Theorem: \( X, Y \in \text{metric spaces}, f : E \to Y \) \( p \) limit point of \( E \)

Then \( \lim_{x \to p} f(x) = \delta \) iff \( \lim_{n \to \infty} f(p_n) = \delta \)

\( \forall p_n \in p \Rightarrow p_n \to p, \lim_{n \to \infty} p_n = p \)

Proof: trivial

Definition: \( X, Y \) metric spaces, \( E \subset X \), \( p \in E \) \( f : E \to Y \)

\( f \) is continuous at \( p \) iff for every \( \epsilon > 0 \) \( \exists \delta > 0 \) \( d_Y(f(x), f(p)) < \epsilon \) \( \forall x \in E \) \( \delta < d_X(x, p) \)

Theorem: Assume that \( p \) is a limit point of \( E \). Then \( f \) is continuous at \( p \) iff \( \lim_{x \to p} f(x) = f(p) \).

Theorem: \( X, Y, Z \) metric spaces, \( E \subset X \) \( f : E \to Y \), \( g : f(E) \to Z \), \( h : E \to Z \) given by

\( h(x) = g(f(x)) \). If \( f \) is continuous at \( p \in E \) \( g \) is continuous at \( f(p) \), then \( h \) is continuous at \( p \).
Proof: Immediate

**Theorem:** A mapping \( f : X \to Y \) is continuous iff \( f^{-1}(V) \) is open in \( X \) for every open set \( V \subseteq Y \).

**Proof:** Suppose that \( f \) is continuous on \( X \) and \( V \) is open in \( Y \).

Suppose that \( p \in X \) and \( f(p) \in V \). Since \( V \) is open, \( \exists \delta > 0 \) such that \( \forall y \in V \), if \( d_Y(f(p), y) < \delta \), which implies that \( \exists \delta > 0 \), \( d_Y(f(p), y) < \delta \), if \( d_X(x, p) < \delta \). Thus \( x \in f^{-1}(V) \) as soon as \( d_X(x, p) < \delta \).

Conversely, suppose that \( f^{-1}(V) \) is open in \( X \) for every \( V \) open in \( Y \).

Fix \( p \in X \) and \( \epsilon > 0 \), let \( V = \{ y \in Y : d_Y(y, f(p)) < \epsilon \} \).

Then \( V \) is open, so \( f^{-1}(V) \) is open, so \( \exists \delta > 0 \) such that \( x \in f^{-1}(V) \) as soon as \( d_X(x, p) < \delta \). But if \( x \in f^{-1}(V) \), \( f(x) \in V \), so \( d_Y(f(x), f(p)) < \epsilon \).
Corollary: \( f : X \to Y \) continuous if and only if \( f^{-1}(C) \) is closed for every \( C \) closed in \( Y \).

**Theorem:** \( f_1, \ldots, f_k \) functions on \( X \),
\[ f : X \to \mathbb{R}^k \quad f(x) = (f_1(x), \ldots, f_k(x)) \quad x \in X \]
Then \( f \) is continuous if and only if each \( f_i \) is continuous.
\[ f, g \text{ continuous } \implies f \circ g, f + g \text{ continuous} \]

**Proof:** Metric comparison.

**Examples:** Boundless...

**Definition:** \( f : E \to \mathbb{R}^k \) bounded if \( \exists M \geq 0 \) such that \( |f(x)| \leq M \quad \forall x \in E \).

**Theorem:** Suppose that \( f \) is a continuous mapping of a compact metric space \( X \) to \( Y \). Then \( f(X) \) is compact.

**Proof:** Let \( \{ V_\alpha \} \) be an open cover of \( f(X) \). Since \( f \) is continuous, \( f^{-1}(V_\alpha) \) is open. By compactness,
\[ X \subset f^{-1}(V_\alpha) \cup \cdots \cup f^{-1}(V_{\alpha_n}) \]
Since \( f(f^{-1}(E)) \subseteq E \), \( f(X) \subseteq V_\alpha \cup \cdots \cup V_{\alpha_n} \) and we are done!
Theorem: If \( f \) is a continuous mapping of a compact metric space \( X \) into \( \mathbb{R}^k \), then \( f(X) \) is closed and bounded. Thus \( f \) is bounded.

Theorem: If continuous real valued on a compact metric space and \( M = \sup_{p \in X} f(p) \), \( m = \inf_{p \in X} f(p) \).

Then \( \exists \ p, q \in X \in f(p) = M \) \& \( f(q) = m \).

Theorem: \( f \) continuous 1-1 mapping of \( X \) compact to \( Y \) metric. Then \( f \) is a continuous mapping of \( Y \) to \( X \).

Proof: It suffices to prove that \( f^{-1}(V) \) is open if \( V \) is open. Since \( V \) is closed, \( f^{-1}(V) \) is compact and hence closed in \( Y \). Since \( f \) is 1-1, \( f^{-1}(V) \) is a complement of \( f^{-1}(V^c) \).

Definition: We say that \( f \) is uniformly continuous if for every \( \varepsilon > 0 \) \( \exists \ \delta > 0 \) \( \forall \) \( p, q \in X \) for which
\[
\forall (p, q) \in f \quad d_X(p, q) < \delta \quad \Rightarrow \quad d_Y(f(p), f(q)) < \varepsilon.
\]
Theorem: \( f : X \rightarrow Y \), metric \( X \) compact

Then \( f \) is uniformly continuous.

**Proof:** Given \( \varepsilon > 0 \), \( \exists \delta(p) \) for each \( p \in X \)

\( \exists \in X, \ d(x, \varepsilon) < \delta(p) \Rightarrow d_y(f(p), f(\varepsilon)) < \varepsilon \)

Let \( J(p) = \{ x \in X : d(x, \varepsilon) < \frac{1}{2} \delta(p) \} \)

an open cover. Now argue by compactness.