# VALUATIONS OF V-ADIC POWER SUMS AND ZERO DISTRIBUTION FOR GOSS' V-ADIC ZETA FOR $\mathbb{F}_q[t]$

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*Dedicated to Jean-Paul Allouche on his 60th Birthday*

Abstract. We study the valuation at an irreducible polynomial $v$ of the $v$-adic power sum, for exponent $k$ (or $-k$), of polynomials of a given degree $d$ in $\mathbb{F}_q[t]$, as a sequence in $d$ (or $k$). Understanding these sequences has immediate consequences, via standard Newton polygon calculations, for the zero distribution of corresponding $v$-adic Goss zeta functions. We concentrate on $v$ of degree one and two and give several results and conjectures describing these sequences. As an application, we show, for example, that the naive Riemann hypothesis statement which works in several cases, needs modifications, even for a prime of degree two. In the appendix, we give an elementary proof of (and generalize) a product formula of Richard Pink for the leading term of the Goss zeta function.

## 1. Introduction

In [T09], we investigated valuation sequences at the infinite place of the power sums of polynomials in $\mathbb{F}_q[t]$, with positive and negative exponents. We gave (i) a simple recipe to find these, (ii) a duality between valuations for positive and negative exponents, (iii) a simple recursion in case $q$ was prime, and (iv) applications to the zero distribution of the Goss zeta function [G96, G00, T04] (giving a cleaner approach to the Riemann hypothesis in this case, due to Wan and Sheats [W96, DV96, S98]) and to the study of the multizeta values.

In this paper, we look at finite places $v$ of $\mathbb{F}_q[t]$ and study $v$-adic valuations of power sums with $v$-factor removed. We give conjectural formulas for them describing very interesting patterns, for $v$ of degree one and two, proving full results in special cases. We then give applications to the zero distribution of the Goss $v$-adic zeta function, showing that in the degree two case, the Riemann hypothesis type statement, holding at the infinite degree one place, needs modification. We also calculate a particular power sum, which is a leading term of the Goss zeta, when $q$ is a prime, and in particular, give an elementary proof of (and generalize) Richard Pink's nice product formula for the same when $q = 2$.

Patterns of these valuation sequences exhibit symmetries of remarkably similar type to those occurring in several works of Allouche, Shallit, Mendès France, Lasjaunias etc., as well as those found by the author for continued fractions for analogs of $e$, Hurwitz numbers, and some algebraic quantities.

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## 2. Power sums

### 2.1. **Notation.**

$$\begin{aligned}
\mathbb{Z} &= \{\text{integers}\}\\
\mathbb{Z}_+ &= \{\text{positive integers}\}\\
\mathbb{Z}_{\geq 0} &= \{\text{nonnegative integers}\}\\
q &= \text{a power of a prime } p,\ q = p^f\\
A &= \mathbb{F}_q[t]\\
A+ &= \{\text{monics in } A\}\\
A_d+ &= \{\text{monics in } A \text{ of degree } d\}\\
K &= \mathbb{F}_q(t)\\
K_v &= \text{completion of } K \text{ at the place } v \text{ of } K\\
C_v &= \text{the completion of an algebraic closure of } K_v\\
[n] &= t^{q^n} - t\\
d_n &= \textstyle\prod_{i=0}^{n-1}(t^{q^n} - t^{q^i})\\
\ell_n &= \textstyle\prod_{i=1}^{n}(t - t^{q^i})\\
\ell(k) &= \text{sum of the digits of the base } q \text{ expansion of } k\\
\deg &= \text{function assigning to } a \in A \text{ its degree in } t,\ \deg(0) = -\infty
\end{aligned}$$

While in the notation above, we let $v$ be any place of $K$, we will use $v$ for the finite places, i.e., those corresponding to irreducible polynomials of $A$ and we will use $\infty$ for the usual infinite place of $K$, i.e., the place corresponding to the valuation coming from the degree in $t$.

2.2. **Power sums.** For $k \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\geq 0}$, and $v$ a prime of $A$, write

$$S_d(k) := \sum_{a \in A_d+} \frac{1}{a^k} \in K, \quad S_{d,v}(k) := \sum_{\substack{a \in A_d+\\ (v,a)=1}} \frac{1}{a^k} \in K.$$

2.3. **Goss Zeta and $v$-adic zeta.** For $k \in \mathbb{Z}$, put

$$\zeta(k) := \sum_{d=0}^{\infty} S_d(k) \in K_\infty, \quad \zeta_v(k) := \sum_{d=0}^{\infty} S_{d,v}(k) \in K_v.$$

More generally, we have two variable Goss zeta functions defined as follows. Define exponent spaces $S_\infty := C_\infty^* \times \mathbb{Z}_p$ and $S_v := C_v^* \times \lim \mathbb{Z}/(q^{\deg v} - 1)p^j\mathbb{Z}$.

For $s = (x, y) \in S_\infty$, put

$$\zeta(s) = \sum_{d=0}^{\infty} x^d \sum_{a \in A_d+} (a/t^d)^y \in C_\infty.$$

Note that for $y$ an integer, the coefficient of the $d$-th term in this power series in $x$ is nothing but $S_d(-y)t^{-dy}$, hence $\zeta(k)$ as above is $\zeta(t^{-k}, -k)$.

For $s = (x, y) \in S_v$, put

$$\zeta_v(s) = \sum_{d=0}^{\infty} x^d \sum_{\substack{a \in A_d+\\ (a,v)=1}} a^y \in C_v.$$

Note that for $y$ an integer, the coefficient of the $d$-th term in this power series is nothing but $S_{d,v}(-y)$, hence $\zeta_v(k)$ above is $\zeta(1, -k)$.

See [C35, G96, T04] and references there for more details on these interpolations, properties.

We begin with definitions and basic results on the valuations of these power sums.

2.4. **Valuations.** For a finite prime $v$ of $A$, and the usual place at infinity, put

$$s_d(k) := \mathrm{val}_\infty(S_d(k)) = -\deg(S_d(k)), \quad v_d(k) := \mathrm{val}_v(S_{d,v}(k)).$$

2.5. **Valuation sequence at $\infty$.** In this paper, we concentrate on valuations at finite primes, but for comparison only, we mention some results on valuations at infinity, and give some remarks.

In [T09], we have described interesting behavior of $s_d(k)$, For example, when $q$ is a prime, we proved [T09, Thm. 1]

$$s_d(k) = s_{d-1}(s_1(k)) + s_1(k) = \sum_{i=1}^{d} s_1^{(i)}(k),$$

where $s_1^{(i)}$ is the $i$-th iteration of $s_1$. There are several other formulas, and applications in [T09], including Riemann hypothesis type zero distribution results for the Goss two variable zeta function for the place at infinity.

2.6. **Remarks.** (1) On the other hand, for non-prime $q$, $s_d(k)$ is not even determined by $s_1(k)$. Example: For $q = 4$, $s_1(75) = s_1(93) = 96$, whereas $s_2(75) = 348$ and $s_2(93) = 480$. For general $q$, [T09, Thm. 2] gives a sort of 'duality' connection between the values at positive and negative $k$, linking values at $-k$ and $q^n - k$ under certain conditions. Dual example to the one above is $q = 4$, $s_1(-181) = s_1(-163) = -160$ and $s_2(-181) = -164$, whereas $s_2(-163)$ is infinite.

(2) The recursion above when $q$ is a prime and the duality [T09, Thm. 2] for $s_d(k)$ for general $q$, also give a fast way of calculating these valuations at positive $k$. Note that the case of negative $k$ (even when $d = 1$) is much easier polynomial calculation than rational calculation at positive $k$.

(3) We do not know whether there is any duality of actual power sums $S_d(k)$ or zeta values $\zeta(k)$ themselves giving some kind of functional equation. For the two parameter special family $k = q^n - q^r$, with $r < n$, we have the following nice relation, for $n - r \geq d$,

$$S_d(q^r - q^n)/S_d(q^r) = (q^{n-r}!)^{q^r}/(q^{n-(r+d)}!)^{q^{r+d}},$$

using Carlitz factorial [T04, pa. 102] and Carlitz results (see [T04, Sec.5.6]). While all the factorial-gamma values are monomials in $[i]$'s, the ratios of power sums for dual exponents are not such monomials for general $k$, so we do not know what, if any, a correct generalization of such phenomena would be.

(4) In [T09, 2.2.5 (ii)], it is shown that if $S_d(k_1) = S_d(k_2)$ holds for $d = 1$, then it holds for all $d$. While true, it should be pointed out that, in addition to the proof there, in case of positive $k$, it is vacuously true also, because [T09, 2.3] shows that $S_d(k)\ell_d^k$ is a polynomial congruent to 1 modulo $t$, so the equality above for any $d$ and $k_2 > k_1 > 0$ would imply $\ell_d^{k_2-k_1}$ is congruent to 1 modulo $t$, whereas it is divisible by $t$. On the other hand, interestingly, the statement is true, when $q$ is prime, with $S_d$ replaced by $s_d$ by [T09, Thm.1], and not in general as we saw above, for example. (In my editing gaffe, these motivating remarks were left off in [T09]).

2.7. **Valuation sequence at $v$.** We have $v_d(k) \geq 0$ for all $k \in \mathbb{Z}$, with it being infinite when the corresponding power sum is zero. Note $v_d(pk) = pv_d(k)$, so without loss of generality, we can restrict to $k$ prime to $p$.

Another simple remark is that two primes related by automorphisms $t \to t + \theta$, ($\theta \in \mathbb{F}_q$) of $A$ (e.g., any two degree one primes) give the same valuation sequence.

2.8. **Non-vanishing of power sums.** The power sums $S_{d,v}(k)$ are non-zero for $k > 0$, as can be seen by choosing a monic prime $P$, unequal to $v$ and of degree $d$ (which can be done unless $q = 2$, $v$ is of degree 2 and $d = 2$, which we can check separately), and noticing that except for $a = P$ term, all other terms in the power sum $S_{d,v}(k)$ have valuation 0 at $P$.

For $k \leq 0$, these power sums can be zero. In fact, $S_{d,v}(k)$ is zero if $d > \ell(-k)/(q-1)+\deg(v)$, by the Carlitz result [T04, Cor. 5.6.2] that $S_d(k) = 0$ if $d > \ell(-k)/(q-1)$ (with the converse of the latter holding also for $q$ prime). We have not investigated the exact conditions corresponding to the non-vanishing. For a similar necessary and sufficient condition for vanishing of $S_d(k)$, for general $q$, due to Carlitz, Sheats and Böckle, see [T09, A5] and [B12].

## 3. Some evaluations and simple bounds for $k > 0$

We recall [T04, Sec. 5.6] the formulas for power sums and valuations of $[n]$ and $\ell_n$.

$$S_d(r) = \frac{1}{\ell_d^r}, \quad 0 < r \leq q, \quad S_d(q+1) = \frac{[1] - [d]^q}{[1]\ell_d^{q+1}}.$$

Note that $[n]$ is the product of all monic irreducible polynomials of degree dividing $n$. When $q = 2$ and $v = t^2 + t + 1$, we have

$$[d] = v^{2^{d-1}} + v^{2^{d-2}} + \cdots + v, \quad \text{or } [d] = v^{2^{d-1}} + \cdots + v^2 + v + 1$$

according to whether $d$ is even or odd.

Hence, the valuation of $[n]$ ($\ell_n$ respectively) at a degree one prime is 1 ($n$ respectively, whereas at the degree 2 prime $v$, when $q = 2$, the valuation is 1 or 0 according as $n$ is even or odd ($\lfloor n/2 \rfloor$ respectively).

**3.1. Claim** If $v$ is a prime of degree 1, then $v_d(1) = q^d - (d+1)$.

*Proof.* Without loss of generality, using automorphisms $t \to t + \theta$ ($\theta \in \mathbb{F}_q$) of $\mathbb{F}_q[t]$ which preserve sign, we may assume that $v = t$. Then

$$S_{d,v}(1) = \frac{1}{\ell_d} - \frac{1}{t\ell_{d-1}} = \frac{t - (t - t^{q^d})}{t\ell_d}.$$

□

**3.2. Claim** If $q = 2$, $v$ is a prime of degree 1, then $v_d(3) = 2^{d+1} - 3d - 1$, for $d \geq 1$.

*Proof.* Since $S_d(3) = (1 + [d]^q/[1])/\ell_d^3 = [d+1]/([1]\ell_d^3)$, we have $S_{d,v}(3) = S_d(3) - S_{d-1}(3)/t^3 = (t^{2^{d+1}} + t^{2^{d+2}-3})/([1]\ell_d^3)$. □

**3.3. Claim** If $q = 2$, $v$ is the prime of degree 2, namely $v = t^2 + t + 1$, then $v_d(1) = 2^{d-1} - \lfloor (d+1)/2 \rfloor$, for $d \geq 1$.

*Proof.* Proceeding as before, we have

$$S_{d,v}(1) = \frac{v + [d][d-1]}{v\ell_d}$$

so the valuation of the denominator is $1 + \lfloor d/2 \rfloor$. Let us write $M = v^{2^{d-2}} + \cdots + v^2 + v$ temporarily, so that $M^2 + M = v^{2^{d-1}} + v$. The numerator is $v + (v^{2^{d-1}} + M)(M+1) = v^{2^{d-1}} M$ or $v + (v^{2^{d-1}} + M + 1)M = v^{2^{d-1}} + v^{2^{d-1}} M$ according as whether $d$ is even or odd, so its valuation is $2^{d-1} + 1$ or $2^{d-1}$ respectively. □

**3.4. Claim** If $q = 2$, $v = t^2 + t + 1$, then $v_d(3) = 2^d - 3\lfloor d/2 \rfloor + (-1)^{d-1}$, for $d > 1$. We have $v_1(3) = 1$, $v_0(3) = 0$.

*Proof.* We have

$$S_{d,v}(3) = \frac{[d+1]v^3 + [d-1]^4[d]^3}{v^3[1]\ell_d^3}.$$

The valuation of the denominator is $3 + 3\lfloor d/2 \rfloor$. The valuation of the numerator is $2^d + 2$ for $d$ even, and $2^d + 4$ for $d$ odd, by a straight calculation (we omit the details) as in the proof above. □

**3.5.** More generally, we have the following **conjectural recipe** guessed from a small numerical data computed:

If $q = 2$, $v = t^2 + t + 1$, $d > 1$,

$$v_d(2^n - 1) = 2^{n+d-2} - (2^n - 1)\lfloor d/2 \rfloor + (-1)^{d-1}, \text{ if } n \text{ is even},$$

and

$$v_d(2^n - 1) = 2^{n+d-2} - (2^n - 1)\lfloor d/2 \rfloor - \lfloor (d+1)/2 \rfloor + \lfloor d/2 \rfloor, \text{ if } n \text{ is odd}.$$

**3.6. Trivial lower bound. Claim** If $k > 0$, and $d > m \deg(v)$, then $v_d(k) \geq m$.

*Proof.* The terms in the sum $S_{d,v}(k)$ can be grouped by orbits $1/(n + \theta v^m)^k$, as $\theta$ runs through elements of $\mathbb{F}_q$. The terms of each orbit add to zero modulo $v^m$. □

**3.7. Trivial upper bound. Claim** For $k > 0$, $d \geq \deg(v)$, we have $v_d(k) \leq dk(q^d - q^{d-\deg(v)} - 1)/\deg(v)$.

*Proof.* By definition $S_{d,v}(k)$ is the sum of $q^d - q^{\deg(v)}$ terms of the form $1/n^k$ with $n$ monic and prime to $v$, so with the common denominator the product of $n^k$'s, the numerator is sum of products of $q^d - q^{d-\deg(v)} - 1$ terms of degree $dk$. □

**3.8. Congruences and periodicity.** In contrast to $(\mathbb{Z}/p^n)^*$ which is cyclic for odd prime $p$, the analog $(A/v^n)^*$ is far from cyclic in general, when $n > 1$. If $v$ has degree $D$, then $(A/v^n)^*$ has order $(q^D - 1)q^{D(n-1)}$, but it has exponent

$$e_n = (q^D - 1)p^{\lceil \log_p(n) \rceil}.$$

So $S_{d,v}(k) \equiv S_{d,v}(k + me_n)$ modulo $v^n$. In particular,

$$\text{if } v_d(k) < n, \text{ then } v_d(k) = v_d(k + me_n), \; m \in \mathbb{Z}.$$

(Hence for any fixed $d$, $v_d(k)$ can be small even for large $k$. )

We will see applications resulting in nice patterns below. Also note that as $m$ can be negative, we can replace calculation with rational functions to get $v_d(k)$ at positive $k$, by easier calculation with polynomials to get it from negative $k$.

We know [T09, Thm. 2] says that $s_d(q^n - k) - s_d(-k) = dq^n$, if $s_d(-k) \neq 0$ and $q^n > k > 0$. Using the congruence idea at the infinite prime together with a switching trick, let us now give another proof of this equality claimed, but using a slightly stronger hypothesis that $s_d(-k) \neq 0$ and $q^n > k\ell(k)/(q-1)$.

Let us temporarily denote by $w$ the prime $1/t$ of $\mathbb{F}_q[1/t]$. We have

$$S_d(-k) = t^{dk} \sum (1 + \frac{f_{d-1}}{t} + \cdots + \frac{f_0}{t^d})^k, \quad S_d(q^n - k) = t^{dk-dq^n} \sum (1 + \cdots)^{k-q^n}.$$

Now the first sum is (term-by-term) congruent modulo $w^{q^n}$ to the second sum. Since the first sum is non-zero, we get a trivial upper bound $dk$ on its valuation at $w$. On the other hand, we know by [T04, 5.6.2] that $d \leq \ell(k)/(q-1)$, because $s_d(-k) \neq 0$. Hence the claim follows as before.

## 4. RECIPES AND INTER-RELATIONS FOR DEGREE ONE PRIMES, $k \in \mathbb{Z}$

Let us write $a \oplus b \oplus c + \cdots$ to denote the sum $a + b + c + \cdots$, where this sum has no carry over base $p$. In the Theorem below, by greedy algorithm, we mean first choosing among valid decompositions with $m_d \leq \cdots \leq m_1$, least $m_d$, then among these, the least $m_{d-1}$ and so on.

**Theorem 1.** *(i) Let $k$ be negative and $m = -k$. Then either $s_d(k) = -dm + \min(m_1 + 2m_2 + \cdots + dm_d)$, where $m = m_0 \oplus \cdots \oplus m_d$, $m_i \geq 0$, and for $i \geq 1$, $(q-1)$ divides $m_i > 0$, or $s_d(k)$ is infinite, if there is no such decomposition. When the decompositions exist, the minimum is uniquely given by greedy algorithm.*

*(ii) Let $k$ be positive. Then $s_d(k) = dk + \min(m_1 + \cdots + dm_d)$, with $(k-1) + m = (k-1) \oplus m$ and $m = m_1 \oplus \cdots \oplus m_d$, with $m_i$ positive and divisible by $q-1$. The minimum is uniquely given by the greedy algorithm.*

*(iii) Let $k$ be negative and $m = -k$. Let $v$ be a prime of $A$ of degree one. Then either $v_d(k) = \min(m_1 + \cdots + dm_d)$, where $m = m_0 \oplus \cdots \oplus m_d$, with $(q-1)$ divides $m_i > 0$ for $0 < i < d$, and $q-1$ divides $m_0 \geq 0$; or $v_d(k)$ is infinite, if there is no such decomposition. When the decompositions exist, the minimum is uniquely given by the greedy algorithm. (We see immediately that if $r$ is the least non-negative residue of $m$ modulo $q-1$, then $m_d$ is the least sum of $p$-powers chosen from the $p$-expansion of $m$ which is $r$ mod $q-1$. If $q = p$ a primes, then it is just the sum of the least $r$ of the $p$ powers chosen from the expansion, because $p^n$ is $1$ mod $q-1$ then.)*

*(iv) Let $k$ be positive. Let $v$ be a prime of $A$ of degree one. Then $v_d(k) = \min(m_1 + \cdots + dm_d)$, with $(k-1) + m = (k-1) \oplus m$, $m = m_1 \oplus \cdots m_d$, and $(q-1)$ divides $m_j > 0$ for $j < d$ and $(q-1)$ divides $k + m_d$. The unique minimum is given by the greedy algorithm.*

*Proof.* The part (i) follows from results of Carlitz, Diaz-Vargas, Sheats as explained and referenced in [T04, 5.8] or [T09, Sec.4]. The part (ii) is proved [T09, Sec.4]. The parts (iii) and (iv) follow by exactly the same arguments, once we note a crucial difference as follows. First consider negative $k = -m$, then we have

$$S_d(k) = \sum_{f_i \in \mathbb{F}_q} (t^d + f_{d-1}t^{d-1} + \cdots + f_0)^m = \sum \binom{m}{m_0, \cdots, m_d} (t^d)^{m_d} \cdots (f_0)^{m_0},$$

whereas for $v = t$, $S_{d,v}(k)$ is given by exactly the same sum, except that the condition $f_0 \neq 0$ is added and rather than looking at highest power of $t$ present in the answer, we now look at the lowest power. The parity and positivity conditions come from the well-known fact that $\sum f^n$, (where $f$ runs through elements of $\mathbb{F}_q$ and $n$ is a positive integer) is $-1$ or 0, according as whether $n$ is divisible by $q-1$ or not. The carry over conditions come from Lucas theorem that the binomial coefficient above is non-zero exactly when there is no carry over. Hence our assertions follow easily if the minimum is unique. This is the hard part, proved in Sheats [S98] in general (and by Diaz-Vargas [DV96] for much easier case of prime $q$). So (iii) follows similarly by reduction to the Sheats minimization theorem and then (iv) is deduced exactly the same way as in the proof of Theorem 1 in [T09, Sec. 4]. □

**Corollary 2.** *Let $v$ be a prime of degree one.*

*(i) Let $k$ be positive, then $v_{d+1}(k) = s_d(k) - dk$ if $q-1$ divides $k$.*

*(ii) Let $k$ is negative, then $v_{d+1}(k) = s_d(k) - dk$, if $q-1$ divides $k$ and $s_d(k)$ is finite.*

*(iii) If $q-1$ divides $k$, $v_d(k)$ is also divisible by $q-1$.*

*(iv) If $q$ is prime, $k$ is divisible by $q-1$, we have $v_{d+1}(k) = v_d(v_2(k)+k)+dv_2(k)$.*

*Proof.* The part (i) (part (ii) respectively) follows from comparing (ii) and (iv) ((i) and (iii) respectively) of the previous Theorem. The part (iii) follows from (i) and (ii) and the fact [T09, Thm.6, Thm. 14] that $s_d(k) \equiv dk \mod q-1$. The part (iv) follows from (i), (ii) and the fact [T09, Thm.1] that for $q$ prime, $s_d(k) = s_{d-1}(s_1(k)) + s_1(k)$.

Another direct way to prove this Corollary is as follows. For $k$ divisible by $q-1$, we have

$$t^{-dk} S_d(k) = \sum (1 + f_1/t + \cdots + f_0/t^d)^{-k} = \sum_{i=0}^{d} S'_i(k),$$

where we temporarily write $S'_d$ for power sum $S_{d,v}$ for $A = \mathbb{F}_q[1/t]$ for its degree one finite prime $1/t$. Replacing $t$ by $1/t$, and telescoping, we see that $v_{d+1}(k)$ is $s_d(k) - k$ (as it is greater than $s_{d+1}(k) - (d+1)k$, by the last paragraph in the proof of Theorem 4 in [T09], combined with $s_1(k) > k$). □

**4.1. Remarks.** (1) By part (iii), the recursion in (iv) can be continued, so that $v_2(k)$ determines $v_d(k)$, for $d \geq 2$, for a prime $v$ of degree one.

(2) All parts fail, if we drop the divisibility condition.

(3) The second proof mentioned above is achieved by developing further the ideas of Wan [W96] and Goss [G00, Prop. 9].

## 5. $v_d(k)$ WHEN $q$ IS A PRIME, $v$ IS OF DEGREE 1, $k < 0$

**Theorem 3.** *Let $q = p$ be a prime, $v$ a prime of $A$ of degree one and $-m = k < 0$. Write $m = \sum_{i=1}^{\ell} p^{e_i}$, with $e_i$ monotonically increasing and with not more than $p-1$ of the consecutive values being the same (i.e., consider the base $p$-digit expansion sequentially one digit at a time). Also, let $r$ be the least non-negative residue of $m$ modulo $q-1$. Then $v_d(k)$ is infinite if $\ell < (p-1)(d-1)+r$, and otherwise*

$$v_d(k) = d \sum_{s=1}^{r} p^{e_s} + \sum_{j=1}^{d-1} j \sum_{s=1}^{p-1} p^{e_{(d-1-j)(p-1)+r+s}}.$$

*Proof.* Note $p^i \equiv 1 \mod (q-1)$, when $q$ is prime. Hence $p-1$ powers together give divisibility by $q-1$. Hence the recipe in (iii) of the first Theorem in the last section simplifies and has for the minimum the choice $m_d = \sum_{s=1}^{r} p^{e_s}$ as stated there , and $m_{d-1}, \cdots, m_1$ obtained by picking $p-1$ digits each from the base $p$ expansion of $m - m_d$ starting from the lowest digits (and dumping the rest of the expansion, if any, into $m_0$). □

Note that it looks even simpler for $q = 2$, so that $r = 0$ and inner sums are singleton, and also that in this case $v_1(k) = 0$, $v_2(k)$ is, in fact, the valuation of $k$ at 2.

## 6. $v_d(k)$ WHEN $q = 2$, $v$ IS OF DEGREE 1, $k > 0$

First note $v_d(2k) = 2v_d(k)$. So we will focus on $k$ odd.

Next $v_1(k) = 0$ for all $k$ and $v_2(k) = 1$ for all odd $k$.

From [T09, Sec. A.2 (1)] and duality part (i) of Corollary 2, we see that

$$w_n := v_d(2^n - 1) = 2^{d+n-1} - 2^n d + (d-1).$$

We define sequence $f_n$ (it is $v_{n-3}(5) - v_{n-3}(1)$) by

$$f_0 = 0 \text{ and } f_n = 2f_{n-1} + 4n.$$

Here is the recipe: For a given $d$, we describe the sequence of $v_d(k)$ with $k$ odd, so that the $n$-th entry will correspond to $k = 2n - 1$ and thus $w_n$ is $2^{n-1}$-th entry. Let $X_n$ be the vector of the first $2^{n-1} - 1$ entries, and write $X_{n+1} = X_n, w_n, X'_n$, in two halves. In other words, the whole sequence is of the form $X_1, w_1, X'_1, w_2, X'_2, w_3, X'_3, \cdots$ or equivalently,

$$X_n, w_n, X'_n, w_{n+1}, \cdots,$$

**Theorem 4.** *The second half $X'_n$ is obtained from the first half $X_n$ by adding $2^{n-2} f_m$ to the entries with index (i.e., $(k+1)/2$) having the base two expansion*

$$\sum_{i=w}^{n-2} b_i 2^i, \ b_w \neq 0, \text{with exactly } d-3-m \text{ of the } b_i\text{'s zero}$$

*where $1 \leq m \leq d-3$.*

*Proof.* We replace $d$ by $d+1$ for convenience. We then need to prove $v_{d+1}(k + 2^n) - v_{d+1}(k) = 2^{n-2} f_m$. By using the duality (i) of the previous Theorem, we are reduced to proving $s_d(k + 2^n) - s_d(k) = d2^n + 2^{n-2} f_m$, for $2^n - 1 > k$ odd and $m$ as in the Theorem, but with $d$ replaced by $d+1$. By [T09, 3.3], we have

$$s_d(k) - dk = d * 2^{e_0} + \cdots + 1 * 2^{e_{d-1}},$$

where we write the base 2 expansion of $k$ as

$$k = \cdots 0_{e_{t+1}} 0_{e_t} 1 \cdots 1 0_{e_{t-1}} 1 \cdots 1 \cdots 0_{e_2} 1 \cdots 1 0_{e_1} 1 \cdots 1_{e_0} 0 \cdots 0.$$

Since, for us, $k$ is odd, less than $2^n$, (with $n = e_t + r$, $r \geq 0$) $k + 2^n$ has expansion of the form

$$\cdots 0_{e_{t+r}} 1 0_{e_{t+r-1}} \cdots 0_{e_t} 1 \cdots 1 0_{e_{t-1}} 1 \cdots 1 \cdots 0_{e_2} 1 \cdots 1 0_{e_1} 1 \cdots 1_{e_0}.$$

Observe that $(d+1)-3-m = t+r-2$, by counting the relevant zeros in the expansion of $(k+1)/2$. If $m \leq 0$, the relevant $e_i$'s are the same for $k$ and for $k+2^n$, and hence the left side of the first formula is $d2^n$, which agrees with the right side as $f_m = 0$ then. Now we proceed by an induction on $m$. Now the $e_i$'s which are different are

$e_{d-m}, \cdots, e_{d-1}$ which are $n, \cdots, n+(m-1)$ for $k$ whereas $n+1, \cdots, n+m$ for $k+2^n$ resulting in the difference $X_m := (2^{n+m} + 2 * 2^{n+m-1} + \cdots + m2^{n+1}) - (2^{n+m-1} + \cdots + m2^n)$. Hence $X_{m+1} - 2X_m = (m+1) * 2^n = 4 * 2^{n-2}(m+1)$, matching the recursion for $2^{n-2}f_m$'s. $\square$

**6.1. Remark.** Using duality, we have converted the recursion in $d$ for valuations at infinity into proving nice fast 'doubling' pattern for a fixed $d$.

**6.2. Examples.** For $d \leq 3$, $X'_n = X_n$, so there is block repetition after new entries $w_n$. The cases $d = 0, 1$ respectively, correspond to vectors with all entries zero, one respectively, as mentioned above, whereas for $d = 3$, we get

$$\overline{4}, \overline{6}, 4, \overline{10}, 4, 6, 4, \overline{18}, 4, 6, \cdots, 4, \overline{34}, 4, 6, 4, \cdots$$

where the over-lined entries are $w_n$'s.

For $d = 4$, $X'_n$ is obtained by keeping its first half the same and adding $2^n = 2^{n-2} * 4$ to the half-way, half of the next half-way etc. entries (i.e., $\sum_w^{n-2} 2^i$-th entries).

More generally, $X_n$ and $X'_n$ share their first $X_{n-(d-3)}$ part. This also follows another way from 3.8.

## 7. When $q = 3$, $v$ is of degree one, $k > 0$

We list valuation sequence $v_d(k)$ for $k$ not divisible by 3.

For $d = 1$, $v_d(k)$ is zero for even $k$ and 1 for odd $k$. So the sequence is periodic of period $1, 0, 0, 1$ of length 4, when only $k$ not divisible by 3 are used.

**Conjectural recipe for** $d = 2$:

The valuation sequence is of the form $X_1, a_1, X_2, a_2, X_3, a_3, \cdots$ where

(i) $X_1 = X_{3n} = X_{3n+1} = [6, 4, 2, 14, 12, 10, 2, 8, 6, 4, 2]$,

(ii) $X_{3n+2}$ is the same as $X_1$ except $a_{3n+1} - a_{3n+2}$ is added to fourth, fifth and sixth terms of $X_1$ to get the corresponding entries,

(iii) The $a_n$'s (which correspond to $k = 17 + 18(n-1)$) look like $3^r + 3^s + 2$, with $r \geq s \geq 2$ (and $a_{3m+2} = 20$, so that $r = s = 2$ and for $n$ of the form $3m+1$, we have $s = 2$). More precisely, we describe the full sequence $[a_1, a_2, \cdots]$ as follows: $a_{3^n} = 3^{n+3} + 3^{n+2} + 2$, $a_{2*3^n} = 3^{n+2} + 3^{n+2} + 2$, the block between $a_{2*3^n+1}$ to $a_{3^{n+1}-1}$ is exactly the block between $a_1$ to $a_{3^n-1}$, whereas you get the block between $a_{3^n+1}$ to $a_{2*3^n-1}$ by taking the block between $a_1$ to $a_{3^n-1}$ and replacing the entries $3^{n+2} + 3^k + 2$ by the new entries $3^{n+3} + 3^k + 2$.

So the sequence is $X_1$ followed by 38 followed by 6, 4, 2, 32, 30, 28, 2, 8, 6, 4, 2, followed by 20, $X_1$, 110, $X_1$, 92, followed by 6, 4, 2, 86, 84, 82, 2, 8, 6, 4, 2, followed by 20, $X_1$, 56, $X_1$, 38, ....

## 8. When $q = 4$, $v$ is of degree one, $k > 0$

Here is the **conjectural recipe for** $d = 1$:

We describe the sequence $v_1(k)$ with $k$ odd, so that the $n$-th entry will correspond to $k = 2n - 1$ and thus $w_n$ is $2^{n-1}$-th entry:

The whole sequence is limit of vectors $X_n$ (of increasing sizes with initial portion being $X_{n-1}$) with

(i) $X_1$ being $[2, 0, 1, 8, 0, 1]$ and

(ii) $X_n$ consisting of $X_{n-1}$ followed by $X'_{n-1}$, where the entries of $X'_{n-1}$ are the same as that of $X_{n-1}$, except one entry is changed as follows: If $n$ is odd, change the

$k = 2^n - 1$-th entry (which is $2^n$) to $2^{n+2}$ and if $n$ is even, change $k = 2^{n+1} - 1$-th entry (which is $2^{n+1}$) to $2^n + 1$.

So the sequence is $X_1$ followed by 2, 0, 1, 5, 0, 1, 2, 0, 1, 32, 0, 1, 2, 0, 1, 5, 0, 1, $X_1$, ....

## 9. When $q = 2$, $v$ is of degree 2, $k > 0$

For a given $d$, we describe the sequence of $v_d(k)$ with $k$ odd, so that the $n$-th entry will correspond to $k = 2n - 1$.

By definition, it is easy to see that $v_1(k) = 1$ or 0 respectively and $v_2(k) = 0$ or 1 respectively, according as $k$ is divisible by 3 or not. Let us check this for $d = 1$: Numerator of $1/t^k + 1/(t+1)^k$ is $t^k + (t+1)^k \equiv t^k + (t^2)^k \equiv t^k(1 + t^k)$ modulo $v$, but $t^{3m} \equiv 1$ and $(t^{3m} + 1)/(v(t+1)) \equiv t^{3m-3} + t^{3m-6} + \cdots + 1 \equiv m \not\equiv 0$ modulo $v$ (as $m$ is odd).

Now we fix $d \geq 3$, so it will be dropped from the notation sometimes.

We write the sequence in the form $X_1, w_1, X_1', w_2, X_2', w_3, X_3', \cdots$ where $X_{n+1} = X_n, w_n, X_n'$. In other words, For every $n$, we write the sequence in the form

$$X_n, w_n, X_n', w_{n+1}, \cdots ,$$

where $X_n$ and $X_n'$ are vectors of entries of length $3 * 2^{n-1} - 1$, with $X_n$ containing entries (odd $k$'s) from $v_d(1)$ to $v_d(3*2^n - 3)$ and $w_n = v_d(3*2^n - 1)$. We can further subdivide $X_n$ in 'thirds'. More precisely, $X_n = (A_n, B_n, C_n)$ with $A_n$ consisting of first $2^{n-1}$ entries, namely for (odd) $k = 1$ to $k = 2^n - 1$, i.e. those with at most $n$ (base 2) digits, with $B_n$ consisting of entries with $k$ from $2^n + 1$ to $2^{n+1} - 1$, i.e., with $n+1$ digits and with $C_n$ consisting of entries with $k$ from $2^{n+1} + 1$ to $3*2^n - 3$, i.e. with those entries in $X_n$ with $k$ of $n+2$ digits. We write $X_n' = (A_n', B_n', C_n')$ in the obvious fashion.

Then as we have already proved in Section 3, we have

(i) **Initial value** $X_1 = [2^{d-1} - \lfloor (d+1)/2 \rfloor, 2^d + (-1)^{d-1} - 3\lfloor d/2 \rfloor]$,

Here is the **conjectural recipe** for the rest:

(ii) **The sequence** $w_n = w_{n,d}$ For $d$ odd, put

$$w_1 = 2^d - (5d-1)/2, \quad w_{n+1} = 2w_n - (d-1)/2,$$

so that $w_{n,1} = 0$, put $w_{n,2} = 1$, and for $d > 2$ even, put

$$w_1 = 2^d + 6 - 5d/2, \quad w_{2n} = 2w_{2n-1} - d/2 + 5, \quad w_{2n+1} = 2w_{2n} - 4 - d/2.$$

We now give conjectural description of the value of

$$t_k := t_{k,n,d} := v_d(k + 3 * 2^n) - v_d(k)$$

depending on whether $k$ belongs to $A_n, B_n, C_n$ respectively:

(iii) **Description of $A_n$ to $A_n'$ transition**: $t_k = v_{d-i}(2^{n+2} - 1) - v_{d-i}(2^n - 1)$ when $k$ has $n - i$ ones, with $0 \leq i \leq d - 3$, otherwise $t_k = 0$.

In other words, add $v_{d-i}(2^{n+2} - 1) - v_{d-i}(2^n - 1)$ (which is $3 * 2^n(2^{d-i-2} - \lfloor (d-i)/2 \rfloor)$ by conjecture above) to the entry in $A_n$ with $k$ having $n - i$ ones, with $0 \leq i \leq d - 3$, to get the corresponding entry in $A_n$. All other entries remain unchanged.

(iv) **Description of $B_n$ to $B_n'$ transition**: Let us write temporarily $f(n, d) := v_d(5 * 2^n - 1) - v_d(2^{n+1} - 1)$.

Then $t_{k,n,d} = -f(n, d - i - 1)$ and 0 respectively, if the base 2 expansion of $k$ has $n - i$ number of ones (there is exactly one term with $i = -1$, otherwise $i \geq 0$), with $i < d-4$ and $i > d-4$ respectively. The special case $i = d-4$ has $t_k < 0$ and is described later.

(v) **Description of $C_n$ to $C'_n$ transition**: Put $g(n,i) := w_{n+1,i} - w_{n,i}$, when $i \geq 0$ and 0 otherwise, and $r(n,i) := (-1)^{n+i-1} * 3 * 2^{n-i}$, when $1 \leq i \leq n-1$. Note $g(n,i) = 0$ for $i \leq 3$.

Let $k$ have $n-m$ number of ones in its base two expansion, so that $0 \leq m \leq n-2$. If $m \geq d-4$, then $t_k = 0$. So fix $m > d-4$. If $d$ and $m$ have the same parity, then $t_k = g(n, d-(m+1))$. If $d$ and $m$ have opposite parity, list such $k$'s (there are $\binom{n-1}{n-m-2} = \binom{n-1}{m+1}$ of these, as out of the $n+2$ digits, first two are $1,0$ and the last is 1) in increasing order and add to corresponding entries the amounts

$$g(n, d-(m+1)) + r(n,i_1) + r(n,i_2) + \cdots + r(n,i_{m+1}),$$

with $1 \leq i_1 < \cdots < i_{m+1} \leq n-1$, in lexicographic order, where smaller $i$ (i.e., larger absolute value of $r(n,i)$) comes first.

**9.1. Remarks on consequences.** (I) In the $A_n$ to $A'_n$ and $C_n$ to $C'_n$ transition to $t_k \geq 0$, while in $B_n$ to $B'_n$ transition, $t_k \leq 0$.

(II) The first $2^{n-d+2}-1$ entries of $X_n$ are unchanged (i.e., with $t_k = 0$) in $X'_n$. More generally, those entries in $A_n$ ($B_n$, $C_n$ respectively) with $k$ having at most $n+2-d$ ($n+3-d$, $n+4-d$ respectively) ones in the (base 2) digit expansion remain unchanged in the corresponding places of $X'_n$. (Some other entries also remain unchanged, so this is not if and only if condition.)

The first consequence can be proved from the hypothesis that the maximum of $v_d(k)$ with odd $k \leq 2^n - 1$ occurs at $k = 2^n - 1$ and the conjectural formula of $v_d(2^n - 1)$ as well as the congruence noted above. (Another way to see this is to note $1/a^k + 1/a^{k+3*2^n} = (a^{3*2^n}+1)/a^{k+3*2^n}$, for $a$ prime to $v$, so that $a^3 \equiv 1 \mod v$ implies the sum has valuation at least $2^n$.). In more detail, if $k \leq 2^{n-d+2}-1$, then by these hypotheses we have $v_d(k) \leq v_d(2^{n-d+2}-1) < 2^n$, so that $v_d(k) = v_d(k + m * 3 * 2^n)$.

(III) In (iii) the bound $d-3$ could have been changed to $d$, since the addition amount for $d-2 \leq i \leq d$ is zero.

(IV) In (iv) we also conjecture that $f(n,d)$ is $(3d-6)2^n$ for $d$ even and $(3d-15)*2^n$ for $d > 3$ odd, while for $d = 3$, it is $2^n$ for $n$ odd and $2^n - 1$ for $n$ even. Note $f(n,5) = 0$, so that by (iv) more entries are unchanged than listed in (II).

**Special case (iv), $i = d-4$:** Since $k$ is odd with $n+1$ digits, we have $3 \leq d \leq n+2$ and there are $\binom{n-1}{d-3}$ such entries in $B_n$. Let $a_n$ denote $2^n - 1$, if $n$ is even and $2^n$, if $n$ is odd.

For $d = 3$ ($d = n+2$ respectively), we have a unique such $k$ and we have $t_k = -a_n$ ($t_k = -2^n$ respectively).

For $d = 4$, the $n-1$ differences $-t_k - a_n$ are $(-1)^{i-1}(2^{n-i}-1)$ with increasing $1 \leq i \leq n-1$, if $n$ is even and they are $2^{n-1}+1, -2^{n-2}, \cdots, 2^2-1, -2$ if $n$ is odd.

For $d = n+1$, the differences $-t_k - a_n$ are $0, 2^3, -(2^2-1), 2^5, \cdots, 2^{n-1}, -(2^{n-2}-1)$ if $n$ is even and $2^2+1, -2, \cdots, 2^{n-1}+1, -2^{n-2}$ if $n$ is odd.

For $d = 5$, $-t_k - a_n$ are given as follows. Write these differences in $d = 4$ case described above as $c_{n-1}, \cdots, c_1$, then in $d = 5$ case, the first $n-2$ differences are (for $k$'s in the special case written in increasing order) $0, c_{n-1}+c_{n-3}, c_{n-1}+c_{n-2}+c_{n-3}+c_1, c_{n-1}+c_{n-5}, c_{n-1}+c_{n-4}+c_{n-5}+c_1, \cdots$, (thus ending with $c_{n-1}+c_4+c_3+c_1, c_{n-1}+c_1$ when $n$ is even and with $c_{n-1}+c_2, c_{n-1}+c_3+c_2+c_1$

when $n$ is odd. These are followed by $n-3$ copies of $-a_{n-2}$, followed by the differences at $(n-2, d)$ level (there are $\binom{n-3}{2}$ of them.)

For $d \geq 3$, for $k$ written in increasing order, for the $\binom{n-1}{d-3}$ $k$'s that we consider at $(n, d)$-level, differences $t_{k,n,d} - a_n$ are given by repeating these differences ($\binom{n-3}{d-5}$ of them) at $(n-2, d-2)$ level, followed by portion denoted by $Q$, as we have not been able to guess it yet, (it consists of $\binom{n-3}{d-4}$ entries), followed by $-a_{n-2}$ repeated $\binom{n-3}{d-4}$ times, followed by the differences at $(n-2, d)$ level ($\binom{n-3}{d-3}$ of them).

The portion $Q$ is $2^{n-1} + (-1)^{n-1}$ for $d = 4$. For $d = 5$, it is sum of first and third differences at $(n, 4)$ level, followed by $3 * 2^{n-3}+$ top entries ($n-4$ of them) from $(n-2, 5)$ level. First term of the portion $Q$ in $d = 6$ case is $2^{n-1} + (-1)^{n-1}$. For $d = n+1$, the portion $Q$ is $2^{n-1} + 1$ and $2^{n-1}$ according as $n$ is odd or even respectively. More generally, the first (last respectively) entry of the portion $Q$ is $2^{n-1} + (-1)^{n-1}$ ($2^{n-1}$ respectively) according as $d$ is even ($d$ is odd and $n$ is even respectively).

**Special cases of low** $d$ We consider the $v_d(k)$ sequence for $k$ odd.

($d = 1$) The pattern is 0, 1, 0, 0, 1, 0, ..., periodic with period 3.

($d = 2$) The pattern is 1, 0, 1, 1, 0, 1, ... , periodic with period 3.

($d = 3$) $w_n = 1$ for $d = 3$, $X'_n$ is obtained from $X_n$ by adding $3*2^n$ to the $2^{n-1}$-th entry (i.e., $k = 2^n - 1$) and subtracting from $2^n$-th entry (i.e., $k = 2^{n+1} - 1$) either $2^n$ or $2^n - 1$ respectively, as $n$ is odd or even.

So the pattern is 2, 6, 1, 8, 4, 1, 2, 18, 1, 5, 4, 1, 2, 6, 1, 32, 4, 1, 2, 10, ....with $k$ being 5, 11, 1, 9 modulo 12 giving entries 1, 1, 2, 4 respectively.

## 10. Zero distribution of Goss zeta functions

For a given $y$, $\zeta(x, y)$ is a power series in $x$ with coefficients in $K_\infty$. Daqing Wan [W96] noticed and proved using Newton polygon calculation, using estimates of $s_d(-y)$, that its zeros are simple and always lie in $K_\infty$, when $q$ is a prime. This was then generalized by Sheats to any $q$. See [T04, Sec. 5.8] and [G96, G00] for the references for this development and discussion of higher genus case.

Noting that $K_\infty$ and $C_\infty$ are analogs of real and complex number fields respectively, this restrictive behavior reminds one of Riemann hypothesis situation [G96, G00, T04]. This looks even more remarkable, if one notices further that algebraic degree of $C$ over $R$ is just 2, whereas in our case, it is infinity.

We now turn to the $v$-adic case and the zero distribution for the power series $\zeta_v(x, y)$ in $x$ with coefficients in $K_v$, for a given $y$. We can ask whether its zeros are in $K_v$ and whether they are simple. Our results can be used to calculate Newton polygons and the zero distribution. For this we can approach the $p$-adic integers $y$ through sequence of positive $k$'s, or through negative $k$, or indeed through any dense subsequence (See [T04, 5.8] and [T09]). We leave this for a future paper and note here only two simple applications requiring only a few things we have proved.

(I) When degree of $v$ is one, and $q = 2$, Wan [W96] (see also [G00, Prop. 9]) already showed that the zeros are simple and in $K_v$. For general $q$, one has the same results for $y \in (q-1)S_v$. (Note that when $q = 2$, non-zero is the same as monic, otherwise we restrict to 'even' $k$ to kill the signs.) This can be also derived immediately from Corollary 2, which in fact provides much more precise information to calculate the Newton polygon.

We remark that, when $q = 2$, the one unit part of $S_d(k)$ when you substitute $1/t$ for $t$ is $S_{\leq d}(k)$ with $t$ factor removed. So $s_d(k) - k \geq \min v_i(k)$ $(i \leq d)$ with equality if there are no clashes. (But there can be clashes, even for $d = 1$, $k$ odd.)

To do the general $q$ case, without restriction of 'evenness', for degree one primes, we need to use information provided by Theorems 1 and 3. We leave this for a future paper.

(II) We now show that when $q = 2$, $v = t^2 + t + 1$, the zeros need not be in $K_v$.

For the first example, let $k = 5$. In fact, we saw that $v_d(5) = 0, 0, 1, 1, 12, 20, \ldots$. We do not need the full pattern. We can say that the first two slopes are 0 and 1/2 because degree of $v$ being 2, we have easy estimate $v_d(k) \geq m$, if $d > 2m$, by the trivial lower bound described above. (In fact, since the valuations are not zero infinitely often, the slope would be at most 1/2 in any case.)

Here is a second example: $q = 2$, $v = t^2 + t + 1$, and $k = 3$. Now $v_d(3)$ for $d = 0, 1, 2, \cdots$ is $0, 1, 0, 6, 9, 27, \ldots$, so that the first two slopes are 0 and 9/2. To see this, trivial lower bounds above are not sufficient, but in this case, we know $v_d(3)$, by Section 3, so a straight calculation justifies this.

## 11. Appendix: Leading term formulas

It follows [T04, Cor. 5.6.2] from Carlitz' work that for $k > 0$, $S_d(-k) = 0$ if and only if $d > \ell(k)/(q-1)$, for $q$ prime. (See [B12] and [T09, A.5] for the general situation). Hence, for $k > 0$, $S_{\lfloor \ell(k)/(q-1) \rfloor}(-k)$ is the leading term of the Goss zeta series, at least when $q$ is prime, and also for general $q$, when $\ell(k)$ is the minimum of $\ell(p^i k)$.

**Theorem 5.** *Let $q$ be any prime power. Let $k > 0$ and $\ell(k) = (q-1)d + r$, with $0 \leq r < (q-1)$, so that $d = \lfloor \ell(k)/(q-1) \rfloor$. Write the base $q$-expansion $k = \sum_1^{d(q-1)+r} q^{k_i}$. Then*

$$S_d(-k) = (-1)^d \sum t^{\sum_{i=1}^{d-1} i \sum_1^{q-1} q^{k_j} + d \sum_1^r q^{k_m}}$$

*where the sum is over all assignments to $i$'s of groups of $q-1$ of the powers $q^{k_j}$'s corresponding to indices in partitions of $d(q-1)+r$ indices into $d$ groups of $q-1$ each and one group of $r$ powers.*

*Proof.* Let us first see the simplest $q = 2$ case, then $r = 0$.:

$$\begin{aligned} S_d(-k) &= \sum_{f_0, \ldots, f_{d-1} \in \mathbb{F}_q} (t^d + f_{d-1} t^{d-1} + \cdots + f_0)^k \\ &= \sum \prod_{i=1}^{d(q-1)} (t^{dq^{k_i}} + f_{d-1} t^{(d-1)q^{k_i}} + \cdots + f_0) \end{aligned}$$

Keep the sum and consider the terms obtained by expanding the product. Any term not containing all $f_i$'s will vanish after summing over that missing $f_i$ (compare proof of [T04, Thm. 5.1.2]). So terms that matter are of the form $f_0 \cdots f_{d-1} t^{\sum_1^{d-1} i q^{k_{j_i}}}$, where there is really only one non-zero term corresponding to $f_i = 1$, so that $S_d(-k)$ is exactly the sum of these $t$ powers, over $j_i$'s which are permutations of 1 to $d$.

Now consider the general $q$ case. Then, as before, we have exactly the same displayed expression, and as before, when we expand the product, all $f_i$'s need to be there and each $f_i$ with minimal power $q-1$ to get the non-zero sum, so only terms that matter have coefficient $(f_0 \cdots f_{d-1})^{q-1}$ (as the relation between $d$ and $\ell(k)$ shows) so that we get the expression as claimed. □

**11.1. Example.** Let $q = 3, k = 38 = 27 + 9 + 1 + 1$, so that $d = 2$ and our our formula gives $S_d(-k) = t^{27+9} + 2 * t^{27+1} + 2 * t^{9+1} + t^2$.

**Corollary 6.** *With the notation as in the Theorem, when $q = 2$, we have a product formula*

$$S_d(-k) = \prod_{d \geq n > m} (t^{2^{k_n}} + t^{2^{k_m}}).$$

*More generally, for any $q$, but for the special family $k = (q-1)\sum_1^d q^{k_i} > 0$ (with $k_i$ distinct) we have the leading term*

$$S_d(-k) = (-1)^d \prod_{d \geq n > m} (t^{q^{k_n}} - t^{q^{k_m}})^{q-1}.$$

*Proof.* Put $T_n = t^{q^{k_n}}$. The product formula follows immediately, when $q = 2$, by simple counting of monomials in $\prod(T_n + T_m)$. For general $q$, one has to only note, in addition, that $\binom{q-1}{i} = (-1)^i$, so that $(T_n - T_m)^{q-1} = \sum T_n^a T_m^b$, where the sum is over $a, b$ with $a + b = q - 1$. □

**11.2. Remarks.** (1) The product formula in the $q = 2$ case was obtained earlier by Richard Pink using cohomological formula for the leading power sum. See [B12, 7.1] for this as well as the proof of the Corollary using the Vandermonde determinantal formula combined with the cohomological machinery.

(2) When $q > 2$, we do not have a product formula involving only monomials in $[n]$'s, in the general case, for the leading term, even if $q$ is a prime. For example, when $q = 3$, $k = 13$, $S_1(-13) = -(t^3 - t)(t^3 - t + 1)(t^3 - t - 1)$. On the other hand, for many families of $q, k$, we can prove the product expression (for the leading term $S_d(-k)$ as above)

$$c \prod (t^{q^j} - t^{q^i})^{r_{i,j}},$$

where $c \in \mathbb{F}_q$ expressed in terms of multinomial coefficient, product being over $i < j$ such that $k_i + k_j = q - 1 + r_{i,j}$, with $r_{i,j} > 0$, where $k = \sum k_j q^j$ is the base $q$ expansion of $k$. We leave it to the future paper to investigate the exact scope of when it works, and the cohomological explanation of the prime factors which enter in terms of the $p$-ranks of the Jacobians (components) of the corresponding cyclotomic extensions.

(3) When $q = 2$, the product has $d(d-1)/2$ terms of two terms, so when you expand it has $2^{(d-1)d/2}$ terms, the sum we have has $d!$ terms (some can cancel), whereas if we just use the definition there are $2^d$ terms to be added each consisting of $(d+1)^k$ (or rather $(d+1)^{\ell(k)}$ using $p-$th powers) terms.

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