 1).$$

In particular, ξ does not belong to support of V.

We review the language of Drinfeld modules again in this notation.

Let $\iota : A \to F$ be an embedding of A in F. By a Drinfeld A-module ρ relative to ι, we will mean such a ρ normalized with respect to sgn, of rank one and generic characteristic, but we will drop these words. Let H be the Hilbert class field. Write its exponential as $e(z) = \sum e_i z^{q^i}$.

Fix a transcendental point $\xi \in X(F)$. Then evaluation at ξ induces an embedding of A into F. In our earlier notation, $\xi = \iota$. By solving the corresponding equation on the Jacobian of X, we see that for some divisor V, $V^{(1)} - V + (\xi) - (\infty)$ is principal. A Drinfeld divisor V relative to ξ is defined to be an effective divisor of degree g such that $V^{(1)} - V + (\xi) - (\infty)$ is principal. From the theorem and Riemann-Roch, it follows that Drinfeld divisor is the unique effective divisor in its divisor class. (In particular, there are h such divisors.) Hence there exists a unique function $f = f(V)$ with $\mathrm{sgn}(f) = 1$ and such that $(f) = V^{(1)} - V + (\xi) - (\infty)$. By abuse of terminology, we call f a Shtuka. (In fact, in our context, Shtuka is a line bundle $\mathcal{L}$ on $\overline{X}$ with $\mathcal{L}^{(1)}$ being isomorphic to $\mathcal{L}(-\xi + \overline{\infty})$ and in our case, with $\mathcal{L} = \mathcal{O}_{\overline{X}}(V)$, f realizes this isomorphism.)

Drinfeld bijection: The set of Drinfeld divisors V (relative to ξ) is in natural bijection with the set of Drinfeld A-modules ρ (relative to ξ) as follows. (See [Mum78] for details of the proof.) Let $f = f(V)$ be as in above. Then

$$1, f^{(0)}, f^{(0)} f^{(1)}, f^{(0)} f^{(1)} f^{(2)}, \cdots$$

is an F-basis of the space of sections of $\mathcal{O}_{\overline{X}}(V)$ over $\overline{X}-(\infty\times_{\mathbb{F}_q}F)$. Define $\rho_{a,j}\in F$ by the rule

$$a := \sum_j \rho_{a,j} f^{(0)}\cdots f^{(j-1)}.$$

Then the ρ corresponding to V is given by $\rho_a := \sum \rho_{a,j}\tau^j$.

Theorem 4.7. *We have*

$$e(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{(f^{(0)}\dots f^{(n-1)})|_{\xi^{(n)}}}.$$

Proof. To see that the right hand side satisfies the correct functional equations for $e(z)$, divide both sides of the equation above defining $\rho_{a,j}$ by $f^{(0)}\cdots f^{(n-1)}$ and evaluate at $\xi^{(n)}$. □

Given ρ, we can recover f and V from exponential.

Theorem 4.8. *We have*

$$l(z) = \sum_{n=0}^{\infty} (\mathrm{Res}_\xi \frac{\omega^{(n+1)}}{f^{(0)}\dots f^{(n)}}) z^{q^n}.$$

Examples 4.1. For Carlitz module for $\mathbb{F}_q[t]$, the genus is zero and thus the Drinfeld divisor is empty, so that $f = t - t|_\xi$ and $\omega = dt$. The reader should verify the formula for coefficients of the exponential and the logarithm.

We give one more example in 4.7. For more examples, see [T04, 8.2].

4.5. **Some details: Definition of new gamma.** We can write $e_\rho(z) = \sum z^{q^i}/d_i$.

Definition 4.1. *For $n\in\mathbb{N}$, we define the factorial $\Pi(n)$ of n as follows: Write $n=\sum n_i q^i$, $0\le n_i<q$ and put $\Pi(n) := \prod d_i^{n_i}$.*

Let $\wp$ be a prime of A of degree d and let θ be an $\overline{\mathbb{F}_q}$- point of $\overline{X}$ above $\wp$. If $w\in K_\wp$ is a local parameter at θ, we put

$$\tilde{d}_i := \tilde{d}_{i,w} := \frac{d_i}{d_{i-d}w^{l_i}}$$

where l_i is chosen so that $\tilde{d}_i$ is a unit at θ.

We can show that as i tends to infinity, $\overline{d}_i$ (unit part with respect to chosen uniformizer at infinity) tends to one, ∞-adically and the degree of d_i tends to zero, p -adically. So we can interpolate the one-unit part and also put the degree back to get the gamma interpolation at ∞ exactly as in the first lecture. We just note here that when $d_\infty > 1$, there are several sign issues to be taken care of, the coefficients now are in field bigger than the Hilbert class field etc.

It can be shown that if u is any local parameter at θ, there is a nonzero $c\in\mathbf{F}_{q^d}$ such that with $w=cu$, $\tilde{d}_i$ tends to one θ-adically as i tends to infinity.

Let w be a local parameter at θ such that $\tilde{d}_i$ is a one unit for large i, then this implies that $\tilde{d}_i$ tends to one.

Definition 4.2. *Define $\wp$-adic factorial $\Pi_\wp(z) := \Pi_w(z)$ for $z\in\mathbb{Z}_p$ as follows. Write $z=\sum z_i q^i$, $0\le z_i<q$ and put $\Pi_\wp(z) := \prod \tilde{d}_i^{z_i}$.*

When $h_A = 1$, if we choose w to be a monic prime $\wp$ of A of degree d, then $-\tilde{d}_i \to 1$ as i tends to infinity.

4.6. **Some details: Gauss sums.** First we recall definition of Gauss and Jacobi sums. Let $\wp$ be a prime of A of degree d. Choose an A-module isomorphism $\psi : A/\wp \to \Lambda_\wp$ (an analog of additive character) and let χ_j ($j \mod d$) be $\mathbb{F}_q$-homomorphisms $A/\wp \to F$, indexed so that $\chi_j^q = \chi_{j+1}$ (special multiplicative characters which are q^j-powers of 'Teichmüller character'). We can identify $\chi_j(z)$ with $a|_{\theta^{(j)}}$, for some geometric point θ above $\wp$, if $a \in A$ is such that a mod $\wp$ is z. Then we define basic Gauss sums

$$g_j := g(\chi_j) := -\sum_{z\in(A/\wp)^*} \chi_j(z^{-1})\psi(z).$$

The g_j are nonzero. We define Jacobi sums J_j by $J_j := g_{j-1}^q/g_j$. Then J_j is independent of the choice of ψ. We put $J := J_0$.

For the Carlitz module, it was shown that $J_j = -(t - \chi_j(t))$ and Stickelberger factorization of g_j (mentioned in proof of theorem 10) was easily obtained from this. Note for example that $g_j^{q^d-1} = J_j J_{j-1}^q \cdots J_{j-d+1}^{q^{d-1}}$.

We now describe Anderson's explicit Gauss-Jacobi sums [ATp, A92] in geometric case, and the corresponding Stickelberger theorem already mentioned in 3.4. (The Stickelberger theorem in this context was first proved by Hayes [H85], but with a different construction). We use the notation of 4.6. Let $f \in A_+$ be of positive degree, $x \in K$ be of form $x = a/f$, with $a \in A$, $(a, f) = 1$, and let $\wp$ be a prime of $A[\mathbf{e}(1/f)]$ not dividing f. Let χ_j be the q^j-th power of the Teichmuller character at $\wp$ as above. Let a_i, b_i be f-dual family as in 4.6. We consider Gauss and Jacobi sums

$$g_j(\wp, x) := 1 - \sum_{i=1}^{\deg f} \chi_j(\mathbf{e}_*(a_i/f))\mathbf{e}(b_i x)) \in \overline{\mathbb{F}}_q A[\zeta_f],\ g(\wp, x) := \prod_j g_j(\wp, x) \in A[\zeta_f].$$

For a fractional ideal I of $A[\mathbf{e}(1/f)]$ supported away from f, we define $g(I, x)$ multiplicatively from the above sums for primes. Let $G_f = \mathrm{Gal}(K(\mathbf{e}(1/f))/K)$ and $\sigma_{f,a} \in G_f$ be the element corresponding to a under the usual identification. Then we have Stickelberger type result [ATp]

$$(g(I, x)) = I^{\sum \sigma_{f,ab^{-1}}},$$

where the sum is over $b \in A_+$ prime to f and of degree less than $\deg f$

4.7. **Some details: Shtuka connection and Analog of Gross-Koblitz.** Now we show that the Jacobi sums made up from $\wp$ torsion of a Drinfeld module can be interpreted as specializations at geometric points above $\wp$ of a meromorphic function, obtained from the Shtuka corresponding to the Drinfeld module, on curve cross its Hilbert cover. Hence the strange factorizations of the Gauss sums get related to the divisor of this function and since the divisor is encoding cohomology jumps, to theta divisor.

Theorem 4.9. *Let V and f be the Drinfeld divisor and the Shtuka respectively, corresponding to a sgn-normalized Drinfeld module ρ (of rank one and generic characteristic) via the Drinfeld bijection. Then with the Jacobi sums J_j defined using $\wp$-torsion of ρ and normalized as above, we have*

$$f|_{\theta^{(j)}} = J_j.$$

We put $\overline{x} := x|_\xi$ and $\overline{y} := y|_\xi$. While giving the divisor of J, by abuse of notation, instead of the quantities corresponding to θ, we use those corresponding ξ.

(i) $A = \mathbb{F}_2[x,y]/y^2 + y = x^3 + x + 1$: We have

$$f = \frac{\overline{x}(x+\overline{x}) + y + \overline{y}}{x + \overline{x} + 1}.$$

If $\xi + 1$ is the point where x is $\overline{x}+1$ and y is $\overline{x}+\overline{y}+1$, then $V = (\xi+1)$. (Note that this point corresponds to the automorphism σ above.) By the recipe above it follows that

$$(J) = 2(\underline{\xi+1})^{(-1)} - (\underline{\xi+1}) + (\xi) - 2(\infty)$$

where $\underline{\xi+1}$ is the point where x is $\overline{x}+1$ and y is $\overline{x}+\overline{y}$.

We showed how in $g = d_\infty = 1$ case to write down f 'parametrically' in terms of the coefficients of sgn-normalized ρ.

Let F be an algebraic closure of $K_\wp$, ξ be the tautological point, i.e., the F-valued point of X corresponding to $K \hookrightarrow K_\wp$ and let θ be the Teichmüller representative in the residue disc of ξ. Note that even though we used the ∞-adic completion earlier, the Taylor coefficients of $e(z)$, being in H, can be thought of as elements of F.

Theorem 4.10. *Let $0 \le j < d$. If μ is the valuation of g_j at ξ, then we have $g_j = \zeta w^\mu/\Pi_\wp(q^j/(1-q^d))$, where ζ is a q^d-1-th root of unity.*

4.8. **Some details: Value for ∞-adic gamma.** Let us use the short-form $f_{ij} := f^{(i)}|_{\xi^{(j)}}$. In the Carlitz case of genus 0, we have switching-symmetry $f_{ij} = -f_{ji}$. In general, up to simple algebraic factors arising from residue calculations, $\tilde{\pi}^{-1}$ is $\prod_{i=1}^{\infty} f_{i0}$ and $\Pi_\infty(-1)^{-1}$ is $\prod_{j=1}^{\infty} f_{0j}$, as can be seen from formulas giving d_i and l_i in terms of specializations of f. So the switching-symmetry mentioned above connects the two in the Carlitz case.

4.9. **Some details: Non-vanishing of exponential coefficients.** We proved (for $d_\infty = 1$) that the coefficients of z^{q^i} in the expansion of $e(z)$ are non-zero, so that we can consider d_i as the reciprocal coefficient. In general, this non-vanishing is equivalent to the nice geometric condition that $\xi^{(i)}$ does not belong to the support of the Drinfeld divisor, and is known for many situations [T04, 8.3.1] but not in full generality, when $d_\infty > 1$.

4.10. **General A: arithmetic and geometric gamma.** The gamma functions we defined earlier for general A are expected to have good properties, and we have established some of them, such as period connections for simple values. But we saw very strong complete results only for $A = \mathbb{F}_q[t]$ case. In general, the hypotheses (Hi) have not been scrutinized much in general. For simple examples of higher genus solitons, in arithmetic as well as geometric case, see [T04, Sec. 8.7]. But to get general results, one needs to expand both the technology of A-modules and solitons for general A. The crucial step of providing solitons for general A, at least conjecturally with a lot of evidence, has been taken by Greg Anderson in another spectacular paper [A06] 'A two variable refinement of the Stark conjecture in the function field case', building on his earlier work [A94, A96]. It uses the adelic framework of Tate's thesis.

I hope that somebody (hopefully from this audience!) takes up this issue and settles the general case soon!

5. Zeta values: Definitions, first properties and relations with cyclotomy

The Riemann zeta function, the Dedekind zeta function in the number field case, and Artin's analog in the function field case, can all be defined as $\zeta(s) = \sum \mathrm{Norm}(I)^{-s} \in \mathbb{C}$, where the sum is over non-zero ideals and $s \in \mathbb{C}$, with $\mathrm{Re}(s) > 1$. Artin's zetas satisfy many analogies. But they are simple rational functions of $T := q^{-s}$-variable. For example, $\zeta(s) = 1/(1-qT)$, for $A = \mathbb{F}_q[t]$, as you can see by counting and summing the resulting geometric series. So the special values theory in this case is completely trivial.

The norm depends only on the degree and we thus ignore all the information except the degree. We can take the actual ideal or polynomial or a relative norm to the base instead: Carlitz used another function field analogy and considered special zeta values $\zeta(s) := \sum n^{-s}$, where now n runs through monic polynomials in $\mathbb{F}_q[t]$, i.e., monic generators of non-zero ideals. (With this basic idea, we can then consider various zeta and L-values, for example, those attached to finite characteristic valued representations by using product of characteristic polynomials of 'Frobenius at $\wp$ multiplied by $\wp^{-s}$'. We refer to [G96] and references therein for various such definitions.)

Now polynomials can be raised to integral powers, and in particular, if s is a natural number, then the sum converges (as the terms tend to zero) to a Laurent series in K_∞.

Goss showed that if s is non-positive integer, then just grouping the terms for the same degree together, the sum reduces to a finite sum giving $\zeta(s) \in A$.

Examples 5.1. For $A = \mathbb{F}_3[t]$, we have $\zeta(0) = 1 + 3 + 3^2 + \cdots = 1 + 0 + 0 + \cdots = 1$ and $\zeta(-3^n) = \zeta(-1)^{3^n} = 1$ because

$$\zeta(-1) = 1 + (t + (t+1) + (t-1)) + \cdots = 1 + 0 + \cdots = 1.$$

For $A = \mathbb{F}_2[t]$ on the other hand, $\zeta(-2^n) = 0$ because $\zeta(-1) = 1 + (t + (t+1)) + 0 + \cdots = 1 + 1 = 0$. We leave it to the reader to verify $\zeta(-5) = 1 + t - t^3$ for $A = \mathbb{F}_3[t]$ using either the bound in the lemma below or the formula below. Note that these examples fit with the naive analogy with Riemann zeta, which vanishes at negative integers exactly when they are even, as 1 is 'even' exactly when $q = 2$.

In fact, the weaker version consisting of the first 3 lines of the proof of the following Theorem, which is essentially due to Lee, implies that grouped terms vanish for large degree. (See [T90, T95] for the relevant history and [She98, T09a] for more recent results.)

For a non-negative integer $k = \sum k_i q^i$, with $0 \le k_i < q$, we let $\ell(k) := \sum k_i$, i.e., $\ell(k)$ is the sum of base q digits of k. Here is very useful general vanishing theorem.

Theorem 5.1. *Let W be a $\mathbb{F}_q$-vector space of dimension d inside a field (or ring) $\mathcal{F}$ over $\mathbb{F}_q$. Let $f \in \mathcal{F}$. If $d > \ell(k)/(q-1)$, then $\sum_{w \in W} (f+w)^k = 0$.*

Proof. Let $w_1, \cdots, w_d$ be a $\mathbb{F}_q$-basis of W. Then $(f+w)^k = (f + \theta_1 w_1 + \cdots + \theta_d w_d)^k$, $\theta_i \in \mathbb{F}_q$. When you multiply out the k brackets, terms involve at most k of θ_i's, hence if $d > k$, the sum in the theorem is zero, since we are summing over some θ_i, a term not involving it, and $q = 0$ in characteristic p. The next observation is that in characteristic p, $(a+b)^k = \prod (a^{q^i} + b^{q^i})^{k_i}$, hence the sum is zero, if $d > \ell(k)$, by the argument above. Finally, note that $\sum_{\theta \in \mathbb{F}_q} \theta^j = 0$ unless $q-1$ divides j.

Expanding the sum above by the multinomial theorem, we are summing multiples of products $\theta_1^{j_1}\cdots\theta_d^{j_d}$ and hence the sum is zero; because the sum of the exponents being $\ell(k) < (q-1)d$, all the exponents can not be multiples of $q-1$. □

For the applications, note that $A_{<i} = \{a \in A : \deg(a) < i\}$ form such $\mathbb{F}_q$ vector spaces whose dimensions are given by the Riemann-Roch theorem. Similarly, $A_{i+} = \{a \in A : \deg(a) = i, a \text{ monic}\}$ are made up of affine spaces as in the theorem.

Remarks 5.1. Following Goss, we can interpolate the zeta values above to $\zeta(s)$ with $s = (x, y)$ in the much bigger space $C_\infty^* \times \mathbb{Z}_p$, by defining $a^s := x^{\deg a}\langle a\rangle^y$, with $\langle a\rangle$ denoting the unit part of a, which can be raised to p-adic y-th power, just as in the gamma case. These p-adic power homomorphisms are the only [J09] locally analytic endomorphisms of one-unit groups. At least for an integral y, the usual integral power a^y is recovered as $a^{(t^y, y)}$. In general, $\zeta(s)$ is a power series in the x-variable, which keeps track of the degrees. See for more on these analytic issues [G96] or Böeckle's lecture series notes.

Focusing on just the special values at integers, we use the theorem above and can ignore the convergence questions. Thus we work in the following simpler set-up.

Let L be a finite separable extension of K and let $\mathcal{O}_L = \mathcal{O}$ denote the integral closure of A in L. Now we define the relevant zeta functions.

Definition 5.1. *For $s \in \mathbb{Z}$, define the 'absolute zeta function':*

$$\zeta(s, X) := \zeta_A(s, X) := \sum_{i=0}^{\infty} X^i \sum_{a \in A_{i+}} \frac{1}{a^s} \in K[[X]]$$

$$\zeta(s) := \zeta_A(s) := \zeta(s, 1) \in K_\infty.$$

By theorem above $\zeta(s) \in A$ for integer $s \leq 0$. The results on the special values and the connection with Drinfeld modules later on, justifies the use of elements rather than ideals even when the class number is more than one. Also, we can look at a variant where a runs through elements of some ideal of A or we can sum over all ideals by letting s to be a multiple of class number and letting a^s to be the generator of I^s with say monic generator. In the latter case, we have Euler product for such s.

If L contains H (note that this is no restriction if $hd_\infty = 1$, eg., for $A = \mathbb{F}_q[t]$), then it is known that the norm of an ideal $\mathcal{I}$ of $\mathcal{O}$ is principal. Let Norm$\mathcal{I}$ denote the monic generator. Let us now define the relative zeta functions in this situation.

Definition 5.2. *For $s \in \mathbb{Z}$, define the 'relative zeta function':*

$$\zeta_{\mathcal{O}}(s, X) := \zeta_{\mathcal{O}/A}(s, X) := \sum_{i=0}^{\infty} X^i \sum_{\deg(Norm\ \mathcal{I})=i} \frac{1}{Norm\ \mathcal{I}^s} \in K[[X]]$$

$$\zeta_{\mathcal{O}}(s) := \zeta_{\mathcal{O}/A}(s) := \zeta_{\mathcal{O}}(s, 1) \in K_\infty.$$

Finally, we define the vector valued zeta function, which generalizes both definitions above and works without assuming that L contains H. We leave the simple task of relating these definitions to the reader.

Definition 5.3. *Let* $\mathcal{C}_n$ *(*$1 \le n \le hd_\infty$*) be the ideal classes of* A *and choose an ideal* I_n *in* $\mathcal{C}_n^{-1}$*. For* $s \in \mathbb{Z}$*, define the vector* $Z_{\mathcal{O}}(s, X)$ *by defining its* n*-th component* $Z_{\mathcal{O}}(s, X)_n$ *via*

$$Z_{\mathcal{O}}(s,X)_n := Z_{\mathcal{O}/A}(s,X)_n := \sum_{i=0}^{\infty} X^i \sum \frac{1}{(I_n \mathit{Norm}\, \mathcal{I})^s} \in K[[X]]$$

where $I_n \mathit{Norm}\, \mathcal{I}$ *stands for the monic generator of this ideal and the second sum is over* $\mathcal{I}$ *whose norm is in* $\mathcal{C}_n$ *and is of degree* i*.*

$$Z_{\mathcal{O}}(s)_n := Z_{\mathcal{O}/A}(s)_n := Z_{\mathcal{O}}(s,1)_n \in K_\infty.$$

Also note that Z depends very simply on the choice of I_n's, with components of Z, for different choices of I_n's, being non-zero rational multiples of each other. In particular, the questions we are interested in such as, when it (i.e., all the components) vanishes, when it is rational or algebraic etc. are independent of such choice.

We use X as a deformation parameter for otherwise discretely defined zeta values and hence we define the order of vanishing ord_s of ζ, $\zeta_{\mathcal{O}}$ or $Z_{\mathcal{O}}$ at s to be the corresponding order of vanishing of $\zeta(s, X)$, $\zeta_{\mathcal{O}}(s, X)$ or $Z_{\mathcal{O}}(s, X)$ at $X = 1$. This procedure is justified by the results described below. The order of vanishing is the same for s and ps, as the characteristic is p.

Using these basic ideas, we can immediately define L-functions in various settings and can study their values and analytic properties [G96], but we focus here on the simplest case.

5.1. **Values at positive integers.** In contrast to the classical case where we have a pole at $s = 1$, here $\zeta(1)$, which can now be considered as an analog of Euler's constant γ, makes sense.

Euler's famous evaluation, $\zeta(m) = -B_m(2\pi i)^m/2(m!)$ for even m, has following analog:

Theorem 5.2. *(Carlitz* [Car35]*): Let* $A = \mathbb{F}_q[t]$*. Then for 'even'* m *(i.e., a multiple of* $q-1$*),* $\zeta(m) = -B_m\tilde{\pi}^m/(q-1)\Pi(m)$*.*

Proof. First note that $q - 1 = -1$ in the formula. Multiplying the logarithmic derivative of the product formula for $e(z)$ by z, we get

$$\frac{z}{e(z)} = 1 - \sum_{\lambda \in \Lambda - \{0\}} \frac{z/\lambda}{1 - z/\lambda} = 1 - \sum_{n=1}^{\infty} \sum_{\lambda} (\frac{z}{\lambda})^n = 1 + \sum_{n \text{ `even'}} \frac{\zeta(n)}{\tilde{\pi}^n} z^n$$

since $\sum_{c \in \mathbb{F}_q^\times} c^n = -1$ or 0 according as n is 'even' or not. But $z/e(z) = \sum B_n z^n/\Pi(n)$. □

For general A and corresponding sign-normalized ρ, noting that now the coefficients of the exponential are in H, we get, similarly

Theorem 5.3. *Let* s *be a positive 'even' (i.e., a multiple of* $q-1$*) integer, then* $\zeta_A(s)/\tilde{\pi}^s \in H$*.*

Examples 5.2. By comparing the coefficients of z^{q-1} in the equation above, we get $\zeta(q-1)/\tilde{\pi}^{q-1} = -1/d_1 \in H$.

Now we turn to the relative zeta functions.

Theorem 5.4. *Let L be an abelian totally real (i.e., split completely at ∞) extension of degree d of K containing H and let s be a positive 'even' (i.e., a multiple of $q-1$) integer. Then $R_s := \zeta_{\mathcal{O}}(s)/\tilde{\pi}^{ds}$ is algebraic, and in fact, $R_s^2 \in H$.*

The idea of the proof is to factor such zeta value as product of L-values which are linear combinations of partial zeta values, which are handled as in the absolute case above, and use Dedekind determinant formula to get better control of the fields involved. There is more refined version when $d_\infty > 1$. We refer to [T04, 5.2.8].

Remarks 5.2. (1) We can not [T95] replace H in the theorem, even when $d_\infty = 1$, by K, in general. Classically, for arbitrary (not necessarily abelian over $\mathbb{Q}$) totally real number field F, it is known that $(\zeta_F(s)/(2\pi i)^{r_1 s})^2 \in \mathbb{Q}$. This uses Eisenstein series.

(2) For the history and references on this general as well as the abelian case using L-functions, as we have done here, see references in [T95] and see [G92] for ideas about carrying over the proof in the general case. On the other hand, such an algebraicity result for the ratio of the relative zeta value with an appropriate power of the period $2\pi i$ is not expected for number fields which are not totally real. But in our case, we can have such a result even if L is not totally real, as was noted in [G87]: Let L be a Galois extension of degree p^k of K, then since all the characters of the Galois group are trivial, the L-series factorization shows that $\zeta_{\mathcal{O}}(s) = \zeta(s)^{p^k}$ and the result follows then from the theorem above on the absolute case. More elementary way to see this, when the degree is p and $\mathcal{O}$ is of class number one is to note that for $\alpha \in \mathcal{O} - A$, there are p conjugates with the same norms which then add up to zero, whereas for $\alpha \in A$, the norm is α^p.

5.2. **Values at non-positive integers.** We have seen how just grouping together the terms of the same degree gives $\zeta(-k) = \sum_{i=0}^{\infty}(\sum_{n \in A_i+} n^k) \in A$, for $k > 0$. Hence we have stronger integrality rather than rationality in the number field case, reflecting absence of pole at $s = 1$ in our case.

We also have the following vanishing result, giving the 'trivial zeros':

Theorem 5.5. *For a negative integer s, $\zeta((q-1)s) = 0$.*

Proof. If $k = -(q-1)s$, then

$$\zeta(-k) = \sum_{i=0}^{\infty} \sum_{a \in A_{i+}} a^k = -\sum_{i=0}^{\infty} \sum_{a \in A_i} a^k = 0$$

where the second equality holds since $\sum_{\theta \in \mathbb{F}_q} \theta^{q-1} = -1$ and the third equality is seen by using that the sum is finite and applying Theorem above with $W = A_{<m}$ and $f = 0$, for some large m. □

For $A = \mathbb{F}_q[t]$, another proof giving a formula as well as non-vanishing result parallel to the case of the Riemann zeta function can be given: For $k \in \mathbb{Z}_+$, $\zeta(-k) = 0$ if and only if $k \equiv 0 \mod (q-1)$. Also $\zeta(0) = 1$.

The proof [G79, T90] follows by writing a monic polynomial n of degree i as $th + b$ with h of degree $i-1$ and $b \in \mathbb{F}_q$ and using the binomial theorem to get the induction formula

$$\zeta(0) = 1, \quad \zeta(-k) = 1 - \sum_{f=0,(q-1)|(k-f)}^{k-1} \binom{k}{f} t^f \zeta(-f),$$

which shows $\zeta(-k) \in A$ since $\zeta(0) = 1$. If $(q-1)$ divides k, then induction shows $\zeta(-k) = 1 - 1 + 0 = 0$. If $(q-1)$ does not divide k, then there being no term in the summation corresponding to $f = 0$, $\zeta(-k) = 1 - tp(t) \neq 0$ where $p(t) \in A$.

We do not know for general A whether the values at odd integers are non-zero. But we have

Theorem 5.6. *Let s be a negative odd integer. Then $\zeta(s)$ is non-zero if A has a degree one rational prime $\wp$ (e.g. if $g = 0$ (this reproves $\mathbb{F}_q[t]$ case) or if q is large compared to the genus of K) or if $h = 1$. (Together they take care of the genus 1 case.) In fact, in the first case, $\zeta(s)$ is congruent to 1 mod $\wp$.*

Proof. The first part follows by looking at the zeta sum modulo $\wp$ and noticing that $\sum_{\theta \in \mathbb{F}_q} \theta^{-s} = 0$, hence the only contribution is 1 from the $i = 0$ term. Apart from $g = 0$, there are only 4 other A's with $hd_\infty = 1$ (these have no degree 1 primes), which we check by calculation omitted here. □

Remarks 5.3. This can be easily improved, but the full non-vanishing is not known. Note that for number fields case, the Dedekind zeta functions have Euler product thus showing non-vanishing in the right half-plane, and the functional equation (missing here!) concluding the non-vanishing at negative odd integers by analyzing the relevant gamma factors.

Let us now turn to the relative zeta functions.

Theorem 5.7. *(Goss [G92]) For a negative integer s, $\zeta_{\mathcal{O}}(s) \in A$. In fact, $Z_{\mathcal{O}}(s)_n \in A$ for every n. For a negative integer s, $\zeta_{\mathcal{O}}((q-1)s) = 0$. In fact, $Z_{\mathcal{O}}((q-1)s)_n = 0$ for every n.*

The idea is to decompose the sum carefully in $\mathbb{F}_q$-vector spaces and use Theorem above to show that for large k, the k-th term of the zeta sum is zero. The second part follows as before.

Remarks 5.4. The values of the Dedekind zeta function at negative integers are all zero, if the number field is not totally real. By the remarks above, similar result does not hold in our case. In fact, we do not even need degree to be a power of the characteristic, as will be seen from examples below. The ramification possibilities for the infinite places are much more varied in the function field case.

For $A = \mathbb{F}_q[t]$, Goss defined modification $\beta(k) \in A$ of $\zeta(-k)$, for $k \in \mathbb{Z}_{\geq 0}$, as follows:

$$\beta(k) := \zeta(-k) \text{ if } k \text{ is odd}, \quad \beta(k) := \sum_{i=0}^{\infty} (-i) \sum_{n \,\in\, A_i+} n^k \ \text{ if } k \text{ is even}.$$

In other words, the deformation $\zeta(-k, X)$ of $Z(-k, 1) = \zeta(-k)$ has a simple zero at $X = 1$ if k is even and hence one considers $\frac{d\zeta}{dx}(-k, X)\,\big|_{X=1} = \beta(k)$ instead. For $k \in \mathbb{Z}_+, \beta(k) \neq 0$ and in fact

$$\beta(0) = 0, \quad \beta(k) = 1 \; - \sum_{f=0,(q-1)|(k-f)}^{k-1} \binom{k}{f} \, t^f \; \beta(f).$$

For general A, Goss similarly defines $\beta(k)$ by removing the 'trivial zero' at k, of order d_k say (see the next section for the discussion of this order) by $\beta(k) := (1 -$

$X)^{-d_k}\zeta_A(-k,X)|_{X=1}$. One gets Kummer congruences and $\wp$-adic interpolations for $\beta(k)$'s, as for zeta values.

Since $\zeta(-k)$ turns out to be a finite sum of n^k's, by Fermat's little theorem we see that $\zeta(-k)$'s satisfy Kummer congruences enabling us to define a $\wp$-adic interpolation $\zeta_\wp$ from $s \in \mathbb{Z}_p$, by removing the Euler factor at $\wp$. For interpolations at $\wp$ and ∞ on much bigger places, analytic properties etc, we refer to [G96].

Instead of the kummer congruences, the B_m's satisfy analogs of the von-Staudt congruences and the Sylvester-Lipschitz theorem. We have now two distinct analogs of B_k/k: $-\zeta(-k+1)$ for $k-1$ 'odd' on one hand and $\Pi(k-1)\zeta(k)/\tilde{\pi}^k$, with k 'even' on the other. But the shift by one does not transform 'odd' to 'even' unless $q=3$, and we do not know any reasonable functional equation linking the two.

5.3. **First relations with cyclotomic theory: At positive integers.** Next we give the connection of these Bernoulli numbers with the class groups of cyclotomic fields, giving some analogs of Herbrand-Ribet theorems. We restrict to $A = \mathbb{F}_q[t]$.

Let $\wp$ be a monic prime of A of degree d. Recall analogies

$$\Lambda_\wp = e(\tilde{\pi}/\wp) \leftrightarrow \zeta_p = e^{2\pi i/p}, \quad K(\Lambda_\wp) \leftrightarrow \mathbb{Q}(\zeta_p).$$

One also has 'maximal totally real' subfield

$$K(\Lambda_\wp)^+ = K\left(\prod_{\theta \in \mathbb{F}_q^*} e\left(\frac{\theta\tilde{\pi}}{\wp}\right)\right) \leftrightarrow \mathbb{Q}(\zeta_p)^+ = \mathbb{Q}\Big(\sum_{\theta \in \mathbb{Z}^*} e^{\theta 2\pi i/p}\Big).$$

The classical cyclotomic Galois group $(\mathbb{Z}/p\mathbb{Z})^*$ now gets replaced by $(A/\wp A)^*$ and the Frobenius action is similar from this identification, so that it is straightforward to write down the splitting laws for primes. Note that we now have two kinds of class groups, one traditionally done in algebraic geometry, that is class group (divisors of degree zero modulo principal) for the complete curve, and the other the class group for the integral closure of A in these cyclotomic fields (which typically has many points at infinity). Hence we make the following definition.

Definition 5.4. *Let C (C^+, $\tilde{C}$, $\tilde{C}^+$ resp.) denote the p-primary component of the class group of complete curve for $K(\Lambda_\wp)$ ($K(\Lambda_\wp)^+$, ring of integers of $K(\Lambda_\wp)$, of $K(\Lambda_\wp)^+$ respectively).*

Let W be the ring of Witt vectors of $A/\wp$. Then we've decomposition into isotypical components $C \otimes_{\mathbb{Z}_p} W = \oplus_{0 \le k < q^d-1} C(w^k)$ according to characters of $(A/\wp)^*$, where w is the Teichmüller character. (Similarly for C^+, $\tilde{C}$, $\tilde{C}^+$.)

Theorem 5.8. *(Okada, Goss* [Oka91, G96]*): Let $A = \mathbb{F}_q[T]$. Then for $0 < k < q^d - 1$, k 'even', if $\tilde{C}(w^k) \neq 0$, then $\wp$ divides B_k.*

Proof. (Sketch) We define analogs of Kummer homomorphisms $\psi_i \colon \mathcal{O}_F^* \to A/\wp$ ($0 < i < q^d - 1$) (note that p-th powers map to zero) by $\psi_i(u) = u_{i-1}$, where u_i is defined as follows. Let $u(t) \in A[[t]]$ be such that $u = u(\lambda)$ and define u_i to be $\Pi(i)$ times the coefficient of z^i in the logarithmic derivative of $u(e(z))$. Using the definition of the Bernoulli numbers, we calculate that the i-th Kummer homomorphism takes the basic cyclotomic unit $\lambda^{\sigma_a - 1}$ to $(a^i-1)B_i/\Pi(i)$. If $\tilde{C}(w^k) \neq 0$, then by (the component-wise version due to Goss-Sinnott) of Galovich-Rosen theorem (analog of Kummer's theorem) that class number of ring of integers of $K(\Lambda_\wp^+)$ is the index of cyclotomic units in full units of that field, we have $\psi_k(\lambda^{\sum w^{-k}(\sigma)\sigma^{-1}}) = 0$. Hence the calculation above implies that $\wp$ divides B_k. ∎

The converse is false and Gekeler has suggested the following modification (see also [Ang01]) using the Frobenius action restriction:

Conjecture (Gekeler [Gek90]) *If* $\wp | B_{k'}$ *for all* k' *such that* $k' \equiv p^m k(q^d - 1)$ *with* $0 < k,\ k' < q^d - 1,\ k \equiv 0(q-1)$ *then* $\tilde{C}(w^k) \neq 0$.

5.4. **First relations with cyclotomic theory: At non-positive integers.** For the zeta values at non-positive integers, we have the following story.

Theorem 5.9. *(Goss- Sinnott* [GS85]*): For* $0 < k < q^d - 1$, $C(w^{-k}) \neq 0$ *if and only if* p *divides* $L(w^k, 1)$.

Proof. (Sketch) The duality between the Jacobian and the p-adic Tate module T_p transforms the connection between the Jacobian and the class group in the Tate's proof of Stickelberger Theorem [T04, Cha.1] to $T_p(w^{-k})/(1-F)T_p(w^{-k}) \cong C(w^{-k})$. On the other hand, we have a Weil type result: $\det(1 - F : T_p(w^{-k})) = L_u(w^k, 1)$. Here L_u is the unit root part of the L-function and hence has the same p-power divisibility as the complete L-function. Hence $\mathrm{ord}_p(L(w^k, 1))$ is the length of $C(w^{-k})$ as a $\mathbb{Z}_p[G]$-module. □

For a proof without going through the complex L-function, we refer to the lecture notes of Böckle.

The interpretation above of the L-function as the characteristic polynomial in the above is precisely the result on which Iwasawa's main conjecture is based. Since this is already known, the Gras conjecture giving the component-wise results above, which follows classically from the main conjecture, is known here. This was recognized in [GS85].

Comparison with the corresponding classical result shows that we are looking at divisibility by p, the characteristic, rather than the prime $\wp$ relevant to the cyclotomic field. To bring $\wp$ in, we need to look at the finite characteristic zeta function.

Theorem 5.10. *(Goss-Sinnott* [GS85]*)* $C(w^{-k}) \neq 0$ *if and only if* $\wp | \beta(k), 0 < k < q^d - 1$.

Proof. The identification $W/pW \cong A/\wp$ provides us with the Teichmüller character $w\colon (A/\wp)^* \to W^*$ satisfying $w^k(n \mod \wp) = (n^k \mod \wp) \mod p$. Hence the reduction of L value (since it also has 'trivial zero' factors missing) in the Theorem above modulo p is $\beta(k) \mod \wp$. □

Remarks 5.5. (1) Recall that for $A = \mathbb{F}_q[t]$ and 'odd' k, $\beta(k) = \zeta(-k)$, so the result is in analogy with Herbrand-Ribet theorem, but for 'even k' we get a new phenomenon. This is connected with the failure of Spiegelgungsatz for Carlitz-Drinfeld cyclotomic theory. Classically the leading terms at even k are conjectured to be transcendental, here they are rational, even integral. For values at positive integers on the other hand, the situation seems to be as expected with naive analogies.

(2) For $A = \mathbb{F}_q[t]$, both analogs of B_n/n mentioned in 5.2 thus connect to arithmetic of related class groups suggesting a stronger connection between the two analogs than what is currently understood.

5.5. **Orders of vanishing mystery.** Now we turn to the question of the order of vanishing. Classically, the answer is simple: The Euler product representation shows there are no zeros in the region where it is valid and hence the orders of vanishing of the trivial zeros (namely those at the negative even integers) are easily proved by looking at poles of the gamma function factors in the functional equation. (For the number field situation in general, these are predicted by motives and these orders of vanishing are connected to K-theory and extensions of motives, so many structural issues and clues are at stake in this simple question.) We do not have functional equations. As described below, Goss gave a lower bound for orders of vanishing, when L contains H and mentioned as an open question whether they are exact. The lower bounds match the naive analogies. But we will see that they are not exact orders and indeed that the patterns of extra vanishing are quite surprising in terms of the established analogies. The full situation is still not understood, even conjecturally.

The main idea of Goss (already mentioned in the proof of Theorem 5.10) is to turn the similarity in the definition of our zeta function with the classical one into a double congruence formula, using the Teichmüller character, and to use the knowledge of the classical L-function Euler factors to understand the order of vanishing. This is done as follows:

Let $\wp$ be a prime of A and let W be the Witt ring of $A/\wp$. The identification $W/pW \cong A/\wp$ provides us with the Teichmüller character $w : (A/\wp)^* \to W^*$ satisfying $w^k(a \mod \wp) = (a^k \mod \wp) \mod p$.

Now let $\Lambda_\wp$ denote the $\wp$-torsion of rank one, sgn-normalized Drinfeld module ρ of generic characteristic. Let L contain H and let G be the Galois group of $L(\Lambda_\wp)$ over L. Then G can be thought of as a subgroup of $(A/\wp)^*$ and hence w can be thought of as a W-valued character of G. Let $L(w^{-s}, u) \in W(u)$ be the classical L-series of Artin and Weil in $u := q^{-sm}$, where m is the extension degree of the field of constants of L over $\mathbb{F}_q$.

Let $S_\infty := \{\infty_j\}$ denote the set of the infinite places of L and let G_j denote the Galois group of $L_{\infty_j}(\Lambda_\wp)$ over L_{∞_j}. Then $G_j \subset \mathbb{F}_q^*$. Given s, let $S_s \subset S_\infty$ be the subset of the infinite places at which w^{-s} is an unramified character of G. Then S_s does not depend on $\wp$. Put

$$\tilde{\zeta}_{\mathcal{O}}(s, X) := \zeta_{\mathcal{O}}(s, X) \prod_{\infty_j \in S_s} (1 - w^{-s}(\infty_j) X^{\deg(\infty_j)})^{-1}.$$

Theorem 5.11. *(Goss* [G92]*) Let L contain H and let s be a negative integer. Then $\tilde{\zeta}_{\mathcal{O}}(s, X) \in A[X]$.*

Proof. (Sketch) Tracing through the definitions, the property of w mentioned above gives the double congruence formula $L(w^{-s}, X^m) \mod p = \tilde{\zeta}_{\mathcal{O}}(s, X) \mod \wp$ for infinitely many $\wp$. But as the L function is known to be a polynomial, the result follows. □

This gives the following lower bound for the order of vanishing:

Theorem 5.12. *(Goss* [G92]*) Let L contain H and let s be a negative integer, then the order of vanishing of $\zeta_{\mathcal{O}}(s)$ is at least*

$$V_s := \mathrm{ord}_{X=1} \prod_{\infty_j \in S_s} (1 - w^{-s}(\infty_j) X^{\deg(\infty_j)}).$$

Remarks 5.6. Note that V_s depends on s only through its value modulo $q-1$ and $V_s \leq [L:K]$. This is analogous to the properties of the exact orders of vanishing in the classical case, but not in our case as we will see.

Examples 5.3. (i) $L = K = H$: Then $V_s = 0$ or 1, according as s is 'odd' or 'even'. We have seen that for $A = \mathcal{O} = \mathbb{F}_q[t]$, the order of vanishing is V_s.

(ii) L is totally real extension of degree d of K containing H, i.e., ∞ splits completely in L: V_s is 0 or d according as whether s is 'odd' or 'even'. (Note that the bounds V_s in (i) and (ii) are in analogy with the orders of vanishing in the classical case.)

(iii) $L = K(\Lambda_\wp)$: $V_s = (q^{\deg(\wp)}-1)/(q-1)$ for all s.

(iv) $L = \mathbb{F}_{q^n}(t)$ and $K = \mathbb{F}_q(t)$: $V_s = 0$ when s is 'odd', and for s 'even', $V_s = p^k$, if $n = p^k l$, $(l,p) = 1$.

(v) $L = \mathbb{F}_5(\sqrt{-t})$ and $K = \mathbb{F}_5(t)$: V_s is 1 or 0 according as whether 2 (not 4) does or does not divide s.

Let O_s denote the order of vanishing of the relevant zeta function at s. Then simple exact calculations using the vanishing theorem bounds show that $O_s = V_s$ for small s in the several examples, such as (i) $A = \mathbb{F}_3[x,y]/y^2 = x^3 - x - 1$, (ii) $A = \mathbb{F}_3[t]$ with $\mathcal{O} = \mathbb{F}_3[t,y]/y^2 = t^3 - t - 1$, (iii) $A = \mathbb{F}_3[t]$ with $\mathcal{O} = \mathbb{F}_9[t]$.

But the following simple calculation shows that the $V_s \neq O_s$ in general.

Examples 5.4. Let (a) $A = \mathbb{F}_2[x,y]/y^2+y = x^3+x+1$, (b) $A = \mathbb{F}_2[x,y]/y^2+y = x^5+x^3+1$. In both these situations, $hd_\infty = 1$ and hence $K = H$. For small s we compute $\zeta(s,X)$ as follows: The genus of K is 1 and 2 for (a) and (b) respectively. Hence by the bounds obtained from the vanishing Theorem and the Riemann-Roch theorem shows that for (a) and (b), $\zeta(-1,X) = 1 + 0 * X + (x + (x+1)) * X^2 = 1 + X^2 = (1+X)^2$. Hence the order of vanishing is 2 for $s = -2^n$. Similar simple calculation shows that the order of vanishing is 1 if $2^n \neq -s \leq 9$ for (a) and is 1 for $s = -7$ and 2 for other s with $-s \leq 9$ for (b).

In fact, more generally, we have

Theorem 5.13. *If $d_\infty = 1$, $q = 2$ and K is hyper-elliptic, then the order of vanishing of $\zeta(s)$ at negative integer s is 2 if $l(-s) \leq g$, where g is the genus of K.*

Proof. Let $x \in A$ be an element of degree 2 and let n be the first odd non-gap for A at ∞. Then the genus g of K is seen to be $(n-1)/2$. Let $S_A(i)$ denote the coefficient of X^i in $\zeta_A(s,X)$. A simple application of our general vanishing Theorem and the Riemann-Roch theorem shows that for $i > l(-s) + g$, $S_A(i) = 0$. Hence

$$\zeta_A(s,X) = \sum_{i=0}^{(n-1)/2} S_A(2i) X^{2i}$$

since $l(-s) \leq g = (n-1)/2$. On the other hand, as x has degree 1 in $\mathbb{F}_q[x]$, we have $S_A(2i) = S_{\mathbb{F}_q[x]}(i)$, for $0 \leq i \leq (n-1)/2$. Further, by vanishing Theorem, $S_{\mathbb{F}_q[x]}(i) = 0$, if $i > (n-1)/2$, as $(n-1)/2 \geq l(-s)$. Hence, $\zeta_A(s,X) = \zeta_{\mathbb{F}_q[x]}(s,X^2)$. (This works only for the s as above, and is not an identity of zeta functions). Hence by Example (i), the order of vanishing is $2 = 2 * 1$ as required. □

Remarks 5.7. (i) We do not need a restriction on the class number of K, so apart from (a) and (b) this falls outside the scope of the theorem giving the lower bounds.

But there are other examples of this phenomenon, with $q > 2$, as well as for full ideal zeta function for higher class numbers. See [T95] and University of Arizona thesis (1996) of Javier Diaz Vargas and [DV06].

(ii) Analogies between the number fields and the function fields are usually the strongest for $A = \mathbb{F}_q[y]$. But even in that case, there are examples of extra vanishing. For example, when $A = \mathbb{F}_4[y]$ and $\mathcal{O} = \mathbb{F}_4[x, y]/y^2 + y = x^3 + \zeta_3$, the order of vanishing of $\zeta_{\mathcal{O}}$ at $s = -1$ is 2, as can be verified by direct computation, whereas the lower bound is 1.

Remarks 5.8. These results, when combined with Goss-Sinnott results above, imply relative class group components non-vanishing for all primes and raise important open questions about the meaning of the real leading terms.

6. Period interpretations and complete relations between values

For $A = \mathbb{F}_q[t]$, we saw in 3.1 that a binomial coefficient, also related to terms of gamma product, was given as (i) a ratio of natural products related to A, as we all as (ii) a coefficient coming from a series involving Drinfeld exponential and logarithms for rank one sgn-normalized Drinfeld A-modules. If you generalize to A, the two properties do not agree and give rise to two binomial coefficients notions, expressed by round and curly brackets respectively. The following result shows this coincidence of the two notions allows one to get nice generating function for the terms of the zeta sums for $\zeta(-k)$, namely the power sums $S_d(k) = \sum a^k$, where the sum is over monic elements of degree d in A.

The basic idea is that power sums are symmetric functions of a's and can be calculated via Newton's formulas from the elementary symmetric functions, which are coefficients of $\prod(x-a)$ related to binomial coefficients, where one uses $x \to x-t^d$ to move from $\mathbb{F}_q$-vector space of all a's of degree less than d to the affine space of monic of degree d. We can use the reciprocal polynomial for $1/a$'s.

6.1. **Generating function.** Here is [T04, Thm. 5.6.3] and its proof with signs corrected.

Theorem 6.1. *(Carlitz) If $\mathcal{B}_d(x)$ denotes the binomial coefficients $\left\{ {x \atop q^d} \right\} = \binom{x}{q^d}$ for $\mathbb{F}_q[t]$, then the quantity $-\frac{\mathcal{B}_d'(0)}{1-\mathcal{B}_d(x)}$ has Laurent series expansion $\sum_{n=0}^{\infty} S_d(n)x^{-n-1}$ at $x = \infty$ and $-\sum_{n=0}^{\infty} S_d(-n-1)x^n$ at $x = 0$.*

Proof. Since $1 - \mathcal{B}_d(x)$ has all monic polynomials of degree d as its simple zeros, by taking the logarithmic derivative to convert products into sums, we see that

$$\sum_{a \in A_{d+}} \frac{1}{x-a} = -\frac{\mathcal{B}_d'(0)}{1-\mathcal{B}_d(x)}.$$

Developing the left hand side as geometric series in two different ways, we see that the Laurent series expansion is $\sum_{n=0}^{\infty}(\sum_{a \in A_{d+}} a^n)x^{-n-1}$ at $x = \infty$ and $-\sum_{n=0}^{\infty}(\sum_{a \in A_{d+}} a^{-n-1})x^n$ at $x = 0$. □

We note $\exp_1 x \log_1 = \sum \mathcal{B}_d(x)\tau^d$, and $\mathcal{B}_d'(0) = 1/\ell_d$.

In particular, we see that $S_d(-1) = 1/\ell_d$ and so $\zeta(1) = \log(1)$, implying in particular, that $\zeta(1)$ is transcendental. When $q > 2$, 1 is 'odd' value. For the Riemann zeta, the transcendence of only even values is known because of Euler's result. We will show below how the t-motives allow us to handle all values for $A = \mathbb{F}_q[t]$.

Remarks 6.1. Let us give a simplification of the argument, avoiding the full evaluation of binomial coefficients, to prove directly the simplest case $S_d(-k) = 1/\ell_d^k$, for $1 \le k \le q$. Let a (b respectively) run through polynomials of degree $< d$ (monic of degree d respectively). Then $\prod(x-a)$ has x-coefficient $(-1)^d(D_0D_1 \cdots D_{d-1})^{q-1}$. Since this polynomial is $\mathbb{F}_q$-linear, $\prod(x-t^d-a) = \prod(x-b)$ has the same x-coefficient (only constant coefficient is different), which is now the sum over b of products of all monics except b of degree d, that is $(\prod b)(\sum 1/b) = D_dS_d(-1)$. Comparison gives the claim for $k = 1$. The $\mathbb{F}_q$-linearity of the polynomial makes appropriate coefficients zero so that the Newton formula for power sums implies that the k-th power sum is k-th power of the first, for the given range of k.

In the other direction, relating explicitly the two different notions of binomial coefficients mentioned above, this relation between zeta and logarithms of Drinfeld modules was generalized [T92b] (see also [T04, 4.15, 5.9] to some higher genus A. For example, for $A = \mathbb{F}_3[x,y]/y^2 = x^3 - x - 1$, we have $\zeta(1) = \log_\rho(y-1)$ for the corresponding sgn-normalized (x and y having signs 1) Drinfeld A-module ρ. We omit the details and move back to $A = \mathbb{F}_q[t]$ case and $\zeta(n)$ for any n.

6.2. **n-th tensor power of the Carlitz module.** By definition, the tensor product of t-motives is the tensor product over $F[t]$ on which τ acts diagonally. Thus ranks multiply under tensor product and so the Carlitz-Tate motive $C^{\otimes n}$ is of rank one. It is uniformizable, simple, pure t-motive of dimension n and it has no complex multiplications. Now the Carlitz module is rank one over $F[t]$ with basis m so that $\tau m = (t-\theta)m$. Thus $C^{\otimes n}$ is free rank one over $F[t]$ with basis $m_1 = m^{\otimes n}$. With $m_i := (t-\theta)^{i-1}m_1$ as its $f[\tau]$-basis, we see that if we write $[a]_n \in \mathrm{End}(\mathbb{G}_a^n)$ for the image of a, then. $[t]_n(x_1, \cdots, x_n) = (tx_1 + x_2, \cdots, tx_{n-1} + x_n, tx_n + x_1^q)$. Here by abuse of notation, we have identified $\theta = t$. If we write $d[a]_n$ for the coefficient of τ^0, then it is the matrix representing the endomorphism of $\mathrm{Lie}(\mathbb{G}_a^n)$ induced by $[a]_n$ and in particular, we have $d[a]_n(*, \cdots, *, x) = (*, \cdots, *, ax)$: The largest quotient of $\mathrm{Lie}(\mathbb{G}_a^n)$ on which the derivative and multiplication actions of A coincide is isomorphic to $\mathbb{G}_a$ with the isomorphism $\ell_n : \mathrm{Lie}(\mathbb{G}_a^n) \to \mathbb{G}_a$ given by the last co-ordinate.

Note that by the definition of diagonal τ-action, the matrix Φ for $C^{\otimes n}$ is just the scalar matrix $(t-\theta)^n$.

There is $\lambda \in C_\infty^n$ with the last entry $\tilde{\pi}^n$ such that the period lattice of $C^{\otimes n}$ is $\{d[a]_n\lambda : a \in A\}$.

By solving the functional equations for the exponential and logarithm, we can find expansions for these series. In particular, the canonical last co-ordinate ℓ_n of $\log_n(X)$ is given by $\sum_{i=0}^{n-1}(-1)^i \Delta^i \mathcal{L}_n(x_{n-i})$, where $\mathcal{L}_n(x) = \sum_{i=0}^{\infty}(-1)^{in}x^{q^i}/L_i^n$ is naive poly-logarithm corresponding to the Carlitz logarithm and Δ is our analog of zd/dz operator, which is commutator with t acting on linear functions $f(z)$ giving $f(tz) - tf(z)$. In particular, the last co-ordinate seems to be a good deformation of the naive poly-log $\ell_n(\log_n(0, \cdots, 0, x)) = \mathcal{L}_n(x)$. Note that

$$\Delta^k \mathcal{L}_n(x) = \sum \frac{[i]^k x^{q^i}}{\ell_i^n} = \sum \binom{k}{i} t^{k-i} \mathcal{L}_n(t^i x).$$

Also note the differences with the usual complex logarithms and multilogarithms, that in our case, $\log(1-x) = \log(1) - \log(x)$, $\mathcal{L}_n(x)$ is naive generalization of series for $\log(x)$ rather than for $\log(1-x)$, and that $\Delta^i\mathcal{L}_n(x)$ is as above compared to the usual $(zd/dz)^i \log_n(x) = \log_{n-i}(x)$, for $i < n$.

6.3. **Period interpretation for zeta.** We now use the logarithmic derivatives trick of generating function with binomial coefficients as explained above to turn the solitons giving terms of gamma products into terms giving zeta sums and give an explicit algebraic incarnation of (transcendental) Carlitz zeta value $\zeta(n)$ (and its v-adic counterparts), for n a positive integer, on the Carlitz-Tate motives $C^{\otimes n}$.

For classical polylog $\log_n$ and Riemann ζ, we have a simple connection $\zeta(n) = \log_n(1)$, which follows directly from definition and is of not much use to prove irrationality or transcendence results for these zeta values. For $\mathbb{F}_q[t]$ case, the situation is quite different. The relation as above holds for $n = p^r m$, with $m \leq q$, as we have seen in 6.1, but in general the relation is much more complicated, but because of the direct motivic connection to abelian t-modules, such as $C^{\otimes n}$, we can draw much stronger transcendence and algebraic independence conclusions.

Let us temporarily write V_n for the vector $(0, \cdots, 0, 1)$ of length n.

Theorem 6.2. *Let $A = \mathbb{F}_q[t]$. There exists (constructed explicitly below) A-valued point Z_n of $C^{\otimes n}$ such that the canonical last co-ordinate of $\log_n(Z_n)$ is $\Gamma(n)\zeta(n)$.*

If we put $Z_{n,v} := [v^n - 1]_n Z_n$, for a monic irreducible $v \in A$, then the canonical last co-ordinate of $\log_{n,v}(Z_{n,v})$ is $v^n\Gamma(n)\zeta_v(n)$.

The point Z_n is a torsion point, if and only if n is 'even'.

Proof. We use the connection between the binomial coefficients and the power sums giving the terms of the zeta functions explained earlier to construct the point Z_n as follows.

Let $G_n(y) := \prod_{i=1}^{n}(t^{q^n} - y^{q^i})$, so that $G_i(t^{q^k}) = (l_k/l_{k-i})^{q^i}$ and so for the binomial coefficient notation before, $\mathcal{B}_k(x) = \sum_{i=0}^{k} G_i(t^{q^k})/d_i(x/l_k)^{q^i}$. Hence if we define $H_n(y) \in K[y]$ by

$$\sum_{n=0}^{\infty} \frac{H_n(y)}{n!}x^n = (1 - \sum_{i=0}^{\infty} \frac{G_i(y)}{d_i}x^{q^i})^{-1},$$

then the fact that $n!m!$ divides $(n+m)!$ for the Carlitz factorial, implies first of all that with $H_{n-1}(y) = \sum h_{ni}y^i$, the coefficients h_{ni} belong to A. Further by generating function recalled above, we see that

$$H_n(t^{q^k})/l_k^{n+1} = n!S_k(-(n+1)).$$

This connection with multilogarithms is what we need for later applications.

We define $Z_n \in C^{\otimes n}(A) = A^n$ by

$$Z_n := \sum [h_{ni}]_n(t^i V_n).$$

Straight degree estimates [AT90] show that for $i \leq \deg H_{n-1}$, $t^i V_n$ is in the region of convergence for $\log_n$. Hence we have

$$\begin{aligned} \ell_n(\log_n(Z_n)) &= \sum \ell_n(d[h_{ni}]_n \log_n(t^i V_n)) &&= \sum h_{ni}\ell_n(\log_n(t^i V_n)) \\ &= \sum h_{ni}\frac{t^{iq^k}}{l_k^n} &&= \Gamma(n)\zeta(n). \end{aligned}$$

Notice that $\log_n$ is multi-valued function. With $z_n := \sum d[h_{ni}]_n(t^i V_n) \in K_\infty^n$, we have $\exp_n(z_n) = Z_n$.

If Z_n were a torsion point, when n is 'odd', then since $\tilde{\pi}^n$ is the last co-ordinate logarithm of zero, $\zeta(n)$ would be a rational multiple of $\tilde{\pi}^n$. But we know (e.g., from fractional degree $q/(q-1)$ of $\tilde{\pi}$) that $\zeta(n)/\tilde{\pi}^n$ is not in K_∞, when n is 'odd'. Conversely, if n is 'even' and Z_n is not a torsion point, then by Jing Yu's Hermite-Lindemann type transcendence result for $C^{\otimes n}$, we would have $\zeta(n)/\tilde{\pi}^n$ transcendental, contradicting the Carlitz result mentioned above. (In [ATp], we now have a direct algebraic proof of this without appealing to the transcendence theory).

Notice that the calculation proceeds degree by degree, so that instead of dealing with $\zeta(s) \in K_\infty$, we can also work with $\zeta(s, X) \in K[[X]]$, so multiplication by $1 - v^n$, which is just removing the Euler factor at $s = -n$, then gives the corresponding sum where the power sums are now over a prime to v, thus giving v-adic zeta value (see [AT90] for detailed treatment paying attention to convergence questions). □

Examples 6.1. For $n = rp^k$, with $0 < r < q$, we have $Z_n = [\Gamma(n)]_n(0, \cdots, 0, 1)$ and $Z_{q+1} = (1, 0, \cdots, 0) + [t^q - t]_{q+1}V_{q+1}$.

6.4. **Transcendence and algebraic independence results.** Proving analog for $C^{\otimes n}$ of Hermite-Lindemann theorems (and v-adic counterparts) about transcendence of values of logarithm for $C^{\otimes n}$, Jing Yu [Y91] concluded from above theorem that

Theorem 6.3. *Let $A = \mathbb{F}_q[t]$. Then $\zeta(n)$ is transcendental for all positive integers n and for all 'odd' n, $\zeta(n)/\tilde{\pi}^n$ and $\zeta_v(n)$'s are transcendental.*

More recently, directly using multilogarithms and using the techniques of [ABP04, P08], much stronger results (at least at ∞) were proved by calculations of dimensions of relevant motivic groups. We now turn to explaining these.

Theorem 6.4. *(1) [CY] Only algebraic relations between $\zeta(n)$'s come from the Carlitz-Euler evaluation at 'even' n, and $\zeta(pn) = \zeta(n)^p$. In particular, for n 'odd', $\zeta(n)$ and $\tilde{\pi}$ are algebraically independent and the transcendence degree of the field $K(\tilde{\pi}, \zeta(1), \cdots, \zeta(n))$ is $n + 1 - \lfloor n/p \rfloor - \lfloor n/(q-1) \rfloor + \lfloor n/(p(q-1)) \rfloor$.*

(2) [CPYa] Only algebraic relations between $\zeta_\ell(n)$'s for all ℓ and n, where ζ_ℓ denotes Carlitz zeta over $\mathbb{F}_{q^\ell}[t]$, are those as above coming from n 'even' or divisible by p. The periods $\tilde{\pi}_\ell$ of the Carlitz modules for $\mathbb{F}_{q^\ell}[t]$ are all algebraically independent.

(3) [CPYb] Only algebraic relations between $\zeta(n)$'s and geometric $\Gamma(z)$'s at proper fractions are those between zeta above and bracket relations for gamma.

(4) [CPTY] Only algebraic relations between $\zeta(n)$'s and arithmetic gamma values at proper fractions are those for zeta mentioned above and those for gamma coming from the bracket relations, and thus the transcendence degree of the field

$$K(\tilde{\pi}, \zeta(1), \cdots, \zeta(s), (c/(1-q^\ell))!)_{1 \le c \le q^\ell - 2}$$

is $s - \lfloor s/p \rfloor - \lfloor s/(q-1) \rfloor + \lfloor s/(p(q-1)) \rfloor + \ell$.

We quickly describe relevant motives and corresponding algebraic groups, (which can often be calculated by using recipes in [P08] or using Pink's results [Pin97] on images of Galois representations) leaving the details to the references above.

For Carlitz module M, we have $\Phi = (t - \theta)$, $\Psi = \Omega$ and $\Gamma_M = \mathbb{G}_m$. This follows from the fact that $\mathrm{Hom}(C^{\otimes m}, C^{\otimes n})$ is $\mathbb{F}_q(t)$ or zero, according as $m = n$ or not, and thus that the category generated by M is equivalent to the $\mathbb{Z}$-graded category of vector spaces over $\overline{\mathbb{F}_q(\theta)}(t)$ with fiber functor to category of vector spaces over $\mathbb{F}_q(t)$.

For its n-th tensor power (with $n \in \mathbb{Z}$, with $n = -1$ corresponding to the dual) M, we have $\Phi = (t - \theta)^n$, $\Psi = \Omega^n$, and $\Gamma_M = \mathbb{G}_m$.

The rank two Drinfeld module M given by $\rho_t = t + g\tau + \tau^2$ is in analogy with elliptic curves. We give the set-up here, higher rank situation being straight generalization. It corresponds to size two square matrix $\Phi = [0, 1; t - \theta, -g^{(-1)}]$ and the corresponding period matrix $\Psi^{-1}(\theta) = [w_1, w_2; \eta_1, \eta_2]$, where w_i are periods, and $\eta_i = F_\tau(w_i)$ are quasi-periods, and where the quasi-periodic function F_τ satisfies $e_\rho(z)^q = F_\tau(\theta z) - \theta F_\tau(z)$. (See [T04, Sec. 6.4].) Recall the elliptic curve analog $\eta_i = 2\zeta(w_i/2)$, with Weierstrass ζ related to $\wp$ by $\zeta' = -\wp$. For non-CM case, we get the group Gl_2.

It is clear how to generalize to define the matrices for higher rank Drinfeld modules, and we can calculate the motivic groups by appealing to Pink's results.

In particular, for getting Carlitz module over base $\mathbb{F}_{q^\ell}[t]$, which is a rank ℓ Drinfeld module over A, we have size ℓ-matrix $\Phi_\ell := (t-\theta)$ if $\ell = 1$, and otherwise

$$\Phi_\ell := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ (t-\theta) & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let ξ_ℓ be a primitive element of $\mathbb{F}_{q^\ell}$ and define $\Psi_\ell := \Omega_\ell$ if $\ell = 1$, and otherwise let

$$\Psi_\ell := \begin{bmatrix} \Omega_\ell & \xi_\ell \Omega_\ell & \cdots & \xi_\ell^{\ell-1}\Omega_\ell \\ \Omega_\ell^{(-1)} & (\xi_\ell\Omega_\ell)^{(-1)} & \cdots & (\xi_\ell^{\ell-1}\Omega_\ell)^{(-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_\ell^{(-(\ell-1))} & (\xi_\ell\Omega_\ell)^{(-1)} & \cdots & (\xi_\ell^{\ell-1}\Omega_\ell)^{(-(\ell-1))} \end{bmatrix},$$

where Ω_ℓ is just obtained from Ω by replacing q^n-th powers by $q^{\ell n}$-th powers.

The associated motivic Galois group is the Weil restriction of scalars of $\mathbb{G}_m$ from the constant field extension in question, and hence a torus of dimension ℓ. (See [CPTY]).

This already allows us to handle the arithmetic gamma values as follows. We have

$$\frac{(\frac{1}{1-q^\ell})!}{(\frac{q^{\ell-1}}{1-q^\ell})!^q} \sim \Omega_\ell(\theta), \qquad \frac{(\frac{q^j}{1-q^\ell})!}{(\frac{q^{j-1}}{1-q^\ell})!^q} \sim \Omega_\ell^{(-(\ell-j))}(\theta) \quad (1 \le j \le \ell-1),$$

where the first is just the Chowla-Selberg analog we proved for arithmetic gamma and the second its quasi-period analog established similarly.

Now any gamma value with denominator $1-q^\ell$ (note that any denominator is of this form without loss of generality by Fermat's little theorem), by integral translation by integer of the argument resulting in harmless algebraic factor, a monomial in basic values $(q^j/(1-q^\ell))!$'s $(0 \le j \le \ell-1)$, thus the field generated by all these values is ℓ-dimensional, by the dimensional calculation mentioned above. In particular, $(q^j/(1-q^\ell))!$'s $(0 \le j \le \ell-1)$ are algebraically independent.

To handle zeta values, we use the multilogarithm connection mentioned above and first handle a set of multilogarithm $\mathcal{L}_n$ at algebraic α_i's.

For $\alpha \in \bar{k}^\times$ with $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$, the power series

$$L_{\alpha,n}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t-\theta^q)^n(t-\theta^{q^2})^n \cdots (t-\theta^{q^i})^n},$$

satisfies $L_{\alpha,n}(\theta) = \mathcal{L}_n(\alpha)$. It is easy to check the functional equation

$$(\Omega^n L_{\alpha,n})^{(-1)} = \alpha^{(-1)}(t-\theta)^n\Omega^n + \Omega^n L_{\alpha,n}. \tag{1}$$

Thus after choosing

$$\Phi_n = \Phi(\alpha_{n0}, \ldots, \alpha_{nk}) := \begin{bmatrix} (t-\theta)^n & 0 & \cdots & 0 \\ \alpha_{n0}^{(-1)}(t-\theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{nk}^{(-1)}(t-\theta)^n & 0 & \cdots & 1 \end{bmatrix},$$

$$\Psi_n = \Psi(\alpha_{n0}, \ldots, \alpha_{nk}) := \begin{bmatrix} \Omega^n & 0 & \cdots & 0 \\ \Omega^n \mathcal{L}_{n0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega^n \mathcal{L}_{nk} & 0 & \cdots & 1 \end{bmatrix},$$

the functional equation translates to $\Psi_n^{(-1)} = \Phi_n \Psi_n$, and Φ_n defines a t-motive which is an extension of the $k+1$-dimensional trivial t-motive 1^{k+1} over $\bar{k}(t)$ by $C^{\otimes n}$ and its motivic Galois group is extension of $\mathbb{G}_m$ by a vector group. (See [CY] Lemma A.1). Using this, [CY] proves that the linear independence over K implies algebraic independence over $\bar{K}$ for n-th multilogarithms at algebaraic quantities, generalizing Papanikolas [P08] result for $n = 1$ by the same method, which generalizes earlier result of Jing Yu giving analog of Baker's theorem, giving linear independence over $\bar{K}$.

To deal with the algebraic independence issue for the zeta values $\zeta(n)$ for $n \leq s$, one takes direct sums of such appropriate multilog motives for appropriate 'odd' $n \leq s$ not divisible by p and α_i's coming from the relation before and calculate [CY] dimensions.

For geometric gamma, because of complex multiplications from cyclotomic function field, the motivic Galois group of the corresponding 'gamma motive' is a torus inside certain finite product of the Weil restriction of scalars of $\mathbb{G}_m$ from the cyclotomic function field. The direct sum with the 'zeta motive' has motivic Galois group an extension of a torus by vector group. The dimension can be written down explicitly and gives what we want.

7. Cyclotomic and class module, special points, zeros and other aspects

In this section, we review some new developments without proofs, just stressing the new objects, ideas and interconnections. We begin by mentioning new connections of the special zeta value theory with cyclotomic theory.

7.1. **Log-algebraicity, Cyclotomic module, Vandiver.** In our first look at the relations with cyclotomic theory, the p-divisibilities of orders of groups related to the class groups for $\wp$-th cyclotomic fields were linked to the $\wp$-divisibilities of the zeta values. This mixture of p and $\wp$ is not satisfactory analog. Also, we see that naive analogs of the Kummer's theorem that 'p does not divide $h(\mathbb{Q}(\zeta_p))^-$ implies p does not divide $h(\mathbb{Q}(\zeta_p)^+)$' and of Vandiver conjecture that 'p does not divide $h(\mathbb{Q}(\zeta_p)^+)$' both fail for simple examples. The p-divisibilities or even Norm($\wp$)-divisibilities do not work. To get good analog [T94, pa. 163] so that we can talk about $\wp$ divisibilities, we want A-modules and not abelian groups which are just $\mathbb{Z}$-modules.

In a very nice important work [A96], Greg Anderson provided 'cyclotomic A-module, class polynomial', and an analog of Vandiver conjecture! To do this, he generalized the log-algebraicity phenomena that we have been looking at in the previous lectures, as follows.

The cyclotomic units come from $1 - \zeta = \exp(\log(1 - \zeta)) = \exp(\sum -\zeta^n/n)$. Instead of logarithm, which is just inverse function of exponential, to get more, one should view the series occurring at right as the harmonic series or the zeta. For x a fraction, we consider, following Anderson, the triviality $\zeta^n = \exp(2\pi i n m x) = \exp(2\pi i n x)^m$ and look at analog $e_C(\sum e_C(ax)^m/a)$. By such mixing of the Carlitz action by $a \in A = \mathbb{F}_q[T]$ and usual multiplicative action by $m \in \mathbb{Z}$ (thus getting the usual powers and polynomials in the game using functional equations for the exponential), Anderson considered, for $A = \mathbb{F}_q[T]$,

$$S_m(t, z) := \sum_{i=0}^{\infty} \frac{1}{d_i} \sum_{a \in A_+} (\frac{C_a(t)^m}{a})^{q^i} z^{q^{i+\deg a}} \in K[[t, z]]$$

and proved the following 'log-algebraicity' theorem:

Theorem 7.1. *(Anderson [A96]) We have $S_m(t, z) \in A[t, z]$.*

Using this theorem, Anderson then defines cyclotomic module (an analog of the group of the cyclotomic units in $\mathbb{Q}(\zeta_p)$) $\mathcal{C}$ to be the A-sub-module of $\mathcal{O} := \mathcal{O}_{K(\Lambda_\wp)}$ (under the Carlitz action) generated by 'special points' $S_m(e_C(\tilde{\pi}b/\wp), 1) = e_C(\sum e_C(ab\tilde{\pi}/\wp)^m/a)$'s. He proved that the cyclotomic module $\mathcal{C}$ is Galois stable module of rank $(\text{Norm}(\wp) - 1)(1 - 1/(q-1))$.

His first proof [A94], which works in more generality of any A with $d_\infty = 1$, uses his soliton ideas of deformation theory to get explicit interpolating functions on powers of the cyclotomic cover similar to what we have seen for solitons, and then showing directly by analysis of divisors and zeros that the coefficients for large enough power of z are zero. His second proof [A96] is more elementary using the Dwork's v-adic trick (for all finite v) to deduce integrality of coefficients and then using analytic theory to get degree estimates to show that coefficients tend to zero eventually, so that they are zero from some point on-wards and the power series is really a polynomial.

We record a family of examples. Let $m = \sum_{e=0}^{k} q^{\mu_e}$, with $k < q$. We have [T04, pa. 300]

$S_i(m-1) = (\prod[\mu_e]_i)/l_i$, where $[h]_i = d_h/d_{h-i}^{q^i}$.

$$S_m(t,z) = \sum_{l_1=0}^{\mu_1} \cdots \sum_{l_k=0}^{\mu_k} C_{P_{(\mu_j,l_j)}(T)}(z \prod_{e=1}^{k} C_{T^{l_e}}(t))$$

where

$$P_{(\mu_j,l_j)}(T) = (-1)^{\sum(\mu_e - l_e)} \sum T^{q^{j_{1,1}}+\cdots+q^{j_{1,\mu_1-l_1}}+\cdots+q^{j_{k,1}}+\cdots+q^{j_{k,\mu_k-l_k}}}$$

with $0 \leq j_{e,1} < j_{e,2} < \cdots j_{e,\mu_e-l_e} \leq \mu_e - 1$, for $1 \leq e \leq k$.

This suggests that there is more underlying structure to these special polynomials and it would be desirable to get such an expression for general m.

Anderson also expresses $L(1, w^i)$ as algebraic linear combination of logarithms of these 'special points' (and a v-adic analog). (Papanikolas in a recent preprint has shown using these and ideas from $C^{\otimes n}$ section that $L(m,\chi)$ (for $m \leq q$ and χ character of $(A/f)^*$ which is m-th power) can be expressed as explicit algebraic linear combination of m-th polylog at algebraic arguments).

Now Kummer-Vandiver conjecture that p does not divide the class number of $\mathbb{Q}(\zeta_p)^+$ can be rephrased as the canonical map taking the cyclotomic units mod the p-th power to all units modulo p-th power is injective. Replacing $\mathcal{O}^*$ which are integral points for the multiplicative group by $\mathcal{O}$ which are integral points for the additive group (with Carlitz action), analog of Anderson for Vandiver conjecture would be that map $\mathcal{C}/\wp\mathcal{C} \to \mathcal{O}/\wp\mathcal{O}$ is injective. It would be nice to settle this.

As an analog of the Kummer's theorem that p does not divide h^- implies p does not divide h^+, Anderson proved that the map above is injective, if $\wp$ does not divide $\zeta(1-i)$ for $1 \leq i \leq Norm(\wp)$, and $1-i$ 'odd'.

If $\sqrt{\mathcal{C}}$ is the divisible closure of $\mathcal{C}$ in $\mathcal{O}$ thought of as $\mathcal{O}$-points of the Carlitz module, then since the units are divisible closure of cyclotomic units, the analogies suggest that analog of the class number h^+ is the class polynomial giving the (Fitting) index of A-module $\sqrt{\mathcal{C}}/\mathcal{C}$.

Using the analogy with cyclotomic units and the relation between class numbers and index of cyclotomic units, we thus get a polynomial as some kind of A-module version of class number in this special case. How about getting A-module 'class group' concept in general?

This leads us to the discussion of the recent beautiful work of Taelman.

7.2. **Taelman class number formula conjecture.** In a very nice recent work [Ta10], Lenny Taelman gave a good analog of Dirichlet unit theorem in the context of Drinfeld modules and gave a conjectural 'class number formula' for Dedekind zeta $\zeta_{\mathcal{O}/A}(1)$, for $A = \mathbb{F}_q[t]$. We now describe some of the ideas.

Let $A = \mathbb{F}_q[t]$. We use the notation of the zeta section, with finite extension L of K and $R = \mathcal{O}$ being the integral closure of A in it. Define $L_\infty := L \otimes K_\infty$ and L_∞^{sep} similarly. Note that these are products of fields. Let $G = \mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$.

Let ρ be Drinfeld A-module with coefficients in R. Taking the G-invariants of the G-equivariant short exact sequence $\Lambda \hookrightarrow \mathrm{Lie}_\rho(L_\infty^{sep}) \twoheadrightarrow \rho(L_\infty^{sep})$ (given by $\exp_\rho$) of A-modules, we get $\Lambda^G \hookrightarrow \mathrm{Lie}(L_\infty) \to \rho(L_\infty) \twoheadrightarrow H^1(G,\Lambda)$ by additive Hilbert 90. Taelman proves

Theorem 7.2. *The cokernel (kernel respectively) of* $\rho(R) \to H^1$ *is finite (finitely generated respectively) and the inverse image under* $\exp_\rho$ *of* $\rho(R)$ *is a discrete and co-compact sub-A-module of* $Lie_\rho(L_\infty)$.

He defines the cokernel to be the class module H_R, and the kernel to be the Mordell-Weil group U_R, and shows that $U_R = \sqrt{\mathcal{C}}$, when L is the cyclotomic extension of K, and ρ is the Carlitz module.

By the theorem, the natural map $\ell : \exp^{-1}(\rho(R)) \otimes_A K_\infty \to \mathrm{Lie}_\rho(R) \otimes_A K_\infty$ induced by $\exp^{-1}(\rho(R)) \to \mathrm{Lie}_\rho(L_\infty)$ is an isomorphism of K_∞-vector spaces. With respect to any A-basis of $\exp^{-1}(\rho(R))$ and $Lie_\rho(R)$, the map has well-defined determinant in K_∞^*/A^*, call its monic representative in K_∞^* the regulator Reg_R. Finally, for A-module M, let $|M|$ denote monic generator of its first fitting ideal, i.e. the product of monic polynomials $f_i \in A$ such that $M \equiv \oplus A/(f_i)$.

Now let ρ be the Carlitz module. Then Taelman's conjectured class number formula is

$$\zeta_{R/A}(1) = Reg_R * |H_R|.$$

He proves it for $R = A = \mathbb{F}_q[t]$ and gives numerical evidence. The author has proved the naive generalization of this for $A = R$ of class number one, having already calculated the left side as a logarithm [T92b] as we mentioned above. For details, we refer to [Tp]. Note that Anderson, in the work already mentioned above, already proves $\zeta_{R/A}(1)$ to be some explicit combination of logarithms of special points for abelian extensions.

Let us now look at the example $A = \mathbb{F}_q[t]$ and $R = \mathbb{F}_{q^2}[t]$. Then $\zeta_{R/A}(1) = \zeta(1)L(1)$, where $L(1) = \sum \chi(a)/a$, where $\chi(a) = (-1)^{\deg(a)}$, so that if ζ is a $q-1$-th root of -1, then $L(1) = \log(\zeta)/\zeta$, as can be verified from formulas we have seen for logarithms and power sums, and is obvious for $q = 2$. The class number formula in this case can be verified using this.

Remarks 7.1. While $\mathbb{C}$ is a local field and extension of degree 2 of $\mathbb{R}$, C_∞ is not local and is of infinite degree over K_∞. But the infinitude of the degree has virtue of allowing lattices, and thus Drinfeld modules, of arbitrarily large ranks. Similarly, Taelman's work above shows another virtue by getting finitely generated quantities by taking invariance with respect to infinite $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$.

Update: In recent preprints [Tap1, Tap2], Taelman has proved his conjecture and has given several nice interpretations in the Carlitz case.

7.3. **Valuations of the power sums and the zero-distribution for the Zeta.** We will go in reverse chronological order to explain this topic. Let $A = \mathbb{F}_q[t]$ to start with.

For $k \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\geq 0}$, let $S_d(k) = \sum a^{-k}$, where the sum is over $a \in A_{d+}$. (Note we changed sign from the earlier notation) and $s_d(k)$ be its valuation (i.e., negative of the degree) at the infinite place. While the individual terms have valuations dk, which are monotonic in either d or k, if we fix the other, constant (k or d respectively) jumps, the cancellations in summation makes $s_d(k)$ a very erratic function, not monotonic in k and monotonic in d, but for not so obvious reasons.

For $q = p$ a prime, in [T09a], we found a nice but strange recursion (another proof by Böckle using his cohomological formula can be found in the lecture notes of the parallel lecture series) $s_d(k) = s_{d-1}(s_1(k)) + s_1(k)$, which immediately implies that $s_d(k)$ is monotonic in d and the jumps $s_{d+1}(k) - s_d(k)$ are also strictly monotonic in d, just by straight induction on d, the recursion relation reducing the statement for d to $d-1$, and the initial case being obvious. (The recursion works for both k positive and negative and so you can approach p-adic integer y below both from positive and negative integers).

Now, for a fixed $y \in \mathbb{Z}_p$, the zeroes of Goss zeta function $\zeta(y, X)$ in $X \in C_\infty$ can be calculated by the Newton polygon of this power series, but the above monotonicity in jumps exactly translates to the slopes of the polygon increasing at each vertex, so that horizontal width of each segment is one, and thus all these zeros, which are a priori 'complex' are simple and 'real', i.e., lie on the 'real line' $K_\infty \subset C_\infty$. (Also, there is at most one zero for each valuation). This zero distribution result is the Riemann hypothesis for the Goss zeta function, first proved by Daqing Wan, in case $\mathbb{F}_p[t]$. The author noticed that the statement can be quickly reduced to a combinatorial unproved assertion made by Carlitz. This was then proved for $q = p$ by Javier Diaz-Vargas [DV96] in his Univeristy of Arizona thesis. But the general non-prime case proved to be much more difficult and after the initial proof by Bjorn Poonen for $q = 4$, Jeff Sheats, then a postdoc in combinatorics at University of Arizona, gave a complete but quite complicated proof of the general case, proving the same statement for $A = \mathbb{F}_q[t]$.

It would be nice if this proof can be simplified in any way, for example, by finding some simple recursion similar to that above for any q, which gives the slope statement as immediately. For more on power sums, references to the literature, see [T09a].

What happens for general A? Notice that our example in the extra vanishing phenomena were of the type $\zeta_A(s, X) = \zeta_{\mathbb{F}_q[x]}(s, X^p)$, for some s and some $x \in A$, and since these polynomials are often non-trivial, we get zeroes giving inseparable extensions of K_∞. (The simplest example being A of class number one, genus 2 (there is unique such, given as example (b) before theorem 44) and $s = -3$). So the naive generalization of Riemann hypothesis as above does not work. But David Goss [G00, G03] has made some interesting conjectures for the general case of L-functions and also at v-adic interpolations, regarding the finiteness of maximal separable extension when you adjoin zeroes, absolute values of the zeroes etc. For precise statements, a discussion and examples, we refer to these papers.

Wan already noticed that v-adic situation for degree one primes can be handled similarly, at least partially.

The author, in ongoing work, has found interesting patterns (partially proved, partially conjectured) in valuation tables of $S_d(k)$ at finite primes of higher degree. This should lead to some interesting consequences for the zero distribution of v-adic zeta functions of Goss. We also saw some interesting computationally observed phenomena in higher genus case in Böckle's lecture series.

7.4. **The Zeta measure.** The polynomials taking integers to integers are integral linear combinations of the binomial coefficients, just as in the $\mathbb{Z}$ case, for $A = \mathbb{F}_q[t]$ with the binomial coefficients we talked about.

Now Mahler's theorem that continuous functions from $\mathbb{Z}_p$ to itself are exactly the functions of the form $f(x) = \sum f_k \binom{x}{k}$, with $f_k \to 0$ as k tends to infinity, has exact analog obtained by replacing $\mathbb{Z}_p$ by $A_\wp$ and replacing usual binomial coefficients by Carlitz binomial coefficients we introduced above. It was proved by Carlitz' student Wagner.

A $\mathbb{Z}_p$-valued p-adic measure on $\mathbb{Z}_p$ is just a $\mathbb{Z}_p$-linear map μ: Conti $(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathbb{Z}_p$. One writes $\mu(f)$ symbolically as $\int_{\mathbb{Z}_p} f(x)\, d\mu(x)$.

In the binomial coefficient basis, such a measure μ is uniquely determined by the sequence $\mu_k := \int_{\mathbb{Z}_p} \binom{x}{k} d\mu(x)$ of elements of $\mathbb{Z}_p$. Any sequence μ_k determines a $\mathbb{Z}_p$-valued measure μ by the formula $\int_{\mathbb{Z}_p} f d\mu = \sum\limits_{k\geq 0} f_k\mu_k = \sum\limits_{k\geq 0} (\triangle^k f)(0)\mu_k$.

Convolution $*$ of measures μ, μ' is defined by $\int f(x)\, d(\mu * \mu')(x) = \int\int f(x + t)\, d\mu(x) d\mu'(t)$.

In other words, if one identifies measure μ with $\sum \mu_k X^k$ then convolution $*$ on measures is just multiplication on the corresponding power series. This identification of $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p$ with power series is called Iwasawa isomorphism.

In the $\mathbb{F}_q[t]$ case, with function field binomial coefficient analogs, an $A_\wp$-valued measure μ on $A_\wp$ (called just measure μ for short) is uniquely determined by the sequence $\mu_k = \int_{A_\wp} \binom{x}{k} d\mu$ of elements of $A_\wp$. Any sequence μ_k determines measure μ by $\int_{A_\wp} f\, d\mu = \sum_{k\geq 0} f_k\, \mu_k$.

Because of different properties of this analog, if we identify measure μ with the divided power series $\sum \mu_k (X^k/k!)$ (recall the formal nature of $X^k/k!$ in the context of divided power series), then the convolution $*$ on measures is multiplication on the corresponding divided power series.

Classically, Nick Katz showed that the measure μ whose moments $\int_{\mathbb{Z}_p} x^k d\mu$ are given by $(1 - a^{k+1})\zeta(-k)$ for some $a \geq 2$, $(a, p) = 1$ has the associated power series $(1 + X)/(1 - (1 + X)) - a(1 + X)^a/(1 - (1 + X)^a)$. We need a twisting factor in front of the zeta values to compensate for the fact that the zeta values are rational rather than integral, in contrast to our case. It would be quite interesting to understand the comparison of this result with the following result [T90]. (This reference contains references and details about all the results mentioned in this subsection).

Theorem 7.3. *For $A = \mathbb{F}_q[t]$, the divided power series corresponding to the zeta measure μ, namely the measure whose i-th moment $\int_{A_\wp} x^i d\mu$ is $\zeta(-i)$, is given by $\sum \mu_k (X^k/k!)$ with*

$$\mu_k = (-1)^m \quad \textit{if} \quad k = cq^m + (q^m - 1),\ 0 < c < q - 1, \textit{and } \mu_k = 0 \textit{ otherwise.}$$

7.5. **Multizeta values, period interpretation and relations.** We briefly mention work [T04, Sec. 5.10] and more recent works [AT09, T09b, T10, Lr09, Lr10, Tp] on multizeta values. For detailed discussion, bigger context in the number field and function field situation, full results and proofs, these references should be consulted.

There has been a strong recent interest in multizeta values which are iterated sums

$$\zeta(s_1, \cdots, s_k) := \sum_{n_1 > \cdots n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

initially defined by Euler, through their appearance in Grothendieck-Ihara program [I91] of understanding the absolute Galois group of $\mathbb{Q}$ through the algebraic fundamental group of the projective line minus zero, one and infinity, and related interesting mathematical and mathematical physics structures in diverse fields. The connection comes through integral representation from them by iterated integral of holomorphic differentials dx/x and $dx/(1 - x)$ on this space giving a period for its fundamental group. The relations they satisfy such as sum shuffle and integral shuffle relations have many structural implications. We will just mention the simplest sum shuffle

$$\zeta(s_1)\zeta(s_2) = \sum 1/n_1^{s_1} \sum 1/n_2^{s_2} = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2),$$

which just follows from $n_1 > n_2$ or $n_1 < n_2$ or $n_1 = n_2$, the generalization being obtained by shuffling the orders, when you multiply multizeta values, of terms or of differential forms to write the expression as the sums of some multizeta values. Thus the span of multizeta values is an algebra.

For $s_i \in \mathbb{Z}+$, we define multizeta value $\zeta(s_1, \cdots, s_r)$ by using the partial order on $A+$ given by the degree, and grouping the terms according to it:

$$\zeta(s_1, \cdots, s_r) = \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty,$$

where the second sum is over $a_i \in A_{d_i}+$ satisfying the conditions as in the first sum.

We say that this multizeta value (or rather the tuple $(s_1, \cdots, s_r)$) has depth r and weight $\sum s_i$. Note we do not need $s_1 > 1$ condition for convergence as in the classical case.

Since there are many polynomials of given degree (or norm), the usual proof of sum shuffle relations fails. In fact, it can be seen that naive analogs of sum or integral shuffle relations fail. The Euler identity $\zeta(2, 1) = \zeta(3)$ fails in our case, for simple reason that degrees on both sides do not match.

On the other hand, we showed that any classical sum-shuffle relation with fixed s_i's works for q large enough.

More interestingly, we can show that linear span of multizeta values is an algebra, by completely different type of complicated 'shuffle relations' such as:

If b is odd, $\zeta(2)\zeta(b) = \zeta(2 + b) + \sum_{1 \leq i \leq (b-3)/2} \zeta(2i + 1, 1 + b - 2i)$, and if b is even, $\zeta(2)\zeta(b) = \zeta(2 + b) + \sum_{1 \leq i \leq b/2-1} \zeta(2i, b + 2 - 2i)$.

There are very interesting recursive recipes [T09b, Lr09, Lr10] which are conjectural. To give a flavor of combinatorics, we will just say that for q prime, given expression of $\zeta(a)\zeta(b)$ as sum of multizeta, one can get expression for $\zeta(a)\zeta(b + (q-1)p^m)$, where m is the smallest integer such that $a \leq p^m$, by adding precisely described $t_a = \prod (p-j)^{\mu_j}$ new multizeta terms, where μ_j is the number of j's in the base p expansion of $a - 1$.

There are identities as above with $\mathbb{F}_p$-coefficients (understood in some precise sense) and identities (not yet well-understood) with $\mathbb{F}_p(t)$ (or K) coefficients, such as $\zeta(1, 2) = \zeta(3)/\ell_1 = \zeta(3)/(t - t^3)$, for $q = 3$ as analog of Euler identity.

In any case, in [AT09], the period interpretation was provided.

Theorem 7.4. *Given multizeta value $\zeta(s_1, \cdots, s_r)$, we can construct explicitly iterated extension of Carlitz-Tate t-motives over $\mathbb{F}_q[\theta]$ which has as period matrix entry this multizeta value (suitably normalized).*

Recall from 6.3, the polynomials $H_n(y) \in K[y] = \mathbb{F}_q(t)[y]$. We write them as $H_n(y, t)$ below.

Let $s = (s_1, \ldots, s_r) \in \mathbb{Z}_+^r$, and consider matrices (the first one is a diagonal matrix)

$$D = [(t-\theta)^{s_1 + \cdots + s_r}, (t-\theta)^{s_2 + \cdots + s_r}, \cdots, (t-\theta)^{s_r}, 1],$$

$$Q = \begin{bmatrix} 1 & & & \\ H_{s_1-1}(\theta,t) & 1 & & \\ & \ddots & \ddots & \\ & & H_{s_r-1}(\theta,t) & 1 \end{bmatrix}.$$

Then $\Phi = Q^{(-1)}D$ defines the mixed t-motive of rank $r+1$ whose period matrix (obtained by specializing Ψ^{-1} at $t=\theta$) contains the depth r multizeta value $\zeta(s_1, \ldots, s_r)$, for $A = \mathbb{F}_q[\theta]$. For the details of calculation, we refer to the paper.

In this case, the situation is quite complicated and one is far from understanding the motivic Galois group or the corresponding transcendence degrees unlike the cases studied in the last section.

There is interesting Galois side to this motivic study and we achieved the first step of constructing [ATp] analogs of Ihara power series, Deligne-Soule cocycles which connects zeta, gamma, gauss sums, cyclotomic modules etc. in towers and big Galois actions.

Update: In the recent preprint [Lrp] based on his PhD thesis work with the author, Alejandro Lara-Rodriguez has proved many of the explicit conjectures [Lr09, Lr10, T09b] that the author and he made on the multizeta relations with $\mathbb{F}_p$-coefficients, by making the effective recipe of [T10] very explicit.

7.6. **Cohomology approach.** Taguchi-Wan [TW96, TW97], Böckle-Pink [BöP, Böc02] have developed nice cohomological approach getting determinantal formulas (see also [A00] and [T04, Sec. 5.11]) for zeta special values, developing much general L-functions, rationality results, meromorphic continuations, criterion for analyticity etc. by Dwork style or Crystalline methods respectively. Fortunately, we got a nice exposition about some of this in the parallel lecture series by Böckle. So we can just refer to those lecture notes.

Recently, Vincent Lafforgue [L09] proved some Bloch-Kato type results at positive integers using Fontaine-Perrin-Riou methods, and he also defined good notion of extensions which inspired Taelman's work mentioned above. Soon after these CRM lecture series, Taelman applied these determinantal formula directly to the special value and proved his class number formula conjecture mentioned in 7.1.

7.7. **Open questions.** We collect some important open questions at the end.

Overall the present results bring the special value theory of the arithmetic, geometric gamma functions and zeta function for $A = \mathbb{F}_q[\theta]$ at infinite place to a very satisfactory state; however, similar questions for (i) v-adic interpolations (see [T04, Sec. 11.3] for very partial results about v-adic gamma values using automata methods and [Y91] for transcendence of v-adic zeta values), (ii) generalizations to other rings 'A' in the setting of Drinfeld modules [T04, 4.5, 8.3], (iii) values of the two variable gamma function of Goss [T04, 4.12], [T91a, Sec. 8], [G88], are still open.

Let us gather some other important open questions and projects, apart from generalizations, that we looked at.

(1) What is the arithmetic nature of $\Gamma(0)$ for the gamma function in 4.5?

(2) What are general conjectures for special values of L-functions, generalizing Taelman? What are the arithmetic meanings and applications of new class modules etc.? What are general conjectures about orders of vanishing (good computational data can be easily obtained here with some effort), and what would be

the arithmetic significance of new leading terms, or of non-trivial zeros of Goss zeta functions? Is the zeta non-vanishing at 'odd' integers a general phenomena?

(3) Connection of the zeta values at the positive and negative integers with class groups suggests some connection between them. How can we make it more explicit?

(4) For $d_\infty > 1$, there is a question referred in 4.9 on non-vanishing of e_n's and equivalently whether $\xi^{(n)}$ does not belong to support of the Drinfeld divisor V.

(5) Values of zeta for non-abelian extensions at positive values and whether there is a modular forms connection as in the classical case. Better understanding of the zeta measure result is also highly desirable.

(6) Simplification of the Sheats proof of the zero distribution, by possible simple recursive in d relation on the valuations.

Guide to the literature: We give some comments on the references missing in the text. Good references for pre-2003 material are [G96, T04], which contain more details and extensive references. Except for the comments and developments after 2003 that are mentioned, and except for most of sections 3, 6, 7, the other material is based on [T04], which in turn is based on many original papers accessible from author's homepage.

In more detail, most of the material in the first two sections can be found (except for citations there) in [T87, T88, T91a] or [T04, Cha. 4], that in section three in [T99] or [T04, Cha. 7, 8] and that in the fourth section in [T96, T98, T93b] or [T04, Cha. 8, 11]. For section 5, see [T95, G96] and [T04, Cha. 5] and references there. For section 6, see [A86, ABP04], [T04, Cha. 7].

Having focused on relatively new results with a particular transcendence method and period connections, the references given here are limited. For surveys with many other references and overlapping results on transcendence, in particular, Mahler method, automata method, see [Wal90, Bro98, G96, Pe07], [T04, Cha. 10].

Note added, October 2012: For many interesting related developments since the lectures were delivered, we refer to papers by Chang, Lara Rodriguez (multizeta relations and independence); author (multizeta, congruences); Angles, Taelman (cyclotomic and class modules, Herbrand-Ribet, Vandiver analogs, L-values); Papanikolas (L-values); and Pellarin, Perkins (deformations of L-values).

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