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Hypergeometric Functions for Function Fields

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We introduce and study analogues of hypergeometric functions in the setting of function fields over finite fields. We show analogues of the differential equations, integral representations, transformation formulae, and continued fractions and show how analogues of various special functions and orthogonal polynomials occur as their specializations. There are two analogues: one with characteristic zero domain and one with characteristic p domain.

INTRODUCTION

In a series of original papers (e.g., [C1, C2, C3, C4, C5]) Carlitz introduced analogues for $\mathbf{F}_q[t]$ of the classical exponential, logarithm, and Bessel functions, factorial and zeta values, Bernoulli numbers and polynomials, cyclotomic polynomials, some orthogonal polynomials, binomial coefficients, and $2\pi i$.

Our aim in this paper is to introduce and study analogues of hypergeometric functions for $\mathbf{F}_q[t]$ and for general function fields over finite fields. As there are multiple analogies between the number fields and function fields, various naive candidates can be introduced using the ingredients defined by Carlitz and they may satisfy some analogues of the properties of the classical or the basic q-hypergeometric functions, but not others. In fact, we define below two candidates, one with characteristic zero domain and one with characteristic p domain. We will see that analogues of several crucial properties are shared between them, giving us some confidence that these are good analogues. Also, there are connections with

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tensor products (3.8) and solitons (5.4), concepts which have proved to be very fruitful in the function field arithmetic. The appearance of two distinct analogues is very much in the same spirit as for the gamma functions for the function fields (see [T4]). In fact, even those gamma functions make their appearance here in quite unexpected fashion.

Out of the many equivalent ways to look at a particular classical concept, the way which translates well in the function field case may be thought of as a more fundamental one. Hence the analogues help to sharpen our understanding of the classical concepts many times.

The plan of the paper is as follows: First, we introduce some notation and some analogies, then we define an analogue of the hypergeometric function and prove several properties for it. Next, we study another analogue. For the most part, we restrict ourselves to the case of $\mathbf{F}_q[t]$, for simplicity as well as for the fact that the analogies are the strongest in this case. Finally, we indicate how the theory generalizes in the setting of the general function fields. We hope to give a detailed treatment of this as well as some other aspects such as more complete treatment of the special functions in the near future. For the classical material, we refer to and use the notation of [S1]. We use the same notation for the analogues defined below, but the context will make clear what we are referring to.

1. The Ingredients

1.1. We will start by defining several objects and then proceed to explain their significance. Let $\mathbf{F}_q$ be a finite field of characteristic p consisting of q elements. Let $A := \mathbf{F}_q[t]$, $K := \mathbf{F}_q(t)$, $K_\infty := \mathbf{F}_q((t^{-1}))$, and Ω be the completion of an algebraic closure of K_∞. Then A, K, K_∞, Ω are well-known analogues of $\mathbf{Z}$, $\mathbf{Q}$, R, C, respectively. Next, we let

$$[n] := t^{q^n} - t, \qquad n \in \mathbf{Z} \tag{1}$$

$$d_0 := l_0 := 1, \qquad d_n := [n]d_{n-1}^q, \qquad l_n := -[n]l_{n-1},\ n \geq 1 \tag{2}$$

$$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i}, \qquad l(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{l_i} \tag{3}$$

$$\left\{ \begin{matrix} z \\ q^i \end{matrix} \right\} := \sum_{k=0}^{i} \frac{z^{q^k}}{d_k l_{i-k}^{q^k}} \tag{4}$$

$$C_a(z) := \sum_{i=0}^{\deg a} \left\{ \begin{matrix} a \\ q^i \end{matrix} \right\} z^{q^i}, \qquad a \in A \tag{5}$$

$$e_i(z) := \prod_{\substack{a \in A \\ \deg a < i}} (z - a). \tag{6}$$

1.2. Carlitz [C1] defined these quantities (with different normalizations of signs and slightly different notation and not necessarily in the same way as we have defined them) and proved the following results. (See the references for the details and justifications of analogies mentioned).

(1) d_n (respectively $(-1)^n l_n$) is the product (respectively the least common multiple) of the monic polynomials of A of degree n. Note that if $n \geq 1$, then $[n]$ is the product of the monic irreducible polynomials of A of degree dividing n. Also d_i is a good analogue [T2, T4] of the factorial of q^i.

(2) The functions (we ignore the information about the convergence, etc. in this summary) e and l are good analogues [T2, T3] of the classical exponential and logarithm functions, respectively. The quantities $\{{z \atop q^i}\}$ and $C_a(z)$ are analogues of the binomial coefficients [T0] and the cyclotomic polynomials [C3] $(1+z)^n - 1$, respectively. This C_a is the Carlitz module, which is the simplest Drinfeld module. In the references, the reader can find more on this subject.

(3) Carlitz then proved the following relations:

$$\left\{ {z \atop q^i} \right\} = \frac{e_i(z)}{d_i} \tag{7}$$

$$e(az) = C_a(e(z)),\ al(z) = l(C_a(z)),\ C_{ab}(z) = C_a(C_b(z)),\ a, b, \in A. \tag{8}$$

1.3. Let us point out another connection between d_i's and l_i's: Let $\bar{e}(z) := \sum z^{q^{-i}}/d_i^{q^{-i}}$, $\bar{l}(z) := \sum z^{q^{-i}}/l_i^{q^{-i}}$, and $\overline{C_a}(z) := \sum \{{z \atop q^i}\}^{q^{-i}} z^{q^{-i}}$. Then it is easy to see that

$$\bar{l}(az) = \overline{C_a}(\bar{l}(z)), \quad a\bar{e}(z) = \bar{e}(\overline{C_a}(z)), \quad \overline{C_{ab}}(z) = \overline{C_a}(\overline{C_b}(z)), \quad a, b, \in A. \tag{9}$$

Remark. Drinfeld's concept of Shtuka is flexible enough to accommodate qth root map as well as qth power map, hence the definitions and the equations fit in its framework. See also 6.1. Also note that $\bar{e}$ appears in [C5]. It turns out that $\bar{e}$, $\bar{l}$, *and* $\overline{C_a}$ are the instances of the adjoints of Ore [O] (see [G] also), for e, l, and C_a. The equations above connecting the exponential for the adjoint to the adjoint of the logarithm and showing the multiplicative property of adjoints also follow from the basic formalism of adjoints.

2. The First Analogue

2.1. Let n be a nonnegative integer and let $a \in \mathbf{Z}$. We define $(a)_n$ (which should really be $(a)_{q^n}$) as follows:

$$(a)_n := \begin{cases} d_{n+a-1}^{q^{-(a-1)}} & \text{if } a \geq 1 \\ 1/l_{-a-n}^{q^n} & \text{if } n \leq -a \geq 0 \\ 0 & \text{if } n > -a \geq 0. \end{cases} \tag{10}$$

We have $(1)_n = d_n$ as well as

$$(a)_{n+1} = [n + a]^{q^{-(a-1)}} (a)_n^q \tag{11}$$

$$(a)_n = (a + 1)_{n-1}^q \qquad \text{for } a \neq 0 \tag{12}$$

$$(a + 1)_n = [n + a]^{q^{-a}} (a)_n \qquad \text{for } a \neq 0. \tag{13}$$

For a_i, $b_i \in Z$, for which it makes sense (see 2.2), we define

$$_rF_s := {}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n \, d_n} z^{q^n}. \tag{14}$$

2.2. When $b_j > 0$, the terms are well-defined and when $a_i \leq 0$ the series terminates, just as in the classical case. When a_i, $b_j > 0$, the degree of the coefficient of z^{q^n} is easily seen to be $nq^n(r - s - 1) + q^n(\Sigma(a_i - 1) - \Sigma(b_j - 1))$. So the series then converges for $z = 0$ only, for all z, or for z with $\deg z < \Sigma(b_j - 1) - \Sigma(a_i - 1)$ according to whether $r > s + 1$, $r < s + 1$, or (the balanced case) $r = s + 1$, respectively. This division into the three cases is parallel to the classical case. For example, Gauss series $_2F_1$ is balanced, and if, e.g., $a + b = c + 1$, then it converges for $\deg z < 0$. Below, when we state some formulae, we will leave it to the reader to figure out the domain of validity.

2.3. Now we explain some motivation, analogies, and the splicing at $a = 0$ of $(a)_n$ we have used. Classically, $(a)_n = a(a + 1) \cdots (a + n - 1) = (a + n - 1)!/(a - 1)!$ With the analogue of the factorial defined by Carlitz [C2, T2], the right hand side should be replaced by d_{n+a-1}/d_{a-1} if we think of $a + n - 1$ and $a - 1$ taking the place of q^{a+n-1} and q^{a-1}, respectively. Here, some extra twistings of q-powers come up in shifting the factorial and also we do not divide to get linear term normalized to be z. This is done for simplicity and to get good specializations (see 3.5). If we do normalize the linear term to be z, we get the radius of convergence to be one in the balanced case, parallel to the classical case. (This will be explored more in a future paper.) This is related to the fact that in the differential equation analogue given below d_F defined below is linear only with scalars in $\mathbf{F}_q$, unlike d/dz which is linear. The factorial function of Carlitz mentioned above has been interpolated from $\mathbf{N}$ to $\mathbf{Z}_p$ by Goss and is a good analogue of the Euler factorial function (see [T2]), but it does not have poles, unlike its classical counterpart.

For our purposes, it seems that an appropriate analogy is the following: Think of d_n, which is the Carlitz factorial at q^n, as (some other) factorial at n and try to use the functional equation (2) to extend the definition for negative n. We immediately run into problems of dividing by zero, which we interpret to be poles. Then, we do have poles at negative integers, analogous to the classical situation. The same problem is encountered in extending $(a)_n$ to nonpositive a's. We then renormalize or reinterpret $(a)_n$ by picking "residue" or in other words specifying $(0)_0 = 1$ and then continue using the relations (11)–(13) from that point onwards. This explains the exceptions in (11)–(13). The process is justified, a posteriori, by the results in the next section proving analogues of various classical properties, and it also fits with the classical normalizations $(1)_0 = (0)_0 = 1$ coming from $(a)_{-n} = (-1)^n/(1 - a)_n$. Another option of just keeping the poles is discussed in 3.5.

3. Properties of the First Analogue

3.1. *Differential Equation.* We want to show that ${}_rF_s$ satisfies an analogue of the Gauss differential equation. The convenient form for the differential equation turns out to be the product form (see 1.2.5 of [S1]). We begin by explaining analogues of d/dz and zd/dz in our case (see 1.2 of [T2] for more on this).

For $a \in \mathbf{Z}$, consider the operations (Δ and $\Delta^{(w)}$ appear in [C5])

$$\Delta_a g(z) := g(tz) - t^{q^{-a}} g(z), \qquad \Delta := \Delta_0,\ \Delta^{(w)} := \Delta_0 \cdots \Delta_{1-w} \tag{15}$$

$$d_F := \Delta^{1/q}, \qquad d_F^{(w)} := (\Delta^{(w)})^{1/q^w}. \tag{16}$$

Analogies are as follows: We consider series of the form $\sum a_i z^{q^i}$ instead of $\sum a_i z^i$ in the classical case. Hence we consider the "qth power" operator, which takes z^{q^i} to $z^{q^{i+1}}$, as analogous to the "multiplication by z" operator in the classical case. We consider Δ_a to be an analogue of $zd/dz + a$, d_F as an analogue of d/dz, and $d_F^{(w)}$ as an analogue of d^w/dz^w because of the analogies explained in 1.2 of [T2] (e.g., Δ is a linear derivation on linear functions, multiplication operation being composition and $d_F e(z) = e(z)$ parallel to the classical $d/dz(e^z) = e^z$), as well as

$$\left(z\frac{d}{dz} + a\right) z^n = (n + a)z^n \Leftrightarrow \Delta_a z^{q^n} = [n + a]^{q^{-a}} z^{q^n} \tag{17}$$

$$\left(z\frac{d}{dz}\right) \cdots \left(z\frac{d}{dz} - w + 1\right) = z^w \frac{d^w}{dz^w} \Leftrightarrow \Delta^{(w)} = (d_F^{(w)})^{q^w}. \tag{18}$$

Also note that $\Delta_a g(z) = (\Delta g(z)^{q^a})^{q^{-a}}$ is an analogue of classical $(zd/dz + a)g(z) = 1/z^a(zd/dz)(z^a g(z))$.

The analogue of the Gauss differential equation is

$$\prod \Delta_{a_i}\, {}_rF_s = d_F \prod \Delta_{b_j - 1}\, {}_rF_s \qquad (a_i, b_j, b_j - 1 \neq 0). \tag{19}$$

Proof of (19). By (2), (12), and (15), if a_i, $b_j \neq 0$, we have

$$\begin{aligned}\Delta_r F_s &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{t^{q^n} - t}{d_n} z^{q^n} \\ &= \sum_{n=1}^{\infty} \frac{(a_1 + 1)_{n-1}^q \cdots (a_r + 1)_{n-1}^q}{(b_1 + 1)_{n-1}^q \cdots (b_s + 1)_{n-1}^q} \left(\frac{z^{q^{n-1}}}{d_{n-1}}\right)^q \\ &= {}_rF_s(a_1 + 1, \ldots, a_r + 1; b_1 + 1, \ldots, b_s + 1; z)^q\end{aligned}$$

Hence, by (16), we have

$$d_F {}_rF_s(\{a_i\}; \{b_j\}; z) = {}_rF_s(\{a_i + 1\}; \{b_j + 1\}; z). \tag{20}$$

On the other hand, if $a_i \neq 0$, by (13) and (15), we have

$$\Delta_{a_i} {}_rF_s = {}_rF_s(a_1, \ldots, a_i + 1, \ldots, a_r; \{b_j\}; z) \tag{21}$$

and if $b_j \neq 1$, by (13) and (15), we have

$$\Delta_{b_j - 1} {}_rF_s = {}_rF_s(\{a_i\}; b_1, \ldots, b_j - 1, \ldots, b_s; z). \tag{22}$$

Hence (19) follows. ■

It is straightfoward to verify another solution $({}_2F_1(a + 1 - c, b + 1 - c; 2 - c; z)^{q^{1-c}}$ parallel to the classical $z^{1-c}{}_2F_1(a + 1 - c, b + 1 - c; 2 - c; z)$ in addition to ${}_2F_1(a, b; c; z)$ in the case of Gauss ${}_2F_1$ case. More extensive treatment of the solutions from the differential equation point of view will be given in another paper.

3.2. *Ratios of Terms.* The ratios c_{n+1}/c_n of consecutive terms of ${}_rF_s$ are rational functions of n, q^n, respectively, in the classical and q-analogue cases, respectively. In our case, c_{n+1}/c_n^q is a q-power power of a rational function of t^{q^n} by (11).

3.3. *Linear Relations between Contiguous Functions.* We see another interesting justification of the analogies we have described in the linear relations between the contiguous (those with all parameters same, except for one pair, which differs by one) hypergeometric functions. See [S1, pp. 13, 14] for the fifteen such relations given by Gauss. We choose the following two for illustration: $(b - a){}_2F_1 = b{}_2F_1(b + 1) - a{}_2F_1(a + 1)$ and $(1 - z){}_2F_1 + (c - b)c^{-1}z{}_2F_1(c + 1) - {}_2F_1(a - 1) = 0$, which is linear in z. An analogue, easy to verify term-wise, of the first one is

$$[b - a]^{q^{-b}}\, {}_2F_1 = {}_2F_1(b + 1) - {}_2F_1(a + 1). \tag{23}$$

An analogue, easy to verify term-wise, of the second one is

$$ {}_2F_1 - {}_2F_1^q + [c-b]^{q^{-c+1}} {}_2F_1(c+1)^q + [-a+1] {}_2F_1(a-1) = 0. \quad (24) $$

Recalling that multiplication by z corresponds to raising to the qth power, we can write this more suggestively to show that it is "linear in Z" (the symbol Z denoting the raising to the qth power operator) as follows:

$$ (1-Z) {}_2F_1 + [c-b]^{q^{-c+1}} Z {}_2F_1(c+1) + [-a+1] {}_2F_1(a-1) = 0. \quad (25) $$

3.4. *Integral Representation.* Our analogue is a formal reinterpretation of Barnes contour integral formula [S1, p. 24], rather than the Euler formula and is in the spirit of the integral formula for the gamma function given in 1.1 of [T2]. The proof of the Barnes formula consists of calculating the contour integral by the residue theorem. Since $\Gamma(-s)$ has poles at $s = n$, for nonnegative integers n, with residues $(-1)^{n+1}/n!$, the sum of residues of $\Gamma(a+s)\Gamma(b+s)\Gamma(-s)(-1)^s z^s/\Gamma(c+s)$ turns out to be $-{}_2F_1$ except for the scalar multiple normalization factor consisting of $\Gamma(a)$'s (which we avoid: see 2.3).

In our case, the role of $\Gamma(a+n)/\Gamma(a)$ is played by $(a)_n$ as described before. We now show how to find an analogue of the $\Gamma(-s)$ occurring in the Barnes formula having poles at $s = n$ with residues $(-1)^{n+1}/d_n$. This then implies, as in the classical case, that the sum of residues of $(a)_s(b)_s(1)_{-s-1}(-1)^s z^{q^s}/(c)_s$ is $-{}_2F_1$:

First, as explained in 2.3, in our case, the role of $\Gamma(-n)$ is played by $d_{-n-1} = (1)_{-n-1}$ for nonpositive n. We want to make sense of this even for positive n.

In the classical case, we have similarly $n! = (1)_n$ for nonnegative n and we define $(a)_{-n} := (-1)^n/(1-a)_n$, thus enabling us, for example, to develop bilateral series. This would mean that $(1)_{-n-1}$ should be formally considered as $(-1)^{n+1}/(0)_{n+1}$, which can be thought of as having a pole with residue $(-1)^{n+1}/n!$, because $(0)_{n+1} = 0.1.2. \cdots n$.

In our case, on the other hand, repeated application of (11) gives $(0)_{n+1} = 0.d_n^q$. Since d_n is an analogue of the factorial, a comparison with the classical case described above suggests that we consider $(1)_{-n-1}$ rather as $(-1)^{n+1}/(0)_{n+1}^{q^{-1}}$, which then can be thought of as having a pole with the residue as claimed. (If we think of $(a)_n$ as a replacement for $\Gamma(a+n)$ instead, and take $(0)_{-n}$ for $\Gamma(-n)$, the classical transformation $(0)_{-n} = (-1)^n/(1)_n$ gives the same value $(-1)^n/d_n$, but it is not clear why one should think of this as a residue.) Using ideas of 1.3, we have recently defined analogues of $(a)_{-n}$ and of the bilateral series, which will be described in another paper.

3.5. *Specializations.* Analogues of many classical special functions defined by Carlitz [C5] are specializations of the hypergeometric functions defined above. (Here we note that there are many sign differences with the references, because of the different normalizations.) We get the Carlitz exponential by default:

$$e(z) = {}_0F_0(-;\ -;\ z). \tag{26}$$

We get the binomial coefficients as

$$\begin{Bmatrix} z \\ q^m \end{Bmatrix} = {}_2F_1(-m,\ k;\ k;\ z) = {}_1F_0(-m;\ -;\ z). \tag{27}$$

The connection with the analogues of the Bessel functions of the first and second kind is the following. For $k \geq 1$,

$$J_{-(k-1)}(z) = J_{k-1}^{q^{-(k-1)}}(z) = Y_{k-1}^{q^{-(k-1)}}(z) = {}_0F_1(-;\ k;\ z). \tag{28}$$

We note here that our J here is a Carlitz modified Bessel function I and the reason is different normalization of signs: More precisely, isomorphism by conjugation of the $(q-1)$th root of -1 (analogue of i, which occurs classically connecting the two) switches our $C_t = t + F$ to $t - F$, J to I.

The Jacobi, Legendre polynomials analogues come via

$$(P_m^{(n,k)})^{q^{-n-k}}(z)/d_m = {}_2F_1(n+k+1,\ -m;\ k+1;\ z) \tag{29}$$

$$(M_m^{(n)}(z))^{q^{-n}}/d_m = {}_2F_0(n+1,\ -m;\ -;\ z). \tag{30}$$

Remark. We note that if we had defined the hypergeometric functions without splicing and with $(a)_n$ considered as a pole for negative a, i.e., with $1/(a)_n = 0$, then we could have avoided powers in some of the expressions, but some other expressions would have been impossible. Using coefficients of $\tilde{e}(z)$ of 1.3 rather than $e(z)$ one can consider variations of the hypergeometric functions, whose specialization would be the analogue of Laguerre polynomials $L_m^{(n)}$. In [C5], Carlitz extends the definition of $[n]$'s to some fractional n's, introducing another variant worth studying and allowing many more interesting specializations. We hope to come back to these issues in a future paper.

3.6. *Continued Fraction.* In [T3], we obtained a continued fraction for $e(z)$, with a nice (but quite different from the classical analogue) pattern. Here we just point out that by (11) and the remark (1) of p. 154 of [T3], we get similar continued fractions for the hypergeometric analogues. These are especially simpler for $q = 2$, with the partial quotients involving the products of (q-power powers of) $[n]$'s compared to the products of n's appearing in the classical result of Gauss [S1, p. 15].

At the suggestion of the referee, we provide an example of ${}_2F_1$, when $q = 2$. Put $r_{-1} := (d_{c-1}^{q^{1-c}} z^{-1})/(d_{a-1}^{q^{1-a}} d_{b-1}^{q^{1-b}})$ and for $i \geq 0$, put $r_i := [c+i]^{q^{1-c}}/([a+i)^{q^{1-a}}[b+i]^{q^{1-b}})$. Then we have the following simple continued fraction with the doubling (see [T3]) pattern:

$$ {}_2F_1(a, b; c, z) = [0, r_{-1}, r_0, r_{-1}, r_1, r_{-1}, r_0, r_{-1}, r_2, \cdots]. $$

We note here that usual transformations allow us to write a generalized continued fraction and also that if there are any fractional powers of $[n]$'s are involved, we can get rid of them by raising the hypergeometric function to a suitable q^kth power.

3.7. *Summation Formula.* The usual trick [S1, p. 27] of putting $z = 1$ to cancel terms in the linear relations does not work. Instead, we proceed as follows: Let us write

$$ \sum_{j=0}^{\mu} f_j(t)T^j := (T - t^{q^{-(k+\mu-1)}}) \cdots (T - t^{q^{-k}}). $$

Then using (2) and (4), we see that for $m, k, \mu > 0$ we have

$$ \begin{aligned} {}_2F_1(-m, k+\mu; k; 1) &= \sum_{r=0}^{m} \frac{d_{r+k+\mu-1}^{q^{-(k+\mu-1)}}}{d_{r+k-1}^{q^{-(k-1)}} l_{m-r}^{q^r} d_r} \\ &= \sum_{r=0}^{m} \frac{[r+k+\mu-1]^{q^{-(k+\mu-1)}} \cdots [r+k]^{q^{-k}}}{l_{m-r}^{q^r} d_r} \\ &= \sum_{r=0}^{m} \frac{(t^{q^r} - t^{q^{-(k+\mu-1)}}) \cdots (t^{q^r} - t^{q^{-k}})}{l_{m-r}^{q^r} d_r} \\ &= \sum_{j=0}^{\mu} f_j(t) \left\{ \begin{matrix} t^j \\ q^m \end{matrix} \right\}. \end{aligned} $$

By (6) and (7), the terms are zero for $j < m$. In particular, this vanishes when $\mu < m$, parallel to implication of Vandermonde's evaluation [S1, p. 2] $(-\mu)_m/(k)_m$ of the left hand side in the classical case. When $\mu = m$, we get the value 1. When $\mu = m + 1$, we get the value $-\sum_{i=0}^{m}[-k-2i]^{q^i}$.

3.8. *Connection with Tensor Products.* In [AT, p. 174], the appearance of the Carlitz analogue of the Bessel function $J_0(z)$ in the formula for the exponential for the second tensor power of the Carlitz module was pointed out. We have seen in 3.5 that $J_0(z) = {}_0F_1(-; 1; z)$. In fact, it can be verified by a direct computation, using 2.2.3 of [AT], that the exponential $\exp_m$ of the mth tensor power of the Carlitz module evaluated at the column vector (which has canonical meaning) with the top entry z and the other entries zero is the column vector whose jth entry is ${}_0F_{m-1}(-; 1, \cdots, 1; z)$ if $j = 1$ and ${}_0F_{m-1}(-; 2, \cdots, 2, 1, \cdots, 1; z)^q$ otherwise, where there are $j - 1$ 2's. For example,

$$\exp_4 \begin{pmatrix} z \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} {}_0F_3(-; 1, 1, 1; z) \\ {}_0F_3(-; 2, 1, 1; z)^q \\ {}_0F_3(-; 2, 2, 1; z)^q \\ {}_0F_3(-; 2, 2, 2; z)^q \end{pmatrix}.$$

4. The Second Analogue

4.1. Let n be a nonnegative integer and let $a \in \Omega$. We define $(a)_n$ to be $e_n(a)$. Now, $(t^n)_n = d_n$. (Note that a and n now belong to the rings of different characteristics.)

For $a_i, b_i \in \Omega$ for which it makes sense (see below) we define

$${}_r\mathcal{F}_s := {}_r\mathcal{F}_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n d_n} z^{q^n}. \tag{31}$$

4.2. If $b_j \notin A$, then the terms are well-defined and $a_i \in A$ gives the terminating case. Note that we do not need any sign condition in contrast to the classical case and the first analogue. If $a_i, b_j \notin A$, it is easy to calculate the radius of convergence from the degrees of $e_i(z)$. We then see that the degree of the coefficient of z^{q^n} *is* $nq^n(r - s - 1) - (r - s)q(q^n - 1)/(q - 1)$. So the series then converges for $z = 0$ only, for all z, or for z with $\deg z < q/(q - 1)$ according to whether $r > s + 1$, $r < s + 1$, or (the balanced case) $r = s + 1$, respectively. This division into three cases is parallel to the classical case. In paticular, we see that ${}_2\mathcal{F}_1$ converges if $\deg z < q/(q - 1)$, which is exactly the radius of convergence of the logarithm and in that sense similar to the classical case.

4.3. Now we explain the motivation for the definition. Classically, $(a)_n = (-1)^n n! \binom{-a}{n}$, hence 1.2 (2) and (7) motivate our definition.

5. Properties of the Second Analogue

5.1. *Specializations.* An analogue of the binomial series $(1 + z)^n = \exp(n(\log(1 + z)))$ is $e(al(z))$. In particular, for $a \in A$, we get $C_a(z)$. These are specializations of ${}_2\mathcal{F}_1$ in a manner parallel to the classical case. By (3), (4), and (7), we have

$$_2\mathcal{F}_1(a, b; b; z) = e(al(z)). \tag{32}$$

This provides a formula for the exponential similar to the classical case:

$$\lim_{\deg a\to\infty} {}_2\mathcal{F}_1\left(a, b; b; \frac{z}{a}\right) = e(z). \tag{33}$$

We have, of course, the ready-made specialization $_0\mathcal{F}_0(-; -; z) = e(z)$. Similarly, we have expression for the 0th Bessel function as

$$\lim_{\deg a,b\to\infty} {}_3\mathcal{F}_1\left(a, b, c; c; \frac{z}{ab}\right) = J_0(z). \tag{34}$$

The following expression for the logarithm seems different than the usual classical expression: $_2F_1(1, 1; 2; z) = \log(1 - z)$.

$$\lim_{a\to 0} \frac{_2\mathcal{F}_1(a, b; b; z)}{a} = \lim \sum \frac{e_i(a)}{ad_i} z^{q^i} = \sum \frac{e_i'(0)}{d_i} z^{q^i} = l(z). \tag{35}$$

5.2. *Euler and Kummer Transformations.* Keeping in mind the analogue of the binomial series explained above and the fact that we are dealing with an additive rather than a multiplicative situation, an analogue of the Euler transformation formula $_2F_1(c - a, c - b; c; z) = (1 - z)^{c-a-b}$ $_2F_1(a, b; c; z)$ in our case is

$$_2\mathcal{F}_1(c - a, c - b; c; z) = e((c - a - b)l(z)) + {}_2\mathcal{F}_1(a, b; c; z). \tag{36}$$

Similarly for a closely related analogue of Kummer's theorem for the confluent hypergeometric function $e^{-z}\, {}_1F_1(a; b; z) = {}_1F_1(b - a; b; -z)$ in the classical case, we have the analogue

$$e(-z) + {}_1\mathcal{F}_1(a; b; z) = {}_1\mathcal{F}_1(b - a; b; z). \tag{37}$$

5.3. *Sum Formula.* The fact that we get a sum instead of the product as in the classical case of Euler transformation implies that we get the following sum formulae as analogues of the product formulae [S1, p. 13]: If $u - x - y = 1 + n + v + w - m$, then

$$\begin{aligned} &{}_2\mathcal{F}_1(x, y; u; z) + {}_2\mathcal{F}_1(1 - n - v, 1 - n - w; 1 - n - m; z) \\ &= {}_2\mathcal{F}_1(u - x, u - y; u; z) + {}_2\mathcal{F}_1(v - m, w - m; 1 - n - m; z). \end{aligned} \tag{38}$$

5.4. *Ratios of Terms.* Analogous with 3.2, if c_n denotes the nth term in the definition of ${}_r\mathcal{F}_s$ and if $a_i, b_j \in K - A$, then c_{n+1}/c_n^q is an algebraic function of t^{q^n} by (2) of this paper, (13) of [T5], and Theorem 2 of [A] (see also [T7]). These algebraic functions are related to the solitons of Anderson [A].

At the suggestion of the referee, we give two simple examples (I and II of Section 2 of [T7]): Write $T := t^{q^{n+1}}$. Then we have

$$\frac{\left(\frac{1}{t}\right)_{n+1}}{\left(\frac{1}{t}\right)_n^q} = 1 - \frac{T}{t} \tag{39}$$

$$\frac{\left(\frac{1}{t^2}\right)_{n+1}}{\left(\frac{1}{t^2}\right)_n^q} = 1 - \frac{T^2}{(x(TX + X^q) - X(tx + x^q))^{q-1}}, \tag{40}$$

where X (x respectively) is a primitive T^2 (t^2 respectively) torsion of the Carlitz module. More explicitly, X satisfies the equation $X^{q^2} + (T + T^q)X^q + T^2 X = 0$ and x satisfies the similar equation in lower case letters.

6. General Case

6.1. Drinfeld and Hayes generalized some aspects of the Carlitz theory from the rational function field to any function field. In particular, there are generalizations of $e(z)$, $l(z)$, d_i, and l_i coming naturally from the theory of Drinfeld modules and in particular from the sign-normalized Drinfeld modules of Hayes (see [GHR]). We can hence proceed to define hypergeometric functions in analogous fashion using these generalized ingredients. Many of the properties we have shown do generalize. For example, all the results of the previous section work exactly the same if we use the definition (4) in general, rather than (7), to define ${}_r\mathcal{F}_s$. For the first candidate, the place of $[n]$'s is assumed by the appropriate specializations of Shtukas, see [T6]. (In particular, 0.3.6–0.3.8, 5.8, and 5.9 reflect dual nature of d_i and l_i, coming from Serre duality (in addition to connection with adjoints as in 1.3)). In fact, [T4], [T5], and [T6] show that there are two interesting generalizations of d_i's and the binomial coefficients. The interested reader can refer to these references for some explicit examples also. We hope to address these issues in full in another paper.

6.2. We take this opportunity to correct some errors. In [T1] p. 246, line -14, χ_{j-1} should be replaced by χ_{j+1}. In [T5] p. 196, line 2, $\mathbf{F}_4$ should be replaced by $\mathbf{F}_q$ and in [T5] p. 188, line -7, $g - 1$ should be replaced by $2g - 1$. In [T6] p. 559, line 3, the first $\bar{X}$ should be replaced by X and in 5.1 to 5.3 the additional hypothesis that $\delta = 1$ should be inserted. The second to last statement in (5) of [T4] p. 85 should be deleted.

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It is with great pleasure and respect that we dedicate this paper to L. Carlitz, whose mathematical influence, especially through [C1] and [C5], can be traced throughout this paper. We would also like to pay our respect to the fond memory of Anna, whose influence, though non-mathematical, and blessings were at least as important.

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