PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 8, August 2013, Pages 2603–2608
S 0002-9939(2013)11538-8
Article electronically published on April 17, 2013

HIGHER DIOPHANTINE APPROXIMATION EXPONENTS AND CONTINUED FRACTION SYMMETRIES FOR FUNCTION FIELDS II

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(Communicated by Matthew A. Papanikolas)

ABSTRACT. We construct families of non-quadratic algebraic laurent series (over finite fields of any characteristic) which have only bad rational approximations so that their rational approximation exponent is as near to 2 as we wish and, at the same time, have very good quadratic approximations so that the quadratic exponent is close to the Liouville bound and thus can be arbitrarily large. In contrast, in the number field case, the Schmidt exponent (an analog of the Roth exponent of 2 for rational approximation) for approximations by quadratics is 3. We do this by exploiting the symmetries of the relevant continued fractions. We then generalize some of the aspects from the degree $2(= p^0 + 1)$-approximation to degree $p^n + 1$-approximation. We also calculate the rational approximation exponent of an analog of π.

1. BACKGROUND

We recall [S80, Chapter 8] some basic definitions, facts and conjectures about diophantine approximation of real numbers by rationals or (real) algebraic numbers. (See also [B04, BG06] and [W] for a nice survey of recent developments.)

Definition 1 (Height and higher diophantine approximation exponents)**.** For a non-zero algebraic number β, define $H(\beta)$ to be the maximum of the absolute values of the coefficients of a non-trivial irreducible polynomial with co-prime integral coefficients that it satisfies.

For α an irrational real number not algebraic of degree $\leq d$, define $E_d(\alpha)$ ($E_{\leq d}(\alpha)$ respectively) as $\limsup(-\log|\alpha-\beta|/\log H(\beta))$, where β varies through all algebraic real numbers of degree d ($\leq d$ respectively).

Note that $E_1(\alpha)$ is the usual exponent $E(\alpha) := \limsup(-\log|\alpha - P/Q|/\log|Q|)$.

Then for irrational α, we have $E(\alpha) \geq 2$ by Dirichlet's theorem, whereas for irrational algebraic α of degree d, we have $E(\alpha) \leq d$ by Liouville's theorem and $E(\alpha) = 2$ by Roth's theorem, improving Liouville, Thue, Siegel, and Dyson bounds.

For real α not algebraic of degree $\leq d$, Wirsing (generalizing Dirichlet's result) conjectured $E_{\leq d}(\alpha) \geq d + 1$ and proved a complicated lower bound (for this exponent) which is slightly better than $(d + 3)/2$, whereas Davenport and Schmidt proved his conjecture for $d = 2$. On the other hand, for α of degree $> d$, we have

Received by the editors August 13, 2011 and, in revised form, September 1, 2011 and November 5, 2011.

2010 *Mathematics Subject Classification.* Primary 11J68, 11J70, 11J93.

This work was supported in part by NSA grant H98230-10-1-0200.

the Liouville bound $E_{\leq d}(\alpha) \leq \deg \alpha$. Schmidt (generalizing Roth's result) proved that for real algebraic α of degree greater than d, $E_{\leq d}(\alpha) \leq d+1$.

From now on, unless stated otherwise, we only focus on the function field analogs (see, e.g., [T04] for general background and [T04, Cha. 9], [T09] for diophantine approximation, continued fractions background and references), where the role of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ is played by $A = \mathbb{F}[t], K = \mathbb{F}(t), K_\infty = \mathbb{F}((1/t))$ respectively, where $\mathbb{F}$ is a finite field of characteristic p. With the usual absolute value coming from the degree in t of polynomials or rational functions, we have exactly the same definitions of heights and exponents. Now by rationals, reals, and algebraic, we mean elements of K, K_∞, and algebraic over K respectively. (There should be no confusion between degrees in t (as above) of rationals and algebraic degree over K of irrational algebraics.)

Then analogs of Dirichlet and Liouville theorems hold, but a naive analog of Roth's theorem fails, as shown by Mahler [M49]. For other results, see [dM70, S00, T04] and the references therein, and, for example, results [KTV00] in the Wirsing direction. We also recall that for almost all real α (and for almost all $\alpha \in K_\infty$), $E_d(\alpha) = d+1$. (Here 'almost all' means that the set of exceptions is of measure zero.)

We now deal with the similar phenomena for quadratic approximations. In [T11], for $p = 2$ and any integer $m > 1$, we constructed algebraic elements α of degree at most 2^{m^2} having continued fractions with folding pattern symmetries and a bounded sequence of partial quotients, so that $E(\alpha) = 2$, but with $E_2(\alpha) \geq 2^m > 3$. In this paper, with a different construction, based on some ideas of [T99, T11], we prove:

Theorem 1. *Let p be a prime, q be a power of p and $\epsilon > 0$ be given. Then we can construct infinitely many algebraic α, with explicit equations and continued fractions, such that*

$$q \leq \deg(\alpha) \leq q+1, \quad E(\alpha) < 2+\epsilon, \quad E_2(\alpha) > q-\epsilon,$$

with an explicit sequence of quadratic approximations realizing the last bound.

Theorem 2. *Let p be a prime, q be a power of p and $m, n > 1$ be given. Then we can construct infinitely many algebraic $\alpha_{m,n}$, with explicit equations and continued fractions, such that*

$$\deg(\alpha_{m,n}) \leq q^m + 1, \quad \lim_{n\to\infty} E(\alpha_{m,n}) = 2, \quad \lim_{n\to\infty} E_{q+1}(\alpha_{m,n}) \geq q^{m-1} + \frac{q-1}{(q+1)q},$$

with an explicit sequence of degree $q+1$-approximations realizing the last bound.

A natural question raised by these considerations is whether there are algebraic α's of each degree d, with $E(\alpha) = 2$ (or even with bounded partial quotients) and for which the Liouville bound for the lower degree approximations is attained, or whether some of these requirements need to be relaxed.

2. Continued fractions

Continued fractions are natural tools of the theory of diophantine approximation. See [dM70, BS76, S00, T04] for the basics in the function field case.

Let us review some standard notation. We write $\alpha = a_0 + 1/(a_1 + 1/(a_2 + \cdots))$ in the short-form $[a_0, a_1, \cdots]$. We write $\alpha_n = [a_n, a_{n+1}, \cdots]$ so that $\alpha = \alpha_0$. Let us define p_n and q_n as usual in terms of the partial quotients a_i so that p_n/q_n is the n-th convergent $[a_0, \cdots, a_n]$ to α. Hence $\deg q_n = \sum_{i=1}^{n} \deg a_i$.

Following the basic analogies mentioned above, we use the absolute value coming from the degree in t to generate the continued fraction in the function field case, and we use the 'polynomial part' in place of the 'integral part' of the 'real' number $\alpha \in K_\infty$. In the function field case, for $i > 0$, a_i can be any non-constant polynomial, and so the degree of q_i increases with i, but a_i or q_i need not be monic. As usual, we have

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}, \qquad \alpha = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}, \tag{1}$$

implying the usual basic approximation formula

$$\alpha - p_n/q_n = (-1)^n / ((\alpha_{n+1} + q_{n-1}/q_n) q_n^2), \tag{2}$$

which because of the non-archimedean nature of the absolute value, now implies

$$|\alpha - p_n/q_n| = 1/(|a_{n+1}||q_n|^2). \tag{3}$$

If we know the continued fraction for α, the equation allows us to calculate the exponent, using $\deg q_n = \sum_1^n \deg a_i$, as

$$E(\alpha) = 2 + \limsup \frac{\deg a_{n+1}}{\sum_{i=1}^{n} \deg a_i}. \tag{4}$$

3. Proof of Theorem 1

First, we note that if $\gamma = [a_0, \cdots, a_m, b, \cdots]$ (with arbitrary entries after b), then, by (1),

$$\alpha - \gamma = \frac{\alpha_{m+1} p_m + p_{m-1}}{(\alpha_{m+1} q_m + q_{m-1})} - \frac{\gamma_{m+1} p_m + p_{m-1}}{(\gamma_{m+1} q_m + q_{m-1})} = \pm \frac{\alpha_{m+1} - \gamma_{m+1}}{(\cdots)(\cdots)}.$$

Hence the non-archimedean nature of the absolute values implies that

$$|\alpha - \gamma| = \frac{1}{|b q_m^2|} \quad \text{if } \deg(a_{m+1} - b) = \deg a_{m+1}. \tag{5}$$

Next, we normalize the absolute value and the logarithm so that $\log |a|$ is a degree of a in t and denote by $h := \log H$ the resulting logarithmic height and give a simple bound on the height of a quadratic irrational θ in terms of the degrees of the partial quotients of its continued fraction, which is eventually periodic by analogy (see, e.g., [S00]) of Lagrange's theorem. Thus consider $\theta = [b_0, \cdots, b_j, \mu]$, where $\mu = [\overline{X}]$ is purely periodic continued fraction obtained by repeating period (tuple) $X = (a_0, \cdots, a_n)$. Thus $\mu = (\mu p_n + p_{n-1})/(\mu q_n + q_{n-1})$ implies $h(\mu) \leq \deg(p_n) = \sum \deg a_i$, where the sum runs over i from 0 to n. For $a \in A$, comparing the polynomials satisfied by quadratic γ and $a + 1/\gamma$, we see that $h(a + 1/\gamma) \leq h(\gamma) + 2\deg(a)$, whose repeated application gives

$$h(\theta) \leq 2 \sum_{i=0}^{j} \deg b_i + \sum_{i=0}^{n} \deg a_i. \tag{6}$$

Next, we proceed to the construction of the α's and their quadratic approximations β_s. By capital letters X, Y, etc., we will denote tuples of partial quotients, and we will denote by X^m the tuple resulting from X by raising each of its entries to the m-th power. Let $n+1 = r\ell$, let $a_0, \cdots, a_{\ell-1} \in A$ be non-constant polynomials, and let $Y = (a_0, \cdots, a_{\ell-1})$ and $X = (Y, \cdots, Y) = (a_0, \cdots, a_n)$, obtained by repeating Y r times. Then

$$\alpha := [X, X^q, X^{q^2}, \cdots] = [X, \alpha^q] = \frac{\alpha^q p_n + p_{n-1}}{\alpha^q q_n + q_{n-1}}$$

so that α is algebraic of degree at most $q+1$. For any $s > 1$, let

$$\beta_s := [X, X^q, \cdots, X^{q^{s-1}}, \overline{Y^{q^s}}].$$

Let $L = \sum_{i=0}^{\ell-1} \deg a_i$ so that $\sum_{i=0}^{n} \deg a_i = Lr$. We see using (6) that $h(\beta_s) \leq 2Lr(q^s-1)/(q-1) + q^s L$, while (5) implies (since $b = a_0^{q^s}$) that $-\log|\alpha - \beta_s| = q^s \deg a_0 + 2(Lr(q^{s+1}-1)/(q-1) - \deg a_0)$. Hence, letting s tend to infinity, we see that $E_2(\alpha) \geq (2Lrq + (q-1)\deg a_0)/(2Lr + L(q-1))$ (when is this an equality?), which is at most q and tends to q if r tends to infinity. On the other hand, by (4), for some i, $E(\alpha) - 2 \leq q^{m+1}\deg(a_i)/(rq^m \deg a_i)$, which tends to 0, as r tends to infinity. Hence, given $\epsilon > 0$, choosing r appropriately large, we satisfy the claims of the theorem, with the Liouville bound implying that $\deg(\alpha) \geq q$, since without loss of generality we can assume that $\epsilon < 1$. This completes the proof.

Finally, we remark that if our equation for α is reducible, then we reach the Liouville bound (of q in that case), at least as ϵ tends to zero. It might also be possible to tighten the inequalities to get a better lower bound for E_2.

4. Proof of Theorem 2

We follow the strategy of the previous section, but now we define

$$\alpha := \alpha_{m,n} := [X, X^{q^m}, X^{q^{2m}}, \cdots, X^{q^{(s-1)m}}, X^{q^{sm}}, \cdots]$$

with $X = (B, B^q, \cdots, B^{q^{n-1}}, C)$, where $B, C \in \mathbb{F}[t]$, with $b := \deg B > 0$, $\deg C = q^n b$, and $\deg(B^{q^n} - C) = q^n b$, which is clearly possible if $\mathbb{F} \neq \mathbb{F}_2$. (We will deal with the case $\mathbb{F} = \mathbb{F}_2$ at the end.) Next we let

$$\beta_s := [X, X^{q^m}, X^{q^{2m}}, \cdots, X^{q^{(s-1)m}}, B^{q^{sm}}, B^{q^{sm+1}}, B^{q^{sm+2}}, \cdots].$$

The mobius transformation expression for α and β_s, as in the proof of the last theorem, shows that $\deg \alpha \leq q^m + 1$ and that $\deg(\beta_s) \leq q+1$. The last inequality is in fact equality by the Liouville theorem by calculating its exponent by (4).

Write

$$D := b\frac{q^{n+1}-1}{q-1}\frac{q^{sm}-1}{q^m-1}, \quad F := q^{ms} b \frac{q^{n+1}-1}{q-1},$$

which are the sums of the degrees of entries in $(X, X^{q^m}, \cdots, X^{q^{(s-1)m}})$ and $X^{q^{sm}}$, respectively. Then a straight calculation using (5) shows that

$$-\log|\alpha - \beta_s| = 2(D - b + F) - q^{ms+n} b.$$

Now the height of $\theta := [B^{q^{sm}}, B^{q^{sm+1}}, \cdots] = B^{q^{sm}} + 1/\theta^q$ is clearly $q^{sm} b$. For $a \in A$, by comparing heights of γ and $a + 1/\gamma$ for a γ satisfying an equation of the form $\gamma = (P\gamma^q + Q)/(R\gamma^q + S)$, we see that $h(a + 1/\gamma) \leq h(\gamma) + (q+1)\deg(a)$, whose repeated application gives

$$h(\beta_s) \leq (q+1)D + q^{ms} b.$$

Taking the limit of the ratio as s tends to infinity, we get

$$E_{q+1}(\alpha) \geq (2(\frac{q^{n+1}-1}{q-1})(\frac{q^m}{q^m-1}) - q^n)/((q+1)(\frac{q^{n+1}-1}{q-1})(\frac{1}{q^m-1}) + 1),$$

and taking the limit of the right side, as n tends to infinity, we get the lower bound claimed in the theorem.

Finally, using (4), we see that

$$E(\alpha) = 2 + \lim_{s\to\infty} \frac{q^{ms}b}{D-b} = 2 + \frac{(q-1)(q^m-1)}{q^{n+1}-1},$$

implying the claim on the limit as n tends to infinity.

Finally, we consider the case $\mathbb{F} = \mathbb{F}_2$. We can choose $B, C \in \mathbb{F}[t]$, with $\deg B = b$, $\deg(C) = q^n b$ and $\deg(B^{q^n} - C) = q^n b - 1$. The whole asymptotic analysis is the same, as b tends to infinity, leading to the same bounds. This finishes the proof.

5. Exponent for an analog of π

In [T11, Sec. 7], we calculated the exponent of an analog of π for $\mathbb{F}_q[t]$. (For this section, we take $\mathbb{F} = \mathbb{F}_q$.) But as discussed in [T04, pp. 47-48], there are a couple of good candidates for analogs of π (up to rational multiples, which do not change exponents). We now consider $\pi_1 := \prod(1 - [j]/[j+1]) \in K_\infty$, where $[j] := t^{q^j} - t$ and the product is over j from 1 to ∞. As explained in the reference above, the Carlitz period (good analog of $2\pi i$) is then $(-[1])^{1/(q-1)}\pi_1$.

Theorem 3. *For π_1 as above, $E(\pi_1) \geq (q-1)^2/q$, with equality when $q > 5$.*

Proof. Note that $[j+1] - [j] = [1]^{q^j}$ and $[1]$ divides $[n]$ with the quotient coprime to $[1]$. Thus the truncation of the product at $j = N-1$, which equals $[1]^{q+q^2+\cdots+q^{N-1}}/([2][3]\cdots[N])$, has a denominator of degree $q^2 + q^3 + \cdots + q^N - (N-1)q$, whereas it approximates π_1 with error of degree $q^N - q^{N+1}$ (resulting from $1 - (1 - [N]/[N+1])$), showing that the inequality claimed, as the limit of the ratio of these two quantities as N tends to ∞, tends to $(q-1)^2/q$. When $q > 5$, we have $(q-1)^2/q > \sqrt{q} + 1$, and this implies by a proposition of Voloch (see [V88, Prop. 5] or [T04, Lemma 9.3.3]) the equality claimed. ☐

Acknowledgments

The author thanks Liang-Chung Hsia and the National Center for Theoretical Sciences, Hsinchu, Taiwan, for their hospitality and for the invitation to speak on [T11] in July 2011, the preparation for which led to the first theorem. The author also thanks Professor Michel Waldschmidt for spotting a notational clash in the first draft.

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