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APPROXIMATION EXPONENTS FOR FUNCTION FIELDS

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Dedicated to Professor Klaus Roth

Abstract. For diophantine approximation of algebraic real numbers by rationals, Roth's celebrated theorem settles the issue of the approximation exponent completely in the number field case. But in spite of strong analogies between number fields and function fields, its naive analogue fails in function fields of finite characteristic, and the situation is not even conjecturally understood. In this short survey, we describe the recent progress on this and the related issues of explicit continued fractions for algebraic quantities.

1. Exponents of Diophantine Approximation

How well we can approximate an irrational real number α by rationals a/b, compared to their complexity, traditionally measured by the size of b, is a central question of diophantine approximation theory. Thus we define the *approximation exponent* of α by

$$E(\alpha) := \limsup \left(-\frac{\log |\alpha - a/b|}{\log |b|} \right).$$

A simple application of the box principle or the basic continued fraction theory shows that $E(\alpha) \geqslant 2$. On the other hand, if further α is algebraic of degree d, by applying the mean value theorem to $f(\alpha) - f(a/b)$ where f is the minimal polynomial for α, Liouville showed that $E(\alpha) \leqslant d$. By this, or by the periodicity of quadratic continued fractions, we know that $E(\alpha) = 2$ for $d = 2$. For $d > 2$, Thue improved Liouville's estimate and deduced important finiteness results in the theory of diophantine equations. After improvements by Siegel and Dyson, finally in 1955, Roth [20] settled this question completely by proving the fundamental result that $E(\alpha) = 2$ for any real irrational algebraic number α.

A much simpler measure theoretic calculation due to Khintchine shows that the exponent is 2 for almost all real numbers. So Roth's theorem shows that none of the members of the very special countable class of algebraic real numbers is "special" in this respect.

We now focus on the situation in the function field case, leaving to other parts in this volume the implications of Roth's theorem to diophantine geometry, its generalizations by Schmidt and others, its relations with the ABC conjecture, and significance of the analogies with Nevanlinna theory observed by Osgood and Vojta leading eventually to (another) proof by Vojta of the Mordell conjecture in function fields and number fields.

Let F be a field. We consider $F[t]$, $F(t)$ and $F((1/t))$ as analogues of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively. Thus we focus on $\alpha \in F((1/t))$ algebraic over $F(t)$.

We now consider well-approximating a "real" function $\alpha \in F((1/t))$ by rational functions a/b, where $a, b \in F[t]$. Using the usual absolute value coming from the degree in t of a polynomial, we can use the same definition as above for the exponent in this situation. Mahler proved analogues of Dirichlet and Liouville bounds by essentially the same proofs. An analogue of Khintchine's theorem giving the behaviour of "almost all" functions can also be proved similarly. D. Fenna (Manchester thesis 1956) and Uchiyama [30] proved the analogue of Roth's theorem in the function field case with F of characteristic zero, that $E(\alpha) = 2$ for irrational $\alpha \in F((1/t))$ algebraic over $F(t)$.

Analogies with the number theory are usually even better, when F is a finite field, so that there are only finitely many remainders when one divides in $F[t]$, analogous to what happens when one divides in $\mathbb{Z}$. So the finite characteristic case is of great interest. Because of deep partial analogies between number fields, i.e. finite extensions of $\mathbb{Q}$, and function fields over finite fields, i.e. finite extensions of $\mathbb{F}_p(t))$ benefitting the study of both, the number theorists develop them together as "global fields".

In this article, either at the beginning of a section or locally we will make it clear when we specialize F to a field of zero or finite characteristic or to a finite field.

Let F be of characteristic $p > 0$, and q be a power of p. Then, as Mahler observed, $E(\alpha) = q$ for $\alpha = \sum t^{-q^i}$, by a straightforward estimate of approximation by truncation of this series. Now $\alpha^q - \alpha - t^{-1} = 0$, so that α is algebraic of degree q over $F(t)$, and hence the Liouville upper bound is best possible in this case. Mahler suggested, and it was claimed to have been proved in a published paper and believed for a while, that such phenomena may be special to the degrees divisible by the characteristic, but Osgood, and Baum and Sweet, gave examples in each degree for which Liouville's exponent is the best possible. A few more isolated examples were proved after extensive computer searches.

We will try to describe below what is known and what we would like to know about the distribution of the exponents of algebraic quantities in finite characteristic. We include some remarks in zero characteristic situation also. For more details and references, we refer the reader to [28, Chapter 9], and to [29, 22].

2. Differential Degree and Exponent Bounds

In this section, we deal with general F, unless noted otherwise.

In a function field $F(t)$ of any characteristic, we can differentiate with respect to t, in contrast to the number field situation. Maillet, Kolchin and Osgood [17, 18] used this to obtain better and/or effective bounds for diophantine approximation.

Kolchin's idea was to use the Liouville argument and replace the minimal polynomial of α with a "small" differential polynomial that kills α. Frequently, this gives a smaller exponent and we obtain good effective bounds by more refined work of Osgood. Hence, even in characteristic zero, where the optimal exponent is known, one gets an improvement because of the effective bounds. We will only concentrate on the exponents and will not discuss questions of effectiveness.

We denote the m-th derivative of y with respect to t by $y^{(m)}$ and also write y' for $y^{(1)}$ following usual practice. For a vector $\mathbf{e} = (e_0, \ldots, e_k)$ of non-negative integers, we write $y^{\mathbf{e}}$ as an abbreviation for the differential monomial $y^{e_0}(y^{(1)})^{e_1} \ldots (y^{(k)})^{e_k}$.

Consider a differential polynomial

$$P(y) = \sum p_{\mathbf{e}} y^{\mathbf{e}}.$$

Note that the j-th derivative of a/b has the power b^{j+1} in the denominator. We define the *denomination* $\overline{d}(P)$ to be the maximum of

$$\sum_{j=0}^{k} (j+1)e_j$$

corresponding to $\mathbf{e}$ such that $p_{\mathbf{e}} \neq 0$.

If $P(a/b) \neq 0$, then $|P(a/b)| \geqslant 1/|b|^{\overline{d}(P)}$, so $\overline{d}(P)$ replaces the degree in the Liouville argument. Let $\overline{d}(\alpha)$ denote the smallest possible $\overline{d}(P)$ for P satisfying $P(\alpha) = 0$.

Here α can be differentially algebraic. If, in fact, it is also algebraic, then

$$\overline{d}(\alpha) \leqslant \max\{\deg(\alpha) - 1, 2\}.$$

Note that differentiating the minimal polynomial $P(x)$ for α we obtain the equation $\alpha' P_x(\alpha) + P'(\alpha) = 0$. Simplifying, we obtain

$$\alpha' = \sum_{j=0}^{w} a_j(t)\alpha^j,$$

with $w < \deg \alpha$.

Kolchin's analogue [7] of Liouville's theorem is as follows.

Theorem 1. *Given an irrational α which is differentially algebraic over a function field of characteristic zero, there is a constant $c > 0$ such that*

$$\left|\alpha - \frac{a}{b}\right| > \frac{c}{|b|^{\overline{d}(\alpha)}}.$$

The proof is by the Liouville argument, except that the differential minimal polynomial P has, in general, infinitely many zeros and, *a priori*, some approximations a/b can be among those. Kolchin shows that this is impossible by showing that the other zeros cannot come close to α by an estimation of Wronskians. It uses an inequality of the form $c_1|a| \leqslant |a'| \leqslant c_2|a|$ for some positive constants c_1 and c_2, which is true in a function field of characteristic zero, but fails in one of characteristic p; e.g., for a non-zero p-th power a.

How small can we get the denomination for numbers α of algebraic degree n? Since any $n+1$ elements of $K(t,\alpha)$ are dependent over $K(t)$, a simple count shows that for large n, as there are more than $n+1$ differential monomials $y^{\mathbf{e}}$ satisfying

$$\sum (j+1)e_j \leqslant (\log n)^2,$$

we can achieve $\overline{d}(\alpha) \leqslant (\log n)^2$ for large n, in the characteristic zero case. In the characteristic p case, since $y^{(p)} = 0$, we can only choose $\mathbf{e}$ with $k < p$, so for a fixed p the denomination cannot be improved to order less than $n^{1/p}$, for large n. For p large compared to n, we can reach near the characteristic zero bound. Also small denomination is not useful, unless there is a corresponding non-vanishing.

We achieve smallest denomination 2 for irrational α satisfying the *Riccati equation* $y' = ay^2 + by + c$ with rational functions as coefficients. So for such elements, for instance, any element of degree 3 or any irrational n-th root of a rational, we have an (effective) Roth estimate – there is even no ϵ. Since we did not give the full proof of Kolchin's theorem, let us see how Osgood proved this special case.

We only need to show that other roots β of the Riccati equation cannot come arbitrarily close to the root α. Fix two other roots γ and δ. We use the well-known fact, easy to verify, that the cross-ratio of any 4 roots of the Riccati equation is a constant function, to deduce that the cross ratio $(\alpha-\gamma)(\delta-\gamma)^{-1}(1+(\alpha-\delta)(\beta-\alpha)^{-1})$ is a constant function. But this implies that $|\beta-\alpha|$ cannot come arbitrarily close to zero, as required.

Remark. In contrast, in the characteristic p case, α, which is a rational Möbius transformation of its p^n-th power, satisfies the Riccati equation, and we will see below that in this case the Riccati examples can have any rational exponent within Dirichlet and Liouville bounds, at least for some degrees. In characteristic p, the "constant function" in the previous paragraph has to be replaced by "function with zero derivative", which now includes the p-th power function, for example.

In fact, Osgood [17, 18] proved the following very interesting theorem.

Theorem 2. *In the case of function fields of finite characteristic, the exponent bound can be reduced from the Liouville bound to the Thue bound*

$$E(\alpha) \leqslant \left\lfloor \frac{\deg(\alpha)}{2} \right\rfloor + 1$$

for any non-Riccati α.

The relevance of the Riccati equation to this question is clearly brought out by the theorem of Osgood and Schmidt [21].

Theorem 3. *If $y'B(y)+A(y)=0$, where A and B are coprime polynomials with integral coefficients,*[1] *then all its rational solutions have height bounded in terms of those of A and B, as long as the equation is not Riccati, i.e. we do not have* $\deg(B)=0$ *and* $\deg(A)\leqslant 2$.

This theorem implies that the rational approximations close enough will not be the roots and hence the Liouville-Thue type argument goes through when applied to $y'B(y)+A(y)$, proving Theorem 2.

More precisely, for any $0 \leqslant d < n = \deg(\alpha)$, the $n+1$ elements

$$\alpha', \alpha'\alpha, \ldots, \alpha'\alpha^d \quad \text{and} \quad 1, \alpha, \ldots, \alpha^{n-d-1}$$

being linearly dependent over $K(t)$, we obtain A and B with $\deg(A) \leqslant n-d-1$ and $\deg(B) \leqslant d$, such that $P = y'B + A$ vanishes at α. Since P has denomination $\max\{d+2, n-d-1\}$, we obtain the Thue bound for optimum $d = \lfloor (n-2)/2 \rfloor$.

Solutions $1/t^k$ (respectively $1/(t^k-1)$) to the Riccati equation $ty' = -ky$ (respectively $ty' = -ky(y+1)$), where k is a natural number, so that $|k| = 1$ in the function field absolute value, shows that the analogous statement fails for Riccati equations, the situation being even worse in characteristic p, since then $1/t^{k+np}$ is also a solution to the first equation for any n.

[1] The coefficients are polynomials in t.

Since we did not prove Schmidt's theorem, let us prove, following Osgood, an easier result, that for a non-Riccati α of degree n, the exponent bound n can be improved to $n-1$, without using Schmidt's theorem.

If P is the monic minimal polynomial for α, then α satisfies two differential polynomials of denomination at most $n-1$, namely

$$y' - \sum_{j=0}^{m} a_j y^j,$$

with $n-1 \geqslant m > 2$ and $a_m \neq 0$, obtained by differentiating, and

$$(P - y^n) + a_m^{-1} y^{n-m} \left(y' - \sum_{j=0}^{m-1} a_j y^j \right).$$

Since any approximation vanishing for both these differential polynomials has to satisfy P, which has only finitely many roots, we obtain the result.

Finally, a rational Riccati equation in characteristic $p > 2$, but not for $p = 2$, has infinitely many rational solutions, if it has a non-quadratic solution. See [12] for a nice direct calculation proving the case $p = 3$. The general case follows from the classification in [19, pp. 369–370].

3. Continued Fraction Expansions

In this section, we deal with general F, unless noted otherwise.

By good rational approximations to α, whether we mean those which are closer than any with lower complexity, or whether we mean those with errors much smaller in comparison to the complexity, we are led directly to the approximations given by truncations of the continued fraction expansions of α, just as in the real number case. Continued fraction study for function fields over finite fields began with Emil Artin's thesis. We refer the reader to [28, 22] for references and details.

Let us review some standard notation. We use the abbreviation

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots = [a_0, a_1, a_2, \ldots].$$

We also write $\alpha_n = [a_n, a_{n+1}, \ldots]$, so that $\alpha = \alpha_0$. Let us define p_n and q_n as usual in terms of the partial quotients a_i, so that p_n/q_n is the n-th *convergent* to α.

Following the basic analogies mentioned above, we use the absolute value coming from the degree in t. To generate the continued fractions in the function field case, we use the "polynomial part" in place of the "integral part" of the "real number" $\alpha \in F((1/t))$.

Let us mention some contrasts and comparisons with the real case.

(i) The absolute value now takes discrete jumps.

(ii) In the real case, for $i > 0$, we have $a_i > 0$ and so q_i is positive and increases with i. In the function field case, for $i > 0$, a_i can be any non-constant polynomial and so the degree of q_i increases with i, but a_i or q_i need not be monic.

(iii) We have, e.g. $|\alpha| = |p_n/q_n|$.

(iv) There are many denominators (even monic) of the same size, unlike natural order on positive denominators in the real case.

We have the usual basic approximation formula

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{(\alpha_{n+1} + q_{n-1}/q_n)q_n^2}. \tag{1}$$

In the real case, this leads to

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2},$$

whereas in the function field case, we deduce the fundamental formula

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{|a_{n+1}||q_n|^2} \tag{2}$$

because of the non-archimedean absolute value.

One also studies the *intermediate convergents*

$$c_{n,r} := \frac{rp_{n-1} + p_{n-2}}{rq_{n-1} + q_{n-2}},$$

where $0 < r < a_n$ in the real case and $0 \leqslant \deg r \leqslant \deg a_n$, $0 \neq r \neq a_n$ in the function field case. We have

$$\alpha - c_{n+1,r} = \pm \frac{\alpha_{n+1} - r}{(\alpha_{n+1}q_n + q_{n-1})(rq_n + q_{n-1})}.$$

With these preliminaries, we are now ready to compare the situation with the classical case. To avoid special cases, in what follows, we assume that $n > 1$, $|b| > |a_1|$ and that a, b are relatively prime.

Following the traditional terminology [4], the approximation a/b to α, where, without loss of generality, a and b are relatively prime, is called *best* (respectively *good*), if

$$|b\alpha - a| < |b'\alpha - a'| \qquad \left(\text{respectively } \left|\alpha - \frac{a}{b}\right| < \left|\alpha - \frac{a'}{b'}\right|\right)$$

for $0 < |b'| \leqslant |b|$ and $a/b \neq a'/b'$. In the function field case, note that there can be more than one b' having the same absolute value as $|b|$. In both cases, the best approximations are precisely those given by the convergents.

There are subtle differences in the two situations. In the function field case, it makes quite a difference where the inequalities used are strict or not.

Let us say that a/b is *good in a weak sense* if

$$\left|\alpha - \frac{a}{b}\right| \leqslant \left|\alpha - \frac{a'}{b'}\right|$$

whenever $|b'| \leqslant |b|$, unless $a/b = a'/b'$. Let us also say that a/b is *fair* if

$$\left|\alpha - \frac{a}{b}\right| < \left|\alpha - \frac{a'}{b'}\right|$$

whenever $|b'| < |b|$.

Clearly, "best" implies "good", which in turn implies "good in a weak sense" and "fair".

In the real case [4], clearly "good" equals "fair" equals "good in weak sense", but "best" is stronger. In fact, a/b is best if and only if it is a convergent. It is good if and only if it is a convergent or an intermediate convergent $c_{n,r}$ with either $r > a_n/2$, or $r = a_n/2$ and $[a_n, a_{n-1}, \dots, a_1] > \alpha_n$. Also, $|\alpha - a/b| < 1/2b^2$ implies that a/b is a convergent and the coefficient 2 is best possible. In fact, $|\alpha - a/b| < 1/b^2$ implies that a/b is a convergent or an intermediate convergent $c_{n,r}$ with $r = 1$ or $r = a_n - 1$ and in fact it is a "good approximation from its side".

In the function field case, these concepts are related by the following result [28, Theorem 9.2.3].

Theorem 4. *In the function field case, the following hold:*

(i) *The properties "best", "good", "good in weak sense" and "convergent" are equivalent, and equivalent to an approximation a/b with error less than $1/|b|^2$, i.e. $\leqslant 1/|tb^2|$.*
(ii) *An approximation a/b is "fair" and not "good" if and only if it is an intermediate convergent $c_{n+1,r}$ with $|a_{n+1} - r| < |a_{n+1}|$; in particular, $|r| = |a_{n+1}|$.*
(iii) *We have $|\alpha - a/b| = 1/|b|^2$ if and only if for some n, a/b is the intermediate convergent $c_{n,r}$ with $r \in F^*$.*

Remarks. (1) The equivalence of "best", "convergent" and "with error less than $1/|b|^2$" in part (i) was proved in [13], while part (iii) can be obtained from an easy adaptation of the proof in [1] for the case $F = \mathbb{F}_2$.

(2) As is the case classically, the tails of the continued fraction expansion are the same if and only if the numbers are related by integral Möbius transformation of determinant ± 1. Eventually, periodic continued fraction expansion for α immediately implies that α is its own Möbius transformation with integral coefficients, and so quadratic. The converse is true in the real case and for function fields over finite fields. For other function fields, we have an analogue of Abel's theorem. So the analogies are stronger for function fields over finite fields. See [22, 13] for a discussion and development of the basics of continued fraction theory. For comparison with Hurwitz, Markoff results, Lehmer conjecture, *etc.*, see [28, chapter 9].

(3) If we know the continued fraction for α, the equation (1) allows us to calculate the exponent as

$$E(\alpha) = 2 + \limsup \frac{\deg a_{n+1}}{\sum_{i=1}^{n} \deg a_i}.$$

If we just have sum or product expansion of α, we can still calculate the exponent as follows under certain conditions, from a good sequence of approximations.

Lemma 1 (Voloch). *If $a_n/b_n \to \alpha$, with relatively prime polynomials a_n and b_n, satisfying*

$$\limsup \frac{\deg(b_{n+1})}{\deg(b_n)} = b \quad \text{and} \quad \frac{\log |\alpha - a_n/b_n|}{\log |b_n|} \to a,$$

where $a > b^{1/2} + 1$, then $E(\alpha) = a$.

We refer the reader to [33] for a nice proof using just triangle inequalities for our non-archimedean absolute value.

4. Frobenius and Möbius Transformations

In this section, we restrict to fields F of characteristic p. Let q be a power of p.

We have seen that the best approximations come through the truncations of continued fractions. Though the effect of addition and multiplication is complicated on continued fraction expansions, Möbius transformations of determinant ± 1 just shift the expansion. In characteristic p, raising to q-th power also has a transparent effect on partial quotients. If $\alpha = [a_0, a_1, \ldots]$, then $\alpha^p = [a_0^p, a_1^p, \ldots]$.

If $A_i(t) \in F[t]$ are any non-constant polynomials, the remark above shows that

$$\alpha := [A_1, \ldots, A_k, A_1^q, \ldots, A_k^q, A_1^{q^2}, \ldots] \tag{3}$$

is algebraic over $F(t)$, since it satisfies the algebraic equation

$$\alpha = [A_1, \ldots, A_k, \alpha^q].$$

So we have a variety of explicit equations with explicit continued fractions, from which we can compute their exponents, in terms of the degrees of A_i, and determine their possible values. It was thus proved independently by Schmidt [22] and the author [26, 28] that

Theorem 5.

(i) *Let the degree of A_i be d_i, and let*

$$r_i := \frac{d_i}{(d_1 + \cdots + d_{i-1})q + d_i + \cdots + d_k}.$$

Then for α given in (3)*, we have $E(\alpha) = 2 + (q-1)\max\{r_1, \ldots, r_k\}$.*

(ii) *Given any rational μ between*[2] *$q^{1/k} + 1$ and $q + 1$, we can construct a family of elements α as in* (3) *with $E(\alpha) = \mu$ and* $\deg(\alpha) \leqslant q + 1$.

Remarks. (1) We have thus also produced[3] explicit continued fractions for explicitly given algebraic families of degree more than 2. In contrast, the continued fraction expansion is not known even for a single algebraic real number of degree more than 2.

(2) If α satisfies $\alpha = (A\alpha^q + B)/(C\alpha^q + D)$, for $A, B, C, D \in F[t]$ with non-zero determinant $AD - BC$, α is said to be of *class I*, and if further $AD - BC \in F^*$, then it is said to be of *class IA*. Since, for $f \in F^*$, we have

$$f[a_0, a_1, a_2, \ldots] = [fa_0, f^{-1}a_1, fa_2, \ldots],$$

the examples above take care of continued fractions of all α of class IA. The pattern of continued fractions for general α of class I is an interesting open question, with interesting isolated examples and results given by, e.g. Baum, Sweet, Mills, Robbins, Buck, Voloch, de Mathan, Lasjaunias, Ruch and Schmidt. See [3, 9, 8, 22, 29] and a recent survey by Lasjaunias [10] for references and some explicit continued fractions of class I, but not of class IA.

(3) From this theorem, it seems therefore reasonable to guess that the set of exponents of algebraic elements is just the set of rational numbers in the Dirichlet–Liouville range. To show that all rationals occur, we need to control the exact degrees. By the analogue of Liouville's theorem for $d = p^n + 1$, we do have explicit families with all rational exponents between d and $d - 1$. But even for such values of d, we need to check irreducibility of the equation to obtain the full range. This should not be too difficult, given the wide choice for k and A_i, but has been done only in a few small cases. It seems that settling the case of general d with this method will require much more combinatorial effort.

[2] Note that $q^{1/k} + 1$ tends to 2 as k tends to infinity.

[3] The first such examples were due to Baum and Sweet [1, 2].

That this countable set of the exponents of algebraic Laurent series does not contain any irrational is not known in any generality except for a result of de Mathan [14].

Theorem 6. *For any α of class I, $E(\alpha)$ is rational.*

Corollary 1. *If $p = 2$, then the set of $E(\alpha)$ with α of degree 3 is exactly the set of all rational numbers in the closed interval $[2, 3]$.*

Note that every α of degree 3 is of class I and in general, every α of class I is easily seen to satisfy the Riccati equation. In fact, Lasjaunias and de Mathan [11, 12] established the following generalization of Osgood's theorem, which was conjectured in [33].

Theorem 7. *In finite characteristic, the exponent bound can be reduced from the Liouville bound to the Thue bound $E(\alpha) \leqslant \lfloor \deg(\alpha)/2 \rfloor + 1$ for any α not of class I.*

5. Non-Riccati Algebraic Continued Fractions

In [27, 26] different types of families of explicit continued fractions for algebraic quantities, in finite characteristic, were produced where the pattern of the sequence of partial quotients is based on block reversal, very similar to the pattern observed [23, 24, 25] for (transcendental!) analogues of Euler's "e" and Hurwitz numbers $(a\mathrm{e}^{2/n} + b)/(c\mathrm{e}^{2/n} + d)$ in the setting of Carlitz–Drinfeld modules for $\mathbb{F}_q[t]$.

The new examples are based on the following simple lemma, due to Mendes France [15] and which has been rediscovered many times.

Lemma 2. *Let $[a_0, a_1, \ldots, a_n] = p_n/q_n$, with p_n, q_n normalized as usual. Then*

$$[a_0, a_1, \ldots, a_n, y, -a_n, \ldots, -a_1] = \frac{p_n}{q_n} + \frac{(-1)^n}{y q_n^2}.$$

We will refer to this pattern as "a signed block reversal pattern with the new term y".

Now, if

$$\alpha := \sum f_i t^{-n_i} \in F((1/t)),$$

where n_i is an increasing sequence of integers satisfying $n_{i+1} > 2n_i$, for $i \geqslant i_0$ say, then repeated application of the lemma, starting with the continued fraction of the rational function obtained by truncating at i_0-th power, gives the complete continued fraction of α consisting of signed block reversals, with the new values of y being $t^{n_{i+1}-2n_i}$, up to elements of F which are easy to calculate from the lemma. As before, with the continued fraction expansion in hand, the calculation of the exponent and the determination of its range are routine.

When F has finite characteristic, it is often easy to construct such α which are algebraic over $F(t)$. First we give the main examples [27]. By taking linear combinations of Mahler's example above, we know that any

$$\alpha = \sum_{i=1}^{k} f_i \sum_{j=0}^{\infty} t^{-m_i q^j + b_i},$$

where $m_i \geqslant 0$ and b_i are rational numbers so that the exponents are integers,[4] is algebraic. And it is easy to write down conditions on the coefficients to ensure that

[4]With integral coefficients, we can write the exponent as $a_i q^j + b_i(q^j - 1)/(q - 1) + c_i$.

$n_{i+1} > 2n_i$ for large i. For example, $m_{i+1} > 2m_i$ for $1 \leqslant i < k$ and $qm_1 > 2m_k$ are clearly sufficient, but not necessary. With this condition, as in [26, Theorem 2], we see that

$$E(\alpha) = \max\left\{\frac{m_2}{m_1}, \dots, \frac{m_k}{m_{k-1}}, \frac{qm_1}{m_k}\right\},$$

and that it takes any rational value between $q/2^{k-1}$ and $q^{1/k}$, if further that $q > 2^k$.

The algebraic equation for each term, corresponding to a fixed i, is immediate, since it is just a multiple of Mahler's example. So the polynomial equation satisfied by α follows.

The flexibility in the choice of m_i and b_i can be used to produce many families of elements α not satisfying the rational Riccati equation.

Any α as above with $q = 2^k$, $m_i = 2^{i-1}$ and $b_i > b_{i+1}/2$, for i modulo k, will produce an explicit continued fraction with bounded sequence of partial quotients in characteristic 2.

Let us show that most of these do not satisfy the rational Riccati equation, and so are of degree more than 3. Take $f_i = 1$ for simplicity, and write α_i for the i-th term of the sum expression for α above. Then $\alpha = \alpha_1 + \cdots + \alpha_k$, and $\alpha_i = \alpha_1^{2^{i-1}} p_i$ with $p_i := t^{b_i - 2^{i-1} b_1}$. Again for simplicity, take b_1 odd and b_i even if $i \neq 1$, so that $\alpha' = \alpha_1/t + t^{b_1-2}$. If α were to satisfy the rational Riccati equation $\alpha' = a\alpha^2 + b\alpha + c$, then we would have

$$\frac{\alpha_1}{t} + t^{b_1-2} = a(\alpha_1^2 + \alpha_1^4 p_1^2 + \cdots + \alpha_1^{2^k} p_k^2) + b(\alpha_1 + \alpha_1^2 p_1 + \cdots + \alpha_1^{2^{k-1}} p_k) + c.$$

But by the degree comparison, this equation has to be the Mahler type irreducible equation $\alpha_1^{2^k} = t^{b_1(q-1)}\alpha_1 + t^{qb_1-1}$, which is clearly impossible for most choices of p_i for $k > 2$. The same construction in characteristic $p > 2$, with say exponent p, gives examples (now $k > 1$ is fine) which are non-Riccati; in fact, not of the form α' equals polynomial in α of degree $\leqslant p$.

When F is a finite field of characteristic p, the elements

$$\sum f_i t^{-i} \in F((1/t))$$

are algebraic over $F(t)$ if and only if the sequence f_i is produced by p-automata, by a theorem of Christol. See, e.g. [28, Chapter 11] for definitions, examples and applications to transcendence theory. Using this, it has been established [27] that

(i) if α satisfying our general conditions has bounded sequence of partial quotients, then the characteristic p is 2; and
(ii) any of our examples with exponent 2 has bounded sequence of partial quotients.

More generally, a classification of those α satisfying our conditions is established in [27] using this automata classification of algebraic power series.

6. Deformation Hierarchy versus Exponent Hierarchy

We now describe the results of [6] linking the exponents of α to the deformation possibilities of certain curves over function fields that we associate to α. We show that the bounds on the rank of the Kodaira–Spencer map of these curves imply the bounds on the diophantine approximation exponents of the power series α, with more "generic" curves, in the deformation sense, giving lower exponents. If we transport Vojta's conjecture on height inequality to finite characteristic, modifying it by adding a suitable deformation theoretic condition, then we see that the exponents of those α giving rise to general curves approach the Roth bound.

Voloch [34] observed that the condition that α satisfies the rational Riccati equation is equivalent to the condition that the cross ratio of any four conjugates of α have zero derivative which in turn is equivalent to the vanishing of the Kodaira-Spencer class of projective line minus conjugates of α.

This suggests that it might be possible to successively improve on Osgood's bound, if we throw out some further classes of differential equations coming from the conditions that some corresponding Kodaira–Spencer map, or say the vector space generated by derivatives of the cross-ratios of conjugates of α, has rank not more than some integer. Note that even though the Kodaira–Spencer connection holds in characteristic zero, an analogue of Roth's theorem holds in the complex function field case.

In the light of Theorems 2 and 7, the differential equation hierarchy suggested above might have some corresponding more refined Frobenius equation hierarchy.

First let us recall the setup [5, 6] of deformation theory and height inequalities.

Let X be a smooth projective surface over a perfect field k. Assume that X admits a map $f : X \to S$ to a smooth projective curve S defined over k, with function field L in such a way that the fibers of f are geometrically connected curves and the generic fiber X_L is smooth of genus $g \geqslant 2$.

Consider algebraic points $P : T \to X$ of X_L, where T is a smooth projective curve mapping to S, such that the triangle commutes. Define the canonical height of P to be

$$h(P) := \frac{\deg P^*\omega}{[T : S]} = \frac{\langle P(T), \omega\rangle}{[K(P(T)) : L]},$$

where $\omega = \omega_X := K_X \otimes f^*K_S^{-1}$ denotes the relative dualizing sheaf for $X \to S$. Define the relative discriminant to be

$$d(P) := \frac{2g(T) - 2}{[T : S]} = \frac{2g(P(T)) - 2}{[K(P(T)) : L]}.$$

The Kodaira–Spencer map is constructed on any open set $U \subset S$ over which f is smooth from the exact sequence

$$0 \to f^*\Omega^1_U \to \Omega^1_{X_U} \to \Omega^1_{X_U/U} \to 0,$$

by taking the coboundary map

$$KS : f_*(\Omega^1_{X_U/U}) \to \Omega^1_U \otimes R^1 f_*(\mathbf{0}_{X_U}).$$

Theorem 8.

(i) *If the rank of the kernel of the Kodaira–Spencer map is $\leqslant i$, then*[5]

$$h(P) \leqslant \left(\max\left\{\frac{2g-2}{g-i}, 2\right\} + \epsilon\right) d(P) + O(1), \quad 0 \leqslant i < g.$$

(ii) *In particular, if the Kodaira–Spencer map of X/S has maximal rank, i.e. $i = 0$, then $h(P) \leqslant (2+\epsilon)d(P) + O(1)$.*

[5] Note that the maximum is 2 only for $i = 0, 1$.

The inequality in (ii) was proved by Vojta [32] in the characteristic 0 function field analogue without any hypothesis. He also conjectured earlier [31] the stronger inequality with 2 replaced by 1 in the number field case, and presumably also in the characteristic 0 function field case. In [6], we look at the *Hypothesis H* in the function field case, that

$$h(P) \leqslant (1+\epsilon)d(P) + O(1), \quad \text{if the Kodaira-Spencer map has maximal rank,}$$

and also consider conjectural improvements on the height inequalities in (i).

We now apply this result to the "Thue curve" X and "super-elliptic" curves X_k associated to α as follows.

Let $f(x) = f_0 + f_1 x + \ldots + f_d x^d$ be an irreducible polynomial with $\alpha = \alpha(t)$ as a root and with $f_i \in F[t]$ being relatively prime. Let $F(x,y) = y^d f(x/y)$ be its homogenization.[6]

Assume p does not divide d. Let X be the projective curve with its affine equation $F(x,y) = 1$. Given a rational approximation x/y to α, reduced in the sense that $x, y \in F[t]$ are relatively prime, with $F(x,y) = m(t)$, we associate the algebraic point $P = (x/m^{1/d}, y/m^{1/d})$ of X.

Remark. Note that if x/y is an approximation approaching the exponent bound, then the degree of the polynomial $m(t)$ is asymptotically $(d - E(\alpha))\deg(y)$, as $\deg(y)$ tends to infinity. The exponent occurs in the calculation through this.

Let X_k have affine equation $y^k = f(x)$, with k relatively prime to p and d. Corresponding to a (reduced) approximation x/z, let

$$P = (x/z, (m/z^d)^{1/k}),$$

where $m = F(x,z)$.

Theorem 9. *Corresponding to given upper bounds on the rank of the kernel of the Kodaira–Spencer map of X or X_k, through the height inequalities, we have explicit upper bounds on $E(\alpha)$. In particular, under the Hypothesis H above,*

(i) *applying Theorem 8 to X in the case of maximal rank leads to the estimate $E(\alpha) \leqslant 2d/(d-1)$, which tends to the Roth bound 2 as d tends to infinity; while*
(ii) *applying Theorem 8 to X_k in the case of maximal rank leads to the estimate $E(\alpha) \leqslant 2 + 2/(k-1)$, which tends to the Roth bound as k tends to infinity.*

Over the complex numbers, the maximal Kodaira-Spencer rank is a generic phenomenon, but we are looking at a special countable class. For more discussion, and for proofs, we refer the reader to [28, 6] where new unconditional results in this direction on approximation by algebraics, rather than by rationals, are also given.

7. Open Problems and Speculations

In this section, we only focus on function fields over finite fields which is the main case of interest to a number theorist.

For a given finite field F of q elements and of characteristic p, and given $d > 2$, let E_d be the set of approximation exponents of $\alpha \in F((1/t))$, algebraic of degree d over $F(t)$.

The first main question concerns what this set is.

[6] There should be no confusion with the field F.

Theorem 5 suggests that E_d might consist precisely of all the rational numbers between 2 and d. That it does contain all these rationals has been worked out only in a few cases for $d = p^n + 1$; see the Remark (3) after Theorem 5. That it does not contain any irrational is known only for $d = 3$; see Theorem 6.

The second main question concerns the distribution of α, for given F and d, with respect to their approximation exponents, in terms of their heights or deformation behaviour, etc., and what the exponent hierarchies are.

Using Theorem 5(i), most elements α of class IA seem to have exponents near 2, since most have large k and large d_i. The precise asymptotic needs to be worked out for the exponents in a given range for these elements in terms of heights. How about class I? For $q = 2$ and $d = 3$, every α is of class I. Now Möbius transforms preserve the exponents. If we can get all these elements as Möbius transforms of class IA and some other variants[7] whose exponent distribution is well understood, we might hope to settle the exponent distribution completely in this simplest test case.

We would like to settle[8] the Hypothesis H of Section 6 and find the corresponding best height inequality hierarchy and exponent hierarchy, improving Theorems 8 and 9. Is there a simpler differential equation/Frobenius equation hierarchy implying an exponent bound hierarchy generalizing Osgood's theorem?

The third main question asks for the nature of the sequence of partial quotients of the continued fractions.

We have a complete understanding of the patterns in class IA and know many examples of families in class I, and in non-Riccati families above, which can be classified [27] using automata theory as the sparsest density algebraic families. Most of these involve patterns of iterated p^n powers, block reversals and block repetitions.

What is the general description of patterns for α of class I? The method of automata and transducers [16] generates such expansions, but it has not led to direct description of the patterns yet. It would be of interest to know whether all (non-degenerate) Möbius transformations of the examples above also show similar patterns, as is the case for Hurwitz numbers; see the first paragraph in Section 5. Hurwitz proved that any non-degenerate integral Möbius transformation of any real number, having the pattern of partial quotients consisting eventually of finitely many arithmetic progressions, has the same kind of pattern. While we know such a result in the particular case of analogue of Hurwitz numbers mentioned in the first paragraph in Section 5, we do not know the analogue of the general Hurwitz result mentioned above in the function field case. It is an interesting challenge to settle it. Note that it can be checked and disproved (if false) by computer experimentation.

How close to the random behaviour of the partial quotients do we get for "general" algebraic elements α of degree $d > 2$, in analogy with experimentation with the real algebraic numbers? Note here though that in the case $q = 2$ and $d = 3$, every α has a nice pattern described in Section 4.

Do we have unbounded partial quotients for almost all values of α? Bounded partial quotients are rare, in the imprecise sense of difficulty of constructing examples with bounded sequences. The examples we give in Section 5, as well as those obtained by

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[7]See [29] for discussion and also [28, Theorem 9.3.4] for another variant.

[8]See [6, Section 6] for a discussion.

degeneration techniques in [26], are all of characteristic 2. For $p \neq 2$, it seems harder to get such examples. Can this be made precise?

What does this suggest for real algebraic numbers? It is not known even for a single real algebraic number of degree more than 2, if the sequence of its partial quotients is bounded or unbounded. In view of the numerical evidence and a belief that the real algebraic numbers are like most real numbers in this respect, it is often conjectured that the sequence is unbounded.

From the function field analogy, thus it is conceivable that a very thin set of real algebraic numbers might have bounded partial quotient sequence.

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