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# Integrable systems and number theory in finite characteristic☆

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Dedicated to Vladimir Zakharov on his 60th birthday

**Abstract**

The purpose of this paper is to give an overview of applications of the concepts and techniques of the theory of integrable systems to number theory in finite characteristic. The applications include explicit class field theory and Langlands conjectures for function fields, effect of the geometry of the theta divisor on factorization of analogs of Gauss sums, special values of function field Gamma, zeta and $L$-functions, analogs of theorems of Weil and Stickelberger, control of the intersection of the Jacobian torsion with the theta divisor. The techniques are the Krichever–Drinfeld dictionaries and the theory of solitons, Akhiezer–Baker and tau functions developed in this context of arithmetic geometry by Anderson. © 2001 Elsevier Science B.V. All rights reserved.

*Keywords:* Drinfeld modules; Soliton; Gamma; Zeta; Theta; Shtuka

## 1. Introduction

This paper is an overview of some relatively recent work in number theory finding some of its inspiration in integrable systems theory. In the early 19th century, Abel and Jacobi, in their work on differential equations, used special theta function solutions to linearize flows of completely integrable Hamiltonian systems. Thus, for several classical systems, one gets linear flows on a real sub-torus inside the Jacobian (a complex torus) associated to a Riemann surface. In the early 20th century, Burchnall, Chaundy and Baker studied commuting pairs of ordinary differential operators using transcendental function theory (Akhiezer–Baker functions) associated to the Jacobians to give explicit constructions of the additive structure on these tori — what is more generally referred to nowadays as Bäcklund transformations. This was rediscovered and extended in 1970s by Krichever to apply to the so-called integrable partial differential equations or soliton equations. He explored infinitesimal deformations of such pairs of commuting operators and used them to give algebro-geometric solutions in terms of theta functions to partial differential equations such as KdV (see [17]).

Around the same time, Drinfeld had introduced elliptic modules, called Drinfeld modules now (studied in the special case by Carlitz in 1930s). He saw the analogy with the Krichever ideas and formulated a precise version of this correspondence between geometric data and operator data. This is commonly referred to as the Krichever–Drinfeld dictionary. Drinfeld's motivation was to attack an analog of conjectures of Langlands in number theory. More recently, Anderson has used this dictionary to great effect to develop geometric tools for studying objects in number theory such as values of gamma and zeta functions, which occur in the context of Drinfeld module theory. We would like to give a succinct overview of these exciting

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developments. For more details, references are provided, e.g. [11,16].

We will start with a very brief mention of what number theory in finite characteristic is about, just to give some perspective and to fix some notation (more will come in the discussion of applications). Then, we will describe the Krichever and Drinfeld dictionaries and some of their applications. This will be followed by a discussion of Anderson's ideas and their applications. Finally, we will provide a little more detail on the connections with techniques and concepts of integrable systems theory, such as Akhiezer–Baker functions and solitons. This paper is primarily addressed to those who know the theory of integrable systems, but we also explain some key points informally and give some basic references for those with only number theory background.

## 2. Number theory in finite characteristic

Number theory in finite characteristic is also called arithmetic of function fields or arithmetic of curves over finite fields depending on what aspects we want to emphasize. Here, as analogs to the basic objects of number theory:

- $\mathbb{Q}$: the field of rational numbers;
- $\infty$: the infinite place corresponding to the usual absolute value;
- $\mathbb{Z}$: the ring of integers;
- $\mathbb{R}$: the field of real numbers;
- $\mathbb{C}$: the field of complex numbers;

we look at:

- $\mathbb{F}_q(t)$: the field of rational functions in a variable $t$ over a finite field $\mathbb{F}_q$;
- $\infty$: the 'infinite' place corresponding to the degree;
- $\mathbb{F}_q[t]$: the ring of polynomials in $t$ over $\mathbb{F}_q$;
- $\mathbb{F}_q((1/t))$: the field of Laurent series in $1/t$;
- $C_\infty$: the completion of an algebraic closure of $\mathbb{F}_q((1/t))$.

Number fields are the finite extensions of $\mathbb{Q}$, whereas the function fields are the finite extensions $K$ of $\mathbb{F}_q(t)$. We can choose an arbitrary place $\infty$ of $K$. (Here the infinite place should be thought of as the place where we allow poles or denominators. In general, 'place' in number theory corresponds to a notion of size or absolute value and in addition to the usual notion of absolute value, there are also $p$-adic absolute values for each prime $p$.) Let $A$ be the ring of functions with poles only at $\infty$, let $K_\infty$ be the completion and let $C_\infty$ be the completion of an algebraic closure of $K_\infty$. These number fields and function fields are collectively called global fields and are usually studied together, because, e.g. the property of having all of the absolute values (sizes) linked through a product formula and having only finitely many remainders (finite residue fields), when you divide, characterize these fields. (The last property is the reason for restricting consideration to function fields defined over finite fields rather than over $\mathbb{C}$.) The ring $A$ is a Dedekind domain (which essentially means that factorization theory works in a fashion analogous to the number field situation) which sits discretely in $K_\infty$ with a compact quotient, similar to the number field situation, $\mathbb{Z}$ sitting in $\mathbb{R}$, for example. In fact, the basic algebraic number theory and class field theory for both go in parallel. Multiple, imperfect but close analogies exist at various levels: tools are different, but many theorems are parallel, so in addition to the intrinsic interest, it is also a very good testing ground for conjectures in the number field situation. For example, the strongest evidence for the Riemann hypothesis is the analog proved by Weil (and Deligne, etc.) whose techniques in turn help develop the Iwasawa theory. A similar situation exists (or existed) for the conjectures of Mordell, Stark, Birch and Swinnerton-Dyer (and Langlands as we will see next) (see references in [12]).

## 3. The Krichever and Drinfeld dictionaries

Now we will recall (in the simplest case of 'rank' and 'dimension' one) what in Mumford's words [16] is 'A remarkable dictionary discovered by Krichever based on suggestions in the work of Zakharov and Shabat, where they attempted to find a common formalism for the inverse scattering method of integrating certain nonlinear partial differential equations'.

*Krichever dictionary*. Let $k$ be a field of characteristic 0. Then there is a natural bijection between the sets of data '$X$ a complete (possibly singular) curve over $k$, $P$ a smooth point on it, a fixed isomorphism of the tangent space at $P$ with $k$ and a torsion-free, rank one sheaf $\mathcal{F}$ (i.e. a line bundle in the non-singular case) on $X$ with $h^0 = h^1 = 0$' and the data 'commutative subring $R$ (containing $k$) of $k[[t]][\mathrm{d}/\mathrm{d}t]$ having two operators of relatively prime orders, and with $R_1$ and $R_2$ identified if there is $u \in k[[t]]^\times$ with $R_1 = uR_2u^{-1}$'.

(For those unfamiliar with the language of sheaves, bundles or cohomology, the phrase 'torsion-free ... $h^0 = h^1 = 0$' may be replaced, at least in the non-singular case, by 'a non-special divisor on $X$ of degree $g-1$, where $g$ is the genus of X'.) Inspired by these ideas Drinfeld [11,16] explained the following.

*Drinfeld dictionary*. Let $k$ be a field (of finite characteristic) on which $F(x) = x^q$ is an automorphism of infinite order (with fixed field $k_0 = \mathbb{F}_q$). Then there is a natural bijection between the sets of data '$X_0$ a complete curve over $k_0$, $P_0$ a smooth $k_0$-rational point on it, $\mathcal{F}$ a torsion-free rank one sheaf (called a Shtuka) on $X := X_0 \times_{k_0} k$ with $h^0 = h^1 = 0$ and isomorphism $(1_{X_0} \times F)^*\mathcal{F} \equiv \mathcal{F}(P_0 - P_1)$ for some smooth point $P_1$ (distinct from $P_0$) on $X$' and the data 'commutative subring $R$ of $k\{F\}$ (the non-commutative polynomial ring in 'Frobenius' $F$) containing $k_0$, having two operators of relatively prime degree (in $F$) and with $R_1$ and $R_2$ identified, if there is $u \in k^\times$ with $R_1 = uR_2u^{-1}$'.

(Note that when we are in characteristic $p$, we have $(x+y)^q = x^q + y^q$, when $q$ is a power of $p$. So the $q$th power operator is then a linear operator. Just as the solution space for the usual differential equations is a vector space over $\mathbb{C}$, the solution space for polynomial equations in $F$ is a vector space over $\mathbb{F}_q$. Of course, this analogy does not work too literally and the point of the dictionary is to make precise the points of contact.)

Now we sketch very briefly (see [9,11,16,17,20] for more details) some ideas behind (a) the Krichever dictionary and (b) how we express the coefficients of commuting operators in terms of the Akhiezer–Baker function $\psi$ and theta functions. (We do this now, so that a reader familiar with these techniques, but possibly in a different setting, can make connections to the setting here. We do not need (b) until the section on techniques and concepts.)

(a) From $R$ we recover $X - P$ as Spec($R$) (defined to be the set of prime ideals of $R$ — the classical points are the maximal ideals of $R$) and the map taking the operator $r \in R$ to its degree is the map giving the valuation corresponding to $P$ (i.e. the order of the operator is the order of pole at $P$). Finally, the joint eigenspaces of the operators glue to give $\mathcal{F}$. To go back, we construct a (most trivial isospectral) deformation $\mathcal{F}^*$ over $X \times_k k[[t]]$ of $\mathcal{F}$, by gluing $\mathcal{F}$ to itself on a small punctured disc around $P$ by $\exp(t/z)$, where $z$ is a local coordinate at $P$, and also a differential operator $\nabla : \mathcal{F}^* \to \mathcal{F}^*(P)$ by using $\mathrm{d}/\mathrm{d}t$ on the restriction to $X - P$. Then, we use a trivialization outside a small neighborhood of $P$ to get a section of $\mathcal{F}^*(P)$ which will be the joint eigenfunction $\psi(z, x)$ and then the desired embedding of $R = H^0(X - P, \mathcal{O}_X)$ into $k[[t]][\mathrm{d}/\mathrm{d}t]$. In fact, for any $r \in R$, $r\psi$ is then seen to be of the form $\sum a_j(t)\nabla^j\psi$ and $r$ maps to $\sum a_j(t)(\mathrm{d}/\mathrm{d}t)^j$. (The full isospectral deformation is obtained by replacing $k[[t]]$ by $k[[t_1, t_2, \dots]]$ and $\exp(t/z)$ by $\exp(\sum t_j/z^j)$ in the above.)

(b) The joint spectrum $S$ of the commuting operators $D_1, D_2$ is a curve (with the same function field as $X$, if the operators have relatively prime degrees) in $\mathbb{C}^2$ given by the equation $\det(D_2 : V_{1,\lambda} - \mu\,\mathrm{Id}) = 0$, where $V_{1,\lambda}$ is the λ-eigenspace of $D_1$. We can describe poles and the essential singularity behavior near $\infty$ of the joint eigenfunction $\psi(z, x)(z \in S,\ x \in \mathbb{C})$, which allows us to express it in terms of theta functions. We can also go in the opposite direction and reconstruct the differential operators of which $\psi$ is the joint eigenfunction. Writing down the eigen-equation for this known function then allows us to express the coefficients of the operators in terms of theta or tau.

## 4. Applications

(I) *Explicit class field theory and Langlands conjectures*. Such an embedding (possibly of higher rank in more generality) $A \subset k\{F\}$ (say $a \to \rho_a$) is basically a Drinfeld $A$-module [10] ($A$ acts on the additive group of $k$ via non-trivial action via $\rho$). For example, the simplest (Carlitz) $\mathbb{F}_q[t]$-module is given by $\rho_t = t + F$.

For $\rho$ defined over $C_\infty$, we can define the 'exponential' function $e = e_\rho$ to be the entire function defined on $C_\infty$ and given by a power series $e(z) = z + \sum e_i z^{q^i}$ satisfying $e(az) = \rho_a(e(z))$, a defining property analogous to $e^{nz} = (e^z)^n$ for the usual exponential. (Using $e(tz) = (t+F)\circ(e(z)) = te(z)+e(z)^q$, we find recursively that $e_i^{-1} = (t^{q^i}-t)\cdots(t^{q^i}-t^{q^{i-1}})$.) The exponential is periodic with period lattice of (analytic) rank $r$ over $A$, where $r$ is also the 'algebraic' rank obtained via $F$-degrees of $\rho_a$'s and is also the 'geometric' rank of $\mathcal{F}$. This coming together of algebra, analysis, geometry and arithmetic is the strength of the subject. For example, the Carlitz module is rank one and so its period lattice can be written as $\tilde{\pi}A$ for some $\tilde{\pi}C_\infty$. This lattice can be thought of as analog of $2\pi\mathrm{i}\mathbb{Z}$. (The analog $\tilde{\pi}$ of $2\pi\mathrm{i}$ we thus get is well known to be transcendental over $K$ just as $2\pi\mathrm{i}$ is transcendental over $\mathbb{Q}$.) In this case (and much more generally in the *rank one* case, after suitable normalizations), adjoining the torsion $\{z \in C_\infty : \rho_u(z) = 0\}$ (expressed analytically as $\{e(a\tilde{\pi}/u) : a \in A\}$ in analogy with $\exp(2\pi\mathrm{i}k/n)$) to $K$ gives abelian extensions which are analogs of cyclotomic fields. (Recall that the cyclotomic fields are fields gotten by adjoining roots of unity to $\mathbb{Q}$.) This is the explicit class field theory (see [12] and references therein) of Carlitz–Drinfeld–Hayes.

For example, the Hilbert class field for $A$, i.e. the maximal abelian unramified extension of $K$ split at $\infty$, is just the smallest field of definition of $\rho$. (The Hilbert class field of $\mathbb{Q}$ is $\mathbb{Q}$ itself. For an imaginary quadratic field, its Hilbert class field can be obtained by adjoining appropriate values of modular $j$-function. In general, there is no such simple recipe for number fields.) Also, if using the division algorithm in $k\{F\}$ we define $\rho_I$, for $I$ an ideal of $I$, then its first coefficient is a generator of $I$ in the Hilbert class field. Thus, we see explicitly how all the ideals become principal (i.e. generated by one element) when extended to the Hilbert class field. The maximal abelian extension is obtained by taking the compositum of these cyclotomic ones (where we have to use all (or at least two distinct) places at infinity).

Krichever and Drinfeld dictionaries generalize [16] to all ranks and the arithmetic of rank $n$ objects is linked with $\mathrm{Gl}_n$. (Rank 2 objects are close to elliptic curves, modular forms, etc.) Drinfeld looked at the moduli of these objects (first of Drinfeld modules and then of more general Shtukas motivated by these dictionaries) and showed that on their etale cohomologies we get Galois and automorphic representations connected via the Langlands correspondence. Thus, in the 1970s, Drinfeld proved the Langlands conjectures for $\mathrm{Gl}_2$ (where the moduli is a curve, so the geometry is easier) over function fields and outlined the general approach. Deligne and Drinfeld also proved local Langlands for $\mathrm{Gl}_2$. This was generalized to $\mathrm{Gl}_n$ by Laumon, Rappoport, Stuhler in the early 1990s. After a lot of work (see [14,15] for references) spread over the last 25 years by Drinfeld, Deligne, Kazdhan, Flicker, Pink, Laumon, etc. on compactification and the trace formula in finite characteristic, finally the proof of global Langlands for $\mathrm{Gl}_n$ over function fields has been announced by Lafforgue in the summer of 1999.

(II) *Factorizations of Gauss sums for function fields*. Mixing the cyclotomic theory above with the traditional cyclotomic theory of roots of unity (constant field extensions), I had defined Gauss sums and proved analogs of classical factorization (known as the Stickelberger theorem) and many other results. In the function field theory of Artin–Weil as well as for Drinfeld's theory mentioned above, all function fields are basically on an equal footing in analogy with $\mathbb{Q}$. But for the factorization problem, the higher genus $K$ turned out to be very different in that the Gauss sums made up from $\wp$ torsion had primes not above $\wp$ in their factorizations. These non-classical factorizations were mysterious, at first. Then using the Drinfeld dictionary it was proved [23] that the function $f$ (arising in the isomorphism of the dictionary, on $X(k)$ with divisor $V^F - V + \xi - \infty$, where $V$ is an effective divisor of degree equal to genus of $X$ and corresponds to $\mathcal{F}$ and $\xi$, i.e. $P_1$, is a generic point of $X$) specializes at a geometric point above $\wp$ to the corresponding Jacobi sum. So the prime factorization gets related to $V$ or $\mathcal{F}$. For such Shtukas $\mathcal{F}$, Drinfeld had proved that the Euler characteristic being zero already implies $h^0 = h^1 = 0$, but the cohomology jumps are codified in the theta divisor. So the upshot was that the geometry of the theta divisor explained the prime factorization for general $X$. As an application, an analog of the Gross–Koblitz formula (whose proof classically used crystalline techniques) connecting the Gauss sums to values at fractions of $\wp$-adic gamma was also proved.

These two applications so far used objects motivated by the theory of integrable systems, but not much of the techniques of this theory. Anderson's idea [2–5] was to borrow the deformation techniques, the Jacobian flow techniques used in integrable systems theory for solving soliton equations to number theory in finite characteristic. Anderson developed amazing machinery generalizing the dictionaries to higher dimensions and making systematic use of generic points for deformations.

(III) *Special values of Gamma functions and the Brumer–Stark conjectures*. For the $C_\infty$-valued gamma function $\Gamma(z) := z^{-1}\prod(1+z/n)^{-1}$ on $C_\infty$ (where $n$ runs through monic polynomials in $\mathbb{F}_q[t]$), we know [21] functional equations, interpolations, special cases of Chowla–Selberg relations (expressing the periods of elliptic curves admitting complex multiplications in terms of gamma values at fractions) analogs and transcendence of some values at fractions (by connecting them to the periods of Drinfeld modules). Anderson realized the potential of soliton theory and developed it. By efforts of Anderson, his student Sinha [2,18,19] and by more recent work of Brownawell and Papanikolas [7], we know that all the gamma values at proper fractions occur as periods or quasi-periods of certain $t$-motives (analogs of Jacobians of quotients of Fermat curves), which are higher-dimensional generalizations of Drinfeld modules developed by Anderson [1]. It follows by transcendence theory (see [25] and references therein) of Jing Yu, that all these values are transcendental and in fact, all the linear relations between them over $\bar{K}$ are known. (For the usual gamma function, on the other hand, only the fractions with denominators dividing 4 or 6 have been handled. Chudnovsky [8, p. 8], in his ICM survey, called the generalization the most important and difficult problem in transcendence theory.) At the same stroke, using his soliton tools, Anderson [2] proved a two-dimensional version of the Stickelberger theorem giving ideal class annihilators and showed that the resulting Stark units align into a Hecke character. (This was also proved independently by Hayes [13].)

(IV) *Special values of zeta- and L-functions*. Carlitz had proved an analog of Euler's result on the Riemann zeta function $\zeta_{\mathrm{R}}$, which states that $\zeta_{\mathrm{R}}(2m)/(2\pi\mathrm{i})^{2m} \in \mathbb{Q}$, by showing that for $\zeta(s) := \sum n^{-s} \in \mathbb{F}_q((1/t))$, $s \in \mathbb{Z}_{>0}$ (where the sum is over the monic polynomials $n$ in $\mathbb{F}_q[t]$), we have $\zeta(s)/\tilde{\pi}^s \in \mathbb{F}_q(t)$, for $s$ even (meaning in this context that $s$ is a multiple of $q-1$). (See [12] for more on arithmetic and analytic properties of this $\zeta$ and generalizations.) In [6], it was proved that for any positive integer $s$, $\zeta(s)$ is essentially a logarithm (associated to the $s$th tensor power of the Carlitz module, an analog of the $s$th tensor power $\mathbb{Z}(s)$ of the Tate motive) evaluated at an explicit algebraic quantity which is torsion exactly when $s$ is even. Hence, by an analog (see [25] and references therein), due to Jing Yu, of Hermite–Lindemann theorems on transcendence of the logarithm, we see that $\zeta(s)$, $\zeta(s)/\tilde{\pi}^s$ and also the interpolated values $\zeta_\wp(s)$ are all transcendental for odd $s$. (Here, for a prime $\wp$ (an irreducible polynomial), $\zeta_\wp$ denotes the function of a $\wp$-adic variable obtained from interpolating suitable values of $\zeta$.) For the Riemann zeta function only the irrationality of $\zeta_{\mathrm{R}}(3)$ is known. Again, Anderson [3,5] realized the applicability of the soliton techniques and generalized to zeta- and $L$-functions for general $A$ (the class number one situation was handled in [22] by more ad hoc methods). The development has given rise to new analogs of cyclotomic units, Kummer–Vandiver conjecture with many consequences.

(V) *Torsion points on theta divisors*. Anderson [4] also developed $p$-adic soliton theory, in addition to the finite characteristic theory mentioned above (Drinfeld's dictionary [11,16] generalizes to any characteristic) and proved a quantitative result on the question of the torsion points of the Jacobian (of a quotient of Fermat curve) that lie on the theta divisor. The finiteness (in much more generality) was conjectured by Lang, proved by Raynaud and the later quantitative results of Coleman, etc. used quite different techniques. The connection of this technique with [20] is explained in the introduction of [4]: Calculation of order of contact (modeled on [20]) between the theta divisor and a one-parameter subgroup $P$ obtained by exponentiating the tangent space to the curve at the base-point has a $p$-adic analog, where $P$ basically becomes the torsion point (and Dwork's exponential $\exp((-p)^{1/(p-1)}(t - t^p))$ figures as a one-parameter family of loops).

## 5. Techniques and concepts

*Techniques used in the applications (III) and (IV).* The key point in (III) is to interpolate using the soliton theory the partial gamma products (from the product for $\Gamma(a/f))P_i := \prod(1 + a/(fn))$, where $n$ runs through monic polynomials of degree $i$. Arithmetic of $\Gamma(a/f)$ is closely linked with that of the cyclotomic cover $X_f$ of conductor $f$ of $X_0$, which can be obtained by pullback under isogenies of embedding of $X_0$ in generalized Jacobians. Identifying closed points of $X$ with $k$-valued points of $X_0$, we can make use of generic points, e.g. function field of $X_0$-valued points of $X_0$, etc. if $k$ is chosen large enough. So, e.g. the functions on the product of $X_0$ with itself can be viewed as functions on $X$. Anderson's soliton $\phi = \phi(x, y)$ is a function on $X_f \times X_f$ such that restriction of $\phi$ to graph of $F^i$ ($i$th power of Frobenius) is $P_i$. Description of the soliton $\phi$ in terms of $\theta$ (or Sato's $\tau$ function) allows calculation of its divisor (i.e. the local multiplicities), which shows that there is a related multi-dimensional Shtuka with period essentially $\prod P_i$ as required. (For details on this part, see [18,19] or the informal description in the appendix of [24]). About (IV), we only note that it uses similar interpolation technique for partial zeta sums $S_i := \sum n^{-s}$, where now $n$ runs through monic polynomials of degree $i$. This interpolation problem is connected in some sense to that of $P_i$ by logarithmic derivatives turning products into sums (Newton identities relating power sums to symmetric functions of roots).

Let us see why the basics of the theory of integrable systems makes existence of such a $\phi$ plausible. I will try to explain it in two (necessarily oversimplified) ways. (But, I should stress that Anderson's work is highly non-trivial and not a matter of just following a simple dictionary as I will make it appear to be.) The first way is for those familiar with Krichever's original approach (see [16] or introduction to [17] or the brief sketch below) and the second one for those more familiar with Sato's approach (see [9,20]).

1. In Krichever's approach, we express coefficients of commuting operators (and their flows or deformations) in terms of the Akhiezer–Baker function or the Riemann theta function. The coefficients of $F^i$ in Drinfeld's commuting operators $\rho_a$ are specializations at $x = a$ of the Carlitz binomial coefficients

   $$\left\{ \begin{matrix} x \\ q^i \end{matrix} \right\},$$

   whereas $P_i$ are their specializations [21,22] at $x = a/f$ and hence can be expressed in terms of theta functions by same technology once we introduce generic points to take care of deformations algebraically. Here if we denote the logarithm function, namely the (normalized) inverse function to the Carlitz exponential, by $l(z)$, then the Carlitz binomial coefficients can be defined by

   $$e(x\, l(z)) = \sum \left\{ \begin{matrix} x \\ q^i \end{matrix} \right\} z^{q^i}$$

   in analogy with

   $$\exp(x \log(1 + z)) = (1 + z)^x = \sum \binom{x}{n} z^n.$$

2. The products $\prod_{i \le n} P_i$ can be handled by Moore's determinant identity [2,21]:

   $$|x_i^{q^{j-1}}|_{n\times n} = \prod_{m=1}^{n} \prod_{a_i \in F_q} \left( x_m + \sum_{i=1}^{m-1} a_i x_i \right).$$

   Such determinants fit in the $\tau$ function framework (theta has a determinantal formula), giving $P_i$ as a ratio of $\tau$'s. (There are analogs of the Jacobi–Trudi identity, Schur functions, etc. Basically, classical $n$ is replaced by $q^n$ in $q$-analogs, whereas it is replaced by $t^{q^n}$ in $\mathbb{F}_q[t]$ symmetric function theory.) If you consider the product as a ratio of a product over a coset $\{a + fn\}$ by a product over $\{fn\}$, then the soliton $\phi = \phi_{w+W}$ is thus realized as $\tau_{w+W}/\tau_W$. Here $W$ are $\mathbb{F}_q$-subspaces of $\mathbb{F}_q((1/t))$

which are discrete and co-compact (i.e. projection to $\mathbb{F}_q((1/t))/\mathbb{F}_q[[1/t]]$ having finite kernel and co-kernel (i.e. Fredholm)). The Akhiezer–Baker function $\psi_W \in W((x, y))^{\sim}$ (which is some kind of a completion that we will not specify) is defined as the unique element such that $t^{\text{index}(W)-1} E\psi_W \in 1 + (1/t)\mathbb{F}_q[[1/t]]((x, y))^{\sim}$, where $E(x, y, t) = \prod_1^{\infty}(1 - x^{q^i}t)^{-1}(1 - y^{q^i}t)$. Such a condition is equivalent to an infinite linear system (only certain powers of $t$ are allowed, so the rest are zero), whose determinant is $\tau$. The fact that a soliton occurs as a residue at $t = \infty$ of $\psi$ is Cramer's rule in this setting. (It is hoped that this incomplete description is still helpful to those familiar with [20] approach to give some indication of the connection of the ideas.)

Why is $\phi$ a soliton? We do not understand any analogy with waves having the soliton property (solitary waves asymptotically preserving shapes and velocity under collisions). But $\phi$ does arise via 'soliton theory' as the term is widely used. Also, as explained in [16], the Jacobian flows of Krichever data on singular forms of $\mathbb{P}^1$ with $n$ ordinary double points lead to $n$-solitons. Here we are looking at singular forms of $\mathbb{P}^1$ with modulus $f$: points in the support of $f$ occur with multiplicities dictated by $f$ and (generalized Jacobians are connected with $X_f$) a deformation calculation is giving rise to $\phi$. The partial Frobenius equations (37), Section 6.1.6 of Ref. [2] and equations on pages 306–308, and 313 of Ref. [24] are in some sense analogs of partial differential equations and theta descriptions of solitons, but there is no perfect analog of KdV that I know of.

Finally, in view of connections of the *Painlevé theory* with integrable systems, I will just mention the recent work of Katz and Sarnak where they connected the distribution of eigenvalue spacings of random matrices to that of spacings of zeros of (characteristic zero-valued) zeta functions attached to function fields of finite characteristic. Painlevé theory also seems relevant in some diophantine geometry and diophantine approximation results for function fields.

*Explicit Jacobian of a genus one curve*. We finish by mentioning another recent application of these ideas. Given a genus one curve $X$ defined over an algebraically closed field (e.g. $\mathbb{C}$), $X$ can be provided with a group law, turning it into an elliptic curve, by decreeing a suitable point on $X$ as the identity. This elliptic curve is the Jacobian abelian variety of $X$. Given a genus one curve $X$ over an arbitrary field $K$, the problem of explicitly describing its Jacobian variety (which is an elliptic curve having a point over $K$) becomes much more subtle. This was recently solved by Anderson (unpublished), by adapting the tau-function recipe for the Weierstrass $\wp$ function, which arises as a solution to KdV equation in a genus one setting, by using its determinantal formula and paying close attention to the rationality issues (which are absent over $\mathbb{C}$). Anderson also provides an explicit map from $X$ to its Jacobian. At present, this has been worked out for characteristic more than three.

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