Multizeta in function field arithmetic

Dinesh S. Thakur*

Abstract. This is a brief report on recent work of the author (some joint with Greg Anderson) and his student on multizeta values for function fields. This includes definitions, proofs and conjectures on the relations, period interpretation in terms of mixed Carlitz-Tate t-motives and related motivic aspects. We also verify Taelman's recent conjectures in special cases.

2010 Mathematics Subject Classification. Primary 11M32, Secondary 11G09.
Keywords. Mixed motives, Extensions, Drinfeld modules, shuffle.

1. Introduction

Euler's multizeta values have been pursued recently again with renewed interest because of their emergence, for example in Grothendieck-Ihara program to study the absolute Galois group through the fundamental group of projective line minus three points and related studies of iterated extensions of Tate motives.

Two types of multizeta were defined [T04, Sec 5.10] for function fields, one complex valued (generalizing Artin-Weil zeta function) and the other with values in Laurent series over finite fields (generalizing Carlitz zeta values). For the $\mathbb{F}_q[t]$ case, the first type was completely evaluated in [T04] (see [M06] for more detailed study in the higher genus case). We focus on the second analog in this report.

In contrast to the classical division between the convergent versus the divergent (normalized) values, all the values are convergent in our case. In place of the sum or the integral shuffle relations, we have different kinds of relations: the shuffle type relations with $\mathbb{F}_p$-coefficients and the relations with $\mathbb{F}_p(t)$-coefficients. (Classically, of course, there is no such distinction, the rational number field being the prime field in that case). The first kind of relations have been understood (though not with a satisfying structural description) and show that the product of multizeta values can also be expressed as a sum of some multizeta values, so that the $\mathbb{F}_p$-span of all multizeta values is an algebra. While [T09, Lr09, Lr10] conjectured and proved, in the special case $A = \mathbb{F}_q[t]$, many such interesting relations, combinatorially quite involved to describe unlike the classical case, the proofs [T10] give the existence directly (for general A, defined below) rather than proving those conjectures. We only have examples of second kind of relations so far.

As for the analogs of interconnections mentioned in the first paragraph, we can connect to absolute Galois group (through analog of Ihara power series [ATp]) and fundamental group approach in the Grothendieck-Ihara program only through

*Supported by NSA grants H98230-08-1-0049, H98230-10-1-0200.

the mixed motives [A86, AT09]. We describe some of these motivic aspects and relation with recent work of V. Lafforgue and L. Taelman.

2. Multizeta values for function fields: Definitions

2.1. Notation.

$$\begin{aligned}
\mathbb{Z} &= \{\text{integers}\} \\
\mathbb{Z}+ &= \{\text{positive integers}\} \\
q &= \text{a power of a prime } p \\
\mathbb{F}_q &= \text{a finite field of } q \text{ elements} \\
K &= \text{a function field of one variable with field of constants } \mathbb{F}_q \\
\infty &= \text{a place of } K \text{ of degree one} \\
K_\infty &= \mathbb{F}_q((1/t)) = \text{the completion of } K \text{ at } \infty \\
\mathbb{C}_\infty &= \text{the completion of an algebraic closure of } K_\infty \\
A &= \text{the ring of elements of } K \text{ with no poles outside } \infty \\
A_d+ &= \{\text{monic elements of } A \text{ of degree } d\} \\
[n] &= t^{q^n} - t \\
\ell_n &= (-1)^n \textstyle\prod_{i=1}^n [i] \\
\text{`even'} &= \text{multiple of } q-1
\end{aligned}$$

The simplest case is when $A = \mathbb{F}_q[t]$ and $K = \mathbb{F}_q(t)$, with the usual notions of infinite place, degree and sign (in t).

2.2. Definition of Multizeta values. First we define the power sums. Given $s \in \mathbb{Z}+$ and $d \geq 0$, put

$$S_d(s) = \sum_{a \in A_d+} \frac{1}{a^s} \in K,$$

and given integers $s_i \in \mathbb{Z}_+$ and $d \geq 0$ put

$$S_d(s_1, \cdots, s_r) = S_d(s_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2) \cdots S_{d_r}(s_r) \in K.$$

For $s_i \in \mathbb{Z}+$, we define multizeta value $\zeta(s_1, \cdots, s_r)$ following [T04, Sec. 5.10] (where it was denoted by ζ_d to stress the role of the degree) by using the partial order on $A+$ given by the degree, and grouping the terms according to it:

$$\zeta(s_1, \cdots, s_r) = \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty,$$

where the second sum is over $a_i \in A_{d_i}+$ with d_i's satisfying the conditions as in the first sum.

We say that this multizeta value (or rather the tuple $(s_1, \cdots, s_r)$) has depth r and weight $\sum s_i$. Note we do not need $s_1 > 1$ condition for convergence as in the classical case. This definition generalizes, in one way, the $r = 1$ case corresponding to the Carlitz zeta values. For discussion, references, interpolations and analytic theory, we refer to [G96, T04]. In [T04], we discuss interpolations of multizeta at finite and infinite primes.

3. First kind of relations between multizeta

Recall that Euler's multizeta values ζ (we will use this clashing same notation only in this paragraph) are defined by $\zeta(s_1, \cdots, s_r) = \sum (n_1^{s_1} \cdots n_r^{s_r})^{-1}$, where the sum is over positive integers $n_1 > n_2 > \cdots > n_r$ and s_i are positive integers, with $s_1 > 1$. We then have 'sum shuffle relation'

$$\zeta(s_1)\zeta(s_2) = \sum \frac{1}{n_1^{s_1}} \sum \frac{1}{n_2^{s_2}} = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2),$$

just because $n_1 > n_2$ or $n_1 < n_2$ or $n_1 = n_2$.

Since there are many polynomials of given degree (or norm), this usual proof of the sum shuffle relations fails. Theorem 3 below shows that in place of the three multizeta on the right of the displayed equation, in out case there can be arbitrarily large number of multizeta, depending on s_i's. In fact, it can be seen that naive analogs of the sum or integral shuffle relations fail. The Euler identity $\zeta(2, 1) = \zeta(3)$ fails in our case, for simple reason that degrees on both sides do not match.

3.1. Examples. However, the multizeta values satisfy many interesting combinatorially involved new relations [T09] which we now describe first.

Theorem 1. *(1)* $\zeta(ps_1, \cdots, ps_k) = \zeta(s_1, \cdots, s_k)^p$.

(2) (Carlitz) If $q - 1$ *divides* s, $\zeta(s)/\tilde{\pi}^s \in K$, *where* $\tilde{\pi}$ *is a fundamental period of the Carlitz module,*

(3) Any classical sum-shuffle relation with fixed s_i*'s works for* q *large enough. For example, if* $s_1 + s_2 \leq q$, *we have* $\zeta(s_1)\zeta(s_2) = \zeta(s_1 + s_2) + \zeta(s_1, s_2) + \zeta(s_2, s_1)$.

Now we describe simplest examples when the hypothesis of part (3) is violated.

Theorem 2. *(1) When* $a, b \leq q$ *and* $a + b > q$, *we have*

$$\zeta(a)\zeta(b) = \zeta(a + b) + \zeta(a, b) + \zeta(b, a) + (a + b)\zeta(a + b - q + 1, q - 1).$$

(2) When $1 \leq b \leq q$, $q \neq 2$, *we have*

$$\zeta(b)\zeta(2q) = \zeta(2q+b)+\zeta(b, 2q)+\zeta(2q, b)+b\zeta(q+b+1, q-1)+\binom{b+1}{2}\zeta(b+2, 2q-2).$$

(3) $\zeta(q^n - 1)\zeta((q - 1)q^n) = \zeta(q^{n+1} - 1) + \zeta(q^n - 1, (q - 1)q^n)$.

Note in the special cases, such as when a or b is $q - 1$ or when p divides $a + b$, three depth 2 multizetas in part (1) can mix or disappear giving difference appearance to the identities.

Theorem 3. *Let $q = 2$. We have (1) $\zeta(1)\zeta(a) = \zeta(1+a) + \sum_{i=1}^{a-1} \zeta(i, a+1-i)$,*
(2) If b is odd, $\zeta(2)\zeta(b) = \zeta(2+b) + \sum_{1 \leq i \leq (b-3)/2} \zeta(2i+1, 1+b-2i)$,
and if b is even, $\zeta(2)\zeta(b) = \zeta(2+b) + \sum_{1 \leq i \leq b/2-1} \zeta(2i, b+2-2i)$.

3.2. Conjectural recursive recipe. In [T09], for $q = 2$ a full conjectural description of how the product of two zeta values can be described as the sum of multizetas is given. Here we just give an example of the recursive recipe: In the notation of the first theorem of the next subsection, we let $q = 2$, so that $f_i = 1$. Let $a = 19$. Then the a_j's for b replaced by $b + 32$ are given by those for b and $b+32, b+31, b+28, b+27, b+24, b+23, b+20, b+19$. In other words, at each recursion step of 32, eight new multizeta values described get added.

In [Lr09, Lr10] these conjectures were partially generalized, giving full recursive step (but not the initial values) when q is a prime, and a partial description for any q as follows. Again in the notation of the first theorem of the next subsection, we have a recipe giving f_i and a_i, given (a, b). For a fixed a it is recursive in b of recursion length $(q-1)p^m$, where m is the smallest integer such that $a \leq p^m$, and at each recursive step one adds $t_a = \prod(p-j)^{\mu_j}$ new multizeta terms, where μ_j is the number of j's in the base p expansion of $a - 1$.

David Goss has recently stressed the role of the 'digit expansion permutation symmetries (ρ_* below)' in the theory of Carlitz-Goss zeta function, with respect to its zeros, orders of vanishing etc. If ρ denotes arbitrary permutation of the set of non-negative integers, then we have resulting action $\rho_*(\sum n_i q^i) := \sum n_i q^{\rho(i)}$ on digit expansions base q. Note that, with $q = p$, and given any ρ and a, the same recipe works for both $a - 1$ and $\rho_*(a-1)$ for the recursion length and for the number of multizetas to be added, if we do not insist on the smallest recursion length. Thus a strong form of such symmetry shows up in the theory of multizeta values.

In [T09], it was also described how the 'soliton' technology allows us to prove any such relation (for fixed s_i's). This is much simplified by the next theorem. It seems plusible that the complicated combinatorial recipe above can be deudced from the next theorem, but this has not been done yet.

3.3. General theorem. In [T10], we proved all relations of the first kind, namely with coefficients in the prime field, bypassing nice explicit or recursive relations conjectured above. This is done as follows.

First we consider

$$S_d(a)S_d(b) - S_d(a+b) = \sum f_i S_d(a_i, a+b-a_i), \qquad (*)$$

with $f_i \in \mathbb{F}_p$.

Theorem 4. *(1) Let $A = \mathbb{F}_q[t]$. Given $a, b \in \mathbb{Z}+$, there are $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}+$, so that (*) holds for $d = 1$.*

(2) Fix q. If (*) holds for some $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}+$ for $d = 1$ and $A = \mathbb{F}_q[t]$, then (*) holds for all $d \geq 0$ and for all A (corresponding to the given q). In this

case, we have the shuffle relation

$$\zeta(a)\zeta(b) - \zeta(a+b) - \zeta(a,b) - \zeta(b,a) = \sum f_i \zeta(a_i, a+b-a_i). \qquad (**)$$

(3) $S_d(a_1, \cdots, a_r) S_d(b_1, \cdots, b_k)$ *can be expressed as* $\sum f_i S_d(c_{i1}, \cdots, c_{im_i})$*, with* $f_i \in \mathbb{F}_p$*,* c_{ij}*'s and* m_i*'s being independent of* d*, and with* $\sum a_i + \sum b_j = \sum_j c_{ij}$ *and* $m_i \le r + k$*.*

(4) For any A*, the product of multizeta values can be expressed as a sum of some multizeta values, such an expression preserving total weight and keeping depth filtration. In particular, the* $\mathbb{F}_p$*-span of all the multizeta values is an algebra.*

Note that this theorem gives an effective procedure for expressing a given product of multizeta values as a sum of multizeta values. The resulting proof of such an expression is much simpler than the process mentioned above.

4. Second kind of relations between multizeta

In place of the Euler identity, we have

Theorem 5. *When* $q = 3$*, we have* $\zeta(1,2) = \zeta(3)/\ell_1 = \zeta(3)/(t - t^3)$*. More generally, for any* q*, we have*

$$\zeta(m, m(q-1)) = \zeta(mq)/\ell_1^m, \quad m \le q$$

$$\zeta(1, q^2 - 1) = \zeta(q^2)(1/\ell_2 + 1/\ell_1)$$

Remark When $q = 3$ (when 'even' agrees with even), in comparison with the Euler identity we have an order switch. But, using the sum shuffle identity 2.3 for $\zeta(1)\zeta(2)$, we can express $\zeta(2,1)$ in terms of $\zeta(3) = \zeta(1)^3$ and $\tilde{\pi}$.

Here are some expressions involving logarithms of algebraic quantities.

Theorem 6. *(1) We have*

$$\zeta(1, q^3 - 1) = \zeta(q^3)(\frac{1}{\ell_3} + \frac{1}{\ell_2} + \frac{t}{\ell_2}) - \frac{1}{\ell_2}(\log(t^{1/q}))^{q^3}$$

(2) $\zeta(1, q^n - 1)$ *is (explicit)* $\zeta(q^n)$ *times a rational plus linear combination of* q*-power powers of logarithms of* q*-power roots of polynomials.*

5. Period interpretation and motivic aspects

In [AT09], the following theorem was proved.

Theorem 7. *Given multizeta value* $\zeta(s_1, \cdots, s_r)$*, we can construct explicitly iterated extension of Carlitz-Tate* t*-motives over* $\mathbb{F}_q[t]$ *which has as period matrix entry this multizeta value (suitably normalized).*

This generalizes result [AT90] connecting $\zeta(s)$ to the logarithm of and explicit algebraic point on Carlitz-Tate t-motive $C^{\otimes s}$, or equivalently to the period of one step extensions of such t-motives. In [T92, A94, A96], these were generalized somewhat to higher genus and L-function situation.

In [ATp] Ihara power series theory is developed. (It is meta-abelian etale aspect of the Grothendieck-Ihara program [I91], whereas the multizeta values should be DeRham-Betti aspect at nilpotent level.) In studying the big Galois representation it provides, complicated digit combinatorics [A07], just as what we encountered above in describing the relations between multizeta, enters the picture. The extension giving zeta values is also linked to [ATp] analog of Deligne-Soule cocycles, which have connections with 'cyclotomic unit module' of [A96] in addition to zeta values (though no K-theory link yet).

While all these connections with analogs of motives are exciting, and concrete, and while the natural constructions here lead to much stronger transcendence results than in the number field case, their larger perspective is not yet fully understood even conjecturally, and we lack a good analogous description to that of Deligne (and others) linking zeta and multizeta values to motivic extensions and K-theory.

Recent exciting works by (i) V. Lafforgue [L09] giving an analog of Bloch-Kato, Fontaine-Perrin-Riou work relating p-adic L-values and motivic extensions; and (ii) L. Taelman [Ta10] (which I learned about at this Banff workshop) defining a notion of good extensions (and of class module that we were after [T94, pa. 163] for long!) and conjecturing a class number formula for the L-values at the infinite place; represent excellent steps in this direction.

We end this short report by giving some calculations, inspired by these results, verifying Taelman's conjectures (as hinted in remark 1 of [Ta10]) in the special case of higher genus, class number one [T92] results mentioned above. In the notation of [Ta10], we deal with $R = A$, where A (below) is the base generalizing $A = k[t]$ there and ρ (below) generalizing the Carlitz module E there in conjecture 1.

We deal with examples A-D on pages 192-194 [T92] and recalled below. These are known to be all the examples A having class number one, and positive genus. The sign normalized rank one Drinfeld A-modules ρ's for these A's are given there explicitly, for the sign function with the sign of x and y to be 1. Let $\log_\rho(z)$ and $e(z) := \exp_\rho(z) = \sum z^{q^i}/d_i$ be the corresponding logarithm and exponential respectively.

Theorem 8. *With the notation as above, for the examples (A)-(C),*

$$\zeta_A(1) := \sum_{d=0}^{\infty} \sum_{a \in A_d+} \frac{1}{a} = \log_\rho(g),$$

where g is the unique generator (with its exponential being one-unit at infinity) of Taelman's $\rho(A) := e(K_\infty) \cap A$, which is a rank one A-module under ρ. Further, Taelman's class module is trivial in each case, and thus the zeta value is class module order times the regulator.

Proof. Example (A) is $A = \mathbb{F}_3[x,y]/y^2 = x^3 - x - 1$.

We show [T92, Thm VI] that $\zeta(1) = \log_\rho(y-1)$.

Now we claim that $y-1$ generates $\rho(A)$ as A-module under ρ. From the result quoted above, $y - 1 \in \rho(A)$. Using the functional equations given by explicit description of ρ, we calculate the d_i's and see that the degree of d_i is $(i-2)3^i$, for $i > 0$, and is 0 for $i = 0$. If $z \in K_\infty$ has degree d, then we thus see that degree of $e(z)$ is less than -1 for $d < -1$, 3^{d+1} for $d \geq 0$, and 0 for $d = -1$. So it is immediate, for example, that the degree two elements $x, x \pm 1$ do not belong to $\rho(A)$. Using the $\mathbb{F}_q$-linearity of e, it is enough to show that 1 does not belong to $\rho(A)$. If $e(z) = 1$, z is of the form $\pm x/y+$ terms of the lower degree. Using $d_1 = 1/y$ and degrees of d_i given above, ignoring degrees less than -1, we see that $1 = x/y + x^3/y^2 = x/y + x^3/(x^3 - x - 1)$ and thus we have an element of degree -1 in $e(K_\infty)$ contradicting the estimates above. (Another way to show this is to use the explicit ρ and degree estimates and show that $y-1$ is not in the module, if it is not a generator). Thus this rank one module is generated by $y-1$, which is in fact the unique generator whose exponential is a one-unit at infinity.

Example (B) is $A = \mathbb{F}_4[x,y]/y^2 + y = x^3 + \zeta_3$.

We show [T92, Thm. VIII] that $e(\zeta(1)) = x^8 + x^4 + x^2 + x$.

The calculation as above, shows that degree of d_i is now $(i-3)4^i$ for $i > 0$ and 0 for $i = 0$, so that if degree of $z \in K_\infty$ is d, then degree of $e(z)$ is ≤ -3 for $d \leq -3$, $= 4^{d+2}$ for $d \geq -1$, and 0 for $d = -2$. Thus without loss of generality, by degree considerations, the generator has degree $0, 4$ or 16. Again, a straight degree (and $\rho_x(1) = x^8 + x^2 + x + 1$) calculation shows that the first two possibilities make it impossible for the module to contain $x^8 + x^4 + x^2 + x$. Thus $x^8 + x^4 + x^2 + x$ is generator of $\rho(A)$, again a unique generator whose exponential is one-unit at infinity.

Example (C) is $A = \mathbb{F}_2[x,y]/y^2 + y = x^3 + x + 1$.

We show [T92, Thm. X] that $e(\zeta(1)) = 0$ and in fact that $\zeta(1)$ is the fundamental period (value of logarithm of zero), as $d_1 = 1$.

Thus $\rho(A)$ is the Torsion module generated by zero. (Compare [Ta10] with the Carlitz module case, where it is generated by 1, as $\zeta(1) = \log(1)$, which was essentially [AT90, p.181] proved by Carlitz. In that case, when $q = 2$, it is torsion module $\{0, 1, t, t+1\}$ generated by $t^2 + t$-torsion point 1).

This takes care of the first part of the theorem. As explained in [Ta10] to show that the class module is trivial, it is enough to show the

claim: $X := e(K_\infty) + A = K_\infty$.

It is enough to show that $K_\infty \subset X$. For example A (B respectively), by the calculation of degree d_i's above (or using that $\log(z)$ converges for z of degree less than $-3/2$ ($-8/3$ respectively)), we see that $e(K_\infty)$ contains all elements in K_∞ of each degree less than -1 (-2 respectively) and A contains elements of all non-negative degrees except 1. Hence it remains to show that X contains elements of degree $-1, 1$ ($-2, -1, 1$ respectively). We take care of the remaining degrees as follows.

For example A: $e(x/y)$ is x/y (of degree -1) plus $y^3(y/x)^3 = x^3 \in A$ plus an element of degree less than -8 (and thus in K_∞), so that degree -1 is also taken. Also, $e(y/x)$ is y/x (of degree 1) plus $y^3(y/x)^3 = x^6 - 1 - 1/x^3 \in X$) plus $(y^9+y^{13}(x^3-x))/(x^9(x^9-x))$ (which is y^3+y plus element of degree -3 and thus in X) plus (sum of) terms of degree non-positive (and thus in X). This proves the claim in example A.

For example B: Since $e(1/x)$ is $1/x$ (of degree -2) plus $(x^4+x)/x^4 \in X$ plus terms of degree less than -15 (and thus in X), we get degree -2. The expansions of $e(x/y)$ ($e(y/x)$ respectively) consist of degree -1 (1 respectively) plus two (four respectively) terms which are rational functions of x (and thus subtracting an appropriate polynomial in x contribute degree ≤ -2 and thus in X) plus terms of degree ≤ -2 (and thus in X). This proves the claim in example B.

For example C: Calculation is similar and even simpler. Now degree of d_i is $(i-1)2^i$, for $i \geq 1$, and thus (alternately, logarithm converges for elements of degree less than zero) all elements of degree less than 0 are in the image of exponential. Only degree one is missing in elements of A, but as above we see that $e(y/x)+x$ is of degree one, proving the claim. □

Example (D) is $A = \mathbb{F}_2[x,y]/y^2+y = x^5+x^3+1$ of genus 2. Here $\zeta(1)$ is x^2+x times the fundamental period. The only x^2+x-torsion is zero, thus class module should have order x^2+x, but we have not yet verified this directly.

6. Updates added on 23 August 2011

In his doctoral thesis work with the author, Alejandro Lara Rodriguez has now proved [Lr10, Lr11] most of the conjectures mentioned in 3.2 and has also proved [Lr11, Thm 7.1, Cor 7.2] the following theorem by making the recipe of Theorem 4 explicit.

Theorem 9. *Let q be a power of prime p, a, b be positive integers and m be the smallest integer such that $a+b \leq p^m$. Then we have*

$$\zeta(a)\zeta(b) - \zeta(a+b) - \zeta(a,b) - \zeta(b,a) = \sum_{i=0}^{b-1} f_i \zeta(b-i, a+i) + \sum_{j=0}^{a-1} g_j \zeta(a-j, b+j),$$

where, if $q=2$, we have

$$f_i = \binom{2^m - a}{i}, \qquad g_j = \binom{2^m - b}{j},$$

and more generally, for q arbitrary, with $H_{a,b}(t)$ given by

$$H_{a,b}(t) = \frac{1}{t^a}\left(-(t^{q-1}-1)^{p^m-a} \sum_{\theta \in \mathbb{F}_q^*} \left((t+\theta)^{q-1} - 1\right)^a \bmod t^{a+b}\right),$$

we have $f(t) := f_0 + f_1 t + \cdots + f_{b-1}t^{b-1} = H_{a,b}(t)$, $g(t) := g_0 + g_1 t + \cdots + g_{a-1}t^{a-1} = H_{b,a}(t)$, where $H_{b,a}$ is obtained from $H_{a,b}(t)$ by interchanging a and b.

7. Updates added on 5 February 2013

Using similar, but better partial fraction decomposition formula, Huei Jeng Chen has simplified the above recipe considerably to

$$\zeta(a)\zeta(b)-\zeta(a+b)-\zeta(a,b)-\zeta(b,a)=\sum((-1)^{a-1}\binom{j-1}{a-1}+(-1)^{b-1}\binom{j-1}{b-1})\zeta(a+b-j,j),$$

where the sum is over j which are multiples of $q-1$ and $0<j<a+b$.

Lara's PhD thesis of January 2013 has also investigated which shuffle relations survive in Theorem 4, if we drop the condition on A that the infinite place is of degree one.

Good understanding of all the multizeta relations with $\mathbb{F}_p(t)$-coefficients is still an interesting open problem, though many more such relations have since been discovered, and systematic numerical investigation of linear dependencies, parallel to the one performed by Zagier, is underway. Chieh-Yu Chang has proved [Cp] interesting general transcendence theorems for the multizeta values, making use of Theorem 7 and the transcendence criterion of Anderson, Brownawell and Papanikolas.

In a recent preprint, Kirti Joshi has constructed a neutral, tannakian, F-linear category of mixed t-motives, and also of mixed Carlitz-Tate t-motives containing all those mentioned in Theorem 7, thus providing a natural playground for multizeta and setting the stage for exploring analogs of various recent motivic works related to multizeta.

Acknowledgments I would like to thank BIRS and the organisers of this 2009 conference. I would like to thank Lenny Taelman for explaining me his nice work. I am grateful to my collaborator Greg Anderson for teaching me many interesting things over the years.

References

[A86] G. W. Anderson, t-motives, Duke Math. J. **53**1985, 457-502.

[A94] G. W. Anderson, Rank one elliptic A-modules and A-harmonic series, Duke Math. J. 73 (1994), 491-542.

[A96] G. W. Anderson, Log-algebraicity of twisted A-harmonic series and special values of L-series in characteristic p, J. Number Theory, 60 (1996), 165-209.

[A07] G. W. Anderson, Digit patterns and the formal additive group, Israel J. Math. 161 (2007), 125-139.

[AT09] G. W. Anderson, D. S. Thakur, Multizeta values for $F_q[t]$, their period interpretation and relations between them, International Mathematics Research Notices IMRN (2009) no. 11, 2038-2055.

[AT90] G. W. Anderson and D. S. Thakur, Tensor powers of the Carlitz module and zeta values, Ann. of Math. (2) **132** (1990), 159–191.

[ATp] G. W. Anderson and D. S. Thakur, Ihara power series for $\mathbb{F}_q[t]$, Preprint.

[Cp] C.-Y. Chang, Linear independence of monomials of multizeta values in positive characteristic, Preprint, arXive:1207.2326.

[G96] D. Goss, *Basic structures of function field arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **35**, Springer-Verlag, Berlin, 1996.

[I91] Y. Ihara, Braids, Galois groups and some arithmetic functions, Proceedings of International Congress of Mathematicians, Kyoto 1990, 99-120 Math. Soc. Japan, Tokyo 1991.

[L09] V. Lafforgue, Valueurs spéciales des fonction L en caractéristique p, J. Number Theory, 129 (2009), 2600-2634.

[Lr09] A. Lara Rodriguez, Some conjectures and results about multizeta values for $F_q[t]$, Masters thesis for The Autonomous University of Yucatan, May 7 (2009).

[Lr10] A. Lara Rodriguez, Some conjectures and results about multizeta values for $F_q[t]$, J. Number Theory 130 (2010) 1013-1023.

[Lr11] A. Lara Rodriguez, Some conjectures and results about multizeta values for $F_q[t]$, J. Number Theory 131 (2011) 2081-2099.

[Lr10] A. Lara Rodriguez, Special relations between function field multizeta values and parity results, J. Ramanujan Math. Soc. 27(3) (2012), 275-293.

[M06] R. Masri, Multiple zeta values over global function fields, Proc. Symp. Pure Math 75 (2006), 157-175.

[Ta10] L. Taelman, A Dirichlet unit theorem for Drinfeld modules, Math. Ann. 348 (2010), no. 4, 899-907.

[T92] D. Thakur, Drinfeld modules and arithmetic in function fields, Duke IMRN 9 (1992), 185-197.

[T94] D. Thakur, Iwasawa theory and cyclotomic function fields, Arithmetic Geometry, Contemp. Math. 174 (1994), 157-165. Amer. Math. Soc.

[T04] D. Thakur, *Function Field Arithmetic*, World Scientific Pub., 2004.

[T09] D. Thakur, Relations between multizeta values for $F_q[t]$, International Mathematics Research Notices IMRN (2009) no. 11, 2038-2055.

[T10] D. Thakur, Shuffle relations for function field multizeta values, International Mathematics Research Notices IMRN (2010) no. 11, 1973-1980.

Dinesh S. Thakur, Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

E-mail: thakur@math.arizona.edu