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Shuffle Relations for Function Field Multizeta Values

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Despite the failure of naive analogs of the sum shuffle or the integral shuffle relations, we prove the existence of "shuffle" relations for the multizeta values (for a general A, with a rational place at infinity) introduced by the author [9] in the function field context. This makes the $\mathbb{F}_p$-span of the multizeta values into an algebra. We effectively determine and prove all the $\mathbb{F}_p$-coefficient identities (but not the $\mathbb{F}_p(t)$-coefficient identities).

1 Introduction

Multizeta values introduced and studied originally by Euler have been pursued recently again with renewed interest because of their emergence in studies in mathematics and mathematical physics connecting diverse viewpoints. See, for example, introduction to [11] and references there. This paper is sequel to [11].

The author defined and studied two types of multizeta [9, Section 5.10] for function fields, one complex valued (generalizing the Artin–Weil zeta function) and the other with values in Laurent series ring over finite fields (generalizing the Carlitz zeta values). (For general background on function field arithmetic, we refer to [5, 9].) For the $\mathbb{F}_q[t]$ case, the first type was completely evaluated in [9] (see [8] for more detailed study in the higher genus case). For the second type, the failure of sum and integral shuffle identities was noted, but different combinatorially involved identities were established or conjectured in [9, 11] as well as in the Master's thesis work [6, 7] of Jose Alejandro

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Lara Rodriguez done at the University of Arizona. Also, period interpretation for these multizeta values was given in [1] in terms of explicit iterated extensions of the Carlitz–Tate t-motives.

In contrast to the classical division between the convergent versus the divergent (normalized) values, all the values are convergent in our case. In place of the sum or the integral shuffle relations, we have different kinds of relations: the shuffle type relations with $\mathbb{F}_p$-coefficients and the relations with $\mathbb{F}_p(t)$-coefficients. (Classically, of course, there is no such distinction, the rational number field being the prime field in that case).

In this paper, we show the existence of shuffle type relations proving that the product of multizeta values can also be expressed as a sum of some multizeta values, so that the $\mathbb{F}_p$-span of all multizeta values is an algebra. While [6, 7, 11] conjectured and proved many such interesting relations (in the special case $A = \mathbb{F}_q[t]$), which are combinatorially quite involved to describe unlike the classical case, here we prove the existence directly (for general A, defined below) rather than proving those conjectures.

2 Multiple Zeta Values for Function Fields

2.1 Notation

$$
\begin{aligned}
\mathbb{Z} &= \{\text{integers}\}\\
\mathbb{Z}+ &= \{\text{positive integers}\}\\
q &= \text{a power of a prime } p\\
\mathbb{F}_q &= \text{a finite field of } q \text{ elements}\\
K &= \text{a function field of one variable with field of constants } \mathbb{F}_q\\
\infty &= \text{a place of } K \text{ of degree one}\\
K_\infty &= \mathbb{F}_q((1/t)) = \text{the completion of } K \text{ at } \infty\\
\mathbb{C}_\infty &= \text{the completion of an algebraic closure of } K_\infty\\
A &= \text{the ring of elements of } K \text{ with no poles outside } \infty\\
A+ &= \text{monics in } A\text{, for some fixed sign function}\\
A_d &= \{\text{elements of } A \text{ of degree } d\}\\
A_{<d} &= \{\text{elements of } A \text{ of degree less than } d\}\\
A_d+ &= A_d \cap A+\\
A_{<d}+ &= A_{<d} \cap A+\\
[n] &= t^{q^n} - t\\
\text{“even”} &= \text{multiple of } q-1
\end{aligned}
$$

The simplest case is when $A = \mathbb{F}_q[t]$ and $K = \mathbb{F}_q(t)$, with the usual notions of infinite place, degree and sign (in t).

2.2 Definition of multiple zeta values

First, we define the power sums. Given $s \in \mathbb{Z}+$ and $d \geq 0$, put

$$S_d(s) = \sum_{a \in A_d+} \frac{1}{a^s} \in K,$$

and given integers $s_i \in \mathbb{Z}_+$ and $d \geq 0$ put

$$S_d(s_1, \cdots, s_r) = S_d(s_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2) \cdots S_{d_r}(s_r) \in K.$$

For $s_i \in \mathbb{Z}+$, we define multizeta value $\zeta(s_1, \cdots, s_r)$ following [9, Section 5.10] (where it was denoted by ζ_d to stress the role of the degree) by using the partial order on $A+$ given by the degree, and grouping the terms according to it:

$$\zeta(s_1, \cdots, s_r) = \sum_{d_1 > \cdots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty,$$

where the second sum is over $a_i \in A_{d_i}+$ satisfying the conditions as in the first sum.

We say that this multizeta value (or rather the tuple $(s_1, \cdots, s_r)$) has depth r and weight $\sum s_i$. Note, we do not need $s_1 > 1$ condition for convergence as in the classical case. This definition generalizes, in one way, the $r = 1$ case corresponding to the Carlitz zeta values [4, 5, 9]. For interpolations and analytic theory, we refer to [5, 9].

3 Relations Between Multizeta Values

First, we consider

$$S_d(a)S_d(b) - S_d(a+b) = \sum f_i S_d(a_i, a+b-a_i), \tag{$*$}$$

with $f_i \in \mathbb{F}_p$.

Theorem 1. Let $A = \mathbb{F}_q[t]$. Given $a, b \in \mathbb{Z}+$, there are $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}+$, so that (*) holds for $d = 1$. □

Proof. Since $[1] = t^q - t$ is the product of all the monic polynomials over $\mathbb{F}_q$ of degree one, we see that $[1]^a S_1(a)$ is in A, in fact, in $\mathbb{F}_p[t]$. Since it is invariant with respect to the automorphisms $t \to t+\theta$, $\theta \in \mathbb{F}_q$ of A, we see that, in fact, it is a polynomial (with coefficients in $\mathbb{F}_q$) in $[1]$ of degree less than a. More directly, $[1]^a S_1(a) = \sum \prod (t+\theta)^a$, where the sum is over $\mu \in \mathbb{F}_q$ and the product is over $\theta \neq \mu$. Hence, only the $\mu = 0$ term in the sum is not divisible by t, and hence the constant term of the polynomial above is $(\prod \theta)^a = (-1)^a$.

In other words, $S_1(a) = \sum_{i=1}^{a} f_i/[1]^i$, $f_i \in \mathbb{F}_p$ and $f_a = (-1)^a$.

In fact, by specializing [11, 3.3] to $d = 1$, we see more explicitly that

$$S_1(k+1) = \frac{(-1)^{k+1}}{[1]^{k+1}} \left(1 + \sum_{k_1=1}^{\lfloor k/q \rfloor} \binom{k - k_1(q-1)}{k_1} (-1)^{k_1} [1]^{k_1(q-1)} \right).$$

So, $S_1(a)S_1(b) - S_1(a+b)$ is again a $\mathbb{F}_p$- linear combination of $1/[1]^i$'s with $i < a + b$. Now we proceed by induction on the largest i in such sums. We can keep on lowering such i (till the sum is vacuous) by subtracting $f_i S_1(a_i)$, with a_i being the largest such i in such a sum, for appropriate $f_i \in \mathbb{F}_p$. ∎

Example: An example should make this more transparent. Let $q = 3$, $A = \mathbb{F}_q[t]$, $a = 4$, and $b = 5$. Then by direct calculation (or by applying [11, 3.3.1]), we see that

$$\begin{aligned} S_1(4)S_1(5) - S_1(9) &= \left(\frac{1}{[1]^4} - \frac{1}{[1]^2}\right)\left(-\frac{1}{[1]^5} - \frac{1}{[1]^3}\right) - \frac{1}{[1]^9} \\ &= \frac{1}{[1]^5} \\ &= -S_1(5) - \frac{1}{[1]^3} \\ &= -S_1(5) + S_1(3). \end{aligned}$$

Theorem 2. Fix q. If (*) holds for some $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}+$ for $d = 1$ and $A = \mathbb{F}_q[t]$, then (*) holds for all $d \geq 0$ and for all A (corresponding to the given q). In this case, we have the shuffle relation

$$\zeta(a)\zeta(b) - \zeta(a+b) - \zeta(a,b) - \zeta(b,a) = \sum f_i \zeta(a_i, a+b-a_i). \qquad (**)$$

□

Proof. Fix a general A. Consider $n, n' \in A_{d}+$, $m, m' \in A_{<d}+$. Define

$$S_{n,m} := \{(n+\theta m, n+\mu m) : \theta, \mu \in \mathbb{F}_q, \theta \neq \mu\},$$

$$S'_{n,m} := \{n+\theta m, m) : \theta \in \mathbb{F}_q\},$$

$$n \sim_m n' \leftrightarrow n' = n+\theta m, \text{ for some } \theta \in \mathbb{F}_q.$$

It then follows immediately that $S_{n,m}$ equals $S_{n',m'}$ if and only if $S'_{n,m}$ equals $S'_{n',m'}$ if and only if $m = m'$ and $n' \sim_m n$. Otherwise, the two sets S's, or two S''s, respectively, are disjoint. Further, as n's run over $\sim_m$ equivalence classes, $S_{n.m}$'s partition $\{(n_1, n_2) : n_i \in A_d+, n_1 \neq n_2\}$, while $S'_{n.m}$'s partition $\{(n_1, m_1) : n_1 \in A_d+, m_1 \in A_{<d}+\}$.

Fix $d > 0$. For such n and m writing $t = n/m$, we have

$$\begin{aligned}
\sum_{(n_1,n_2)\in S_{n,m}} \frac{1}{n_1^a n_2^b} &= \sum_{\theta\neq\mu\in\mathbb{F}_q} \frac{1}{(n+\theta m)^a (n+\mu m)^b} \\
&= \frac{1}{m^{a+b}} \sum_{\theta\neq\mu\in\mathbb{F}_q} \frac{1}{(t+\theta)^a (t+\mu)^b} \\
&= \frac{1}{m^{a+b}} \sum f_i \sum_{\eta\in\mathbb{F}_q} \frac{1}{(t+\eta)^{a_i}} \\
&= \sum f_i \sum \frac{1}{(mt+\eta m)^{a_i} m^{a+b-a_i}} \\
&= \sum f_i \sum \frac{1}{(n+\eta m)^{a_i} m^{a+b-a_i}} \\
&= \sum f_i \sum_{(x,y)\in S'_{n,m}} \frac{1}{x^{a_i} y^{a+b-a_i}}.
\end{aligned}$$

(Here, the third equality results from the hypothesis and the rest follow from the substitution, definitions and algebraic manipulations). Adding these equations over all m and with all n in the $\sim_m$ equivalence class, we get the claimed result (*). Then, once we note that for $d = 0$ it is trivially true, (**) follows by adding (*) over all $d \geq 0$. ■

Remarks (1) Hence, we see that the product of two Carlitz zeta values can be expressed as a sum of some multizeta values. The exact form of such expressions is proved in many cases and conjectured fully for $q = 2$ in [11] by complicated recursive recipe. The recursion step conjecture part (but not the initial conditions part) of this recipe has been generalized to a conjecture for q a prime in [7, 6] with a lot of evidence proving many special instances, by a soliton method mentioned in [11]. Our theorem makes it much simpler (and applies to much more general situation of any A's) to have mechanical proofs: Now we just have to verify directly the $d = 1$, $\mathbb{F}_q[t]$ case, which is a finite direct computation, to get a proof of the corresponding multizeta identity.

(2) As explained in [11, Section 5], the identities conjectured (and proved in some cases) were discovered by the heuristic passage, "motivic" implies "for all d" implies "for $d = 1$", whose calculation gave the guesses. We just reverse the logic to prove the identities!

(3) The theorem shows that these $\mathbb{F}_p$-shuffle identities are in some sense "universal" for a given q. Just for fun, we mention that $\zeta(1)\zeta(1) = \zeta(2) + 2\zeta(1, 1)$ is even more universal, even holding for all q. This was already noticed for $\mathbb{F}_q[t]$ case in [11, 2.1, 2.3] and also follows in a different way, from what we proved above.

(4) For any A, such identities working for all d are unique, exactly given by the theorem above, because if x is the smallest degree monic element of A, then identity working in its degree is just the degree one identity for $\mathbb{F}_q[x]$. Also, by [11, 5.4], there is unique identity given (a, b).

(5) We refer to [7, 6] for the full description of the conjectural recipe giving f_i and a_i, given (a, b), but only mention that for a fixed a it is recursive in b of recursion length $(q-1)p^m$, where m is the smallest integer such that $a \le p^m$, and at each recursive step one adds $t_a = \prod(p-j)^{\mu_j}$ new multizeta terms, where μ_j is the number of j's in the base p expansion of $a-1$.

Now, we address the higher depth situation.

Theorem 3. (i) $S_d(a_1, \cdots, a_r)S_d(b_1, \cdots, b_k)$ can be expressed as $\sum f_i S_d(c_{i1}, \cdots, c_{im_i})$, with $f_i \in \mathbb{F}_p$, c_{ij}'s, and m_i's being independent of d, and with $\sum a_i + \sum b_j = \sum_j c_{ij}$ and $m_i \le r + k$.

(ii) For any A, the product of multizeta values can be expressed as a sum of some multizeta values, such an expression preserving total weight and keeping depth filtration.

(iii) In particular, the $\mathbb{F}_p$-span of all the multizeta values is an algebra. □

Proof. We prove (i) by induction on the depth $D = r + k$. By the previous theorem, it is true for the smallest depth $D = 2$. Let $D > 2$. Interchanging r and k, if necessary, we can assume that $r > 1$.

Temporarily, for a vector denoted such as V, we write $V=(v_1, V')$. Then, with $A=(a_1,\cdots,a_r)$ and $B=(b_1,\cdots,b_k)$, we have

$$\begin{aligned}
S_d(A)S_d(B) &= S_d(a_1)S_d(B)\sum_{d_1<d} S_{d_1}(A') \\
&= \left(\sum_i g_i S_d(X_i)\right)\sum_{d_1<d} S_{d_1}(A') \\
&= \left(\sum_i g_i S_d(X_{i1})\sum_{d_2<d} S_{d_2}(X_i')\right)\sum_{d_1<d} S_{d_1}(A') \\
&= \sum_i g_i S_d(X_{i1})\left(\sum_{d_2<d} S_{d_2}(X_i', A') + \sum_{d_1<d} S_{d_1}(A', X_i') + \sum_{d_2<d} S_{d_2}(X_i')S_{d_2}(A')\right) \\
&= \sum_i g_i S_d(X_{i1})\left(\sum_{d_2} + \sum_{d_1} + \sum_{d_2}\sum_j h_j S_{d_2}(Y_j)\right) \\
&= \sum_i g_i\left(S_d(X_{i1}, X_i', A') + S_d(X_{i1}, A', X_i') + \sum_j h_j S_d(X_{i1}, Y_j)\right),
\end{aligned}$$

where $h_j, g_i \in \mathbb{F}_p$, the second, and the fifth equalities follow by induction hypothesis, whereas the first, third, and sixth follow from the definitions and the fourth follows by shuffle on d's. This proves the claim, with straight-forward checking of the weight and the depth claim (by tracing through the equalities and induction) being left to the reader.

(ii) follows from (i) by summing over d and (iii) follows from (ii). ■

Remarks (1) We observe that our proofs give, for a product of multizeta values, an effective procedure of expressing it as a sum of multizeta values, with a proof. It would still be desirable to have a good description of all the identities at once.

(2) While in rational number field case, there is a good description of all the identities we expect between multizeta values, we lack such a good description. Also, we know that all the identities are "motivic" (with respect to Anderson's t-motives, see references and terminology in [11, 5.1]), while in the rational number field case, it is only expected (e.g., from the Grothendieck period conjecture).

(3) We can generalize the identities between the quantity parts (without multizeta applications) of our theorems even further from A we have considered to an infinite dimensional vector space over $\mathbb{F}_q$, graded by "degree", and included in a field.

(4) For simplicity, we assumed that the degree of the infinite place is one, so that signs are in $\mathbb{F}_q^*$. When the degree is higher, the definition of multizeta should be modified the same way as in the zeta case, as explained in [9, p. 156].

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