SHARPNESS RESULTS AND KNAPP’S HOMOGENEITY ARGUMENT*

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INTRODUCTION

Let $S$ be a smooth compact hypersurface in $\mathbb{R}^n$. Let

\[ F_S(\xi) = \int_S e^{i\langle x, \xi \rangle} d\sigma(x) \]

denote the Fourier transform of the surface measure carried by $S$.

Let $Rf = f|_S$, the restriction operator. It is well known (see [T], [G], [S]) that if

\[ |F_S(\xi)| \leq C(1 + |\xi|)^{-r}, \quad r > 0, \]

then

\[ \|Rf\|_2 \leq C_p\|f\|_{p'}, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad \text{for} \quad p \leq p_0 = \frac{2(r + 1)}{r + 2}, \]

where $\mathcal{S}(\mathbb{R}^n)$ is the standard Schwartz class.

However, it is not in general known whether this result is sharp. More precisely, it is natural to ask the following.

**Question A.** Does the estimate (3) imply the estimate (2)?

Let

\[ T_f(x, x_n) = \int f(x - y, x_n - \Phi(y))\psi(y)dy, \]

where $x, y \in \mathbb{R}^{n-1}$, $\psi$ is a smooth cutoff function, $\Phi$ is smooth, $\Phi(0, \ldots, 0) = 0$, and $\nabla \Phi(0, \ldots, 0) = (0, \ldots, 0)$.

It is well known (see [Str]) that if the estimate (2) holds, then

\[ \|Tf\|_{p'} \leq C_p\|f\|_{p'}, \quad \text{where} \quad \frac{1}{p} - \frac{1}{2} \leq \frac{r}{2(r + 1)}, \]

where $p'$ denotes the conjugate exponent of $p$.

The key estimate here is

\[ \|Tf\|_{2(r+1)} \leq C\|f\|_{2(r+1)}^{2(r+1)}; \]

the rest follows by interpolation. It is then natural to ask the following.
Question B. Does the estimate (5) imply the estimate (2)?

The purpose of this paper is to answer questions (A) and B affirmatively in the case of the optimal exponents. We shall employ a multiparameter version of Knapp’s homogeneity argument. (See e.g [C] for a similar argument).

More precisely, we will show that if the estimate (3) holds with 
\[ p = \frac{2(n+1)}{n-3}, \]
then the hypersurface has everywhere non-vanishing Gaussian curvature. Similarly, we will show that if the estimate (5) holds with 
\[ p = \frac{n+1}{n-1}, \]
then the hypersurface has non-vanishing Gaussian curvature.

We remark here, on the other hand, that non-vanishing Gaussian curvature implies that the estimate (2) holds with 
\[ r = \frac{n-1}{2}. \]
Thus Question (A) is answered affirmatively in the case 
\[ r = \frac{n-1}{2}. \]
Since the estimate (2) with 
\[ r = \frac{n-1}{2} \]
implies the estimate (3) with 
\[ p = \frac{2(n+1)}{n-3}, \]
Theorem 2 below shows that the optimal decay of the Fourier transform (i.e. 
\[ r = \frac{n-1}{2} \]
implies that the hypersurface has non-vanishing Gaussian curvature.

We will also see that if a hypersurface has \( \leq k \) non-vanishing principal curvatures at each point, then the exponent 
\[ p = \frac{2(n+k+2)}{2n+k+3}, \]
then at least \( k \) principal curvatures are non-zero at each point. (See Theorem 3 below). Similarly, we will show that if the estimate (5) holds with 
\[ p \geq \frac{n+k+4}{2n+k+1}, \]
than at least \( k \) principal curvatures are non-zero at each point.

The sharpness of the estimate (3) is known in some cases. For example, if the hypersurface has non-vanishing Gaussian curvature, Knapp’s homogeneity argument can be used to show that the exponent 
\[ p = \frac{2(n+1)}{n-3} \]
is the best possible. Indeed, non-vanishing Gaussian curvature implies that the hypersurface has contact of order two with its tangent plane at every point.

Let 
\[ f_\delta(x) = \frac{g(\delta^{-1},x,\delta^{-2},x_n)}, \]
where \( x = (x_1,...,x_{n-1}) \), and \( g \) is the characteristic function of the rectangle with sides \((1,...,1,C), C \) large, with the long side normal to the hypersurface.

It is not hard to check that 
\[ ||f_\delta||_p \approx \delta^{1-\frac{1}{p}} (n+1), \]
whereas 
\[ ||Rf_\delta||_2 \approx \delta^{\frac{n-1}{2}}. \] The comparison yields 
\[ p \leq \frac{2(n+1)}{n-3}. \]
It should be noted that the above example does not verify even a special case of question (A). For example, the above argument does not prove that if the estimate (3) holds with 
\[ p = \frac{2(n+1)}{n-3}, \]
then the estimate (2) holds with 
\[ r = \frac{n-1}{2}. \] We will show (see Theorem 2 below) that this is indeed the case.

The sharpness of the estimate (5) can also be verified in some cases. By testing \( T \) against a characteristic function of a small ball it is not hard to check that if \( T \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \), then \( \left( \frac{1}{p}, \frac{1}{q} \right) \) must be contained in the triangle with the endpoints \((0,0),(1,1), \) and \((\frac{n}{n+1}, \frac{1}{n+1})\). However, as before this does not prove that if the estimate (5) holds with 
\[ p = \frac{n+1}{n-1}, \]
then the estimate (2) holds with 
\[ r = \frac{n-1}{2}. \] We will show (see Theorem 5 below) that this is indeed the case.

**Statement of results**

**Theorem 1.** Let \( S = \{(x,x_n) \in \mathbb{R}^n : x_n = \Phi(x)\} \), where \( x = (x_1,...,x_{n-1}) \), \( \Phi \) is a
smooth function which does not vanish on a set of positive measure, \( \Phi(0,\ldots,0) = 0 \), and \( \nabla \Phi(0,\ldots,0) = (0,\ldots,0) \). Suppose that the estimate (3) holds. Let \( G \) be any continuous function which does not vanish on a set of positive measure satisfying \( G(0,\ldots,0) = 0 \). Then

\[
(\delta G(\delta))^r \geq CR(\delta)^{r+1}\delta_1\delta_2\ldots\delta_{n-1},
\]

where \( R(\delta) = \{x \in [-1,1]^{n-1} : |\Phi(\delta_1x_1,\ldots,\delta_{n-1}x_{n-1})| \leq C|G(\delta)|\} \).

Remark. If \( G(\delta) \) is chosen to be \( \Phi(\delta) \), and \( \Phi \) is increasing in each variable separately, Theorem 1 says that the estimate (3) implies that \( (\Phi(\delta))' = C\delta_1\delta_2\ldots\delta_{n-1} \). The same estimate would be true, of course, if we just assume that \( R(\delta) \) is bounded below, which is a much weaker assumption. To prove Theorem 2, Theorem 3, Theorem 5, and Theorem 6 below we shall use Theorem 1 with

\[
G(\delta) = \sup_{\{x \in [-1,1]^{n-1}\}} |\Phi(x_1\delta_1,\ldots,x_{n-1}\delta_{n-1})|.
\]

**Theorem 2.** Suppose that the estimate (3) holds with \( p = \frac{2(n+1)}{n+4} \). Then the hypersurface \( S \) has everywhere non-vanishing Gaussian curvature.

**Theorem 3.** Suppose that the estimate (3) holds with \( p \geq \frac{2n+k-2}{6} \). Then the hypersurface \( S \) has at least \( k \) non-vanishing principal curvatures at each point.

Remark. The conclusion of Theorem 3 can be motivated as follows. If the hypersurface has exactly \( k \) non-vanishing principal curvatures at a point, then after perhaps applying a rotation we can write it as a graph of the function \( x_1^2 + \ldots + x_n^2 + A(x) \), where \( A \) is a higher order remainder. It is not hard to believe that the best possible estimate (2) is obtained if \( A(x) = |x''| \), where \( x'' = (x_{k+1},\ldots,x_{n-1}) \). This gives us the estimate (2) with \( r = \frac{k}{3} + \frac{n-1-k}{3} \). The conclusion of Theorem 3 is the consequence of the fact that

\[
\frac{2(r+1)}{r+2} = \frac{2n+k-2}{6}.
\]

**Theorem 4.** Let \( \delta y = (\delta_1y_1,\ldots,\delta_{n-1}y_{n-1}) \) and \( g_\delta(s) = |\{y \in \text{supp}(\psi) : s - |\Phi(\delta y)/\Phi(\delta)| \leq C\}| \). Suppose that the estimate (5) holds. Then for \( |\delta| \) sufficiently small,

\[
(|\Phi(\delta)|)^r \geq CP_0\|g_\delta\|_{L^r(\delta s)}.
\]

**Theorem 5.** Suppose that the estimate (5) holds with \( r = \frac{n-1}{2} \). Then \( S = \{(x,x_n) : x \in \text{supp}(\psi), x_n = \Phi(x)\} \) is an almost umbilical hypersurface.

**Theorem 6.** Suppose that the estimate (5) holds with \( p \geq \frac{2n+k+4}{2n+k+1} \). Then the hypersurface has at least \( k \) non-vanishing principal curvatures at each point.

(See the remark after Theorem 3 for the motivation of the conclusion of Theorem 6).
Proof of Theorem 1

Let \( \delta x = (\delta_1 x_1, ..., \delta_{n-1} x_{n-1}) \), and \( \delta^{-1} x = (\delta_1^{-1} x_1, ..., \delta_{n-1}^{-1} x_{n-1}) \). Let \( \hat{f}_\delta(x, x_n) = g(\delta^{-1} x, \frac{x_n}{|G(\delta)|}) \), where \( g \) is the characteristic function of a rectangle with sides of length \((1, 1, ..., 1, C)\). Let \( P_\delta = |\delta_1 \delta_2 ... \delta_{n-1}| \). It is not hard to see that

\[
||f_\delta||_p \approx (P_\delta |G(\delta)|)^{(1-1/p)}.
\]

On the other hand,

\[
||R f_\delta||_2^2 = \int \left| g \left( \delta^{-1} x, \frac{\Phi(\delta x)}{|G(\delta)|} \right) \right|^2 dx =
\]

\[
P_\delta \int \left| g \left( x, \frac{\Phi(\delta x)}{|G(\delta)|} \right) \right|^2 dx \approx C P_\delta R(\delta),
\]

where \( R(\delta) \) is defined in the statement of the theorem.

Comparing the estimates (10) and (11) we see that (3) can hold only if

\[
(|G(\delta)|)^r \geq C P_\delta R^{r+1}(\delta),
\]

for \( |\delta| \) sufficiently small. This completes the proof of Theorem 1.

Proof of Theorem 2

Let \( G(\delta) = \sup_{x \in [-1,1]^{n-1}} |\Phi(\delta_1 x_1, ..., \delta_{n-1} x_{n-1})| \). It follows that \( R(\delta) \equiv 1 \), and so

\[
(G(\delta))^r \geq C P_\delta,
\]

where \( r = \frac{n-1}{2} \) by assumption.

After perhaps applying a rotation, we can use Taylor’s theorem to write

\[
\Phi(x) = a_1 x_1^2 + a_2 x_2^2 + ... + a_k x_k^2 + A(x),
\]

where \( A(x) \) is a higher order remainder term, and \( k \leq n - 1 \). If \( k = n - 1 \), then in a sufficiently small neighborhood of the origin the determinant of the Hessian matrix of \( \Phi \) never vanishes, which would verify the claim of Theorem 2. We shall henceforth assume that \( k < n - 1 \).

It is not hard to check that

\[
(G(\delta))^\frac{n-1}{2} \leq (a_1 \delta_1^2 + ... + a_k \delta_k^2 + C|\delta|^3)^\frac{n-1}{2},
\]

We must show that the estimate (13) cannot hold if \( k < n - 1 \). It suffices to show that the right hand side of (15) is not bounded below by \( CP_\delta \). We may assume that \( A(x) \) is not identically 0, and that \( A(x) \) depends on \( x_{n-1} \), for otherwise the contradiction is immediate.

Let \( \delta_j = \delta \frac{2}{n-1} \). If the right hand side were bounded below by \( CP_\delta \), we could use the fact that \( A(x) \) is a higher order remainder term to force an inequality

\[
|\delta_{n-1}|^{\frac{2(n-1)}{n-2}} \geq C|\delta_{n-1}|^{\frac{2n-4}{n-2}},
\]

\( \delta_{n-1} \) small, which is not true. This shows that the estimate (8) cannot hold unless \( k = n - 1 \). This implies that there exists a small neighborhood of the origin where \( S \) has non-vanishing Gaussian curvature. This completes the proof.

**Proof of Theorem 3**

We must show that if \( \Phi \) is as in the estimate (14) above, with \( k \) denoting the number of non-vanishing principal curvatures, then the estimate

\[
(G(\delta))^r \geq CP_\delta
\]

can only hold if

\[
r \leq \frac{k}{2} + \frac{n+1-k}{3} = \frac{2n+k-2}{6}.
\]

Let \( \delta = (\delta', \delta'') \), where \( \delta' = (\delta_1, ..., \delta_k) \), and \( \delta'' = (\delta_{k+1}, ..., \delta_{n-1}) \).

Let \( \delta_j = |\delta''|^j \). The estimate (16) cannot hold if the inequality

\[
|\delta''|^3r \geq C|\delta''|^{\frac{2k}{3}+(n-1-k)}
\]

is not satisfied. However, the estimate (17) can only hold if \( r \leq \frac{k}{2} + \frac{n+1-k}{3} = \frac{2n+k-2}{6} \). This completes the proof.

**Proof of Theorem 4**

Let \( \delta^{-1}y = (\delta_1^{-1}y_1, ..., \delta_{n-1}^{-1}y_{n-1}) \). Let \( f \) denote the characteristic function of the rectangle with sides of length \( (1, 1, ..., 1, C) \), \( C \) large. Let \( \tau_\delta f(x, x_n) = f(\delta x, |\Phi(\delta)|x_n) \), and \( \tau_\delta^{-1} f(x, x_n) = f(\delta^{-1}x, |\Phi(\delta)|^{-1}x_n) \). Let \( f_\delta(x, x_n) = \tau_\delta^{-1} f(x, x_n) \). Let

\[
T_\delta f(x, x_n) = \int f(x - y, x_n - \Phi(y))\psi(\delta^{-1}y)dy.
\]

After making a change of variables we see that

\[
T_\delta f_\delta(x, x_n) = P_\delta T_\delta^{-1}T_\delta f(x, x_n),
\]
where

$$T^*_\delta f(x, x_n) = \int f(x - y, x_n - \Phi(\delta y)/|\Phi(\delta)|)\psi(y)dy.$$  

It is not hard to see that

$$||f_\delta||_p \approx P^\frac{1}{p} (|\Phi(\delta)|)^\frac{1}{p}.$$  

Also,

$$||T_\delta f_\delta||_{p'} = ||P_\delta^{-1}T^*_\delta f||_{p'} = P^\frac{1}{p'} (|\Phi(\delta)|)^\frac{1}{p'} ||T^*_\delta f||_{p'} \approx$$

$$P_\delta P^\frac{1}{p'} (|\Phi(\delta)|)^\frac{1}{p'} ||g_\delta||_{L^{p'}(ds)},$$

where $g_\delta$ is defined above.

Comparing the estimates (22) and (23) yields the assertion of the theorem.

**Proof of Theorem 5 and Theorem 6**

Let $G(\delta) = \sup_{x \in [-1, 1]^n} |\phi(\delta x)|$. The proof of Theorem 4 shows that if the estimate (5) holds then

$$G(\delta)^r \geq CP_\delta.$$  

The proof of Theorem 5 and Theorem 6 now follows in the same way as the proofs of Theorem 2 and Theorem 3.

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