

**MULTIPLICATIVE RELATIONS BETWEEN COEFFICIENTS OF
LOGARITHMIC DERIVATIVES OF \mathbb{F}_q -LINEAR FUNCTIONS
AND APPLICATIONS**

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Dedicated to the memory of Professor Shreeram Abhyankar

ABSTRACT. We prove some interesting multiplicative relations which hold between the coefficients of the logarithmic derivatives obtained in a few simple ways from \mathbb{F}_q -linear formal power series. Since the logarithmic derivatives connect power sums to elementary symmetric functions via the Newton identities, we establish, as applications, new identities between important quantities of function field arithmetic, such as the Bernoulli-Carlitz fractions and power sums as well as their multi-variable generalizations. Resulting understanding of their factorizations has arithmetic significance, as well as applications to function field zeta and multizeta values evaluations and relations between them. Using specialization/generalization arguments, we provide much more general identities on linear forms providing a switch between power sums for positive and negative powers.

1. INTRODUCTION

Let \mathbb{F}_q be a finite field of characteristic p and consisting of q elements. Let F be a field containing \mathbb{F}_q , and let $f(z) = \sum f_i z^{q^i} \in F[[z]]$. Let $g(z) = \sum_{i=0}^{\infty} g_i z^{q^i}$ satisfy $f(g(z)) = z$. Carlitz [C1935, Thm. 6.1] showed that we also have $g(f(z)) = z$.

We consider

$$h(z) = zf'(z)/f(z), \quad a(z) = zf'(z)/(1-f(z))$$

and write $h(z) = \sum h_i z^i \in F[[z]]$, $a(z) = \sum a_i z^i$.

In the special case where $f(z)$ is a polynomial, writing $u = 1/z$, we consider power series expansions in u of h and a , normalising as follows. We write $-h(z) = \sum H_i u^i$, $-a(z) = \sum A_i u^i$.

If $\theta \in \mathbb{F}_q$, $f(\theta z) = \theta f(z)$ implies that $h(\theta z) = h(z)$ for $\theta \neq 0$, so that

$$h_i = H_i = 0 \quad \text{if } i \not\equiv 0 \pmod{q-1}.$$

This can also be seen by the standard geometric series development, which also implies that if f is of degree q^d , then

$$H_i = A_i = 0 \quad \text{if } i < q^d - 1.$$

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In general, the coefficients of $h(z) = zf'(z)/f(z)$ are rather complicated functions of the coefficients of f ; however, certain coefficients of the reciprocal can be expressed very simply.

Carlitz [C1935, 8.04, Thm. 8.1] showed (for $f_0 = 1$, easy to reduce to)

$$(1.1) \quad h_{q^k-1} = f_0^{q^k} g_k, \quad h_{pm} = h_m^p.$$

We claim (with no restriction on f_0) that for $m \geq 0$,

$$h_{pm} = h_m^p, \quad H_{pm} = H_m^p, \quad a_{pm} = a_m^p, \quad A_{pm} = A_m^p,$$

which follows by slight modification of Carlitz's argument: The claim follows immediately from there being no terms with z^{pm} (or u^{pm}) in $h - h^p$ or $a - a^p$, which follows (by factoring h^p or a^p out) from the fact that $(1/h)^{p-1} - 1$ or $(1/a)^{p-1} - 1$ are of the form $(wz^{-1} \pm 1 + \sum_{i>0} u_i z^{q^i-1})^{p-1} - 1$ (with $w = 0$ in the case of h) and thus considering exponents modulo p have no terms of the form claimed.

We will explain below old and new applications of understanding these coefficients in the function field arithmetic, and proceed to prove four families of new multiplicative relations for these four sets (Theorems 1, 3, 4, 6) of coefficients. Theorems 2 and 5 give general identities on linear forms from which these identities can be recovered by appropriate specializations. Our proofs make use of one set of identities to prove the other through Theorems 2 and 5.

Multi-variable power sums of certain types and more general products have occurred in recent interesting works of Federico Pellarin, Rudolph Perkins and Bruno Angles [P2012, Per2013, APp]. We hope that the results and ideas in the present work finds some applications also in these developments.

2. MAIN RESULTS

2.1. $h(z)$ in terms of z .

Theorem 1. *With the notation as above, for $1 \leq \ell \leq q$, and $0 \leq k_j \leq k$, with $1 \leq j \leq \ell$, we have*

$$(2.1) \quad \prod_{j=1}^{\ell} h_{q^k - q^{k_j}} = h_{\sum (q^k - q^{k_j})}.$$

Proof. If $f_0 = 0$, then h is identically zero. Also, if $c \in F^*$, the logarithmic derivative of f and cf is the same, so that without loss of generality we can assume that $f_0 = 1$. It follows that $h_0 = 1$, and thus we can assume $k_j < k$ without loss of generality. Since $f'(z) = 1$, we have $h(z) = 1/(1 - (1 - f(z))/z)$ so that the geometric series development shows that $h_i = 0$, unless i is a multiple of $q - 1$. Equating coefficients of z^m in the identity $h(z)f(z) = z$, we get, for $m > 1$,

$$(2.2) \quad h_{m-1} = - \sum_{i \geq 1} h_{m-q^i} f_i.$$

We prove the theorem by induction on $\ell \leq q$. The claim is trivially true for $\ell = 0$ or 1. Assume it for up to and including $\ell < q$ and we will prove it for $\ell + 1$, in place of ℓ .

Now we do an induction on k . The result is vacuously true for $k = 0$. If all $k_j > 0$, taking out the q -th power by the Carlitz relation, we reduce it to $k - 1$ case and thus prove it, so it is enough to prove

$$(2.3) \quad h_{q^k-1} h_{\ell q^k - q^{k_1} - \dots - q^{k_\ell}} = h_{(\ell+1)q^k - q^{k_1} - \dots - q^{k_\ell} - 1}.$$

Without loss of generality, we can assume that the first r ($0 \leq r \leq \ell$) of the k_j 's are at least one and the next $\ell - r$ of them are zero.

Let \sum (respectively \sum') stand for the sum, possibly empty, over i between 1 to r (respectively, n between 1 to $\ell - r$). Repeated applications of the recursion (2.2) and the p -th power relation in (1.1) give

$$\begin{aligned}
 M &:= h_{q^k-1} h_{\ell q^k - \sum q^{k_i} - (\ell - r)} \\
 &= \left(- \sum_{j=1}^k h_{q^k - q^j} f_j \right) (-1)^{\ell - r} \sum_{j_1, \dots, j_{\ell - r} = 1}^k h_{\ell q^k - \sum q^{k_i} - \sum' q^{j_n} f_{j_1} \cdots f_{j_{\ell - r}}} \\
 &= (-1)^{\ell - r + 1} \sum_{j, j_1, \dots, j_{\ell - r} = 1}^k h_{q^k - 1 - q^j} h_{\ell q^k - 1 - \sum q^{k_i} - \sum' q^{j_n} f_j f_{j_1} \cdots f_{j_{\ell - r}}} \\
 &= (-1)^{\ell - r + 1} \sum_{j, j_1, \dots, j_{\ell - r} = 1}^k h_{(\ell + 1)q^k - 1 - q^j - \sum q^{k_i} - \sum' q^{j_n} f_j f_{j_1} \cdots f_{j_{\ell - r}}} \\
 &= (-1)^{\ell - r + 1} \sum_{j, j_1, \dots, j_{\ell - r} = 1}^k h_{(\ell + 1)q^k - q^j - \sum q^{k_i} - \sum' q^{j_n} f_j f_{j_1} \cdots f_{j_{\ell - r}}} \\
 &= h_{(\ell + 1)q^k - \sum q^{k_i} - (\ell - r + 1)}
 \end{aligned}$$

proving the claim (2.3). In more details, the first equality (ignoring the definition of M in the first line) follows by the recursion applied to the first term and repeatedly ($\ell - r$ times) to the second term, the second by the p -power relation (or rather its consequence, the q -power relation), the third by induction on k , the fourth by q -th power relation again and the last equality by the repeated ($\ell - r + 1$ times) application of the recursion, noting that since $\ell + 1 \leq q$, $k_j \leq k$ still. \square

Remarks. (1) In particular,

$$(2.4) \quad h_{\ell(q^k - 1)} = h_{q^k - 1}^\ell, \text{ for } \ell \leq q.$$

- (2) (2.1) and even (2.4) are false in general, for $\ell = q + 1$. Example is f being the Carlitz exponential for $q = 3$ and $k = 1$. Breaking this as $8 = 2 \times 3 + 2$, there is no carry over of digits even. Also, in contrast to (1.1), for general m , even h_{2m} need not be h_m^2 , unless $p = 2$, a simple example being q and f as above and $m = 4$.
- (3) Federico Pellarin had discovered the Theorem 1, in a slightly different language, in an unpublished work, as the authors learned from him when they circulated the preprint.

2.2. $h(z)$ in terms of u . Equating the coefficients in the defining equation gives recursion

$$f_d H_m = -\delta_{q^d - 1, m} f_0 - \sum_{j < d} f_j H_{m - q^d + q^j},$$

where, as usual, $\delta_{i, j} = 1$ or 0 according as $i = j$ or not.

Thus we not only have the d -term ' \mathbb{F}_q -linear' recursion $H_m = -\sum (f_j / f_d) H_{m - q^d + q^j}$ corresponding to the \mathbb{F}_q -linear polynomial $P_d := z^{q^d} + \sum (f_j / f_d) z^{q^j}$, but also the specific initial conditions above, namely $h_j = 0$ for $j < q^d - 1$ and $h_{q^d - 1} = -(f_0 / f_d)$.

The roots of P_d form a d dimensional vector space V_d , and with the usual correspondence between the recursion relation and the roots of corresponding polynomials, we see that coefficients H_m are linear combination of m -th powers of these roots. The initial conditions mean (see e.g., [T2004, 5.1.2 or 5.6.2]) that $H_m = -\sum v^m$, where the sum is over $v \in V_d$. (Another way to see this directly is by the use of Newton's formulas giving the power sums in terms of the elementary symmetric functions, which exactly does this).

Similarly, if in 2.1, we specialize the power series f to a polynomial of degree q^d , then we get a relation recursion corresponding to a polynomial of the form $\sum z^{q^d - q^i} p_i$. But its reciprocal polynomial is an arbitrary \mathbb{F}_q -linear polynomial in $u = 1/z$, after multiplying by u . Thus the first theorem specializes to multiplicative relations, for $H_m = -\sum v^{-m}$, for special m 's.

We now claim the more general

Theorem 2. *Let F be a field containing \mathbb{F}_q , let $b_1, \dots, b_d \in F$ be \mathbb{F}_q -linearly independent and let $B_{ij} \in F$, for $i = 1$ to d and $j = 1$ to $s \leq q$. Then*

$$\prod_{j=1}^s \sum_{(\theta_1, \dots, \theta_d) \in \mathbb{F}_q^d - \{0\}} \frac{\sum_i \theta_i B_{ij}}{\sum_i \theta_i b_i} = (-1)^{s-1} \sum_{\emptyset} \frac{\prod_j (\sum_i \theta_i B_{ij})}{(\sum_i \theta_i b_i)^s}.$$

Proof. As explained above, the Theorem in 2.1 proves the special cases when b_i is q^k -th power of i -th element of a basis (and we can specialize to linearly dependent ones) of arbitrary V_d , and $B_{ij} = b_i^{q^{kj}}$, with $0 < k_j < k$ arbitrary.

By subtracting one side of the identity from the other and by making a common denominator, we rewrite it as a polynomial identity $\phi(b_1, \dots, b_d, B_{11}, \dots, B_{1d}, \dots, B_{sd}) = 0$, where ϕ is $(s+1)d$ variable polynomial with coefficients in \mathbb{F}_q . So we know that $\phi(b_1, \dots, b_d, b_1^{q^{k_1}}, \dots, b_d^{q^{k_1}}, \dots, b_d^{q^{k_d}}) = 0$. These specializations are enough (The authors thank Ching-Li Chai for immediately providing much more general reference [Ch2008, 3.1]) to conclude that ϕ is identically 0. This is because, if we take k and all the gaps between k, k_i 's sufficiently large, there can be no cancellations between the terms with different $b_j^{q^{k_i}}$'s. Hence, $b_j^{q^{k_i}}$ can be replaced by independent variables B_{ij} . \square

We now use this theorem to prove

Theorem 3. *With the notation as above, for $1 \leq s \leq q$ and $k_i \geq 1$, with $1 \leq i \leq s$, we have*

$$\prod_{i=1}^s H_{q^{k_i} - 1} = H_{q^{k_1 + \dots + q^{k_s} - s}}.$$

Proof. We now specialize the previous theorem for the case where b_i is a basis of arbitrary V_d , and $B_{ij} = b_i^{q^{k_j}}$, with k_j arbitrary positive, giving the required relations for the power sums which represent these coefficients. \square

Remarks. (1) In particular, for $1 \leq s \leq q$ and $k \geq 1$, we have

$$(2.5) \quad H_{q^k - 1}^s = H_{s(q^k - 1)}.$$

(2) Since $H_{q^k - 1} = 0$ if $k < d$, then $H_{q^{k_1 + \dots + q^{k_s} - s}} = 0$ if some $k_i < d$.

- (3) (2.5) is false in general, for $s = q + 1$. Let $q = 3$ and $f(z) = f_0z + f_1z^q + f_2z^{q^2}$ with $f_0 \neq 0$. Then $H_2 = 0$ and $H_8 = -f_0/f_2$. Also, note that (in contrast to the first theorem identities) $H_{q^2-1}H_{q^3-q^2} = H_{q^2-1}H_{q-1}^{q^2} = 0$, while $H_{q^2-1+q^3-q^2} = H_{q^3-1} = f_0f_1^3/f_2^4$.
- (4) Note that the identity in the theorem is equivalent to a similar identity (with the sign $(-1)^{s-1}$ absent) where the sum is only over tuples with $\theta_i = 1$ for largest i for which θ_i is non-zero (the ‘monic’ version). The specializations, $b_i = t^{i-1}$ and $B_{ij} = t^{q^j(i-1)}$, in Theorem 2, gives the power sum identities for $S_{<d}(k)$ ’s recalled in 3.3 below, whereas $b_i = \theta^i$, $B_{ij} = t_j^i$ relates to Rudolph Perkins’ identity [Per2013, Thm. 4.1.2], when combined with Carlitz evaluation of the sums for $m = q^j - 1$ in the first case. But these evaluations can be done in the general setting of Theorem 2, by using the Moore determinant (see e.g., [T2004, 2.11(b)]). So the theorem gives a multi-variable generalization-deformation of the identities for power sums of polynomials. The Theorem 2 should also be provable directly as stated using basic properties of symmetric functions of finite field elements and counting.
- (5) Since for $k > 0$, $\sum \theta^k = -1$ or 0 , where θ runs through elements of \mathbb{F}_q , according to whether k is ‘even’ or not, and $\sum \theta^0 = \sum 1 = 0$, where we interpret $0^0 = 1$, we have

$$\begin{aligned} \sum_{\theta_1, \dots, \theta_d \in \mathbb{F}_q} (\theta_1 b_1 + \dots + \theta_d b_d)^m &= \sum \binom{m}{m_1, \dots, m_d} \prod b_k^{m_k} \sum \prod \theta_k^{m_k} \\ &= \sum_{m_i \text{ ‘even’} > 0} \binom{m}{m_1, \dots, m_d} \prod b_k^{m_k} (-1)^d. \end{aligned}$$

Thus our claim is also equivalent to the multinomial identity, that for $s \leq q$ and $m_i > 0$ ‘even’,

$$\left(\sum_{i=1}^s q^{k_i} - 1 \right)_{m_1, \dots, m_d} = (-1)^{(d-1)(s-1)} \sum'_{(m_1, \dots, m_d) = \sum_i (i_1, \dots, i_d)} \prod_i \binom{q^{k_i} - 1}{i_1, \dots, i_d} \pmod{p},$$

where \sum' denotes the sum over restricted tuples such that $i_j > 0$ are ‘even’.

We have the well-known identity

$$\sum_{j=0}^k \binom{a}{j} \binom{b}{k-j} = \binom{a+b}{k}$$

of similar flavour, for any (variables) a, b . (The identity and its multinomial generalization is obtained by comparing coefficients of $(\sum x_i)^a (\sum x_i)^b = (\sum x_i)^{a+b}$). But this differs because of the omission of $i_k = 0$, restriction to ‘even’, sign in front, specialized a, b and the characteristic.

Let us prove for any q , the simplest case of $d = s = 2$. The definition of binomial coefficients in terms of factorials shows that in characteristic p , we have $\binom{p^m-1}{j} = (-1)^j$ and $\binom{p^m-2}{j} = (-1)^j(j+1)$, which we will use below. Let us assume $a = q^r - 1 < b = q^s - 1$, without loss of generality. We want to prove $\binom{a+b}{k} = -\sum \binom{a}{i} \binom{b}{j}$ modulo p where k is ‘even’ and the sum is over $i, j > 0$ and ‘even’ such that $i + j = k$. The product of the binomial

coefficients on the right is just $(-1)^{i+j} = (-1)^k = 1$, so the sum on the right side is just the number of ‘even’ i ’s satisfying $\max(0, k-b) < i < \min(a, k)$.

This evaluates (modulo p) to $k/(q-1) - 1 \equiv -k - 1$, $(a - (k-b))/(q-1) - 1 \equiv (-1 - 1 - k)/(-1) - 1 \equiv k + 1$ and $a/(q-1) - 1 \equiv -1/(-1) - 1 \equiv 0$, according as $k < a$, $k > b$, and $a < k < b$ respectively. By the Lucas theorem, considering the base q expansion of $a+b$ and $k = \sum_{i=0}^{\ell} k_i$, the left side respectively is $(-1)^{\sum_{i>0} k_i} (-1)^{k_0} (k_0 + 1) \equiv k_0 + 1 \equiv k + 1$, $1 * (-1)^{k_{\ell-1} + \dots + k_1} (-1)^{k_0} (k_0 + 1) \equiv -(k + 1)$ and 0, as required.

- (6) The geometric series development in the defining equation, together with the multinomial expansions of the resulting powers, shows that H_m is the sum of the terms $(-f_0/f_d)^{\binom{\sum m_j}{(m_j)_j}} \prod (-f_j/f_d)^{m_j}$, one for each decomposition $m = (q^d - 1) + \sum m_j(q^d - q^j)$, where $\binom{\sum m_j}{(m_j)_j}$ is the multinomial coefficient, and $m_j \geq 0, 0 \leq j < d$.

If we consider $m = (\sum q^{k_i}) - s = q^d - 1 + \sum b_j(q^d - q^j)$, then considering modulo q , we see that m_0 is $s-1$ plus a positive multiple of q . Transferring the resulting terms to the left side, we see that Theorem 3 follows from the claim that under the notation of Theorem 3, if $(\sum q^{k_i}) - sq^d = \sum a_j(q^d - q^{d-j})$, with $a_j \geq 0$ (note $a_j = b_j$ for $j > 0$ and $a_0 = b_0 - (s-1)$), then there are m_{ij} such that $a_j = \sum m_{ij}$ and $q^{k_i} - q^d = \sum m_{ij}(q^d - q^{d-j})$ and the multinomial coefficients satisfy

$$\binom{\sum b_j}{(b_j)_j} = \prod_i \binom{\sum_j m_{ij}}{(m_{ij})_j}.$$

Note that when $a > b$, the (base q) digit expansion of $q^a - q^b$ consists of all $q-1$ digits followed by all 0 digits. Thus if m is of the form $q^k - q^d$, with $k > d$, the decomposition as above can be obtained by matching (not necessarily in a unique way) its $(q-1)$ digits by those of $q^d - q^j$ ’s by appropriate shifts (which are obtained by multiplication by powers of q) and addition without carry overs, by considering m_j ’s as sums of such powers of q given by their digit expansions.

Let us now first show the special case that if for $\ell > 0$ of i ’s we have $k_i = j < d$, then both the sides are zero. Since m_j ’s are non-negative, for such j ’s, m_j is a positive multiple of q minus one (-1 corresponding to $q^j - q^d$), contributing the 0-th digit $q-1$, so if $\ell > 2$, we have a carry over in adding such m_j ’s resulting in vanishing of the multinomial coefficient in the coefficient recipe above. For the general case, this can also be seen since $s > \ell \geq 1$ and modulo q we have m_0 is $s-1 \geq 1$ gives carry-over and vanishing in any case.

So without loss of generality, we assume that $k_i \geq d$. We need only to look at the case when $\sum b_j$ ’s do not have carry overs, thus $\sum_j m_{ij}$ ’s also do not have carry overs, and the corresponding expansions of $q^{k_i} - q^d$ are just obtained by shifting and patching as explained above, so that for given i , any q^k at most occurs once in m_{ij} ’s. Thus for $s \leq q-1$, there can not be any carry over in the sum of b_j ’s and the required multinomial identity follows by the Lucas theorem.

For $s = q$, it seems to work with those terms with carry over occurring multiple of p times and thus contributing nothing.

This seems to be the way it works, though we have not explained why. Ideas of 4, 5 and 6 might be alternate approaches to the proof, and we plan to pursue them in the future.

2.3. $a(z)$ in terms of z .

Theorem 4. *With the notation as above, for $1 \leq s < q$ and $0 \leq k_i < k$, with $1 \leq i \leq s$, we have*

$$\prod_{i=1}^s a_{q^k - q^{k_i}} = f_0^{(s-1)q^k} a_{q^k - \sum q^{k_i}}.$$

Proof. We follow the method of proof of Theorem 1. Equating the coefficients of z^m in $a - af = f_0z$ gives $a_1 = f_0$ (and thus $a_{q^k} = f_0^{q^k}$) and the recursion

$$a_m - f_0 a_{m-1} - \sum_{i \geq 1} a_{m-q^i} f_i = 0,$$

giving, in particular,

$$f_0 a_{q^k-1} = f_0^{q^k} - \sum_{i \geq 1} a_{q^k-i-1}^i f_i.$$

Iterating the recursion, we get

$$f_0^r a_{m-r} = \sum_{j=0}^r \sum_{i_1, \dots, i_j > 0} a_{m-q^{i_1} - \dots - q^{i_j}} \binom{r}{j} (-1)^j f_{i_1} \cdots f_{i_j}.$$

We write $N = q^k - \sum q^{k_\ell}$. It is enough to prove $a_{q^k-1} a_{N-r} = f_0^{q^k} a_{N-(r+1)}$. We proceed by induction on r and on k as before.

By the iterations above, the right side of the claimed equality is

$$f_0^{q^k-r-1} \sum_{j=0}^{r+1} \sum_{i_1, \dots, i_j > 0} a_{N-\sum q^{i_n}} \binom{r+1}{j} (-1)^j f_{i_1} \cdots f_{i_j},$$

and the left side is

$$f_0^{-r-1} (f_0^{q^k} - \sum a_{q^k-q^i} f_i) \left(\sum_{j=0}^r \sum_{i_1, \dots, i_j > 0} a_{N-\sum q^{i_n}} \binom{r}{j} (-1)^j f_{i_1} \cdots f_{i_j} \right).$$

Now we follow the procedure exactly as in the proof of the first Theorem and apply the q -power relation on the coefficients and induction on k when we multiply out the two sums. The use of the Pascal triangle binomial identity, $\binom{r}{j} + \binom{r}{j-1} = \binom{r+1}{j}$, when we combine the two resulting sums, turns the left side into the right side. \square

Remarks. (1) For $1 \leq s \leq q$, we have $a_s = f_0^s$.

(2) In particular, for $1 \leq s < q$, we have

$$(2.6) \quad a_{q^k-1}^s = f_0^{(s-1)q^k} a_{q^k-s}.$$

(3) In general, (2.6) does not hold for $s = q$. Let $q = 3$, $f(z) = f_0z + f_1z^q + f_2z^{q^2}$, $k = 2$ and $s = q$. Then $a_{q^k-q} = a_{q-1}^q = f_0^6$ and $a_{q^k-1} = f_0^8 - f_0^5 f_1$.

(4) Assume $f_0 = 1$ and let $g = \sum_{i=0}^{\infty} g_i z^{q^i}$ be the compositional inverse of f . Then, from numerical evidence, it seems that for all $k \geq 1$, we have

$$a_{q^k-1} - a_{q^k-1-1} = g_{k-1}.$$

Since we do not know any application of this, we have not attempted a proof.

2.4. $a(z)$ in terms of u . Equating coefficients in the defining equation gives the recursion

$$f_d A_m = \delta_{q^d-1,m} f_0 + A_{m-q^d} - \sum_{j < d} f_j A_{m-q^d+q^j}.$$

Now the polynomial corresponding to this recursion is a \mathbb{F}_q -linear polynomial plus a constant, thus the roots form an affine space (translation of a \mathbb{F}_q -vector space), and the initial conditions again give the coefficient as the power sum of the roots $\mu + \sum \theta_i b_i$, by Newton's formulas or [T2004, 5.1.2 or 5.6.2] as before. Just as in the case of 2.1 and 2.2, now 2.3 corresponds to negative powers while 2.4 corresponds to power sums for positive powers. By similar method of proof how we derived Theorem 3 via a more general Theorem 2 deduced from Theorem 1, we now prove a more general

Theorem 5. *Let F be a field containing \mathbb{F}_q , let $M_j, \mu, b_1, \dots, b_d, B_{ij} \in F$ ($i = 1$ to d and $j = 1$ to $s < q$). Then (assuming no zeros in the denominators, or in other words, make common denominators and look at the polynomial identity for numerators)*

$$\prod_{j=1}^s \sum_{(\theta_1, \dots, \theta_d) \in \mathbb{F}_q^d} \frac{\sum_i (M_j + \theta_i B_{ij})}{\sum_i (\mu + \theta_i b_i)} = \left(\sum_{\theta} \frac{1}{\sum_i (\mu + \theta_i b_i)} \right)^{s-1} \sum_{\theta} \frac{\prod_j (\sum_i (M_j + \theta_i B_{ij}))}{\sum_i (\mu + \theta_i b_i)}.$$

Proof. We proceed as in the case of Theorem 2. The identity specializes to the conclusion of the previous theorem, when $B_{ij} = b_i^{q^{k_j-k}}$ and $M_j = \mu^{q^{k_j-k}}$. (Note that $f_0 = a_1$ is the power sum for -1 -th power.) The rest of the proof proceeds exactly as that of Theorem 2. \square

Theorem 6. *With the notation as above, for $1 \leq s < q$ and $k_i \geq 0$, with $1 \leq i \leq s$, we have*

$$\prod_{j=1}^s A_{q^{k_j-1}} = f_0^{s-1} A_{q^{k_1+\dots+k_s-1}}.$$

Proof. The claimed relations follow by specializing the identity of the previous Theorem to $B_{ij} = b_i^{q^{k_j}}$ and $M_j = \mu^{q^{k_j}}$, since these coefficients represent the power sums mentioned above. We need only to note that the first bracket on the right side of Theorem 5 exactly matches f_0 as claimed. This is because the recursion formula above immediately implies that $A_{q^d-1} = f_0/f_d$, and $A_{2q^d-1} = A_{q^d-1}/f_d$, so that $f_0 = A_{q^d-1}^2/A_{2q^d-1}$ in this case. \square

Remarks. (1) It follows that $A_{q^{k_1+\dots+k_s-1}} = 0$ if some $k_i < d$, because $A_{q^k-1} = 0$ if $k < d$.

(2) For $1 \leq s < q$, we have

$$(2.7) \quad A_{q^k-1}^s = f_0^{s-1} A_{sq^k-1},$$

(3) Here we have an example showing $s \geq q$ does not work in (2.7). Let $q = 3$, $f(z) = f_0 z + f_1 z^q + f_2 z^{q^2}$ with $f_0 \neq 0$, $s = q$ and $k = 2$. Then $A_{qq^k-1} = (-f_0 f_1^3 + f_0 f_2)/f_2^4$ and $A_{q^k-1} = f_0/f_2$.

- (4) Note that the last three theorems are vacuously true for $q = 2$, unlike the first three, and all are vacuously true for $\ell = s = 1$.
- (5) Theorem 5 relates to Perkins' identity [Per2013, Thm. 4.1.2] via $b_i = \theta^i$, $B_{ij} = t_j^i$, $\mu = \theta^{d+1}$ and $M_j = t_j^{d+1}$.

3. APPLICATIONS

In addition to the general interest of these relations, they have many applications to the function field arithmetic [G1996, T2004]. There are well-known (see e.g., [G1996, Cor. 1.2.2]) links of the root spaces of F_q -linear polynomials to F_q -vector spaces. Many finite, infinite dimensional F_q -vector spaces arise naturally in the function field arithmetic as, for example, Riemann-Roch spaces, rings of integers in function fields, sets of such with degree bounded by some constant etc. Sums over these appear as zeta values, finite power sums etc. Thus we have applications to zeta, and multizeta values [T2004, T2009, L2011]. We now describe these in a little more detail.

3.1. Basic factorizations. Let us first recall some useful factorizations and notations (see e.g., [G1996, T2004,]), we know that $[n] := t^{q^n} - t$ is the product of all the monic irreducibles of degree dividing n , $D_m := \prod_{i=0}^{m-1} [m-i]^{q^i}$ is the product of all the monic polynomials of degree m , whereas $L_n := \prod_{i=1}^n [i]$ is the least common multiple of all the monic polynomials of degree n . Write $d_i := D_i$, $\ell_i := (-1)^i L_i$.

3.2. Basic analogies. Let us quickly recall some basics of the number fields - function fields analogy [G1996, T2004] in the simplest case: between $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, the exponential ' e^z ', the logarithm ' $\log(z)$ ', the Euler-Riemann zeta ' $\zeta(s)$ ', the factorial ' $!$ ', the binomial coefficient ' $\binom{x}{n}$ ' on one hand, and $\mathbb{F}_q[t], \mathbb{F}_q(t), \mathbb{F}_q((1/t))$, the Carlitz exponential ' $e(z)$ ', the Carlitz logarithm ' $\ell(z)$ ', the Carlitz-Goss zeta ' $\zeta_c(s)$ ' and the Carlitz factorial ' $!_c$ ', and the Carlitz binomial coefficient ' $\binom{x}{n}_c$ ' respectively. We now define these quantities.

3.3. Basic quantities of function field arithmetic. If a positive integer n has the base q expansion, $n = \sum n_i q^i$, $0 \leq n_i < q$, then $n!_c := \prod d_i^{n_i} \in \mathbb{F}_q[t]$. We have $e(z) = \sum z^{q^i} / d_i = \sum z^{q^i} / q^{i!}_c$ parallel to $e^z = \sum z^n / n!$. We define $\ell(z)$ to be the compositional inverse of $e(z)$, then $\ell(z) = \sum z^{q^i} / \ell_i$. We write $e(x\ell(z)) = \sum \binom{x}{q^d}_c z^{q^d}$, and $\binom{x}{n}_c = \prod \binom{x}{q^i}_c^{n_i}$.

Finally, for $s \in \mathbb{Z}$, $\zeta_c(s) := \sum_{d=0}^{\infty} \sum 1/a^s \in \mathbb{F}_q((1/t))$, where the second sum is over monic polynomials $a \in \mathbb{F}_q[t]$ of degree d . We consider the power sums $S_d(k) = \sum a^{-k}$, where the sum is over monic $a \in A$ of degree d , and $S_{<d}(k) = \sum a^{-k}$, where now the sum is over monic $a \in A$ of degree less than d .

3.4. Application to factorization of Bernoulli-Carlitz fractions. The Bernoulli-Carlitz numbers $\mathcal{B}_n \in \mathbb{F}_q(t)$ are defined analogously by $z/e(z) = \sum \mathcal{B}_n z^n / n!_c$, by analogy with the classical case. They occur in Euler type evaluation due to Carlitz of ζ_c at 'even' positive integers. Their denominators satisfy von-Staudt type theorems and numerators occur in the Kummer-Herbrand-Ribet theorem analogs [T2004, G1996, Ta2012] due to Goss, Sinnott, Okada and Taelman, explaining the significance of the factorization and multiplicative relations. (Recall that the factorizations of the usual Bernoulli numbers are known to be important in many areas of

mathematics, Herbrand-Ribet and Mazur-Wiles theorems, modular forms congruences, stable homotopy and the Kervaire-Milnor formula, just to mention a few). See also [T2012] for the second author's counter-examples to Chowla conjectures, where the knowledge of these factorizations helped.

Carlitz [T2004, Thm. 4.16.1] proved that for $n = q^k - 1$, we have

$$\mathcal{B}_n = \mathcal{B}_n(n-1)!_c/n!_c = (-1)^k \prod_{i=1}^{k-1} [k-i]^{q^i-2}/[k].$$

Our main result with $f(z) = e(z)$ thus generalizes such factorizations to much wider families of n 's. One sees, in particular, that in contrast to the subtleties for general n , whether a prime divides the numerator (or denominator) of \mathcal{B}_n , just depends on the degree of the prime for n 's in these families.

3.5. Applications to power sums. We have

$$\binom{z}{q^d}_c = \sum_{i=0}^d \frac{z^{q^i}}{d_i \ell_{d-i}^{q^i}} = \frac{1}{D_d} \prod_{a \in A, \deg a < d} (z - a).$$

Consider now $f(z) = \ell_d \binom{z}{q^d}_c$. Carlitz proved [C1935, Thm. 7.2] that the inverse is

$$g(z) = \sum_{j=0}^{\infty} \frac{\ell_{d+j-1}}{\ell_j \ell_{d-1}^{q^j}} z^{q^j}.$$

Let $f(z) = \binom{z}{q^d}_c$, then $f_0 = 1/\ell_d$. We have (with the signs corrected) [C1935, p. 160], [T2004, Thm. 5.6.3], [T2009, 3.2]

$$h_k = S_{<d}(k), \quad H_k = S_{<d}(-k), \quad a_k = S_d(k), \quad A_k = S_d(-k),$$

where the first two equalities hold for k 'even' and the last two for any $k \geq 1$. We have, by [C1935, Thm. 9.2], [C1939, Pa. 941], [Le1943, Pa. 283], [C1939, Pa. 941] (as well as [Ge1988, Thm. 4.1]) respectively,

$$h_{q^i-1} = \frac{\ell_{d+i-1}}{\ell_i \ell_{d-1}^{q^i}}, \quad a_{q^i-1} = \frac{\ell_{d+i-1}}{\ell_{i-1} \ell_d^{q^i}},$$

and for $i \geq d$,

$$H_{q^i-1} = \frac{d_{i-1}^q}{\ell_{d-1} d_{i-d}^{q^d}}, \quad A_{q^i-1} = \frac{d_i}{\ell_d d_{i-d}^{q^d}}.$$

These follow from Newton's identities for power sums specialized to these values using the linear equations connected to $e_d(z)$ above as well as to $e_d(t^d + z)$.

Our theorems thus specialize giving several more evaluations in this case with full understanding of their factorizations. We list a few known special cases of this.

In this special situation, Theorem 6 specializes to [Le1943, Thm. 4.1] (also [T2004, 5.6.4 (1)]), and Theorem 3 to [Le1943, Theorem 5.1]. The formulas for $S_{<d}(l(q-1))$ in [T2009, pa. 2329] as well as the formula for the Bernoulli-Carlitz number $\mathcal{B}_{l(q-1)}$ in [Ge1989, Corollary 4.4 (ii), pa. 218] ($l \leq q$), are consequences of Theorem 1. Theorem 4 generalizes (and proves) the conjecture 2.10 in [L2010] about $S_d(mq^i - 1)$.

3.6. Application to multizeta. We refer to [T2009, 3.4], [L2012, pa. 281, 283], [LTp] and [LTp2, 4.1, 4.2] to see how such factorizations are used, by cancellations of appropriate factors when we take products of zeta values or iterated products involved in the definition of multizeta values, to prove new multizeta identities, as well as simplify earlier proofs, which had used special cases proved separately.

We just give one example [LTp]: We have

$$\zeta(q^n - \sum_{i=1}^s q^{k_i}, (q-1)q^n) = \frac{(-1)^s}{\ell_1^{q^n}} \prod_{i=1}^s [n - k_i]^{q^{k_i}} \zeta(q^{n+1} - \sum_{i=1}^s q^{k_i}),$$

where $n > 0$, $1 \leq s < q$, $0 \leq k_i < n$.

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