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Diophantine Approximation and Transcendence in Finite Characteristic

Dinesh S. Thakur*

This paper is dedicated to Professor T. N. Shorey on his 60th birthday. I am glad to be able to express my gratitude to Professor Shorey and to thank him for his kind encouragement starting when I was an undergraduate. I attended then some of his lectures on Diophantine approximation, so it might be appropriate to give here some account of my recent work on this subject and some related results.

For diophantine approximation of algebraic numbers, straight analog of Roth's theorem in the number field case fails in function fields and the situation is not even conjecturally understood. We describe the partial progress on this and related issues of continued fractions for algebraic quantities. We will also describe progress on transcendence and algebraic independence issues, especially for gamma, zeta and multizeta values, using higher dimensional generalizations of Drinfeld modules.

1 Exponents of Diophantine Approximation

We describe the problem and give a quick overview of the recent progress (see [S1, S2, T] for more details and references).

Let F be a field. We consider $F[t]$, $F(t)$ and $F((1/t))$ as analogs of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. As is well-known, the analogies are even better when F is a finite field.

Consider the basic diophantine approximation problem of well-approximating α, a real algebraic number (or 'real' function i.e. $\alpha \in F((1/t))$) by rational numbers (or rational functions) a/b, where $a, b \in Z$ (or $F[t]$ respectively). If we concentrate on the **exponent**

$$E(\alpha) := \limsup \left(-\frac{\log|\alpha - a/b|}{\log|b|} \right)$$

we thus get $2 \leq E(\alpha) \leq \deg(\alpha)$, by theorems of Dirichlet and Liouville (and Mahler in the function field case). After successive improvements (with

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well-known implications to diophantine equations) on the upper bound by Thue, Siegel, Dyson, finally Roth (D. Fenna and Uchiyama in function field case, with F of characteristic zero) showed that $E(\alpha) = 2$, for algebraic irrational α.

From now on, let F be of characteristic p, and q be a power of p. Then, as Mahler observed, $E(\alpha) = q$ for $\alpha = \sum t^{-q^i}$, by a straight estimate of approximation by truncation of this series. Now $\alpha^q - \alpha - t^{-1} = 0$, so that α is algebraic of degree q and hence the Liouville upper bound is best possible in this case. Mahler suggested (and it was claimed and believed for a while) that such phenomena may be special to the degrees divisible by the characteristic, but Osgood gave examples in each degree for which Liouville exponent is the best possible. A few more isolated examples were proved after extensive computer searches.

Now the best approximations come through the truncations of continued fractions. Though the effect of addition and multiplication is complicated on continued fraction expansions, Möbius transformations of determinant ± 1 just shift the expansion. In characteristic p, raising to q power also has a transparent effect on partial quotients.

If $\alpha = [a_0, a_1, \ldots]$, then $\alpha^p = [a_0^p, a_1^p, \ldots]$, where we use a short-form $[a_0, a_1, \ldots]$ for the expansion $a_0 + 1/(a_1 + 1/(a_2 + \cdots))$.

If $A_i(t) \in F[t]$ are any non-constant polynomials, the remark above shows that

$$\alpha := [A_1, \ldots, A_k, A_1^q, \ldots, A_k^q, A_1^{q^2}, \ldots]$$

is algebraic over $F(t)$ because it satisfies the algebraic equation

$$\alpha = [A_1, \ldots, A_k, \alpha^q].$$

So we get a variety of explicit equations with explicit continued fractions, from which we can compute their exponents, in terms of the degrees of A_i and determine its possible values. It was thus proved independently by Wolfgang Schmidt and the author that

Theorem 1.1 ([S2, T2]) *Given any rational* μ *between* $q^{1/k} + 1$ *(which tends to* 2 *as* k *tends to infinity) and* $q + 1$*, we can construct a family of* α*'s as above with* $E(\alpha) = \mu$ *and* $\deg(\alpha) \leq q + 1$.

It seems thus reasonable to guess that the set of exponents of algebraics is just the set of rational numbers in the Dirichlet-Liouville range. That this countable set cannot contain any irrational number is known (due to de Mathan) for degree 3, but not in general. To show that all rationals occur, we need to control the exact degrees. This has been done only in a few cases.

Note that we have thus also produced explicit continued fractions (first such examples were due to Baum and Sweet) for explicitly given algebraic families. In contrast, continued fraction expansion is not known even for a single algebraic real number of degree more than two.

Now, just as in the case of real numbers, the continued fractions with the same tails are related by integral Möbius transformations of determinant ± 1, but in contrast to the real number case, numbers thus related can have the same tail or tails related by alternate signs. Since, for $f \in F^*$, we have $f[a_0, a_1, a_2, \ldots] = [fa_0, f^{-1}a_1, fa_2, \ldots]$, more generally, the integral Möbius transformations of non-zero constant (i.e., unit) determinant are exactly (see [S2]) the shifts of the continued fraction expansion tail, up to multiplication by constants in the manner above. Thus the examples above take care of exactly those of the form

$$\alpha = \frac{A\alpha^q + B}{C\alpha^q + D}, \qquad (*)$$

with determinant $AD - BC \in F^*$.

Here are some simple variants of algebraic families with non-constants determinants:

(1) For $a \in F(t)^*$, we have $a[a_1, a_2, a_3, \ldots] = [aa_1, a_2/a, aa_3, a_4/a, \ldots]$. Hence, if k is even (which we can assume without loss of generality, by doubling the length of the initial segment and replacing q by q^2, if k is odd) and A_{2i}/a and aA_{2i-1} are non-constant (see [T2, T3] for how to handle degenerate case of constant) polynomials (eg.,if $a \in F^*$), we get explicit regular continued fractions for $a\alpha$. The corresponding determinant is $(AD - BC)a^{q+1}$, if $a \in F[t]$.

(2) Another way (see [S2, lemma 10]) is to use this trick in a different fashion and to generalize under certain simple conditions to $\alpha = [A_1, \ldots, A_k, a\alpha^q]$. If k is odd and numerator of a divides A_{2i}^q and the denominator of a divides A_{2i+1}^q, then we have regular continued fraction

$$\alpha = [A_1, \ldots, A_k, aA_1^q, A_2^q/a, \ldots, aA_k^q, a^{q-1}A_1^{q^2}, \ldots, a^{q^2-q+1}A_1^{q^3}, \ldots, \\ a^{q^3-q^2+q-1}A_1^{q^4}, \ldots],$$

whereas if k is even, with exponent q replaced by $q-1$ in divisibility conditions above, we have regular continued fraction

$$\alpha = [A_1, \ldots, A_k, aA_1^q, \ldots, A_k^q/a, a^{q+1}A_1^{q^2}, \ldots, a^{q^2+q+1}A_1^{q^3}, \ldots].$$

The determinant is now product of numerator and denominator of a times ± 1.

(3) Conditions in (1) and (2) are not the only ones. For example, using the identity

$$a[A_1, P_2a+1, P_3a-1, \mu] = [aA_1, P_2, a, -P_3, -a\mu]$$

repeatedly, we see that algebraic families such as $a[A_1, P_2a+1, P_3a-1, A_1^q, \ldots]$, with $a \in F[t] - F$ have interesting pattern

$$[aA_1, P_2, a, -P_3, -aA_1^q, -P_2^q a^{q-1}, -a, P_3^q a^{q-1}, aA_1^{q^2}, \ldots].$$

Similar variants then can also be used as in (2) to create more interesting algebraic examples starting with continued fractions.

The pattern of continued fractions for general α of the type $(*)$ above with any determinant is an interesting open question, with interesting isolated examples and results given by eg., Baum, Sweet, Mills, Robbins, Buck, Voloch, de Mathan, Lasjaunias and Schmidt.

Now any α of degree 3 is of type $(*)$. Since we understand (as described above, exponents of α's of type $(*)$ with constant determinant, and exponents do not change under any integral Möbius transformation of non-zero determinant, question can be raised whether we can represent any degree 3 'real', as an integral Möbius transformation of type $(*)$ with constant determinant. There is a determinant obstruction unless $q = 2$. There is a local obstruction when $q = 2$. (I thank Bjorn Poonen for his help in determining the last point). It would be complicated, but worth a try to get all, using variants of type (1)–(3) also to start with.

Now α's of type $(*)$ satisfy rational Riccati equation: $d\alpha/dt = a\alpha^2 + b\alpha + c$, with $a, b, c \in F(t)$. The relevance of this equation is Osgood's result that the Liouville/Mahler bound can be improved to $E(\alpha) \leq \lfloor \deg(\alpha)/2 \rfloor + 1$, for α not satisfying such rational Riccati differential equation.

We refer to [T2, T3] where we constructed, by appropriate linear combinations of Mahler type examples, explicit continued fractions for some non-Riccati families of algebraic (finite-tailed) Laurent series, with patterns exhibiting 'block reversals and repetitions'. The techniques and the patterns were exactly those used earlier by the author in case of transcendental quantities: analogs of Euler's e and Hurwitz numbers in Drinfeld module setting!

Classically, the pattern of partial quotient sequence for the continued fraction of $e^{2/n}$ consists eventually of finitely many arithmetic progressions. Hurwitz proved that any non-degenerate integral Möbius transformation of any number having this kind of pattern has the same kind of pattern. While we do not know an analog of Hurwitz result for patterns mentioned above in the function field case, it is an interesting challenge to settle it. Note that it can be checked and disproved (if false) by computer experimentation.

It is not known even for a single algebraic real number of degree more than 2, if the sequence of its partial quotients is bounded or unbounded. Because of the numerical evidence and a belief that algebraic real numbers are like most real numbers in this respect, it is often conjectured that the sequence is unbounded.

While most of the examples constructed above have unbounded sequence of partial quotients, we use degeneration techniques [T2, T3] to give several algebraic non-quadratic examples with bounded sequence, especially in characteristic 2. It is relatively hard to get examples with bounded sequence.

From the function field analogy, thus it is conceivable that a very thin set of algebraic real numbers might have bounded partial quotient sequence.

Finally, we briefly mention some other recent related results (obtained in joint work with Kim and Voloch) establishing the connection between diophantine approximation and the **deformation theory**.

Following up on the initial result of Osgood mentioned above, in [KTV], we study the influence of differential equations on diophantine approximation properties. We give diophantine approximation exponent bound hierarchy corresponding to the rank hierarchy of Kodaira-Spencer map (which controls deformation theory) for some curves, such as Thue curve $P(x, y) = 1$ or curves $y^k = p(x)$, associated to α. Here $p(x)$ is the minimal polynomial of α and $P(x, y)$ is its homogenization.

Roughly speaking, if α corresponds to a curve which is 'general', in the sense of having many deformations, then its exponent is low, approaching the Roth bound of two, assuming (as we suggest) that Vojta height inequalities hold under maximal deformation assumptions. (In this connection, it might be worth pointing out that a simple calculation with explicit exponent formulas shows that for 'most' α's of the Theorem 1, exponents are near 2).

We also have exponent bound hierarchy results for approximations by algebraic quantities of bounded degree.

There are many open questions left about precise bounds and hierarchies and exponent distribution with respect to heights and degrees of α's, as well as whether there are simpler generalizations of Riccati equations generalizing Osgood's result by pushing the exponent bound down, if their solutions are excluded.

2 Transcendence and Algebraic Independence Results on Special Values

Now we study global function fields. From now on, F is a finite field of q elements, q being a power of the characteristic p. We let $A = F[t]$, $K = F(t)$, $K_\infty = F((1/t))$ and C_∞ be the completion of an algebraic closure of K_∞. As mentioned before K is analog of Q and we consider transcendence over K. We also denote by A_+ the set of monic polynomials in A, and we consider it as analog of the set $\mathbb{Z}_+$ of positive integers.

It is an interesting contrast that while natural numbers coming up in number theory are usually not easy to approximate well enough for the Roth theorem to give their transcendence, in function field arithmetic, usually we get good approximations for natural quantities a little easier, but analog of the Roth theorem fails, as we have seen above.

In addition to analogs, due mainly to Jing Yu, (see [T, Chapter 10] for results and references) of usual transcendence theorems, such as those of Hermite-Lindemann, Gelfond-Schneider, Baker, Wüstholz; two other very successful techniques in function field transcendence theory are (i) automata techniques (see [T, Chapter 11] for results and references), and (ii) period techniques, where there is new spectacular progress, on which we will report.

In the function field arithmetic connected with Drinfeld modules and their higher dimensional generalizations, there are good $A = F[t]$-analogs (see [T] for analogies, interrelations and earlier results) of exponential, logarithm, gamma, zeta, multizeta (and hypergeometric) functions. We now give some definitions and latest results; and then explain how they come about:

(I) The **exponential function** is given by

$$e(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{(t^{q^i} - t)(t^{q^i} - t^q)\dots(t^{q^i} - t^{q^{i-1}})}.$$

Let $\tilde{\pi}$ denote its fixed fundamental period. (It is thus analog of $2\pi i$).

Theorem 2.1 ([P]) *Let $\ell_1, \dots, \ell_r \in C_\infty$ be such that $e(\ell_i) \in \overline{K}$. If ℓ_i are linearly independent over K, then they are algebraically independent over $\overline{K}$.*

Note that these are algebraic independence results on logarithms improving linear independence results due to Jing Yu of Baker type.

(II) **Arithmetic gamma function** $\Gamma_a : \mathbb{Z}_p \to K_\infty$, closely connected with cyclotomic theory of constant field extensions, is defined by

$$\Gamma_a(z+1) = \prod_i \left(\frac{\prod_{j=0}^{i-1}(t^{q^i} - t^{q^j})}{t^{iq^i}}\right)^{z_i},$$

where $z = \sum z_i q^i$ with $0 \leq z_i < q$.

Theorem 2.2 ([T]) *Certain explicitly given monomials* $\prod \Gamma_a(r_i)^{n_i}$*, where* r_i *are* p*-integral proper fractions, are algebraic and the rest are all transcendental.*

See [T] for the precise description of algebraic monomials and for the references. We will only say here that this was proved by the author by techniques of automata (with contribution by Allouche, and also with simplification and generalization to non-fractions due to Mendès-France and Yao), when the period techniques yielded only very weak results exactly comparable to what was (and is) known for the classical gamma function. I think that techniques used for the next theorem may yield another proof and even stronger algebraic independence result, but this has not been done.

(III) **Geometric gamma function** is a meromorphic function $\Gamma_g : C_\infty \to C_\infty$ defined by

$$\Gamma_g(z) = \frac{1}{z} \prod_{a \in A_+} \left(1 + \frac{z}{a}\right)^{-1}.$$

It is closely connected with cyclotomic theory of Drinfeld cyclotomic extensions. By Γ-monomial we will mean an element of the subgroup of C_∞^* generated by $\tilde{\pi}$ and values of Γ_g at proper fractions in K.

The following result determines all algebraic relations between the gamma values at fractions.

Theorem 2.3 ([ABP]) *A set of* Γ*-monomials is* $\overline{K}$*-linearly dependent exactly when some pair of* Γ*-monomials is. Pairwise* $\overline{K}$*-linear dependence is decided by explicit combinatorial criterion.*

In particular, the transcendence degree of the field extension of $\overline{K}$ *generated by* $\tilde{\pi}$ *and gamma values at proper fractions with denominators dividing* $a \in A$ *over* $\overline{K}$ *is* $1 + (1 - 1/(q-1))|(A/a)^*|$.

We remark here that there is (see [T, 4.12]) a unified Galois-theoretic description in classical as well as the two cases above of the 'explicit algebraic monomials' mentioned, and that in the classical as well as in geometric gamma case (but not in arithmetic case) they follow from reflection and multiplication formulas.

(IV) The **zeta values** are defined by

$$\zeta(s) = \sum_{a \in A_+} \frac{1}{a^s} \in K_\infty, \quad s \in \mathbb{Z}_+.$$

We consider these special values now. Note that these values are highly transcendental in contrast to simple rational Artin-Weil zeta values obtained by replacing a by its norm, which just uses degree information from a polynomial.

From the definition, we have $\zeta(sp) = \zeta(s)^p$. Carlitz proved the analog of Euler's theorem that $\zeta((q-1)m)/\tilde{\pi}^{(q-1)m} \in K$. The following result says that these generate all algebraic relations.

Theorem 2.4 ([CY]) *For s 'odd', i.e., not divisible by $q-1$, $\zeta(s)$ and $\tilde{\pi}$ are algebraically independent. In fact, the transcendence degree of the field*

$$K(\tilde{\pi}, \zeta(1), \ldots, \zeta(n))$$

over K is

$$n + 1 - \lfloor n/p \rfloor - \lfloor n/(q-1) \rfloor + \lfloor n/(p(q-1)) \rfloor.$$

(V) **Multizeta values** are defined by

$$\zeta(s_1, \ldots, s_k) = \sum \frac{1}{n_1^{s_1} \ldots n_k^{s_k}} \in K_\infty, \quad s_i \in \mathbb{Z}_+.$$

where the sum is over monic n_j's with degree of n_i less than that of n_{i+1}. These are so called non-degenerate values. We also define similarly degenerate multizeta by allowing some degeneration in the degree inequalities above. This immediately gives sum shuffle relations, by shuffling over the degrees and thus we get an algebra of multizeta values.

Theorem 2.5 ([APT]) *Multizeta values are transcendental.*

Certain dependence and independence results are known among them. Sum shuffle and integral shuffle identities that hold classically, both fail in this case. There are relations known (all motivic) which depend in a complicated way on combinatorics of q-base digits of s_i. Such dependence is also seen in other aspects of function field arithmetic. For example, the divided power series corresponding to the Zeta measure, conditions implying extra vanishing multiplicities of zeta at trivial zeros, Gamma functions and Gauss sums functional equations, study of Galois group using Ihara power series etc. all exhibit this interesting phenomenon.

It seems quite plausible that if we can give good conjectural description of all the relations, we can prove them by techniques described below. Classically, the situation seems exactly the opposite!

All of these functions/values have **v-adic counterparts**, where v is a prime of A. We refer to [T] for their definitions and properties. Jing Yu proved transcendence results (see [T, Chapter 10]) for v-adic logarithms. The author and Yao proved (see [T, Chapter 11]) by automata techniques that for v of degree one, v-adic interpolation of the arithmetic gamma takes algebraic (respectively transcendental) values at proper fractions with denominator dividing (respectively, not dividing) $q-1$. The general case for higher degree v, as well as the case for geometric gamma function is open. For the zeta case, Jing Yu (see [T, Chapter 10]) proved that v-adic zeta values at 'odd' s (i.e., s not divisible by $q-1$) are transcendental, using his transcendence results on higher logarithms results combined with (IV) of the next section.

In the next two sections, we sketch how these results come about. There are two ingredients. The first is construction of suitable T-motives (objects related to Drinfeld modules and Shtukas that have been successfully used by Drinfeld, Lafforgue and others to settle Gl_n Langlands conjectures in the function field case) whose periods are essentially the logarithm, gamma, zeta, multizeta values that we are interested in. (This already implies several transcendence results, as described, for example, in [T, Chapter 10]). The second is a breakthrough transcendence result of Anderson, Brownawell and Papanikolas that the relations between the periods of these motives are reflections of relations between the motives themselves and thus can be computed in principle by linear algebra.

We will try to convey the main ideas in words and refer to [T, Chapter 7], references there and below, for technical details.

3 Special Values as Periods of Anderson-Drinfeld Motives

The exponential above is in fact the exponential associated to the simplest Drinfeld A-module: the Carlitz module, which is dimension $d = 1$ and rank $r = 1$ object. Anderson defined higher dimensional generalizations of Drinfeld modules: **T-motives** (or abelian T-modules). Here T stands for a non-trivial endomorphism of d-th power of the additive group and such a motive M (defined over $\overline{K}$) is certain module over non-commutative ring

$\overline{K}[T,\tau]$ (where T commutes with τ and scalars, but $\tau k = k^q \tau$), of τ-rank d and T-rank r such that $(T-t)$ acts nilpotently on $M/\tau M$.

Given a function $f(T)$ with C_∞ coefficients, by its k-th twist $f^{(k)}$ we mean the function obtained by raising scalars by q^k-th powers.

We just note here that [ABP], [P] etc. deal with Anderson's new variant of T-motives called **dual T-motives**, which basically exploits Ore-Elkies-Poonen duality obtained by replacing τ by τ^{-1}. Using q-th roots instead of powers is not a problem over perfect fields such as $\overline{K}$ which are sufficient for transcendence applications. The new set-up allows certain technical simplifications in that it connects immediately to points of the underlying algebraic group and its Lie algebra (See [T, 7.1.12]) and makes the period recipe (See [T, 7.4]) simpler by replacing residue operation explained there by just evaluation as described below. We will gloss over these technical points in the differences between these two notions.

In particular, the τ action can be described by $r \times r$ matrix Φ with coefficients in $\overline{K}[T]$ and the period matrix (for 'uniformizable' motives) is evaluation at $T = t$ of the solution Ψ to the equation

$$\Psi^{(-1)} = \Phi\Psi.$$

The **tensor products** exist in this category (take tensor power over $\overline{K}[T]$ and let τ act diagonally) and simple algebra operations give 'cohomology realizations' and comparison isomorphisms. (Hence the word 'motive', though these are not Grothendieck motives).

Now let us describe how the special values of the previous sections are periods of suitable motives.

(I) Parallel to the classical case, the logarithms in (I) of the previous sections are periods of the suitable extension of the Carlitz module by a trivial module.

(III) When z is a fraction with denominator f, there is a function ('**solitons**' of Anderson, see 8.4 of [T, 8.4] or [ABP]) on product of 2 copies of f-th cyclotomic cover of projective line over F which when specialized to the graph of the d-th power of Frobenius gives the partial product corresponding to a's of degree d in the definition of Γ_g. Out of these 'two variable functions' (say t and T are the two variables) one constructs suitable Φ and the corresponding motive, whose period is then given by the full gamma product as can be (vaguely) seen from the (formal) solution

$$\Psi = \prod_{i=1}^{\infty} \Phi^{(i)}$$

to the equation above.

These motives can be compared to quotients of **Jacobians of Fermat curves** whose periods classically give the Beta values (but not all gamma values as in our case) for which we have transcendence result due to Wüstholz.

(IV) Logarithmic derivative turns partial gamma product into essentially partial zeta. In [AT1], such gadgets were used to provide algebraic points on s-th tensor power (these points are torsion precisely when s is multiple of $q-1$) of Carlitz module whose logarithm is essentially $\zeta(s)$. In other words, $\zeta(s)$ were realized as periods of the extension of this tensor power by the trivial module. Corresponding results connecting v-adic zeta values, for v a prime of A, to v-adic logarithms at related algebraic points were also proved in [AT1].

(V) Multizeta values were obtained by using the same gadgets and now making iterated extensions of s_i-th tensor powers whose periods are $\zeta(s_1, \ldots, s_k)$. These extensions realizing the non-degenerate multizeta are defined over $F[t]$, but the extensions realizing the degenerate ones are defined over $F[t^{1/q^w}]$ for some w's.

Classically, multizeta values arise in DeRham-Betti comparison for mixed tate motives arising in the study of the algebraic fundamental group of the projective line over $\mathbb{Q}$ minus three points. This study arose from Belyi's theorem giving injection of absolute Galois group of $\mathbb{Q}$ into this group. In our case, any curve can be mapped to projective line ramified only at infinity, using Abhyankar's map and possibility of wild ramification, and we do not have fundamental group connection similarly, but we do have multizetas as well as analogs of Ihara power series [AT2] occurring in étale picture of this fundamental group study. At the zeta level, the extensions mentioned above are given by analogs of Deligne-Soulé cocycles [AT2] which come up in formula for coefficients of logarithmic derivative of the Ihara power series in analogous way to the classical case.

4 Period Relations are Motivic

We remark that unfortunately we have to switch roles of t and T in [ABP] below to stay consistent with the notation of this paper.

Let $C_\infty\{T\}$ be the ring of power series over C_∞ convergent in closed unit disc.

Theorem 4.1 ([ABP]) *Consider* $\Phi = \Phi(T) \in Mat_{r\times r}(\overline{K}[T])$ *such that* $\det \Phi$ *is a polynomial in* T *vanishing (if at all) only at* $T = t$ *and* $\psi = \psi(T) \in Mat_{r\times 1}(C_\infty\{T\})$ *satisfying* $\psi^{(-1)} = \Phi\psi$.

If $\rho\psi(t) = 0$ *for* $\rho \in Mat_{1\times r}(\overline{K})$*, then there is* $P = P(T) \in Mat_{1\times r}(\overline{K}[T])$ *such that* $P(t) = \rho$ *and* $P\psi = 0$.

Thus $\overline{K}$-linear relations between the periods are explained by $\overline{K}[T]$-level linear relations (which in our set-up are the motivic relations and thus 'algebraic relations between periods are motivic', as analog of Grothendieck's conjecture for motives he defined.). In terms of special functions of our interest, this makes the vague hope that 'there are no accidental relations and the relations between special values come from known functional equations' precise and proves it.

Motives are thus linear algebra objects and have tensor products via which algebraic relations between periods i.e., linear relations between powers and monomials in them reduce to linear relations between periods (of some other motives). In this sense, the new [ABP] criterion below is similar to Wüstholz type sub-t-module theorem proved by Jing Yu, as remarked in [ABP, 3.1.4]. The great novelty is of course the direct simple proof as well as its perfect adaptation to the motivic set-up here.

Using this, the following theorem goes one step further in the quantitative direction:

Theorem 4.2 ([P]) *If M is uniformizable t-motive over $\overline{K}$, then transcendence degree of the field extension of $\overline{K}$ generated by its periods is the dimension of the motivic Galois group of M (i.e., the group corresponding to Tannakian category generated by M).*

Papanikolas further gives description of the motivic Galois group as 'difference equations Galois group' for the '**Frobenius-difference equation**' $\Psi^{(-1)} = \Phi\Psi$. This allows the calculation of the dimensions and proofs of theorems in section two.

Here is the proof from [ABP, Theorem 7] for the simplest $r = 1$ case. Without loss of generality we can assume $\rho \neq 0$, so that we have to conclude ψ vanishes identically from $\psi(t) = 0$. But for $v \geq 0$, we have

$$(\psi(t^{q^{-v}}))^{q^{-1}} = \psi^{(-1)}(t^{q^{-v-1}}) = \Phi(t^{q^{-v-1}})\psi(t^{q^{-v-1}}),$$

But $\Phi(t^{q^{-v-1}}) \neq 0$, and hence ψ has infinitely many zeros $t^{q^{-v}}$ in the disc $|T| \leq |t|$ and thus vanishes identically.

The general case makes similar beautiful use of functional equation of the hypothesis by manipulating suitable auxiliary function to vanish identically (so as to recover P). This is done by applying the standard transcendence theory tools such as Siegel lemma (to solve system of linear equations thus arising) and Schwarz-Jensen, Liouville inequalities (to estimate bounds needed). We refer the reader to the clean treatment in [ABP].

Finally we mention that we have restricted to genus zero case of $F[t]$, but Drinfeld modules theory and many of the special functions generalize (see [T] for definitions and some simple examples where the special values can be handled) to any curve over finite field. Anderson has recently generalized [A] to this setting a lot of the ingredients in associating motives to the special values.

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University of Arizona, Tucson
E-mail: thakur@math.arizona.edu