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An Alternate Approach to Solitons for $\mathbb{F}_q[t]$

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We describe an alternate approach to the solitons for $\mathbb{F}_q[t]$ introduced by Anderson. The product of the monic polynomials in $\mathbb{F}_q[t]$ of degree i in a given congruence class can be expressed in a compact form by the solitons "for all i at once." They also link the arithmetic of $\mathbb{F}_q[t]$ gamma values at fractions to the analogues of Jacobians of Fermat curves in the setting of higher dimensional generalizations of Drinfeld modules. We will calculate explicit equations, formulae and bound the support of the divisors of the solitons. We also show how the generalization of this approach differs from the one suggested by Anderson's approach. © 1999 Academic Press

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1. BACKGROUND

Carlitz [C1] showed that the product of the monic polynomials in $\mathbb{F}_q[t]$ of degree i is given by

$$D_i = (t^{q^i} - t)(t^{q^i} - t^q) \cdots (t^{q^i} - t^{q^{i-1}}).$$

He also defined an $\mathbb{F}_q[t]$ analogue of the factorial and showed that D_i is its value at q^i. For more on the story, including interpolations to a gamma function, connections with periods and analogues of Gauss sums, we refer to [T4]. We can ask for a formula for the product of the monic polynomials in $\mathbb{F}_q[t]$ of degree i in a given congruence class and this is essentially the same as asking for a formula for the term in the definition of the gamma function (see (8) below) at a fraction. The latter turns out to be a nice compact formula (see (12), (13) for examples) in a sense explained below and uses the analogue of the cyclotomic theory developed by Carlitz [C2]. In special cases, such formulae were obtained in [T3] Section 9.

In a very original paper [A2], Anderson introduced a concept of a "soliton" in the arithmetic of function fields and used it with the related

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tools of the deformation theory to prove a two dimensional analogue of the Stickelberger theorem and to get algebraic Hecke characters. Then the solitons were used by Sinha in his University of Minnesota thesis to relate the periods of corresponding t-motives (analogues of Jacobians of Fermat curves) to the gamma values at fractions. (See [S1, S2], and also [A3, A4, A5] for more on function field (or p-adic) solitons.)

Our modest aim in this paper is to indicate another approach to the solitons. Instead of giving a direct rational construction (reviewed in Section 3) of soliton in an extension constructed using cyclotomic theory, we will give explicit algebraic equation (assuming just that it is algebraic: the fact proved by Anderson) for soliton and then identify the extension it generates. We have succeeded in identifying the extension only under a hypothesis. We will describe the Galois action and show how we can read-off some important information about the divisors of the solitons, calculated in the main result (Theorem 2) of [A2] (and more precisely in Sinha's thesis). We will supplement [A2], rather than give full details from scratch, as Anderson has already given an elegant complete treatment. Our main result (Theorem 3) was obtained in 1991, after we had been told by Anderson of the existence of solitons.

Finally, we will indicate how the natural generalization of this approach provides a different kind of interesting information than that provided by generalization of solitons.

In the appendix, we sketch how solitons are used to relate gamma values to periods of some t-motives and raise some questions about the nature of some special values.

There are various (essentially equivalent) ways (see [M, D1, D2, A1–A5]) to approach this subject: geometric approach of Shtukas and theta functions, which Anderson follows, algebraic approach of Drinfeld modules, which is described here, and analytic approach of lattices, exponentials and tau functions, which also can be traced in both.

We begin by describing the relevant facts about Drinfeld modules, gamma functions, solitons and their interrelations. For motivation and various analogies with the number field case, see [D1, G1–2, H1–3, T1–6]. For first three sections, we only use the simplest Drinfeld module, namely the Carlitz module for $\mathbb{F}_q[t]$ introduced below.

Notation.

q: a power of a prime p.

K: a function field of one variable over its field of constants $\mathbb{F}_q$.

∞: a place of K of degree one (there should be no confusion with the usual usage of ∞).

A: the ring of elements of K integral outside ∞.

K_∞: the completion of K at ∞.

Ω: the completion of an algebraic closure of K_∞.

h: the class number of K.

g: the genus of K.

A choice of an uniformizer u at ∞ allows us to express $z \in K_\infty^\times$ uniquely as $z = \operatorname{sgn}(z) \times \bar{z} \times u^{\deg(z)}$, where the sign $\operatorname{sgn}(z) \in \mathbb{F}_q^\times$, $\bar{z}$ is a one unit at ∞ and the degree $\deg(z)$ is an integer. We make a choice of such a sgn function and call monic the elements of $\operatorname{sgn} = 1$. Let A_+ be the set of monic elements of A. Let A_i (respectively A_{i+}) denote the set of elements (respectively monic elements) of A of degree i. By convention, we also assign degree $-\infty$ to 0. Let H be the maximal abelian unramified extension of K split completely at ∞ and let B be the integral closure of A in H.

Let $B\{F\}$ denote the noncommutative ring generated by elements of B and by a symbol F, satisfying the commutation relation $Fb = b^q F$ for all $b \in B$. We identify the elements of $B\{F\}$ with additive polynomials as usual, namely $\sum b_i F^i(z) := \sum b_i z^{q^i}$. Note that the multiplication in $B\{F\}$ corresponds to composition of the corresponding additive polynomials.

Definition. By a Drinfeld A-module ρ (in fact "sgn-normalized of rank one and of generic characteristic over B", but we will drop these words), we will mean an injective homomorphism $\rho: A \to B\{F\}$ (we write image of a by ρ_a) such that for nonzero $a \in A$, (1) the degree of ρ_a as a polynomial in F is $\deg(a)$, (2) the coefficient of F^0 in ρ_a is a and (3) the top degree coefficient in ρ_a is $\operatorname{sgn}(a)$.

Example. Let $A = \mathbb{F}_q[t]$ and choose sgn so that $\operatorname{sgn}(t) = 1$. Then t being a generator of A, $\rho_t := t + F$ defines a ρ. One has, e.g., $\rho_{t^2} = t^2 + (t + t^q) F + F^2$. This ρ, also denoted by C, is the Carlitz module.

There are h such Drinfeld A-modules and they can be obtained from one another by $\operatorname{Gal}(H/K)$ conjugation. We assume that such a ρ is given and drop it from notation, when convenient.

Associate to ρ, we have the (Drinfeld) exponential $e(z)$, defined to be the entire function (i.e., function from $\bar{K}_\infty$ to $\bar{K}_\infty$ given by an everywhere convergent power series) with the linear term z and satisfying

$$e(az) = \rho_a(e(z)), \qquad a \in A. \tag{1}$$

We define d_i by the expression $e(z) = \sum z^{q^i}/d_i$.

The kernel Λ of the exponential $e(z)$ is a A-lattice of rank one. We can write $\Lambda = \tilde{\pi}_I I$ for $\tilde{\pi}_I \in \Omega$ and some ideal I of A. For simplicity we assume that ρ corresponds to the principal ideal class (i.e., the equality above holds for some principal ideal), we put $\tilde{\pi} = \tilde{\pi}_A$. Note that $\tilde{\pi}$ is then determined up to multiplication by a nonzero element of $\mathbb{F}_q$. It can be considered as an analogue of $2\pi i$.

It is easy to see that

$$e(z) = z \prod_{\lambda \in A - \{0\}} (1 - z/\lambda). \tag{2}$$

Let log be the (multivalued) inverse function of e and let $l(z) = \sum z^{q^i}/l_i$ be the power series, representing the branch of log vanishing at zero, convergent in some neighborhood of zero. We have

$$a \log(z) = \log(\rho_a(z)). \tag{3}$$

We also define $\left\{ {x \atop q^i} \right\} \in H(x)$ by

$$e(xl(z)) = \sum_{i=0}^{\infty} \left\{ {x \atop q^i} \right\} z^{q^i}. \tag{4}$$

This is a $\mathbb{F}_q$-linear polynomial in x of degree q^i. Let

$$e_i(x) =: \psi_i(x) =: \prod_{\substack{a \in A \\ \deg(a) < i}} (x - a) \in A[x], \tag{5}$$

$$D_i := \prod_{a \in A_{i+}} a, \tag{6}$$

$$\binom{x}{q^i} := \frac{e_i(x)}{D_i}. \tag{7}$$

Theorem 1 ([C1, T5]). *For $A = \mathbb{F}_q[t]$ and ρ as in our example,*

$$\left\{ {x \atop q^i} \right\} = \binom{x}{q^i}, \qquad D_i = d_i.$$

The two quantities in the theorem are analogues of binomial coefficients and factorials at q^i respectively. For general A, the equalities do not hold, so we get two candidates for the analogues. For more on the story, see [T1, T4–T6].

Remark. We digress to record some interesting facts: For $A = \mathbb{F}_q[t]$, we have $D_i = \prod_{0 \leqslant j < i} (t^{q^i} - t^{q^j})$. With $[i] := t^{q^i} - t$, we have $[i] - [j] = t^{q^i} - t^{q^j}$. Comparing the building blocks t^{q^i}'s or $[i]$'s with integers i naively (see 1.2 of [T3]), we see an analogy with factorial of i being $i(i-1) \cdots (i-(i-1))$. In fact, in a very nice paper [B], Manjul Bhargava associates a factorial to a subset X of a Dedekind ring R. For $X = R = \mathbb{F}_q[t]$, one recovers the Carlitz factorial, whose value at q^i is D_i. But if $X = \{t^{q^j}: j \geqslant 0\}$ or $X = \{[j]: j \geqslant 0\}$ and $R = \mathbb{F}_q[t]$, then a simple calculation shows that D_i is in fact the value at i of the corresponding factorial.

The connection with the gamma function comes via

$$\Gamma(z) := \frac{1}{z} \prod_{n \in A+} \left(1 + \frac{z}{n}\right)^{-1} = \frac{1}{z} \prod_{i=0}^{\infty} \left(1 + \left(\frac{z}{q^i}\right)\right)^{-1}, \tag{8}$$

where the ith term of the product in the last expression is, in fact, the product of the terms corresponding to n's of degree i in the previous expression.

For various analogies and results on this gamma function, see [G1, G2, T3, T4].

For the rest of this paper except for the last section, we let $A = \mathbb{F}_q[t]$ and $\rho = C$ be the Carlitz module described above.

For any $a \in A$, let d_a denote its degree in t. There should be no confusion with the coefficients of $e(z)$. For $f \in A_+$, $\zeta_f := e(\tilde{\pi}/f)$ generates the fth cyclotomic extension of K. For an irreducible $v \in A_+$ not dividing f, the Artin symbol $\sigma_v \in G := \mathrm{Gal}(K(\zeta_f)/K)$ acts by $\sigma_v(e(\tilde{\pi}a/f)) = e(\tilde{\pi}av/f)$, for $a \in A$. The Artin symbol gives isomorphism $a \mapsto \sigma_a$: $(A/f)^* \cong G$.

Let X be a smooth projective curve over $\mathbb{F}_q$ with its function field isomorphic to $K(\zeta_f)$, i.e., the fth cyclotomic cover of the projective line over $\mathbb{F}_q$. Let U and V be the complements in X of ∞ and the support of f respectively. For $a \in (A/f)^*$, let $[a]$ denote the automorphism of X given by σ_a and let $Z_a^i := graph([a]\ Frob^i: X \to X)$. If we take t and T to be two independent variables, with the obvious abuse of notation,

$$L_f := \mathbb{F}_q(t, \zeta_{f(t)}, T, \zeta_{f(T)})$$

can be thought of as the function field of the product of two copies of X over $\mathbb{F}_q$. If f is understood, we put $\zeta = \zeta_{f(t)}$ and $Z = \zeta_{f(T)}$. Also, $\sigma_a(Z)$ will mean $\sigma_{a(T)}(Z)$.

Now we describe Anderson's main result (Theorem 2 of [A2]).

THEOREM 2. *There exists a unique rational function $\phi \in L_f$ which is regular on $V \times U$ such that for all $a \in A$ prime to f, with $d_a < d_f$ and for all positive integers N, we have*

$$1 - \phi(t, \zeta, T, \sigma_a(Z))|_{T = t^{q^N}, Z = \zeta^{q^N}} = \prod_{n \in A_+, d_n = N-1} \left(1 + \frac{a}{fn}\right).$$

Also, we have

$$\text{divisor}(\phi|_{V\times V})$$
$$= \infty \times V - V \times \infty + \sum_{a\in A,\,(a,f)=1,\,d_a<d_f} \sum_{i=0}^{d_f-d_a-2} Z_a^i \cap (V\times V),$$
$$\text{divisor}((1-\phi)|_{V\times X})$$
$$= -V\times\infty + \sum_{a\in A_+,\,(a,f)=1,\,d_a<d_f} Z_a^{d_f-d_a-1} \cap (V\times X).$$

The function ϕ, called a "soliton" by Anderson, is thus a function on the cyclotomic curve cross itself and specializes at the graph of the Nth power of the Frobenius to essentially the Nth term in the product (8) representing the gamma value at a fraction. Specializing the solitons at appropriate geometric points, Anderson has proved two dimensional version of Stickelberger's theorem. (See [A2].) For connection with the arithmetic of zeta values and theta functions, see [A3, T5]. The reason for the name soliton is that the way it arises in the theory of Drinfeld modules or Shtukas, when dealing with the projective line with some points identified, is analogous to the way the soliton solutions occur in Krichever's theory of algebro-geometric solutions of differential equations, as explained in [M, pp. 130–145, A2, A3].

2. EQUATIONS AND DIVISORS OF SOLITONS

Now let us get an algebraic equation over $L_1 = \mathbb{F}_q(t, T)$, for a soliton.

Let us write $d := d_f$. Let $C_{f(t)} = \sum_{i=0}^{d} f_i(t)\, F^i$. The relation $C_t C_f = C_f C_t$ gives the recursion formula

$$f_0(t) = f(t), \qquad f_i(t) = \frac{f_{i-1}^q(t) - f_{i-1}(t)}{t^{q^i} - t}.$$

For $i \geqslant 0$, we define ϕ_i recursively via

$$\phi_0 := \phi, \qquad \phi_i := \frac{\phi_{i-1}^q - \phi_{i-1}}{T^{q^{i-1}} - t}.$$

We define $c_i \in L_1$ via

$$\sum_{i=0}^{d} c_i \phi^{q^i} = \sum_{i=0}^{d} f_{d-i} \phi_i^{q^{d-i}}.$$

Then we have

THEOREM 3. *The soliton ϕ satisfies*

$$\sum_{i=0}^{d} c_i \phi^{q^i} = 0. \tag{$*$}$$

Proof. Now ϕ is characterized by its specializations at the graph of Nth power of Frobenius being

$$\left\{ \begin{matrix} a/f \\ q^{N-1} \end{matrix} \right\},$$

by (8) and Theorems 1, 2. We will see below that ϕ_i is characterized by its specializations at the graph of Nth power of Frobenius being

$$\left\{ \begin{matrix} a/f \\ q^{N+i-1} \end{matrix} \right\}.$$

Let $\Phi_{a/f} := \sum_{i=0}^{\infty} \left\{ \begin{smallmatrix} a/f \\ q^i \end{smallmatrix} \right\} F^i$. Then by (4), we have

$$\Phi_{a/f}(z) = \sum \left\{ \begin{matrix} a/f \\ q^i \end{matrix} \right\} z^{q^i} = e(al(z)/f).$$

By (1) and (3), $\rho_f(e(al(z)/f)) = \rho_a(z)$, for small z, so $\rho_f \Phi_{a/f} = \rho_a$. Equating coefficients of F^{d+k}, we get

$$\sum f_{d-i} \left\{ \begin{matrix} a/f \\ q^{i+k} \end{matrix} \right\}^{q^{d-i}} = 0$$

for each non-negative k. These infinitely many equations satisfied by the specializations $\left\{ \begin{smallmatrix} a/f \\ q^i \end{smallmatrix} \right\}$'s of ϕ show that (note that the union of graphs of Frobenius powers is Zariski dense) ϕ satisfies $(*)$, except that we have to justify that the ϕ_i characterized by its specializations as above is given by the recursion defined above.

Now even though $B\{F\}$ is a noncommutative ring, Drinfeld A-module provides a way to get its commutative subring isomorphic to A: $\rho_{a_1} \rho_{a_2} = \rho_{a_2} \rho_{a_1}$ for all $a_1, a_2 \in A$. Hence for $c \in A$,

$$\rho_f \rho_c \Phi_{a/f} = \rho_c \rho_f \Phi_{a/f} = \rho_c \rho_a = \rho_a \rho_c = \rho_f \Phi_{a/f} \rho_c,$$

which by cancellation of ρ_f gives

$$\rho_c \Phi_{a/f} = \Phi_{a/f} \rho_c, \qquad c \in A. \tag{9}$$

Applying this to $c = t$ and comparing coefficients of F^j, we get

$$t \begin{Bmatrix} a/f \\ q^j \end{Bmatrix} + \begin{Bmatrix} a/f \\ q^{j-1} \end{Bmatrix}^q = t^{q^j} \begin{Bmatrix} a/f \\ q^j \end{Bmatrix} + \begin{Bmatrix} a/f \\ q^{j-1} \end{Bmatrix}.$$

(We note that this equation is equivalent to the functional Eq. (37) of [A2] and also that it gives a simple recursion formula for the terms in the product (8).)

If now one puts $j = N + i - 1$, then since $t^{q^j} = (t^{q^N})^{q^{i-1}}$ the characterization of ϕ_i stated in the beginning of the proof gets reconciled with the recursive definition for it.

Hence the theorem follows. ∎

Remarks. (a) In fact, what we have shown is that any element ϕ in an finite extension of L_1 (without assuming that it belongs specifically to L_f) which specializes on the graph of Nth power of Frobenius to the right hand side of the first equation of the Theorem 2 (we will denote such an element by $\phi_{a/f}$) satisfies $(*)$. Hence different solutions correspond to the different a's (not necessarily prime to f) with $d_a < d_f$. There is always the trivial solution $\phi = 0$ corresponding to $a = 0$. We will call the (non-trivial) solutions solitons. Note that solutions $\phi_{a/f}$, with a not prime to f, have lower "conductor". This will be explained in more detail in the sub-section on irreducible factors.

(b) Note that ϕ_i defined by the recursion above has another description: ϕ_i is obtained by acting on ϕ by ith power of partial Frobenius in the "T variable". This description makes sense for negative i also. Similarly, we define $\hat{\phi}_i$ to be the function defined by acting on ϕ by ith power of partial Frobenius in the "t variable". The, clearly, $\phi^{q^i}_{-i} = \hat{\phi}_i$.

(c) Our proof, in fact, gives infinite hierarchy of equations: $\sum f_{d-i} \phi_{i+m}^{q^{d-i}} = 0$, one for each integer m. It should be noted that m can be negative: One has to just use specializations of ϕ_{i+m} at the graphs of sufficiently high powers of Frobenius. For the same reason, the recursion formula for ϕ_i given before the Theorem is now justified for all integers i, not just the positive integers.

(d) It is known [H1] that the completion of $K(\zeta_f)$ at an infinite place is $\mathbb{F}_q((1/\zeta_t))$. (Note $\zeta_t^{q-1} = -t$.) Hence we can represent ϕ also by a power series in ζ_t, ζ_T over $\mathbb{F}_q$ and then the theorem gives a recursion relation for the coefficients.

(e) In addition to the equation in the Theorem, we can also get, by the same method, algebraic recursion equations: For example, from (9) we get

$$T\phi_{a/ft}^q - t\phi_{a/ft} = (T - t)(\phi_{a/f})_1. \tag{10}$$

EXAMPLES OF THE EQUATIONS. (I) Let $f(t)=t$, then Theorem 3 gives

$$\phi^q + t\phi_1 = \phi^q + t\,\frac{\phi^q-\phi}{T-t} = 0.$$

Throwing out the trivial solution $\phi=0$, we see that $\phi=(t/T)^{1/q-1}=\zeta_t/\zeta_T$. This is nothing but the formula in 9.1.1 in [T3]. In Section 9 of [T3], the additivity of the solitons (as functions of a/f) was established using Moore determinants and this formula was used (essentially the method of partial fractions) to get formulae for (what is now understood as) the solitons when f was a product of distinct linear factors. In the next section, we will give general formulae of Anderson.

(II) Let $f(t)=t^2$, then Theorem 3 gives

$$\phi^{q^2} + (t+t^q)\,\frac{\phi^{q^2}-\phi^q}{(T-t)^q} + t^2\,\frac{\dfrac{\phi^{q^2}-\phi^q}{(T-t)^q} - \dfrac{\phi^q-\phi}{T-t}}{T^q-t} = 0.$$

To get a rational expression, it is easier to use (10), say with $f=t$. With $\phi=(\zeta_t/\zeta_T)\,\psi$, (10) becomes $\psi^q-\psi=1/T-1/t$, hence $\psi=\zeta/\zeta_t - Z/\zeta_T$ gives an obvious solution. So $\phi=(\zeta C_T(Z)-ZC_t(\zeta))/C_T(Z)^2$. Full space of solutions of the displayed equation is then seen to be $\phi\,\mathbb{F}_q+(\zeta_t/\zeta_T)\,\mathbb{F}_q$.

Let us spell out more leisurely what these formulae say. Carlitz [C1] proved that the product of monic polynomials of degree i in t over $\mathbb{F}_q$ is

$$D_i = \prod_{a\in A_{i+}} a = (t^{q^i}-t)(t^{q^i}-t^q)\cdots(t^{q^i}-t^{q^{i-1}}). \tag{11}$$

The first example shows that

$$\Bigg(\prod_{\substack{a\in A_{i+}\\ a\equiv 1 \bmod t}} a\Bigg)\Bigg/(t^{q^{i-1}}D_{i-1}) = 1-\frac{\zeta_t}{\zeta_t^{q^i}} = 1-(-t)^{-(q^i-1)/(q-1)} \tag{12}$$

and the second example shows that

$$\Bigg(\prod_{\substack{a\in A_{i+}\\ a\equiv 1 \bmod t^2}} a\Bigg)\Bigg/((t^2)^{q^{i-2}}D_{i-2})$$
$$= 1-\frac{\zeta_{t^2}(-t)^{q^{i-1}/(q-1)} - \zeta_{t^2}^{q^{i-1}}(-t)^{1/(q-1)}}{(-t)^{2q^{i-1}/(q-1)}} \tag{13}$$

Irreducible Factors and Galois Action. Just as the cyclotomic polynomial x^n-1 factors into factors of degree equal to Euler's Phi function of d, for d dividing n, the interpolation property of the soliton shows that the situation for $(*)$ is analogous, with the function field analogue of Euler's function, with d dividing the polynomial f. For d a divisor of f, let us call the factor made up of all roots $\phi_{a/d}$, with a prime to d primitive factor (for d). We will see that these primitive factors are irreducible.

First let us see that the splitting field of $(*)$ is $L_1(\phi)$; for any primitive (i.e., with a prime to f) root ϕ. In fact, since $\left\{{ab/f \atop q^k}\right\} = \sum a_i(t) \left\{{b/f \atop q^{k-i}}\right\}^{q^i}$, where $C_a = \sum a_i(t)\, F^i$, we see that

$$\phi_{ab/f} = \sum a_i(t) (\phi_{b/f})^{q^i}_{-i} = \sum a_i(t) (\hat{\phi}_{b/f})_i.$$

This gives the Galois action explicitly.

We will prove that for $i \geqslant 0$, $\hat{\phi}_i \in L_1[\phi]$.

First, note that the fact $\hat{\phi}_i^{q^{d-1}} \in L_1[\phi]$ follows immediately by induction on i, from the fact that for non-negative i, $\phi_i \in L_1[\phi]$ and using $m = -1$, -2, ... successively in Remark (c) above.

Now the equations in Remark (c) do not have coefficients in L_1, if $i + m$ is negative, because of the q-power roots involved in the recursion. But taking appropriate q-power powers, we see that the corresponding equations for $\hat{\phi}_i$ have coefficients in L_1 and that these equations show that $\hat{\phi}_i$ are separable over L_1 (eg. take the derivative). Hence $\hat{\phi}_i \in L_1[\phi]$ as claimed.

The displayed equation for Galois action then implies that $\phi_{a/f} \in L_1[\phi]$ so that $L_1(\phi)$ is the splitting field of the equation.

Now, if a is prime to f, we let b to be its inverse modulo f, then the fact that

$$\left\{{ab/f \atop q^k}\right\} = \left\{{1/f \atop q^k}\right\},$$

for large k, implies that the root $\phi_{a/f}$ generates the same field as $\phi_{1/f}$. Hence the primitive factors are irreducible, by a degree count.

From the Galois action described above, we see that the Galois group is $(A/f)^*$, just as for the Carlitz cyclotomic case (note that the degrees match giving the full Galois group).

Anderson construct $\phi \in L_f$, so the fact that the cyclotomic Galois group G is isomorphic to $(A/f)^*$ immediately shows that ϕ has automorphisms from $(A/f)^* \times (A/f)^*/\Delta$, where Δ is the diagonal, so that ϕ belongs to the fixed field of L_f by Δ. (Via $(a, b) \to ab^{-1}$, this group is isomorphic to $(A/f)^*$ the Galois group obtained via our approach.)

Note that Anderson's description of Galois action is "internal": One acts on Z as in Theorem 2; whereas our description is "external": one gets the Galois conjugate of a soliton explicitly as (q^{d-1}th root of) a polynomial in the soliton.

To pin down the field of rationality for ϕ, we then use class field theory and the information on ramification of $L_1(\phi)$ over L_1 obtained by calculation of the discriminant from $(*)$: Since the equation is $\mathbb{F}_q$-linear, the roots, $\phi^{[i]}$ say, form a $\mathbb{F}_q$-vector space, so that the discriminant $\prod (\phi^{[i]} - \phi^{[j]})^2$ is just power of $\prod \phi^{[k]}$, where now the product is over the non-zero roots. Hence the discriminant is power of $\pm c_0/c_d$. Using the definition for calculation c_0 and the calculation in (v) below to calculate c_d, we see that

$$\frac{c_0}{c_d} = \frac{f(t)}{f(T)^{q^{d-1}}} \prod_{i=0}^{d-1} \frac{(T^{q^{d-1}} - t^{q^i})}{(T^{q^i} - t)}.$$

Hence the ramification of the covering (of the projective line cross itself) obtained from ϕ is only along horizontal, vertical divisors or on graphs of powers of Frobenius.

We now make an assumption that $L_1(\phi) \subset K_1 \otimes K_2$ where K_1 and K_2 are just finite extensions of $\mathbb{F}_q(t)$ and $\mathbb{F}_q(T)$ respectively. Then, since the discriminant shows that the ramification is only over $f(t)$ and $f(T)$, by the class field theory we see that we can take K_i to be $K(\zeta_{f^y})$ for some y. If we now make an further assumption that f is square-free, then since the degree of our extension is then prime to p, we see that we can then take K_i to be $K(\zeta_f)$. So under these assumptions $L_1(\phi) \subset L_f$ and then we see, as above, that the splitting field is $L_1(\phi) = L_f^{\Delta}$.

Divisor of the Soliton

Now let us see what we can read off easily about the divisors of ϕ and $1-\phi$ (or in fact, $\theta - \phi$, where $\theta \in \mathbb{F}_q^*$. Note that $1 - \phi^{q-1} = -\prod_{\theta \in \mathbb{F}_q^*} (\theta - \phi)$.), by looking at $(*)$.

(i) If $\phi = 0$, either the constant term of $(*)$ is zero, or the coefficients of earlier terms have poles to cancel the constant term, i.e., $f(t) = 0$ or $t = T^{q^i}$ $(i \leqslant d_f - 2)$ or $T = \infty$ or $t = \infty$ (we will see that this latter case does not occur).

(ii) By $(*)$, $T = \infty$ implies $\phi = 0$.

(iii) $\theta - \phi = 0$, $\theta \in \mathbb{F}_q^*$ implies that substitution $\phi = \theta$ makes ϕ_i vanish, which implies $t = T^{q^i}$ $(i \leqslant d_f - 1)$ or $t = \infty$. Conversely, it can be verified by straight substitution that $t = T^{q^i}$ produces zeros in the appropriate ranges.

(iv) When $t = \infty$, the constant term becomes nonzero finite and if $\phi \neq \infty$, all other terms are zero. (This is because $\deg(f_i) = (d-i)\, q^i$ and there are hence two terms of the same degree and opposite signs in the numerator of coefficient of each ϕ^{q^i} and hence the numerator is of smaller degree than the denominator.) This contradiction implies $t = \infty$ gives a pole for ϕ.

(v) If $\phi=\infty$ (then $\theta-\phi=\infty$ also for $\theta\in\mathbb{F}_q$), and if $t, T\neq\infty$, then the equation for $1/\phi$ has constant coefficient 0, i.e., $c_d=0$ in (*). Making a common denominator we see that

$$c_d=\frac{\sum f_i(t)(T^{q^{d-1}}-t)(T^{q^{d-1}}-t^q)\cdots(T^{q^{d-1}}-t^{q^{i-1}})}{(T^{q^{d-1}}-t)(T^{q^{d-1}}-t^q)\cdots(T^{q^{d-1}}-t^{q^{d-1}})}.$$

Claim. *Numerator is $f(T)^{q^{d-1}}$. Hence the pole of ϕ occurs at the support of $f(T)$.*

Proof of the Claim. It is enough to prove

$$\sum f_i(t)(U-t)(U-t^q)\cdots(U-t^{q^{i-1}})=f(U).$$

Think of this as a development of the degree d polynomial $f(U)$ in 1, $(U-t)$, ... Hence it is enough to verify this at $U=t, t^q, ..., t^{q^d}$. (In fact, it is easy to see for $t=0$ also, since $f_i(t)$ at $t=0$ is the coefficient of t^i in $f(t)$). So we have to prove that $f+f_1(t^{q^k}-t)+\cdots+f_k(t^{q^k}-t)\cdots(t^{q^k}-t^{q^{k-1}})=f^{q^k}$. (In fact f can be replaced by an unknown, with f_i defined by recursion as before.) This follows by induction on k: To get the $k+1$th equation, take qth power of the kth equation and replace f_i^q by $f_{i+1}(t^{q^{i+1}}-t)+f_i$. This proves the claim.

Another way to see that the support of $f(T)$ contains the polar divisor (except for $t=\infty$) is to use results of Carlitz (see 5.8 of [T1]), which imply that $f^{q^i}(\frac{a/f}{q^i})$ in integral, i.e., lies in A. ▮

In fact, the full description of the divisor (on X cross X, rather than on V cross V as in Theorem 2) and application to ramification properties of Hecke characters of [A2] appears in Sinha's thesis. To get this full result giving the divisor, rather than the image (on projective line cross itself) of its support, we need to analyze the action of covering automorphisms and calculate the orders of vanishing. We see in (v) that, if α is a root of $f(x)$ of multiplicity m, then c_d vanishes to multiplicity mq^{d-1} at $T=\alpha$. Vanishing multiplicities of c_i can also be similarly analyzed and Newton Polygon calculation can be used to understand divisor of ϕ, even in the support of $f(T)$ or $f(t)$, but this has not been carried out.

3. EXPLICIT FORMULAE FOR SOLITONS

Now we describe Anderson's explicit construction (inside L_f) of the solitons. For the proof, refer to 6.1 of [A2]. It would be interesting to show directly that the following recipe does satisfy the equation (*) of our Theorem 3.

Let $f = f(t) = \sum_{i=0}^{d} f_i t^i$ with $f_d = 1$. For $1 \leqslant i \leqslant d$, define $w_i \in \mathbb{F}_q$, $m_{d-i} \in \mathbb{F}_q[\zeta, t]$, M_{d-i}, $\bar{M}_{d-i} \in \mathbb{F}_q[Z, T]$ by recursion as follows:

$$w_1 := -f_{d-1}, \qquad w_i := w_1 w_{i-1} - f_{d-i},$$

$$m_{d-i} := C_{t^{i-1}}(\zeta) - \sum_{j=1}^{i-1} m_{d-i+j} w_j,$$

$$M_{d-i} := C_{T^{i-1}}(Z) - \sum_{j=1}^{i-1} M_{d-i+j} w_j,$$

$$\left(\sum_{i=0}^{d-1} M_i t^i\right)\left(\sum_{i=0}^{d-1} \bar{M}_i t^i\right) \equiv 1 \qquad \bmod f(t).$$

Here the $\bar{M}_i$ can be obtained from the last equation (by say Cramer's rule) when we rewrite it as a system of linear equations. With these definitions we have

$$\phi = \sum_{i=0}^{d-1} m_i \bar{M}_{d-1-i}. \tag{14}$$

The simplest example is

EXAMPLE. $f = t^d$. We have $\phi = \sum_{i=0}^{d-1} C_{t^{d-i-1}}(\zeta)\, \bar{M}_{d-i-1}$, where

$$\bar{M}_0 := C_{T^{d-1}}(Z)^{-1}, \qquad \bar{M}_i := -\bar{M}_0 \sum_{k=0}^{i-1} C_{T^{d+k-i-1}}(Z)\, \bar{M}_k.$$

Remarks. These explicit formulae reveal some interesting phenomena. First of all, q occurs as a parameter in the exponents, just as in the examples (12) and (13). Secondly, the formulae can be just thought of as specializations from universal formulae, e.g., for $f(t) \in \mathbb{Z}[t]$ if $q = p$, and hence their form is independent of questions such as whether f is a prime in A, though the properties of course depend on such questions. The dependence of ϕ on f_0 is implicitly through ζ and Z only.

4. GENERALIZATIONS

Now we relax the restriction that A be $\mathbb{F}_q[t]$ and consider general A.

As remarked earlier, the Theorem 1 does not generalize, giving two candidates for the binomial coefficients and the factorials for general A. See [T1, T4] for more on this. In particular, it is know [T3, T6] that the gamma functions based on both generalizations of the factorials are interesting, satisfy various analogies and are intimately connected with the arithmetic of Drinfeld modules. Natural generalization with Anderson's approach would be based on interpolations of $\binom{x}{q^i}$, whereas our approach deals with the other candidate $\left\{ {x \atop q^i} \right\}$. The gamma function based on the first is shown to have interesting arithmetical properties in [T3]. Hence, it was suggested in [T5] that it might be worthwhile to study the gamma function based on the second. We address this below. We write $\{x, i\} := \left\{ {x \atop q^i} \right\}$ for convenience.

Let us see how to find algebraic equations for the function ϕ obtained by interpolating $\{a/f, i\}$, for general A: If we write $\rho_f = \sum f_i F^i$ for $f \in A$ of degree d, the proof of the Theorem 3 works word by word till (including) (9), with the general meaning to $\{x, i\}$ now. In particular, we get $\sum f_{d-i} \{a/f, i+k\}^{q^{d-i}} = 0$. Since we do not have t, to find a substitute for the last equation in the proof of the Theorem 3, we use (9) with c being two (or more) elements of A and by elimination from the resulting equations find a relation of the form $p_i(\{a/f, i\}) = q_i(\{a/f, i-1\})$, where p_i and q_i are specializations of polynomials p and q, which are nonzero polynomials with coefficients in two copies of H, on the graph of ith power of Frobenius. (This is the F-function relation in terminology of [T5].) I believe, but do not yet know how to prove, that the elimination will always give such an equation. We will illustrate the process below by an example. On the other hand, when we do get such an equation, we can find algebraic equation for ϕ: namely, $\sum f_{d-i} \phi_i^{q^{d-i}} = 0$, with ϕ_i being algebraic functions of ϕ given by the recursion relation thus obtained.

EXAMPLE. $A = \mathbb{F}_2[x, y]/y^2 + y = x^3 + x + 1$ has class number 1 and the corresponding Drinfeld module ρ is explicitly described in [H1] by

$$\rho_x = x + (x^2 + x) F + F^2, \qquad \rho_y = y + (y^2 + y) F + x(y^2 + y) F^2 + F^3.$$

Using (9), with $c = x$, y, and with shortform $c_i := \{a/f, i\}$ we get,

$$x c_i + (x^2 + x) c_{i-1}^2 + c_{i-2}^4 = c_i x^{2^i} + c_{i-1}(x^2 + x)^{2^{i-1}} + c_{i-2}$$

and a similar equation for y. The first explicit equation is of the form $c_i = g_i(c_{i-1}, c_{i-2})$, where g_i is a polynomial of the form above. Namely, with $X = x^{2^{i-2}}$, $x_1 = x^2 + x$ and $X_1 = x_1^{2^{i-2}}$, we have

$$c_i = (X_1^2 c_{i-1} + x_1 c_{i-1}^2 + c_{i-2}^4 + c_{i-2})/(X^4 + x).$$

Hence we can express c_i and c_{i-1} in terms of c_{i-2} and c_{i-3}. Straight substitution in the equation obtained from y gives the relation between c_{i-2} and c_{i-3} of the form claimed. We do not see, at present, any point in writing down the resulting huge expression.

The equations obtained in general seem to be complicated enough to be of use in calculations as in the last section and better techniques are needed to understand the divisor and other properties.

From (4), we have

$$\{z, i\} = \sum_{k=0}^{i} z^{q^k}/d_k l_{i-k}^{q^k} \in H[z]. \tag{15}$$

By (1), (3), and (4) we see that for $a \in A$, $\{a, i\}$ is the coefficient of F^i in ρ_a. In particular, $\{x, i\}$ is a B (which is A if $h=1$) valued polynomial function on A analogous to the binomial coefficients. Also, it follows that $\{a, i\}$ is 0, if $\deg(a) < i$, and it is $\mathrm{sgn}(a)$, if $\deg(a) = i$. Since the degree of $\{x, i\}$, as a polynomial in x, is q^i, by the Riemann–Roch theorem, it has more zeros if the genus g is more than zero. (See [T5] for more on this point.)

THEOREM 4. *Let g be 1 and let $i > 1$ and let ρ correspond to the principal ideal class. Then the set of zeros of $\{x, i\}$ is the union of $S_i := \{a \in A\colon \deg(a) < i\}$ and $w_i + S_i$ for some w_i of degree i.*

Proof. We have seen that S_i is a subset of zeros. Since the zeros form a $\mathbb{F}_q$-vector space, the set of the other zeros will be of the form $w_i + S_i$ by the Riemann–Roch theorem, and it remains to prove that the degree of w_i is i. Comparing the linear terms of the displayed equation in the proof of the Theorem III in [T5] (note $g=1$ and $w_i = t_{i1}$ there), we see that

$$\frac{d_i}{l_i} = \pm (D_0 D_1 \cdots D_{i-1})^{q-1} e_i(w_i)^{q-1}. \tag{16}$$

By the Riemann–Roch theorem, $\deg(D_i) = iq^{i-1}$. An easy inductive computation involving the Theorem V of [T5], shows that $\deg(l_i) = (q^i - 1)(1 - q + 1/(q-1))$, $\deg(d_i) = (i + 1 - q)\, q^i$. Hence $\deg(e_i(w_i)) = iq^{i-1}$ and hence the theorem follows from (5). ∎

Remarks. Clearly, $w_i \notin A$. Straight computation using (16) shows that in the example of A above, w_2 can be taken to be $x + \zeta_3$, but w_3 is not an algebraic integer over A. It would be of interest to understand these extra zeros in general. ∎

COROLLARY. *Let $g = 1$ and let $a \in A$ be of degree d. Then, for $i \leqslant d$, the coefficient of F^i in ρ_a is non-zero, for any ρ.*

Now we turn to the gamma function. From the analogies described above, parallel to the definition (8), we want to look at

$$\Gamma_h(z) := \frac{1}{z} \prod_{i=0}^{\infty} \left(1 + \left\{\begin{matrix} z \\ q^i \end{matrix}\right\}\right)^{-1}. \tag{17}$$

Let us first see when this makes sense. If $z \in A$, the product is a finite product and so if it is well-defined at z, then $\Gamma_h(z) \in H$. If $\mathrm{sgn}(z) = -1$ or $z = 0$, then there is a zero in the denominator, which we will think of as a pole.

Let $g = 1$. Then this pole is a simple pole and there are no poles at other integers. (The condition on ρ is not necessary for this point, since for $k \in K$, $\{k, i\}$ for different ρ's are Galois conjugates.) This is because, if $1 + \{z, i\} = 0$, then $\{p_i + z, i\} = 0$, where p_i is a monic element of the degree i, which exists if $i > 1$ and we get a contradiction to the Theorem 4. In other words, $\Gamma_h(z)$ takes values in H at integers, except for the simple poles at zero and integers with sign -1. By (15) and the formulae for the degrees of d_i and l_i in the proof of Theorem 4, it is easy to see that $\deg(\{z, i\}) = \deg(z) + 1 - q + 1/(q-1)$, if $\deg(z) \ll 0$. Hence for such z the product in (17) does not converge in contrast to (8), where the convergence is everywhere (except for the obvious poles).

We hope to investigate the question of good interpolations and of the arithmetic significance of the special values in a future paper.

5. APPENDIX

In this appendix, written at the suggestion of David Goss, we first explain informally the connection between soliton and the period of the corresponding Shtuka, by making an analogy with the one-dimensional case of Drinfeld modules, and in particular with the Carlitz module. For details and a rigorous treatment, the reader should look at the papers of Anderson and Sinha. At the end, we discuss some open questions.

We assume that the reader is familiar (see [G2] for example) with the notion of t-motives of [A1] and the Drinfeld dictionary, as explained in [D2], [M], or [T6] Section 0. We will follow the notation of [T6] Section 0, but will not repeat the explanations there.

Let X be a curve over $\mathbb{F}_q$ of genus g, with point ∞ on it, assumed rational for simplicity. Let A be the co-ordinate ring of $X - \infty$. Let M be an universal field and $\xi: A \to M$ be an embedding. Finally, let $\bar{X}$ be the base change of X to M.

Shtuka, in [T6], is a function on $\bar{K}$ with divisor of the form $V^{(1)} - V + \xi - \infty$, where V is an effective divisor of genus g and $V^{(1)}$ its Frobenius twist (if you think of $\bar{X}$ as constraining two independent copies of X, the qth power map acts on one copy).

Shtuka f corresponds to a normalized rank one Drinfeld A-module, with the corresponding exponential given by

$$e(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{(f^{(0)} \cdots f^{(n-1)})|_{\xi^{(n)}}}.$$

For example, the Carlitz module for $A = \mathbb{F}_q[T]$ with $t := T|_{\xi}$ corresponds to Shtuka $f = T - t$, with empty V, so that $f^{(i)} = T - t^{q^i}$ and $f^{(i)}|_{\xi^{(n)}} = t^{q^n} - t^{q^i}$ leads to the Carlitz exponential. (Some other examples have been worked out in [T6].)

When we work at the two variable (on two copies of the curve, as in the Shtuka or the soliton approach) level, after interpreting the tangent space and the points of the Drinfeld module suitably, the algebraic description for the exponential turns out to be $\alpha \mapsto \alpha - \alpha^{(1)}/f$. The formal solution $\alpha = (\prod_{i=0}^{\infty} f^{(i)})^{-1}$ to the equation $\alpha - \alpha^{(1)}/f = 0$ gives the period at two variable level. Retracing through he interpretations, we get the usual period by specializing to one variable by taking its residue, which is also the leading term, at ξ.

Let us make this vague description precise in the case of Carlitz module: Let $E(T) := \sum e(\tilde{\pi}/t^{i+1})\, T^i$. Using the functional equation $e(z)^q + te(z) = e(tz)$ of e and the fact that $\tilde{\pi}$ is a period of e, we see that $E^{(1)}(T) + tE(T) = TE(T)$, that is $(T - t)\, E(T) = E^{(1)}(T)$. So $E(T)$ is $\prod (T - t^{q^i})^{-1}$. On the other hand, from the definition of $E(T)$, it is clear that the residue of $E(T)$ at $T = t$ is $\tilde{\pi}$. Comparing with the product expression above gives the formula claimed above.

Formally, $\prod_{i=1}^{\infty} f^{(i)} = \prod (T - t^{q^i}) = \prod (-t)^{q^i} \prod (1 - T/t^{q^i})$, since $\sum q^i = q/(1-q)$, we get the familiar expression $\tilde{\pi} = (-t)^{q/(q-1)} \prod (1 - t^{1-q^i})^{-1}$.

Note that in the sense of this paper, this f interpolates all $d_i/d_{i-1}^q = t^{q^i} - t = -l_i/l_{i-1}$. We also have $D_i = d_i$, $L_i = l_i$. Note that period is $\tilde{\pi} = \Gamma(0) = \prod D_i^{q-1} = \prod (D_i/D_{i-1}^q)^{-1} = \prod (t^{q^i} - t)^{-1}$. These equalities make sense once we remove the degree part from terms and recover at the end, as explained in [T3]. The period is also given by $\tilde{\pi} = \lim (t - t^q)^{q^i/(q-1)}/l_i$.

How does all this generalize to soliton level? We are trying to generalize from this absolute (unramified) case to the relative (ramified with conductor the monic polynomial P (the notation f, for the conductor, of the main body of the paper unfortunately clashes with the notation above, hence we use P here)) case of the Pth cyclotomic over. (This is where the theta technology for the generalized jacobian for conductor P shows up in Anderson's papers.) The integral closure B of A, which has several places above infinity now enters the picture.

Now note that with line bundle $\mathscr{L} := \mathcal{O}_{\bar{X}}(V)$, we have the isomorphism $\mathscr{L}^{(1)} \equiv \mathscr{L}(-\xi + \infty)$ realised by multiplication by f. This is the connection with the usual notation of Shtuka in [D], [M]. And allowing the twist by more than one zeros and poles in the twisting carries one to higher dimensional generalizations of rank one Drinfeld modules: rank one t-motives. Once one defines soliton ϕ by interpolations and calculates its divisor, it has the right form as above with several zeros and infinities, giving a rise to (generalized) Shtuka and corresponding rank one (over B, which has several infinite places) t-motive, whose period is then essentially $(\prod \phi^{(i)})^{-1}$ and gives the gamma value exactly as above. This rank one multiplication (complex multiplication) by cyclotomic integers is the reason for analogy with Fermat motives.

Finally, we look at analogy of Section 4 again, contrasting the genus zero case of the Carlitz module above with general Drinfeld modules over higher genus base: As mentioned before, D_i and d_i are different in general and in [T6], we looked at gamma function with d_i's replacing D_i's of [T3]. We interpolated it at finite places v and to the infinite place to get say, Γ_v and Γ_∞ respectively. We connected special values of Γ_v to Gauss sums of [T2].

What is the arithmetic significance of the special values of Γ_∞? In particular, what is $\Gamma_\infty(0)$? This is an important open question. In the Carlitz case, it is just the period. In general, it is not.

Let us say a few more words about this question: Let us use the short-form $f_{ij} := f^{(i)}|_{\xi^{(j)}}$. In the Carlitz case of genus 0, we have switching-symmetry $f_{ij} = -f_{ji}$. In general, up to simple algebraic factors arising from residue calculations, $\tilde{\pi}^{-1}$ is $\prod_{i=1}^{\infty} f_{i0}$ and $\Gamma_\infty(0)^{-1}$ is $\prod_{j=1}^{\infty} f_{0j}$, as can be seen from formulas giving d_i and l_i in terms of specializations of f. So the switching-symmetry mentioned above connects the two in the Carlitz case.

How are these two quantities obtained by the switch of the two copies of X related? Are we getting some new transcendental invariant of A or are they simply related algebraically? In the second case, how? For example, is $\tilde{\pi}/\Gamma_\infty(0)$ or $\tilde{\pi}^{q^g}/\Gamma_\infty(0)$ algebraic? Is there a similar situation for the gamma function introduced in the Section 4? At present, we have no evidence pointing either way.

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