SUGAWARA'S CONSTRUCTION OF VIRASORO ALGEBRA FOR $c = 1,\ h = 0$

LUKE CYBULSKI

Contents

1. Introduction

Group theory provides a useful way of mathematically studying the symmetries of physical systems. Most common are finite dimensional Lie groups such as SU(n),U(n), and SO(n) which contribute to the framework of rotations. Additionally, finite groups have been used extensively in crystallography (Sternberg [1]). These examples maintain a strong focus on the finite dimensional symmetry and have many applications to quantum mechanics. However, to pass into quantum field theory, where it is important to consider systems with an infinite number of degrees of freedom, algebras such as the Virasoro and Kac-Moody algebras are necessary for infinite dimensional symmetries.

The Virasoro algebra arises naturally in various applications of physics. In a mathematical formulation, the Virasoro algebra can be viewed as coming from the algebra of the conformal group in one or two dimensions. Two-dimensional conformally invariant structures are present in many areas of physics, including 2-dimensional statistical systems of spins on lattices and in finding a mathematically consistent theory of particle interactions (Goddard and Olive [2]). Furthermore, the Virasoro algebra becomes useful in analyzing mass-less fermionic and bosonic field theories.

Due to its presence in numerous areas of physics, it is important to be able to construct representations for the Virasoro algebra. The Virasoro algebra is often associated with observable quantities. In particular, the algebra determines the mass spectrum in string theory, and also energy spectra in 2-dimensional quantum field theories made up of two copies of the algebra (Goddard and Olive [2]). In this sense, it is important for the representation of the Virasoro algebra to be highest weight, so that the energy spectrum is positive (or at least bounded below).

In addition to this requirement, it is necessary that the representations be unitary. In quantum mechanics, unitarity is important as it preserves probability. Furthermore, if unitarity is not imposed in quantum field theories, ghost-states, ie. states with negative norm, may appear and lead to non-physical properties, such as space-like momenta or negative probabilities. From representation theory, it is known that unitary representations may be written as direct integrals of irreducible representations. Hence, it is important to analyze irreducible representations as they provide fundamental building blocks to other representations. Furthermore, in a physical sense, a reducible representation corresponds to non-interacting parts. As many systems in physics deal strictly with interaction of particles, it is thus necessary to consider irreducible representations.

This paper will begin with discussing how the Virasoro algebra arises as a nontrivial central extension of the Witt algebra. We will then use the Heisenberg algebra to perform a Sugawara construction of the Virasoro algebra in the case of $c = 1$ and $h = 0$ (representing a free, mass-less bosonic field theory) that is irreducible, unitary, and highest weight. A brief discussion will then be given on how to generalize the Sugawara construction to other c values.

2. Central Extensions of Lie Algebras

Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras over a field $\mathbb{F}$ of characteristic zero and $\mathfrak{a}$ an abelian Lie algebra over $\mathbb{F}$.

Definition 2.1. Consider the short exact sequence of Lie algebra homomorphisms

$$0 \to \mathfrak{a} \to \mathfrak{h} \to \mathfrak{g} \to 0$$

This sequence is a *central extension* of $\mathfrak{g}$ by $\mathfrak{a}$ if $\mathfrak{a} \subseteq Z(\mathfrak{h}) = \{x \in \mathfrak{h} \mid [x, \mathfrak{h}] = 0\}$, where we identify $\mathfrak{a}$ as subalgebra of $\mathfrak{h}$.

The condition that $[\mathfrak{a}, \mathfrak{h}] = \{0\} \subseteq \mathfrak{a}$ then implies that we may regard $\mathfrak{a}$ as an ideal of $\mathfrak{h}$.

Definition 2.2. A short exact sequence of Lie algebra homomorphisms

$$0 \to \mathfrak{a} \to \mathfrak{h} \xrightarrow{p} \mathfrak{g} \to 0$$

splits if there exists a Lie algebra homomorphism $s : \mathfrak{g} \to \mathfrak{h}$ such that $p \circ s = 1_{\mathfrak{g}}$. We call s a *splitting map* and if such a map exists, then $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$. Moreover, if the sequence is a central extension, then we say that it is a *trivial* extension.

Definition 2.3. A bilinear map $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ that satisfies the following conditions 1 and 2 $\forall\, x, y, z \in \mathfrak{g}$ is called a 2-*cocycle*.

(1) $\phi(x, y) = -\phi(y, x)$
(2) $\phi(x, [y, z]) + \phi(y, [z, x]) + \phi(z, [x, y]) = 0$

Lemma 2.1. *The* 2*-cocycle* $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ *generates a central extension* $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$ *of* $\mathfrak{g}$ *by* $\mathfrak{a}$.

Proof.
Define a Lie bracket on $\mathfrak{h}$ by

$$[(x_1, a_1), (x_2, a_2)] = ([x_1, x_2], \phi(x_1, x_2))$$

This is indeed a Lie bracket as may be readily verified.

(1)

$$\begin{aligned}[(x_1, a_1), (x_2, a_2)] &= ([x_1, x_2], \phi(x_1, x_2)) = (-[x_2, x_1], \phi(x_2, x_1)) \\ &= -([x_2, x_1], \phi(x_2, x_1)) = -[(x_2, a_2), (x_1, a_1)]\end{aligned}$$

(2)

$$\begin{aligned}&[(x_1, a_1), [(x_2, a_2), (x_3, a_3)]] + [(x_2, a_2), [(x_3, a_3), (x_1, a_1)]] + [(x_3, a_3), [(x_1, a_1), (x_2, a_2)]] \\ &= [(x_1, a_1), ([x_2, x_3], \phi(x_2, x_3))] + [(x_2, a_2), ([x_3, x_1], \phi(x_3, x_1))] + [(x_3, a_3), ([x_1, x_2], \phi(x_1, x_2))] \\ &= ([x_1, [x_2, x_3]], \phi(x_1, [x_2, x_3])) + ([x_3, [x_1, x_2]], \phi(x_3, [x_1, x_2])) + ([x_3, [x_1, x_2]], \phi(x_3, [x_1, x_2])) \\ &= ([x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]], \phi(x_1, [x_2, x_3]) + \phi(x_2, [x_3, x_1]) + \phi(x_3, [x_1, x_2])) \\ &= (0, 0)\end{aligned}$$

Clearly $\mathfrak{a} \in Z(\mathfrak{h})$ when we identify $\mathfrak{a}$ as a subalgebra of $\mathfrak{h}$ and thus $\mathfrak{h}$ is a central extension of $\mathfrak{g}$ by $\mathfrak{a}$. □

Lemma 2.2 (see [3, Lemma 4.5])**.** *Let* $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ *be a* 2*-cocycle and* $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$*. Then the central extension* $\mathfrak{h}$ *of* $\mathfrak{g}$ *by* $\mathfrak{a}$

$$0 \to \mathfrak{a} \to \mathfrak{h} \xrightarrow{p} \mathfrak{g} \to 0$$

where p *is the projection map onto* $\mathfrak{g}$*, is trivial if and only if there exists a* $\mathbb{C}$*-linear map* $f : \mathfrak{g} \to \mathfrak{a}$ *such that*

$$\phi(x, y) = f([x, y])$$

Proof.
($\Leftarrow$) Suppose there exists a linear map $f : \mathfrak{g} \to \mathfrak{a}$ such that $\phi(x, y) = f([x, y])$. Then, define a map $\beta : \mathfrak{g} \to \mathfrak{h}$ by

$$\beta(x) = (x, f(x))$$

Clearly $p \circ \beta = 1_{\mathfrak{g}}$. Now, we define the following Lie bracket on $\mathfrak{h}$.

$$[(x_1, a_1), (x_2, a_2)] := ([x_1, x_2], \phi(x_1, x_2))$$

Using this Lie bracket on $\mathfrak{h}$ we find that β is a Lie algebra homomorphism.

$$\beta([x, y]) = ([x, y], f([x, y])) = ([x, y], \phi(x, y)) = [(x, f(x)), (y, f(y))] = [\beta(x), \beta(y)]$$

Hence, the sequence splits and so the central extension is trivial.
($\Rightarrow$) Suppose the central extension is trivial. Then there exists a Lie algebra homomorphism $\beta : \mathfrak{g} \to \mathfrak{h}$ such that $p \circ \beta = 1_{\mathfrak{g}}$. Let $\beta(x) = (x', a')$, for some $x' \in \mathfrak{g}$ and $a' \in \mathfrak{a}$. Then, $p(\beta(x)) = p(x', a') = x'$ but also $p(\beta(x)) = x$ and so we find $\beta(x) = (x, \pi(x))$ for $\pi : \mathfrak{g} \to \mathfrak{a}$. Furthermore, β being a Lie algebra homomorphism imposes the condition that π is a linear map.

Then, from lemma 2.1, ϕ generates a central extension $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$ of $\mathfrak{g}$ by $\mathfrak{a}$ where the Lie bracket on $\mathfrak{h}$ is defined by

$$[(x_1, a_1), (x_2, a_2)] = ([x_1, x_2], \phi(x_1, x_2)])$$

Using this definition along with the fact that β is a Lie algebra homomorphism, we find that for $x, y \in \mathfrak{g}$ that

(1) $[\beta(x), \beta(y)] = \beta([x, y]) = ([x, y], \pi([x, y]))$
(2) $[\beta(x), \beta(y)] = [(x, \pi(x)), (y, \pi(y))] = ([x, y], \phi(x, y))$.

Hence, it must be that $\phi(x, y) = \pi([x, y])$. □

Definition 2.4. A 2-*coboundary* on $\mathfrak{g}$ is a 2-cocycle $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ such that $\phi(x, y) = f([x, y])$ for some $\mathbb{C}$-linear map $f : \mathfrak{g} \to \mathfrak{a}$.

From the definition of a 2-coboundary on $\mathfrak{g}$ we see that lemma implies a 2-cocycle ϕ on $\mathfrak{g}$ generates a trivial extension if and only if ϕ is a 2-coboundary. Hence, we define $C^2(\mathfrak{g}; \mathfrak{a})$ as the set of 2-cocycles on $\mathfrak{g}$ and $B^2(\mathfrak{g}; \mathfrak{a})$ as the set of 2-coboundaries on $\mathfrak{g}$. It is then of importance to consider the space of nontrivial extensions of $\mathfrak{g}$ by $\mathfrak{a}$. Hence, we consider the equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$, specifically the second cohomology group of $\mathfrak{g}$ by values in $\mathfrak{a}$.

$$H^2(\mathfrak{g}; \mathfrak{a}) = C^2(\mathfrak{g}; \mathfrak{a})/B^2(\mathfrak{g}; \mathfrak{a})$$

3. Witt Algebra

We define the the Witt algebra $\mathscr{W}$ as a subalgebra of the algebra of smooth vector fields on the circle S^1, $Vect(S^1)$. An arbitrary element in $Vect(S^1)$ is of the form $f(\theta)\frac{d}{d\theta}$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth periodic function on $\mathbb{R}$. Hence, we we may expand it as a complex Fourier series. To avoid problems with convergence, we then further require that the Fourier series is finite, ie. is represented by a finite sum. With these restrictions, $\mathscr{W}$ is defined as

$$\mathscr{W} := \{f(\theta)\frac{d}{d\theta} \mid f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, a_n = 0 \text{ for all but finitely many } n\}.$$

Note that as $f : \mathbb{R} \to \mathbb{R}$ we require that $f^* = f$, which gives $a_n^* = a_{-n}$.

Now define $L_n \in \mathscr{W}$ as $L_n = ie^{in\theta}\frac{d}{d\theta}$. From the definition of $\mathscr{W}$ it is clear that $\{L_n\}_{n\in\mathbb{Z}}$ forms a basis for $\mathscr{W}$. Recall the definition of the Lie bracket on $Vect(S^1)$.

$$[f\frac{d}{d\theta}, g\frac{d}{d\theta}] = \left(f\frac{dg}{d\theta} - \frac{df}{d\theta}g\right)\frac{d}{d\theta}$$

We restrict this Lie bracket to the space $\mathscr{W}$ and we find that

$$[f_n, f_m] = [ie^{in\theta}\frac{d}{d\theta}, ie^{im\theta}\frac{d}{d\theta}] = -(e^{in\theta}ime^{im\theta} - ine^{in\theta}e^{im\theta})\frac{d}{d\theta} = (n-m)ie^{i(n+m)\theta}\frac{d}{d\theta} = (n-m)f_{n+m}.$$

Hence, we see $\mathscr{W}$ has a basis $\{f_n\}_{n\in\mathbb{Z}}$ such that $[f_n, f_m] = (n-m)f_{n+m}$.

In the contexts of physics, the element $f_0 \in \mathscr{W}$ plays the role of energy, which we force to take positive values. In addition, the f_0 can determine mass spectrum in string theory, and so we wish it to be non-negative.

For these reasons, we are interested in particular representations ρ of $\mathscr{W}$ such that the spectrum of $\rho(f_0)$ is non-negative (or at least bounded below). To simplify notation, we shall henceforth denote $\rho(f_n) = L_n$ for an arbitrary representation ρ.

Definition 3.1. A representation for which L_0 has a spectrum that is bounded below, ie. has a state of lowest energy, is called a *highest weight representation*. A lowest energy state $|\psi_0\rangle$ in such a representation has the following properties

(1) $L_0|\psi_0\rangle = h|\psi_0\rangle$
(2) $L_n|\psi_0\rangle = 0, \quad n > 0$

Any such state $|\psi_0\rangle$ satisfying this property is called a *vacuum state*.

In addition to highest weight representations, we are further interested in unitary representations. We begin by considering antilinear involutions on a Lie algebra $\mathfrak{g}$.

Definition 3.2. An *antilinear involution* on a Lie algebra $\mathfrak{g}$ is a map $\omega : \mathfrak{g} \to \mathfrak{g}$ that satisfies the following properties for all $x, y \in \mathfrak{g}$ and for all $a \in \mathbb{C}$.

(1) $\omega(\omega(x)) = x$
(2) $\omega(ax) = a^*\omega(x)$
(3) $[\omega(x), \omega(y)] = \omega([y, x])$

where * denotes the complex conjugate.

Definition 3.3. A *Hermitian form* $\langle ., . \rangle$ on a vector space V is a map $V \times V \to \mathbb{C}$ such that

(1) $\langle v, w\rangle = \langle w, v\rangle^*$
(2) $\langle v, au + bw\rangle = a\langle v, u\rangle + b\langle v, w\rangle$
(3) $\langle av + bu, w\rangle = a^*\langle v, w\rangle + b^*\langle u, w\rangle$

for all $u, v, w \in V$ and $a, b \in \mathbb{C}$. Furthermore, $\langle ., . \rangle$ is *non-degenerate* if

$$\langle u, v\rangle = 0 \ \forall u \in V \Rightarrow v = 0$$

Definition 3.4. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ with an antilinear involution $\omega : \mathfrak{g} \to \mathfrak{g}$. Let V be a vector space equipped with a non-degenerate Hermitian inner product $\langle ., . \rangle$ and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Then ρ is an *unitary representation* of $\mathfrak{g}$ if

$$\langle \rho(x)(u), v\rangle = \langle u, \rho(\omega(x))(v)\rangle$$

for all $x \in \mathfrak{g}$ and $u, v \in V$. Equivalently, this implies ρ is unitary if

$$\rho(x)^\dagger = \rho(\omega(x))$$

for all $x \in \mathfrak{g}$.

Consider the map $\omega : \mathscr{W} \to \mathscr{W}$ defined by

$$\omega(f_n) = f_{-n}$$

As $\mathscr{W}$ is a $\mathbb{R}$-vector space, it is clear conditions (1) and (2) are satisfied in the definition of an antilinear involution. We further find that condition (3) is also satisfied.

$$\begin{aligned}\omega([f_n, f_m]) &= \omega((n-m)f_{n+m}) = (n-m)\omega(f_{n+m}) \\ &= (n-m)f_{-(n+m)} = [f_{-m}, f_{-n}] = [\omega(f_m), \omega(f_n)]\end{aligned}$$

Hence, the condition for a unitary representation of $\mathscr{W}$ becomes

$$L_n^\dagger = \omega(f_n) = L_{-n}$$

Theorem 3.1. *The Witt algebra $\mathscr{W}$ has no nontrivial, unitary highest weight representation.*

We first prove that the following Lemma.

Lemma 3.1. *Let V be a vector space equipped with a Hermitian form $\langle ., . \rangle$. Let $u, v \in V$. Then, the matrix M given by*

$$M := \begin{pmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle v, u\rangle & \langle v, v\rangle \end{pmatrix}.$$

is positive semidefinite.

Proof.
Let $x_1, x_2 \in \mathbb{C}$. Then,

$$\begin{aligned}\begin{pmatrix} x_1^* & x_2^* \end{pmatrix} \begin{pmatrix} \langle u,u\rangle & \langle u,v\rangle \\ \langle v,u\rangle & \langle v,v\rangle \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} x_1^* & x_2^* \end{pmatrix} \begin{pmatrix} x_1\langle u,u\rangle + x_2\langle u,v\rangle \\ x_1\langle v,u\rangle + x_2\langle v,v\rangle \end{pmatrix} \\ &= |x_1|^2 \langle u,u\rangle + x_1^* x_2 \langle u,v\rangle + x_1 x_2^* \langle v,u\rangle + |x_2|^2 \langle v,v\rangle \\ &= \|x_1 u + x_2 v\|^2 \geq 0\end{aligned}$$

□

Proof of Theorem 3.1.
Consider a unitary highest weight representation for $\mathscr{W}$ with vacuum state $|\psi_0\rangle$ normalized to unity and highest weight h. Define $|\Psi_{n,m}\rangle = L_{-n}^m|\psi_0\rangle$. Let $N \in \mathbb{Z}_{>0}$ and define the matrix Λ as

$$\Lambda := \begin{pmatrix} \langle \Psi_{2N,1}|\Psi_{2N,1}\rangle & \langle \Psi_{N,2}|\Psi_{2N,1}\rangle \\ \langle \Psi_{2N,1}|\Psi_{N,2}\rangle & \langle \Psi_{N,2}|\Psi_{N,2}\rangle \end{pmatrix}.$$

Using the definition of the Witt algebra and the condition that $L_n^\dagger = L_{-n}$ for a unitary representation, we may calculate the matrix elements.

(1)

$$\begin{aligned}\langle \Psi_{2N,1}|\Psi_{2N,1}\rangle &= \langle \psi_0|L_{-2N}^\dagger L_{-2N}|\psi_0\rangle \\ &= \langle \psi_0|L_{2N} L_{-2N}|\psi_0\rangle \\ &= \langle \psi_0|[L_{2N}, L_{-2N}] + L_{-2N}L_{2N}|\psi_0\rangle \\ &= 4N\langle \psi_0|L_0|\psi_0\rangle = 4Nh\end{aligned}$$

(2)

$$\begin{aligned}\langle \Psi_{N,2}|\Psi_{2N,1}\rangle &= \langle \psi_0|(L_{-N}^2)^\dagger L_{-2N}|\psi_0\rangle \\ &= \langle \psi_0|L_N L_N L_{-2N}|\psi_0\rangle \\ &= \langle \psi_0|L_N([L_N, L_{-2N}] + L_{-2N}L_N)|\psi_0\rangle \\ &= 3N\langle \psi_0|L_N L_{-N}|\psi_0\rangle \\ &= 6N^2 h\end{aligned}$$

(3)

$$\begin{aligned}\langle \Psi_{2N,1}|\Psi_{N,2}\rangle &= \langle \psi_0|L_{-2N}^\dagger L_{-N}L_{-N}|\psi_0\rangle \\ &= \langle \psi_0|([L_{2N}, L_{-N}] + L_{-N}L_{2N})L_{-N}|\psi_0\rangle \\ &= 3N\langle \psi_0|L_N L_{-N}|\psi_0\rangle + \langle \psi_0|L_{-N}L_{2N}L_{-N}|\psi_0\rangle \\ &= 6N^2 h + \langle \psi_0|L_{-N}([L_{2N}, L_{-N}] + L_{-N}L_{2N})|\psi_0\rangle \\ &= 6N^2 h + \langle \psi_0|3NL_{-N}L_N + L{-}NL_{2N}|\psi_0\rangle \\ &= 6N^2 h\end{aligned}$$

(4)

$$\begin{aligned}\langle \Psi_{N,2}|\Psi_{N,2}\rangle &= \langle \psi_0|(L_{N}^2)^\dagger L_{-N}L_{-N}|\psi_0\rangle \\ &= \langle \psi_0|L_N([L_N, L_{-N}] + L_{-N}L_N)L_{-N}|\psi_0\rangle \\ &= \langle \psi_0|2NL_N L_0 L_{-N} + L_N L_{-N} L_N L_{-N}|\psi_0\rangle \\ &= \langle \psi_0|2NL_N(NL_{-N} + L_{-N}L_0)|\psi_0\rangle + 4N^2h^2 \\ &= (2N^2 + 2Nh)\langle \psi_0|L_N L_{-N}|\psi_0\rangle + 4N^2h^2 \\ &= (2N^2 + 2Nh)(2Nh) + 4N^2h^2 = 4N^3h + 8N^2h^2\end{aligned}$$

We thus find that

$$\Lambda := \begin{pmatrix} 4Nh & 6N^2h \\ 6N^2h & 4N^3h + 8N^2h^2 \end{pmatrix}.$$

This matrix must be positive semidefinite by the preceding lemma. A consequence of this is that $\det \Lambda \geq 0$.

$$\det(\Lambda) = 4N^3h^2(8h - 5N) \geq 0$$

Then, for $N \gg 1$, we have that $\det(\Lambda) < 0$. Thus, it must be that $h = 0$. With $h = 0$, we have that $\|L_{-n}|\psi_0\rangle\| = 2nh = 0$ for all $n \in \mathbb{Z}_{>0}$. Combining this with the definition of highest weight representation, we have $L_n|\psi_0\rangle = 0$ for all $n \in \mathbb{Z}$. This means that the representation is the trivial representation. Thus, we find there does not exist a nontrivial, unitary highest weight representation of $\mathscr{W}$. ☐

4. Virasoro Algebra

Theorem 3.1 shows that $\mathscr{W}$ has no nontrivial, unitary, highest weight representation. Thus, we consider nontrivial central extensions of the $\mathscr{W}$ that have a nontrivial unitary highest weight representation. For this purpose define a map $\phi : \mathscr{W} \times \mathscr{W} \to \mathbb{C}$ by

$$\phi(f_n, f_m) = \frac{1}{12}n(n^2 - 1)\delta_{n,-m}$$

The following two lemmas will then be sufficient to determine the central extensions of $\mathscr{W}$.

Lemma 4.1 (see [4, Theorem 5.1])**.** *The map ϕ generates a nontrivial central extension $\mathscr{V} = \mathscr{W} \oplus \mathbb{C}$ of $\mathscr{W}$ by $\mathbb{C}$.*

Proof. It is clearly sufficient to prove the following two facts.

(1) $\phi \in C^2(\mathscr{W}; \mathbb{C})$
(2) $\phi \notin B^2(\mathscr{W}; \mathbb{C})$

We first consider (1).

- $\phi(f_n, f_m) = \frac{1}{12}n(n^2 - 1)\delta_{n,-m} = \frac{1}{12}(-m)(m^2 - 1)\delta_{m,-n} = -\frac{1}{12}m(m^2 - 1)\delta_{m,-n} = -\phi(f_m, f_n)$
-

$$\begin{aligned}
&\phi(f_n, [f_m, f_k]) + \phi(f_m, [f_k, f_n]) + \phi(f_k, [f_n, f_m]) \\
&= \phi(f_n, (m-k)f_{m+k}) + \phi(f_m, (k-n)f_{k+n}) + \phi(f_k, (n-m)f_{n+m}) \\
&= \frac{1}{12}n(m-k)(n^2-1)\delta_{n,-(m+k)} + \frac{1}{12}m(k-n)(m^2-1)\delta_{m,-(k+n))} + \frac{1}{12}k(n-m)(k^2-1)\delta_{k,-(n+m)} \\
&= \frac{1}{12}\delta_{n+m+k,0}[n(m-k)(n^2-1) + m(k-n)(m^2-1) + k(n-m)(k^2-1)] \\
&= \frac{1}{12}[n(n+2m)(n^2-1) - m(2n+m)(m^2-1) - (n^2-m^2)((n+m)^2-1)] = 0
\end{aligned}$$

Hence we find that $\phi \in C^2(\mathscr{W}; \mathbb{C})$. Next, suppose that $\phi \notin B^2(\mathscr{W}; \mathbb{C})$. Then, by Lemma 2, $\phi(x, y) = f([x, y])$ for some linear map $p : \mathscr{W} \to \mathbb{C}$. Then,

$$\phi(f_n, f_{-n}) = p([f_n, f_{-n}) = 2np(f_0).$$

Thus, we find that for all $n \in \mathbb{Z} \backslash \{0\}$

$$p(f_0) = \frac{1}{24}(n^2 - 1).$$

However, this cannot hold for all $n \in \mathbb{Z} \backslash \{0\}$, leading to a contradiction. Thus, we find that $\phi \notin B^2(\mathscr{W}; \mathbb{C})$. ☐

Lemma 4.2 (see [4, Theorem 5.1])**.** *Let $\phi : \mathscr{W} \times \mathscr{W} \to \mathbb{C}$ be a 2-cocycle on $\mathscr{W}$. Then, it must be that, up to a 2-coboundary, ϕ has the form*

$$\phi(f_m, f_n) = \frac{c}{12}m(m^2 - 1)\delta_{m,-n}$$

for some $c \in \mathbb{C}$.

Proof.
Let $n, m, k \in \mathbb{Z}$. As ϕ is a 2-cocycle, we have

$$\begin{aligned}&\phi(f_n, [f_m, f_k]) + \phi(f_m, [f_k, f_n]) + \phi(f_k, [f_n, f_m])\\&= (m-k)\phi(f_n, f_{m+k}) + (k-n)\phi(f_m, f_{k+n}) + (n-m)\phi(f_k, f_{n+m})\\&= 0.\end{aligned}$$

Letting $k = 0$ then gives that

$$m\phi(f_n, f_m) - n\phi(f_m, f_n) + (n-m)\phi(f_0, f_{n+m}) = (n+m)\phi(f_n, f_m) + (n-m)\phi(f_0, f_{n+m}) = 0.$$

This relation then reduces down to

$$\phi(f_n, f_m) = \frac{m-n}{n+m}\phi(f_0, f_{n+m})\ , \text{ for } n+m \neq 0$$

Next, we define a $\mathbb{C}$-linear map $p : \mathscr{W} \to \mathbb{C}$ by

$$p(f_n) = \begin{cases} -\frac{1}{2}\phi(f_1, f_{-1}) & n = 0 \\ \frac{1}{n}\phi(f_0, f_n) & n \neq 0 \end{cases}$$

We then consider the 2-cocycle $\widetilde{p} : \mathscr{W} \times \mathscr{W} \to \mathbb{C}$ defined by $\widetilde{p}(f_n, f_m) = p([f_n, f_m])$, and the map $\Phi := \phi + \widetilde{p}$. We note that $\Phi - \phi \in B^2(\mathscr{W}; \mathbb{C})$. Thus, it suffices to look at Φ.

Now let $n + m \neq 0$. Then,

$$\begin{aligned}\Phi(f_n, f_m) &= \phi(f_n, f_m) + \widetilde{p}(f_n, f_m)\\&= \frac{m-n}{n+m}\phi(f_0, f_{n+m}) + p([f_n, f_m])\\&= \frac{m-n}{n+m}\phi(f_0, f_{n+m}) - (m-n)p(f_{n+m})\\&= \frac{m-n}{n+m}\phi(f_0, f_{n+m}) - \frac{m-n}{n+m}\phi(f_0, f_{n+m})\\&= 0.\end{aligned}$$

Hence, we find that $\Phi(f_n, f_m)$ is proportional to $\delta_{n+m,0}$ and so we may write $\Phi(f_n, f_m)$ as

$$\Phi(f_n, f_m) = d(n, m)\delta_{n+m,0} \equiv d(n)\delta_{n+m,0}$$

where $d : \mathbb{Z} \to \mathbb{C}$. First, as Φ is a 2-cocycle, it must be antisymmetric. This then implies

$$\begin{aligned}d(n)\delta_{n+m,0} &= \Phi(f_n, f_m)\\&= -\Phi(f_m, f_n)\\&= -d(m)\delta_{n+m,0}\\&= -d(-n)\delta_{n+m,0}.\end{aligned}$$

Hence, we find

$$d(n) = -d(-n).$$

We may thus consider $n \in \mathbb{Z}_{>0}$. Then, using this property of $d(n)$, we find

$$\begin{aligned}&\Phi(f_n, [f_m, f_k]) + \Phi(f_m, [f_k, f_n]) + \Phi(f_k, [f_n, f_m])\\&= (m-k)\Phi(f_n, f_{m+k}) + (k-n)\Phi(f_m, f_{k+n}) + (n-m)\Phi(f_k, f_{n+m})\\&= [(m-k)d(n) + (k-n)d(m) + (n-m)d(k)]\delta_{n+m+k,0}\\&= (2m-n)d(n) - (2n+m)d(m) + (n-m)d(-n-m)\\&= (2m-n)d(n) - (2n+m)d(m) - (n-m)d(n+m)\\&= 0\end{aligned}$$

We then plug in $m = 1$ into the above relation to obtain a recursive formula for $d(n)$.

$$(n-1)d(n+1) - (n+2)d(n) + (2n+1)d(1) = 0$$

Evaluating $\Phi(L_1, L_{-1})$ explicitly gives

$$\begin{aligned}\Phi(f_1, f_{-1}) &= \phi(f_1, f_{-1}) + \widetilde{p}(f_1, f_{-1}) \\ &= \phi(f_1, f_{-1}) + p([f_1, f_{-1}]) \\ &= \phi(f_1, f_{-1}) + 2p(f_0) \\ &= \phi(f_1, f_{-1}) - \frac{2}{2}\phi(f_1, f_{-1}) \\ &= 0\end{aligned}$$

Thus, $\Phi(f_1, f_{-1}) = d(1) = 0$. The recursive formula then reduces down to

$$(n-1)d(n+1) - (n+2)d(n) = 0$$

Define $d(2) = \widetilde{c} \in \mathbb{C}$. We then claim that $d(n) = \frac{\widetilde{c}}{6}n(n^2-1)$ for all $n \in \mathbb{Z}$. Evidently $d(n) = -d(-n)$ and so we may still consider $n \in \mathbb{Z}_{>0}$. Thus, we proceed by induction on n. As $d(1) = 0$, we find this formula is satisfied for $n = 1$. Furthermore, $d(2) = \frac{6\widetilde{c}}{6} = \widetilde{c}$. Suppose now that this relation holds for $n \in \mathbb{Z}_{>1}$. Then,

$$\begin{aligned}d(n+1) &= \frac{n+2}{n-1}d(n) \\ &= \frac{n+2}{n-1}\frac{\widetilde{c}}{6}n(n^2-1) \\ &= \frac{\widetilde{c}}{6}\frac{n+2}{n-1}n(n+1)(n-1) \\ &= \frac{\widetilde{c}}{6}n(n+1)(n+2) \\ &= \frac{\widetilde{c}}{6}(n+1)[(n^2+2n+1)-1] \\ &= \frac{\widetilde{c}}{6}(n+1)[(n+1)^2-1]\end{aligned}$$

Thus, we confirm that $d(n) = \frac{\widetilde{c}}{6}n(n^2-1)$. As $\widetilde{c} \in \mathbb{C}$ was arbitrary, we relabel $\widetilde{c} = c/12$ for $c \in \mathbb{C}$. This then gives that

$$\Phi(f_n, f_m) = \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

□

Theorem 4.1. $H^2(\mathscr{W}; \mathbb{C}) \simeq \mathbb{C}$

Proof.
Consider $\phi : \mathscr{W} \times \mathscr{W} \to \mathbb{C}$, where

$$\phi(f_m, f_n) = \frac{1}{12}m(m^2-1)\delta_{m,-n}.$$

From Lemma 4.1 this is a nontrivial central extension of $\mathscr{W}$. Then, suppose $\pi \in C^2(\mathscr{W}; \mathbb{C})$. As shown in Lemma 4.2, $\pi = t\phi$ for some $t \in \mathbb{C}$, so that $H^2(\mathscr{W}; \mathbb{C}) \simeq \mathbb{C}$. □

Thus, the Witt algebra has a nontrivial central extension, and we call it the Virasoro algebra, $\mathscr{V}$.

Definition 4.1. The *Virasoro* algebra, $\mathscr{V}$, has basis $\{f_n\}_{n\in\mathbb{Z}} \cup \{c\}$ such that

$$\begin{aligned}[f_n, f_m] &= (n-m)f_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} \\ [c, f_n] &= 0 \text{ for all } n \in \mathbb{Z}\end{aligned}$$

As in the Witt algebra, define $\rho(f_n) = L_n$ for an arbitrary representaton ρ of $\mathscr{V}$. We may further define an antilinear involution $\omega : \mathscr{V} \to \mathscr{V}$ by

$$\begin{aligned}\omega(f_n) &= f_{-n} \\ \omega(c) &= c\end{aligned}$$

and thus a representation is unitary if $L_n^\dagger = L_{-n}$ and $c^\dagger = c$, ie. $c \in \mathbb{R}$.

We can characterize a highest weight representation of the Virasoro algebra by its central charge c and highest weight h. Hence, it is often convenient to write $\mathscr{V}(c, h)$ for the Virasoro algebra. Furthermore, as h is an eigenvalue of L_0 and represents energy or some observable quantity, it must be that $h \in \mathbb{R}$. By imposing that a highest weight representation of the Virasoro algebra be unitary, we may find restrictions on c and h.

Proposition 4.1. For a unitary highest weight representation of $\mathscr{V}(c, h)$, it must be that $c \geq 0$ and $h \geq 0$.

Proof. Let $|\psi_0\rangle$ denote the vacuum state and $n \in \mathbb{Z}_{>0}$. Then,

$$\begin{aligned}
\| L_{-n}|\psi_0\rangle \|^2 &= \langle \psi_0 | L_{-n}^\dagger L_{-n} | \psi_0 \rangle \\
&= \langle \psi_0 | L_n L_{-n} | \psi_0 \rangle \\
&= \langle \psi_0 | [L_n, L_{-n}] + L_{-n} L_n | \psi_0 \rangle \\
&= \langle \psi_0 | 2nL_0 + \frac{c}{12} n(n^2 - 1) | \psi_0 \rangle \\
&= (2nh + \frac{c}{12} n(n^2 - 1)) \| |\psi_0\rangle \|^2
\end{aligned}$$

This then gives the restriction that

$$2nh + \frac{c}{12} n(n^2 - 1) \geq 0$$

First suppose that $n = 1$. Then, the above inequality gives that $2h \geq 0$, or that $h \geq 0$. Suppose then that $c < 0$. As $h > 0$ and $n(n^2 - 1)$ grows faster than n, there must exist a $N \in \mathbb{Z}_{>0}$ such that $2Nh + \frac{c}{12} N(N^2 - 1) < 0$, leading to a contradiction. Hence, we find that $c \geq 0$. □

Proposition 4.2. Any unitary highest weight representation of $\mathscr{V}$ is irreducible.

Hence, by considering unitary highest weight representations of $\mathscr{V}$ we necessarily have irreducibility.

5. Heisenberg Algebra and its Fock Representation

The algebra associated with the simple harmonic oscillator can be made more general into the Heisenberg algebra.

Definition 5.1. The *Heisenberg algebra* $\mathfrak{h}$ has basis $\{t_n \mid n \in \mathbb{Z}\} \cup \{k\}$ such that

$$\begin{aligned}
[t_n, t_m] &= nk\delta_{n+m,0} \\
[k, t_n] &= 0, \text{ for all } n
\end{aligned}$$

For an arbitrary representation ρ of $\mathfrak{h}$ we shall henceforth denote $\rho(t_n) = \phi_n$. As in the cases for $\mathscr{W}$ and $\mathscr{V}$, we may define an antilinear involution ω on $\mathfrak{h}$ such that

$$\begin{aligned}
\omega(t_n) &= t_{-n} \\
\omega(k) &= k.
\end{aligned}$$

Hence, a representation $\mathfrak{h}$ is unitary if $\phi_n^\dagger = \phi_{-n}$ and $k \in \mathbb{R}$. As in Appendix A, define the Fock space $\mathscr{F}$ by

$$\mathscr{F} = \{ |N_1 N_2 N_3 \cdots \rangle \mid N_k = 0 \text{ for all but finitely many } N_k \}.$$

It is possible to construct a representation of $\mathfrak{h}$ on the space $\mathscr{F}$ by identifying ϕ_n with the creation and annihilation operators from the simple harmonic oscillator as discussed in Appendix A. Let $n \in \mathbb{Z}_{>0}$ and identify ϕ_n and ϕ_{-n} as

$$\begin{aligned}
\phi_{-n} &= \sqrt{nk}\, a_n^\dagger \\
\phi_n &= \sqrt{nk}\, a_n.
\end{aligned}$$

Furthermore, define $\phi_0|N\rangle = 0$ and $k = id$, for a state $|N\rangle \in \mathscr{F}$. It is easy to check that this identification yields the appropriate Lie brackets on $\mathfrak{h}$. Let $n, m \in \mathbb{Z}_{>0}$.

(1) $[\phi_n, \phi_m] = k\sqrt{nm}\, [a_n, a_m] = 0$
(2) $[\phi_n, \phi_{-m}] = k\sqrt{nm}\, [a_n, a_m^\dagger] = k\sqrt{nm}\delta_{n,m} = kn\delta_{n,m}$
(3) $[\phi_n, k] = k\sqrt{nk}\, [a_n, 1] = 0$
(4) $[\phi_{-n}, k] = k\sqrt{nk}\, [a_n^\dagger, 1] = 0$

And as ϕ_0 annihilates each state, it evidently commutes with all ϕ_n and k. Hence, we have that $[\phi_n, \phi_m] = kn\delta_{n+m,0}$ and $[\phi_n, k] = 0$, which is precisely the Heisenberg Algebra. With this identification we find that for $n \in \mathbb{Z}_{>0}$ and $|N_1 N_2 \cdots\rangle \in \mathscr{F}$

(1) $\phi_n|N_1 N_2 \cdots 0_n \cdots\rangle = 0$

(2) $\phi_n|N_1 N_2 \cdots N_n \cdots\rangle = \sqrt{nk}\sqrt{N_n}\,|N_1 N_2 \cdots N_n - 1 \cdots\rangle$

(3) $\phi_{-n}|N_1 N_2 \cdots N_n \cdots\rangle = \sqrt{nk}\sqrt{N_n + 1}\,|N_1 N_2 \cdots N_n + 1 \cdots\rangle$

Using these actions on $\mathscr{F}$, we have a representation of $\mathfrak{h}$ on $\mathscr{F}$, called the *Fock representation*. Moreover, as in the simple harmonic oscillator case, we may construct an arbitrary state in $\mathscr{F}$ using $a_n^\dagger$. We may similarly do so with ϕ_{-n} for $n \in \mathbb{Z}_{>0}$.

$$|N_1 N_2 \cdots\rangle \prod_{n=1}^{\infty} \frac{\phi_{-n}^{N_n}}{\sqrt{N_n!}\sqrt{nk^{N_n}}}$$

Proposition 5.1. The representation constructed for $\mathfrak{h}$ is unitary.

Proof.
We need only show that $\phi_n^\dagger = \phi_{-n}$ for all $n \in \mathbb{Z}\backslash\{0\}$.

Case 5.1.1. Suppose $n \in \mathbb{Z}_{>0}$. Then, as $k \in \mathbb{R}$,

$$\phi_n^\dagger = (\sqrt{nk}\,a_n)^\dagger = \sqrt{nk}\,a_n^\dagger = \phi_{-n}$$

Case 5.1.2. Suppose $n = -m$ for $m \in \mathbb{Z}_{<0}$. Then

$$\phi_n^\dagger = \phi_{-m}^\dagger = (\sqrt{mk}\,a_m^\dagger)^\dagger = \sqrt{mk}\,a_m = \phi_m = \phi_{-n}$$

□

Now recall the definition of a nilpotent Lie algebra.

Definition 5.2. A Lie algebra $\mathfrak{g}$ is *nilpotent* if for some fixed $n \in Z_{n>0}$,

$$\mathrm{ad}_{x_1}\mathrm{ad}_{x_2} \cdots \mathrm{ad}_{x_n} y = 0$$

for all $x_1, x_2, \ldots, x_n, y \in \mathfrak{g}$.

Proposition 5.2. The Heisenberg algebra $\mathfrak{h}$ is nilpotent. Specifically, all double brackets vanish.

Proof.
It clearly suffices to show that $[[\phi_n, \phi_m], \phi_s] = 0$ for any $n, m, s \in \mathbb{Z}$. But this is easy as k commutes with all ϕ_n.

$$[[\phi_n, \phi_m], \phi_s] = [nk\delta_{n+m,0}, \phi_s] = nk\delta_{n+m,0}[k, \phi_s] = 0$$

□

6. Sugawara's Construction of $\mathscr{V}(1,0)$

We will use a specific case of Sugawara's construction to formulate the Fock representation of $\mathscr{V}(1,0)$ from the Heisenberg algebra representation already constructed (see Goddard and Olive [2], Kac [5], Schlichenmaier [6], Schlichenmaier and Sheinman [7]). Thus, consider the Heisenberg algebra spanned by $\{\phi_n \mid n \in \mathbb{Z}\} \cup \{k\}$ as above. We then define elements L_n, known as the Sugawara operators, as

$$L_n = \frac{1}{2k} \sum_{j \in \mathbb{Z}} : \phi_{-j}\phi_{j+n} :$$

where $: \phi_{-j}\phi_{j+n} :$ denotes *normal ordering*, defined by

$$: \phi_{-j}\phi_{j+n} := \begin{cases} \phi_{-j}\phi_{j+n} & -j \leq j+n \\ \phi_{j+n}\phi_{-j} & -j \geq j+n \end{cases}.$$

To see the importance of normal ordering, consider L_0 without normal ordering acting on the vacuum state denoted by $|0\rangle$ in the space $\mathscr{F}$ (ie. the state for which $N_j = 0$ for all j).

$$
\begin{aligned}
L_0|0\rangle &= \frac{1}{2k}\sum_{j\in\mathbb{Z}} \phi_{-j}\phi_j|0\rangle \\
&= \frac{1}{2k}\left(\sum_{j=0}^{\infty} \phi_{-j}\phi_j|0\rangle + \sum_{j=-\infty}^{-1} \phi_{-j}\phi_j|0\rangle\right) \\
&= \frac{1}{2k}\sum_{j=1}^{\infty} \phi_j\phi_{-j}|0\rangle \\
&= \frac{\sqrt{k}}{2k}\sum_{j=1}^{\infty} \sqrt{j}\,\phi_j|0\,0\cdots 1_j\cdots\rangle \\
&= \frac{1}{2}\sum_{j=1}^{\infty} j|0\rangle \to \infty|0\rangle
\end{aligned}
$$

Thus, we find that without normal ordering we get an eigenvalue of ∞ which is not desirable. Conversely, with normal ordering incorporated we find that

$$
\begin{aligned}
L_0|0\rangle &= \frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{-j}\phi_j:|0\rangle \\
&= \frac{2}{2k}\sum_{j=1}^{\infty} \phi_{-j}\phi_j|0\rangle = 0
\end{aligned}
$$

We will now show that the definition of L_n above gives rise to the Virasoro algebra $\mathscr{V}(1,0)$.

Proposition 6.1. $[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}n(n^2-1)\delta_{n,-m}$

Proof.
First, from Proposition 5.2, the Heisenberg algebra is nilpotent and all double brackets vanish. Using this, we find for $a, b, c, d \in \mathbb{Z}$,

$$
\begin{aligned}
[\phi_a\phi_b, \phi_c\phi_d] &= [[\phi_a, \phi_b] + \phi_b\phi_a, \phi_c\phi_d] \\
&= [[\phi_a, \phi_b], \phi_c\phi_d] + [\phi_b\phi_a, \phi_c\phi_d] \\
&= \phi_c[[\phi_a\phi_b], \phi_d] + [[\phi_a, \phi_b], \phi_c]\phi_d + [\phi_b\phi_a, \phi_c\phi_d] \\
&= [\phi_b\phi_a, \phi_c\phi_d]
\end{aligned}
$$

A consequence of this result is that we may ignore normal ordering in $[L_n, L_m]$. We now compute explicitly $[L_n, L_m]$.

Case 6.1.1. Suppose that $m + n \neq 0$.

$$
\begin{aligned}
[L_n, L_m] &= [\frac{1}{2k}\sum_{i\in\mathbb{Z}} :\phi_{-i}\phi_{i+n}:, \frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{-j}\phi_{j+m}:] \\
&= \frac{1}{4k^2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}[:\phi_{-i}\phi_{i+n}:, :\phi_{-j}\phi_{j+m}:] \\
&= \frac{1}{4k^2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}[\phi_{-i}\phi_{i+n}, \phi_{-j}\phi_{j+m}] \\
&= \frac{1}{4k^2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left\{\phi_{-i}(\phi_{-j}[\phi_{i+n}, \phi_{j+m}] + [\phi_{i+n}, \phi_{-j}]\phi_{j+m}) + (\phi_{-j}[\phi_{-i}, \phi_{j+m}] + [\phi_{-i}, \phi_{-j}]\phi_{j+m})\phi_{i+n}\right\} \\
&= \frac{1}{4k^2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left\{\phi_{-i}\phi_{-j}(i+n)k\delta_{i+n,-j-m} + \phi_{-i}\phi_{j+m}(i+n)k\delta_{i+n,j} - \phi_{-j}\phi_{i+n}ik\delta_{-i,-j-m} - \phi_{j+m}\phi_{i+n}ik\delta_{-i,j}\right\} \\
&= \frac{1}{2k}(\sum_{i\in\mathbb{Z}}(i+n)\phi_{-i}\phi_{i+n+m} - \sum_{i\in\mathbb{Z}} i\phi_{-i+m}\phi_{i+n})
\end{aligned}
$$

Then, replace i by $i+m$ in the second summation to obtain

$$[L_n, L_m] = \frac{n-m}{2k} \sum_{i\in\mathbb{Z}} \phi_{-i}\phi_{i+n+m}.$$

However, as $n+m \neq 0$ we have that $[\phi_{-i}, \phi_{i+n+m}] = 0$ and so the summation is equivalent to the normal ordered form. Hence,

$$[L_n, L_m] = \frac{n-m}{2k} \sum_{i\in\mathbb{Z}} :\phi_{-i}\phi_{i+n+m} := (n-m)L_{n+m}$$

Case 6.1.2. Now suppose that $m+n=0$. Notice that $[L_0, L_0] = 0$ and $[L_{-n}, L_n] = -[L_n, L_{-n}]$ so we may assume that $n > 0$ without loss of generality. Then, we may write L_n as

$$L_n = \frac{1}{2k}(\sum_{j\geq 0} \phi_{-j}\phi_{j+n} + \sum_{j>0} \phi_{-j+n}\phi_j).$$

From a similar calculation as above, we find

$$\begin{aligned}
[L_n, L_{-n}] &= \frac{2k}{4k^2} \sum_{j>0} \{(j+n)\phi_{-j}\phi_j - j\phi_{-j-n}\phi_{j+n} + j\phi_{-j+n}\phi_{j-n} + (n-j)\phi_{-j}\phi_j\} \\
&= \frac{1}{2k}(2knL_0 + \sum_{j>0} j\phi_{-j+n}\phi_{j-n} - \sum_{j>0} j\phi_{-j-m}\phi_{j+m})
\end{aligned}$$

Replacing j by $j+n$ in first summation and j and $j-n$ in the second summation, we obtain

$$\begin{aligned}
[L_n, L_{-n}] &= \frac{1}{2k}(2knL_0 + \sum_{j>-n} (j+n)\phi_{-j}\phi_j + \sum_{j>n} (n-j)\phi_{-j}\phi_j) \\
&= \frac{1}{2k}(2knL_0 + \sum_{j=0}^{\infty} (j+n)\phi_{-j}\phi_j + \sum_{j=-n+1}^{-1} (j+n)\phi_{-j}\phi_j + \sum_{j=1}^{\infty} (n-j)\phi_{-j}\phi_j - \sum_{j=1}^{n-1} (n-j)\phi_{-j}\phi_j) \\
&= \frac{1}{2k}(2knL_0 + \sum_{j=1}^{n-1} (n-j)[\phi_j, \phi_{-j}] + \sum_{j=0}^{\infty} (j+n)\phi_{-j}\phi_j + \sum_{j=-\infty}^{-1} (j+n)\phi_j\phi_{-j}) \\
&= \frac{1}{2k}(2knL_0 + k\sum_{j=1}^{n-1} j(n-j) + \sum_{j\in\mathbb{Z}} :\phi_{-j}\phi_j:) \\
&= \frac{1}{2k}(4knL_0 + k\sum_{j=1}^{n-1} j(n-j)).
\end{aligned}$$

The finite sum is calculated as

$$\begin{aligned}
\sum_{j=1}^{n-1} j(n-j) &= n\sum_{j=0}^{n-1} j - \sum_{j=0}^{n-1} j^2 \\
&= \frac{n^2(n-1)}{2} - \frac{n(n-1)(2n-1)}{6} \\
&= \frac{n(n-1)(n+1)}{6} \\
&= \frac{n(n^2-1)}{6}.
\end{aligned}$$

Hence, we obtain

$$[L_n, L_{-n}] = 2nL_0 + \frac{1}{12}n(n^2-1).$$

Combining the two above cases gives that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}n(n^2-1)\delta_{n,-m}$$

□

Then, defining $c = id$ on $\mathscr{F}$ and using L_n defined in the above way, we obtain a representation for the Virasoro algebra with $c = 1$ on the space (known as the Fock representation).

Proposition 6.2. The representation constructed for $\mathscr{V}$ is a unitary highest weight representation with vacuum state $|0\rangle$ and $h = 0$.

Proof.
First, by the above computation, we found that $L_0|0\rangle = 0$. Then, let $n \in \mathbb{Z}_{>0}$ and consider $L_n|0\rangle$.

Case 6.2.1. Suppose n is even. Then, from the definition of normal ordering, we find

$$\begin{aligned} L_n|0\rangle &= \frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{-j}\phi_{j+n}: |0\rangle \\ &= \frac{1}{2k}(\sum_{j=-\infty}^{-n/2} \phi_{j+n}\phi_{-j}|0\rangle + \sum_{j=-n/2+1}^{\infty} \phi_{-j}\phi_{j+n}|0\rangle) \\ &= \frac{1}{2k}(\sum_{j=-\infty}^{-n/2} \phi_{j+n}(0) + \sum_{j=-n/2+1}^{\infty} \phi_{-j}(0)) = 0 \end{aligned}$$

where the second sum goes to zero since $j \geq -n/2 + 1$ and so $j + n \geq n/2 + 1 > 0$.

Case 6.2.2. Suppose now that n is odd. Then,

$$\begin{aligned} L_n|0\rangle &= \frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{-j}\phi_{j+n}: |0\rangle \\ &= \frac{1}{2k}(\sum_{j=-\infty}^{-\frac{n-1}{2}} \phi_{j+n}\phi_{-j}|0\rangle + \sum_{j=-\frac{n-1}{2}+1}^{\infty} \phi_{-j}\phi_{j+n}|0\rangle) \\ &= \frac{1}{2k}(\sum_{j=-\infty}^{-\frac{n-1}{2}} \phi_{j+n}(0) + \sum_{j=-\frac{n-1}{2}+1}^{\infty} \phi_{-j}(0)) = 0 \end{aligned}$$

where again the second sum goes to zero since $j \geq -(n-1)/2 + 1$ and so $j + n \geq 0$.

Thus, this is a highest weight representation with $h = 0$.

To illustrate unitarity of the representation, recall the following antilinear involution on $\mathfrak{h}$.

$$\omega(t_n) = t_{-n}$$
$$\omega(k) = k$$

By replacing k with c and t_n with f_n we also get the antilinear involution on $\mathscr{V}$ from before. Hence, it suffices to show that $L_n^\dagger = L_{-n}$. But this is trivial since $:\phi_{-j}\phi_{j+n}: = :\phi_{j+n}\phi_{-j}:$ and $k \in \mathbb{R}$.

$$\begin{aligned} L_n^\dagger &= (\frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{-j}\phi_{j+n}:)^\dagger \\ &= \frac{1}{2k}\sum_{j\in\mathbb{Z}} :(\phi_{-j}\phi_{j+n})^\dagger: = \frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{j+n}^\dagger\phi_{-j}^\dagger: \\ &= \frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{-j-n}\phi_j: = \frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{j-n}\phi_{-j}: \\ &= \frac{1}{2k}\sum_{j\in\mathbb{Z}} :\phi_{-j}\phi_{j-n}: = L_{-n} \end{aligned}$$

∎

7. Generalized Sugawara's Construction of the Virasoro Algebra

It is possible to derive the above construction of $\mathscr{V}(1,0)$ from a more general Sugawara construction (see Goddard and Olive [2], Kac [5], Schlichenmaier [6]) of affine Lie algebras. This construction will then give rise to the Virasoro algebra for other c values.

Let $\mathfrak{g}$ be a Lie algebra with $\dim \mathfrak{g} = d < \infty$. Suppose $\mathfrak{g}$ has a symmetric, non-degenerate bilinear form $\langle .,. \rangle$. Define the *affine Lie algebra* $\hat{\mathfrak{g}}$ as a central extension of $\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ by $\mathbb{C}$, ie. as a vector space $\hat{\mathfrak{g}}$ is

$$\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]) \oplus \mathbb{C}t.$$

Suppose V is a vector space such that $v \in V$ is annihilated by $y \otimes z^n$ for all $y \in \mathfrak{g}$ for n large enough. A vector space satisfying this property is an *admissible representation* of $\hat{\mathfrak{g}}$. Furthermore, suppose that $tv = cv$ for some scalar c and $\{u_i\}_{i=1}^d$ and $\{u^i\}_{i=1}^d$ are bases for $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively (* represents dual space). Recall that a Casimir element of a Lie algebra is an element in the center and that the Casimir element of $\mathfrak{g}$ is given by

$$C_{\mathfrak{g}} = \sum_{i=1}^{d} u_i u^i$$

Further suppose its action is scalar multiplication on the adjoint representation. In this case we define κ (the *dual Coxeter number*) as

$$\kappa = \frac{1}{2} \sum_{i=1}^{d} ad_{u_i} ad_{u^i}.$$

Suppose that $c + \kappa \neq 0$ and define operators T_n by

$$T_n := -\frac{1}{2(c+\kappa)} \sum_{j \in \mathbb{Z}} \sum_{i=1}^{d} : u_{i(-j)} u^{i(j+n)} :$$

where normal ordering is as before.

Theorem 7.1. *Consider the Virasoro algebra with basis $\{f_n\}_{\in \mathbb{Z}} \cup \{t\}$. Then, the mapping defined by*

$$f_n \mapsto T_n$$
$$t \mapsto id_V$$

is a representation for the Virasoro algebra with central charge

$$\frac{c \dim \mathfrak{g}}{c + \kappa}$$

References

[1] Sternberg, S., *Group Theory and Physics*, Cambridge University Press, Cambridge, pp. 16-21.
[2] Goddard, P., Olive, D., *Kac-Moody and Virasoro Algebras in Relation to Quantum Physics*, Int. J. Mod. Phys. A **1** (1986), pp. 303-414
[3] Schottenloher, M., *Central Extensions of Lie Algebras and Bargmann's Theorem*, Lect. Notes Phys. **759** (2008), pp. 63-73.
[4] Schottenloher, M., *The Virasoro Algebra*, Lect. Notes Phys. **759** (2008), pp. 75-85.
[5] Kac, V.G., 1990, *Infinite Dimensional Lie Algebras*, Cambridge University Press, Cambridge, pp. 228-234.
[6] Schlichenmaier, M., *Sugawara Construction for Higher Genus Riemann Surfaces*, Rept. Math. Phys. **43** (1999), pp. 323-339.
[7] Schlichenmaier, M., Sheinman, O.K., *Sugawara Construction and Casimir Operators for Krichever-Novikov Algebras*, Journal of Mathematical Sciences, **92** (1998), pp. 3807-3834.

Appendix A. Simple Harmonic Oscillator of Countable Degrees of Freedom

Consider a simple harmonic oscillator system of countable degrees of freedom. It is then possible to define annihilation and creation operators a_k and $a_k^\dagger$, respectively, for $k \in \mathbb{Z}_{>0}$.

$$a_k = \sqrt{\frac{m\omega}{2\hbar}} (x_k + \frac{i}{m\omega} p_k)$$
$$a_k^\dagger = \sqrt{\frac{m\omega}{2\hbar}} (x_k - \frac{i}{m\omega} p_k)$$

Using these definitions of the annihilation and creation operators, we find that the Hamiltonian $\mathscr{H}_k$ for the k^{th} degree of freedom is thus given by

$$\mathscr{H}_k = \frac{p_k^2}{2m} + \frac{1}{2}m\omega x_k^2 = \hbar\omega(a_k^\dagger a_k + \frac{1}{2})$$

where the total Hamiltonian is $\mathscr{H} = \sum \mathscr{H}_k$. However, with this definition of the Hamiltonian, the ground state energy is already infinite (each degree of freedom contributes $\frac{1}{2}\hbar\omega$). Hence we simply rescale $\mathscr{H}_k$ so that

$$\mathscr{H} = \sum_{k=1}^{\infty} \mathscr{H}_k = \sum_{k=1}^{\infty} \hbar\omega a_k^\dagger a_k$$

It is clear that this rescaling is acceptable as only the energy difference between states is of interest. Furthermore, using the canonical commutation relations $[x_i, p_j] = i\hbar\delta_{i,j}$ and $[x_i, x_j] = [p_i, p_j] = 0$, we find the only non-vanishing commutators are

$$[a_k, a_l^\dagger] = \delta_{k,l}$$

Then, let $N_k \in Z_{\geq 0}$ and consider the set of states $\mathscr{F}$ given by

$$\mathscr{F} = \{|N_1 N_2 N_3 \cdots\rangle \mid N_k = 0 \text{ for all but finitely many } N_k\}$$

This space is known as the oscillator *Fock space*. It is then well known from quantum theory that a_k and $a_k^\dagger$ acting on a given state in $\mathscr{F}$ gives

(1) $a_k|N_1 N_2 \cdots N_k \cdots\rangle = \sqrt{N_k}|N_1 N_2 \cdots N_k - 1 \cdots\rangle$

(2) $a_k^\dagger|N_1 N_2 \cdots N_k \cdots\rangle = \sqrt{N_k + 1}|N_1 N_2 \cdots N_k + 1 \cdots\rangle$

Furthermore, for a state $|N_1 N_2 \cdots 0_k \cdots\rangle$, where 0_k represents a zero in the k^{th} component,

$$a_k|N_1 N_2 \cdots 0_k \cdots\rangle = 0.$$

The state for which all $N_i = 0$ is denoted the ground state of the system. It is then possible to build each state in $\mathscr{F}$ from the ground state using the action of $a_k^\dagger$ on arbitrary states.

$$|N_1 N_2 \cdots\rangle = \prod_{k=1}^{\infty} \frac{(a_k^\dagger)^{N_k}}{\sqrt{N_k!}}\, |0\,0 \cdots\rangle$$

As this state belongs to $\mathscr{F}$, only finitely many N_k are nonzero.