TRANSCENDENCE IN POSITIVE CHARACTERISTIC AND SPECIAL VALUES OF HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We prove a simple transcendence criterion suitable for function field arithmetic. We apply it to show the transcendence of special values at non-zero rational arguments (or more generally, at algebraic arguments which generate extension of the rational function field with less than q places at infinity) of the entire hypergeometric functions in the function field (over $\mathbb{F}_q$) context, and to obtain a new proof of the transcendence of special values at non-natural p-adic integers of the Carlitz-Goss gamma function. We also characterize in the balanced case the algebraicity of hypergeometric functions, giving an analog of the result of F. R. Villegas, based on Beukers-Heckman results in the classical hypergeometric case.

0. INTRODUCTION

Number theorists study the number fields and the function fields (over finite fields) together as global fields in view of the strong analogies that exist between them.

Transcendental number theory originated with Liouville's result of 1844 (see [17]) that an algebraic irrational cannot be "well approximated" by rationals. K. Mahler [18] showed in 1949 that Liouville's theorem holds for function fields over any field, and also that the Liouville's inequality is best-possible if the ground field k is of positive characteristic p, by considering $\alpha = \sum t^{-p^i} \in k((1/t))$, which is algebraic of degree p, as well as of the approximation exponent p. In particular, naive analog of Roth's theorem fails. (See [30, Cha. 9] for a survey of general situation which is not even conjecturally fully understood yet).

So we need stronger approximations in function fields than in number fields counterparts to conclude transcendence by Diophantine approximation. On the other hand, for many naturally occurring quantities in the function field arithmetic related to Drinfeld modules and Anderson's t-motives, such as periods, special values of exponential, logarithm, gamma, and zeta functions etc., different techniques that exist (algebraic group techniques, more developed motivic machinery, automata method, strong diophantine approximation criterion of special kind etc.) in function fields have been used to prove very powerful results. See [30, Cha. 10, 11] for a survey of older results and [4, 21, 22, 7] for some recent results.

Many quantities of interest that occur naturally in the function field arithmetic have expansions that are $\mathbb{F}_q$-linear (or closely related to such) so that they are

Date: May 12, 2010.
1991 *Mathematics Subject Classification.* Primary 11J81, 11J93; Secondary 11T55.
Key words and phrases. hypergeometric functions, transcendence, algebraicity, gamma, Drinfeld module.

spread out with only exponents being q-powers. Further, for proof by contradiction assuming algebraicity, we can assume that algebraic equation is also $\mathbb{F}_q$-linear naturally by looking at the $\mathbb{F}_q$-span of the roots as roots of a new equation. This was used successfully by Wade [33, 34, 35] to prove several transcendence results.

The present work is divided into three parts and organized as follows.

In the first part consisting of Sections 1-4, we first record and prove in Section 1 a simple but general transcendence criterion (Theorem 1) which is essentially a quantitative and generalized version of the Wade's modification of the classical method. This is suitable for many naturally occurring quantities of interest in function field arithmetic. Then in the second section, we give an application of Theorem 1 proving transcendence of values of hypergeometric function [25, 30] in the function field arithmetic, in the (non-trivial) entire case, at any non-zero algebraic argument of degree less than q, or more generally, at any non-zero algebraic argument which generates extension of the rational function field that has less than q places at infinity. In this case, it is not yet known whether one can apply the highly successful period techniques of [4, 21, 22, 7], which when they can be used, often give stronger algebraic independence results.

In Section 3, independent of all these transcendence methods and criteria, we characterize directly in the balanced case the algebraicity of these hypergeometric functions, giving an analog of the result of F. R. Villegas [32] which is based on Beukers-Heckman [6] results in the classical hypergeometric case.

In Section 4, we prove a few results on algebraicity of the second analog [25, 30] of hypergeometric functions using the "solitons" of Anderson. (See [4, 30, 27] and especially [29] for the background on the name and references to Anderson's work). These results are partial compared to the complete results for the first kind and we hope to return to this issue in a future paper. We remark here that in the function field arithmetic, even when we deal with $\mathbb{F}_q[T]$, there are typically two analogs of cyclotomic theories, gamma functions, zeta functions, hypergeometric series etc. (see [30]), one parametrized by characteristic zero numbers and the other by finite characteristic numbers.

We now give a brief comparison with the known results in the classical case and refer to Section 2 for comparison with the known results in the function field case. The determination of which hypergeometric functions with rational parameters are algebraic was made by Beukers-Heckman (following Schwartz in the simplest case). There are also many important results eg., by Siegel, Shidlovsky, Wolfart, Cohen, Wustholz, Brownawell, Beukers about the description, finiteness or infinitude of the set of special values which are algebraic when these functions are transcendental. We refer the reader to the excellent surveys (given at Arizona Winter School 2008 available as lecture notes at the website http://swc.math.arizona.edu) by Beukers and Tretkoff for more details and references. In particular, in the entire case (in fact, in much more general setting of E-functions) Beukers [5] gave algebraic independence results improving the results of Siegel and Shidlovsky, and described explicitly the finite set of possible algebraic special values in terms of the "denominators of relevant differential equations". In some sense, the analog [25, 30] of the differential equation in our setting has no "denominators" at the parameters we handle and so naive analog (unproved so far) of Beukers result would imply that their values at non-zero algebraic values should be transcendental. We achieve a weaker transcendence result mentioned above, but with more direct approximation methods.

In the second part (Section 5), we give another application of Theorem 1 by presenting a new proof of the result [20] (originally proved by automata method) proving transcendence of the Carlitz-Goss gamma values at non-natural p-adic integers. Note that the special case giving the transcendence of the values at fractions treated in [26, 3] by automata method has been vastly generalized to determination of full algebraic relations between the values at fractions in [7], but the period method of [4, 21, 7] does not apply to non-fractional p-adic integers.

A preliminary version of Theorem 1 was firstly announced in [41] and many previously known criteria due to Wade, Spencer, de Mathan, Denis, Hellegouarch, Laohakosol *et al.* can be also deduced relatively easily from it by making suitable choices of multipliers that we record. For many of these, although quite different in appearance, the proofs are indeed variants of the original proofs as they are based on the same principles, but are obtained in a uniform way by using Theorem 1. All these are treated and discussed in the third part (Section 6), just as promised in [41]. Roughly, Wade's method is a general technique for proving transcendence, and each criterion in Diophantine approximation is the mathematical reformulation of this technique related to a special type of multipliers.

In more detail, to show by Wade's method the transcendence of a formal power series α, we often proceed as follows. Firstly, suppose by contradiction that α satisfies a non-trivial equation, which can be assumed to be $\mathbb{F}_q$-linear, by considering the equation satisfied by elements in the $\mathbb{F}_q$-span of the roots. The q-th power being a nice operation in characteristic p, where q is a power of p, this largely preserves the arithmetic structure of the coefficients of the original series. Secondly, multiply the equation by an appropriate polynomial (called multiplier), taking into account this structure, to obtain $I+Q=0$, where I is a polynomial different from zero, and Q is a formal power series with negative degrees only. Thus we arrive at a contradiction, exactly as in many classical proofs, where I and Q represent an integer and a proper fraction respectively. So the main point of each criterion mentioned above consists just in choosing a special type of "good" multipliers fit for the problem in hand. Due to the specialty of the chosen multipliers, each criterion cited above has its limitations. For example, the multipliers for Theorem 8 and Theorem 9 are Q_n's, but they are not good for Theorem 7, which needs $L_{2n}^{q^r}/L_n$ as its multiplier. It seems that our Theorem 1 has enough flexibility because it does not impose any stringent condition on the multipliers, and only has one restriction on the approximation's accuracy. Without Theorem 1, it seems hard to show, by any other criterion cited in the work, the transcendence of values of hypergeometric function treated in Theorem 2.

Some results of the present work are announced without proof in the Note [31].

1. A new diophantine criterion

Fix $p \geqslant 2$ a prime number and $q = p^w$ with $w \geqslant 1$ an integer. Let $\mathbb{L}$ be a valued field of characteristic p, endowed with a non-archimedean absolute value $|\cdot|$. Let $\mathbb{A}$ be a subring of $\mathbb{L}$. We denote by $\mathbb{K}$ the fraction field of $\mathbb{A}$. For example, one can take $\mathbb{A} = \mathbb{F}_q[T]$, $\mathbb{K} = \mathbb{F}_q(T)$, $\mathbb{L} = \mathbb{F}_q((T^{-1}))$, and $|\cdot| = |\cdot|_\infty$ the canonical ∞-adic absolute value, where $\mathbb{F}_q$ is a finite field with q elements.

The following criterion improves and generalizes Theorem 1 in [41].

Theorem 1. *Let $\alpha \in \mathbb{L}$. Then α is transcendental over $\mathbb{K}$ if and only if there exists a sequence $(\alpha_n)_{n\geqslant 0}$ in $\mathbb{L}$ satisfying the following two conditions:*

(1) *There exists a sequence $(\delta_n)_{n\geqslant 0}$ of positive real numbers such that*

$$|\alpha - \alpha_n| \leqslant \delta_n$$

for all integers $n \geqslant 0$;

(2) *For all integers $t \geqslant 1$, there exist $t+1$ integers $0 \leqslant \sigma_0 < \cdots < \sigma_t$ such that for every $(t+1)$-tuple $(A_0, \ldots, A_t)$ of not all zero elements in $\mathbb{A}$, there exist an infinite set $S \subseteq \mathbb{N}$ and $t+1$ sequences $\theta_j = (\theta_j(n))_{n\geqslant 0}$ $(0 \leqslant j \leqslant t)$ of positive integers increasing to $+\infty$ such that for all integers j $(0 \leqslant j \leqslant t)$ with $A_j \neq 0$, we have*

$$\lim_{S \ni n \to +\infty} \frac{|\beta_n|}{\delta_{\theta_j(n)}^{q^{\sigma_j}}} = +\infty,$$

where β_n is defined by

$$\beta_n = \sum_{j=0}^{t} A_j \alpha_{\theta_j(n)}^{q^{\sigma_j}}.$$

Remark: Condition (1) is indeed a definition of $(\delta_n)_{n\geqslant 0}$. Only Condition (2) is truly important. It is also worthy to point out that we do not suppose $\alpha_n \in \mathbb{K}$, and in practice we can thus use for example algebraic approximations instead of rational approximations. In most of our applications, often we need only consider the simplest case where $\sigma_j = j$ and $\theta_j(n) = n - j$ $(0 \leqslant j \leqslant t)$.

Proof of Theorem 1.

Assume that α is transcendental over $\mathbb{K}$. For all integers $n \geqslant 0$, set

$$\delta_n = 2^{-n}, \text{ and } \alpha_n = \alpha.$$

Then for every $(t+1)$-tuple $(A_0, \ldots, A_t)$ of not all zero elements in $\mathbb{A}$, we have

$$\beta_n := \sum_{j=0}^{t} A_j \alpha_n^{q^{\sigma_j}} \equiv \sum_{j=0}^{t} A_j \alpha^{q^{\sigma_j}} \neq 0,$$

for α is transcendental. The conclusion comes directly.

Now we show the sufficiency by contradiction. Suppose that α were algebraic of degree t over $\mathbb{K}$. Then the $t+1$ elements $\alpha^{q^{\sigma_0}}, \alpha^{q^{\sigma_1}}, \ldots, \alpha^{q^{\sigma_t}}$ are linearly dependent over $\mathbb{K}$, and thus we can find $A_0, \ldots, A_t$ in $\mathbb{A}$, not all zero, such that

$$\beta := \sum_{j=0}^{t} A_j \alpha^{q^{\sigma_j}} = 0.$$

In the following we shall give two proofs of the sufficiency.

Proof by Diophantine approximation: For all $n \geqslant r$ $(n \in S)$, we have

$$\begin{aligned} |\beta - \beta_n| &= \left| \sum_{j=0}^{t} A_j (\alpha - \alpha_{\theta_j(n)})^{q^{\sigma_j}} \right| \\ &\leqslant \max_{\substack{0 \leqslant j \leqslant t \\ A_j \neq 0}} |A_j| \left| \alpha - \alpha_{\theta_j(n)} \right|^{q^{\sigma_j}} \\ &\leqslant \max_{\substack{0 \leqslant j \leqslant t \\ A_j \neq 0}} |A_j| \, \delta_{\theta_j(n)}^{q^{\sigma_j}} \\ &\leqslant \max_{0 \leqslant j \leqslant t} |A_j| \cdot \max_{\substack{0 \leqslant j \leqslant t \\ A_j \neq 0}} \delta_{\theta_j(n)}^{q^{\sigma_j}}. \end{aligned}$$

But $\beta = 0$, thus by Condition (2) of Theorem 1, we obtain

$$+\infty = \lim_{S \ni n \to +\infty} \frac{|\beta_n|}{\max\limits_{\substack{0 \leqslant j \leqslant t \\ A_j \neq 0}} \delta_{\theta_j(n)}^{q^{\sigma_j}}} \leqslant \max_{0 \leqslant j \leqslant t} |A_j| \,,$$

which is absurd. So α is transcendental over $\mathbb{K}$. □

Proof by Wade's method: For all integers $n \geqslant t$, define

$$\beta_n = \sum_{j=0}^{t} A_j \alpha_{\theta_j(n)}{}^{q^{\sigma_j}}, \text{ and } J_n = \sum_{j=0}^{t} A_j (\alpha - \alpha_{\theta_j(n)})^{q^{\sigma_j}}.$$

So $0 = \beta = \beta_n + J_n$. Thus by virtue of Condition (2) of Theorem 1, for $0 \leqslant j \leqslant t$ with $A_j \neq 0$, we have

$$\lim_{S \ni n \to +\infty} \frac{|\beta_n|}{\delta_{\theta_j(n)}^{q^{\sigma_j}}} = +\infty.$$

In particular $\beta_n \neq 0$ for all sufficiently large $n \in S$. For such an $n \in S$, put

$$F_n = \frac{J_n}{\beta_n}, \text{ and } M_n = \frac{1}{\beta_n}.$$

Then $0 = M_n \beta = 1 + F_n$. Furthermore, we also have, for $S \ni n \to +\infty$,

$$|F_n| \leqslant \sum_{j=0}^{t} |A_j| \left| \frac{(\alpha - \alpha_{\theta_j(n)})^{q^{\sigma_j}}}{\beta_n} \right| \leqslant \sum_{j=0}^{t} \frac{|A_j| \, \delta_{\theta_j(n)}^{q^{\sigma_j}}}{|\beta_n|} \to 0.$$

Thus there exists at least an $n \in S$ such that $|F_n| < 1$. This is absurd. Consequently α is transcendental over $\mathbb{K}$. □

Remark: In the case that $\mathbb{K} = \mathbb{F}_q(T)$ and $\mathbb{L} = \mathbb{F}_q((T^{-1}))$, we can assume $A_0 \neq 0$ in Theorem 1. For this, it suffices to apply Cartier operators, and proceed as in [1]. Since we need this point in the proof of Theorem 9, we explain this now.

For all integers k $(0 \leqslant k < q)$, and for all $f = \sum_{n=n_0}^{+\infty} u(n) T^{-n} \in \mathbb{L}$, define

$$\tau_k(f) = \sum_{n=[(n_0-k)/q]}^{+\infty} u(qn+k) T^{-n}.$$

The Cartier operator τ_k is additive and for all $f, g \in \mathbb{L}$, we have $\tau_k(fg^q) = \tau_k(f)g$. Suppose that we have $\alpha \in \mathbb{L}$, and $A_\rho, \ldots, A_t$ in $\mathbb{F}_q[T]$ with $A_\rho \neq 0$ and $\rho \geqslant 1$ such that

$$\sum_{j=\rho}^{t} A_j \alpha^{q^j} = 0.$$

Since $\sum_{k=0}^{q-1} T^{-k}\big(\tau_k(A_\rho)\big)^q = A_\rho \neq 0$, there exists an integer ℓ $(0 \leqslant \ell < q)$ such that $\tau_\ell(A_\rho) \neq 0$, and

$$\sum_{j=\rho}^{t} \tau_\ell(A_j)\alpha^{q^{j-1}} = \tau_\ell\left(\sum_{j=\rho}^{t} A_j \alpha^{q^j}\right) = 0.$$

Repeating the above procedure, we finally arrive at the case where $A_0 \neq 0$.

2. Application I: Hypergeometric functions for $\mathbb{F}_q[T]$

We first introduce some notation and quantities which occur in the arithmetic related to Carlitz module. For motivation and various properties, see [11, 30].

Let $\mathbf{C}_\infty$ be the topological completion of a fixed algebraic closure of $\mathbb{F}_q((T^{-1}))$. It is topologically complete and algebraically closed, and plays the role of $\mathbb{C}$ in our study. However, unlike $\mathbb{C}$, $\mathbf{C}_\infty$ is not locally compact.

For all $z \in \mathbf{C}_\infty$, set

$$\deg z := \log_q |z|_\infty := \frac{\log |z|_\infty}{\log q}, \text{ and } v_\infty(z) := -\deg z.$$

Note that v_∞ is the ∞-adic valuation associated with $|\cdot|_\infty$, and that $\deg z$ is just the usual degree of z if $z \in \mathbb{F}_q[T]$. Finally we say that $\alpha \in \mathbf{C}_\infty$ is algebraic or transcendental if α is algebraic or transcendental over $\mathbb{F}_q(T)$.

For each integer $j \geqslant 0$, put

$$[j] = T^{q^j} - T, \quad D_j = \prod_{k=0}^{j-1} [j-k]^{q^k}, \text{ and } L_j = \prod_{k=1}^{j} [k].$$

Then for each integer $j \geqslant 1$, $[j]$ is the product of all the monic prime polynomials in $\mathbb{F}_q[T]$ whose degree divides j, and its zeros form the finite field $\mathbb{F}_{q^j}$, D_j is the product of all the monic polynomials of degree j in $\mathbb{F}_q[T]$, and L_j is the least common multiple of all the polynomials of degree j in $\mathbb{F}_q[T]$. Finally we have

$$D_j = [j]D_{j-1}^q, \text{ and } L_j = [j]L_{j-1}.$$

All these polynomials $[j]$, D_j, and L_j are fundamental for the arithmetic on $\mathbb{F}_q[T]$ and also occur in the formulas for Carlitz module structure, and in the expansions of its exponential, logarithm, etc.

Now let us recall ${}_rF_s$, the hypergeometric function for function fields introduced in [25]. For motivation, various properties such as solutions of differential-difference equations, specializations, etc., we refer to [25, 28] and [30, §6.5].

Fix $a \in \mathbb{Z}$. For all integers $n \geqslant 0$, define

$$(a)_n = \begin{cases} D_{n+a-1}^{q^{-(a-1)}} & \text{if } a \geqslant 1, \\ (-1)^{-a-n} L_{-a-n}^{-q^n} & \text{if } n \leqslant -a \text{ and } a \leqslant 0, \\ 0 & \text{if } n > -a \geqslant 0. \end{cases}$$

For all integers $r, s \geqslant 0$ and for all $a_i, b_j \in \mathbb{Z}$ ($1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$) with $b_j > 0$ (so that there are no zero denominators below), consider the formal power series

$${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{D_n (b_1)_n \cdots (b_s)_n} z^{q^n}.$$

We often denote it by ${}_rF_s(z)$, when the parameters are well understood.

Examples. *If $r = s = 0$, then we obtain the Carlitz exponential*

$${}_0F_0(z) = e_C(z) := \sum_{n=0}^{+\infty} \frac{z^{q^n}}{D_n}.$$

Let $r = s = p = \operatorname{Char} \mathbb{F}_q$, $b_j = a_j + 1 = 2$ $(1 \leqslant j \leqslant p)$. Then

$$\begin{aligned} ({}_rF_s(z))^q &= \sum_{n=0}^{+\infty} \frac{(1)_n^{pq}}{D_n^q (2)_n^{pq}} z^{q^{n+1}} \\ &= \sum_{n=0}^{+\infty} \frac{z^{q^{n+1}}}{[n+1]^{p-1} D_{n+1}} = \frac{e_C^{(p-1)}(z)}{(p-1)!}, \end{aligned}$$

where $e_C^{(p-1)}(z)$ is the $(p-1)$-th derivative of $e_C(z)$ with respect to T. For $r = 0$, $s = 1$, and $b_1 = m + 1$, we get the Bessel-Carlitz function

$$J_m(z) := \sum_{n=0}^{+\infty} \frac{z^{q^{m+n}}}{D_{m+n} D_n^{q^m}}, \text{ and } {}_0F_1(-; m+1; z) = J_m^{q^{-m}}.$$

We note that the normalization here is slightly different from that of [10].

If there exists some integer i ($1 \leqslant i \leqslant r$) such that $a_i \leqslant 0$, then for $n \geqslant -a_i$, we have $(a_i)_n = 0$. In this case ${}_rF_s$ is a polynomial in z.

As we are interested in algebraicity and transcendence questions, we avoid this triviality, and in the following we shall always suppose

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \text{ and } 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s.$$

By direct calculation, the radius of convergence R of ${}_rF_s$ satisfies

$$R = \begin{cases} 0 & \text{if } r > s+1, \\ q^{-\sum_{i=1}^r (a_i - 1) + \sum_{j=1}^s (b_j - 1)} & \text{if } r = s+1, \\ +\infty & \text{if } r < s+1. \end{cases}$$

If $R = +\infty$, then ${}_rF_s$ is entire, and not a polynomial by our hypothesis, so by the classical method of comparing its growth estimates on expanding circles and getting incompatibility if it is assumed to satisfy a polynomial equation with polynomial coefficients (see for example Theorem 5 of [36] for details), we see that ${}_rF_s$ is a transcendental function.

We characterize the algebraicity of the functions in the "balanced case" $r = s+1$ in the next section, and in the next theorem we look at its special values, in the entire case.

Theorem 2. *Let $r, s \geqslant 0$ be integers such that $r < s+1$, and let*

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \text{ and } 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

be integers. Then

$${}_rF_s(\gamma) := {}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \gamma) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{D_n (b_1)_n \cdots (b_s)_n} \gamma^{q^n}$$

is transcendental for all $\gamma \in \mathbf{C}_\infty \backslash \{0\}$ algebraic over $\mathbb{F}_q(T)$ and such that $\mathbb{F}_q(T, \gamma)$ has less than q places above the infinite place of $\mathbb{F}_q[T]$ (In particular, γ can be any non-zero rational or non-zero algebraic of degree less than q).

Proof. Let $\gamma \in \mathbf{C}_\infty \backslash \{0\}$ be algebraic. Put $\mathbf{K} = \mathbb{F}_q(T, \gamma)$ and let $\mathcal{O}_{\mathbf{K}}$ be the integral closure of $\mathbb{F}_q[T]$ in $\mathbf{K}$. We denote by d the number of places of $\mathbf{K}$ above the infinite place of $\mathbb{F}_q[T]$, and suppose $d < q$.

Set $\ell = \max(a_r, b_s)$. Then for all integers $n \geqslant 0$ and $a \geqslant 1$, we have

$$(a)_n^{q^\ell} = D_{n+a-1}^{q^{\ell-(a-1)}} = \left(\prod_{j=1}^{n+a-1} [j]^{q^{n+a-1-j}} \right)^{q^{\ell-(a-1)}} = \prod_{j=1}^{n+a-1} [j]^{q^{\ell+n-j}}.$$

In particular, we obtain

$$\deg (a)_n^{q^\ell} = q^{\ell-(a-1)}(n+a-1)q^{n+a-1} = (n+a-1)q^{n+\ell}.$$

Set $b_0 = 1$. For all integers $n \geqslant 0$, we have $(b_0)_n = D_n$, and then

$$\begin{aligned}
\prod_{k=1}^{r} (a_k)_n^{q^\ell} &= \prod_{k=1}^{r} \prod_{j=1}^{n+a_k-1} [j]^{q^{\ell+n-j}} \\
&= \prod_{j=1}^{n+a_1-1} [j]^{rq^{\ell+n-j}} \prod_{k=2}^{r} \prod_{j=n+a_{k-1}}^{n+a_k-1} [j]^{(r-k+1)q^{\ell+n-j}}, \\
D_n^{q^\ell} \prod_{l=1}^{s} (b_l)_n^{q^\ell} &= \prod_{l=0}^{s} (b_l)_n^{q^\ell} = \prod_{l=0}^{s} \prod_{j=1}^{n+b_l-1} [j]^{q^{\ell+n-j}} \\
&= \prod_{j=1}^{n+b_0-1} [j]^{(s+1)q^{\ell+n-j}} \prod_{l=1}^{s} \prod_{j=n+b_{l-1}}^{n+b_l-1} [j]^{(s-l+1)q^{\ell+n-j}}.
\end{aligned}$$

For $j \in \mathbb{Z}$, put

$$\begin{aligned}
a(j) &= r-k+1, & &\text{if } a_{k-1} \leqslant j \leqslant a_k - 1, \\
b(j) &= s-l+1, & &\text{if } b_{l-1} \leqslant j \leqslant b_l - 1, \\
c(j) &= a(j) - b(j), \\
c^-(j) &= \max(0, -c(j)),
\end{aligned}$$

where by convention, we set $a_0 = b_{-1} = -\infty$ and $a_{r+1} = b_{s+1} = +\infty$.

From the definition, we obtain immediately

$$\begin{aligned} a(j) &= b(j) = 0, \text{ for all integers } j \geqslant \ell, \\ c^{-}(j) &\leqslant b(j) \leqslant s+1. \end{aligned}$$

Moreover we also have

$$\begin{aligned} \alpha &:= ({}_rF_s(\gamma))^{q^\ell} = \sum_{n=0}^{+\infty} \frac{\prod_{k=1}^r (a_k)_n^{q^\ell}}{D_n^{q^\ell} \prod_{l=1}^s (b_l)_n^{q^\ell}} \gamma^{q^{n+\ell}} \\ &= \sum_{n=0}^{+\infty} \left(\prod_{j=1}^{n+\ell-1} [j]^{c(j-n)q^{n+\ell-j}} \right) \cdot \gamma^{q^{n+\ell}}. \end{aligned}$$

For all integers $m \geqslant 0$, define

$$\begin{aligned} w_m &= \frac{\prod_{k=1}^r (a_k)_m^{q^\ell}}{D_m^{q^\ell} \prod_{l=1}^s (b_l)_m^{q^\ell}} \cdot \gamma^{q^{m+\ell}} = \prod_{j=1}^{m+\ell-1} [j]^{c(j-m)q^{m+\ell-j}} \cdot \gamma^{q^{m+\ell}}, \\ \alpha_m &= \sum_{n=0}^{m} w_n, \\ \delta_m &= |\alpha - \alpha_m|_\infty . \end{aligned}$$

In particular, we have

$$\deg w_m = q^{m+\ell}[(r-s-1)m + \sum_{k=1}^r (a_k - 1) - \sum_{l=1}^s (b_l - 1) + \deg \gamma].$$

Then for all integers

$$m > \left(\sum_{k=1}^r (a_k - 1) - \sum_{l=1}^s (b_l - 1) + \deg \gamma \right) / (s+1-r),$$

we obtain

$$\begin{aligned} \delta_m &= \left| \sum_{n=m+1}^{+\infty} w_n \right|_\infty \leqslant \sup_{n \geqslant m+1} |w_n|_\infty \\ &\leqslant \sup_{n \geqslant m+1} q^{q^{n+\ell}[(r-s-1)n + \sum_{k=1}^r (a_k-1) - \sum_{l=1}^s (b_l-1) + \deg \gamma]} \\ &= q^{q^{m+1+\ell}[(r-s-1)(m+1) + \sum_{k=1}^r (a_k-1) - \sum_{l=1}^s (b_l-1) + \deg \gamma]}. \end{aligned}$$

Fix $t \geqslant 1$ an integer, and $A_0, A_1, \ldots, A_t \in \mathbb{F}_q[T]$ not all zero. We denote by ρ the least integer such that $A_\rho \neq 0$. To simplify the notation, we suppose without loss of generality $\rho = 0$. The general case can be proved similarly and directly but with much more complicated notation.

For all integers $m \geqslant t$, set

$$\beta_m = \sum_{j=0}^{t} A_j \alpha_{m-j}^{q^j}.$$

Since $a_1 \geqslant 1$ and $b_0 = 1$, we have

$$a(0) = r \text{ and } b(0) = s+1,$$

so $c(0) = r - (s+1) < 0$. Since for $j \geqslant \ell$, $a(j) = b(j) = 0$ and thus also $c(j) = 0$, we can find a greatest integer j_0,

$$m \leqslant j_0 < m + \ell$$

such that $c(j_0 - m) < 0$ and $c(j - m) \geqslant 0$ for all integers $j > j_0$.

Let $f \in \mathbb{F}_q[T]$ be monic and irreducible of degree j_0. Then f depends on j_0, thus on m. We denote by v_f the associated f-adic valuation over $\mathbb{F}_q(T)$, and extend it over $\mathbf{K}$. Choose an integer

$$N \geqslant \ell + \max_{0 \leqslant j \leqslant t} \deg A_j$$

such that for all integers $m > N$, we have $v_f(\gamma) = 0$.

Fix $m > N$ an integer. For all integers $0 \leqslant n \leqslant m$, we have

$$\begin{aligned} v_f(w_n) &= v_f\left(\prod_{j=1}^{n+\ell-1} [j]^{c(j-n)q^{n+\ell-j}} \cdot \gamma^{q^{n+\ell}}\right) \\ &= \sum_{j=1}^{n+\ell-1} c(j-n)q^{n+\ell-j} v_f([j]) + q^{n+\ell} v_f(\gamma) \\ &= \sum_{j=1}^{+\infty} c(j-n)q^{n+\ell-j} v_f([j]) \\ &= c(j_0 - n)q^{n+\ell-j_0}, \end{aligned}$$

since $c(j-n) = 0$ if $j - n \geqslant \ell$ (note that $m > \ell$, thus $2j_0 > m + \ell$), and $f \mid [j]$ if and only if $\deg f$ divides j, and in the latter case we have $v_f([j]) = 1$.

If $0 \leqslant n < m$, then $j_0 + m - n > j_0$, and thus we have $c(j_0 - n) \geqslant 0$. Consequently, for all integers $0 \leqslant k < m$, we have

$$\begin{aligned} v_f(\alpha_k) &= v_f\left(\sum_{n=0}^{k} w_n\right) \\ &\geqslant \min_{0 \leqslant n \leqslant k} v_f(w_n) \\ &= \min_{0 \leqslant n \leqslant k} c(j_0 - n)q^{n+\ell-j_0} \geqslant 0, \end{aligned}$$

and for $k = m$, we have

$$v_f(\alpha_m) = v_f\left(\sum_{n=0}^{m} w_n\right) = c(j_0 - m)q^{m+\ell-j_0} < 0,$$

since for all integers $0 \leqslant n < m$, we have

$$v_f(w_n) = c(j_0 - n)q^{n+\ell-j_0} \geqslant 0 > v_f(w_m) = c(j_0 - m)q^{m+\ell-j_0}.$$

As a result, we obtain

$$v_f(\beta_m) = v_f\left(\sum_{j=0}^{t} A_j \alpha_{m-j}^{q^j}\right) = v_f(A_0 \alpha_m) = v_f(\alpha_m) < 0.$$

So $\beta_m \neq 0$. Let $E \in \mathbb{F}_q[T]\backslash\{0\}$ be such that $E\gamma \in \mathcal{O}_{\mathbf{K}}$. Set

$$H_m = E^{q^{m+\ell}} \prod_{j=1}^{m+\ell-1} [j]^{G_{m,j}}, \quad \text{with } G_{m,j} = \max_{\substack{0\leqslant n\leqslant m-l\\ 0\leqslant l\leqslant t}} c^-(j-n)q^{n+\ell+l-j}.$$

Note that

$$\begin{aligned}\beta_m &= \sum_{l=0}^{t} A_l \alpha_{m-l}^{q^l} = \sum_{l=0}^{t} A_l \left(\sum_{n=0}^{m-l} w_n\right)^{q^l}\\ &= \sum_{l=0}^{t} A_l \sum_{n=0}^{m-l}\left(\prod_{j=1}^{n+\ell-1} [j]^{c(j-n)q^{n+\ell+l-j}}\right)\cdot \gamma^{q^{n+\ell+l}}.\end{aligned}$$

Thus $H_m\beta_m \in \mathcal{O}_{\mathbf{K}} \setminus \{0\}$. Denote by $v^{(i)}$ $(1 \leqslant i \leqslant d < q)$ the valuations at infinite places of $\mathbf{K}$ which extend the usual infinite place of $\mathbb{F}_q[T]$. Then by the product formula, we obtain immediately

$$\sum_{i=1}^{d} v^{(i)}(H_m\beta_m) \leqslant 0.$$

But $v^{(i)}(\beta_m)$ $(1 \leqslant i \leqslant d)$ is bounded below independently of m, so there exists a constant $c > 0$ such that for all sufficiently large integers m, we have

$$v_\infty(\beta_m) \leqslant d \deg H_m + c \leqslant (q-1)\deg H_m + c,$$

where v_∞ is the valuation corresponding to $\mathbf{C}_\infty$.

Fix $0 \leqslant l \leqslant t$. For all integers

$$1 \leqslant j \leqslant m - t - \log_q \frac{s+1}{s+1-r},$$

we have $j \leqslant m-l$, and

$$(s+1-r)q^m \geqslant (s+1)q^{j+t}.$$

However by definition, for all integers $n \geqslant j$, we have

$$c(j-n) = a(j-n) - b(j-n) = r - (s+1) < 0.$$

So $c^-(j-n) = s+1-r$, and then

$$\begin{aligned}&\max_{0\leqslant n\leqslant m-l} c^-(j-n)q^{n+\ell+l}\\ =\ &\max\left(\max_{0\leqslant n<j} c^-(j-n)q^{n+\ell+l}, \max_{j\leqslant n\leqslant m-l} c^-(j-n)q^{n+\ell+l}\right)\\ =\ &\max\left(\max_{0\leqslant n<j} c^-(j-n)q^{n+\ell+l}, (s+1-r)q^{m+\ell}\right)\\ =\ &(s+1-r)q^{m+\ell},\end{aligned}$$

for we have $c^-(j-n) \leqslant s+1$ $(0 \leqslant n < j)$. In conclusion, we obtain

$$\max_{\substack{0\leqslant n\leqslant m-l\\ 0\leqslant l\leqslant t}} c^-(j-n)q^{n+\ell+l} = (s+1-r)q^{m+\ell}, \text{ for } 1 \leqslant j \leqslant m-t-\log_q \frac{s+1}{s+1-r}.$$

On the other hand, if $j > m - t - \log_q \frac{s+1}{s+1-r}$, then

$$\max_{\substack{0\leqslant n\leqslant m-l\\ 0\leqslant l\leqslant t}} c^-(j-n)q^{n+\ell+l} \leqslant \max_{\substack{0\leqslant n\leqslant m-l\\ 0\leqslant l\leqslant t}} (s+1)q^{n+\ell+l} = (s+1)q^{m+\ell}.$$

By combining the above two inequalities, finally we obtain

$$\begin{aligned}
v_\infty(\beta_m) &\leqslant (q-1)\deg H_m + c\\
&= (q-1)\left(q^{m+\ell}\deg E + \sum_{j=1}^{m+\ell-1} \max_{\substack{0\leqslant n\leqslant m-l\\ 0\leqslant l\leqslant t}} c^-(j-n)q^{n+\ell+l}\right) + c\\
&\leqslant (q-1)\Bigg[q^{m+\ell}\deg E + (s+1-r)q^{m+\ell}\left(m - t - \log_q \frac{s+1}{s+1-r}\right)\\
&\quad +(s+1)q^{m+\ell}\left(\ell + t + \log_q \frac{s+1}{s+1-r}\right)\Bigg] + c\\
&\leqslant (q-1)q^{m+\ell}\Bigg(\deg E + (s+1-r)(m-t)\\
&\quad +(s+1)(\ell+t) + r\log_q \frac{s+1}{s+1-r}\Bigg) + c,
\end{aligned}$$

and then for all integers $0 \leqslant j \leqslant t$, we have, when $m \to +\infty$,

$$\begin{aligned}
&v_\infty(\beta_m) + q^j \log_q \delta_{m-j}\\
\leqslant\ & (q-1)q^{m+\ell}\left(\deg E + (s+1-r)m + (s+1)(\ell+t) + r\log_q \frac{s+1}{s+1-r}\right) + c\\
&+q^{m+1+\ell}\left(\deg\gamma + (r-s-1)(m-j+1) + \sum_{k=1}^{r}(a_k-1) - \sum_{l=1}^{s}(b_l-1)\right)\\
\sim\ & -q^{m+\ell}(s+1-r)m \to -\infty,
\end{aligned}$$

which just means

$$\lim_{m\to+\infty} \frac{|\beta_m|_\infty}{\delta_{m-j}^{q^j}} = +\infty.$$

So $({}_rF_s(\gamma))^{q^\ell}$ (and thus ${}_rF_s(\gamma)$) is transcendental, by virtue of Theorem 1. □

As a direct consequence of Theorem 2 (see examples of specializations at the beginning of this Section), we obtain the following Theorem.

Theorem 3. *For all algebraic $\gamma \in \mathbf{C}_\infty\backslash\{0\}$ as in the Theorem above, all the $e_C(\gamma)$, $e_C^{(p-1)}(\gamma)$, and $J_m(\gamma)$ are transcendental, where $m \geqslant 0$ is an integer.*

Remarks: (1) For $e_C(\gamma)$ and $J_m(\gamma)$, this is a special case (where γ is restricted as in our theorem) of the known result (where γ is allowed to be any non-zero algebraic). But for $e_C^{(p-1)}(\gamma)$, it is stronger than the known result by L. Denis [9] where γ was assumed rational. Also note that since the derivative of an algebraic element is algebraic, we are proving the transcendence of the lower derivatives also, and that higher derivatives are zero as we are in characteristic p.

(2) It is interesting to note that the original proof of L. I. Wade [33] for e_C relied heavily on the functional equation satisfied by e_C which defines the Carlitz module

structure, and the proofs of L. Denis [10] for J_m or [8] for $e_C^{(1)}(\gamma)$ were based on the difference equations satisfied by them which define a t-module structure and gave in fact algebraic independence. Indeed all the three proofs used implicitly or explicitly the ideas of Drinfeld modules and the classical procedure to prove the transcendence or algebraic independence over an algebraic group. However, it seems difficult to apply these techniques in the case of the hypergeometric function.

(3) In the function field case setting of Drinfeld modules and higher dimensional t-motives, Jing Yu [37] introduced an analog of E-functions, called E_q-functions. He proved an analog of Schneider-Lang result for them, which implies in particular, that given a finite extension L of $\mathbb{F}_q(T)$, a transcendental E_q-function takes values outside L for the arguments in L except for finitely many possible exceptions. We note here that our hypergeometric functions in the entire case are E_q-functions. This can be deduced by checking all the conditions in the definition of E_q-functions by straight manipulations from our explicit formula. In more details, we use the estimates for $\deg w_m$ and for degrees of denominators, and the divisibility properties [30, 4.13] of D_i (in particular, that it is a good analog of factorial of q^i making the multinomial coefficients integral). Thus Jing Yu's result complements our result, neither implying the other.

3. Characterization of the algebraic functions in the balanced case

For the balanced case $r = s + 1$, we have the following theorem.

Theorem 4. *Let $r, s \geqslant 0$ be integers such that $r = s+1$, and let*

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \text{ and } 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

be integers. With the notation as above, the following properties are equivalent:

1. *For all integers j $(1 \leqslant j \leqslant r)$, we have $a_j \geqslant b_{j-1}$;*
2. *For all $j \in \mathbb{Z}$, we have $c(j) \geqslant 0$;*
3. $\left({}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z)\right)^{q^\ell} \in \mathbb{F}_q[T][[z]]$;
4. ${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z)$ *is an algebraic function.*

Proof. Note that our formula for the hypergeometric function coefficient shows that (2) is equivalent to (3).

(1)⇒(2): From the definitions, we have

$$c(j) = a(j) - b(j) = 1 - k + l,$$

where $a_{k-1} \leqslant j \leqslant b_l - 1 \leqslant a_{l+1} - 1$, thus $k - 1 < l + 1$ and so $c(j) \geqslant 0$.

(2)⇒(1): Fix $1 \leqslant i \leqslant r$ and $j = a_i$. By definition, we have $a(j) \leqslant r - i$. Thus

$$b(j) = a(j) - c(j) \leqslant a(j) \leqslant r - i.$$

So by definition again, we obtain $a_i = j \geqslant b_{s-b(j)} \geqslant b_{s-r+i} = b_{i-1}$.

(2)⇒ (4): Note that for all integers $n \geqslant 0$ and $1 \leqslant k \leqslant n$, we have

$$a(k-n) = r, \text{ and } b(k-n) = s+1,$$

and thus $c(k-n)=0$. Then

$$\begin{aligned}\frac{\prod_{k=1}^{r}(a_k)_n^{q^\ell}}{D_n^{q^\ell}\prod_{l=1}^{s}(b_l)_n^{q^\ell}} &= \prod_{j=1}^{n+\ell-1}[j]^{c(j-n)q^{n+\ell-j}} = \prod_{j=n+1}^{n+\ell-1}[j]^{c(j-n)q^{n+\ell-j}} \\ &= \prod_{j=n+1}^{n+\ell-1}\left(T^{q^{n+\ell}}-T^{q^{n+\ell-j}}\right)^{c(j-n)} = \prod_{j=1}^{\ell-1}\left(T^{q^{n+\ell}}-T^{q^{\ell-j}}\right)^{c(j)}\end{aligned}$$

which is a sum of $2^{\sum_{j=0}^{\ell-1}c(j)}$ terms of the form $\pm T^{kq^{n+\ell}+h}$, with $k,h\geqslant 0$ bounded. So $({}_rF_s(z))^{q^\ell}$ is a finite $\mathbb{F}_q[T]$-linear combination of the formal power series

$$g_k(z)=\sum_{n=0}^{+\infty}T^{kq^{n+\ell}}z^{q^{n+\ell}}.$$

However we have

$$(g_k(z))^q=\sum_{n=0}^{+\infty}T^{kq^{n+1+\ell}}z^{q^{n+1+\ell}}=\sum_{n=1}^{+\infty}T^{kq^{n+\ell}}z^{q^{n+\ell}}=g_k(z)-T^{kq^\ell}z^{q^\ell}.$$

Then $g_k(z)$ is an algebraic function, and so is ${}_rF_s(z)$.

(4)⇒ (3): By definitions of $(a)_n$ and ℓ, we see that $({}_rF_s(z))^{q^\ell}\in\mathbb{F}_q(T)[[z]]$. To show that the coefficients are polynomials rather than rational functions, we proceed as in [32] and notice that by argument as in Eisenstein theorem, there is $a\in\mathbb{F}_q[T]\setminus\{0\}$ such that a^n times the n-th coefficient is integral (i.e., a polynomial in T). Next we use the results due to L. Carlitz that D_n and L_n are the product and least common multiple of all monic polynomials of degree n respectively (see eg. [30, §2.5]). The expression of the coefficients in terms of D_n's and L_n's now shows that if the coefficients are not integral, then for large n, the denominator of the coefficient of z^{q^n} will be divisible by a prime of degree (note [30, p. 14] that there are primes of every degree n) larger than the degree of a and thus cannot be cancelled by multiplication by any power of a, contradicting the integrality mentioned above.

Another way to see this implication is to use the characterization of algebraic functions proved independently by H. Sharif and C. Woodcock [23] and by T. Harase [12] generalizing the automata criterion of Christol (see also [2]). Indeed we only need the following special case of their criterion (see [39, Theorem 4]):

Let $\overline{\mathbb{F}_q(T)}$ be the algebraic closure of $\mathbb{F}_q(T)$ in $\mathbf{C}_\infty$, and $u=(u(n))_{n\geqslant 0}$ be a sequence with terms in $\mathbb{F}_q(T)$. Then the formal power series $\sum_{n=0}^{+\infty}u(n)z^{q^n}$ is algebraic over $\mathbb{F}_q(T)(z)$ if and only if the $\overline{\mathbb{F}_q(T)}$-vector space generated the family of sequences $\left(u^{1/q^k}(n+k)\right)_{n\geq 0}$ $(k\in\mathbb{N})$ has a finite dimension over $\overline{\mathbb{F}_q(T)}$.

Assume that ${}_rF_s(z)$ is an algebraic function. For all integers $n\geqslant 0$, set

$$u(n)=\frac{\prod_{k=1}^{r}(a_k)_n^{q^\ell}}{D_n^{q^\ell}\prod_{l=1}^{s}(b_l)_n^{q^\ell}}=\prod_{j=1}^{n+\ell-1}[j]^{c(j-n)q^{n+\ell-j}}.$$

Then by the above result, the $\overline{\mathbb{F}_q(T)}$-vector space generated by the family of sequences $\left(u^{1/q^k}(n+k)\right)_{n\geq 0}$ $(k\in\mathbb{N})$ is of finite dimension. Thus there exists an integer $t \geqslant 1$ such that for all integers $k \geqslant 0$, we can find $A_{k,j} \in \overline{\mathbb{F}_q(T)}$ $(0 \leqslant j \leqslant t)$ such that

$$u^{1/q^k}(n+k) = \sum_{j=0}^{t} A_{k,j} u^{1/q^j}(n+j), \quad \text{for all integers } n \geqslant 0.$$

For all integers $n \geqslant 1$, let $f_n \in \mathbb{F}_q[T]$ be a fixed monic and irreducible polynomial of degree n. We denote by ν_n the associated f_n-adic valuation over $\mathbb{F}_q(T)$, and then extend it over $\overline{\mathbb{F}_q(T)}$. If $({}_rF_s(z))^{q^\ell} \notin \mathbb{F}_q[T][[z]]$, then $\exists j_0 \in \mathbb{Z}$ $(1 \leqslant j_0 \leqslant \ell - 1)$ such that $c(j_0) < 0$. Fix $k > t + \ell - j_0$. Note that $f_{n+k+j_0} \mid [m]$ if and only if $\deg f_{n+k+j_0}$ divides m, and in the latter case $\nu_{n+k+j_0}([m]) = 1$. Thus for all integers $n \geqslant 0$,

$$\nu_{n+k+j_0}(u(n+k)) = c(j_0) q^{\ell - j_0}, \text{ and } \nu_{n+k+j_0}(u(n+j)) = 0 \, (0 \leqslant j \leqslant t),$$

But for all integers j $(0 \leqslant j \leqslant t)$, if $A_{k,j} \neq 0$, then $\nu_{n+k+j_0}(A_{k,j}) = 0$ for all large integers n, and then

$$0 > c(j_0) q^{\ell - j_0 - k} = \nu_{n+k+j_0}(u^{1/q^k}(n+k)) \geqslant \min_{0 \leqslant j \leqslant t} \nu_{n+k+j_0}(A_{k,j} u^{1/q^j}(n+j)) = 0.$$

Absurd. So we must have $({}_rF_s(z))^{q^\ell} \in \mathbb{F}_q[T][[z]]$. □

Remarks: (1) Unlike the classical hypergeometric specializations of interest at rational arguments a_i, b_j, this hypergeometric function is defined only at the integral arguments. But $q = p^w$ with $p \geqslant 2$ a prime and $w \geqslant 1$ an integer, then we can extend the definition of the hypergeometric function to rational arguments with denominators dividing w (as suggested in [25, p. 226], but not published) as follows: As L. Carlitz noticed, the definition

$$[n] := T^{q^n} - T$$

for an integer n can be extended by the same formula to n now being a rational with denominator dividing w. Now extend the definition of D_n for n rational with denominator dividing w by the formula

$$D_n = \prod_{j=0}^{\lceil n \rceil - 1} [n-j]^{q^j}$$

and use the same formula for $(a)_n$ and the hypergeometric functions with now arguments being such rationals. They still satisfy the hypergeometric equations, there are some interesting specializations, and characterization of algebraic functions among them is now an open question, which we hope to address in future.

(2) As explained in [25, 28, 30], the hypergeometric function satisfies the analog of Gauss differential equation [30, p. 229] where the analogs of operators zd/dz and $zd/dz + a$ are the operators Δ and Δ_a respectively, in these references. A. Kochubei [14] has suggested using the interpretation $\Delta_a = \Delta - [-a]$ mentioned in [30, p. 229] and [28, p. 46]. He considers this as $\Delta_{[-a]}$ instead and redefines for $a \in \mathbf{C}_\infty$, $\Delta_a = \Delta - a$ and solves the corresponding Gauss equation to get a hypergeometric function with parameters $a \in \mathbf{C}_\infty$, specializing to the one considered above when the parameters are of special form $[-a]$. This looses some analogies that we have, but retains some other properties after this reinterpretation (see [14]). This is another direction in which the study can be pursued.

4. The second analog of hypergeometric functions in function field arithmetic

In [25, 30], there is another analog of hypergeometric functions ${}_r\mathcal{F}_s$ defined (called the second analog there) where the arguments are in $\mathbf{C}_\infty$ a priori. For these analogs, $(a)_n := e_n(a) := \prod(a-f)$, with f running over all polynomials of degree less than n. Note that this is zero for large n, when a itself is a polynomial, so that the bad or trivial cases now are when $a \in \mathbb{F}_q[T]$. So we focus on a's which are in $\mathbb{F}_q(T) \setminus \mathbb{F}_q[T]$. Question now is for what such parameters do we get an algebraic function. Let us start with some simple facts and collect some tools.

From the definition, we see that $(\theta a)_n = \theta(a)_n$ for $\theta \in \mathbb{F}_q^*$ and $(a)_n = (a+i)_n$ for large n where $i \in \mathbb{F}_q[T]$.

(i) So for algebraicity question at rational parameters, we can ignore integral translations and sign (in $\mathbb{F}_q^*$) changes.

(ii) Since $(a)_n$ is now additive (see [30, p. 45]) in a, if ${}_r\mathcal{F}_s(a, a_i; b_j; z)$ and ${}_r\mathcal{F}_s(a', a_i; b_j; z)$ are algebraic, then so is ${}_r\mathcal{F}_s(a+a', a_i; b_j; z)$.

In the balanced case, the coefficient of z^{q^n} is a monomial in terms of the form $(a)_n/D_n$. These terms are described by specializations of "two variable" algebraic functions called solitons by G. W. Anderson [4, 27, 29, 25, 30]. We recall the relevant results now.

First we note special cases [27, p. 309]: with ζ_a being the primitive (i.e., a generator of A-sub-module giving) Carlitz a-torsion, for $a \in \mathbb{F}_q[T]$, we have

(iii) $(1/T)_n/D_n = -1/(-T)^{(q^{n+1}-1)/(q-1)} = -\zeta_T/\zeta_T^{q^{n+1}}$,

(iv) $(1/T^2)_n/D_n = (\zeta_{T^2}\zeta_T^{q^{n+1}} - \zeta_{T^2}^{q^{n+1}}\zeta_T)/\zeta_T^{2q^{n+1}}$. We write this as $ad^{q^{n+1}} + bc^{q^{n+1}}$, with $a = \zeta_T$, $b = \zeta_{T^2}$, $c = 1/a$. Note $d = -b/a^2$.

More generally, we have Anderson's formula:

(v) $(a)_n/D_n = \sum \alpha_i^{q^n}\beta_i$, for some algebraic α_i and β_i and with i running from 1 to the degree of the denominator of a (α_i, β_i are explicit in the original theorem of Anderson (see [30, Theorem 8.4.4]), but we do not need the explicit form right now). We write $\binom{a}{q^n}$ for $(a)_n/D_n$ because of the analogy with binomial coefficients (see [30, §4.14]).

Theorem 5. *(1) The function ${}_1\mathcal{F}_0(a/b;;z)$ is algebraic if a/b is a proper fraction.*

(2) If ${}_{s+1}\mathcal{F}_s(a_j; b_i; z)$ is algebraic, then ${}_{s+2}\mathcal{F}_{s+1}(a_j, a; b_i, b; z)$ is algebraic, where a is a proper fraction and b is a proper fraction with denominator of degree one.

(3) If ${}_{s+1}\mathcal{F}(a_1, \cdots, a_{s+1}; b_1, \cdots, b_s, z)$ and ${}_{r+1}\mathcal{F}_r(a_1', \cdots, a_{r+1}'; b_1', \cdots, b_r'; z)$ are algebraic, then ${}_{s+r+2}\mathcal{F}_{s+r+1}(a_h, a_i'; b_j, b_k', c; z)$ is algebraic if c is fraction with degree one denominator.

(4) If you stay in the balanced case, you can add or remove parameters having degree one denominators retaining algebraicity. In other words, if ${}_{s+1}\mathcal{F}_s(a_j; b_i; z)$ is algebraic, and a, b (a_{s+1}, b_s respectively) are fractions with degree one denominators, then ${}_{s+2}\mathcal{F}_{s+1}(a_j, a; b_i, b; z)$ (${}_s\mathcal{F}_{s-1}(a_j; b_i; z)$ respectively) is algebraic.

Proof. We know [30, p. 234] that ${}_1\mathcal{F}_0(a/b;;z) = e_C(\frac{a}{b}\log_C(z))$ is an algebraic function $F(z)$, since it satisfies polynomial equation $C_b(F(z)) = C_a(z)$, where C is the Carlitz module with $\log_C$ the corresponding logarithm. This implies part (1).

By (iii) (with T replaced by degree one prime $T+c$, $c \in \mathbb{F}_q$, as $\mathbb{F}_q[T] = \mathbb{F}_q[T+c]$) and by (v), the part (2) follows because of the following: If $f(z) = \sum f_n z^{q^n}$ is algebraic, then

$$\sum f_n (a)_n / D_n z^{q^n} = \sum \beta_i f(\alpha_i z)$$

is algebraic.

By the result of Furstenburg, Harase, Sharif-Woodcock [23, 12, 2], we know that Hadamard products of algebraic functions are algebraic, so that we can combine two algebraic parameter sets, while adding or removing fractions with degree one denominators using (iii), to keep the balanced case hypergeometric form. This gives part (3).

The last part again follows from (iii), with T replaced by degree one prime. □

Corollary 1. *Any function ${}_{s+1}\mathcal{F}_s(a_i; b_j; z)$, with a_i being any proper fractions and b_j being fractions with denominators of degree one, is algebraic.*

Remark: In contrast to the situation in the last section, the algebraicity is not equivalent to the integrality in this case, as we can see by using fractions $1/(T-f)$'s, $f \in \mathbb{F}_q$, in numerator and denominators and using (iii).

Theorem 6. *(1) The function ${}_2\mathcal{F}_1(1/T, 1/T; 1/T^2; z)$ is transcendental.*

(2) The function ${}_{q+1}\mathcal{F}_q(1/T, (\theta_1 T+1)/T^3, \cdots, (\theta_q T+1)/T^3; 1/T^2, \cdots, 1/T^2; z)$ is transcendental, where we denote the elements in $\mathbb{F}_q$ by $\theta_1, \ldots, \theta_q$.

(3) When $q = 2$, ${}_{q+1}\mathcal{F}_q(1/T^2, (\theta_1 T+1)/T^3, \cdots, (\theta_q T+1)/T^3; 1/T^2, \cdots, 1/T^2; z) = {}_2\mathcal{F}_1(1/T^2 + 1/T^3, 1/T^3; 1/T^2; z)$ and ${}_2\mathcal{F}_1(1/T^3, 1/T^3; 1/T^2; z)$ are algebraic.

(4) When $q = 2$, ${}_2\mathcal{F}_1(1/T^2, 1/T; 1/(T+1)^2; z)$ and ${}_2\mathcal{F}_1(1/T^2, 1/T^2; 1/(T+1)^2; z)$ are transcendental.

Proof. By simple manipulation using (iii) and (iv), the function in (1) reduces to

$$\frac{\zeta_T^2}{\zeta_{T^2}} \sum_{n=0}^{\infty} \frac{(z/\zeta_{T^2}^q)^{q^n}}{(\zeta_T/\zeta_{T^2})^{q^{n+1}} - (\zeta_T/\zeta_{T^2})}$$

which is seen to be transcendental by comparison to

$${}_2F_1(1, 1; 2, z)^q = \sum_{n=0}^{\infty} \frac{(z^q)^{q^n}}{(T^{q^{n+1}} - T)}$$

which is transcendental by Theorem 4. This proves (1).

Note that $\sum f_n z^{q^n}$ is algebraic if and only if $\sum \alpha \beta^{q^n} f_n z^{q^n}$ is algebraic, where α, β are non-zero algebraic. We shall write such a relation between the sequence of coefficients of these series by $\equiv$ equivalence.

The sequence of coefficients of $z^{q^{n-1}}$ of the function in part (2) is

$$\binom{1/T}{q^{n-1}} \frac{D_n}{D_{n-1}^q} \cdot \frac{\prod_{\theta \in \mathbb{F}_q} ((\theta T + 1)/T^3)_{n-1}}{(1/T^2)_n} \cdot \frac{\binom{1/T^2}{q^n}}{\binom{1/T^2}{q^{n-1}}^q},$$

which is easily seen, using (iii), (iv) and the definition of $(a)_n$, to be $\equiv$ equivalent to

$$(T^{q^n} - T) \frac{\binom{1/T^2}{q^n}}{\binom{1/T^2}{q^{n-1}}^q} = (T^{q^n} - T) \frac{a d^{q^{n+1}} + b c^{q^{n+1}}}{a^q d^{q^{n+1}} + b^q c^{q^{n+1}}}.$$

Subtracting off $(T^{q^n}-T)/a^{q-1}$ (which is a sequence of coefficients in an algebraic series) and using $b/a=-d/c$, we are left with

$$(T^{q^n}-T)\frac{(-b^q+ba^{q-1})c^{q^{n+1}}}{a^{q-1}(a^q d^{q^{n+1}}+b^q c^{q^{n+1}})}\equiv\frac{(T^{q^n}-T)}{((d/c)^{q^n}-(d/c))^q}.$$

Put $x=d/c$. Now using Carlitz module torsion definitions, we see that $x^q-x=1/T$. Hence our quantity becomes

$$\frac{T^{q^n+1}(1/T-1/T^{q^n})}{x^{q^{n+1}}-x^q}\equiv\frac{(x^q-x)-(x^q-x)^{q^n}}{x^{q^{n+1}}-x^q}=-1+\frac{1}{(x^{q^n}-x)^{q-1}}.$$

The corresponding function is seen to be transcendental by comparison to the function ${}_qF_{q-1}(1,\cdots,1;2,\cdots,2;z)$, which in turn is transcendental by Theorem 4. This proves (2).

When $q=2$, by the calculation above in case of part (3), we reduce to the coefficient

$$(T^{q^n}-T)\frac{ad^{q^{n+1}}+bc^{q^{n+1}}}{ad^{q^n}+bc^{q^n}}.$$

Subtracting off $(T^{2^n}+T)(c^{2^n}+d^{2^n})$ corresponding to an algebraic function, we reduce to

$$\frac{(T^{2^n}+T)(a+b)(cd)^{2^n}}{ad^{q^n}+bc^{q^n}}\equiv\frac{(T^{2^n}+T)}{ad^{q^n}+bc^{q^n}}\equiv\frac{1/T^{2^n}+1/T}{x^{2^n}+x}=1+x^{2^n}+x.$$

The corresponding function is algebraic by comparison to ${}_1F_0(2;;z)$ dealt in Theorem 4 (with T replaced by x). The algebraicity of the second function then follows by part (1) of Theorem 5 and (ii). This proves (3).

Let F denote the first function in (4). Let $x=\zeta_{T^2}/T$ and $y=\zeta_{(T+1)^2}/(T+1)$. Then $x^2+x=1/T$ and $y^2+y=1/(T+1)$. By (iv), we see just as before that the coefficient $f(n)$ (ignoring $\binom{1/T}{q^n}$ using $\equiv$ and (iii) as before) of z^{q^n} in F is

$$\frac{x+x^{2^{n+1}}}{y+y^{2^{n+1}}}\quad=\quad\frac{1/T+1/T^2+\cdots+1/T^{2^n}}{1/(T+1)+1/(T+1)^2+\cdots+1/(T+1)^{2^n}}$$

Let

$$P_n=\sum_{j=0}^{n}T^{2^n-2^j},\qquad\text{and}\qquad Q_n=\sum_{j=0}^{n}(T+1)^{2^n-2^j}=P_n(T+1).$$

If F were algebraic, then as in the proof of Theorem 4, the $\overline{\mathbb{F}_2(T)}$-vector space generated by the sequences $\left(\left(f(n+k)\right)^{1/2^k}\right)_{n\geq 0}$ $(k\in\mathbb{N})$ would be of finite dimension, and thus there exists some integer $d\geqslant 1$ such that for all integers $k\geqslant 0$, we can find $A_{k,j}\in\overline{\mathbb{F}_2(T)}$ $(0\leqslant j\leqslant d)$ with

$$\big(f(n+k)\big)^{1/2^k}=\sum_{i=0}^{d}A_{k,i}\big(f(n+i)\big)^{1/2^i},\quad\text{for all integers } n\geqslant 0,$$

which just means that we have

$$\left(\frac{(T+1)^{2^{n+k}}}{T^{2^{n+k}}}\cdot\frac{P_{n+k}}{Q_{n+k}}\right)^{1/2^k}=\sum_{i=0}^{d}A_{k,i}\left(\frac{(T+1)^{2^{n+i}}}{T^{2^{n+i}}}\cdot\frac{P_{n+i}}{Q_{n+i}}\right)^{1/2^i}.$$

In particular, for all integers $n \geqslant 0$,

$$\frac{P_{n+d+1}}{Q_{n+d+1}} = \sum_{i=0}^{d} A_{d+1,i}^{2^{d+1}} \left(\frac{P_{n+i}}{Q_{n+i}} \right)^{2^{d+1-i}}.$$

Using the analog [15] of the Dirichlet theorem on primes in arithmetic progressions, we see that there are primes of large enough degree which are congruent to one modulo $(T+1)^2$, but not congruent to one modulo T^2. Using the definition of $(a)_n$ for $a = 1/T^2$, we see that $P_n = (T^{2^n-1}/D_n) \prod b$, where b runs through the polynomials congruent to one modulo T^2 and of degree at most $n+1$. We thus get existence of primes $\wp_{n+d+1}$ dividing Q_{n+d+1} but coprime with P_{n+d+1} such that $\deg \wp_n$ goes to ∞ with n. For all integers $0 \leqslant i \leqslant d$, we have

$$Q_{n+d+1} - Q_{n+i}^{2^{d+1-i}} = (T+1)^{2^{n+d+1}-2^{d-i}} Q_{d-i}.$$

Denote by v_n the $\wp_{n+d+1}$-valuation extended over $\overline{\mathbb{F}_2(T)}$. Then for all large integers n, $\wp_{n+d+1}$ does not divide Q_{d-i}, and so it does not divide Q_{n+i} either. Thus $v_n(Q_{n+i}) = 0$ $(0 \leqslant i \leqslant d)$. We can also suppose $v_n(A_{d+1,i}) = 0$ $(0 \leqslant i \leqslant d)$. Then

$$\begin{aligned} -1 &\geqslant -v_n(Q_{n+d+1}) = v_n\left(\frac{P_{n+d+1}}{Q_{n+d+1}}\right) = v_n\left(\sum_{i=0}^{d} A_{d+1,i}^{2^{d+1}} \left(\frac{P_{n+i}}{Q_{n+i}}\right)^{2^{d+1-i}}\right) \\ &\geqslant \min_{0 \leqslant i \leqslant d} v_n\left(A_{d+1,i}^{2^{d+1}} \left(\frac{P_{n+i}}{Q_{n+i}}\right)^{2^{d+1-i}}\right) = \min_{0 \leqslant i \leqslant d} 2^{d+1-i} v_n\left(P_{n+i}\right) \geqslant 0. \end{aligned}$$

This contradiction proves that F is transcendental. The same argument shows that the second function in (4) (whose coefficients lead to the same denominator but the square of the numerator) is also transcendental. □

Remarks: (1) Our results so far are consistent with the statement (with naive analogy with Theorem 4) that if ${}_r\mathcal{F}_s$ (with $r = s+1$) is an algebraic function, then the degree of the denominator of a_j is at least the degree of the denominator of b_{j-1} (or b_j), when arranged in order. But the last part of Theorem 6 shows that degree equalities can still lead to transcendental functions in contrast.

(2) Some transcendence results are already known in the entire case. For example, ${}_0\mathcal{F}_0(z) = e_C(z) = {}_1\mathcal{F}_1(a; a; z)$. Some more examples follow from Theorem 2 and connections between the two analogs (for some special parameters and with change of variables) mentioned in the proofs above.

(3) In [6] monodromy techniques are used to understand the algebraicity issue in the classical case. We have proceeded differently above, but it is plausible that Berkovitch's new non-archimedean monodromy techniques, once sufficiently developed, might solve in future the algebraicity issue in our case in a way parallel to the classical case.

5. Application II: The Carlitz-Goss gamma values

For all integers $j \geqslant 0$, put

$$\overline{D}_j = D_j / T^{\deg D_j}.$$

Then $\left|\overline{D}_j - 1\right|_\infty = q^{-(q-1)q^{j-1}}$, for all $j \geqslant 1$. Hence $\overline{D}_j$ tends to 1 in $\mathbb{F}_q((T^{-1}))$ when j goes to $+\infty$. So for all p-adic integer $n \in \mathbb{Z}_p$ with

$$n = \sum_{j=0}^{+\infty} n_j q^j \ (0 \leqslant n_j \leqslant q-1),$$

the following infinite product

$$\overline{\Pi}(n) := \prod_{j=0}^{+\infty} \overline{D}_j^{n_j} := \lim_{k \to +\infty} \prod_{j=0}^{k} \overline{D}_j^{n_j}$$

converges in $\mathbb{F}_q((T^{-1}))$. This is the Carlitz-Goss factorial function (See [11, 30]. The gamma function can be defined by translating the argument by one, as in the classical case). See [11, 30] for various properties such as functional equations, interpolations at finite primes, relations to periods and Gauss sums, etc.

If $n \in \mathbb{N}$, one sees at once $\overline{\Pi}(n) \in \mathbb{F}_q(T)$. By automata theory, M. Mendès France and J.-Y. Yao showed in [20] that $\overline{\Pi}(n)$ is algebraic over $\mathbb{F}_q(T)$ if and only if $n \in \mathbb{N}$ (see [20, 40] and [30, Cha. 10, 11] for a quick review about this subject). By an argument of J.-P. Allouche [3], this result is a corollary of Theorem 7 below, which is a special case of Theorem 2 in [16] and can be deduced from Theorem 2 in [40], proved by Wade's method.

Theorem 7. *Let $u = (u(m))_{m \geqslant 1}$ be a sequence in $\mathbb{F}_q$ which is not ultimately zero. Then the following formal power series*

$$\alpha = \sum_{m=1}^{+\infty} \frac{u(m)}{[m]}$$

is transcendental over $\mathbb{F}_q(T)$.

Proof. For all integers $n \geqslant 1$, set

$$\alpha_n = \sum_{m=1}^{n} \frac{u(m)}{[m]},$$

from which we obtain at once

$$\begin{aligned} |\alpha - \alpha_n|_\infty &= \left| \sum_{m=n+1}^{+\infty} \frac{u(m)}{[m]} \right|_\infty \leqslant \sup_{m \geqslant n+1} \frac{1}{|[m]|_\infty} \\ &= \sup_{m \geqslant n+1} q^{-q^m} = q^{-q^{n+1}}, \end{aligned}$$

where the last term will be denoted by δ_n.

Let $r \geqslant 1$ be an integer, and let $A_0, \ldots, A_r$ in $\mathbb{F}_q(T)$ with $A_r \neq 0$. Up to taking a q-th power, we can always choose the parity of r so that the following set

$$S = \{n \in \mathbb{N} \mid n > r + \deg A_r \text{ and } u(2n - r) \neq 0\}$$

is infinite. For all integers $n \geqslant 1$, $0 \leqslant j < r$, and $1 \leqslant s \leqslant r-j$, put

$$\begin{aligned}
\beta_n &= \sum_{k=0}^{r} A_k \alpha_{n-k}^{q^k},\\
H_n &= \frac{L_{2n}^{q^r}}{L_n}\beta_{2n+r} = \frac{L_{2n}^{q^r}}{L_n}\sum_{k=0}^{r} A_k \sum_{m=1}^{2n+r-k}\frac{u(m)}{[m]^{q^k}},\\
B_{n,j,s} &= \sum_{k=0}^{r-s-j-1}(-1)^{k+1}\prod_{i=0}^{k}[s+i]^{q^{r-s-i}}\cdot[2n+s]^{q^{r-s-k-1}-q^j}L_{2n-k-2}^{q^r},\\
C_{n,j,s} &= (-1)^{r-s-j+1}\prod_{i=0}^{r-s-j}[s+i]^{q^{r-s-i}}\cdot\frac{L_{2n-r+s+j-1}^{q^r}}{[2n+s]^{q^j}}\\
I_n &= \frac{L_{2n}^{q^r}}{L_n}\sum_{k=0}^{r} A_k \sum_{m=1}^{2n}\frac{u(m)}{[m]^{q^k}}\\
&\quad+\sum_{k=0}^{r-1}A_k\sum_{s=1}^{r-k}u(2n+s)[2n+s]^{q^{r-s}-q^k}\frac{L_{2n-1}^{q^r}}{L_n}.\\
&\quad+\sum_{k=0}^{r-1}A_k\sum_{s=1}^{r-k}u(2n+s)\frac{B_{n,k,s}}{L_n},\\
Q_n &= H_n - I_n.
\end{aligned}$$

Then by Lemma 4.1 in [33] (see also [40]), we obtain

$$Q_n = \sum_{j=0}^{r-1}A_j\sum_{s=1}^{r-j}u(2n+s)\frac{C_{n,j,s}}{L_n}.$$

From now on, we assume $n \in S$. Then $2n-r > n$, and $L_n[2n-r] \mid L_{2n-1}^{q^r}$. For $k \geqslant 1$, $[k]$ divides $[2k] = [k]^{q^k} + [k]$. So $L_n[k]^{q^j} \mid L_{2n}^{q^r}$ for $0 \leqslant j \leqslant r$ and $1 \leqslant k \leqslant 2n$. Indeed if ℓ is the greatest integer such that $2^\ell k \leqslant 2n$, then $2^\ell k > n$, and $L_n[2^\ell k]^{q^r} \mid L_{2n}^{q^r}$. Now $L_n[2n-r] \mid L_{2n-k-2}^{q^r}$ for $0 \leqslant k \leqslant r-s-j-1$, since in this case $q^r \geqslant 2$ and $k+2 \leqslant r$. Thus

$$L_n[2n-r] \mid B_{n,j,s}\ (0 \leqslant j < r, 1 \leqslant s \leqslant r-j),$$

from which we deduce $I_n \in \mathbb{F}_q[T]$, and we have

$$\begin{aligned}
I_n &\equiv \frac{L_{2n}^{q^r}}{L_n}\sum_{j=0}^{r}A_j\sum_{k=1}^{2n}\frac{u(k)}{[k]^{q^j}} \pmod{[2n-r]}\\
&\equiv A_r\frac{L_{2n}^{q^r}}{L_n}\sum_{k=1}^{2n}\frac{u(k)}{[k]^{q^r}} \pmod{[2n-r]},
\end{aligned}$$

since $L_n[k]^{q^j} \mid L_{2n}^{q^j}$, $[2n-r] \mid L_{2n}$, and $q^r - q^j \geqslant 1$ for $0 \leqslant j < r$.

Let k be an integer such that $1 \leqslant k \neq 2n-r \leqslant 2n$. Then we have

$$L_n \mid \frac{L_{2n}}{[k]} \text{ and } [2n-r] \mid \frac{L_{2n}}{[k]},$$

and thus $L_n[2n-r][k]^{q^r} \mid L_{2n}^{q^r}$, for we have $q^r \geqslant 2$. As a result, we obtain

$$\begin{aligned} I_n &\equiv A_r \frac{L_{2n}^{q^r}}{L_n} \frac{u(2n-r)}{[2n-r]^{q^r}} \pmod{[2n-r]} \\ &\equiv u(2n-r) A_r L_r^{q^r} \frac{L_{2n-r-1}^{q^r}}{L_n} \pmod{[2n-r]}, \end{aligned}$$

since for all integers $k \geqslant 1$, we have

$$[2n-r+k] \equiv [k] \pmod{[2n-r]}.$$

Therefore $I_n \neq 0$ for we have $u(2n-r) \neq 0$, and $n > \deg A_r$.

Let j, s be two integers satisfying $0 \leqslant j < r$ and $1 \leqslant s \leqslant r-j$. Then

$$\begin{aligned} & \deg \left(\frac{L_{2n-r+s+j-1}^{q^r}}{L_n [2n+s]^{q^j}} \right) \\ = \;& \frac{q^{r+1}\left(q^{2n-r+s+j-1}-1\right)}{q-1} - \frac{q\left(q^n-1\right)}{q-1} - q^{2n+s+j} \\ \leqslant \;& -\frac{q\left(q^n-1\right)}{q-1} \to -\infty \text{ (as } n \to +\infty). \end{aligned}$$

So there exists an integer $N > 0$ such that for all integers $n \geqslant N$, every term of Q_n is of negative degree. Hence for all $n \in S$ with $n \geqslant N$, we obtain

$$\deg H_n = \deg(Q_n + I_n) = \deg I_n \geqslant 0,$$

which implies immediately that we have

$$\deg \beta_{2n+r} \geqslant -\deg \frac{L_{2n}^{q^r}}{L_n} = \frac{q\left(q^n-1\right)}{q-1} - \frac{q^{r+1}\left(q^{2n}-1\right)}{q-1}.$$

Finally for all integers $0 \leqslant v \leqslant r$, we have, for $S \ni n \to +\infty$,

$$\begin{aligned} \deg \beta_{2n+r} - q^v \log_q \delta_{2n+r-v} &\geqslant \frac{q\left(q^n-1\right)}{q-1} - \frac{q^{r+1}\left(q^{2n}-1\right)}{q-1} + q^{2n+r+1} \\ &\geqslant \frac{q\left(q^n-1\right)}{q-1} \to +\infty. \end{aligned}$$

Consequently α is transcendental over $\mathbb{F}_q(T)$ by virtue of Theorem 1. □

6. Application III: Some classical transcendence criteria

In this section we show that many classical and well-known transcendence criteria are indeed special case of our criterion Theorem 1, just as promised in [41].

Let K be a fixed field of characteristic p, and let $\Delta \in K[T]$ be monic and irreducible. For each $P \in K[T]\backslash\{0\}$, we denote by $v_\Delta(P)$ the greatest integer $n \geqslant 0$ such that P is divisible by Δ^n in $K[T]$. By convention, we put $v_\Delta(0) = +\infty$. Then we extend v_Δ over $K(T)$ by setting

$$v_\Delta(P/Q) = v_\Delta(P) - v_\Delta(Q),$$

for all $P, Q \in K[T]$ with $Q \neq 0$, and we call it *the* Δ*-adic valuation* on $K(T)$.

For all $P, Q \in K[T]$ with $Q \neq 0$, we define

$$v_\infty(P/Q) = \deg(Q) - \deg(P), \text{ and } \left|\frac{P}{Q}\right|_\infty = q^{-v_\infty(P/Q)},$$

and we call v_∞ and $|\cdot|_\infty$ respectively *the ∞-adic valuation* and *the ∞-adic absolute value* on $K(T)$. Finally we denote by $K((T^{-1}))$ the topological completion of $K(T)$ relative to $|\cdot|_\infty$, and we set $\deg \alpha = -v_\infty(\alpha)$, for all $\alpha \in K((T^{-1}))$.

Motivated by Wade's method and some considerations for Diophantine approximation in positive characteristic, B. de Mathan proved in [19] a very powerful transcendence criterion, which has many interesting applications in the study of transcendence of formal power series arising from the Carlitz module. Inspired by this result and automata theory but ignorant of Theorem 8 below, the third author gave in [38] a series of transcendence criteria, including a particular one, which had an important application in the study of the automaticity of the partial quotients of the formal power series of Baum-Sweet.

Much more generally, L. Denis established in [9] the following criterion which also has many other interesting applications, for example, the proof of the transcendence of the special values at non-zero rational arguments of the Bessel-Carlitz function.

Theorem 8. *Let $\alpha \in K((T^{-1}))$. Suppose that there exist two sequences $(P_n)_{n\geqslant 0}$ and $(Q_n)_{n\geqslant 0}$ in $K[T]\backslash\{0\}$ satisfying the following four conditions:*

(1) *For all integers $n \geqslant 1$, $\exists\ \Lambda_n \in K[T]$ such that $Q_n = \Lambda_n Q_{n-1}^q$;*

(2) *There exist two real numbers $c_2, c_3 > 0$ such that for all integers $n \geqslant 1$,*

$$\deg(\Lambda_n) \leqslant c_2 \deg(\Lambda_{n+1}) + c_3;$$

(3) *There exist a real number $c_1 > 0$ and a sequence $(t_n)_{n\geqslant 0}$ of real numbers bounded below by a constant $\eta > 0$ such that for all integers $n \geqslant 0$,*

$$\left|\alpha - \frac{P_n}{Q_n}\right|_\infty \leqslant \frac{c_1}{|Q_n|_\infty^{t_n}};$$

(4) *For every $(r+1)$-tuple $(A_0, \ldots, A_r)$ of elements not all zeros in $K[T]$, there exist an infinite set S of integers n such that*

$$C_n := A_0 P_n + A_1 \Lambda_n P_{n-1}^q + \cdots + A_r \Lambda_n \Lambda_{n-1}^q \cdots \Lambda_{n-r+1}^{q^{r-1}} P_{n-r}^{q^r}$$

is different from zero, and for all couple of integers $\upsilon, \tau \geqslant 0$,

$$\lim_{S\ni n\to+\infty} \frac{\deg C_n + (t_{n-\upsilon} - 1)\deg Q_n}{t_{n-\tau}(\deg \Lambda_n + 1)} = +\infty.$$

Then α is transcendental over $K(T)$.

The present version of condition (3) corrects a minor error (which did not alter any applications) in [9].

Proof. Let $r \geqslant 1$ be an integer, and let $A_0, \ldots, A_r$ be elements in $K[T]$ with $A_r \neq 0$. For all integers $n \geqslant 0$, we define

$$\alpha_n := \frac{P_n}{Q_n}, \quad \delta_n := \frac{c_1}{|Q_n|_\infty^{t_n}},$$

$$\beta_n := \sum_{j=0}^{r} A_j \left(\frac{P_{n-j}}{Q_{n-j}}\right)^{q^j} \quad (n \geqslant r),$$

$$C_n := \beta_n Q_n = A_0 P_n + A_1 \Lambda_n P_{n-1}^q + \cdots + A_r \Lambda_n \Lambda_{n-1}^q \cdots \Lambda_{n-r+1}^{q^{r-1}} P_{n-r}^{q^r} \ (n \geqslant r).$$

By Condition (2) of Theorem 8, we have, for all integers $n \geqslant 1$,

$$\deg \Lambda_n \leqslant c_2 \deg \Lambda_{n+1} + c_3,$$

from which we deduce immediately, for all integers $j \geqslant 1$,

$$\deg \Lambda_n \leqslant c_2^j \deg \Lambda_{n+j} + \left(\sum_{i=0}^{j-1} c_2^i \right) c_3 \leqslant c_2^j \deg \Lambda_{n+j} + \frac{(c_2+1)^j}{c_2} c_3.$$

Notice also that for all integers n, v $(n > v \geqslant 0)$,

$$Q_{n-v}^{q^v} = Q_n / \prod_{j=0}^{v-1} \Lambda_{n-j}^{q^j}.$$

So we have

$$\begin{aligned} q^v \deg Q_{n-v} &= \deg Q_n - \sum_{j=0}^{v-1} q^j \deg \Lambda_{n-j} \\ &\geqslant \deg Q_n - \sum_{j=0}^{v-1} q^j \left(c_2^j \deg \Lambda_n + \frac{(c_2+1)^j}{c_2} c_3 \right) \\ &= \deg Q_n - a_v \deg \Lambda_n - b_v, \end{aligned}$$

where we set

$$a_v = \sum_{j=0}^{v-1} (qc_2)^j, \text{ and } b_v = \sum_{j=0}^{v-1} \frac{(q(c_2+1))^j}{c_2} c_3.$$

Then we obtain

$$\begin{aligned} & v_\infty(Q_n / C_n Q_{n-v}^{q^v t_{n-v}}) \\ = \; & \deg C_n + q^v t_{n-v} \deg Q_{n-v} - \deg Q_n \\ \geqslant \; & \deg C_n + (t_{n-v} - 1) \deg Q_n - t_{n-v}(a_v \deg \Lambda_n + b_v) \\ \geqslant \; & \deg C_n + (t_{n-v} - 1) \deg Q_n - d_v t_{n-v} (\deg \Lambda_n + 1), \end{aligned}$$

with $d_v = \max(a_v, b_v)$. So we have, by noting that $t_{n-v}(\deg \Lambda_n + 1) \geqslant t_{n-v} \geqslant \eta$,

$$\begin{aligned} & v_\infty(Q_n / C_n Q_{n-v}^{q^v t_{n-v}}) \\ \geqslant \; & \eta \frac{\deg C_n + (t_{n-v} - 1) \deg Q_n - d_v t_{n-v} (\deg \Lambda_n + 1)}{t_{n-v}(\deg \Lambda_n + 1)} \\ = \; & \eta \frac{\deg C_n + (t_{n-v} - 1) \deg Q_n}{t_{n-v}(\deg \Lambda_n + 1)} - \eta d_v. \end{aligned}$$

Finally by Condition (4) of Theorem 8, we obtain

$$\lim_{S \ni n \to +\infty} v_\infty(Q_n / C_n Q_{n-v}^{q^v t_{n-v}}) = +\infty,$$

which is equivalent to say

$$\lim_{S \ni n \to +\infty} |\beta_n| / \delta_{n-v}^{q^v} = +\infty.$$

So α is transcendental over $K(T)$, by virtue of Theorem 1. □

In a manner quite different from that of L. Denis, Y. Hellegouarch [13] obtained the following criterion which also generalizes the result of B. de Mathan [19].

Theorem 9. *Let $\mathbb{F}_q$ be the finite field with q elements. Let $\alpha \in \mathbb{F}_q((T^{-1}))$. Suppose that there exist two sequences $(P_n)_{n \geqslant 0}$ and $(Q_n)_{n \geqslant 0}$ of non-zero elements in $\mathbb{F}_q[T]$ satisfying the following three conditions:*

(1) *For all integers* $n \geqslant 1$*, there exists* $\Lambda_n \in \mathbb{F}_q[T]$ *such that* $Q_n = \Lambda_n Q_{n-1}^{q^{s_n}}$*, where* $(s_n)_{n\geqslant 0}$ *is a bounded sequence of positive integers;*

(2) *There exist some normalized absolute values* $|\cdot|_n \neq |\cdot|_\infty$ *such that for all integers* $i \geqslant 1$*,*

(a) $\lim_{n\to+\infty} \frac{|P_n|_n}{|P_{n-i}|_n^{q^{s_n+\cdots s_{n-i+1}}}} = +\infty$ *and* $\lim_{n\to+\infty} |P_n|_n = 0$*,*

respectively

(b) $\lim_{n\to+\infty} \frac{|P_n/Q_n|_n}{|P_{n-i}/Q_{n-i}|_n^{q^{s_n+\cdots s_{n-i+1}}}} = +\infty$*.*

(3) *For all integers* $n \geqslant 0$*,* $|Q_n\alpha - P_n|_\infty \leqslant \varepsilon_n$*, where* $(\varepsilon_n)_{n\geqslant 0}$ *satisfies*

(a) *For all integers* $h, k, \ell \geqslant 0$*,*

$$\varepsilon_n = o\left(|\Lambda_{n+h}|_\infty^{-k} |P_{n+\ell}|_{n+\ell}^{-q^{-(s_{n+\ell}+\cdots s_{n+1})}}\right) \ (n \to +\infty),$$

respectively

(b) *For all integers* $h, k \geqslant 1$*,* $\varepsilon_n = o\left(|\Lambda_{n+h}|_\infty^{-k}\right)$ $(n \to +\infty)$*.*

Then α *is transcendental over* $\mathbb{F}_q(T)$*.*

This version corrects some minor errors in [13].

Y. Hellegouarch presented in [13] many interesting applications of his criterion, in particular a new proof of the result of [26] about the transcendence of the special values of Carlitz-Goss gamma function at some non-zero rational arguments, proved initially by automata theory.

Proof. Let $r \geqslant 1$ be an integer, and $A_0, \ldots, A_r \in \mathbb{F}_q[T]$ not all zero. We denote by ρ the least integer such that $A_\rho \neq 0$. To simplify the notation, we suppose $\rho = 0$. The general case can be proved similarly and directly but with much more complicated notation. Moreover just like what was explained in the Remark following the proof of Theorem 1, the general case can also be reduced easily to the special case $\rho = 0$ with the help of Cartier operators.

For all integers n, i $(n \geqslant i \geqslant 1)$, define

$$\eta(n, i) = s_n + \cdots + s_{n-i+1}.$$

By convention, we set $\eta(n, 0) = 0$. Since $(s_n)_{n\geqslant 0}$ is bounded, then there exists an infinite set S of integers $n \geqslant r$ such that the $(r+1)$-tuple

$$\big(\eta(n, 0), \eta(n, 1), \ldots, \eta(n, r)\big)$$

is equal to a constant $(\sigma_0, \sigma_1, \ldots, \sigma_r)$.

For each integer $n \geqslant 0$, put

$$\alpha_n := \frac{P_n}{Q_n}, \quad \delta_n := \frac{\varepsilon_n}{|Q_n|_\infty},$$

$$\beta_n := \sum_{j=0}^{r} A_j \left(\frac{P_{n-j}}{Q_{n-j}}\right)^{q^{\eta(n,j)}}, \quad C_n := \beta_n Q_n \in \mathbb{F}_q[T] \quad (n \geqslant r).$$

In particular, for all integers $n \in S$, we have

$$\beta_n = \sum_{j=0}^{r} A_j \left(\frac{P_{n-j}}{Q_{n-j}}\right)^{q^{\sigma_j}}.$$

According to the statement of Theorem 9, we should distinguish two cases.

In Case (a), we obtain, from Condition (2.a) of Theorem 9 and the fact that the absolute value $|\cdot|_n$ is bounded on $\mathbb{F}_q[T]$, that

$$\max_{1\leqslant j\leqslant r}\left|Q_nA_j\left(\frac{P_{n-j}}{Q_{n-j}}\right)^{q^{\eta(n,j)}}\right|_n=\max_{1\leqslant j\leqslant r}\left|A_jP_{n-j}^{q^{\eta(n,j)}}\prod_{i=0}^{j-1}\Lambda_{n-i}^{q^{\eta(n,i)}}\right|_n<|A_0P_n|_n\,,$$

for all sufficiently large integers n. Consequently

$$|C_n|_n=|A_0P_n|_n=|A_0|_n\,|P_n|_n\,.$$

Note that the sequence $(|A_0|_n)_{n\geqslant 0}$ only takes a finite number of values, and that all the absolute values (except the ∞-adic one) over $\mathbb{F}_q[T]$ are bounded by 1. Then by the product formula of absolute values, we can find a constant $c>0$ such that

$$|C_n|_\infty\geqslant\frac{c}{|P_n|_n},$$

for all sufficiently large integers n.

Recall that for all integers $n\geqslant 0$, we have $\delta_n=\varepsilon_n/\,|Q_n|_\infty$. Then

$$\left|\alpha-\frac{P_n}{Q_n}\right|_\infty\leqslant\delta_n.$$

Now by Condition (3.a) of Theorem 9, we have, for all integers $h,k,\ell\geqslant 0$,

$$\varepsilon_n=o\left(|\Lambda_{n+h}|_\infty^{-k}\,|P_{n+\ell}|_{n+\ell}^{-q^{-(s_{n+\ell}+\cdots s_{n+1})}}\right)\ (n\to+\infty).$$

Hence for all integers $v,h,k\geqslant 0$,

$$\varepsilon_{n-v}=o\left(|\Lambda_{n-v+h}|_\infty^{-k}\,|P_n|_n^{-q^{-\eta(n,v)}}\right)\ (n\to+\infty).$$

In particular, by choosing appropriately $h,k\geqslant 0$, we obtain

$$\varepsilon_{n-v}^{q^{\eta(n,v)}}=o\left(|P_n|_n^{-1}\prod_{i=0}^{v-1}\left|\Lambda_{n-i}^{q^{\eta(n,i)}}\right|_\infty^{-1}\right)\ (n\to+\infty),$$

and thus,

$$\begin{aligned}\liminf_{S\ni n\to+\infty}\frac{|\beta_n|_\infty}{\delta_{n-v}^{q^{\eta(n,v)}}}&=\liminf_{S\ni n\to+\infty}\frac{|Q_{n-v}|_\infty^{q^{\eta(n,v)}}\,|C_n|_\infty}{|Q_n|_\infty\,\varepsilon_{n-v}^{q^{\eta(n,v)}}}\\&\geqslant c\lim_{n\to+\infty}\left(\varepsilon_{n-v}^{q^{\eta(n,v)}}\,|P_n|_n\prod_{i=0}^{v-1}|\Lambda_{n-i}^{q^{\eta(n,i)}}|_\infty\right)^{-1}\\&=+\infty.\end{aligned}$$

So α is transcendental over $\mathbb{F}_q(T)$, by virtue of Theorem 1.

In Case (b), we deduce from Condition (2.b) of Theorem 9 that

$$\max_{1\leqslant j\leqslant r}\left|A_j\left(\frac{P_{n-j}}{Q_{n-j}}\right)^{q^{\eta(n,j)}}\right|_n<\left|A_0\frac{P_n}{Q_n}\right|_n,$$

for all sufficiently large integers n. Consequently

$$|C_n|_n=|Q_n\beta_n|_n=|A_0P_n|_n\neq 0,$$

and $C_n \neq 0$. So $|C_n|_\infty \geqslant 1$, for all sufficiently large integers n. Now for all $v \geqslant 0$, by choosing appropriately $h, k \geqslant 0$ in Condition (3.b) of Theorem 9, we obtain

$$\varepsilon_{n-v}^{q^{\eta(n,v)}} = o\left(\prod_{i=0}^{v-1} |\Lambda_{n-i}^{q^{\eta(n,i)}}|_\infty^{-1}\right) \ (n \to +\infty),$$

and thus

$$\begin{aligned} \liminf_{S \ni n \to +\infty} \frac{|\beta_n|_\infty}{\delta_{n-v}^{q^{\eta(n,v)}}} &\geqslant \liminf_{S \ni n \to +\infty} \frac{|Q_{n-v}|_\infty^{q^{\eta(n,v)}}}{|Q_n|_\infty \, \varepsilon_{n-v}^{q^{\eta(n,v)}}} \\ &= \lim_{n \to +\infty} \left(\varepsilon_{n-v}^{q^{\eta(n,v)}} \prod_{i=0}^{v-1} |\Lambda_{n-i}^{q^{\eta(n,i)}}|_\infty\right)^{-1} = +\infty. \end{aligned}$$

Once again by Theorem 1, we obtain that α is transcendental over $\mathbb{F}_q(T)$. □

More generally, for a fixed integer $\varkappa \geqslant 1$, we denote by $\mathbf{A} = K[T_1, \dots, T_\varkappa]$ the integral domain of polynomials in $T_1, \dots, T_\varkappa$ with coefficients in K, and by $\mathbf{K} = K(T_1, \dots, T_\varkappa)$ the fraction field of $\mathbf{A}$. For all polynomials $F \in \mathbf{A}$, we write $\deg F$ for the total degree of F, and define, for all $P, Q \in \mathbf{A}$ with $Q \neq 0$,

$$\left|\frac{P}{Q}\right|_\infty := q^{\deg(P/Q))} := q^{\deg P - \deg Q}.$$

The topological completion of $\mathbf{K}$ relative to $|\cdot|_\infty$ is noted $\mathbf{K}_\infty$.

The following result intersects Theorem 1 in [16], and generalizes slightly the criterion of S. M. Spencer, Jr. (which generalizes in turn Wade's criterion [33]), proved by Wade's method in [24].

Theorem 10. *Let $(G_n)_{n\geqslant 0}$ be a sequence in $\mathbf{A}\backslash\{0\}$ such that $G_n^q \mid G_{n+1}$ in $\mathbf{A}$ for all integers $n \geqslant 0$. Let $(B_n)_{n\geqslant 0}$ be a sequence not ultimately zero in $\mathbf{A}$. Suppose that there exists a sequence of real numbers $(d_n)_{n\geqslant 0}$ satisfying*

(1) $d_n \to +\infty$ $(n \to +\infty)$,
(2) $\max(\deg B_{n+1}, 0) \leqslant (q-1)h_n q^n - d_{n+1}q^{n+1}$, *for all integers* $n \geqslant 0$,
(3) $d_{n-r} - h_n + h_{n-r} > \eta$, *for all* $n \geqslant n(r)$ *and* $r \geqslant 1$, *with* $\eta > 0$ *a constant,*

where $h_n = \deg G_n / q^n$ for all integers $n \geqslant 0$. Then the following series

$$\alpha = \sum_{n=0}^{+\infty} \frac{B_n}{G_n}$$

is transcendental over the field $\mathbf{K}$.

Proof. Let $r \geqslant 1$ be an integer, and $A_0, \dots, A_r$ in $\mathbf{A}$ not all zero. Without loss of generality, we suppose $A_0 \neq 0$. Indeed if σ is the least integer such that $A_\sigma \neq 0$, we can study α^{q^σ} in the place of α by noting that α^{q^σ} also satisfies the hypotheses of our theorem. However we can also work directly with α, but the notation will become much heavier and we should replace A_0 by A_σ everywhere.

For all integers $n \geqslant 0$, put

$$\Lambda_{n+1} := G_{n+1}/G_n^q, \quad Q_n := G_n, \quad P_n := Q_n \sum_{j=0}^{n} \frac{B_j}{G_j},$$
$$\alpha_n := \frac{P_n}{Q_n}, \quad \delta_n := q^{v(n+1)},$$
$$\beta_n := \sum_{j=0}^{r} A_j \left(\frac{P_{n-j}}{Q_{n-j}}\right)^{q^j} \qquad (n \geqslant r).$$

Then $\Lambda_{n+1}, P_n \in \mathbf{A}$, $Q_{n+1} = \Lambda_{n+1} Q_n^q$, and $Q_n\beta_n \in \mathbf{A}$, since $G_n^q \mid G_{n+1}$ in $\mathbf{A}$ for all integers $n \geqslant 0$. .

Firstly notice that the above series α converges in $\mathbf{K}_\infty$. Indeed for $n \to +\infty$,

$$\begin{aligned} \deg(B_n/G_n) &= \deg B_n - \deg G_n \leqslant (q-1)\deg G_{n-1} - d_n q^n - \deg G_n \\ &\leqslant -\deg G_{n-1} - d_n q^n \to -\infty. \end{aligned}$$

So we have $|B_n/G_n|_\infty \to 0$ $(n \to +\infty)$.

Secondly notice that $(h_n)_{n\geqslant 0}$ is increasing since $G_n^q \mid G_{n+1}$ for all $n \geqslant 0$.

For all integers $n \geqslant 1$, set $v(n) = \max(\deg B_n, 0) - \deg G_n$. Then $(v(n))_{n\geqslant 0}$ is ultimately decreasing. Indeed for all large integers n,

$$\begin{aligned} v(n) - v(n+1) &\geqslant \deg G_{n+1} - \deg G_n - \max(\deg B_{n+1}, 0) \\ &\geqslant \deg G_{n+1} - \deg G_n - (q-1)\deg G_n + d_{n+1} q^{n+1} \\ &\geqslant d_{n+1} q^{n+1} \geqslant 0, \end{aligned}$$

since $G_n^q \mid G_{n+1}$, $\deg B_{n+1} \leqslant (q-1)\deg G_n - d_{n+1}q^{n+1}$, and $d_{n+1} \to +\infty$.

As a result, we obtain, for all large integers n,

$$\left|\alpha - \frac{P_n}{Q_n}\right|_\infty = \left|\sum_{j=n+1}^{+\infty} \frac{B_j}{G_j}\right|_\infty \leqslant \sup_{j\geqslant n+1} \left|\frac{B_j}{G_j}\right|_\infty \leqslant \sup_{j\geqslant n+1} q^{v(j)} = q^{v(n+1)} = \delta_n.$$

Let S be the set of all integers $n \geqslant r$ such that $\beta_n \neq 0$. We distinguish below two different cases.

Case I: the set S is finite. Then $\exists\, \rho > r$ such that $\beta_n = 0$ for all $n \geqslant \rho$, *i.e.*

$$0 = Q_n\beta_n = A_0 P_n + A_1 \Lambda_n P_{n-1}^q + \cdots + A_r \Lambda_n \Lambda_{n-1}^q \cdots \Lambda_{n-r+1}^{q^{r-1}} P_{n-r}^{q^r}. \tag{1}$$

For all integers i, j $(i \geqslant j \geqslant 1)$, put

$$\Lambda_{i,j} = \prod_{l=j}^{i} \Lambda_l^{q^{i-l}}, \text{ and } \Lambda_{i,i+1} = 1.$$

Then $\Lambda_{i,j} = G_i/G_{j-1}^{q^{i-j+1}}$, and we have

$$\deg \Lambda_{i,j} = \sum_{l=j}^{i} q^{i-l}\left(q^l h_l - q^l h_{l-1}\right) = q^i(h_i - h_{j-1}).$$

Notice also that we have $\Lambda_{i,l} = \Lambda_{i,j}\Lambda_{j-1,l}^{q^{i-j+1}}$ for $1 \leqslant l \leqslant j \leqslant i$.

Now Relation (1) becomes

$$\sum_{i=0}^{r} A_i \Lambda_{n,n-i+1} P_{n-i}^{q^i} = 0, \tag{2}$$

from which we obtain $A_0P_n \equiv 0 \pmod{\Lambda_{n,n}}$. So $\Lambda_n = \Lambda_{n,n} \mid A_0P_n$ for all $n \geqslant \rho$. Multiplying both sides of (2) by A_0^q, we get

$$A_0^{1+q}P_n + A_1\Lambda_n(A_0P_{n-1})^q + \sum_{i=2}^{r} A_r\Lambda_{n,n-i+1}A_0^qP_{n-i}^{q^i} = 0. \tag{3}$$

Hence $\Lambda_{n,n-1} \mid A_0^{1+q}P_n$ for all $n \geqslant \rho+1$ since $\Lambda_{n-1} \mid A_0P_{n-1}$. Now multiplying both sides of (3) by $A_0^{q^2}$, we obtain

$$\begin{aligned} A_0^{1+q+q^2}P_n + A_1\Lambda_n(A_0^{1+q}P_{n-1})^q + A_0^qA_1\Lambda_{n,n-1}(A_0P_{n-2})^{q^2} & \\ + \sum_{i=3}^{r} A_r\Lambda_{n,n-i+1}A_0^{q(1+q)}P_{n-i}^{q^i} &= 0. \end{aligned}$$

Therefore $\Lambda_{n,n-2} \mid A_0^{1+q+q^2}P_n$ for all $n \geqslant \rho+2$, since we have

$$\Lambda_{n-1,n-2} \mid A_0^{1+q}P_{n-1}, \text{ and } \Lambda_{n-2} \mid A_0P_{n-2}.$$

By induction, we obtain finally $\Lambda_{n,n-i} \mid A_0^{\frac{q^{i+1}-1}{q-1}}P_n$ for all $n \geqslant i+\rho$. In particular, we obtain $\Lambda_{n,\rho} \mid A_0^{\frac{q^{n-\rho+1}-1}{q-1}}P_n$ for all $n \geqslant \rho$. So

$$H_n := A_0^{\frac{q^{n-\rho+1}-1}{q-1}}P_n/\Lambda_{n,\rho} \in \mathbf{A}.$$

Note that for all integers $n \geqslant \rho+1$,

$$P_n = Q_n\sum_{j=0}^{n}\frac{B_j}{G_j} = B_n + G_n\sum_{j=0}^{n-1}\frac{B_j}{G_j} = B_n + \Lambda_n\,(G_{n-1})^{q-1}\,P_{n-1}.$$

Thus we have

$$B_n = P_n - \Lambda_nG_{n-1}^{q-1}P_{n-1} = P_n - G_{\rho-1}^{(q-1)q^{n-\rho}}\Lambda_n\Lambda_{n-1,\rho}^{q-1}P_{n-1}.$$

Multiplying both sides of this relation by $A_0^{\frac{q^{n-\rho+1}-1}{q-1}}$, we get

$$\begin{aligned} A_0^{\frac{q^{n-\rho+1}-1}{q-1}}B_n &= \Lambda_{n,\rho}H_n - A_0^{q^{n-\rho}}G_{\rho-1}^{(q-1)q^{n-\rho}}\Lambda_n\Lambda_{n-1,\rho}^{q-1}\Lambda_{n-1,\rho}H_{n-1} \\ &= \Lambda_{n,\rho}\left(H_n - A_0^{q^{n-\rho}}G_{\rho-1}^{(q-1)q^{n-\rho}}H_{n-1}\right), \end{aligned}$$

for $\Lambda_{n,\rho} = \Lambda_n\Lambda_{n-1,\rho}^q$. Therefore $\Lambda_{n,\rho} \mid A_0^{\frac{q^{n-\rho+1}-1}{q-1}}B_n$.

Let E be the set of all integers n such that $B_n \neq 0$. Then E is infinite and for all integers $n \geqslant \rho+1$ in E, we have

$$\begin{aligned} \deg B_n &\geqslant \deg\Lambda_{n,\rho} - \frac{q^{n-\rho+1}-1}{q-1}\deg A_0 \\ &= q^n(h_n - h_{\rho-1}) - \frac{q^{n-\rho+1}-1}{q-1}\deg A_0. \end{aligned}$$

However by hypothesis, we also have

$$\deg B_n \leqslant (q-1)q^{n-1}h_{n-1} - d_nq^n.$$

Consequently we obtain

$$\begin{aligned} q^n(h_n - h_{\rho-1}) - \frac{q^{n-\rho+1}-1}{q-1}\deg A_0 &\leqslant (q-1)q^{n-1}h_{n-1} - d_n q^n \\ &\leqslant q^n h_{n-1} - d_n q^n, \end{aligned}$$

which implies immediately

$$d_n + (h_n - h_{n-1}) - h_{\rho-1} \leqslant \deg(A_0) < +\infty.$$

But $h_n \geqslant h_{n-1}$ and $d_n \to +\infty$ $(n \to +\infty)$. Thus we arrive at a contradiction. Hence, Case I cannot hold and S is infinite.

Case II: the set S is infinite. Let $0 \leqslant j \leqslant r$ be an integer. For $n \to +\infty$,

$$\begin{aligned} & q^j v(n-j+1) + \deg Q_n \\ = \;& q^j[\max(\deg B_{n-j+1}, 0) - \deg G_{n-j+1}] + \deg G_n \\ \leqslant \;& (q-1)h_{n-j}q^n - d_{n-j+1}q^{n+1} - q^{n+1}h_{n-j+1} + q^n h_n \\ = \;& -q^{n+1}(d_{n-j+1} - h_n + h_{n-j+1}) - (q-1)q^n(h_n - h_{n-j}) \\ \leqslant \;& -q^{n+1}\eta \to -\infty. \end{aligned}$$

Therefore, for all integers $0 \leqslant j \leqslant r$, we have

$$\lim_{n\to+\infty} |Q_n|_\infty \, \delta_{n-j}^{q^j} = \lim_{n\to+\infty} q^{q^j v(n-j+1)+\deg Q_n} = 0.$$

But $\deg(Q_n\beta_n) \geqslant 0$ for all integers $n \in S$. So

$$\lim_{S\ni n\to+\infty} \frac{|\beta_n|_\infty}{\delta_{n-j}^{q^j}} = +\infty,$$

and thus α is transcendental over $\mathbf{K}$ by Theorem 1. □

Remark: V. Laohakosol *et al.* obtained five criteria in [16] by Wade's method, which generalizes respectively the five classical results of L. I. Wade [33, 34, 35]. Although apparently all of them are quite different from Theorem 1, they can however be deduced from the latter. These five results can be divided into two classes. The first class contains the first, the fourth, and the fifth criterion, of which all the involved formal power series converge relatively quickly, and thus are easy to treat. The second class consists of the second, and the third criterion. This time the formal power series in question converge rather slowly, and thus are difficult to settle down. Since the underlying ideas are the same and to avoid some technical and routine verifications, we have only established by Theorem 1 some special cases (see Theorem 10 and Theorem 7) of the first two criteria in [16], leaving the reader to check the rest. See also [41] for the last two criteria.

Acknowledgments. Part of the work was done while Jia-Yan Yao visited the Laboratoire de Recherche en Informatique (Orsay) during 2002-2003, and he would like to thank his colleagues, in particular, Jean-Paul Allouche, for their generous hospitality and interesting discussions. He would like also to thank the National Natural Science Foundation of China (Grant no. 10671104), the Chern Institute of Mathematics (Tianjin), the Morningside Center of Mathematics (CAS), and the China Scholarship Council for partial financial support. He would like equally to thank Jean-Paul Allouche for having drawn his attention to the work [16]. The first author was supported by NSA grant H98230-08-1-0049. Finally we would like to thank very heartily the anonymous referee for his careful reading and valuable suggestions.

References

[1] J.-P. Allouche, *Automates finis en théorie des nombres.* Exposition. Math. **5** (1987), 239-266.

[2] J.-P. Allouche, *Note sur un article de Sharif et Woodcock.* Sém. de Théorie des Nombres de Bordeaux **1** (1989), 163-187.

[3] J.-P. Allouche, *Transcendence of the Carlitz-Goss gamma function at rational arguments.* J. Number Theory **60** (1996), 318-328.

[4] G. W. Anderson, W. D. Brownawell, and M. A. Papanikolas, *Determination of the algebraic relations among special* Γ*-values in positive characteristic.* Ann. of Math. (2) **160** (2004), 237–313.

[5] F. Beukers, *A refined version of the Siegel-Shidlovsky theorem.* Ann. of Math. (2) **163** (2006), 369-379.

[6] F. Beukers and G. Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$. Invent. Math. **95** (1989), 325-354.

[7] C.-Y. Chang, M. Papanikolas, D. Thakur, and J. Yu, *Algebraic independence of arithmetic gamma values and Carlitz zeta values.* Adv. Math. **223** (2010), 1137-1154.

[8] L. Denis, *Transcendance et dérivées de l'exponentielle de Carlitz.* In: Séminaire de Théorie des Nombres (Paris, 1991-92), S. David (Ed.), Progr. Math. **116**, Birkhäuser (1993), 1-21.

[9] L. Denis, *Un critère de transcendance en caractéristique finie.* J. Algebra **182** (1996), 522-533.

[10] L. Denis, *Valeurs transcendantes des fonctions de Bessel-Carlitz.* Ark. Mat. **36** (1998), 73-85.

[11] D. Goss, *Basic Structures of Function Field Arithmetic.* Second edition. Springer (1998).

[12] T. Harase, *Algebraic elements in formal power series rings.* Israel J. Math. **63** (1988), 281-288.

[13] Y. Hellegouarch, *Une généralisation d'un critère de de Mathan.* C. R. Acad. Sci. Paris, Sér. I, Math., **321** (1995), 677-680.

[14] A. Kochubei, *Evolution equations and functions of hypergeometric type over fields of positive characteristic*, Bull. Belg. Math. Soc. Simon Stavin **14** (2007), 947-959.

[15] H. Kornblum, *Uber die Primfunktionen in einer Arithmetischen progression.* Math. Z. **5** (1919), 100-111.

[16] V. Laohakosol, K. Kongsakorn, and P. Ubolsri, *Some transcendental elements in positive characteristic.* Science Asia **26** (2000), 39-48.

[17] J. Liouville, *Sur des classes très étendues de quantités dont la valeur n'est ni rationnelle ni même réductible à des irrationnelles algébriques.* C. R. **18** (1844), 883-885, 910-911.

[18] K. Mahler, *On a theorem of Liouville in fields of positive characteristic.* Canadian J. Math. **1** (1949), 397-400.

[19] B. de Mathan, *Irrationality measures and transcendence in positive characteristic.* J. Number Theory **54** (1995), 93-112.

[20] M. Mendès France and J.-Y. Yao, *Transcendence and the Carlitz-Goss gamma function.* J. Number Theory **63** (1997), 396-402.

[21] M. Papanikolas, *Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms.* Invent. Math. **171** (2008), 123-174.

[22] F. Pellarin, *Aspects de l'indépendance algébriques en caractéristique non nulle, d'Après Anderson, Brownawell, Denis, Papanikolas, Thakur, Yu.* Séminaire Bourbaki, no. **973**, March 2007.

[23] H. Sharif and C. F. Woodcock, *Algebraic functions over a field of positive characteristic and Hadamard products.* J. Lond. Math. Soc. **37** (1988), 395-403.

[24] S. M. Spencer, Jr., *Transcendental numbers over certain function fields.* Duke Math. J. **19** (1952), 93-105.

[25] D. S. Thakur, *Hypergeometric functions for function fields.* Finite Fields Appl. **1** (1995), 219-231.

[26] D. S. Thakur, *Transcendence of gamma values for* $\mathbb{F}_q[T]$. Ann. of Math. (2) **144** (1996), 181-188.

[27] D. S. Thakur, *An alternate approach to solitons for* $\mathbb{F}_q[T]$. J. Number Theory 76 (1999), 301-319.

[28] D. S. Thakur, *Hypergeometric functions for function fields II.* J. Ramanujan Math. Soc. **15** (2000), 43-52.

[29] D. S. Thakur, *Integrable systems and number theory in finite characteristic.* Advances in nonlinear mathematics and science. Physica D. **152/153** (2001), 1-8.

[30] D. S. Thakur, *Function Field Arithmetic.* World Scientific Publishing Co., Inc. (2004).

[31] D. S. Thakur, Z.-Y. Wen, J.-Y. Yao, and L. Zhao, *Hypergeometric functions for function fields and transcendence.* C. R. Acad. Sci. Paris, Sér. I **347** (2009), 467-472.

[32] F. R. Villegas, *Integral ratios of factorials and algebraic hypergeometric functions.* arXiv:math/0701362.

[33] L. I. Wade, *Certain quantities transcendental over $GF(p^n, x)$.* Duke Math. J. **8** (1941), 701-720.

[34] L. I. Wade, *Certain quantities transcendental over $GF(p^n, x)$, II.* Duke Math. J. **10** (1943), 587-594.

[35] L. I. Wade, *Two types of function field transcendental numbers.* Duke Math. J. **11** (1944), 755-758.

[36] L. I. Wade, *Remarks on the Carlitz ψ-functions.* Duke Math. J. **13** (1946), 71-78.

[37] J. Yu, *Transcendence and Drinfeld modules.* Invent. Math. **83** (1986), 507-517.

[38] J.-Y. Yao, *Critères de non-automaticité et leurs applications.* Acta Arith. **80** (1997), 237-248.

[39] J.-Y. Yao, *Some transcendental functions over function fields with positive characteristic.* C. R. Acad. Sci. Paris, Sér. I **334** (2002), 939-943.

[40] J.-Y. Yao, *Carlitz-Goss gamma function, Wade's method, and transcendence.* J. Reine Angew. Math. **579** (2005), 175-193.

[41] J.-Y. Yao, *A transcendence criterion in positive characteristic and applications.* C. R. Acad. Sci. Paris, Sér. I **343** (2006), 699-704.

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