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Iwasawa theory and Cyclotomic Function Fields

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Abstract. We will describe and put in the perspective of Drinfeld's theory, some therorems and conjectures relating class numbers and zeta values at positive and negative integers (as we will see, these are two distinct theories in contrast to the classical case), analogues of results and conjectures of Kummer and Vandiver, growth rates of class numbers, zeta measures and other aspects of Iwasawa theory.

Iwasawa theory started as an attempt by Iwasawa to carry out analogue for number fields of a well-developed theory for function fields, due to Andre Weil and others. This theory related the zeta function of the function field to the characteristic polynomial of Frobenius acting on p-power order torsion in the Jacobian of the corresponding curve. Here p can be any prime unequal to the characteristic. In this theory, to get good structural results, one needs to take algebraic closure of the finite field of constants of the function field and hence consideration of the tower of constant field extensions enters naturally. Since the constant field extensions are just the extensions obtained by adjoining roots of unity, to get a better analogy between p-power order torsion in the Jacobian on the function field side and the p-Sylow subgroup of the class group on the number field side, Iwasawa considered the tower of number fields obtained by adjoining p-power roots of unity or more generally $\mathbf{Z}_p$-extensions for some fixed prime p.

Over $\mathbf{Q}$, the cyclotomic extensions (i.e. (subfields of) the extensions obtained by adjoining the roots of unity) coincide with the abelian extensions by the Kronecker-Weber theorem. Over a function field this is far from the case. Indeed, the cyclotomic extensions are then just the constant field extensions and there are, of course, many more abelian extensions, for example, various Kummer and

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Artin-Schreier extensions. Carlitz, Drinfeld and Hayes [C2, D, H1] developed other families of abelian extensions of function fields, which also can be thought of as 'cyclotomic' because of the strong analogies with the classical case. We will now briefly describe this 'cyclotomic' theory. For more details see [H1,H2]. For the corresponding classical cyclotomic theory, see the books by S. Lang and L. Washington.

At the most basic level we have, as analogues of $\mathbf{Q}$, the archimedean place ∞, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ respectively, their counterparts: a function field K with a finite field $\mathbf{F}_q$ (of characteristic p) of constants, any place ∞ of K (we will assume it to be rational for simplicity), the ring A of integers outside ∞, K_∞ and the completion Ω of an algebraic closure of K_∞ respectively. The simplest example, where the analogies also turn out to be the strongest, is when $K = \mathbf{F}_q(T)$ and $A = \mathbf{F}_q[T]$.

Now the roots of unity can be interpreted as the torsion of the rank one object '$\mathbf{Z}$ inside the endomorphism ring of the multiplicative group' (where n is viewed as the n-th power map). In finite characteristic, we have a shorter supply of multiplicative functions, (i.e., there is no nontrivial 'exponential' from additive to multiplicative groups in characteristic p: we would have $e(0) = e(px) = e(x)^p$, for all x, for such an exponential e) but have a larger supply of additive functions. The endomorphism ring of the additive group in characteristic p is a huge non-commutative ring of polynomials in the Frobenius endomorphism. Hence Drinfeld considered, as an analogue of roots of unity, the torsion of a rank one object 'A inside this ring' (where $a \in A$ is viewed as the map $u \mapsto \rho_a(u) = \sum_{i=0}^{\deg a} a_i u^{q^i}$, with the normalizations $a_0 = a$ and $a_{\deg a} = \mathrm{sgn}(a)$). Here deg and sgn are the degree and a fixed sign function on A. The conditions on $\rho_a(u)$ assure that the a-torsion $\Lambda_a := \{u \in \Omega : \rho_a(u) = 0\}$ is an $A/(a)$ module of rank one. The simplest example is the 'Carlitz module' : $A = \mathbf{F}_q[T]$, $\rho_T(u) = Tu + u^q$. (Exercise: The T^2-th cyclotomic equation is $u^{q^2} + (T + T^q)u^q + T^2 u = 0$).

To the 'Drinfeld module' (*we have restricted the terminology more than in the original definition in* [D]) ρ one associates the exponential function $e\colon = e_\rho\colon \Omega \to \Omega$ defined to be an entire additive function $e(z) = \sum_{i=0}^\infty e_i z^{q^i}$ satisfying $e(az) = \rho_a(e(z))$ (in analogy with $e^{nz} = (e^z)^n$) for all $a \in A$ and normalized by $e_0 = 1$. The kernel of e, which can be thought of as an analogue of $2\pi i\mathbf{Z}$, can be written as $\tilde{\pi}A$ (if the class number of A is greater than one, it can be $\tilde{\pi}I$ for some ideal I of A and in fact there are class number many Drinfeld modules corresponding to these rank one A-lattices). In terms of these analogues of the exponential and $2\pi i$, the a-torsion can be written as $e(\tilde{\pi}a'/a)$ for some $a' \in A$ and can be thought of as analogue of n-th root of unity ζ_n or $1 - \zeta_n$. (See the next paragraph).

To illustrate these close analogies a little further, we note that if K is of class number one, and $\wp$ is a prime of A, then $K(\Lambda_\wp)$ is an extension of K with Galois group $(A/\wp)^*$, in which $\wp$ is totally ramified (so that the extension is geometric in contrast to the constant field extension), all other finite primes are unramified, and for $\lambda \in \Lambda_\wp - \{0\}$, (λ) is a prime above $\wp$. (Comparing this with the fact that

$1-\zeta_p$ is a prime above p in $\mathbf{Q}(\zeta_p)$, we see that λ is an analogue of $1-\zeta_p$. This reflects the fact that we are now dealing with the additive group rather than the multiplicative group.) The Galois action is given by $\mathrm{Frob}_v(\lambda) = \rho_v(\lambda)$. (For I an ideal of A, ρ_I is defined to be the monic generator of the ideal generated by ρ_i, $i \in I$). If we denote the 'maximal totally real (i.e., ∞ splits completely) subfield' by $K(\Lambda_\wp)^+$, the Galois group over it is $A^* = \mathbf{F}_q^*$. Compare to $\mathbf{Z}^* = \{\pm 1\}$ of the classical case. The respective cardinalities, namely $q-1$ and 2 play an analogous role and we call multiples of $q-1$ 'even'.

We remark that the arbitrary choice of the infinite place ∞, which made possible this strong analogy also makes all the abelian extensions obtained by adjoining the a-torsion tamely ramified at ∞ by the above. To get the maximal abelian extension of K one needs to play the whole game again by switching to a different infinite place and by taking the compositum of all such extensions.

We now turn to some deeper aspects of the cyclotomic theory. Let $L = K(\Lambda_a)$ and $F = K(\Lambda_a)^+$, where $a \in A$ is nonconstant. The intersection of the subgroup of L^* generated by the elements of $\Lambda_a - \{0\}$ with $\mathcal{O}_L^*$ is the group E of cyclotomic units. We denote by $h(R)$ the class number of R, for R a Dedekind domain or a field. We then have the following analogue of Kummer's theorem.

Theorem 1. (Galovich-Rosen [G-R]): *Let $A = \mathbf{F}_q[T]$ and $a = \wp^n$, where $\wp$ is a prime of A. Then $h(\mathcal{O}_F) = [\mathcal{O}_L^* : E]$.*

Remark: This was later generalized to the case of any a (i.e., an analogue of Sinnott's result). Shu has announced [S1] generalization to any A.

Sketch of the proof. We have $\mathcal{O}_L^* = \mathcal{O}_F^*$ just as in the classical case. Let $S = \{\infty_i\}$ denote the set of the infinite primes of F. Let $\mathrm{Div}^0(S) \supset P(S) \supset \mathcal{E}$ be the groups of divisors of degree zero supported on S, divisors of elements of $\mathcal{O}_F^*$ and the divisors of cyclotomic units respectively. It is elementary to see that the index in the theorem is equal to that of $\mathcal{E}$ in $\mathrm{Div}^0(S)$. The calculation of the divisors of cyclotomic units using the basis $\infty_i - \infty_0$ of $\mathrm{Div}^0(S)$ allows us to express this index as determinant, which by the Dedekind determinant formula, can be expressed as a product of certain character sums. Finally, the analytic class number formula for the Artin-Weil zeta and L-functions for the function fields identifies this product as the class number $h(F)$ of F. The theorem follows by noticing that $[\mathrm{Div}^0(S) : P(S)] = h(F)/h(\mathcal{O}_F)$. $\square$

Next we discuss Tate's proof of the analogue of the Stickelberger theorem. Let K' be a geometric (i.e., with the same field of constants) extension over K, with an abelian Galois group G. Let $Cl(K')$ be the class group of K' and let $\theta(T) = \sum_{\sigma \in G} Z(\sigma^{-1}, T)\sigma$ where Z is the partial zeta function for σ. Then the Stickelberger element is $\theta = \theta(1)$.

Theorem 2. (Tate [Ta]): *$(q-1)\theta \in \mathbf{Z}[G]$ kills $Cl(K')$.*

Sketch of the proof. Class group is just the group of $\mathbf{F}_q$- rational points of the Jacobian (of the corresponding curve), i.e., the part of the $\overline{\mathbf{F}_q}$- points of the Jacobian where the Frobenius acts as the identity. Now by fundamental results of Weil, the L-function of a character of G is the characteristic function of the Frobenius on the corresponding component of the Jacobian or rather the Tate module, and hence it kills the component when $T = F$ by the Cayley-Hamilton theorem. Hence $\theta(T)$, which is just a linear combination of L-functions with projection to the components operators, kills the class group when $T = 1$. (One needs $(q-1)$ factor to clear out the denominators to get polynomials, when one makes this sketch precise.) □

The interpretation of the L-function as the characteristic polynomial in the above is precisely the result on which Iwasawa's main conjecture is based. Since this is already known, the Gras conjecture giving the componentwise version of Theorem 1, which follows classically from the main conjecture, is known here. This was recognised in [G-S].

In contrast to the number field case, we have class groups of fields as well as the ring of integers outside the infinite places above the chosen ∞. Let $a = \wp$ be a prime of A of degree d, so that $L = K(\Lambda_\wp)$. Let C, $\tilde{C}$ be the p-primary components of the class groups of L and $\mathcal{O}_L$ respectively. Let W be the Witt ring of $A/\wp$. Then if w denotes the Teichmuller character, we have the decomposition $C \bigotimes_{\mathbf{Z}_p} W = \bigoplus C(w^k)$ into isotypical components according to the characters of $(A/\wp)^*$.

Theorem 3. (Goss- Sinnott [G-S])*: For $0 < k < q^d - 1$, $C(w^{-k}) \neq 0$ if and only if p divides $L(w^k, 1)$*

Sketch of the proof. The duality between the Jacobian and the p-adic Tate module T_p transforms the connection between the Jacobian and the class group in the proof of Theorem II to $T_p(w^{-k})/(1-F)T_p(w^{-k}) \cong C(w^{-k})$. On the other hand, we have a Weil type result: $\det(1 - F : T_p(w^{-k})) = L_u(w^k, 1)$. Here L_u is the unit root part of the L-function and hence has the same p-power divisibility as the complete L-function. Hence $\mathrm{ord}_p(L(w^k, 1))$ is the length of $C(w^{-k})$ as a $\mathbf{Z}_p[G]$-module and the theorem follows. □

Comparison with the corresponding classical result shows that we are looking at divisibilty by p, the characteristic, rather than the prime $\wp$ relevant to the cyclotomic field. To bring $\wp$ in, we need to look at another zeta function introduced by Carlitz and Goss:

Let $\zeta(s) := \sum n^{-s} \in K_\infty$, where the sum is over monic polynomials of A and s is a positive integer. If s is any integer,

$$\zeta(s) := \sum_{i=0}^{\infty} \sum_{\substack{\deg n = i \\ n \text{ monic}}} n^{-s}$$

makes sense and in fact belongs to A for s a negative integer, since in that case the second sum vanishes for large i.

The identification $W/pW \cong A/\wp$ provides us with the Teichmuller character $w\colon (A/\wp)^* \to W^*$ satisfying $w^k(n \mod \wp) = (n^k \mod \wp) \mod p$. Hence the reduction of L value in the theorem 3 modulo p is $\zeta(-k) \mod \wp$. (This works for k 'odd' (i.e. not a multiple of $q-1$), for 'even' k, we need to use 'the leading term' when there are 'trivial zeros'; but we will ignore this aspect below). Hence we get

Theorem 4. (Goss-Sinnott [G-S])*: For k 'odd', $0 < k < q^d - 1$, $C(w^k) \neq 0$ if and only if $\wp$ divides $\zeta(-k)$.*

For simplicity, we will now restrict to the case $A = \mathbf{F}_q[T]$. Classically, Bernoulli numbers occur in the special values of the Riemann zeta function at both positive and negative integers and these values are connected by the functional equation for the Riemann zeta function. In our case, there is no simple functional equation known and in fact we get two distinct analogues of Bernoulli numbers B_k (or rather the more fundamental B_k/k) both connecting to class groups: those coming from the positive values relate to class groups of rings of integers (see Theorem 6 below) in contrast to class groups of fields (as in Theorem 4).

Let us define the factorial function $\Pi(m)$ and Bernoulli numbers B_m by analogy with the classical case: For a positive integer m, define $\Pi(m) := \prod_{\wp} \wp^{m_\wp} \in \mathbf{F}_q[T]$, where $m_\wp := \sum_{e \geq 1} [m/\operatorname{Norm}(\wp)^e]$. Define $B_m \in \mathbf{F}_q(T)$ by the formula $z/e(z) = \sum B_m/\Pi(m) z^m$ (compare with the classical generating function $z/(e^z - 1)$). The connection with the special zeta values at the positive integers is through the following analogue of the Euler's result: $\zeta(m) = -B_m(2\pi i)^m/2(m!)$ for even m.

Theorem 5. (Carlitz [C1])*: For 'even' m, $\zeta(m) = -B_m \tilde{\pi}^m/(q-1)\Pi(m)$.*

Sketch of the proof. First note that $q - 1 = -1$ in the formula. The proof follows by taking the logarithmic derivative of $e(z)$ (to get the Bernoulli numbers through the generating function on one hand and to get zeta values using the geometric sum expansion through the product formula for $e(z)$ on the other hand) and by comparing coefficients. □

Since $\zeta(-k)$ turns out to be a finite sum of n^k's, by Fermat's little theorem, the $\zeta(-k)$'s satisfy Kummer congruences enabling us to define a $\wp$-adic interpolation $\zeta_\wp$. On the other hand, the B_m satisfy analogues of the von-Staudt congruences and the Sylvester-Lipschitz theorem. We have now two distinct analogues of B_k/k: $-\zeta(-k+1)$ for $k-1$ 'odd' on one hand and $\Pi(k-1)\zeta(k)/\tilde{\pi}^k$, with k 'even' on the other. It should be noticed that the shift by one does not transform 'odd' to 'even' unless $q = 3$, and we do not know any reasonable functional equation linking the two.

THEOREM 6. (OKADA, GOSS [O]): *Let $A = \mathbf{F}_q[T]$. Then for $0 < k < q^d - 1$, k 'even', if $\tilde{C}(w^k) \neq 0$, then $\wp$ divides B_k.*

SKETCH OF THE PROOF. We define analogues of Kummer homomorphisms $\psi_i \colon \mathcal{O}_F^* \to A/\wp$ $(0 < i < q^d - 1)$ by $\psi_i(u) = u_{i-1}$, where u_i is defined as follows. Let $u(t) \in A[[t]]$ be such that $u = u(\lambda)$ and define u_i to be $\Pi(i)$ times the coefficient of z^i in the logarithmic derivative of $u(e(z))$. Using the definition of the Bernoulli numbers, we calculate that the i-th Kummer homomorphism takes the basic cyclotomic unit $\lambda^{\sigma_a - 1}$ to $(a^i - 1)B_i/\Pi(i)$. If $\tilde{C}(w^k) \neq 0$, then by the componentwise version of Theorem 1 ('Gras conjecture'), $\psi_k(\lambda^{\sum w^{-k}(\sigma)\sigma^{-1}}) = 0$ and hence the calculation above implies that $\wp$ divides B_k. $\square$

Using her generalization of Theorem 1, Shu has announced [S2] a generalization of Theorem 6 to any A.

Now we describe some results by Anderson and myself [A-T] about the zeta function for $A = \mathbf{F}_q[T]$, defined above. Classical counterparts of these results are not known. To avoid defining a lot more terminology, we describe the result roughly as an expression of $\zeta(n)$ ($\zeta_\wp(n)$ resp.) (where n can be 'odd' as well as 'even') as a logarithm ($\wp$-adic logarithm resp.) of an explicit algebraic point on the n-th tensor power of the Carlitz module. Using this result together with his theorems on the transcendence properties of the exponential (analogues of Hermite- Lindemann, Gelfond-Schneider and Mahler theorems), Jing Yu has proved

THEOREM 7. (YU [Y]): *For a positive integer n, $\zeta(n)$ is transcendental over K and if further n is 'odd', then $\zeta(n)/\tilde{\pi}^n$ and $\zeta_\wp(n)$ are transcendental.*

Now we mention some curious consequence of these results. K. Kato has raised the question of whether, for a given n, $\wp$ divides $\zeta_\wp(n)$ for infinitely many $\wp$'s or not. The expressions in [A-T] mentioned above show that whether $\wp$ divides $\zeta_\wp(1)$ is equivalent to whether $\wp^2$ divides $\rho_{\wp-1}(1)$. This last statement is clearly an analogue of the well-known Wieferich criterion in classical cyclotomic theory: whether $(1+1)^{p-1} - 1 \equiv 0 \mod p^2$. We can then also write down 'higher Wieferich criteria' using higher zeta values. It might be interesting to understand their classical counterparts and their significance. It should be stressed though that classically the zeta function has a pole at $n = 1$, in contrast to the case here.

Two important statements in the cyclotomic theory concerning the class numbers are Kummer's result that p divides $h(\mathbf{Q}(\zeta_p)^+)$ implies that p divides $h(\mathbf{Q}(\zeta_p)^-)$ and the conjecture of Kummer and Vandiver that p does not divide $h(\mathbf{Q}(\zeta_p)^+)$. Ireland and Small [I-S] exhibited a simple example showing that an analogue of both these statements is false. Namely, if $A = \mathbf{F}_q[T]$, with $q = p = 3$ and $\wp = 2 + T^2 + T^4$, then p divides $h(\mathcal{O}_L^+)$ but does not divide $h(\mathcal{O}_L^-)$. We would like to point out that there are various possible analogues that might be explored. First note that we have p and $\wp$ instead of just p as in the classical case. How about checking the divisibilty by the norm of $\wp$? Some examples show that the analogue of the Kummer result still fails if we use the class numbers of fields, but it is not known with the rings of integers. Another naive analogy would be to note that p divides the order of a group if and only if $\mathbf{Z}/p$ sits in the group and we may want to ask whether $A/\wp$ sits in the group: a question a priori different from whether the norm divides the order. Instead of the class group, which is an abelian group or a $\mathbf{Z}$- module we may need to get some A-module to formulate an analogue. (See also [G3] pa. 391).

Let us now look at the class number growth in towers of fields. We can use the class number formula for the zeta function which gives $h = \prod_1^{2g}(1-\alpha_i)$, where α_i are the eigenvalues of Frobenius of absolute value $\sqrt{q}$, by Weil's result. (Analogue of the Riemann hypothesis). This implies that $(\sqrt{q}+1)^{2g} \geq h \geq (\sqrt{q}-1)^{2g}$, where g is the genus of the field. For the constant field extensions, on which the Iwasawa theory was based, we get for the n-th layer of a $\mathbf{Z}_l$ -tower the asymptotics $h_n \sim q^{gl^n}$, because $\prod \alpha_i = q^g$. Note that by class field theory, for $l \neq p$, all $\mathbf{Z}_l$-extensions are essentially constant field extensions, whereas for $l = p$, the characteristic, there are many more $\mathbf{Z}_p$-extensions. Gold and Kisilevsky [G-K] have shown that for geometric $\mathbf{Z}_p$-extensions, log $h_n \geq p^{2(n-n_1)-1}/3$ and in fact they could construct such towers with arbitrarily large growth.

Analogies we have been discussing suggest that we might look at $A_\wp$-towers rather than $\mathbf{Z}_p$-towers. But note that $A_\wp$'s are much too wild to arise as Galois groups in a similar fashion. The cyclotomic tower $K(\Lambda_{\wp^n})$ corresponds to an $A_\wp^*$-extension (a general theory of such extensions or of extensions over 'the first level' is not much developed yet; it is interesting to note that the dependence of the group on $\wp$ is just through its degree) and putting the ramification data mentioned above in the Riemann-Hurwitz formula gives $g_n \sim d(q^d-1)nq^{d(n-1)}$. This implies that log $h_n \asymp n(\mathrm{Norm}(\wp))^n$. A similar asymptotic is not known classically. It is known for the minus part in the classical as well as our case. How about the class numbers of the rings of integers?

We now come to the zeta measure associated to $\mathbf{F}_q[T]$. (See [T1], [G3] and references there). Under the Iwasawa isomorphism, $\mathbf{Z}_p$-valued measures on $\mathbf{Z}_p$ can be identified with power series in such a way that the convolution of measures corresponds to the multiplication of the corresponding power series. In fact, the binomial coefficients $\binom{x}{k}$ give basis of polynomials in x over $\mathbf{Q}$, which map $\mathbf{Z}$ to $\mathbf{Z}$ and the power series associated to the measure μ is just $\sum \mu_k X^k$ where $\mu_k := \int_{\mathbf{Z}_p} \binom{x}{k} d\mu$. The analogue for $A = \mathbf{F}_q[T]$ of $(1+t)^x - 1 = \sum \binom{x}{k} t^k$ is $\rho_x(t) = \sum \left\{ {x \atop q^k} \right\} t^{q^k}$ and in fact $\left\{ {x \atop q^k} \right\}$ gives a basis of 'additive' polynomials over K which map A into A. There is a way to extend this definition of binomial coefficients to any $\left\{ {x \atop k} \right\}$ and if we associate to an $A_\wp$-valued measure μ on $A_\wp$ a divided power series $\sum \mu_k (X^k/k!)$, with μ_k defined analogously, then the convolution corresponds to the multiplication of the divided power series.

Classically, the measure μ whose moments $\int_{\mathbf{Z}_p} x^k d\mu$ are $(1-a^{k+1})\zeta(-k)$ for some $a \geq 2$, $(a,p) = 1$ has the associated power series $(1+X)/(1-(1+X)) - a(1+X)^a/(1-(1+X)^a)$. We need a twisting factor in front of the zeta values to compensate for the fact that the zeta values are rational rather than integral, in contrast to our case. Comparison of this result with the following is not well-understood.

THEOREM 8. (See[T1]): *For $A = \mathbf{F}_q[T]$, the divided power series corresponding to the measure μ whose i-th moment is $\zeta(-i)$ is given by $\sum \mu_k (X^k/k!)$ with μ_k being $(-1)^n$ when $k = cq^n + (q^n - 1)$, $0 < c < q-1$, and $\mu_k = 0$ otherwise.*

The k's for which $\mu_k \neq 0$ can be characterised as those 'odd' k's for which any smaller positive integer has each base q digit no larger than the corresponding base q digit of k. For such k's, all the binomial coeffecients $\binom{k}{i}$ are nonzero modulo q. When q is a prime, this last property characterises such k's among the 'odd' numbers. Some other properties of k's are described in [T1, G3]. More results on the interesting influence of the base q digits on the zeta values and their orders of vanishing will be described elsewhere.

Let us now look at an analogue of Fermat equation, which was, of course, a motivation for the study of cyclotomic fields. The usual Fermat equation is well understood. Writing the Fermat equation as $z^p = y^p((x/y)^p - 1)$ to bring in the analogies with the cyclotomic theory, Goss [G2] looked at the equation in the cyclotomic theory for $A = \mathbf{F}_q[T]$ analogous to the one here, namely $z^{q^d} = y^{q^d} \rho_\wp(x/y)$. He proved various analogous features of the theory and made conjectures about its non-trivial solutions. (There are no non-trivial solutions except for some small exceptions, just as in the classical case). These have been recently settled in a very nice work [De] by Laurent Denis. The idea is to rewrite the equation as $(z/y)^{q^d} = \sum a_i (x/y)^{q^i}$, differentiate with respect to T (note $q = 0$ in characteristic p) and then clear out the denominators by multiplying by y^{q^d} to see that y divides a power of x. But essentially by the usual reductions, y could have been taken prime to x and hence apart from some low cases and trivial solutions, there are no more!

We end this paper by just remarking that there are analogues of Gauss sums, Gamma functions, Gross-Koblitz formula, etc. In fact there is a family of such objects associated to each 'cyclotomic family': one for the constant field extensions and one for the geometric extensions of Drinfeld. There is even more (see [T2] and references there) to the story when K is not $\mathbf{F}_q(T)$.

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