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Diophantine Approximation Exponents and Continued Fractions for Algebraic Power Series

Dinesh S. Thakur[1]

Tata Institute of Fundamental Research, Colaba, Bombay, India; and University of Arizona, Tucson, Arizona 85721
E-mail: thakur@math.arizona.edu

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For each rational number not less than 2, we provide an explicit family of continued fractions of algebraic power series in finite characteristic (together with the algebraic equations they satisfy) which has that rational number as its diophantine approximation exponent. We also provide some non-quadratic examples with bounded sequences of partial quotients. © 1999 Academic Press

Continued fraction expansions of real numbers and laurent series over finite fields are well studied because, for example, of their close connection with best diophantine approximations. In both the cases, the expansion terminates exactly for rationals and is eventually periodic exactly for quadratic irrationals. But continued fraction expansion is not known even for a single algebraic real number of degree more than two. It is not even known whether the sequence of partial quotients is bounded or not for such a number. (Because of the numerical evidence and a belief that algebraic numbers are like most numbers in this respect, it is often conjectured that the sequence is unbounded.) It is hard to obtain such expansions for algebraic numbers, because the effect of basic algebraic operations (except for adding an integer or, more generally, an integral Mobius transformation of determinant ± 1), such as addition or multiplication or even multiple or power, is not at all transparent on the continued fraction expansions.

In finite characteristic p, the algebraic operation of taking pth power has a very transparent effect: If $\alpha = [a_0, a_1, ...]$, then $\alpha^p = [a_0^p, a_1^p, ...]$, where we write $[a_0, a_1, ...]$ as a short form for the expansion $a_0 + 1/(a_1 + 1/(a_2 + \cdots))$.

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Let F be a finite field of characteristic p and let q be a power of p. If $A_i(t) \in F[t]$ are any non-constant polynomials, the remark above shows that

$$\alpha := [A_1, ..., A_k, A_1^q, ..., A_k^q, A_1^{q^2}, ...] \qquad (*)$$

is algebraic over $F(t)$ because it satisfies the algebraic equation

$$\alpha = [A_1, ..., A_k, \alpha^q].$$

So one obtains a variety of explicit continued fractions with explicit equations. By integral Mobius transformations with determinant ± 1 (which keeps the tail of the expansion the same), one can obtain many more such explicit examples. One can let the determinant be any $f \in F^*$ by using $f[a, b, c, d, ...] = [af, b/f, cf, d/f, ...]$.

Since continued fractions give best diophantine approximations, let us analyze what we get for α's as above.

For β an algebraic irrational real number (an element of $F((t^{-1}))$ algebraic irrational over $F(t)$, respectively), define its diophantine approximation exponent $E(\beta)$ by

$$E(\beta) := \limsup \left(-\frac{\log |\beta - P/Q|}{\log |Q|} \right)$$

where P and Q run over integers (polynomials in $F[t]$, respectively), the absolute value is the usual one in each case and the limit is taken as $|Q|$ grows.

The well-known theorems of Dirichlet and Liouville and their analogues for function fields [M] show that $2 \leqslant E(\beta) \leqslant d(\beta)$, where $d(\beta)$ is the algebraic degree of β. (The diophantine approximation results and, in particular, the improvement on the Liouville bound of $d(\beta)$ have interesting implications for the study of related diophantine equations, as the work of Thue, Siegel, Baker, Voloch, Vojta, Faltings, etc., has shown). For the real number case, the well-known theorem of Roth shows that $E(\beta) = 2$, but Mahler [M] showed that $E(\beta) = d(\beta) = q$ for $\beta = \sum t^{-q^i}$ (so that $\beta^q - \beta - t^{-1} = 0$), as a direct estimate of approximation by truncation of this series shows. Contradicting the assertions in the literature that such phenomena only occur when $d(\beta)$ is divisible by p, Osgood [O2] and Baum and Sweet [BS1, 2] gave many examples in various degrees. See References for other examples.

For given $d = d(\beta)$, $E(\beta)$'s form a countable subset of interval $2 \leqslant x \leqslant d$. What is it? Does it contain any irrational number? Does it contain all the rationals in the range? Looking at variations of Mahler and Osgood examples, Voloch [V1] gave a variety of examples (but without continued fractions) showing that $E(\beta)$ can be any rational between $1+\sqrt{q}$ to d, for various d's. On the other hand, non-quadratic algebraic examples of continued fractions with bounded sequence of partial quotients (so that the exponent is 2) were first given in [BS1] and the references below contain more examples for some (isolated) exponents.

THEOREM 1. (1) *Let the degree of A_i be d_i, and put $r_i := d_i/((d_1 + \cdots + d_{i-1})\, q + d_i + \cdots + d_k)$. Then for α as in $(*)$, we have*

$$E(\alpha) = 2 + (q-1)\, MAX(r_1, ..., r_k).$$

(2) *Given any rational μ between $q^{1/k}+1$ (which tends to 2 as k tends to infinity) and $q+1$, we can find a family of α's as in $(*)$, with $E(\alpha)=\mu$ and $d(\alpha) \leqslant q+1$.*

Proof. The convergents p_n/q_n of the continued fractions give best approximations, so that, in the definition of the exponent, one needs to use P/Q coming from the truncations of the continued fraction only. Now we have $\beta - p_n/q_n = \pm 1/(\beta_{n+1} q_n^2 + q_n q_{n-1})$, where β_{n+1} is the complete partial quotient $[a_{n+1}, a_{n+2}, ...]$ if $\beta = [a_0, a_1, ...]$. Hence we have $E(\beta) = 2 + \limsup\, (\deg a_{n+1}/\deg q_n)$.

By the usual recursion formula for q_n, we see that the degree of q_n is the sum of the degrees of $a_1, ..., a_n$. By $(*)$, $\deg a_n = d_{j+1} q^{(n-j)/k}$, where j is the smallest non-negative residue of n modulo k. Hence a straight calculation (see Remark (3)) proves (1). (In more detail, the summation of the resulting geometric series shows that $\deg q_{n-1} = (d_1 + \cdots + d_k)(q^{(n-j)/k} - 1)/(q-1) + (d_1 + \cdots + d_j)\, q^{(n-j)/k} - d_1$. Hence $\deg a_n/\deg q_{n-1}$ tends to r_{j+1}, as n (congruent to j modulo k) tends to infinity).

Let $s_i := (q-1)\, r_i + 1 = ((d_1 + \cdots + d_i)\, q + d_{i+1} + \cdots + d_k)/((d_1 + \cdots + d_{i-1})\, q + d_i + \cdots + d_k)$. Then $1 < s_i < q$ and the telescoping product $s_1 \cdots s_k$ is q, so that $MAX(s_i) \geqslant q^{1/k}$. Now given any rational $u_1, ..., u_{k-1}$, with $1 < u_i < q$ and $q > u_1 \cdots u_{k-1}$ (so that $q > u_k := q/(u_1 \cdots u_{k-1}) > 1$), it is easy to solve for l_i (with $l_1 = 1$) inductively for the system $u_i = ((l_1 + l_2 + \cdots + l_i)\, q + l_{i+1} + \cdots + l_k)/((l_1 + l_2 + \cdots + l_{i-1})\, q + l_i + \cdots + l_k)$, to obtain positive rational solutions $l_i = u_1 \cdots u_{i-1}(u_i - 1)/(u_1 - 1)$, for $2 \leqslant i < k$ and $l_k = (q - u_1 \cdots u_{k-1})/(u_1 - 1)$. (One can directly verify this solution also: The direct substitution of the claimed solution in the right hand side of the system leads to the numerator and denominator (after cancelling the common denominator $u_1 - 1$) which telescope to $qu_1 \cdots u_i - u_1 \cdots u_i$ and $qu_1 \cdots u_{i-1} - u_1 \cdots u_{i-1}$, respectively, giving the ratio u_i as claimed.) Hence with d_1 to be the least common multiple of the denominators of l_i and with $d_i = d_1 l_i$, we get $u_i = s_i$. This proves (2). (In fact, we have shown that all rational limiting behaviours, constrained only by the inequalities above, for

$-\log|\alpha - P/Q|/\log|Q|$, for the continued fraction truncation approximations, can be prescribed by a suitable choice of d_i's.) ∎

Remarks. (1) We will not address the question of exact degree of α, as it is easily addressed for explicit families and also for Voloch's [V1] examples. If $q+1=3$, α has degree 3, because the partial quotient sequence is unbounded. (So the question of the rational numbers in the range of exponents is fully resolved in this case.) More generally, if we arrange α with $E(\alpha) > q$, then the analogue of the Liouville theorem shows that $d(\alpha) = q+1$.

(2) Every individual principle involved in the proof above has appeared in some form or other in the following references. So the only novelty may be starting with continued fractions and obtaining the equations rather than the other way around and putting them all together in a transparent form to obtain many families of examples and analyzing the exponents.

(3) Ironically, in the original submission of this paper, even though the family and the proof till this point were the same, I made a mistake in the "straight calculation" mentioned in the proof, when $k > 2$, leading to

$$MAX\left(\frac{d_1}{d_1 + \cdots + d_k}, \frac{d_k}{(d_1 + \cdots + d_{k-1})\,q + d_k}, \frac{d_{k-1} + d_k}{(d_1 + \cdots + d_{k-2})\,q + d_{k-1} + d_k}, \ldots, \frac{d_2 + \cdots + d_k}{d_1 q + d_2 + \cdots + d_k}\right)$$

rather than $MAX(r_i)$ as in the theorem. This led to wrong, weaker conclusions (this maximization reduces to the case $k=2$), when $k > 2$. Only later when I received the manuscript [Sc2] of Wolfgang Schmidt, who derived the same family of examples independently, did I discover this mistake. So, for $k > 2$, the first correct analysis of the exponent is due to Schmidt. We provide a different proof of part (2), which is his Lemma 11.

We now give an additional interesting family of examples, with continued fractions, for any rational exponent in a certain range approaching the exponent 2 sometimes.

The following lemma appears in [PS], [S] and [MF]. See also [T1–T3] where we use the same techniques for obtaining continued fractions with similar patterns for transcendental numbers related to e, rather than for the algebraic quantities given below.

LEMMA 1. *Let* $[a_0, a_1, \ldots, a_n] = p_n/q_n$, *with the usual notation of continued fractions; then* $[a_0, \ldots, a_n, y, -a_n, \ldots, -a_1] = p_n/q_n + (-1)^n/yq_n^2$.

We will use the short form $[a_0, ..., a_n, x, \leftarrow, y, ...]$ for $[a_0, ..., x, -a_n, ..., -a_1, y, ...]$.

Let $0 < m_1 < m_2 < \cdots < m_k$ be integers such that $m_i/m_{i-1} > 2$ for $2 \leqslant i \leqslant k$ and $qm_1/m_k > 2$. Let q be a power of the cardinality of F. Let $f_i \in F^*$, for $1 \leqslant i \leqslant k$. Put $w_i := -f_i/f_{i-1}^2$ for $2 \leqslant i \leqslant k$ and $w_1 = -f_1/f_k^2$. Put

$$\begin{aligned}\theta := [0, f_1 t^{m_1}, w_2 t^{m_2 - 2m_1}, \leftarrow, w_3 t^{m_3 - 2m_2}, \leftarrow, ..., \\ w_k t^{m_k - 2m_{k-1}}, \leftarrow, w_1 t^{qm_1 - 2m_k}, \leftarrow, \\ w_2 t^{(m_2 - 2m_1)q}, \leftarrow, w_3 t^{(m_3 - 2m_2)q}, \leftarrow, ..., \\ w_k t^{(m_k - 2m_{k-1})q}, \leftarrow, w_1 t^{(qm_1 - 2m_k)q}, \leftarrow, \\ w_2 t^{(m_2 - 2m_1)q^2}, ..., ...].\end{aligned}$$

THEOREM 2. (1) *We have* $\theta = \sum_{j=0}^{\infty} (\sum_{i=1}^{k} 1/f_i t^{m_i})^{q^j}$, *so that* $\theta^q - \theta = \sum_{i=1}^{k} 1/f_i t^{m_i}$.

(2) $E(\theta) = MAX(m_2/m_1, ..., m_k/m_{k-1}, qm_1/m_k)$.

(3) *If* $q > 2^k$, *then given any rational number* μ *between* $q/2^{k-1}$ *and* $q^{1/k}$, *we can find* θ *as above with* $E(\theta) = \mu$. *In particular, rational* μ *arbitrarily close to and greater than 2 can be realised as the exponent for some* θ *for suitable arbitrarily large* q.

Proof. To prove the first part of (1), we inductively apply the lemma to the successive truncations of the continued fraction after each $\leftarrow$, with each new truncation giving the next term of the claimed series expansion: Since any "block followed by a term followed by the reversed block" contains an odd number of terms, the n from the lemma is always odd, so that the new term has to match with $-1/yq_n^2$. Now the sign of q_n, which is the sign of $a_1, ..., a_n$, matters only up to ± 1 (as it is squared) to get y, so that the number of negations in the reversed block does not matter. Hence we see that the sign as well as the degree is chosen appropriately for the application of the lemma. The second part of (1) follows from the series by telescoping cancellation.

Part (2) then follows as in Theorem 1, by straightforward counting of degrees of q_n and a_i's: In fact, as you go through the sequence of the partial quotients, the maximum degree contribution for the exponent can only occur at the new large degree terms introduced at each stage, namely at $a_n = w_i t^{(m_i - 2m_{i-1}) q^j}$ or at $a_n = w_1 t^{(qm_1 - 2m_k) q^j}$, where the corresponding q_{n-1} has degree $m_{i-1} q^j$ or $m_k q^j$, respectively. Hence $2 + \deg a_n / \deg q_{n-1}$ is m_i/m_{i-1} or qm_1/m_k as claimed. This calculation can also be seen from a comparison of the denominator and error in the truncation of the series in (1).

Finally, the observation that all rationals inside MAX are greater than 2 and their product is q (and are otherwise arbitrary) gives the first part of (3). For the second part one has to just take q close to 2^k for large k, so that both the lower and upper bounds for μ, given in the first part, are as close to 2 as we wish. ∎

Remarks. (1) When θ also satisfies Voloch's condition $qm_1/m_k \geqslant \sqrt{q}+1$ in [V1] (or even $qm_1/m_k \geqslant \sqrt{q}$), then we get another proof of Voloch's determination of the exponent.

(2) Osgood [O2] proved that the Liouville/Mahler bound can be improved to (even effectively) $E(\beta) \leqslant \lfloor (d(\beta)+3)/2 \rfloor$ (or rather $\lfloor d(\beta)/2 \rfloor + 1$; see [Sc1] or [LdM]) for β not satisfying the generalized Riccati differential equation $d\beta/dt = a\beta^2 + b\beta + c$, with $a, b, c \in F(t)$. All our and most of the known (see [BR] and [L] for exceptions) examples β, whose continued fraction is known, do satisfy the Riccati equation and indeed β is an integral linear fractional transformation (of determinant ± 1 for the first family) of β^q. See [V1, V2, LdM, dM1, dM2] for more on such elements, e.g., [dM1] shows that the exponent of such a β is rational. For our family, many theorems of these papers can be proved directly from the continued fractions. What is the range of exponents for β not of this form? In [V2], Voloch makes an important observation that the Riccati condition is equivalent to the vanishing of the Kodaira–Spencer class of projective line minus conjugates of β. Hence, it may not be unreasonable to speculate that it might be possible to successively improve on Osgood's bound, if we eliminate some further classes of differential equations coming from the condition that the corresponding Kodaira–Spencer map have rank not more than some integer. It should be also noted that even though the Kodaira–Spencer connection holds in characteristic zero, the analogue of Roth's theorem holds in the complex function field case. Voloch [V1] conjectured, and it was proved in [LdM], that the Osgood bound still holds by eliminating only a subclass given by the "Frobenius" equation $\beta^q = (a\beta + b)/(c\beta + d)$; he further thinks that this might be the best one can get, and that, similarly, the differential equations hierarchy suggested above might have some corresponding, more refined hierarchy. I thank Felipe Voloch for helpful discussions on this remark.

(3) In retrospect, the case $k=1$ of our families can also be deduced from Theorem 6 and Corollary 7 of [BS1] and Theorem C of [L], respectively (where the exponent is the degree, and [BS1] restricts further to characteristic 2): One must simply replace their x by $A_1(t)$ and $1/T$ by $1/f_1 t^{m_1}$, respectively. In fact, in the latter case we can replace $1/T$ by $1/f(t)$, for a non-constant $f \in F[t]$.

All our examples so far have unbounded sequences of partial quotients. To obtain bounded sequence families, we let our family degenerate by choosing all the ratios to be 2. This forces us into characteristic 2, with $q=2^k$, and the equations are $\theta^q-\theta=\sum_{j=0}^{\infty}(\sum_{i=1}^{k} 1/f_i t^{2^i})^{q^j}$, without loss of generality, where of course, t can be replaced by a non-constant $f(t)\in F[t]$ as usual. Often θ is non-quadratic. The continued fraction $[0, f_1 t, w_2, \leftarrow, w_3, \leftarrow, ...]$, while still valid, is no longer proper, because some of the partial quotients are constants. But it is an easy manipulation to obtain the usual continued fraction expansion as follows:

In characteristic 2, we have the identity $[x, f, y]=[x+1/f, f^2y+f]$, which together with $f[a, b, c, ...]=[af, b/f, cf, ...]$, mentioned above, can be used iteratively with $f\in F^*$ to turn our improper continued fraction into a proper one. All the partial quotients are then of degree one and can be easily described inductively. For example,

$$[x, f, y, f', z, ...]=[x+1/f, f^2y+f, f'/f^2, f^2z, ...]$$

$$=[x+1/f, f^2y+f+f^2/f', (f'^2z+f')/f^2, ...].$$

Remarks. (1) One can obtain additional explicit families (with unbounded partial quotients) in other characteristics by choosing some ratios to be 2 and some to be greater than 2, and applying the techniques above.

(2) It is known that $\sum f_n/t^{p^n}$ is algebraic over $F(t)$ if and only if the sequence f_n is ultimately periodic. (This is a direct corollary of Christol's automata criterion for algebraicity, as was pointed out to me by J.-P. Allouche.) If we adjust such an algebraic power series by adding an appropriate rational function of t to make it a purely periodic power series in $1/t$, then the theorem above applies (directly for $q>2$ and by the above manipulation for $q=2$), with q being p raised to the period, yielding its continued fraction expansion.

(3) For $e=\sum 1/d_i$ (where $d_0=1$, $d_i=(t^{q^i}-t)\, d_{i-1}^q$), the analogue (see, e.g., [T1] for its properties) of the usual e for $F_q[t]$, approximations by truncations of the series show that $E(e)\geqslant q$. Application of Proposition 5 of [V1] to this case easily implies that $E(e)=q$, if $q>2$. We note that direct calculations as above from the continued fractions for it established in [T1] show that $E(e)=q$ for all q. This e, however, is known to be transcendental.

Finally, we take this opportunity to point out the sign errors in [T2], [T3], when $q>2$: the formula for w_i on page 253 of [T2] and the same formula quoted in Theorem 1 (where the notation is t_n) must be multiplied by an appropriate element of F_q^*, by taking into account the sign of the denominator of the initial convergent.

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