Contemporary Mathematics

Homotopy groups with coefficients

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Abstract. This paper has two goals. It is an expository paper on homotopy groups with coefficients in an abelian group and it contains new results which correct old errors and omissions in low dimensions. The homotopy groups with coefficients are functors on the homotopy category of pointed spaces. They satisfy a universal coefficient theorem, give long exact sequences when applied to fibrations, and have Hurewicz maps into homology groups with coefficients.

When the coefficient group is finitely generated, homotopy group functors are corepresentable as homotopy classes of maps out of a Peterson space. A Peterson space is a space with exactly one nonzero integral reduced cohomology group which is the coefficient group. Kan and Whitehead showed that Peterson spaces do not exist for the coefficient group of the rational numbers.

Of course, rational homotopy groups can be defined by tensoring the classical homotopy groups with the rationals. For many nonfinitely generated groups, homotopy groups can be defined by a direct limit of homotopy groups with coefficients from finitely generated subgroups. This depends on having sufficient functoriality in the Peterson spaces of the coefficient subgroups. Coefficient group functoriality fails in the presence of 2-torsion.

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2000 *Mathematics Subject Classification.* Primary 55-02, 55P60; Secondary 13-02, 13D07.
Key words and phrases. homotopy groups.

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1. Introduction

Let X be a pointed space, A an abelian group, and n a positive integer. Homotopy groups of X with coefficients in A, denoted $\pi_n(X; A)$, were introduced in the thesis of Frank Peterson [**25**] and further developed in this author's thesis [**21**]. Both of these were heavily influenced by John Moore.

We can define homotopy groups $\pi_n(X; A)$ with coefficients in any finitely generated abelian group A. But if there is two torsion in A or if $n = 3$, these groups are not functors of the coefficient groups. There are intrinsic failures here, especially at the prime 2, but also at odd primes in dimension 3. This has two consequences.

When there is two torsion or if $n = 3$, it is impossible to make a natural definition of homotopy groups with non-finitely generated coefficients. In all other cases, we can define homotopy groups with coefficients $\pi_n(X; A)$ in an arbitrary abelian group A by taking direct limits over all finitely generated subgroups of A.

Failure of functoriality in the coefficient group also leads to the fact that the homotopy groups may have a larger exponent than the coefficient group. We focus on the case of coefficients in cyclic groups of prime power order p^r. These homotopy groups have exponent p^r unless $p^r = 2$ in which case the exponent is 4. For the 3-dimensional homotopy groups, these exponents are determined by means of a mod p^r version of the classical Hopf invariant.

The equality of the exponent for the coefficient group and the homotopy group is equivalent to decomposing the smash product of two Peterson spaces into a bouquet of two Peterson spaces. The smash product decomposition leads to the definition of Samelson products in homotopy groups with coefficients mod p^r unless $p^r = 2$. When p is a prime greater than 3, these Samelson products give the structure of a graded Lie algebra to the homotopy groups of a loop space. When p is 2 or 3, the Jacobi identity may fail. When $p = 3$, we show for the first time here that it may fail in mod 3 homotopy groups. This result was proved in the author's thesis. Unfortunately, in the original version of this paper, I made the mistake of thinking that the same argument would show that the Jacobi identity may fail in mod 3^r homotopy, $r \geq 2$. Brayton Gray pointed out that the Jacobi identity would be valid in mod 3^r homotopy, $r \geq 2$.

When $p = 2$, the Samelson products may fail to be anti-symmetric.

The above decompositions of smash products of Peterson spaces are unique up to composition with Whitehead products. This uniqueness is essential to the proof that the Samelson products give a graded Lie algebra structure in homotopy groups with mod p^r coefficients, $p \geq 3$. The proof of uniqueness given here is different from that given in the author's book [**24**]. It is a more natural argument, depending on the Hilton-Milnor theorem and the geometric Hurewicz map.

If p is an odd prime, the mod p homotopy Bockstein spectral sequence of a loop space $E^r(\Omega X)$ is in all dimensions ≥ 1 a spectral sequence of abelian groups. If $p = 2$ and $r \geq 2$, then it is a spectral sequence of abelian groups in all dimensions.

If p is an odd prime, the mod p homotopy Bockstein spectral sequence is in all dimensions ≥ 1 a spectral sequence of Lie algebras except for the fact that the Jacobi identity and a related triple vanishing identity may fail if $p = 3$.

The mod p homotopy Bockstein spectral sequence $E^r(\Omega X)$ restores the Lie algebra identities for Samelson products. For example, if $p = 3$, the Lie identities are valid in the mod 3 homotopy Bockstein spectral sequence if $r \geq 3$. It was known that, if $p \geq 5$, the Lie identities are valid for all $r \geq 1$.

There is a necessary modification of the some maps related to the homotopy Bockstein spectral sequence at the prime 2 or even at odd primes in dimension 3. For some purposes, the multiples of the identity map on a Peterson space must be replaced by so-called fake multiples. For example, the composition

$$p^{r-1} : Z/p^r Z \to Z/pZ \to Z/p^r Z$$

does not correspond to the geometry of Peterson spaces if $p = 2$ or if p is an odd prime and the Peterson space is in dimension 3. But this modification does not change the form of the final description of the homotopy Bockstein spectral sequence from what was given in the author's book [**24**].

The author hopes that the presentation here will be definitive for homotopy groups with odd primary coefficients. It is an abridgment, modification, and correction of the treatment of homotopy groups with coefficients in the author's book [**24**]. It is said that mistakes in the design of Islamic prayer rugs are intentionally introduced as a show of humility since only Allah can produce perfection. And the Japanese Buddhist aesthetic known as wabi-sabi requires that beauty be imperfect, impermanent, and incomplete. The author has unintentionally followed these quidelines in previous works.

The ultimate goal of the author's thesis [**21**] was applications to stable homotopy theory with the result that the theory was confined to dimensions ≥ 4. Later unstable applications made it desirable to extend the theory to lower dimensions. This was done in the author's AMS memoir [**21**]. In the course of writing the author's book, it was discovered that the identification of the terms of the mod p homotopy Bockstein spectral sequence as

$$E^r(X) = image \quad p^{r-1} : \pi_*(X; Z/p^r Z) \to \pi_*(X; Z/p^r Z)$$

is valid for all primes p and for all dimensions where we have groups, that is, in dimensions ≥ 3. But the argument needed to extend to the prime 2 and, in dimension 3, to odd primes was complicated enough that it was not included in the book. This argument, included in this paper, requires the nontrivial use of Hilton-Hopf invariants to study the distributivity laws for homotopy classes.

The treatment here is certainly not definitive for 2 primary coefficients. Much more exploration of Samelson products and the Bockstein spectral sequence is needed in that case.

For the record, these modifications at odd primes do not occur in a low enough dimension to have any effect on the results in [**4, 5, 6, 22, 23**].

2. What are homotopy groups with coefficients?

What should be the criteria for $\pi_n(X;A)$ to be called the $n-$th homotopy group of X with coefficients in A? First of all, depending on X, there should be some restrictions on n and A in order that $\pi_n(X;A)$ be defined. No matter what A is, $\pi_n(X;A)$ may be undefined for low values of n. Even if it is defined, it may be merely a set and not a group, much less an abelian group.

We make no attempt to characterize by axioms homotopy groups with coefficients, but we wish to indicate a list of desirable criteria for such groups. For large enough n, the criteria for $\pi_n(X;A)$ are:

Functor on the homotopy category: For fixed A, $\pi_n(X;A)$ should be a functor defined from the homotopy category of pointed spaces to the category of abelian groups. $\pi_n(X;A)$ should also be a functor of A restricted to some full subcategory of abelian groups, this will be called coefficient functoriality.

Universal coefficient sequences: There should be short exact sequences $0 \to \pi_n(X)\otimes A \to \pi_n(X;A) \to Tor(\pi_{n-1}(X),A) \to 0$. The maps in the universal coefficient sequences should be natural transformations for X in the homotopy category of pointed spaces and for A in some full subcategory of abelian groups. We regard the existence of the universal coefficient exact sequences as essential.

Long exact sequences of fibrations: If $F \xrightarrow{\iota} E \xrightarrow{p} B$ is a fibration sequence of pointed spaces, there should be natural transformations $\partial : \pi_n(B;A) \to \pi_{n-1}(F,A)$ on the category of fibration sequences which are also natural for A in some full subcategory of abelian groups. These maps should yield long exact sequences

$$\dots \xrightarrow{p_*} \pi_{n+1}(B;A) \xrightarrow{\partial} \pi_n(F;A) \xrightarrow{\iota_*} \pi_n(E;A) \xrightarrow{p_*} \pi_n(B;A) \xrightarrow{\partial} \pi_{n-1}(F;A) \xrightarrow{\iota_*} \dots.$$

The maps in these long exact sequences should be compatible with the maps in the universal coefficient theorem, that is, the usual long exact fibration sequences for π_n should commute with the above when combined with the natural transformations

$$\pi_n(\quad) \to \pi_n(\quad)\otimes A \to \pi_n(\quad;A) \quad and \quad \pi_n(\quad;A) \to Tor(\pi_{n-1}(\quad),A).$$

Hurewicz maps: There should be Hurewicz maps

$$\phi : \pi_n(X;A) \to H_n(X;A)$$

which are natural transformations for pointed spaces X and for A in some full subcategory of abelian groups. The Hurewicz maps should be compatible with the maps in the two universal coefficient sequences.

The problems we must address are the following:
1) the existence of homotopy groups with coefficients.
2) coefficient functoriality.

3) the exponents of homotopy groups with coefficients, that is, does an exponent e for the abelian group A, $e\,A = 0$, imply that $e\,\pi_n(X;A) = 0$ for all pointed spaces X.

4) the existence of Hurewicz maps and the truth of Hurewicz theorems.

5) the existence of Samelson products in homotopy groups with coefficients and the validity of the identities for a graded Lie algebra.

3. Peterson spaces and finitely generated coefficients

If the homotopy groups with coefficients A are corepresentable, that is, if there is a space $P^n(A)$ such that $\pi_n(X;A) = [P^n(A);X]_*$, then that space must be a Peterson space:

DEFINITION 3.1. An n-th Peterson space $P^n(A)$ is a pointed space with exactly one nonzero reduced integral cohomology group, occurring in dimension n, such that that group is isomorphic to A, that is,

$$\overline{H}^k((P^n(A)) = \begin{cases} A, & k = n \\ 0, & k \neq n. \end{cases}$$

Similarly, an n-th Moore space $M_n(A)$ is a pointed space with exactly one nonzero reduced integral homology group, occurring in dimension n, such that that group is isomorphic to A, that is,

$$\overline{H}_k((P^n(A)) = \begin{cases} A, & k = n \\ 0, & k \neq n. \end{cases}$$

We claim that

THEOREM 3.2. *If homotopy with coefficients $\pi_n(X,A)$ is corepresentable by a space $P^n(A)$, that is, if $\pi_n(X,A) = [P^n(A),X]_*$, then $P^n(A)$ is an n-th Peterson space.*

PROOF. The universal coefficient theorem implies that

$$\begin{aligned} H^k(P(A)) &= [P^n(A), K(A,k)]_* = \pi_n(K(Z,k);A) \\ &= \pi_n(K(Z,k)) \otimes A = \begin{cases} A, & k = n \\ 0, & k \neq n \end{cases} \end{aligned}$$

□

Unfortunately, we shall see that Peterson spaces do not exist for some abelian groups, for example, there are no rational Peterson spaces $P^n(Q)$. Hence, homotopy with coefficients $\pi_n(X;A)$ cannot always be corepresentable. Nonetheless, it is corepresentable if the coefficient group A is a finitely generated abelian group and $n \geq 2$. In this case, we use Peterson spaces to show that homotopy groups with coefficients exist and satisfy the four criteria.

Let A be a finitely generated abelian group and let $0 \to F_1 \xrightarrow{\Psi} F_0 \xrightarrow{\epsilon} A \to 0$ be a free resolution of abelian groups with

$$F_0 = \bigoplus_\alpha Z, \qquad F_1 = \bigoplus_\beta Z$$

being finite direct sums with α running over an index set of order a and β running over an index set of order b. Then Ψ is represented by a $b \times a$ matrix M with a

columns and b rows of integer entries $M_{\beta,\alpha}$. Let M^* be the adjoint $a \times b$ matrix. Let

$$S_0 = \bigvee_{\alpha} S^1, \qquad S_1 = \bigvee_{\beta} S^1$$

and let $M^* : S_1 \to S_0$ be any map which induces $\Psi : F_1 \to F_0$ in integral cohomology in dimension 1.

Let $P^2(A)$ be the cofibre of the map $M^* : S_1 \to S_0$ and consider the long cofibation sequence

$$S_0 \xrightarrow{M^*} S_1 \to P^2(A) \to \Sigma S_0 \xrightarrow{M^*} \Sigma S_1 \to \Sigma P^2(A) \to \Sigma^2 S_0 \xrightarrow{M^*} \Sigma^2 S_1 \to \Sigma^2 P^2(A) \to \dots.$$

It is clear that the spaces $P^k(A) = \Sigma^{k-2} P^2(A)$ are Peterson spaces for $k \geq 2$.

For a finitely generated abelian group A and a pointed space X, we define $\pi_k(X;A) = [P^k(A), X]_*, \quad k \geq 2$.

Of course, for a fixed finitely generated abelian group A, $\pi_k(X;A)$ is a functor on the homotopy category with values in the category of sets if $k = 2$, in the category of groups if $k = 3$, and in the category of abelian groups if $k \geq 4$.

If we map the above long cofibration sequence into X, we get the long exact sequence

$$\Pi_\alpha \pi_1(X) \xleftarrow{M} \Pi_\beta \pi_1(X) \leftarrow \pi_2(X;A) \leftarrow \Pi_\alpha \pi_2(X) \xleftarrow{M} \Pi_\beta \pi_2(X) \leftarrow \pi_3(X;A) \leftarrow$$
$$\Pi_\alpha \pi_2(X) \xleftarrow{M} \Pi_\beta \pi_2(X) \leftarrow \pi_3(X;A) \leftarrow \dots$$

where the maps M have degrees on the summands which are compatible with the integers in the matrix, that is, the compositions

$$M_{\beta,\alpha} : \pi_k(X) \to \Pi_\beta \pi_k(X) \xrightarrow{M} \Pi_\alpha \pi_k(X) \to \pi_k(X)$$

are multiplications by the integers $M_{\beta,\alpha}$.

In other words the maps M may be identified with

$$\pi_k(X) \otimes F_1 \xrightarrow{1 \otimes M} \pi_k(X) \otimes F_0.$$

Hence, we get the short exact sequences of the universal coefficient theorem for $k \geq 2$:

$$0 \to \pi_k(X) \otimes A \to \pi_k(X;A) \to Tor(\pi_{k-1}(X), A) \to 0$$

where, if $k = 2$ and $\pi_1(X)$ is not abelian, the Tor just means the kernel of the map M.

It is well known that a sequence of suspensions gives the long exact sequence of a fibration.

$$\dots \to \pi_3(F;A) \to \pi_3(E;A) \to \pi_3(B;A) \xrightarrow{\partial}$$
$$\pi_2(F;A) \to \pi_2(E;A) \to \pi_2(B;A).$$

In general, this terminates in an exact sequence of pointed sets. See George Whitehead's book [**31**]. In particular, there is a map from the path fibration on the base to any other fibration

$$\begin{array}{ccccc} \Omega B & \to & PB & \to & B \\ \downarrow & & \downarrow & & \downarrow 1_B \\ F & \to & E & \to & B \end{array}$$

and the connecting homomorphism is given by $\partial : \pi_{n+1}(X;A) = [\Sigma P^n(A), X]_* = [P^n(A), \Omega X]_* \to [P^n(A), F]_* = \pi_n(F;A)$.

4. Long exact homotopy sequences for fibrations of loop spaces

DEFINITION 4.1. Suppose ΩX is a loop space. We define $\pi_1(\Omega X; A) = \pi_1(\Omega X) \otimes A = \pi_2(X) \otimes A$.

Suppose $F \xrightarrow{\iota} E \xrightarrow{q} B$ is a fibration sequence. We desire hypotheses which will yield the long exact sequence of a looped fibration down to dimension one, that is, there should be a long exact sequence of groups

$$\cdots \to \pi_3(\Omega F; A) \to \pi_3(\Omega E; A) \to \pi_3(\Omega B; A) \xrightarrow{\partial}$$

$$\pi_2(\Omega F; A) \to \pi_2(\Omega E; A) \to \pi_2(\Omega B; A) \xrightarrow{\overline{\partial}}$$

$$\pi_1(\Omega F; A) \to \pi_1(\Omega E; A) \to \pi_1(\Omega B; A) \to 0.$$

THEOREM 4.2. *Suppose that $\pi_1(F)$, $\pi_1(E)$, $\pi_1(B)$ are abelian groups. There is such a long exact sequence if either of the following two hypotheses are satisfied:*

1) $\pi_2(E) \to \pi_2(B)$ is an epimorphism and the sequence of fundamental groups is short exact

$$0 \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 0.$$

2) $\pi_2(E) \to \pi_2(B)$ is an epimorphism and $Tor(\pi_1(F), A) = 0$.

PROOF. : Consider the middle long exact sequence and the vertical short exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \pi_2(F)\otimes A & \to & \pi_2(E)\otimes A & \to & \pi_2(B)\otimes A & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\pi_3(B;A) & \xrightarrow{\partial} & \pi_2(F;A) & \to & \pi_2(E;A) & \to & \pi_2(B;A) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Tor(\pi_1(F),A) & \to & Tor(\pi_1(E),A) & \to & Tor(\pi_1(B),A) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If the composition $\partial : \pi_3(B; A) \xrightarrow{\partial} \pi_2(F; A) \to Tor(\pi_1(F), A)$ is zero, then ∂ can be factored as $\pi_3(B; A) \xrightarrow{\overline{\partial}} \pi_2(F) \otimes A \to \pi_2(F; A)$. This is trivially true under the second hypotheses. Under the first hypotheses, it follows from the fact that the Tor sequence is exact beginning on the left with 0.

Once we know that we can factor ∂ as above, the fact that the desired sequence is long exact is an immediate consequence.

□

5. Peterson spaces and Moore spaces

It is convenient in this section to assume that coefficient groups A are finitely generated abelian groups and hence that the $n-$th Peterson spaces $P^n(A)$ can be assumed to be finite complexes.

If A is a free abelian group, then Peterson spaces are the same as Moore spaces, $P^n(A) = M_n(A)$ up to unnatural isomorphism of A. But, if A is a torsion group, there is a dimension shift, $P^n(A) = M_{n-1}(A)$ up to unnatural isomorphism of A. To rid ourselves of this unnaturality, we need to discuss two kinds of duality.

If F is a finitely generated free abelian group, the $Z-$ dual is $F^* = Hom(F, Z)$. There is an unnatural isomorphism $\beta : F \simeq F^*$ given by choosing a dual basis and a natural isomorphism $\alpha : F \to F^{**}$ given by $\alpha(x)(y) = y(x)$ for $x \in F, y \in F^*$.

Similarly, if T is a finite abelian group, the Q/Z-dual is $T^* = Hom(T, Q/Z)$. There is an unnatural isomorphism $\beta : T \simeq T^*$ and a natural isomorphism $\alpha : T \to T^{**}$ given by $\alpha(x)(y) = y(x)$ for $x \in T, y \in T^*$. Both isomorphisms follow from the fact that they are true for finite cyclic groups, that is, if $T = Z/kZ$, then $\beta(\ell)(r) = \frac{r\ell}{k}$ is an isomorphism and $\alpha : T \to T^{**}$ is a monomorphism between cyclic groups of the same order.

Since Q is a torsion free injective abelian group, the exact sequence $0 \to Z \to Q \to Q/Z \to 0$ gives an isomorphism $Hom(T, Q/Z) \simeq Ext(T, Z)$.

Let $M_n(A)$ denote a Moore space with exactly one nonzero reduced integral homology group isomorphic to A in dimension n. The universal coefficient exact sequence

$$0 \to Ext(H_{n-1}(X), Z) \to H^n(X) \to Hom(H_n(X), Z) \to 0$$

and the fact that dualizations are idempotent implies that

THEOREM 5.1. *Let A be a finitely generated abelian group, then*

$$P^n(A) = \begin{cases} M_n(A^*) & if \quad A \quad is\, free. \\ M_{n-1}(A^*) & if \quad A \quad is\, finite. \end{cases}$$

For example, $P^n(Z) = S^n = M_n(Z)$ and $P^n(Z/kZ) = S^{n-1} \cup_k e^n = M_{n-1}(Z/kZ)$ are both Moore spaces. This and the next lemma give explicit constructions of Peterson spaces.

LEMMA 5.2. *If A and B are abelian groups, then $P^n(A \oplus B) = P^n(A) \vee P^n(B)$ in the sense that the right side of the equation is a candidate for the left side.*

The homotopy uniqueness of Peterson spaces is addressed in

THEOREM 5.3. *If A is a finitely generated abelian group and $n \geq 3$, then the homotopy type of the Peterson space $P^n(A)$ is determined by A, provided it is simply connected.*

PROOF. Let $A = F \oplus T$ where F is free and T is finite.

$$\overline{H}_k(P^n(A)) = \begin{cases} T^* & if \quad k = n-1, \\ F^* & if \quad k = n, \\ 0 & if \quad k \neq n-1 \,\text{or}\, n. \end{cases}$$

In dimension $n-1$ the Hurewicz map is an isomorphism and in dimension n it is an epimorphism.

We pick cyclic generators e_β for F and cyclic generators e_α for T. Then there is a map

$$\bigvee_\alpha S^{n-1} \vee \bigvee_\beta S^n \to P^n(A)$$

which is a homology epimorphism in dimension $n-1$ and a homology isomorphism in dimension n. We can attach $n-$cells to get a map

$$\bigvee_\alpha P^n(Z/k_\alpha Z) \vee \bigvee_\beta S^n \to P^n(A)$$

which is a homology isomorphism in all dimensions.

Since $n \geq 3$ the spaces are all simply connected and this is a homotopy equivalence.

□

The following examples show that there must be some restrictions in order that Peterson spaces have unique homotopy type.

Constructing a fake circle: Let α be an element of the fundamental group and let β be an element of any homotopy group. Denote the left action of α on β by $\alpha * \beta$.

Let $\iota_1 : S^1 \to S^1 \vee S^2$ and $\iota_2 : S^2 \to S^1 \vee S^2$ be the two standard inclusions and let $\gamma = \iota_2 - 2\iota_1 * (\iota_2)$.

Let

$$X = (S^1 \vee S^2) \cup_\gamma e^3$$

be the result obtained by attaching a 3-cell to the bouquet by the map γ. The inclusion $S^1 \to X$ is a homology equivalence but not a homotopy equivalence. To see this, inspect the universal cover of X to see that $\pi_2(X) = Z[\frac{1}{2}] \neq 0$.

Constructing a fake Moore space: Let $\iota_1 : S^1 \to P^2(Z/2Z) \vee S^2$ and $\iota_2 : S^2 \to P^2(Z/2Z) \vee S^2$ be the two standard inclusions. As before, let $\gamma = \iota_2 - 2\iota_1 * (\iota_2)$. and let

$$Y = (P^2(Z/2Z) \vee S^2) \cup_\gamma e^3$$

be the result obtained by attaching a 3-cell to the bouquet by the map γ. The inclusion $P^2(Z/2Z) \to Y$ is a homology equivalence but not a homotopy equivalence. To see this, inspect the universal covers to see that $\pi_2(P^2(Z/2Z)) = Z$ and $\pi_2(Y) = Z \oplus Z/3Z$.

6. Strong coefficient functoriality

Let A and B be finitely generated abelian groups and let $P^n(A)$ and $P^n(B)$ be Peterson spaces. We already know: If $n \geq 2$, these Peterson spaces exist, can be constructed by means of free resolutions, and, if $n \geq 3$ and the Peterson spaces are simply connected, their homotopy types are uniquely determined. We shall show that, if $n \geq 4$, Peterson spaces give functors from the category of finitely generated abelian groups with no 2-torsion to the homotopy category of pointed spaces.

THEOREM 6.1. *If $n \geq 2$ and the Peterson spaces $P^n(A)$ and $P^n(B)$ are constructed from free resolutions, then the assignment of the cohomology class to a map defines a surjection $\Psi : [P^n(B), P^n(A)]_* \to Hom(A, B)$, $\quad [g] \mapsto \Psi(g) = g^*$.*

PROOF. Let

$$\begin{array}{ccccccccc} 0 & \to & F_1 & \xrightarrow{d_0} & F_0 & \xrightarrow{\epsilon} & A & \to & 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ 0 & \to & G_1 & \xrightarrow{\overline{d}_0} & G_0 & \xrightarrow{\overline{\epsilon}} & B \to & & 0 \end{array}$$

be a map of resolutions.

The spaces $P^{n-1}(F_0)$, $P^{n-1}(F_1)$, $P^{n-1}(G_0)$, $P^{n-1}(G_1)$ are bouquets of $n-1$ spheres. Let

$$\begin{array}{ccccccccccc}
P^{n-1}(G_0) & \xrightarrow{\overline{d}_0^*} & P^{n-1}(G_1) & \xrightarrow{\iota} & P^n(B) & \xrightarrow{\overline{\epsilon}^*=q} & \Sigma P^{n-1}(G_0) & \xrightarrow{\Sigma\overline{d}_0^*} & \Sigma P^{n-1}(G_1) & \to \dots \\
\downarrow f_1^* & & \downarrow f_0^* & & \downarrow f^* & & \downarrow \Sigma f_1^* & & \downarrow \Sigma f_0^* & \\
P^{n-1}(F_0) & \xrightarrow{d_0^*} & P^{n-1}(F_1) & \xrightarrow{\iota} & P^n(A) & \xrightarrow{\epsilon^*=q} & \Sigma P^{n-1}(F_0) & \xrightarrow{\Sigma d_0^*} & \Sigma P^{n-1}(F_1) & \to \dots
\end{array}$$

be maps of cofibrations sequences whose cohomology maps realize these maps of resolutions.

Then $[f^*] \mapsto (f^*)^* = f$ shows that Ψ is a surjection.

□

Theorem 6.2. *Let $g : P^n(B) \to P^n(A)$ be a map which induces 0 in integral cohomology, that is, $\Psi(g) = 0$.*

a) if B and A are finitely generated free abelian and $n \geq 2$, then $\Psi : [P^n(B), P^n(A)]_* \to Hom(A, B)$ is a bijection and g is null homotopic.

b) if B if finite abelian, A is finitely generated free abelian, $P^n(A)$ is simply connected, and $n \geq 2$,then $\Psi : [P^n(B), P^n(A)]_* = A^* \otimes B \to Hom(A, B)$ is a bijection and hence g is null homotopic.

c) if B if finitely generated free abelian, $P^n(B)$ is a finite bouquet of n spheres, A is finite abelian, $n \geq 3$, and $h : P^n(B) \to P^n(A)$ is any map, then $h^* = 0$ in integral cohomology, that is, $h = g$ as above and g factors as

$$g : P^n(B) \xrightarrow{\overline{g}} P^{n-1}(F_1) \xrightarrow{\iota} P^n(A).$$

If $n \geq 4$, then $2\,[P^n(B), P^{n-1}(F_1)]_* = 2\,[P^n(B), P^{n-1}(A)]_* = 0$. Hence, if A has odd order, any map is null homotopic and $\Psi : [P^n(B), P^n(A)]_* \to Hom(A, B)$ is a bijection.

d) if B and A are finite abelian, and $n \geq 3$, then g factors as

$$g : P^n(B) \xrightarrow{q} P^n(G_0) \xrightarrow{\overline{g}} P^{n-1}(F_1) \xrightarrow{\iota} P^n(A).$$

If $n \geq 4$, then $2\,[P^n(G_0), P^{n-1}(F_1)]_* = 0$. Hence, if A or B has odd order, g is null homotopic and $\Psi : [P^n(B), P^n(A)]_* \to Hom(A, B)$ is a bijection.

Proof. a) Let

$$B = \bigoplus_\beta Z, \quad A = \bigoplus_\alpha Z$$

be free abelian and $n \geq 2$. Then

$$P^n(B) = \bigvee_\beta S^n_\beta, \quad P^n(A) = \bigvee_\alpha S^n_\alpha$$

and hence

$$[P^n(B), P^n(A)]_* = [\bigvee_\beta S^n_\beta, \bigvee_\alpha S^n_\alpha]_* = \prod_\beta [S^n_\beta, \bigvee_\alpha S^n_\alpha]_* = (\text{since}\, n \geq 2)$$

$$\prod_\beta \prod_\alpha [S^n_\beta, S^n_\alpha]_* = Hom(A, B).$$

b) Since $P^n(A)$ is simply connected, $P^n(A) = \bigvee_\alpha S^n_\alpha$ and

$$[P^n(B), P^n(A)]_* = [P^n(B), \bigvee_\alpha S^n_\alpha]_* = \pi_n(\bigvee_\alpha S^n_\alpha; B) =$$

$$\pi_n(\bigvee_\alpha S^n_\alpha) \otimes B = A^* \otimes B \xrightarrow{\Psi \simeq} Hom(A, B).$$

The last map is known to be a surjection between finite sets of equal cardinality and is therefore a bijection. This proves b).

The following lemma will be used in the proofs of parts c) and d).

Lemma 6.3. *Let A be a finite group with free resolution $0 \to F_1 \to F_0 \to A \to 0$ and let $P^{n-1}(F_0) \to P^{n-1}(F_1) \xrightarrow{\iota} P^n(A) \xrightarrow{q} P^n(F_0) \to P^n(F_1)$ be a cofibration sequence. If F is the homotopy theoretic fibre of $q : P^n(A) \to P^n(F_0)$, there is a factorization of $\iota : P^{n-1}(F_1) \xrightarrow{\gamma} F \to P^n(A)$ such that $\gamma : P^{n-1}(F_1) \to F$ is a $2n-3$ equivalence.*

This is a consequence of the last section on the Serre exact sequence.

c) Given any map $h : P^n(B) \to P^n(A)$ as above with B finitely generated free abelian and A finite, the composition $q \circ h : P^n(B) \xrightarrow{\iota} P^n(A) \xrightarrow{q} P^n(F_0)$ induces the zero map in cohomology since B and F_0 are free and A is finite. Hence, $q \circ h$ is null homotopic and h factors through the homotopy theoretic fibre F of q, that is, $h : P^n(B) \to F \to P^n(A)$. By Lemma 4.3, $\gamma : P^{n-1}(F_1) \to F$ is a $2n-3$ equivalence. If $n \geq 3$, it follows that $n \leq 2n-3$ and h factors as

$$P^n(B) \xrightarrow{\overline{h}} P^{n-1}(F_1) \xrightarrow{\iota} P^n(A).$$

If $n \geq 4$, then $[P^n(B), P^{n-1}(F_1)]_*$ is a direct sum of copies of $\pi_n(S^{n-1})$. Hence, it is a direct sum of copies of $Z/2Z$, for example, $\overline{h} : P^n(B) \to P^{n-1}(F_1)$ has order 2 and is a suspension.

For any integer k and any suspension ΣY, let $k : \Sigma Y \to \Sigma Y$ indicate the $k-$th multiple of the identity vis a vis the co-H structure.

Since $2h$ is represented by $h \circ 2 = \iota \circ \overline{h} \circ 2 = \iota \circ 0 = 0$, where $2 : P^n(B) \to P^n(B)$ is twice the identity, $2\,[P^n(B), P^{n-1}(A)]_* = 0$.

Since $\overline{h}^* : [P^{n-1}(F_1), P^n(A)]_* \to [P^n(B), P^n(A)]_*$ is a homomorphism of groups and $[P^{n-1}(F_1), P^n(A)]_* = A^* \otimes F_1$, it follows that $h = \overline{h}^*(\iota)$ has the order of A. Hence, if A has odd order, then h has both order 2 and odd order, therefore, h is null homotopic.

d) Suppose $n \geq 3$ and we are given a map $g : P^n(B) \to P^n(A)$ such that $g^* = 0 : A \to B$ in integral cohomology H^*. Then $g_* = 0 : A^* \to B^*$ in integral homology H_*. Hence, the composition $P^{n-1}(G_0) \to P^n(B) \to P^n(A)$ is null homotopic and g factors as $g = h \circ q$,

$$g : P^n(B) \xrightarrow{q} P^n(G_0) \xrightarrow{h} P^n(A)$$

and, by part c) above, h factors as $h = \iota \circ \overline{g}$,

$$h : P^n(G_0) \xrightarrow{\overline{g}} P^{n-1}(F_1) \xrightarrow{\iota} P^n(A).$$

Now suppose $n \geq 4$. Since $0 = 2\,\iota \circ \overline{g} = \iota \circ \overline{g} \circ 2$ and $q : P^n(B) \to P^n(G_0)$ is the suspension of $q : P^{n-1}(B) \to P^{n-1}(G_0)$,

$$2\,g = g \circ 2 = \iota \circ \overline{g} \circ q \circ 2 = \iota \circ \overline{g} \circ 2 \circ q = 0 \circ q = 0,$$

that is, g has order 2.

Similarly, if A has odd order k, then $0 = k\,g$ and $g = 0$. Thus, in this case, $\Psi : [P^n(B), P^n(A)]_* \to Hom(A, B)$ is a bijection and both groups have exponent k.

If B has odd exponent k, then g has order 2 and order k, hence $g = 0$.

The fact that $k\,g = 0$ if B has odd exponent k follows from the corollary

Corollary 6.4. *If B is a finite abelian group with odd exponent k and $n \geq 4$, then $0 = k : P^n(B) \to P^n(B)$ and hence $\pi_n(X, B)$ has exponent k, that is, $k\,\pi_n(X; B) = 0$, for all spaces X.*

This follows from the facts that $[P^n(B), P^n(B)]_* \to Hom(B, B)$ is a bijection by the earlier portion of part d) and hence both groups have exponent k.

□

The Hilton-Milnor theorem gives that

$$[P^n(A), P^n(B) \vee P^n(C)]_* = [P^n(A), P^n(B)]_* \times [P^n(A), P^n(C)]_*, \ n \geq 4$$

and hence

Corollary 6.5. *Let B and A be finitely generated abelian groups such that A has no 2-torsion. If $n \geq 4$, then*

$$\Psi : [P^n(B), P^n(A)]_* \to Hom(A, B)$$

is an isomorphism. This situation is called strong coefficient functoriality.

Notice that strong coefficient functoriality is not the same as coefficient functoriality which it implies.

Corollary 6.6. *For all $n \geq 4$, there is a functor $P^n(A)$ from the category of finitely generated abelian groups A with no 2-torsion to the homotopy category of spaces such that, for every homomorphism of such abelian groups $f : A \to B$, the map $P^n(f) : P^n(B) \to P^n(A)$ induces $f = P^n(f)^*$ in integral cohomology. Hence, the groups (or pointed sets) $\pi_n(X; A)$ are well defined functors of both the spaces X and the finitely generated groups A with no 2-torsion.*

7. Global exponents, decompositions of smash products, and fake multiples

Let p be a prime. If $n \geq 3$ and $p^r \neq 2$, we claim that the groups $\pi_n(X; Z/p^r Z)$ are all annihilated by p^r. If p is odd and $n \geq 4$, this is an immediate consequence of strong coefficient functoriality. But if $n = 3$, it requires the use of modular Hopf invariants which come in a later section.

Let ℓ be an integer and, for $n \geq 3$, let $\ell : P^n(Z/kZ) \to P^n(Z/kZ)$ also denote the multiple of the identity defined by the suspension structure, that is, $\ell : P^n(Z/kZ) \to P^n(Z/kZ)$ is defined by $\ell(< t, x >) = < \ell t, x >$ for $t \in I$. This map is called ℓ times the identity and induces multiplication by ℓ on the homotopy groups $\pi_n(X; Z/kZ)$ when these are abelian groups. We sometimes call these maps true multiples to distinguish them from what we call fake multiples, that is, maps which induce multiplication by ℓ in integral cohomology but which are not homotopic to the true multiples. For example, this occurs for all $n \geq 3$ if k is odd or for all $n \geq 4$ if $p = 2$.

Of course, the range of validity for n is increased to one less if X is a loop space, that is, if k is odd, there are no fake multiples on $\pi_3(\Omega Y; Z/kZ) = \pi_4(Y; Z/kZ)$. If these are just groups, then ℓ induces the $\ell-$th power map.

Clearly, a true multiple ℓ induces multiplication $\ell : Z/kZ \to Z/kZ$ on the integral cohomology groups.

THEOREM 7.1. *If p is a prime and $p^r > 2$, then the multiple $p^r : P^3(Z/p^rZ) \to P^3(Z/p^rZ)$ is null homotopic.*

PROOF. We assume here knowledge of the modular Hopf invariants which will be introduced later. The modular Hopf invariant $\mathcal{H}(f)$ is the integer mod p^r defined for self maps $f : P^3(Z/p^rZ) \to P^3(Z/p^rZ)$ which induce zero in mod p^r reduced homology. If C_f is the mapping cone and if $u \in H^2(C_f; Z/p^rZ)$ and $e \in H^4(C_f; Z/p^rZ)$ are generators then $\mathcal{H}(f)$ is defined by the cup product

$$u \cup u = \mathcal{H}(f)e.$$

Two properties of $\mathcal{H}(f)$ are that it is zero if and only if f is null homotopic and that it is zero if f is a suspension.

Since the induced map $p^r : H^3(P^3(Z/p^rZ); Z) = Z/p^rZ \to Z/p^rZ = H^3(P^3(Z/p^rZ) : Z)$ is zero in integral cohomology, it is sufficient to check that the modular Hopf invariant is zero, $\mathcal{H}(p^r) = 0 \in Z/p^rZ$.

The map $S^1 \xrightarrow{p^r} S^1$ defines the multiple $p^r = p^r \wedge 1 : S^1 \wedge P^2(Z/p^rZ) \to S^1 \wedge P^2(Z/p^rZ)$. If

$$S^1 \xrightarrow{p^r} S^1 \to P^2(Z/p^rZ)$$

is a cofibration sequence, then so is the smash of it with $P^2(Z/p^rZ)$,

$$S^1 \wedge P^2(Z/p^rZ) \xrightarrow{p^r} S^1 \wedge P^2(Z/p^rZ) \to P^2(Z/p^rZ) \wedge P^2(Z/p^rZ).$$

Hence, $P^2(Z/p^rZ) \wedge P^2(Z/p^rZ)$ is the mapping cone of $p^r : P^3(Z/p^rZ) \to P^3(Z/p^rZ)$.

Let e_1 and e_2 be 1 and 2 dimensional cochains dual to the cells of $P^2(Z/p^rZ)$ and recall Steenrod's computation [**28**] of the cup product,

$$e_1 \cup e_1 = \frac{p^r(p^r+1)}{2} e_2.$$

Let

$$u = e_1 \otimes e_1 \in H^2(P^2(Z/p^rZ) \wedge P^2(Z/p^rZ); Z/p^rZ),$$
$$e = e_2 \otimes e_2 \in H^4(P^2(Z/p^rZ) \wedge P^2(Z/p^rZ); Z/p^rZ)$$

be generators. Then

$$u \cup u = -\frac{p^{2r}(p^r+1)^2}{4} e.$$

Hence, $\mathcal{H}(p^r) = -\frac{p^{2r}(p^r+1)^2}{4}$ which is zero in Z/p^rZ if $p^r > 2$. □

COROLLARY 7.2. *Let n, $m \geq 2$. If $p^r > 2$, then there is a decomposition of the smash into a bouquet,*

$$P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \simeq P^{n+m-1}(Z/p^rZ) \vee P^{n+m}(Z/p^rZ).$$

If $p^r = 2$, then there is no such decomposition for any such m, $n \geq 2$.

Proof. If $p^r > 2$, then the mapping cone of the null homotopic map $p^r : P^3(Z/p^rZ) \to P^3(Z/p^rZ)$ is

$$P^2(Z/p^rZ) \wedge P^2(Z/p^rZ) \simeq P^3(Z/p^rZ) \vee P^4(Z/p^rZ).$$

We just suspend this to get the decomposition into a bouquet for any $n\,, m \geq 2$.

On the other hand, if $p^r = 2$, then we have a nontrivial square $e = u \cup u = Sq^2 u$ in the mod 2 cohomology of $P^2(Z/2Z) \wedge P^2(Z/2Z)$. Since Steenrod operations are stable, this nontrivial Sq^2 will never vanish via suspension and there can be no bouquet decomposition. □

For the localization away from 2 of any 2-cell complex $P = S^{2m-1} \cup_g e^{2n}$, Brayton Gray [**9**] used the action of the cyclic group of order 2 to construct homotopy idempotents and hence to give a decomposition of spaces

$$\Sigma P \wedge P \simeq \Sigma^{2m} P \vee \Sigma^{2n+1} P.$$

This decomposition generalizes the above decomposition of the smash product $P^n(p^r) \wedge P^m(p^r)$. Away from the primes 2 and 3, both decompositions are commutative and associative as needed in section 15 for the theory of internal Samelson products. The associativity requires looking at the action of the symmetric group on the 3-fold smash. But, if $n = m = 2$, our decomposition does not require a suspension.

Since the decomposition of the smash product is equivalent to the multiple of the identity being null homotopic, we have

Corollary 7.3. *If $p^r > 2$ and $n \geq 3$, the groups $\pi_n(X; Z/2^rZ)$ have exponent p^r. But the groups $\pi_n(P^n(Z/2Z); Z/2Z) = Z/4Z$.*

We now introduce useful maps $\overline{\ell} : P^n(Z/kZ) \to P^n(Z/kZ), \quad n \geq 2$ which induce $\ell : Z/kZ \to Z/kZ$ in integral cohomology and thus look like multiples of the identity.

Write $P^2(Z/kZ) = S^1 \cup_k C(S^1)$ where

$$S^1 = \{z \in \mathcal{C} | |z| = 1\}, \quad k(z) = z^k$$

is the circle of unit complex numbers and

$$C(S^1) = \frac{I \times S_1}{0 \times S^1 \cup I \times 1}, \quad < 1, z > \simeq z^k$$

is the attachment of the reduced cone.

Definition 7.4. The fake $\ell-$th multiple of the identity is the map

$$\overline{\ell} : P^2(Z/kZ) \to P^2(Z/kZ), \quad z \mapsto z^\ell, < t, z > \mapsto < t, z^\ell > .$$

In general, $\overline{\ell} = \Sigma^{n-2}\overline{\ell} : P^n(Z/kZ) \to P^n(Z/kZ)$ is the suspension.

It is interesting to compare the true multiples of the identity with the fake multiples of the identity.

1) The map $\overline{\ell} : P^n(Z/kZ) \to P^n(Z/kZ)$ is defined for $n \geq 2$. The map $\ell : P^n(Z/kZ) \to P^n(Z/kZ)$ is defined only for $n \geq 3$. The map $\overline{\ell}$ is a suspension for $n = 3$. If $n = 3$ and $p^r = 2$, $\ell = 2$ has a nontrivial modular Hopf invariant, hence, it is not a suspension.

2) Both ℓ and $\overline{\ell}$ induce $\ell : Z/kZ \to Z/kZ$ in integral cohomology, thus they induce the same maps in homology and cohomology with any coefficients.

3) If k is odd and $n \geq 4$, then the maps ℓ and $\overline{\ell}$ are homotopic.

4) Both maps satisfy the composition laws $\ell \circ k = \ell k$ and $\overline{\ell} \circ \overline{k} = \overline{\ell k}$.

We claim

LEMMA 7.5. *The fake multiples of the identity* $\overline{k} : P^n(Z/kZ) \to P^n(Z/kZ)$ *are null homotopic.*

PROOF. It is sufficient to check it when $n = 2$. Consider the general situation of maps of mapping cones given by a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \to & Y \cup_f CX \\ \downarrow a & & \downarrow b & & \downarrow b \cup Ca \\ W & \xrightarrow{g} & Z & \to & Z \cup_g CW \end{array}$$

Suppose we have a map $\ell : Y \to W$ such that $a = \ell \circ f$ and $b = g \circ \ell$.

Then we have a factorization

$$b \cup Ca : Y \cup_f CX \xrightarrow{\ell \cup Ca} CW \to Z \cup_g CW$$

and hence $b \cup Ca$ is null homotopic.

In our case, we choose $f = g = a = b = k, \quad \ell = 1$.

□

Since the fake mulitples $\overline{\ell}$ and the actual multiples ℓ induce the same maps in integral cohomology, $\overline{\ell} = \ell + \alpha$ where $\alpha : P^n(Z/kZ) \to P^n(Z/kZ)$ induces zero in integral cohomology. But, in fact more is true:

THEOREM 7.6. *Let p be a prime. Then for all $j \geq 1$, there exist maps δ_j : $P^3(Z/p^rZ) \to P^3(Z/p^rZ)$ which induce zero in integral cohomology such that*

$$\overline{p}^j = p^j + \delta_j \circ p^j = (1 + \delta_j) \circ p^j.$$

REMARK 7.7. The above theorem will be proved in the section on Hilton-Hopf invariants. Note that each map $1 + \delta_j$ is a cohomology isomorphism and therefore a homotopy equivalence. Thus, up to composition with a homotopy equivalence, the fake multiples $\overline{p}^j = \overline{p^j}$ and the actual multiples p^j are the same. By suspension, this is true for Peterson spaces of all dimensions. Hence, their dual maps

$$(p^j)^*, (\overline{p}^j)^* = (p^j)^* \circ (1 + \delta_j)^* : \pi_n(X; Z/p^rZ) \to \pi_n(X; Z/p^rZ)$$

have the same images, which are exactly the elements which are divisible by p^j.

8. Bocksteins, reductions, and inflations

Let k and ℓ be nonzero integers. The canonical maps of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & Z & \xrightarrow{\ell} & Z & \xrightarrow{\rho} & Z/\ell Z & \to & 0 \\ & & \downarrow \rho & & \downarrow \rho & & \downarrow 1 & & \\ 0 & \to & Z/kZ & \xrightarrow{\eta} & Z/k\ell Z & \xrightarrow{\rho} & Z/\ell Z & \to & 0 \end{array}$$

$$\rho(1) = 1, \quad \eta(1) = \ell.$$

suggest important maps of coefficient groups. The maps ρ are called reduction maps and the maps η are called inflation maps.

In this section, we introduce canonical maps between 2-dimensional Peterson spaces which induce these maps in integral cohomology and which make the corresponding dual diagrams of Peterson spaces commute exactly, not just up to homotopy. The maps between higher dimensional Peterson spaces are defined by suspending these maps. The maps between 2-dimensional Peterson spaces are the following:.

$$\overline{\rho} : P^2(Z/kZ) \to S^2$$
$$\overline{\rho} : P^2(Z/kZ) \to P^2(Z/k\ell Z)$$
$$\overline{\eta} : P^2(Z/k\ell Z) \to P^2(Z/kZ)$$

To make these definitions we use the same models for Peterson spaces that we used for defining fake multiples of the identity:

$$P^2(Z/kZ) = S^1 \cup_k C(S^1), \quad k : S^1 \to S^1, \quad k(z) = z^k.$$

DEFINITION 8.1. 1) The coreduction map $\overline{\rho} : P^2(Z/kZ) \to S^2$ is the quotient map q which pinches the bottom circle to a point, $q(z) = *$ for all $z \in S^1$.

2) The coreduction map $\overline{\rho} : P^2(Z/kZ) \to P^2(Z/k\ell Z)$ is the map

$$z \mapsto z^\ell, \qquad < t, z > \mapsto < t, z >$$

3) The coinflation map $\overline{\eta} : P^2(Z/k\ell Z) \to P^2(Z/kZ)$ is the map

$$z \mapsto z, \qquad < t, z > \mapsto < t, z^\ell >$$

It is easy to check that these maps are compatible with the identifications. The coreduction and coinflation maps give commutative diagrams of maps of cofibration sequences

$$\begin{array}{ccccccccc}
S^1 & \xrightarrow{k} & S^1 & \xrightarrow{\iota=\overline{\beta}} & P^2(Z/kZ) & \xrightarrow{q=\overline{\rho}} & S^2 & \xrightarrow{k} & \ldots \\
\downarrow 1 & & \downarrow \ell & & \downarrow \overline{\rho} & & \downarrow 1 & & \\
S^1 & \xrightarrow{k\ell} & S^1 & \xrightarrow{\iota=\overline{\beta}} & P^2(Z/k\ell Z) & \xrightarrow{q=\overline{\rho}} & S^2 & \xrightarrow{k} & \ldots
\end{array}$$

$$\begin{array}{ccccccccc}
S^1 & \xrightarrow{k\ell} & S^1 & \xrightarrow{\iota=\overline{\beta}} & P^2(Z/k\ell Z) & \xrightarrow{q=\overline{\rho}} & S^2 & \xrightarrow{k\ell} & \ldots \\
\downarrow \ell & & \downarrow 1 & & \downarrow \overline{\eta} & & \downarrow \ell & & \\
S^1 & \xrightarrow{k} & S^1 & \xrightarrow{\iota=\overline{\beta}} & P^2(Z/kZ) & \xrightarrow{q=\overline{\rho}} & S^2 & \xrightarrow{k} & \ldots
\end{array}$$

DEFINITION 8.2. The maps $\iota = \overline{\beta} : S^1 \to P^2(Z/kZ)$ are called coBocksteins. As with the coreductions and coinflations they are defined on higher dimensional Peterson spaces by suspension.

The diagrams above help in identifying the following compositions below. The compositions 3) and 4) below are equal to the fake multiples of the identity $\overline{\ell}$ in the previous section. After suspension, they are equal to the true multiples of the identity only if $n \geq 4$ and if p is odd.

LEMMA 8.3. *1)*

$$\overline{\rho} = \overline{\rho} \circ \overline{\rho} : P^2(Z/kZ) \xrightarrow{\overline{\rho}} P^2(Z/k\ell Z) \xrightarrow{\overline{\rho}} P^2(Z/k\ell mZ)$$

2)

$$\overline{\eta} = \overline{\eta} \circ \overline{\eta} : P^2(Z/k\ell mZ) \xrightarrow{\overline{\eta}} P^2(Z/Z/k\ell Z) \xrightarrow{\overline{\rho}} P^2(Z/k)$$

3)

$$\overline{\ell} = \overline{\eta} \circ \overline{\rho} : P^2(Z/k\ell Z) \xrightarrow{\overline{\eta}} P^2(Z/kZ) \xrightarrow{\overline{\rho}} P^2(Z/k\ell Z)$$

4)

$$\overline{\ell} = \overline{\eta} \circ \overline{\rho} : P^2(Z/kZ) \xrightarrow{\overline{\rho}} P^2(Z/Z/k\ell Z) \xrightarrow{\overline{\eta}} P^2(Z/kZ)$$

LEMMA 8.4. *Applying P^2 to the diagram which began this section, reversing arrows, replacing reductions by coreductions, inflations by coinflations, and multiplications by fake multiples produces a strictly commutative diagram.*

REMARK 8.5. The coreduction maps

$$\overline{\rho} = q : P^2(Z/kZ) \to S^2$$

pinch the bottom cell to a point and induce the reduction maps

$$\rho : \pi_2(X) \to \pi_2(X; Z/p^r Z).$$

There is a factorization

$$\overline{\rho} = \overline{\rho} \circ \overline{\rho} : P^2(Z/kZ) \to P^2(Z/k\ell Z) \to S^2$$

and a consequent factorization

$$\rho = \rho \circ \rho : \pi_2(X) \to \pi_2(X; Z/k\ell Z) \to \pi_2(X; Z/kZ).$$

The fact that the fake multiples $\overline{\ell} : P^2(Z/\ell Z) \to P^2(Z/\ell Z)$ are null homotopic yields

THEOREM 8.6. *The compositions*

$$P^2(Z/kZ) \xrightarrow{\overline{\rho}} P^2(Z/k\ell Z) \xrightarrow{\overline{\eta}} P^2(Z/\ell Z)$$

are null homotopic.

PROOF. Let $d = (k, \ell) =$ the greatest common divisor. The above composition is identical to the composition

$$P^2(Z/kZ) \xrightarrow{\overline{\eta}} P^2(Z/dZ) \xrightarrow{\overline{d}} P^2(Z/dZ) \xrightarrow{\overline{\rho}} P^2(Z/\ell Z).$$

□

The point of the above theorem is that short exact sequences of abelian groups should correspond to cofibration sequences. This is rather close to being true.

Consider the natural maps

$$\begin{array}{ccccccccc} P^2(Z/kZ) & \xrightarrow{\overline{\rho}} & P^2(Z/k\ell Z) & \xrightarrow{\iota} & C_{\overline{\rho}} & \xrightarrow{j} & P^3(Z/kZ) & \xrightarrow{\overline{\rho}} & P^3(Z/k\ell Z) \\ \downarrow 1 & & \downarrow 1 & & \downarrow & & \downarrow & & \downarrow 1 \\ P^2(Z/kZ) & \xrightarrow{\overline{\rho}} & P^2(Z/k\ell Z) & \xrightarrow{\overline{\eta}} & P^2(Z/\ell Z) & \xrightarrow{\iota} & C_{\overline{\eta}} & \xrightarrow{j} & P^3(Z/k\ell Z) \end{array}$$

The map $C_{\overline{\rho}} \to P^2(Z/\ell Z)$ induces an isomorphism in integral cohomology (therefore in integral homology) and an isomorphism of fundamental groups. Perhaps it is not a homotopy equivalence but any suspension of it is. Hence, if Y is a loop space, $[P^2(Z/\ell Z), Y]_* \xrightarrow{\cong} [C_{\overline{\rho}}, Y]_*$ is an isomorphism and, from the point of view of maps into Y,

$$P^2(Z/kZ) \xrightarrow{\rho} P^2(Z/k\ell Z) \xrightarrow{\overline{\eta}} P^2(Z/\ell Z) \xrightarrow{\overline{\beta}}$$

$$P^3(Z/kZ) \xrightarrow{\overline{\rho}} P^3(Z/k\ell Z) \xrightarrow{\overline{\eta}} P^3(Z/\ell Z) \to \dots$$

behaves just like a cofibration sequence.

The map $P^3(Z/kZ) \to C_{\overline{\eta}}$ induces an isomorphism in integral cohomology. Since the spaces are simply connected, it is a homotopy equivalence. The map $\overline{\beta} : P^2(Z/\ell Z) \to C_{\overline{\eta}} = P^3(Z/kZ)$ is called the coBockstein.

For dimension reasons it is clear that we have a factorization of the coBockstein as:

$$\overline{\beta} = \overline{\beta} \circ \overline{\rho} : P^2(Z/\ell Z) \to S^2 \to P^3(Z/kZ)$$

where the second map is the inclusion of the bottom cell and is therefore a co-H-map.

The coBockstein and its suspensions induce the Bockstein homomorphisms

$$\beta = \rho \circ \beta : \pi_n(X; Z/kZ) \to \pi_{n-1}(X) \to \pi_{n-1}(X; Z/\ell Z).$$

We get

THEOREM 8.7. *For $n \geq 3$, the coreduction and coinflation maps form a cofibration sequence*

$$P^{n-1}(Z/k\ell Z) \xrightarrow{\overline{\eta}} P^{n-1}(Z/\ell Z) \xrightarrow{\overline{\beta}} P^n(Z/kZ) \xrightarrow{\overline{\rho}} P^n(Z/k\ell Z) \xrightarrow{\overline{\eta}} P^n(Z/\ell Z)$$

REMARK 8.8. Suppose Y is any space. The cofibration sequence above yields the middle horizontal Bockstein sequence which is exact in the first three terms on the left. The bottom sequence is exact by the properties of Tor:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \pi_2(Y) \otimes Z/\ell Z & \to & \pi_2(Y) \otimes Z/k\ell Z & \to & \pi_2(Y) \otimes Z/kZ & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\pi_3(Y; Z/kZ) & \xrightarrow{\beta} & \pi_2(Y; Z/\ell Z) & \to & \pi_2(Y; Z/k\ell Z) & \to & \pi_2(Y; Z/kZ) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Tor(\pi_1(Y), Z/\ell Z) & \to & Tor(\pi_1(Y), Z/k\ell Z) & \to & Tor(\pi_1(Y), Z/kZ) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The connecting homomorphism or Bockstein factors as $\beta : \pi_3(Y; Z/kZ) \to \pi_2(Y) \otimes Z/\ell Z \to \pi_2(Y; Z/\ell Z)$. The map $\pi_3(Y; Z/kZ) \to \pi_2(Y) \otimes Z/\ell Z$ is a homomorphism since $S^2 \to P^3(Z/kZ)$ is a co-H-map.

Hence, we can splice the sequences to get a long exact Bockstein sequence

$$\cdots \to \pi_3(Y; Z/kZ) \to \pi_2(Y) \otimes Z/\ell Z \to \pi_2(Y) \otimes Z/k\ell Z \to \pi_2(Y) \otimes Z/kZ \to 0.$$

In particular, if X is a loop space, the above can be written as the long exact Bockstein sequence

$$\cdots \to \pi_2(X; Z/kZ) \to \pi_1(X; Z/\ell Z) \to \pi_1(X, Z/k\ell Z) \to \pi_1(X; Z/kZ) \to 0$$

which terminates in 0.

REMARK 8.9. Consider the fake multiples $\overline{\ell} : P^2(Z/2Z) \to P^2(Z/2Z)$. If ℓ is even, $\overline{\ell} = \overline{2} \circ 2\overline{k}$, then $\overline{\ell}$ is null homotopic. If ℓ is odd, then $\overline{\ell}$ induces isomorphisms on integral homology and hence also on the fundamental group and integral cohomology. But $\overline{\ell}$ is not a homotopy equivalence if $\ell \neq 1$ since it is covered on the universal cover S^2 by a degree ℓ map. To see this, write the projective plane $P^2(Z/2Z)$ as the quotient of the universal cover, that is, the quotient of the suspension $S^2 = \Sigma S^1 = \{< t, z > \ | t \in [-1, 1], z \in S^1\}$ by the relations $< t, z > \sim < -t, -z >$. Since ℓ is odd, the fake multiple $\overline{\ell}$ is given by $< t, z > \mapsto < t, z^\ell >$. This lifts to the universal cover by $< t, z > \mapsto < t, z^\ell >$, that is, by $\ell : S^2 \to S^2$.

9. Hurewicz maps

In this section, we introduce the mod k Hurewicz maps and prove the mod k Hurewicz theorems, including the results on the equivalence of mod k homology isomorphisms and mod k homotopy isomorphisms.

Recall that $H_n(P^n(Z/kZ), Z/kZ) \simeq Z/kZ$ and let e_n be a generator. Then $H_{n-1}(P^n(Z/kZ), Z/kZ) \simeq Z/kZ$ is generated by the Bockstein $\beta(e_n)$ associated to the short exact sequence $0 \to Z/kZ \to Z/k^2Z \to Z/kZ \to 0$.

DEFINITION 9.1. For $n \geq 2$, the Hurewicz map $\phi : \pi_n(X; Z/kZ) \to H_n(X; Z/kZ)$ is the map defined by $\phi[f] = f_* e_n$.

The Hurewicz map is clearly a natural transformation. It is an exercise to check that the Hurewicz map commutes with reductions, inflations, and Bocksteins.

LEMMA 9.2. *For all $n \geq 2$, the following diagrams commute*

$$\begin{array}{ccccccc} \pi_{n+1}(X; Z/\ell Z) & \stackrel{\beta}{\to} & \pi_n(X; Z/kZ) & \stackrel{\eta}{\to} & \pi_n(X; Z/k\ell Z) & \stackrel{\rho}{\to} & \pi_n(X; Z/\ell Z) \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ H_{n+1}(X; Z/\ell Z) & \stackrel{\beta}{\to} & H_n(X; Z/kZ) & \stackrel{\eta}{\to} & H_n(X; Z/k\ell Z) & \stackrel{\rho}{\to} & H_n(X; Z/\ell Z) \end{array}$$

$$\begin{array}{ccccccc} \pi_n(X) & \stackrel{\rho}{\to} & \pi_n(X; Z/\ell Z) & \stackrel{\beta}{\to} & \pi_{n-1}(X) & \stackrel{\ell}{\to} & \pi_{n-1}(X) \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ H_n(X) & \stackrel{\rho}{\to} & H_n(X; Z/\ell Z) & \stackrel{\beta}{\to} & H_{n-1}(X) & \stackrel{\ell}{\to} & H_{n-1}(X) \end{array}$$

For $n \geq 3$, the Peterson space $P^n(Z/kZ)$ is a suspension and hence the diagonal is given by $\Delta(e_n) = e_n \otimes 1 + 1 \otimes e_n$. Hence,

LEMMA 9.3. *If $n \geq 3$, the mod k Hurewicz map $\phi : \pi_n(X; Z/kZ) \to H_n(X; Z/kZ)$ is a homomorphism.*

PROOF. Given maps f, $g : P^n(Z/kZ) \to X$, the sum is given by the composition

$$P^n(Z/kZ) \to P^n(Z/kZ) \vee P^n(Z/kZ) \xrightarrow{f \vee g} X \vee X \to X$$

and this induces in homology

$$e_n \mapsto e_n \otimes 1 + 1 \otimes e_n \mapsto f_* e_n \otimes 1 + 1 \otimes g_* e_n \mapsto f_* e_n + g_* e_n = \phi(f) + \phi(g).$$

□

If X is a group-like space, then the multiplication gives a group structure on $\pi_2(X; Z/kZ)$. We have

LEMMA 9.4. *If X is a group-like space, then the mod k Hurewicz map ϕ : $\pi_2(X; Z/kZ) \to H_2(X; Z/kZ)$ satisfies*

$$\phi(f + g) = \phi(f) + \phi(g) + \frac{k(k+1)}{2} (\beta\phi(f)) * (\beta\phi(g))$$

where the last term is the Pontrijajgin product. Hence, if k is odd, this Hurewicz map is a homomorphism.

PROOF. The sum in homotopy is the composition

$$P^n(Z/kZ) \xrightarrow{\Delta} P^n(Z/kZ) \times P^n(Z/kZ) \xrightarrow{f \times g} X \times X \to \mu X.$$

Steenrod's computation [**28**] is that $\Delta_*(e_2) = e_2 \otimes 1 + 1 + 1 \otimes e_2 + \frac{k(k+1)}{2}(\beta e_2 \otimes \beta e_2)$. The result follows.

$\square$

REMARK 9.5. Since the Bockstein factors through the subgroup $Tor(\pi_1(X), Z/kZ) \subset \pi_1(X)$, it follows that the above mod k Hurewicz map is a homomorphism if k is even and $Tor(\pi_1(X), Z/kZ)$ is annihilated by $\frac{k}{2}$.

Recall that a pointed space X is called nilpotent if the action of the fundamental group on all the homotopy groups is nilpotent.

Let $\pi = \pi_1(X)$ be the fundamental group and let $Z[\pi]$ be the group ring of π. Suppose for simplicity that π is abelian. Hence, π acts trivially by conjugation on itself.

For all $j \geq 1$, the homotopy group $\pi_j(X)$ is a module over $Z[\pi]$.

Let I be the augmentation ideal, $I = kernel\ \epsilon : Z[\pi] \to Z, \quad \epsilon(g) = 1\, \forall g \ \in \pi$.

DEFINITION 9.6. *X is nilpotent if, for all $j \geq 1$, there is a positive integer n_j such that $I^{n_j}\pi_j(X) = 0$.*

For a nilpotent X, the descending filtration

$$\pi_j(X) \supseteq I\pi_j(X) \supseteq I^2\pi_j(X) \supseteq \dots I^{n_{j-1}}\pi_j(X) \supseteq I^{n_j}\pi_j(X) = 0$$

is finite and terminates at 0. The action on each quotient $\pi_{j,\ell}(X) = I^\ell\pi_j(X)/I^{\ell+1}\pi_j(X)$ is trivial.

The effect of this is that there is a refinement of the fibration sequences in the Postnikov system for X, that is, $K(\pi_{j,\ell}(X), j) \to X_\alpha \to X_{\alpha-1} \to K(\pi_{j,\ell}(X), j+1)$ are all orientable and there is a homotopy equivalence $X \to \lim_{\leftarrow} X_\alpha$.

We adopt the convention that $\pi_1(X; Z/kZ) = \pi_1(X) \otimes Z/kZ$.

THEOREM 9.7 (mod k Hurewicz theorem). *Suppose X is a nilpotent space with abelian fundamental group and $n \geq 1$. Suppose that $Tor(\pi_1(X), Z/kZ) = 0$. If $\pi_\ell(X; Z/kZ) = 0$ for all $\ell < n$, then $\pi_\ell(X; Z/kZ) = 0$ for all $\ell < n$ and the Hurewicz map $\phi : \pi_n(X; Z/kZ) \to H_n(X; Z/kZ)$ is an isomorphism. If $n \geq 2$, then the Hurewicz map $\phi : \pi_{n+1}(X; Z/kZ) \to H_{n+1}(X; Z/kZ)$ is an epimorphism.*

REMARK 9.8. The significant effect of the hypothesis $Tor(\pi_1(X), Z/kZ) = 0$ is that $\pi_2(X; Z/kZ) = \pi_2(X) \otimes Z/kZ$ is a group and hence that all of the homotopy groups with coefficients are groups. In addition, the Hurewicz map is a homomorphism in all dimensions.

Before we begin the proof of the above Hurewicz theorem, we present here a quick summary of the Serre exact sequence. The presentation is heavily influenced by [**19**].

Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be an orientable fibration sequence. Suppose that F is $r-1$ connected and B is $s-1$ connected.

Without changing notation we can regard $E \xrightarrow{p} B$ as an inclusion whenever it is convenient. For example, we have maps of cofibration sequences

$$\begin{array}{ccccccccc}
F & \xrightarrow{\iota} & E & \to & E/F & \to & \Sigma F & \xrightarrow{\Sigma\iota} & \Sigma E \\
 & & \downarrow 1 & & \downarrow & & \downarrow & & \downarrow 1 \\
 & & E & \to & B & \to & B/E & \to & \Sigma E
\end{array}$$

$$\begin{array}{ccccccc}
\to & \Sigma E/F & \to & \Sigma^2 F & \xrightarrow{\Sigma^2\iota} & \Sigma^2 E\dots \\
 & \downarrow & & \downarrow & & \downarrow 1 \\
\xrightarrow{\Sigma p} & \Sigma B & \to & \Sigma B/E & \to & \Sigma^2 E\dots
\end{array}$$

The transgression in the Serre spectral sequence for the fibration $E \xrightarrow{p} B$ is defined by the relation

$$\tau : H_*B \leftarrow H_*E/F \to H_*\Sigma F = H_{*-1}F.$$

The Serre spectral sequence of the pair

$$E^2_{p,q} = H_p(B, *; H_qF) \Longrightarrow H_{p+q}(E, F)$$

shows that the map $H_*(E, F) \to H_*(B, *)$ is an $r + s$ equivalence, that is, an isomorphism for $* \leq r + s - 1$ and an epimorphism for $* = r + s$. Thus, the transgression τ is a well defined map for $* \leq r + s - 1$.

The Serre spectral sequence of the fibration gives the Serre exact sequence with the transgression as the connecting homomorphism

$$H_{r+s-1}F \to H_{r+s-1}E \to H_{r+s-1}B \xrightarrow{\tau} H_{r+s-2}F$$

$$\to H_{r+s-2}E \to H_{r+s-2}B \xrightarrow{\tau} H_{r+s-3}F \to$$

$$\dots \to H_1F \to H_1E \to H_1B$$

With the understanding that $H_*(E, F) \xrightarrow{\simeq} H_*B$ is an isomorphism in the range $* \leq r + s - 1$, the Serre exact sequence is just the exact sequence of the pair (E, F). We know that the Hurewicz maps give a map of the long exact sequence of homotopy groups of a pair to the long exact sequence of homology groups of the pair. Since $\pi_*(E, F) \cong \pi_*B$ for all $*$, the long exact sequence of the homotopy groups of a fibration map to the Serre exact sequence in the range where the Serre exact sequence is valid.

Since $E/F \to B$ is an $r + s$ equivalence, the 5-lemma shows that $\Sigma F \to B/E$ is also an $r + s$ equivalence.

Furthermore, suppose we have a factorization

$$\begin{array}{ccccccccc}
F & \xrightarrow{\iota} & E & \to & E/F & \to & \Sigma F & \xrightarrow{\Sigma\iota} & \Sigma E \\
\downarrow g & & \downarrow 1 & & \downarrow & & \downarrow h = \Sigma g & & \downarrow 1 \\
C & \to & E & \to & B & \to & B/E & \to & \Sigma E
\end{array}$$

where the bottom row is a cofibration sequence up to homotopy. Since h is an $r + s$ equivalence in homology and homotopy, it follows that its desuspension g is an $r + s - 1$ equivalence.

Proof. Let $\mathcal{C}_n(X)$ be the mod k Hurewicz conclusion for a space X and for a fixed integer n and let $\mathcal{H}_n(X)$ be the mod k Hurewicz hypothesis for a space X and for a fixed integer n. We want to prove that $\mathcal{H}_n(X)$ implies $\mathcal{C}_n(X)$ when X is nilpotent with abelian fundamental group.

The proof proceeds by proving the mod k Hurewicz theorem in three successive cases, 1) $X = K(\pi, 1)$, 2) $X = K(A, q)$, A *abelian*, $q \geq 2$, 3) $X =$ an arbitrary nilpotent space.

First of all, we note that, since the fundamental group $\pi_1(X)$ is abelian, the mod k Hurewicz theorem is true for $n = 1$, $\quad H_1(X; Z/kZ) = H_1(X) \otimes Z/kZ = \pi_1(X) \otimes Z/kZ = \pi_1 \otimes Z/kZ$, $\quad \mathcal{H}_1(X)$ implies $\mathcal{C}_1(X)$

Case 1. $X = K = K(\pi, 1)$:

When p is a prime, Cartan's calculation [**2**] is

$$H_*(K(A,1); Z/pZ) = E(A \otimes Z/pZ, 1) \otimes \Gamma(Tor(A, Z/pZ), 2)$$

where E is an exterior algebra and Γ is a divided power algebra. The Hurewicz hypothesis for $n = 2$ implies that $A \otimes Z/pZ = 0$ and hence $H_2(K; Z/pZ) = Tor(A, Z/pZ)$ and $H_3(K; Z/pZ) = 0$ which is the mod p Hurewicz theorem in this case, $\mathcal{H}_2(K)$ implies $\mathcal{C}_2(K)$ when $k = p =$ a prime.

Suppose we know the mod d Hurewicz theorem for $n = 2$ and for all proper divisors of k. We have a map of long exact Bockstein sequences. (Remark: Since K is an abelian topological group, $\pi_2(K, Z/kZ)$ is an abelian group and the mod k Hurewicz map is a homomorphism.)

$$\begin{array}{ccccccc}
\pi_3(K, Z/dZ) & \to & \pi_3(K, Z/kZ) & \to & \pi_3(K, Z/\frac{k}{d}Z) & \to & \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \\
H_3(K, Z/dZ) & \to & H_3(K, Z/kZ) & \to & H_3(K, Z/\frac{k}{d}Z) & \to & \\
 & & & & & & \\
\pi_2(K, Z/dZ) & \to & \pi_2(K, Z/kZ) & \to & \pi_2(K, Z/\frac{k}{d}Z) & \to & 0 \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \\
H_2(K, Z/dZ) & \to & H_2(K, Z/kZ) & \to & H_2(K, Z/\frac{k}{d}Z) & \to & 0
\end{array}$$

The 5-lemma implies that the mod k Hurewicz theorem is true for K and $n = 2$, that is, $\mathcal{H}_2(K)$ implies $\mathcal{C}_2(K)$ for all k.

Now suppose $n \geq 3$. The mod k Hurewicz hypothesis in this case implies that $A \otimes Z/kZ = Tor(A, Z/kZ) = 0$ and that $A \otimes Z/pZ = Tor(A, Z/pZ) = 0$ for all primes dividing k. Hence, $H_*(K(A, Z/pZ)) = Z/pZ$ and then $H_*(K(A,1); Z/kZ) = 0$. Since $\pi_*(K(A,1); Z/kZ) = 0$, the mod k Hurewicz theorem is true in this case, $\mathcal{H}_n(K)$ implies $\mathcal{C}_n(K)$ for all n and k.

Case 2. $X = K = K(A, q)$, $q \geq 2$:

If $q = 2$, we claim that $H_2(K; Z/kZ) = A \otimes Z/kZ$ and $H_3(K; Z/kZ) = Tor(A, Z/kZ))$. This is the mod k Hurewicz theorem when $n = 2$, $\quad \mathcal{H}_2(K)$ implies $\mathcal{C}_2(K)$ for any k when $q = 2$.

If d is a proper divisor of k, the long exact Bockstein sequences associated with the coefficient sequence

$$0 \to Z/dZ \to Z/kZ \to Z/\frac{k}{d}Z \to 0$$

show that it is sufficient to prove the above when $k = p =$ a prime. In detail:

The Eilenberg-Moore spectral sequence [**18, 24**] converging to $H_*(K; Z/pZ)$ has

$$E^2 = Tor^{H_*(K(A,1); Z/pZ)}(Z/pZ, Z/pZ) =$$

$$Tor^{E(A \otimes Z/pZ, 1)}(Z/pZ, Z/pZ) \otimes Tor^{\Gamma(Tor(A, Z/pZ), 2)}(Z/pZ, Z/pZ) =$$

$$\Gamma(A \otimes Z/pZ, 2) \otimes H_*(K(B,3); Z/pZ)$$

where B is a free Z module such that there is an isomorphism $B \otimes Z/pZ \to A$. The identification $Tor^{\Gamma(Tor(A,Z/pZ),2)}(Z/pZ, Z/pZ) = H_*(K(B,3); Z/pZ)$ is a convenient way to describe this computation of Tor in a way that is familiar [**2, 26**]. If p is odd, then $H_*(K(B,3); Z/pZ) = E(B,3) \otimes C$ where C is an algebra generated by elements of degrees $\geq 3 + 2p - 2 = 2p + 1$. If $p = 2$, then $H_*(K(B,3); Z/2Z) = \Gamma(B,3) \otimes D$ where D is an algebra generated by elements of degrees ≥ 5.

Hence, $H_2(K(A,2), Z/pZ) = A \otimes Z/pZ$, $\quad H_3(K(A,2), Z/pZ) = Tor(A, Z/pZ)$, and, provided $A \otimes Z/pZ = 0$, $H_4(K(A,2); Z/pZ) = 0$. Furthermore, if $A \otimes Z/pZ = Tor(A, Z/pZ) = 0$, then $H_*(K(A,2), Z/pA) = Z/pZ$. Thus, we have the Hurewicz theorem for all $n \geq 1$ for $K = K(A,2)$ and $k = p =$ a prime.

Suppose we have the mod k Hurewicz hypotheses for some $n \geq 1$. By the long exact Bockstein sequence, vanishing of tensor and Tor with Z/k implies the same vanishing results for tensor and Tor with Z/dZ for all divisors d of k. Hence, the mod k Hurewicz hypotheses imply the mod d Hurewicz hypotheses for all divisors d. By induction, we can assume the truth of the mod d Hurewicz theorem for proper divisors. Now the 5-lemma applied to the long exact Bockstein sequences implies the mod k Hurewicz theorems for all $n \geq 1$ for $K = K(A,2)$, that is, $\mathcal{H}_n(K)$ implies $\mathcal{C}_n(K)$ for $q = 2$ and for all $n \geq 1$.

If $q \geq 3$, the Serre spectral sequence and induction using the path fibrations $K(A, q-1) \to PK(A,q) \to K(A,q)$ show that $H_q(K(A,q), Z/pZ) = A \otimes Z/kZ$, $\quad H_{q+1}(K(A,q), Z/kZ) = Tor(A, Z/kZ)$, and, provided $A \otimes Z/kZ = 0$, $H_{q+2}(K(A,q); Z/kZ) = 0$. Furthermore, if $A \otimes Z/kZ = Tor(A, Z/kZ) = 0$, then $H_*(K(A,q), Z/kZ) = Z/kZ$. Thus, we have the Hurewicz theorem for all $n \geq 1$ for $K = K(A,q)$ and a integer k, that is, $\mathcal{H}_n(K)$ implies $\mathcal{C}_n(K)$ for all $q \geq 1$ and for all $n \geq 1$.

Case 3. $X =$ **a nilpotent space with abelian fundamental group**

LEMMA 9.9. *Let A be a nilpotent π module and set $A_\ell = I^\ell A / I^{\ell+1} A$ where I is the augmentation ideal of $Z[\pi]$.*

1) $A \otimes Z/kZ = 0$ implies $A_\ell \otimes Z/kZ = 0$ for all ℓ.

2) $A \otimes Z/kZ = Tor(A, Z/kZ) = 0$ implies $A_\ell \otimes Z/kZ = Tor(A_\ell, Z/kZ) = 0$ for all ℓ.

PROOF. Notice that $I^\ell A$ is an epimorphic image of A and so is A_ℓ. Hence, $A \otimes Z/kZ = 0$ implies $I^\ell A \otimes Z/kZ = 0$ and $A_\ell \otimes Z/kZ = 0$ for all ℓ.

Since $I^\ell A$ is simultaneously a submodule and an epimorphic image of A, $A \otimes Z/kZ = Tor(A, Z/kZ) = 0$ implies $I^\ell A \otimes Z/kZ = Tor(I^\ell A, Z/kZ) = 0$ for all ℓ.

Now,

$$0 = I^\ell A \otimes Z/kZ = Tor(I^\ell A, Z/kZ) = I^{\ell-1} A \otimes Z/kZ = Tor(I^{\ell-1} A, Z/kZ)$$

implies $A_\ell \otimes Z/kZ = Tor(A_\ell, Z/kZ) = 0$ for all ℓ.

□

Consider the Serre spectral sequence of an orientable fibration sequence $K \to E \to B$ with $K = K(A, q+1), q \geq 1$. It converges to $H_*(E; Z/kZ)$ with $E^2_{r,s} = H_r(B; H_s(K; Z/kZ))$. Suppose that $H_s(B; Z/kZ) = 0$ for all $s \leq n-1$, then $E^2_{s,t} = H_s(B; H_t(K; Z/kZ)) = 0$ for all s. Hence, there is a Serre exact sequence

$$H_{n+q}(K; Z/kZ) \to H_{n+q}(E; Z/kZ) \to H_{n+q}(B; Z/kZ) \to \dots$$
$$\to H_2(K; Z/kZ) \to H_2(E; Z/kZ) \to H_2(B; Z/kZ) \to 0.$$

We have a similar exact sequence for homotopy groups with coefficients and a map from this homotopy sequence to the Serre exact sequence.

Let

$$X_1 = K(\pi_1(X),1),\ K(A_{2,0},2) \to X_{2,0} \to X_1,\ \dots$$
$$K(A_{q,\ell},q) \to X_\alpha \to X_{\alpha-1},\ \dots$$

where $A_{q,\ell} = I^\ell \pi_q(X)/i^{\ell-1}\pi_q(X)$ is the refinement into orientable fibrations of the Postnikov sequence of fibrations for X.

Lemma 9.10. *$\mathcal{C}_n(X_{\alpha-1})$ and $\mathcal{C}_n(K(A_{q,\ell},q))$ imply $\mathcal{C}_n(X_\alpha)$.*

Proof. Assume $\mathcal{C}_n(X_{\alpha-1})$.

For $s \le n+1$, consider the maps of horizontal exact sequences

$$\begin{array}{ccccccc} \pi_s(K(A_{q,\ell},q);Z/kZ) & \to & \pi_s(X_\alpha;Z/kZ) & \to & \pi_s(X_{\alpha-1};Z/kZ) & \to \\ \downarrow\phi & & \downarrow\phi & & \downarrow\phi & \\ H_s(K(A_{q,\ell},q);Z/kZ) & \to & H_s(X_\alpha;Z/kZ) & \to & H_s(X_{\alpha-1};Z/kZ) & \to \end{array}$$

If $s \le n$, the left and right mod k Hurewicz maps are isomorphisms. The mod k Hurewicz map which is out of the picture to the left is an epimorphism and the one to the right out of the picture is an monomorphism. Hence, a standard 5-lemma argument shows that the mod k Hurewicz map in the middle is an isomorphism.

If $s = 2$, then we remark that our hypotheses guarantee that the homotopy sets are in fact groups and that the Hurewicz maps are homomorphisms. Otherwise, the argument is the same.

If $s = n+1$, then the left and right mod k Hurewicz maps are epimorphisms. The one to the right out of the picture is an monomorphism. Hence, the standard 5-lemma argument shows that the mod k Hurewicz map in the middle is an epimorphism.

The lemma is proved. □

Assume $\mathcal{H}_n(X)$. This implies $\mathcal{H}_n(X_q)$ and $\mathcal{H}_n(K(A_{q,\ell},q))$ for all $q \ge 1$ and all ℓ. Hence, we have $\mathcal{C}_n(K(A_{q,\ell},q))$ for all $q \ge 1$ and all ℓ. The lemma implies that we have $\mathcal{C}_n(X_\alpha)$ for all $q \ge 1$. If we let α be sufficiently large, then we have $\mathcal{C}_n(X)$. □

In particular, the mod k Hurewicz theorem implies

Corollary 9.11. *If X is a simple space (that is, the fundamental group acts trivially on all the homotopy groups) and if $\pi_1(X)\otimes Z/kZ = 0$, then $\pi_2(X;Z/kZ) = H_2(X;Z/kZ)$ is an abelian group. If, in addition, $\pi_2(X;Z/kZ) = 0$, then $\pi_3(X;Z/kZ) = H_3(X;Z/kZ)$ is an abelian group.*

The next section will strengthen this 3-dimensional result when k is an odd integer.

The next result is a strengthening of the mod k Hurewicz theorem in the case of loop spaces. Recall that $\pi_1(\Omega Y;Z/kZ) = \pi_1(\Omega Y)\otimes Z/kZ$ by definition.

Theorem 9.12 (mod k Hurewicz theorem for loop spaces). *Suppose $X = \Omega Y$ is a loop space and $n \ge 1$. If $\pi_\ell(X;Z/kZ) = 0$ for all $\ell < n$, then $H_\ell(X;Z/kZ) = 0$ for all $\ell < n$ and the Hurewicz map $\phi : \pi_n(X;Z/kZ) \to H_n(X;Z/kZ)$ is a bijection. If $n \ge 2$, then the Hurewicz map $\phi : \pi_{n+1}(X;Z/kZ) \to H_{n+1}(X;Z/kZ)$ is a surjection.*

PROOF. First of all, we again recall that the case $n = 1$ is true almost by definition, $\pi_1(X; Z/kZ) = \pi_1(X) \otimes Z/kZ = H_1(X) \otimes Z/kZ = \pi_1(X; Z/kZ)$. Hence we begin with the case $n = 2$. The proof is the same as that of the mod k Hurewicz theorem for nilpotent spaces with three differences. One, all the homotopy sets are automatically groups. Two, the loop space is simple space, that is the fundamental group acts trivially on all the homotopy groups. There is no need to refine the Postnikov system to get orientable fibrations. And the Postnikov system consists entirely of loop spaces and loop maps. These are improvements, not problems. But finally, three, the Hurewicz map $\phi : \pi_2(X) \to H_2(X; Z/kZ)$ may not be a homomorphism. But our hypotheses gurarantee that it is a homomorphism since

$$\phi(f + g) = \phi(f) = \phi(g) + (\beta\phi(f)) * (\beta\phi(g)) = \phi(f) = \phi(g) + (0) * (0),$$
$$(\beta\phi(f)), (\beta\phi(g)) \in H_1(X; Z/kZ) = 0.$$

□

COROLLARY 9.13. *Let $f : X \to Y$ be a map of simply connected spaces. Assume that $Tor(\pi_2(Y)/f * \pi_2(X), Z/kZ) = 0$. Let $n \geq 1$. Then $\pi_j(X; Z/kZ) \to \pi_j(Y; Z/kZ)$ is a bijection for all $j \leq n$ and a surjection for $j = n + 1$, if and only if $H_j(X; Z/kZ) \to H_j(Y; Z/kZ)$ is a bijection for all $j \leq n$ and a surjection for $j = n + 1$.*

REMARK 9.14. The example of the inclusion of the circle into the fake circle shows that we need some hypothesis to insure the truth of this result.

PROOF. We can assume that f is a fibration and that $F \to X \to Y$ is a fibration sequence. Since the base is simply connected, the fibration is orientable. The fibration sequence $\Omega Y \to F \to X \to Y$ shows that F is nilpotent: Let I be the augmentation ideal of the group ring $Z[\pi_1(F)]$. Since X is simply connected,

$$I\pi_n(F) \subseteq image\ \pi_n(\Omega Y).$$

Since $\pi_1(\Omega Y) \to \pi_1(F)$ is a surjection and ΩY is a simple space, that is, its fundamental group acts trivially on all $\pi_n(\Omega Y)$, F is nilpotent of length ≤ 2, that is, $I^2\pi_n(F) = 0$.

Note that $\pi_1(F) = \pi_2(Y)/f * \pi_2(X)$ and hence the preliminary hypotheses of the mod k Hurewicz theorem are satisfied.

Assume the homotopy hypothesis. Then $\pi_j(F; Z/kZ) = 0$ for all $j \leq n$. The mod k Hurewicz theorem implies that $H_j(F; Z/kZ) = 0$ for all $j \leq n$. The homology Serre spectral sequence shows that $H_j(X; Z/kZ) \to H_j(Y; Z/kZ)$ is a bijection for all $j \leq n$ and a surjection for $j = n + 1$.

On the other hand, the same homology Serre spectral sequence shows that the homology hypothesis implies that $H_j(F; Z/kZ) = 0$ for all $j \leq n$. The mod k Hurewicz theorem implies that $\pi_j(F; Z/kZ) = 0$ for all $j \leq n$.

□

A simpler version of the same proof shows

COROLLARY 9.15. *Let $f : X \to Y$ be a loop map of loop spaces and suppose that it is a monomorphism on fundamental groups. Let $n \geq 1$. Then $\pi_j(X; Z/kZ) \to \pi_j(Y; Z/kZ)$ is a bijection for all $j \leq n$ and a surjection for $j = n + 1$, if and only if $H_j(X; Z/kZ) \to H_j(Y; Z/kZ)$ is a bijection for all $j \leq n$ and a surjection for $j = n + 1$.*

The next result removes the requirement that the map be a loop map. The proof is only slightly more difficult.

COROLLARY 9.16. *Let $f : X \to Y$ be a map of loop spaces which is an isomorphism on fundamental groups and suppose that $\pi_2(X) \to \pi_2(Y)$ is a surjection. Then $\pi_j(X; Z/kZ) \to \pi_j(Y; Z/kZ)$ is an isomorphism for all $j \geq 1$ if and only if $H_j(X; Z/kZ) \to H_j(Y; Z/kZ)$ is an isomorphism for all $j \geq 1$.*

PROOF. Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the map of universal covers. The fibration sequences

$$\tilde{X} \to X \to K(\pi_1(X), 1), \tilde{Y} \to Y \to K(\pi_1(Y), 1)$$

are both orientable.

Suppose that $\pi_j(X; Z/kZ) \to \pi_j(Y; Z/kZ)$ is an isomorphism for all j. Since $\pi_2(\tilde{X}; Z/kZ) = \pi_2(X) \otimes Z/kZ, \quad \pi_2(\tilde{Y}; Z/kZ) = \pi_2(Y) \otimes Z/kZ$, it is easy to see that the same equivalence is true for $\pi_j(\tilde{X}; Z/kZ) \to \pi_j(\tilde{Y}; Z/kZ)$. Since these spaces are simply connected, the previous result shows that the same equivalence is true for $H_j(\tilde{X}; Z/kZ) \to H_j(\tilde{Y}; Z/kZ)$.

Since there is an isomorphism on the bases, the Zeeman comparison theorem [**33, 14**] applies to show that the same equivalence is true for $H_j(X; Z/kZ) \to H_j(Y; Z/kZ)$.

The argument is reversible. Hence, we are done.

□

REMARK 9.17. Suppose $f : X \to Y$ is a map of simply connected spaces which are localized at a prime p. Suppose that the homology groups $\overline{H}_*(X; Z) = \overline{H}_*(X; Z_{(p)})$ and $\overline{H}_*(Y; Z) = \overline{H}_*(Y; Z_{(p)})$ are finitely generated $Z_{(p)}$ modules in each degree. Suppose also that there is some $r \geq 0$ such that the induced map $f : H_j(X; Z/p^r Z) \to H_j(Y; Z/p^r Z)$ is an isomorphism for all $j \leq n$ and an epimorphism for $j = n$. In other words, f is a mod p^r $n-$ equivalence. Then, if W is any CW- complex, the map of pointed mapping sets $f_* : [W, X]_* \to [W, Y]_*$ is a bijection if the dimension of W is $< n$ and an epimorphism if the dimension of W is $= n$. The argument is: Since the homologies are of finite type, the map is an $n-$equivalence localized at p, therefore, it is an integral homology $n-$equivalence. The integral Hurewicz theorem for pairs implies that it is an integral homotopy $n-$equivalence. Now the classical theorem of J.H.C. Whitehead implies the result.

10. Abelian homotopy groups in dimension 3

THEOREM 10.1. *If k is odd, then $\pi_3(X; Z/kZ)$ is an abelian group.*

PROOF. Consider the isomorphic group $\pi_2(\Omega X; Z/kZ)$.

The commutator $[\quad , \quad] : \Omega X \times \Omega X \to \Omega X$ is given by $[\omega, \gamma] = \omega\gamma\omega^{-1}\gamma^{-1}$. Since $[\quad , \quad]$ is null homotopic on the bouquet $\Omega X \vee \Omega X$, it factors as

$$\Omega X \times \Omega X \to \Omega X \wedge \Omega X \xrightarrow{\overline{[\ ,\ ]}} \Omega X.$$

Let $f : P^2(Z/kZ) \to \Omega X$ and $g : P^2(Z/kZ) \to \Omega X$ be two maps. The commutator $[f, g]$ is the composition

$$P^2(Z/kZ) \xrightarrow{\Delta} P^2(Z/kZ) \times P^2(Z/kZ) \xrightarrow{f \times g} \Omega X \times \Omega X \xrightarrow{[\ ,\ ]} \Omega X$$

where Δ is the diagonal. If $\overline{\Delta}$ is the reduced diagonal, this is the same as the composition

$$P^2(Z/kZ) \xrightarrow{\overline{\Delta}} P^2(Z/kZ) \wedge P^2(Z/kZ) \xrightarrow{f \wedge g} \Omega X \wedge \Omega X \xrightarrow{\overline{[\ \cdot\ ]}} \Omega X.$$

Let e_1 and e_2 be generators of the reduced homology $\overline{H}_*(P^2(Z/kZ); Z/kZ)$ of respective dimensions 1 and 2. A computation of Steenrod [**28**] asserts that

$$\overline{\Delta}_*(e_2) = \frac{k(k+1)}{2} e_1 \otimes e_1$$

and this equals 0 when k is odd. If k is odd, then the mod k Hurewicz image $\phi(\overline{\Delta}) = 0$.

The Hurewicz theorem implies that $\overline{\Delta}$ is null homotopic.

Hence $[f, g]$ is null homotopic and $\pi_2(\Omega X; Z/kZ)$ is abelian if k is odd. □

Corollary 10.2. *If k is odd, then the homotopy groups with coefficients Z/kZ of a loop space are all abelian groups, that is, $\pi_1(\Omega X; Z/kZ) = \pi_1(\Omega X) \otimes Z/kZ$ and $\pi_n(\Omega X; Z/kZ) = \pi_{n+1}(X; Z/kZ)$, $n \geq 2$ are all abelian groups.*

Let G be a group-like space. If k is odd, the proof above shows that $\pi_2(G; Z/kZ)$ is an abelian group. If k is any integer, then the fact that the composition

$$P^2(Z/kZ) \xrightarrow{\overline{\Delta}} P^2(Z/kZ) \wedge P^2(Z/kZ) \xrightarrow{1 \wedge \overline{\rho}} P^2(Z/kZ) \wedge S^2$$

is null homotopic shows

Theorem 10.3. *If k is any integer, the image of the reduction map $\rho : \pi_2(G) \to \pi_2(G; Z/kZ)$ is a central subgroup.*

Of course, it already follows from the Hurewicz theorem that

Corollary 10.4. *If k is any integer and $\pi_1(\Omega X) \otimes Z/kZ = 0$, then the homotopy groups with coefficients Z/kZ are all abelian groups, that is, $\pi_n(\Omega X; Z/kZ)$ are abelian groups for all $n \geq 2$.*

The fake multiples allow us to prove that certain other subgroups of 3-dimensional homotopy groups mod 2^r are central, that is

Theorem 10.5. *If $r \geq 1$, then the image of the $2-nd$ power*

$$2^* : \pi_3(X; Z/2^r Z) \to \pi_3(X; Z/2^r Z)$$

is a central subgroup.

Proof. Since the image is the same, we consider the dual of the fake multiple $\overline{2}^*$. Images under this map are represented by compositions

$$P^3(Z/2^r Z) \xrightarrow{\overline{2}} P^3(Z/2^r Z) \xrightarrow{f} X.$$

Since the fake multiples are suspensions, we can take the adjoints as follows

$$P^2(Z/2^r Z) \xrightarrow{\overline{2}} P^2(Z/2^r Z) \xrightarrow{\underline{f}} \Omega X$$

where $\underline{f}$ is the adjoint of f. Hence, the images are subgroups.

The desired result follows from the fact that the following maps are null homotopic:

$$P^2(Z/2^r Z) \xrightarrow{\overline{\Delta}} P^2(Z/2^r Z) \wedge P^2(Z/2^r Z) \xrightarrow{\overline{2} \wedge 1} P^2(Z/2^r Z) \wedge P^2(Z/2^r Z) \xrightarrow{f \wedge g} \Omega X \wedge \Omega X \xrightarrow{\overline{[\ \cdot\ ]}} \Omega X.$$

It is sufficient to note that the mod 2^r Hurewicz image is zero. But

$$e_2 \mapsto \frac{2^r(2^r+1)}{2} e_1 \otimes e_1 \mapsto \frac{2^r(2^r+1)}{2} 2e_1 \otimes e_1 = 2^r(2^r+1)e_1 \otimes e_1 = 0$$

mod 2^r if $r \geq 1$. □

11. Classical and modular Hopf invariants

In this section, we revisit the old result of Hopf [**11**] on the nontriviality of $\pi_3(S^2) = Z \neq 0$ and discuss a modular form of the Hopf invariant which can be used to determine whether a map $P^3(Z/kZ) \to P^3(Z/\ell Z)$ is trivial or nontrivial.

The Hopf fibering $\eta : S^3 \to S^2$ has the mapping cone $S^2 \cup_\eta e^4 = CP^2$ and the embedding $CP^2 \subseteq CP^\infty$ yields the fibration sequence

$$\Omega S^3 \xrightarrow{\Omega\eta} \Omega S^2 \xrightarrow{\partial} S^1 \to S^3 \xrightarrow{\eta} S^2 \to CP^\infty.$$

This has the immediate consequences:

1) $\eta : S^3 \to S^2$ represents a generator of the group $\pi_3(S^2) \cong Z$.

2) there is a homotopy equivalence

$$S^1 \times \Omega S^3 \xrightarrow{\iota \times \Omega\eta} \Omega S^2 \times \Omega S^2 \xrightarrow{mult} \Omega S^2.$$

3) if $\overline{\eta} : S^2 \to \Omega S^2$ is the adjoint of η, then the Hurewicz image is $\phi(\overline{\eta}) = \pm x^2$ where $H_*(\Omega S^2; Z) = P(x) =$ the polynomial algebra generated by a 2-dimensional class x.

The following are two equivalent definitions of the classical integral Hopf invariant $H(g)$ of a map $g : S^3 \to S^2$. (The map $\overline{g} : S^2 \to \Omega S^2$ is the adjoint of g.)

Definition 11.1. 1) The Hurewicz image definition:

$$\phi(\overline{g}) = H(g)\phi(\overline{\eta}).$$

2) The cup product definition:

$$u \cup u = H(g)e$$

in the integral cohomology of the mapping cone $C_g = S^2 \cup_g e^4$ where $u \in H^2(C_g; Z)$ and $e \in H^4(C_g; Z)$ are generators.

It is clear that η has Hopf invariant one in both definitions. To see that the two definitions are equivalent, observe that the linearity of the Hurewicz map implies that the first definition of H defines a linear map $H_1 : \pi_3(S^2) \to Z$.

We claim that, for any integer ℓ, the second definition of H is semi-linear, that is, $H_2(kg) = H_2(g \circ k) = kH_2(g)$. This implies that they are equal, $H = H_1 = H_2$.

The justification for semi-linearity is a consequence of the maps of cofibration sequences

$$\begin{array}{ccccccc} S^3 & \xrightarrow{g\circ k} & S^2 & \to & C_{g\circ k} & \to & S^4 \\ \downarrow k & & \downarrow 1 & & \downarrow & & \downarrow k. \\ S^3 & \xrightarrow{g} & S^2 & \to & C_g & \to & S^4 \end{array}$$

Just compute the Hopf invariant via the bottom row and use naturality.

Since $\pi_3(S^2) = Z\eta$, any map $g : S^3 \to S^2$ is a multiple of η, that is, $g = \ell\eta$. Hence,

$$H_2(g) = H_2(\eta \circ \ell) = \ell H_2(\eta) = \ell = \ell H_1(\eta) = H_1(\ell\eta) = H_1(g).$$

In any case, we have the classical result of Hopf [**11**]:

THEOREM 11.2. *The integral Hopf invariant is an isomorphism* $H : \pi_3(S^2) \to Z$ *and* $H(k\eta) = k$ *for all integers* k.

We know for trivial reasons that any suspension $\Sigma h : S^3 \to S^2$ is null homotopic but the classical Hopf invariant gives another proof since nontrivial cup products are zero in the cohomology ring of a suspension $H^*(S^2 \cup_{\Sigma h} e^4; Z)$.

Inspection of the diagram

$$\begin{array}{ccccccc} S^3 & \xrightarrow{g} & S^2 & \to & C_g & \to & S^4 \\ \downarrow 1 & & \downarrow k & & \downarrow & & \downarrow k \\ S^3 & \xrightarrow{k\circ g} & S^2 & \to & C_{k\circ g} & \to & S^4 \end{array}$$

shows the quadratic nature of the Hopf invariant:

LEMMA 11.3. *If* $g : S^3 \to S^2$ *and* k *is an integer, then* $H(k \circ g) = k^2 H(g)$ *and, since the Hopf invariant is an isomorphism,* $k \circ g \simeq k^2 g = g \circ k^2$.

Our next goal is to prove a modular analog of Hopf's theorem.

LEMMA 11.4. *Let* A *and* B *be finite groups. If* $f : P^n(A) \to P^n(B)$ *is a map which induces zero in integral cohomology, that is,* $f^* = 0 : B \to A$, *then* f *induces zero in cohomology and homology with all coefficients.*

PROOF. The map f induces $(f^*)^* = 0 : A^* \to B^*$ in integral homology and the universal coefficient theorems imply that it induces zero in homology and cohomology with all coefficients. □

DEFINITION 11.5. For A and B finite, let

$$K^n(A, B) = \{f \in [P^n(A), P^n(B)]_* \,|f^* = 0 : B \to A\}$$

.

For B finite abelian, let $h : P^n(B) \to K(B^*, n-1)$ be a map such that the integral homology map $h_* : B^* \to B^*$ is an isomorphism and let $F^n(B)$ be the homotopy theoretic fibre of h.

Suppose that $f : P^n(A) \to P^n(B)$ is a map. Note that $h \circ f : P^n(A) \to K(B^*, n-1)$ is null homotopic if and only if $0 = f_* : A^* \to B^*$. Thus, f factors through the fibre $F^n(B)$ if and only if $0 = f^* : B \to A$, that is, if and only if $f \in K^n(A, B)$. The fibration sequence

$$K(B^*, n-2) \to F(B) \to P^n(B) \to K(B^*, n-1)$$

shows that

LEMMA 11.6.

$$\pi_n(F^n(B); A) = K^n(A, B)$$

We compute

THEOREM 11.7. *a) Let* A *and* B *be finite abelian groups with relatively prime order, then* $K^n(A, B) = 0$ *for all* $n \geq 3$.

b) If p *is an odd prime, then*

$$K^3(Z/p^r Z, Z/p^s Z) = Z/p^r Z \otimes Z/p^s Z = Z/(p^r, p^s)Z$$

where $(p^r, p^s) =$ *the greatest common divisor of* p^r *and* p^s.

c) If p is an odd prime and $n \geq 4$, then

$$K^n(Z/p^r Z, Z/p^s Z) = 0$$

d)

$$K^3(Z/2^r Z, Z/2^s Z) = Z/2^r Z \otimes Z/2^{s+1} Z = Z/(2^r, 2^{s+1})Z$$

where $(2^r, 2^{s+1})$ = the greatest common divisor of 2^r and 2^{s+1}.

e) If $n \geq 4$, then

$$K^n(Z/2^r Z, Z/2^s Z) = Z/2Z.$$

PROOF. a) It is easy to see that $\overline{H}_*(K(B^*, 1); Z)$ has the same exponent as the exponent of B = the exponent of B^*. Induction using the Serre spectral sequence shows that every dimension of $\overline{H}_*(K(B^*, n); Z)$ has exponent which involves only the primes which divide the order of B. Since this is true of both $\overline{H}_*(K(B^*, n); Z)$ and $\overline{H}_*(P^n(B); Z)$, it is also true of the fibre $\overline{H}_*(F(B); Z)$. We use here the fact that $n \geq 3$ implies that $K(B^*, n-1)$ is simply connected if $n \geq 3$ and hence the fibration is orientable.

The Hurewicz theorem shows that $\pi_*(F^n(B); Z/p^r) = 0$ for all $r \geq 1$ and for all primes p which are relatively prime to the order of B. Hence, if A is finite of order relatively prime to the order of B, then $\pi_n(F(B); A) = 0$.

b) In dimensions $* \leq 5$, $\overline{H}^*(K(Z/p^s Z, 2); Z/pZ)$ has basis

$$x, \quad y = \beta_s x, \quad x^2, \quad xy$$

where x has degree 2 and with β_s = the $s-$th cohomology Bockstein. There are no Steenrod operations yet. Since $\beta_s(x^2) = 2xy$, it follows that the integral cohomology is

$$\overline{H}^*(K(Z/p^s Z, 2); Z) = \begin{cases} Z/p^s Z & * = 3, 5 \\ 0 & * \neq 3, 5, * \leq 5 \end{cases}$$

Hence, the integral homology is

$$\overline{H}_*(K(Z/p^s Z, 2); Z) = \begin{cases} Z/p^s Z & * = 2, 4 \\ 0 & * \neq 2, 4, * \leq 4 \end{cases}$$

The Serre spectral sequence yields

$$\overline{H}_*(F^3(Z/p^s Z); Z) = \begin{cases} Z/p^s Z & * = 3 \\ 0 & * \neq 3, * \leq 3. \end{cases}$$

It follows that

$$K^3(Z/p^r Z, Z/p^s) = \pi_3(F^3(Z/p^s Z); Z/p^r Z) = Z/p^s Z \otimes Z/p^r Z.$$

c) The strong coefficient functoriality implies that $K^n(Z/p^r Z, Z/p^s Z) = 0$ if p is an odd prime and $n \geq 4$.

d) In dimensions $* \leq 6$, Serre's computations [**26, 19**] show that $\overline{H}^*(K(Z/2^s Z, 2); Z/2Z)$ has basis

$$x, \quad y = \beta_s x, \quad x^2, \quad Sq^2 y, \quad xy, \quad y^2$$

where x has degree 2 and with β_s = the $s-$th cohomology Bockstein. Recall the Adem relation $Sq^1 Sq^2 y = Sq^3 y = y^2$. Since $\beta_r x^2 = 2xy = 0$, Browder's theorem [**1, 10**] asserts that

$$\beta_{r+1} x^2 = \begin{cases} xy + Sq^2 y \quad mod \quad image(Sq^1) = image(\beta_1), \quad s = 1 \\ xy \quad mod \quad image(\beta_r), \quad s \geq 2 \end{cases}$$

The integral cohomology is

$$\overline{H}^*(K(Z/2^sZ,2);Z) = \begin{cases} Z/2^sZ & *=3 \\ Z/2^{s+1}Z & *=5 \\ 0 & *\neq 3,5,\ *\leq 5 \end{cases}$$

The remainder of the argument in this case is the same as in part b).

e) For $n \geq 4$, Serre's computations show that, in dimensions $* \leq n+3$, $\overline{H}^*(K(Z/2^sZ,n-1);Z/2Z)$ has basis

$$x, \quad \beta_r x = y, \quad Sq^2x, \quad Sq^2y, \quad Sq^3x, \quad Sq^3y, \quad and, \quad if\ \ n=4, \quad xy$$

Since $Sq^1Sq^2 = Sq^3$, it follows that the integral homology is

$$\overline{H}_*(K(Z/2^sZ,n-1);Z) = \begin{cases} Z/2^sZ & *=n-1 \\ Z/2Z & *=n+1,n+2 \\ 0 & *\neq n-1,n+1,n+2, \quad *\leq n+2 \end{cases}$$

and the rest follows as before.

□

Remark 11.8. The generator of $Z/2Z$ in e) is clearly given by the composition $\overline{\eta}: P^n(Z/2^sZ) \xrightarrow{q} S^n \xrightarrow{\eta} S^{n-1} \xrightarrow{\iota} P^n(Z/2^sZ), \quad n\geq 4$.

Let $f : P^3(Z/kZ) \to P^3(Z/\ell Z)$ be a map which induces zero in integral cohomology and therefore zero in homology and cohomology with any coefficients. That is, $f \in K^3(Z/kZ, Z/\ell Z)$. For such maps we define the modular Hopf invariant $\mathcal{H}(f) \in Z/kZ \otimes Z/\ell Z = Z/(k,\ell)Z$ as follows:

Definition 11.9. Let C_f be the mapping cone of $f : P^3(Z/kZ) \to P^3(Z/\ell Z)$. Let $j = (k,\ell) =$ the greatest common divisor of k and ℓ and let $u \in H^2(C_f; Z/jZ)$ and $e \in H^4(C_f; Z/jZ)$ be generators. The modular Hopf invariant of f is the integer mod j defined by the formula

$$u \cup u = \mathcal{H}(f)e.$$

We claim that f factors as $P^3(Z/kZ) \xrightarrow{q} S^3 \xrightarrow{g} S^2 \xrightarrow{\iota} P^3(Z/\ell Z)$. The vanishing of f in integral homology shows that f is null on $S^2 \subset P^3(Z/kZ)$ and hence it factors as $P^3(Z/kZ) \xrightarrow{q} S^3 \to P^3(Z/\ell Z)$. Since any composition $S^3 \to P^3(Z/\ell Z) \to qS^3$ is degree zero, it is null homotopic. Now, the Serre spectral sequence shows that the fibre of q is approximated by S^2 up to dimension 3. Hence $S^3 \to P^3(Z/\ell Z)$ factors as $S^3 \to S^2 \xrightarrow{P}{}^3 (Z/\ell Z)$.

If $h = g \circ q$, the commutative diagrams of cofibration sequences

$$\begin{array}{ccccccc} P^3(Z/kZ) & \xrightarrow{f} & P^3(Z/\ell Z) & \to & C_f & \to & P^4(Z/kZ) \\ \downarrow q & & \downarrow 1 & & \downarrow & & \downarrow q \\ S^3 & \xrightarrow{h} & P^3(Z/\ell Z) & \to & C_h & \to & S^4 \\ \uparrow 1 & & \uparrow \iota & & \uparrow & & \uparrow 1 \\ S^3 & \xrightarrow{h} & S^2 & \to & C_g & \to & S^4 \end{array}$$

shows that, if we choose the generator $e \in H^4(C_f; Z/jZ))$ to be compatible with the standard generator of $H^4(S^4; Z/jZ)$, then the modular Hopf invariant $\mathcal{H}(f)$ is the mod j reduction of the classical Hopf invariant $H(g)$.

Hence,

LEMMA 11.10. *For any integer k, $\mathcal{H}(k \circ f) = k^2\mathcal{H}(f)$ and $\mathcal{H}(f \circ k) = k\mathcal{H}(f)$.*

We also get that the modular Hopf invariant is always an epimorphism. Theorem 8.7 gives

LEMMA 11.11. *1) If p is an odd prime, then the modular Hopf invariant $\mathcal{H}: K^3(Z/p^rZ, Z/p^sZ) \to Z/p^rZ \otimes Z/p^sZ$ is an isomorphism.*

2) If $p = 2$, then the modular Hopf invariant $\mathcal{H} : K^3(Z/2^rZ, Z/2^sZ) = Z/2^rZ \otimes Z/2^{s+1} \to Z/2^rZ \otimes Z/2^sZ$ is an epimorphism which is an isomorphism if $r \leq s$ and has kernel $Z/2Z$ if $r \geq s+1$.

COROLLARY 11.12. *Let p be a prime and let $f \in K^3(Z/p^rZ, Z/p^s)$. Consider the following three statements:*

1) f is null homotopic.

2) f is a suspension.

3) $\mathcal{H}(f) = 0 \in Z/p^rZ \otimes Z/p^sZ$.

Then 1) $\implies$ 2) $\implies$ 3), and, if p is odd or if $p = 2$ and $r \leq s$, then 3) $\implies$ 1).

12. The Dold-Thom theorem

The Hurewicz map is given a geometric form as the map $X \to SP^\infty(X)$ where $SP^\infty(X)$ is the infinite symmetric product. Recall that $SP^\infty(X)$ is the free abelian group generated by the points of X subject to the relation that the basepoint is the unit.

It is filtered by the subspaces $SP^q(X)$ consisting of the words of length $\leq q$. Each $Sp^q(X)$ is a quotient space of the $q-$fold product $X \times \cdots \times X$ and $SP^\infty(X) = \lim SP^q(X)$ has the direct limit topology.

If X is connected, Dold and Thom [**8**] proved that $SP^\infty(X) \simeq K(\overline{H}_*(X; Z))$ is a product of Eilenberg-MacLane spaces if X is connected. For $n \geq 1$, the map $\pi_n(X; Z) \to \pi_n(SP^\infty(X); Z) = H_n(X; Z)$ is the Hurewicz map.

The proof proceeds by showing that the functor $X \mapsto SP^\infty(X)$ preserves homotopy,

$$f \simeq g \implies SP^\infty(f) \simeq SP^\infty(g),$$

and preserves colimits

$$SP^\infty(\lim X_n) = \lim SP^\infty(X_n).$$

The key point is that the functor SP^∞ converts cofibrations $A \to X \to X/A$ into quasi-fibrations

$$SP^\infty(A) \to SP^\infty(X) \to SP^\infty(X/A).$$

This last result is quite plausible, since if the infinite symmetric product were an abelian group, then $SP^\infty(X/A) = SP^\infty(X)/Sp^\infty(A)$ would be a homogeneous space and we would expect $SP^\infty(A) \to SP^\infty(X) \to SP^\infty(X/A)$ to be an actual fibration.

The key feature of quasi-fibrations is that their homotopy groups satisfy long exact sequences just like those of fibrations.

Thus the cofibration sequence $X \to CX \to \Sigma X$ gives the isomorphic connecting homomorphism $\pi_n(SP^\infty(\Sigma X)) \to \pi_{n-1}(SP^\infty(X))$.

Now it is easy to see that the functor $X \mapsto \pi_n(SP^\infty X)$ satisfies the axioms for a reduced homology theory. If we can check that it is a connected theory with coefficient group Z, then there must be a natural isomorphism, $\pi_n(SP^\infty(X)) = \overline{H}_n(X;Z)$ for all $n \geq 1$. This follows from

LEMMA 12.1.
$$SP^\infty(S^2) \cong CP^\infty.$$

PROOF. Regard $S^2 = \mathcal{C} \cup \infty$ as the Riemann sphere and define a homeomorphism $\Phi : SP^q(S^2) \to CP^q$ by taking the coordinates of the homogeneous complex polynomial with roots $(b_1, b_2, b_3, \ldots, b_q)$.

That is, if the b_i are all finite,
$$\Phi(b_1, b_2, b_3, \ldots, b_q) = (z + b_1)(z + b_2)(z + b_3)\ldots(z + b_q) =$$
$$a_0 + a_1 z + \cdots + a_{q-1}z^{q-1} + z^q = [a_0, a_1, \ldots, a_q, 1].$$
If $b_i = \infty$, it is defined recursively by omitting the $i-$th factor,
$$\Phi(b_1, b_2, b_3, \ldots, b_q) = \Phi(b_1, b_2, \ldots, b_{i-1}, b_{i+1}\ldots, b_q) =$$
$$(z + b_1)\ldots(z + b_{i-1})(z + b_{i+1})\ldots(z + b_q) =$$
$$a_0 + a_1 z + \cdots + a_{q-2}z^{q-2} + z^{q-1} = [a_0, a_1, \ldots, a_{q-2}, 1, 0].$$
And so on until there are no more $b_i = \infty$.

Thus, $SP^q(S^2) \cong CP^q$. Letting $q \to \infty$ gives the result. □

$SP^\infty(S^2) = K(Z, 2)$ with coefficient group $Z = \pi_2 SP^\infty(S^2)$ and thus
$$\pi_n(SP^\infty(X)) \cong H_n(X;Z)$$
for all connected X.

COROLLARY 12.2. *If $M_n(A)$ is a Moore space with $n \geq 1$, then $SP^\infty(M_n(A)) \simeq K(A, n)$.*

Note the obvious

LEMMA 12.3. $SP^\infty(X \vee Y) = SP^\infty(X) \times SP^\infty(Y)$ *and the analogous result for infinitely many bouquet summands.*

The following result was originally observed by John Moore. With the infinite symmetric product, it becomes very easy.

LEMMA 12.4. *Let G be a strictly commutative connected topological monoid with homotopy groups $\pi_i(G) = \pi_i$. Let*
$$M = M(\pi_*) = \bigvee_i M_i(\pi_i, i)$$
be the corresponding bouquet of Moore spaces. Then there are weak homotopy equivalences
$$SP^\infty(M) \to G, \quad SP^\infty(M) \to K(\pi_*(G)).$$

PROOF. Start with a map $\bigvee S^i \to G$, one sphere for each generator of π_i. Attaching $i+1$ cells to kill the relations gives a map $M_i(\pi_i, i) \to G$ and thus a map
$$M = \bigvee_i M_i(\pi_i, i) \to G.$$
Since G is strictly commutative, we can extend to a multiplicative map $SP^\infty(M) \to G$ which is a weak equivalence.

Applying this to $G = K(\pi_*(G))$ gives the second result above. □

Corollary 12.5. *If X is connected and $M = M(\overline{H}_*(X))$, then there are weak homotopy equivalences*

$$SP^\infty(M) \to SP^\infty(X), \quad SP^\infty(M) \to K(\overline{H}_*(X;Z)).$$

For simply connected X and $n \geq 2$, T. Kobayashi [**13**] observed: for a finitely generated abelian group A and simply connected X, the functor

$$X \mapsto \pi_n(SP^\infty(X); A)$$

satisfies the axioms for a reduced homology theory with coefficient group $A = \pi_2(SP^\infty(S^2); A) = \pi_2(K(Z,2); A)$. Thus

$$\pi_n(SP^\infty X; A) \cong H_n(X; A)$$

for all simply connected X.

Hence, for simply connected X, the map $\pi_n(X;A) \to \pi_n(SP^\infty(X);A) = H_n(X;A)$ can be used to define the Hurewicz map with coefficients in A.

This general result is satisfying but we prefer not to use it since it involves using the fact that a connected homology theory is characterized by the Eilenberg-Steenrod axioms. We prefer to use instead the more direct and specific form of the Hurewicz theorem which we have given for coefficients Z/kZ. One advantage of the latter approach is that one does not have to restrict to simply connected spaces and to dimensions ≥ 2.

13. The fibre of the geometric Hurewicz map and uniqueness of smash decompositions

We can use the idea of a geometric Hurewicz map to study the mod k Hurewicz map and will refer to any map $\phi : X \to K(\overline{H}_*(X;Z))$ as a geometric Hurewicz map if

$$\phi_* : H_n(X;Z) \xrightarrow{\cong} H_n(K(\overline{H}_n(X;Z),n)) \subseteq H_n(K(\overline{H}_*(X;Z)))$$

is the natural inclusion.

The case of a Peterson space is particularly simple. We embed the Peterson space into an Eilenberg-MacLane space by attaching cells to kill all the higher homotopy groups, $P^n(Z/kZ) \xrightarrow{\iota} K(Z/kZ, n-1)$.

Lemma 13.1. *If $f, g : X \to P^n(Z/kZ)$ are any maps, then a) and b) are equivalent where these are:*

a) $f^* = g^* : H^{n-1}(P^m(Z/kZ); Z/kZ) \to H^{n-1}(X; Z/kZ)$.

b) $\iota \circ f \simeq \iota \circ g : X \to P^n(Z/kZ) \xrightarrow{\iota} K(Z/kZ, n-1)$.

c) If, in addition, $X = P^m(Z/kZ)$, then a) and b) are equivalent to $f_* = g_* : H_{n-1}(P^m(Z/kZ); Z/kZ) \to H_{n-1}(P^n(Z/kZ); Z/kZ)$.

Proof. The equivalence of a) and b) follows from the fact that Eilenberg-MacLane spaces classify cohomology.

Suppose $X = P^m(Z/kZ)$. Since the CW chains are

$$C(P^m(Z/kZ)) = < 1, \beta e_m, e_m >, \quad C(P^n(Z/kZ)) = < 1, \beta e_n, e_n >$$

with differentials $d(e_m) = k\beta e_m$, $d(e_n) = k\beta e_n$, it follows that mod k cohomology is dual to mod k homology,

$$H^{n-1}(P^m(Z/kZ); Z/kZ) \cong Hom(H_{n-1}(P^m(Z/kZ); Z/kZ), Z/kZ),$$
$$H^{n-1}(P^n(Z/kZ); Z/kZ) \cong Hom(H_{n-1}(P^n(Z/kZ); Z/kZ), Z/kZ),$$

and it follows that a) is equivalent to c).

□

The above condition on homology is often convenient but, in general, it is more convenient to focus on cohomology, for example

COROLLARY 13.2. *Suppose*

$$f,\, g : X \to \bigvee_{\alpha} P^{n_\alpha}(Z/kZ)$$

are any maps into a finite bouquet. If $f^* = g^*$ in mod k cohomology, then the compositions below are homotopic,

$$\iota \circ f \simeq \iota \circ g : P^m(Z/kZ) \to \bigvee_{\alpha} P^{n_\alpha}(Z/kZ) \xrightarrow{\iota} K = \Pi_\alpha K(Z/kZ, n_\alpha - 1).$$

In other words, the geometric Hurewicz map is a faithful representation in homotopy of mod k cohomology.

Let p be a prime and recall the decompositions of the smash products

$$S_1 = P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \simeq P^{n+m-1}(Z/p^rZ) \vee P^{n+m}(Z/p^rZ)$$

valid for $n, m \geq 2$ and $p^r \neq 2$. We can also smash three of the Peterson spaces and get

$$S_2 = P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \wedge P^q(Z/p^rZ) \simeq$$

$$P^{n+m+q-2}(Z/p^rZ) \vee P^{n+m+q-1}(Z/p^rZ) \vee P^{n+m+q-1}(Z/p^rZ) \vee P^{n+m+q}(Z/p^rZ)$$

valid for $n, m, q \geq 2$ and $p^r \neq 2$.

The main goal of this section is to determine exactly how much cohomology with coefficients mod p^r determines these decompositions when p is an odd prime.

The result we want is the following:

THEOREM 13.3. *Suppose p is a prime greater than 3. Let $S_1 \to K(\overline{H}_*(S_1 : Z))$ and $S_2 \to K(\overline{H}_*(S_2 : Z))$ be geometric Hurewicz maps. Let $f, g : X \to S_1$ and $s, t : X \to S_2$ be maps with $f^* = g^*$ and $s^* = t^*$ in mod p^r cohomology. If $dim X = \ell \leq n + m =$ dimension of S_1, then $f = g + w$ where w is a sum of compositions with Whitehead products. And, if $dim X = \ell \leq n+m+q =$ dimension of S_2, then $s = t + v$ where v is a sum of compositions with Whitehead products.*

If $p = 3$, the term w is as before but, if $\ell = n + m + q$ the term v may also include a summand of order 3.

REMARK 13.4. If $m + n > 5$, there are no nontrivial Whitehead products in the relevant range, so that $f = g$ above. If $m = n + q > 7$, there are no nontrivial Whitehead products in the relevant range, so that $s = t$ above if $p > 3$. On the other hand, if $p = 3$, the element of order 3 in dimension $n + m + q$ is a nontrivial stable element and will always be there. [**20**]

Before we prove this result, we will review Samelson and Whitehead products.

DEFINITION 13.5. If $f : X \to G$ and $g : Y \to G$ are mappings into a group-like space, the (external) Samelson product is the map, unique up to homotopy,

$[f,g] = \overline{[\ \ ,\ \ ]} \circ (f \wedge g) : X \wedge Y \to G$ which factors the group commutator

$$\begin{array}{ccccc} X \times Y & \xrightarrow{f \times g} & G \times G & \xrightarrow{[\ \ ,\ \ ],\quad (x,y) \mapsto xyx^{-1}y^{-1}} & G \\ \downarrow & & \downarrow & & \downarrow 1 \\ X \wedge Y & \xrightarrow{f \wedge g} & G \wedge G & \xrightarrow{\overline{[\ \ ,\ \ ]}} & G \end{array}$$

The existence and uniqueness of the Samelson product follows from the cofibration sequence

$$X \vee Y \to X \times Y \to X \wedge Y \to \Sigma X \vee \Sigma Y \to \Sigma(X \times Y)$$

and the fact that $\Sigma X \vee \Sigma Y \to \Sigma(X \times Y)$ admits a retraction.

Definition 13.6. Let $F : \Sigma X \to Z$ and $G : \Sigma X \to Z$ be maps with respective adjoints $f : X \to \Omega Z$ and $g : X \to \Omega Z$. The (external) Whitehead product $[F, G] : \Sigma X \wedge Y \to Z$ is the adjoint of the Samelson product $[f, g] : X \wedge Y \to \Omega Z$.

The following lemma is immediate:

Lemma 13.7. *If G is a homotopy commutative group-like space, then any Samelson product $[f, g] : X \wedge Y \to G$ is null homotopic. If Z is an H-space, then any Whitehead product $\Sigma X \wedge Y \to Z$ is null homotopic.*

The second half of the above lemma follows from the fact that the loops on an H-space is homotopy commutative.

Definition 13.8. For $p^r \neq 2$, the inclusion into the smash decomposition,

$$\Delta = \Delta_{n,m} : P^{m+n}(Z/p^rZ) \to P^{m+n}(Z/p^rZ) \vee P^{m+n-1}(Z/p^rZ) \simeq P^n(Z/p^rZ) \wedge P^m(Z/p^rZ)$$

is called the coproduct.

Remark 13.9. If p is an odd prime, the two inclusions

$$\Delta_{n,m} : P^{m+n}(Z/p^rZ) \to P^n(Z/p^rZ) \wedge P^m(Z/p^rZ)$$

and

$$\delta_{n,m} : P^{m+n-1}(Z/p^rZ) \to P^n(Z/p^rZ) \wedge P^m(Z/p^rZ)$$

induce in mod p^r homology the maps on generators

$$\Delta_{n,m}(e_{m+n}) = e_n \otimes e_m, \quad \Delta_{n,m}(\beta e_{m+n}) = \beta e_n \otimes e_m + (-1)^n e_n \otimes \beta e_m,$$

$$\delta_{n,m}(e_{m+n-1}) = (-1)^{n+1} \beta e_n \otimes e_m + e_n \otimes \beta e_m, \quad \delta_{n,m}(\beta e_{m+n}) = 2\beta e_n \otimes \beta e_m.$$

We would like these maps to be characterized by these mod p^r homology images, at least up to compositions with Whitehead products. This would follow from the unproved first theorem in this section.

This coproduct allows us to define internal Samelson products and internal Whitehead products:

Definition 13.10. If $f : P^n(Z/p^rZ) \to G$ and $g : P^m(Z/p^rZ) \to G$ are maps into a group-like space, the (internal) Samelson product $[f, g]$ is the composition

$$[f, g] \circ \Delta_{n,m} : P^{m+n}(Z/p^rZ) \to P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \to G.$$

Definition 13.11. If $F : P^{n+1}(Z/p^rZ) \to Z$ and $G : P^{m+1}(Z/p^rZ) \to Z$ are maps into a space, the (internal) Whitehead product $[F, G]$ is the composition

$$[F, G] \circ \Sigma\Delta_{n,m} : P^{m+n+1}(Z/p^rZ) \to \Sigma P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \to Z.$$

Thus, the internal Samelson products and the internal Whitehead products are adjoints of one another. In both cases, we will abuse notation and use the same notation for all these products, Samelson and Whitehead, internal and external. After all, the internal products are merely the restrictions of the external products to bouquet summands of the domain.

REMARK 13.12. Note that, if the coproduct map is characterized up to composition with Whitehead products by its effect in mod p^r homology, then these internal Samelson products and internal Whitehead products are well defined since Whitehead products vanish into an H-space.

The internal Samelson products provide a candidate for a Lie algebra structure in the homotopy groups of a group-like space G when p is a an odd prime. As yet we have proved no Lie identities for this structure or even that it is well defined. But the Hurewicz map provides a representation $\phi : \pi_\ell(G; Z/p^rZ) \to H_\ell(G; Z/p^rZ)$ which is consistent with the Lie algebra structure in the Pontrjagin ring.

LEMMA 13.13. *If $f : P^n(Z/p^rZ) \to G$ and $g : P^m(Z/p^rZ) \to G$ are homotopy classes with internal Samelson product $[f, g] : P^{n+m}(Z/p^rZ) \to G$, then the Hurewicz map is a homomorphism of Lie algebras in the sense that $\phi[f, g] = [\phi f, \phi g]$.*

PROOF. Since the coproduct $\Delta_{n,m} : P^{n+m}(Z/p^rZ) \to P^n(Z/p^rZ) \wedge P^m(Z/p^rZ)$ is $\Delta_*(e_{m+n}) = e_n \otimes e_m$ in mod p^r homology, it suffices to check where $e_n \times e_m$ is sent by the commutator map

$$P^n(Z/p^rZ)\times P^m(Z/p^rZ) \xrightarrow{\Delta\times\Delta} P^n(Z/p^rZ)\times P^n(Z/p^rZ)\times P^n(Z/p^rZ)\times P^n(Z/p^rZ)$$
$$\xrightarrow{f\times f\times g\times g} G \times G \times G \times G \xrightarrow{1\times T\times 1} G \times G \times G \times G$$
$$\xrightarrow{1\times 1\times\iota\times\iota} G \times G \times G \times G \xrightarrow{mult} G$$

where T is the twist map and ι is the inverse.

But

$$e_n \otimes e_m \mapsto (e_n \otimes 1 + 1 \otimes e_n) \otimes (e_m \otimes 1 + 1 \otimes e_m) =$$
$$e_n \otimes 1 \otimes e_m \otimes 1 + e_n \otimes 1 \otimes 1 \otimes e_m + 1 \otimes e_n \otimes e_m \otimes 1 + 1 \otimes e_n \otimes 1 \otimes e_m \mapsto$$
$$f_*e_n\otimes 1\otimes g_*e_m\otimes 1+f_*e_n\otimes 1\otimes 1\otimes g_*e_m+1\otimes f_*e_n\otimes g_*e_m\otimes 1+1\otimes f_*e_n\otimes 1\otimes g_*e_m \mapsto$$
$$f_*e_n\otimes g_*e_m\otimes 1\otimes 1+f_*e_n\otimes 1\otimes 1\otimes g_*e_m+(-1)^{nm}1\otimes g_*e_m\otimes f_*e_n\otimes 1+1\otimes 1\otimes f_*e_n\otimes g_*e_m \mapsto$$
$$f_*e_n\otimes g_*e_m\otimes 1\otimes 1-f_*e_n\otimes 1\otimes 1\otimes g_*e_m-(-1)^{nm}1\otimes g_*e_m\otimes f_*e_n\otimes 1+1\otimes 1\otimes f_*e_n\otimes g_*e_m \mapsto$$
$$(f_*e_n)(g_*e_m) - (f_*e_n)(g_*e_m) - (-1)^{nm}(g_*e_m)(f_*e_n) + (f_*e_n)(g_*e_m) =$$
$$(f_*e_n)(g_*e_m) - (-1)^{nm}(g_*e_m)(f_*e_n) = \phi(f)\phi(g) - (-1)^{nm}\phi(g)\phi(f) =$$
$$[\phi(f), \phi(g)]$$

since the inverse $\iota(x) = -x$ on primitive elements x.

□

Let $F_n \to P^n(Z/p^rZ) \to K(Z/p^rZ, n-1)$ be the fibration sequence of the geometric Hurewicz map with $n \geq 3$. For the remainder of this section, unless specified otherwise, the coefficients in homology will be mod p coefficients.

With coefficients mod p, $H_*(P^n(Z/p^rZ)) =< 1, \beta^r e, e >$ where the degree of e is n and β^r denotes the $r-$th homology Bockstein. Recall

$$H_*(\Omega P^n(Z/p^rZ)) = T(u, v) = UL(u, v)$$

where the degree of v is $n-1$, the degree of $u = \beta^r v$ is $n-2$ and $UL(u, v)$ denotes the universal enveloping algebra on the free Lie algebra $L(u, v)$.

A basic result of [**5**] is the following:

THEOREM 13.14. *Let* $0 \to A \xrightarrow{\iota} B \xrightarrow{j} C \to 0$ *be a short exact sequence of graded Lie algebras over a field in which 2 is a unit. Then the sequence of universal enveloping algebras*

$$UA \to UB \to UC$$

is a short exact sequence of Hopf algebras, in particular, a choice of a lift $s : UC \to UB$ *defines an isomorphism of left* UA *modules*

$$UA \otimes UC \xrightarrow{\iota \otimes s} UB \otimes UB \xrightarrow{mult} UB.$$

Recall that the mod p homology of Eilenberg-MacLane spaces is given by divided power algebras

$$H_*(K(Z/p^r Z, 1)) = \Gamma(u, v)$$

with degree of v equal to 2 and $u = \beta^r v$ has degree 1. (If the degree of a generator is odd, then the divided power algebra on that generator is just the exterior algebra.) If $n \geq 4$,

$$H_*(K(Z/p^r Z, n-2)) = \Gamma(u, v) \otimes \Gamma(x) \otimes \dots$$

where the degree of v is $n-1$, $u = \beta^r v$ has degree $n-2$, and x is the element of degree $n+2p-4$ which represents the dual of the first homology Steenrod operation in the sense that $P_1 x = u$ where P_1 lowers degree by $2p-2$.

We note that the graded commutative algebra $S(u, v) = U < u, v >$ is the universal enveloping algebra of the abelian Lie algebra $< u, v >$ and there is a multiplicative map

$$S(u, v) \to \Gamma(u, v)$$

which is an isomorphism in dimensions $\leq p \; deg(v)$ if the degree of v is even and an isomorphism in dimensions $\leq p \; deg(\beta^r v))$ is the degree of v is odd. In the first case, the divided power $\gamma_p(v)$ is not in the image and the power v^p is in the kernel. In the second case, the divided power $\gamma_p(u)$ is not in the image and the power u^p is in the kernel.

Let L be the kernel of the map of graded Lie algebras $L(u, v) \to < u, v >$. It follows from the above quoted theorem that there is an isomorphism of left UL modules

$$UL \otimes S(u, v) \to T(u, v).$$

Since the Lie elements are all the images of Samelson products and since Eilenberg-Maclane spaces are homotopy commutative, the Lie elements all map into the fibre ΩF_n.

Clearly, the composition $UL \to H_*(\Omega F_n) \to H_*(\Omega P^n(Z/p^r Z))$ is a monomorphism.

In the Serre spectral sequence of the loop fibration sequence

$$\Omega F_n \to \Omega P^n(Z/p^r Z) \to K(Z/p^r Z, n-2),$$

the elements $u = \beta^r v, v$ are infinite cycles. It is easy to see that

THEOREM 13.15.

$$H_*(\Omega F_3) = UL \oplus < \tau\gamma_p(v) >$$

in dimensions $\leq 2p-1$ *where* τ *is the transgression. The element* $\tau\gamma_p(v)$ *has degree* $2p-1$.

This uses the fact that the elements $\gamma_p(v)$ are primitive and must transgress. The same argument applies to the elements x below.

THEOREM 13.16. *If $n \geq 4$*

$$H_*(\Omega F_n) = UL \oplus < \tau x >$$

in dimensions $\leq n+2p-5$ where τ is the transgression. The element τx has degree $n+2p-5$.

In [**5**] the following two results are proved:

THEOREM 13.17. *Over a field in which 2 is a unit, subalgebras of free graded Lie algebras are free.*

THEOREM 13.18. *Suppose $deg(u) = deg(v) - 1$. The kernel L of abelianization $L(u, v) \to < u, v >$ is the free graded Lie algebra*

$$L = L(ad^k(v)[v, u],\ ad^k(v)[u, u])_{k \geq 0}$$

if the degree of v is even, and

$$L = L(ad^k(u)[v, v],\ ad^k(u)[u, v])_{k \geq 0}$$

if the degree of v is odd.

When $p = 3$, $2p-1 = 5$, and the only Lie generators in $L \subseteq H_*(\Omega F_3)$ of degree ≤ 5 are $[u, u]$, $[v, u]$, $[v, [u, u]]$ $[v, [v, u]]$. Since β^r is a derivation on Lie brackets and we have anticommutativity and the Jacobi identity, we compute

$$\beta^r v = u, \quad \beta^r u = 0, \quad \beta^r [v, u] = [u, u],$$

$$\beta^r [v[u, u] = [u, [u, u]] = 0,$$

$$[v, [u, u]] = [[v, u], u] + [u, [v, u]] = 2[u, [v, u]],$$

$$\beta^r [v, [v, u]] = [u, [v, u]] + [v, [u, u]] = 3[u, [v, u]] = \frac{3}{2}[v, [u, u]].$$

Hence, no matter what the odd prime,

$$U < [u, u], [v, u] > \to H_*(\Omega F_3)$$

is an isomorphism in dimensions ≤ 3, $< [u, u], [v, u] >= K = L$ mod Lie brackets of length ≥ 3..

Recall that, if $f : X \to \Omega Z$ and $g : Y \to \Omega Z$ are maps with respective adjoints $F : \Sigma X \to Z$ and $G : \Sigma X \to Z$, the multiplicative extension of the Samelson product $[f, g] : X \wedge Y \to \Omega Z$ is the loop map $\overline{[f, g]} : \Omega\Sigma X \wedge Y \to \Omega\Sigma\Omega Z \to \Omega Z$. As the name indicates, it extends the map on $X \wedge Y$. This map is the loop of the corresponding Whitehead product $\Omega[F, G] : \Omega\Sigma X \wedge Y \to \Omega Z$.

Let $\mu : S^1 \to \Omega P^3(Z/p^r Z)$ and $\nu : P^2(Z/p^r Z) \to \Omega P^3(Z/p^r Z)$ be the elements which are adjoint to the inclusions $\iota_2 : S^2 \to P^3(Z/p^r Z)$ and the identity $1_3 : P^3(Z/p^r Z) \to P^3(Z/p^r Z)$. The mod p Hurewicz image of the internal Samelson product $[\nu, \mu] : P^3(Z/p^r Z) \to \Omega P^3(Z/p^r Z)$ is the Lie commutator $[v, u]$,

$$\phi([\nu, \mu]) = [\nu, \mu]_* e_2 = [v, u].$$

Since the Hurewicz map commutes with the Bockstein differential, the element $[u, u]$ is also in the image of mod p homology via

$$[\nu, \mu]_*(\beta^r e_2) = \beta^r [\nu, \mu]_* e_2 = \beta^r [v, u] = [u, u].$$

The homology map of this Peterson space has an image of rank 2. The multiplicative extension $\overline{[\nu,\mu]} : \Omega\Sigma P^3(Z/p^rZ) \to \Omega P^3(Z/p^rZ)$ gives the homology map

$$H_*(\Omega\Sigma P^3(Z/p^rZ)) = UL([u,u],[v,u]) \to H_*(\Omega F_3).$$

The internal Whitehead product $[1_3, \iota_2] : P^4(Z/p^rZ) \to P^3(Z/p^rZ)$ vanishes when projected to the Eilenberg-MacLane space and thus factors through the fibre F_3. Now it loops to give the map $\Omega P^4(Z/p^rZ) \to \Omega F_3$ which is the same map as the multiplicative extension of the internal Samelson product and which in mod p homology is the map

$$H_*(\Omega P^4(Z/p^rZ)) = UL([u,u],[v,u]) \to H_*(\Omega F_3).$$

This is a mod p homology isomorphism in dimensions ≤ 3. Since these spaces are localized at p, it is a weak equivalence in the sense that it is a homotopy isomorphism in dimensions ≤ 2 and a homotopy epimorphism in dimension 3. Hence,

Lemma 13.19. *a) If p is an odd prime, the Whitehead product $[1_3, \iota_2]$ factors as*

$$P^4(Z/p^rZ) \to F_3 \to P^3(Z/p^rZ)$$

where the first map is a homotopy isomorphism in degrees ≤ 3 and a homotopy epimorphism in degree 4.

Corollary 13.20. *If p is an odd prime, then any multiple $p^s : P^3(Z/p^rZ) \to P^3(Z/p^rZ)$ is homotopic to the fake multiple $\overline{p}^s$.*

Corollary 13.21. *If p is an odd prime and $s \leq r$, then the composition*

$$P^3(Z/p^rZ) \xrightarrow{\overline{\eta}} P^3(Z/p^sZ) \xrightarrow{\overline{\rho}} P^3(Z/p^rZ)$$

is homotopic to the multiple p^{r-s} plus a possible sum of compositions with Whitehead products. Similarly, the composition

$$P^3(Z/p^sZ) \xrightarrow{\overline{\rho}} P^3(Z/p^sZ) \xrightarrow{\overline{\eta}} P^3(Z/p^rZ)$$

is homotopic to the multiple p^{r-s} plus a possible sum of compositions with Whitehead products.

Remark 13.22. If the dimension is ≥ 4 in the two corollaries above, then coefficient functoriality implies that the indicated maps are homotopic without the need of sums of compositions with Whitehead products.

Let

$$P = \begin{cases} P^{2n-1}(Z/p^rZ) & \text{if } n \text{ is even} \\ P^{2n-2}(Z/p^rZ) & \text{if } n \text{ is odd.} \end{cases}$$

Note that P is the domain of the Whitehead product $[1_n, 1_n]$ if n is even, and it is the domain of the Whitehead product $[1_n, \iota_{n-1}]$ if n is odd, where $1_n : P^n(Z/p^rZ) \to P^n(Z/p^rZ)$ is the identity and $\iota_{n-1} : S^{n-1} \to P^n(Z/p^rZ)$ is the inclusion.

Theorem 13.23. *If $p \geq 5$ and $n \geq 4$, the respective Whitehead product factors as $P \to F_n \to P^n(Z/p^rZ)$ where the $P \to F_n$ is an $n+3$ equivalence, that is, a homotopy isomorphism in dimensions $\leq n+4$ and a homotopy epimorphism in dimension $n+5$.*

Remark 13.24. If $p \geq 5$ and $n \geq 9$, then P is $n+5$ connected and hence $* \to F_n$ is an $n+5$ equivalence.

The situation with $p = 3$ is slightly more complicated. Denote by α_1 the first nonzero 3-primary homotopy class $\alpha_1 : S^{n+2} \to S^{n-1} \subset P^n(Z/3^rZ)$. It has order 3.

THEOREM 13.25. *If $p = 3$ and $n \geq 4$, the respective Whitehead product factors as $P \to F_n \to P^n(Z/3^rZ)$ and α_1 factors as $S^{n+2} \to F_n \to P^n(Z/3^rZ)$ which gives an $n+2$ equivalence $P \vee S^{n+2} \to F_n$.*

REMARK 13.26. Just as before, if $p = 3$ and $n \geq 6$, we may omit P and there is an $n+2$ equivalence $S^{n+2} \to F_n$.

PROOF. Assume $p \geq 5$ and $n \geq 4$.
Since
$$H(\Omega F_n) = UL \oplus < \tau x >$$
in dimensions $\leq n+2p-5$, $H(\Omega F_n) = UL$ in dimensions $\leq n+2p-6$. The minimum value of $2p-6$ is 4.

When $K = L$ modulo products of length ≤ 3, $H(\Omega F_n) = UK$ in dimensions $\leq min(3n-6, n+4)$. The minimum value is $n+2$. Hence, $H(\Omega F_n) = UK$ in dimensions $\leq n+2$.

Thus, $\Omega P \to \Omega F_n$ is an $n+2$ equivalence and $P \to F_n$ is an $n+3$ equivalence.

Assume $p = 3$ and $n \geq 4$. Then
$$H(\Omega F_n) = UL \oplus < \tau x >= UK \oplus < \tau x >$$
in dimensions $\leq n+1$.

Recall that the transgression τx detects α_1
$$\begin{cases} S^{n+1} \to \Omega S^{n-1}, & n \text{ even} \\ S^{n+1} \to S^{n-2}, & n \text{ odd} \end{cases}$$
via one of the fibration sequences
$$\begin{cases} F \to \Omega S^{n-1} \to K(Z, n-2), & n \text{ even} \\ F \to S^{n-2} \to K(Z, n-2), & n \text{ odd} \end{cases}$$

It follows from naturality that τx is detecting the factorization of $\alpha_1 : S^{n+1} \to \Omega F_n \to \Omega P^n(Z/3^rZ)$.

Hence, $\Omega(P \vee S^{n+2}) \to \Omega F_n$ is an $n+1$ equivalence and $P \vee S^{n+2} \to F_n$ is an $n+2$ equivalence. □

We are now in a position to use the Hilton-Milnor theorem to prove the main result of this section.

Recall
$$S_1 = P^{n+m-1}(Z/p^rZ) \vee P^{n+m}(Z/p^rZ) = P_1 \vee P_2$$
valid for $n, m \geq 2$ and the geometric Hurewicz map
$$S_1 \to K = K_1 \times K_2 = K(Z/p^rZ, n+m-1) \times K(Z/p^rZ, n+m-2).$$
Recall also
$$S_2 = P^{n+m+q-2}(Z/p^rZ) \vee P^{n+m+q-1}(Z/p^rZ) \vee P^{n+m+q-1}(Z/p^rZ) \vee P^{n+m+q}(Z/p^rZ) = P_1 \vee P_2 \vee P_3 \vee P_4$$
and the geometric Hurewicz map
$$S_2 \to K_1 \times K_2 \times K_3 \times K_4 =$$

$$K(Z/p^rZ, n+m+q-1) \times K(Z/p^rZ, n+m+q-2)\times$$
$$K(Z/p^rZ, n+m+q-2) \times K(Z/p^rZ, n+m+q-3).$$

In all the above cases, let $F_i \to P_i \to K_i$ denote the fibration sequence.

THEOREM 13.27. *Suppose p is a prime greater than 3. Let $h : X \to S_1$ and $j : X \to S_2$ be maps with $h^* = 0$ and $j^* = 0$ in mod p^r cohomology. If $dimX = \ell \leq m+n =$ dimension of S_1, then $h = w$ where w is a sum of compositions with Whitehead products. And, if $dimX = \ell \leq n+m+q =$ dimension of S_2, then $j = v$ where v is a sum of compositions with Whitehead products.*

If $p = 3$, the term w is as before but, if $\ell = n+m+q$ the term v may also include a summand of order 3.

PROOF. One form of the Hilton-Milnor theorem asserts that there is a fibration sequence

$$\bigvee_{k \geq 0} \Sigma Y^{\wedge k} \wedge Z \to \Sigma(Y \vee Z) \to \Sigma Y$$

where the fibre is mapped in by a bouquet of the Whitehead products $ad^k(1_{\Sigma Y})(1_{\Sigma Z})$.

Repeating this gives a fibration sequence

$$\bigvee_{\ell \geq 0, k > 0} \Sigma Z^{\wedge \ell} \wedge Y^{\wedge k} \wedge Z \to \Sigma(Y \vee Z) \to \Sigma Y \times \Sigma Z$$

where the fibre is mapped in by a bouquet of Whitehead products.

Suppose that $h : X \to S_1$ is such that $h^* = 0$ in mod p^r cohomology.

Consider the composition

$$X \xrightarrow{h} P_1 \vee P_2 \xrightarrow{\iota_1 \vee \iota_2} P_1 \times P_2 \xrightarrow{j_1 \times j_2} K_1 \times K_2.$$

Since $h^* = 0$, it follows that the composition

$$(j_1 \times j_2) \circ (\iota_1 \vee \iota_2) \circ h \simeq 0$$

and hence $(\iota_1 \vee \iota_2) \circ h$ factors through the fibre $F_1 \times F_2$. Thus,

$$(\iota_1 \vee \iota_2) \circ h = (y_1, y_2)$$

where y_1 and y_2 are both sums of compositions with Whitehead products.

Hence,

$$(\iota_1 \vee \iota_2) \circ h - \iota_1 \circ y_1 - \iota_2 \circ y_2 = 0$$

and $h - \iota_1 \circ y_1 - \iota_2 \circ y_2$ factors through the fibre of $P_1 \vee P_2 \to P_1 \times P_2$. By the Hilton-Milnor theorem,

$$h - \iota_1 \circ y_1 - \iota_2 \circ y_2 = z$$

where $z =$ a sum of compositions with Whitehead products, that is, $h = \iota_1 \circ y_1 + \iota_2 \circ y_2 + z$ is a a sum of compositions with Whitehead products.

The case of $j^* = 0$ is similar. The iteration of the Hilton-Milnor theorem becomes more complicated but the fibre of the 4-fold bouquet into the 4-fold product is still a bouquet of Whitehead products.

The only extra fact required is that, if $p = 3$ and $dimX = \ell = m+n+q$, then any map $P^{m+n+q}(Z/3^rZ) \to S^{m+n+q}$ is a suspension and thus composition with

$$\alpha_1 : S^{m+n+q} \to S^{m+n+q-3} \subseteq P^{m+n+q_2}(Z/3^rZ)$$

has order 3.

□

14. Samelson products and Lie identities in groups

Let G be a group-like space and let $X, Y,$ and Z be pointed spaces. The homotopy sets $[X, G]$ are groups and there are certain natural subgroups related to the definition of Samelson products, that is,

LEMMA 14.1. *The natural map* $j^* : [X \wedge Y, G]_* \to [X \times Y, G]$ *is a monomorphism and the domain can be identified via this monomorphism with the subgroup* $K \subseteq [X \times Y, G]_*$ *where*

$$K = \mathit{ker}\, [X \times Y, G] \to [X \vee Y, G]_*$$

is the kernel of restriction to the wedge.

Similarly, the natural map $j^* : [X \wedge Y \wedge Z, G]_* \to [X \times Y \times Z, G]$ *is a monomorphism and the domain can be identified via this monomorphism with the subgroup* $K' \subseteq [X \times Y \times, G]_*$ *where*

$$K' = \mathit{ker}\, [X \times Y \times Z, G] \to [(X \times Y \times *) \cup (X \times * \times Z) \cup (* \times Y \times Z), G]_*$$

is the kernel of restriction to the fat wedge.

PROOF. The first statement follows from the cofibration sequence

$$X \vee Y \to X \times Y \to X \wedge Y \to \Sigma X \vee \Sigma Y \to \Sigma(X \times Y)$$

and the fact that $\Sigma X \vee \Sigma Y$ is a retract of

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

The second statement follows from the cofibration sequence

$$(X \times Y \times *) \cup (X \times * \times Z) \cup (* \times Y \times Z) \to X \times Y \times Z \to X \wedge Y \wedge Z \to$$
$$\Sigma((X \times Y \times *) \cup (X \times * \times Z) \cup (* \times Y \times Z)) \to \Sigma(X \times Y \times Z)$$

and the fact that

$$\Sigma((X \times Y \times *) \cup (X \times * \times Z) \cup (* \times Y \times Z))$$

is a retract of $\Sigma(X \times Y \times Z)$.

□

DEFINITION 14.2. If $f : X \to G$ and $g : Y \to G$ are two maps, their Samelson product

$$[X, G]_* \times [Y, G]_* \to [X \wedge Y, G]_* \subseteq [X \times Y, G]_*, \quad (f, g) \mapsto [f, g]$$

has the following two equivalent descriptions in terms of the commutator $c(f, g) = [f, g] : c(f, g)(x, y) = [f(x), g(y)] = f(x)g(y)f(x)^{-1}g(y)^{-1}, \quad (x, y) \in X \times Y.$

a) $[f, g] : X \wedge Y \to G$ is the element, unique up to homotopy, such that

$$c(f, g) = [f, g] \circ j : X \times Y \to X \wedge Y \to G.$$

b) $c(f, g) = [f, g] : X \times Y \to G$ regarded as lying in the subgroup $\mathit{ker}\, [X \times Y, G]_* \to [X \vee Y, G]_*$.

Note that the above is a commutator of length 2 and in two variables x, y.

Similarly, if $f : X \to G$, $g : Y \to G$, and $h : Z \to G$ are three maps, their Samelson product

$$[X, G]_* \times [Y, G]_* \times [Z.G]_* \to [X \wedge Y \wedge Z, G]_* \subseteq [X \times Y \times Z, G]_*, \quad (f, g, h) \mapsto [f, [g, h]]$$

has the following two equivalent descriptions in terms of the commutator $c'(f,g,h) = [f,[g,h]] : c'(f,g,h)(x,y,z) = [f(x),[g(y),h(z)]], \quad (x,y,z) \in X \times Y \times Z$.

a) $[f,[g,h]] : X \wedge Y \wedge Z \to G$ is the element, unique up to homotopy, such that

$$c'(f,g,h) = [f,[g,h]] \circ j : X \times Y \times Z \to X \wedge Y \wedge Z \to G.$$

b) $c'(f,g,h) = [f,[g,h]] : X \times Y \times Z \to G$ regarded as lying in the subgroup

$$ker\,[X \times Y \times Z, G] \to [(X \times Y \times *) \cup (X \times * \times Z) \cup (* \times Y \times Z), G]_*$$

This is a commutator of length 3 in three variables x, y, z.

Remark 14.3. The Samelson products $[f,g] : X \wedge Y \to G$ are sometimes called externel Samelson products in order to distinguish them from the internal Samelson products in the next section.

Definition 14.4. A pointed space X is coabelian if the reduced diagonal map $\overline{\Delta} : X \xrightarrow{\Delta} X \times X \xrightarrow{j} X \wedge X$ is null homotopic.

Remark 14.5. The reduced diagonal should not be confused with the comultiplication $\Delta = \Delta_{n,m} : P^{n+m}(Z/p^rZ) \to P^n(Z/p^rZ) \wedge P^m(Z/p^rZ)$.

Remark 14.6. Examples of coabelian spaces are: suspensions, co-H-spaces, $P^2(Z/kZ)$ with $k = p^r \neq 2$, a power of a prime. The last example is not a co-H-space since:

A factorization $X \to X \vee X \to X \times X$ of the diagonal implies a factorization of the fundamental groups. In the above case, this would be a factorization

$$Z/kZ \to (Z/kZ) * (Z/kZ) \to (Z/kZ) \times (Z/kZ).$$

Since the only elements of finite order in the free product lie in conjugates of the two Z/kZ factors [**15**], this is impossible.

The next result is a generalization of a result on homotopy groups with coefficients, $\pi_2(\Omega W; Z/kZ)$ is an abelian group if $k = p^r \neq 2$.

Theorem 14.7. *G grouplike, X coabelian implies $[X,G]_*$ is an abelian group.*

Proof. If $f; X \to G$ and $g : X \to G$ are maps, then the commutator

$$[f,g] : X \xrightarrow{\overline{\Delta}} X \wedge X \xrightarrow{f \wedge g} G \wedge G \xrightarrow{\overline{[\ ,\ ]}} G, \quad x \mapsto [f(x), g(x)]$$

is null homotopic. □

It is worthwhile reflecting on the above proof. We showed that a length 2 commutator $[f(x), g(x)]$ in one variable vanishes in the group $[X,G]_*$ when X is coabelian. We call a commutator $c(f_1, f_2, \ldots, f_n)$ a simple commutator if each of the $f_i = f_i(x_i)$ is a function of one variable. We call it a commutator in k variables if the number of distinct x_i is k. We allow the possibility that a commutator may have some inverses in it. For example, $[a,b]^{-1}, [a^{-1},b]$, $[a,[b^{-1},c]]$, $[[a,b]^{-1},[c,d]]$ are commutators of respective lengths 2,2,3, and 4.

The same proof as above shows that

Theorem 14.8. *If $X_1, X_2, \ldots, X_n$ are coabelian spaces, then a simple commutator of length $> n$ in n variables vanishes in the group $[X_1 \wedge X_2 \wedge \cdots \wedge X_n, G]_*$.*

PROOF. The simple commutator of length $k > n$ is defined by a composition

$$X_1 \wedge X_2 \wedge \cdots \wedge X_n \to Y_1 \wedge Y_2 \wedge \cdots \wedge Y_k \xrightarrow{f_1 \wedge f_2 \wedge \cdots \wedge f_k} G \wedge G \wedge \cdots \wedge G \xrightarrow{c} G.$$

The Y_i are chosen from among the X_i. Since $k > n$, there must be at least one repetition of the X_i in the list of Y_i and thus at least one occurence of the reduced diagonal in the map

$$X_1 \wedge X_2 \wedge \cdots \wedge X_n \to Y_1 \wedge Y_2 \wedge \cdots \wedge Y_k.$$

Hence, it and all compositions with it are null homotopic. □

For example, $[f(x), [g(x), h(y)]$ is null homotopic in the group $[X \wedge Y, G]_*$ since the length 3 is greater than the number of variables 2. Similarly, $[f(x), [g(y), [h(y), k(z)]]]$ and $[[f(x), g(y)]^{-1}, [h(y), k(z)]]$ are null homotopic in the group $[X \wedge Y \wedge Z, G]_*$.

Serre's book [**27**] contains a list of the identities of Lie type for the commutators in a group.

Let a, b, c be elements of a group G and define

1) The congugate homomorphisms are $a^b = b^{-1}ab$. Recall that $(ab)^c = a^c b^c$ and $(a^b)^{b^{-1}} = a$.

2) The commutators are $[a, b] = aba^{-1}b^{-1}$. Thus, $[a, b]^c = [a^c, b^c]$.

The Lie identities in groups are the following formulas:

LEMMA 14.9. *For elements a, b, c in a group G,*
1) exponentiation modulo a commutator:

$$a^b = a[a^{-1}, b^{-1}]$$

2) inverse of a commutator:

$$[a, b]^{-1} = [b, a], \qquad [a^{-1}, b] = [b, a]^a$$

3) commutativity modulo commutators:

$$ab = [a, b]ba$$

4) bilinearity modulo commutators:

$$[a, bc] = [a, b]\ [a, c]^{(b^{-1})}, \qquad [ab, c] = [b, c]^{(a^{-1})}\ [a, c]$$

5) Jacobi identity modulo commutators

$$[a^{(b^{-1})}, [c, b]]\ [b^{(c^{-1})}, [a, c]]\ [c^{(a^{-1})}, [b, a]] = 1$$

REMARK 14.10. It is difficult to discover some of the above formulas and it is difficult to remember their exact form, but there can be no doubt that they are straightforward to prove. Apply the procedure of reducing a word in a free group to the identity. That is, write them in the form $c = 1$ and reduce the word c to the identity via successive applications of
1) $wdw^{-1} = 1$ if and only if $d = 1$.
2) $ww^{-1} = 1$.

Let G be a group-like space and let X, Y, Z be coabelian spaces. The multiplication in G, $(g, h) \mapsto gh$ induces an abelian operation in each $[X, G]_*$, written additively as $(f, g) \mapsto x + y \equiv xy$. Let

$$f, f_1 : X \to G,\ g, g_1 : Y \to G,\ h : Z \to G$$

be maps. The next result is a variation of a result of George Whitehead [**30**]. The fact that we have domains which are coabelian transforms the above Lie identities for groups into the standard (ungraded) Lie algebra identities for Samelson products.

THEOREM 14.11. *Each $[X, G]_*$ is an abelian group with $x + y \equiv xy$, $-x \equiv x^{-1}$ and $0 \equiv 1$.*

1)

$$[g, f] = -[f, g] \circ T$$

in $[Y \wedge X, G]_*$ where $T : Y \wedge X \to X \wedge Y, \quad y \wedge x \mapsto x \wedge y$ is the twist.

2)

$$[f + f_1, g] = [f, g] + [f_1, g], \quad [f, g + g_1] = [f, g] + [f, g_1]$$

in $[X \wedge Y, G]_*$.

3)

$$[f, [h, g]] + [g, [f, h]] \circ \sigma + [h, [g, f]] \circ \sigma^2 = 0$$

in $[X \wedge Z \wedge Y, G]$ where

$$\sigma : X \wedge Z \wedge Y \to Y \wedge X \wedge Z, \quad x \wedge z \wedge y \mapsto y \wedge x \wedge z$$

is the cyclic permutation.

PROOF. 1) Since $[f(x), g(y)]^{-1} \simeq [g(y), f(x)]$, the maps $[f, g] \circ T : Y \wedge X \to X \wedge Y \to G$ and $[g, f] : Y \wedge X \to G$ are homotopic.

2) Since

$$[f(x), g(y)g_1(y)] = [f(x), g(y)][f(x), g_1(y)]^{g(y)^{-1}} = [f(x), g(y)][f(x), g_1(y)][[f(x), g(y)]^{-1}, g(y)]$$

and

$$[[f(x), g(y)]^{-1}, g(y)] = 0$$

since it is a simple length 3 commutator in 2 variables,

$$[f, g + g_1] = [f, gg_1] = [f, g][f, g_1] = [f, g] + [f, g_1]$$

in $[X \wedge Y, G]_*$.

By 1), this is sufficient, that is,

$$[f + f_1, g] = [g, f + f_1] \circ T = ([g, f] + [g, f_1]) \circ T =$$
$$[g, f] \circ T + [g, f_1] \circ T = [f, g] + [f_1, g].$$

3) We know that

$$[f(x)^{(g(y)^{-1})}, [h(z), g(y)]]\ [g(y)^{(h(z)^{-1})}, [f(x), h(z)]]\ [h(z)^{(f(x)^{-1})}, [g(y), f(x)]] = 1.$$

We claim that

$$[f(x)^{(g(y)^{-1})}, [h(z), g(y)]] = [f(x), [g(y), h(z)]]$$

and, symmetrically,

$$[g(y)^{(h(z)^{-1})}, [f(x), h(z)]] = [g(y), [f(x), h(z)]], \quad [h(z)^{(f(x)^{-1})}, [g(y), f(x)]] = [h(z), [g(y), f(x)]].$$

Then the first line translates to

$$[f, [h, g]] + [g, [f, h]] \circ \sigma + [h, [g, f]] \circ \sigma^2 = 0.$$

in $[X \wedge Z \wedge Y, G]_*$

But

$$[f(x)^{(g(y)^{-1})}, [h(z), g(y)]] = [f(x)[f(x)^{-1}, g(y)^{-1}], [h(z), g(y)]] =$$

$$[[f(x)^{-1}, g(y)^{-1}], [h(z), g(y)]]^{f(x)^{-1}}[f(x), [h(z), g(y)]] = [f(x), [h(z), g(y)]].$$

We use that $[[f(x)^{-1}, g(y)^{-1}], [h(z), g(y)]]$ and all its conjugates are zero since it is a simple commutator of length 4 in 3 variables.

□

15. Internal Samelson products

Let G be a group-like space and p an odd prime.

In this section, we are going to discuss internal Samelson products

$$[\ ,\ ] : \pi_n(G; Z/p^rZ) \otimes \pi_m(G; Z/p^rZ) \to \pi_{n+m}(G; Z/p^rZ)$$
$$[\ ,\ ] : \pi_n(G; Z) \otimes \pi_m(G; Z/p^rZ) \to \pi_{n+m}(G; Z/p^rZ)$$
$$[\ ,\ ] : \pi_n(G; Z/p^rZ) \otimes \pi_m(G; Z) \to \pi_{n+m}(G; Z/p^rZ)$$
$$[\ ,\ ] : \pi_n(G) \otimes \pi_m(G) \to \pi_{n+m}(G).$$

These constructions are based on external Samelson products. In order to avoid confusion, in this section external Samelson products will be distinguished with a subscript, that is,

$$[f, g]_e : P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \to G$$

when $f : P^n(Z/p^rZ) \to G$ and $f : P^m(Z/p^rZ) \to G$. Similarly, we have external Samelson products

$$[f, g]_e : S^n \wedge P^m(Z/p^rZ) \to G$$
$$[f, g]_e : P^n(Z/p^rZ) \wedge S^m \to G$$
$$[f, g]_e : S^n \wedge S^m \to G$$

The passage from external Samelson products to internal Samelson products is via comultiplications

$$\Delta_{n,m} : P^{n+m}(Z/p^rZ) \to P^n(Z/p^rZ) \wedge P^m(Z/p^rZ)$$
$$\Delta_{n,m} : P^{n+m}(Z/p^rZ) \to S^n \wedge P^m(Z/p^rZ)$$
$$\Delta_{n,m} : P^{n+m}(Z/p^rZ) \to P^n(Z/p^rZ) \wedge S^m$$
$$\Delta_{n,m} : S^{n+m} \to S^n \wedge S^m$$

The last three of these coproducts are just the standard homeomorphisms. All four are characterized by the effect in homology:

$$\Delta_*(e_{m+n}) = e_n \otimes e_m.$$

Even in the lowest dimensions, homology characterizes these maps up to homotopy and up to composition with Whitehead products. Thus, the internal product $[f, g]$ which arises by composition, for example,

$$[f, g] = [f, g]_e \circ \Delta : P^{n+m}(Z/p^rZ) \xrightarrow{\Delta} P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \xrightarrow{[f,g]_e} G$$

has codomain in an $H-$space and is well defined.

We have already checked that the Hurewicz map $\phi : \pi_n(G; Z/p^rZ) \to H_n(G; Z/p^rZ)$ is a map of graded Lie algebras, that is, the Hurewicz map is an additive homomorphism (p is odd!) and it respects the Lie products

$$\phi([f, g]) = [\phi f, \phi g] = (\phi f)(\phi g) - (-1)^{deg(f)deg(g)}(\phi g)(\phi f).$$

When 2 is a unit in the ground ring, the definition of a graded Lie algebra is

DEFINITION 15.1. Let R be a commutative ring in which 2 is a unit. A graded Lie algebra $L = L_*$ over R is a graded $R-$module together with bilinear pairings

$$[\quad , \quad] : L_m \otimes L_n \to L_{m+n}$$

such that

1)
$$[x, y] = -(-1)^{deg(x)deg(y)}[y, x].$$

2) if the degree of x is even, the double vanishing identity is valid:
$$[x, x] = 0.$$

3) if the degree of x is odd, the triple vanishing identity is valid:
$$[x, [x, x]] = 0.$$

4)
$$[x, [y, z]] = [[x, y], z] + (-1)^{deg(x)deg(y)}[y, [x, z]].$$

REMARK 15.2. If 2 is a unit in the ring, it is clear that 1) implies 2) since:

$$[x, x] = -[x, x]$$

when x has even degree.

If 3 is a unit in the ring, then 1) and 4) imply 3) since:

$$[x, [x, x]] = [[x, x], x] - [x, [x, x]] = -2[x[x, x]]$$

when x has odd degree.

Let f, f_1 be chosen from either $\pi_n(G; Z/p^rZ)$, $n \geq 2$, or from $\pi_n(G), n \geq 1$. Similarly, let g, g_1 be chosen from either $\pi_m(G; Z/p^rZ), m \geq 2$, or from $\pi_m(G), m \geq 1$, and let h be chosen from either $\pi_q(G; Z/p^rZ), q \geq 2$, or from $\pi_q(G), q \geq 1$. Then

THEOREM 15.3. *a) Internal Samelson products are bilinear:*

$$[f + f_1, g] = [f, g] + [f_1, g], \quad [f, g + g_1] = [f, g] + [f, g_1].$$

b) Samelson products are anti-commutative:

$$[f, g] = -(-1)^{nm}[g, f].$$

c) If $p \neq 3$ the Jacobi identity is satisfied:

$$[f, [g, h]] = [[f, g], h] + (-1)^{nm}[g, [f, h]].$$

If $p = 3$, the Jacobi identity is valid up to an error of order 3.

PROOF. a) Since

$$[f + f_1, g]_e = [f, g]_e + [f_1, g]_e$$

for external products and since the addition is defined by the multiplication in G, the internal products satisfy

$$[f + f_1, g] = [f + f_1, g]_e \circ \Delta = [f, g]_e \circ \Delta + [f_1, g]_e \circ \Delta = [f, g] + [f_1, g].$$

b) The diagram below commutes in mod p^r homology and therefore it

$$\begin{array}{ccc} P^{n+m}(Z/p^rZ) & \xrightarrow{\Delta} & P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \\ \downarrow (-1)^{nm} & & \downarrow T \\ P^{n+m}(Z/p^rZ) & \xrightarrow{\Delta} & P^m(Z/p^rZ) \wedge P^n(Z/p^rZ) \end{array}$$

commutes up to composition with Whitehead products. The diagrams which correspond to the other choices of coefficients also commute.

Hence,

$$[f,g] = [f,g]_e \circ \Delta = -[g,f]_e \circ T \circ \Delta = -(-1)^{nm}[g,f]_e \circ \Delta = -(-1)^{nm}[g,f].$$

c) Write $\underline{\Delta} = (1 \wedge \Delta) \circ \Delta$. If $p \neq 3$, the diagram below commutes in mod p^r homology and therefore it commutes up to composition with Whitehead products. (Here $P^n = P^n(Z/p^rZ)$):

$$\begin{array}{ccccc} P^{n+m+q} & \xrightarrow{\Delta} & P^n \wedge P^{m+q} & \xrightarrow{1\wedge\Delta} & P^n \wedge P^q \wedge P^m \\ \downarrow 1 & & & & \downarrow 1 \\ P^{n+m+q} & \xrightarrow{\Delta} & P^{n+q} \wedge P^m & \xrightarrow{\Delta\wedge 1} & P^n \wedge P^q \wedge P^m \\ \downarrow 1 & & \downarrow T & & \downarrow \sigma \\ P^{n+m+q} & \xrightarrow{\Delta} & P^m \wedge P^{n+q} & \xrightarrow{1\wedge\Delta} & P^m \wedge P^n \wedge P^q. \end{array}$$

Thus, $\sigma \circ \underline{\Delta} = (-1)^{m(n+q)}\underline{\Delta}$. And

$$\sigma^2 \circ \underline{\Delta} = (-1)^{m(n+q)}\sigma \circ \underline{\Delta} = (-1)^{m(n+q)+q(m+n)}\underline{\Delta} = (-1)^{(m+q)n}\underline{\Delta}.$$

After we apply right composition with $\underline{\Delta}$ to the Jacobi identity for external Samelson products, we get the equation

$$[f,[h,g]_e]_e \circ \underline{\Delta} + [g,[f,h]_e]_e \circ \sigma \circ \underline{\Delta} + [h,[g,f]_e]_e \circ \sigma^2 \circ \underline{\Delta} = 0.$$

This is equivalent to

$$[f,[h,g]_e]_e \circ \underline{\Delta} + (-1)^{n+m)q}[g,[f,h]_e]_e \circ \underline{\Delta} + (-1)^{(m+q)n}[h,[g,f]_e]_e \circ \underline{\Delta} = 0.$$

Since $[f,[h,g]] = [f,[h,g]_e]_e \circ \underline{\Delta}$, we get the equation of internal products

$$[f,[h,g]] + (-1)^{n+m)q}[g,[f,h]] + (-1)^{(m+q)n}[h,[g,f]] = 0.$$

But, in the presence of 2) above, this equation is equivalent to

$$[f,[h,g]] = [[f,h],g] + (-1)^{nq}[h,[f,g]].$$

When $p = 3$, there is an error of order 3 in the initial equations for compositions $\sigma \circ \underline{\Delta}$ and $\sigma^2 \circ \underline{\Delta}$. (See the last theorem in Section 13.) This leads to an error of order 3 in the Jacobi identity. □

REMARK 15.4. Since the diagram below commutes up to composition with Whitehead products

$$\begin{array}{ccccc} P^{n+m}(Z/p^rZ) & \xrightarrow{\overline{\rho}} & P^{n+m}(Z/p^rZ) & \xrightarrow{\overline{\rho}} & S^{n+m} \\ \downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta \\ P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) & \xrightarrow{1\wedge\overline{\rho}} & P^n(Z/p^rZ) \wedge S^m & \xrightarrow{\overline{\rho}\wedge 1} & S^n \wedge S^m \end{array}$$

it follows that, for $f : S^n \to G$, $g : S^m \to G$, and $h : P^n(Z/p^rZ) \to G$,

$$\rho[f,g] = [\rho f, g], \quad \rho[h,g] = [h, \rho g].$$

In the diagram below, consider the three horizontal compositions

$$\begin{array}{ccccccccc}
 & & & \longrightarrow & \xrightarrow{\Delta} & \xrightarrow{\cong} & \longrightarrow & S^{n-1} \wedge P^m(Z/p^r Z) \\
 & \cong\nearrow & & & & & & \downarrow \overline{\beta} \wedge 1 \\
P^{n+m-1}(Z/P^r Z) & \xrightarrow{\overline{\rho}} & S^{n-1} & \xrightarrow{\overline{\beta}} & P^{n+m}(Z/p^r Z) & \xrightarrow{\Delta} & & P^n(Z/p^r Z) \wedge P^m(Z/p^r Z) \\
 & \cong\searrow & & & & & & \uparrow 1 \wedge \overline{\beta} \\
 & & & \longrightarrow & \xrightarrow{\Delta} & \xrightarrow{\cong} & \longrightarrow & P^n(Z/p^r Z) \wedge S^{m-1}.
\end{array}$$

The sum $\overline{\beta} \wedge 1 + (-1)^n 1 \wedge \overline{\beta}$ induces the same map in mod p^r homology as the map $\Delta \circ \overline{\beta} \circ \overline{\rho}$. Hence, they are equal up to composition with Whitehead products. This yields the derivation formula for the Bockstein

THEOREM 15.5. *If $f : P^n(Z/p^r Z) \to G$ and $g : P^m(Z/p^r Z) \to G$ are maps, then*

$$\beta[f, g] = [\beta f, g] + (-1)^n [f, \beta g].$$

DEFINITION 15.6. Let 2 be a unit in the ground ring. A differential graded Lie algebra is a graded Lie algebra L together with a degree -1 linear map $d : L_m \to L_{m-1}$ which is a differential, $d^2 = 0$, and a derivation,

$$d[x, y] = [dx, y] + (-1)^{deg(x)} [x, dy].$$

If we set $\pi_1(G; Z/p^r Z) = \pi_1(G) \otimes Z/p^r Z$, and adopt the convention that the Bockstein on a 1 dimensional class is zero, then we get

THEOREM 15.7. *If p is a prime greater than 3 and G is a group-like space, then the composition $d = \rho \circ \beta : \pi_m(G; Z/p^r Z) \to \pi_{m-1}(G) \to \pi_{m-1}(G; Z/p^r Z)$ makes $\pi_*(G; Z/p^r Z)$, $* \geq 1$ into a differential graded Lie algebra over the ring $Z/p^r Z$.*

REMARK 15.8. If $f : S^1 \to G$ and $g : S^1 \to G$ are one-dimensional classes, $[f, g]$ is defined mod p^r as a composition

$$P^2(Z/p^r Z) \xrightarrow{\overline{\rho}} S^1 \wedge S^1 \xrightarrow{h=[f,h]} G.$$

If $g : P^n(Z/p^r Z) \to G$, then $[f, g]$ is defined mod p^r as a composition

$$P^{n+1}(Z/p^r Z) \xrightarrow{\overline{\rho}} S^1 \wedge P^n(Z/p^r Z) \xrightarrow{h=[f.g]} G.$$

In order to justify the above theorem, we need that, if f is a one-dimensional class, then $[p^r f, g] = 0$ mod p^r. In either case,

$$[p^r f, g] = h \circ (p^r \wedge 1) \circ q = h \circ 0 = 0.$$

REMARK 15.9. If $p = 3$, $\pi_*(G; Z/3^r Z)$ is a differential graded Lie algebra except that the Jacobi identity is valid only up to an error of order 3. But the Jacobi identity is valid if any of the three elements in the identity is the reduction of an integral class, that is, if any of the classes factors as

$$P^n(Z/3^r Z) \to S^n \to G$$

or, if $n = 1$, is represented by $S^1 \to G$.

If $p = 3$, the triple vanishing identity $[x, [x, x]] = 0$ for an odd class x is valid up to an error of order 9. The Jacobi identity implies that, if X is an odd class, then $3[x, [x, x]] = 0$. When $p = 3$, the fact that the Jacobi identity contains an error of order 3 implies that $9[x, [x, x]] = 0$.

If p is an odd prime and $s \leq r$, the diagram below commutes up to compositions with Whitehead products since it commutes in mod p^r cohomology

$$\begin{array}{ccc} P^{n+m}(Z/p^sZ) & \xrightarrow{\Delta} & P^n(Z/p^sZ) \wedge P^m(Z/p^sZ) \\ \downarrow \overline{\rho} & & \downarrow \overline{\rho} \wedge \overline{\rho} \\ P^{n+m}(Z/p^rZ) & \xrightarrow{\Delta} & P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \end{array}$$

In fact, they commute on the chain level:

$$\begin{array}{ccc} e_{n+m} & \xrightarrow{\Delta_*} & e_n \otimes e_m \\ \downarrow \overline{\rho}_* & & \downarrow \overline{\rho}_* \otimes \overline{\rho}_* \\ e_{n+m} & \xrightarrow{\Delta_*} & e_n \otimes e_m \end{array}$$

$$\begin{array}{ccc} e_{n+m-1} & \xrightarrow{\Delta_*} & e_{n-1} \otimes e_m + (-1)^n e_n \otimes e_{m-1} \\ \downarrow \overline{\rho}_* & & \downarrow \overline{\rho}_* \otimes \overline{\rho}_* \\ p^{r-s}e_{n+m-1} & \xrightarrow{\Delta_*} & p^{r-s}e_{n-1} \otimes e_m + (-1)^n e_n \otimes p^{r-s}e_{m-1} \end{array}$$

We can replace a Peterson space by a sphere in the above, that is, if $s \leq r$, the following commutes up to compositions with Whitehead products

$$\begin{array}{ccc} P^{n+m}(Z/p^sZ) & \xrightarrow{\Delta} & P^n(Z/p^sZ) \wedge P^m(Z/p^sZ) \\ \downarrow \overline{\rho} & & \downarrow \overline{\rho} \wedge \overline{\rho} \\ P^{n+m}(Z/p^rZ) & \xrightarrow{\Delta=1} & S^n \wedge P^m(Z/p^rZ) \end{array}$$

Similarly, if p is an odd prime and $s \leq r$, the diagram below commutes up to composition with Whitehead products

$$\begin{array}{ccc} P^{n+m}(Z/p^rZ) & \xrightarrow{\Delta} & P^n(Z/p^rZ) \wedge P^m(Z/p^rZ) \\ \downarrow \overline{\eta} & & \downarrow \overline{\eta} \wedge \overline{\eta} \\ P^{n+m}(Z/p^sZ) & \xrightarrow{\Delta \circ p^{r-s}} & P^n(Z/p^sZ) \wedge P^m(Z/p^sZ) \end{array}$$

Hence,

Theorem 15.10. *If p is an odd prime, $s \leq r$ and G is a group-like space, the following diagrams commute*

$$\begin{array}{ccc} \pi_n(G; Z/p^rZ) \otimes \pi_m(G; Z/p^rZ) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(G; Z/p^rZ) \\ \downarrow \rho \otimes \rho & & \downarrow \rho \\ \pi_n(G; Z/p^sZ) \otimes \pi_m(G; Z/p^sZ) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(G; Z/p^sZ) \end{array}$$

$$\begin{array}{ccc} \pi_n(G) \otimes \pi_m(G; Z/p^rZ) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(G; Z/p^rZ) \\ \downarrow \rho \otimes \rho & & \downarrow \rho \\ \pi_n(G; Z/p^sZ) \otimes \pi_m(G; Z/p^sZ) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(G; Z/p^sZ) \end{array}$$

$$\begin{array}{ccc} \pi_n(G; Z/p^sZ) \otimes \pi_m(G; Z/p^sZ) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(G; Z/p^sZ) \\ \downarrow \eta \otimes \eta & & \downarrow p^{r-s}\eta \\ \pi_n(G; Z/p^rZ) \otimes \pi_m(G; Z/p^rZ) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(G; Z/p^rZ) \end{array}$$

16. Universal models and relative Samelson products

If G is a group-like space, Samelson products $\pi_n(G; Z/p^r Z) \otimes \pi_m(G; Z/p^r Z) \to \pi_{n+m}(G; Z/p^r Z)$ are topological analogs of the commutator maps $[\quad , \quad] : G \times G \to G, \quad (x, y) \mapsto xyx^{-1}y^{-1}$ for groups.

If $H \subseteq G$ is a normal subgroup, then the commutator compresses, that is, $[\quad , \quad] : H \times G \to H \subseteq G$. Let $F \to E \to B$ be a fibration sequence.

The topological analogs of the normal subgroups are the the relative Samelson products

$$\pi_n(\Omega F; Z/p^r Z) \otimes \pi_m(\Omega E; Z/p^r Z) \xrightarrow{[\quad , \quad]_r} \pi_{n+m}(\Omega F; Z/p^r Z) \to \pi_{n+m}(\Omega E; Z/p^r Z)$$

associated to the loops on the fibration sequence

$$\Omega F \to \Omega E \to \Omega B.$$

Such a compression of the Samelson product is an easy consequence of the naturality of the Samelson product. But the Lie algebra identities need care. If p is a prime greater than 3, we claim that the map of differential graded Lie algebras $\pi_*(\Omega F; Z/p^r Z) \to \pi_*(\Omega E; Z/p^r Z)$ is an extended differential ideal in the sense of the next definition. It is remarkable that this is a consequence of the existence of a natural Samelson product and of the Hilton-Milnor theorem.

DEFINITION 16.1. Let $L' \to L$ be a morphism of graded Lie algebras. We call L' an extended ideal of L if there are two bilinear pairings (called Lie brackets):

$$[\quad , \quad] : L' \times L \to L'$$
$$[\quad , \quad] : L \times L' \to L'$$

such that

1) the diagram of Lie brackets commutes

$$\begin{array}{ccc} L' \times L' & \xrightarrow{[\ ,\ ]} & L' \\ \downarrow & & \downarrow \\ L' \times L & \xrightarrow{[\ ,\ ]} & L' \\ \downarrow & & \downarrow \\ L \times L & \xrightarrow{[\ ,\ ]} & L \end{array}$$

2) for all x, y, and z in the union of L' and L,

$$[x, y] = -(-1)^{deg(x)\cdot deg(y)}[y, x]$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{deg(x)\cdot deg(y)}[y, [x, z]].$$

An extended differential ideal $L' \to L$ is a morphism of differential graded Lie algebras which is an extended ideal and such that the differential d is a derivation in the sense that, for all x and y in the union of L' and L,

$$d[x, y] = [dx, y] + (-1)^{deg(x)}[x, dy].$$

We use the method of universal models to show that the theory of relative Samelson products is a consequence of the Hilton-Milnor theorem and of the fact that we already have a functorial theory of Samelson products.

We substitute S^1 for the nonexistent $P^1(Z/p^r Z)$ and recall that $\pi_1(\Omega X; Z/p^r Z) = \pi_1(X) \otimes Z/p^r Z$. We shall show that

THEOREM 16.2. *If p is a prime greater than 3, then $\pi_*(\Omega F; Z/p^r Z) \to \pi_*(\Omega E; Z/p^r Z), \quad * \geq 1$ is an extended differential ideal.*

REMARK 16.3. If $p = 3$, the above is true except for an error of order 3 in the Jacobi identity and an error of order 9 in the triple vanishing identity for an odd dimensional class.

A universal 2-variable model for Samelson products is the bouquet $\Sigma P^{n,m} = \Sigma P^n(Z/p^r Z) \vee \Sigma P^m(Z/p^r Z)$.

Given homotopy classes $f : P^n(Z/p^r Z) \to \Omega E$ and $g : P^m(Z/p^r Z) \to \Omega E$, the respective adjoints $\overline{f} : \Sigma P^n(Z/p^r Z) \to E$ and $\overline{g} : \Sigma P^m(Z/p^r Z) \to E$ define a map $\overline{f} \vee \overline{g}; \Sigma P^{n,m} \to E$. If

$$\iota_n : P^n(Z/p^r Z) \to \Omega\Sigma P^n(Z/p^r Z) \subseteq \Omega\Sigma P^{n,m}$$

$$\iota_m : P^m(Z/p^r Z) \to \Omega\Sigma P^m(Z/p^r Z) \subseteq \Omega\Sigma P^{n,m}$$

denote the two standard inclusions, then

$$\Omega(\overline{f} \vee \overline{g})_*([\iota_n, \iota_m]) = [f, g].$$

We adopt the shorthand $P^n = P^n(Z/p^r Z)$.

The universal 2-variable model for relative Samelson products is the sequence of inclusion map and projection map $\Sigma P^n \to \Sigma P^{n,m} \to P^m$. Given homotopy classes $f : P^n \to \Omega F$ and $g : P^m \to \Omega E$, the respective adjoints $\overline{f} : \Sigma P^n \to E$ and $\overline{g} : \Sigma P^m \to E$ define a commutative diagram of maps

$$\begin{array}{ccc} \Sigma P^n & \xrightarrow{\overline{f}} & F \\ \downarrow & & \downarrow \\ \Sigma P^{n,m} & \xrightarrow{\overline{f}\vee\overline{g}} & E \\ \downarrow & & \downarrow \\ \Sigma P^m & \xrightarrow{\overline{g}} & B. \end{array}$$

The standard method of replacing a map $F : X \to Y$ by a fibration is a functorial factorization $F : X \xrightarrow{\iota} E_F \xrightarrow{\pi} Y$ where

$$E_F = \{(x, \omega) \in X \times Y^I \mid fx = \omega(0)\}$$

$$\pi(x, \omega) = \omega(1),$$

$$\iota(x) = (x, \omega_x) \quad \text{where } \omega_x \text{ is the constant path at } x.$$

Note that the inclusion $X \to E_F$ is a strong deformation retraction and hence the maps

$$X \times [0,1] \to X \times [0,1] \cup E_F \times \{0,1\} \to E_F \times [0,1]$$

are all homotopy equivalences.

Applying this to $\Sigma P^{n,m} \to P^m$ yields a fibration sequence $F^{n,m} \to E^{n,m} \to P^m$ and a factorization

$$\begin{array}{ccccc} \Sigma P^n & \to & F^{n,m} & \xrightarrow{\Psi^{n,m}} & F \\ \downarrow & & \downarrow & & \downarrow \iota \\ \Sigma P^{n,m} & \xrightarrow{\simeq} & E^{n,m} & \xrightarrow{\Phi^{n,m}} & E \\ \downarrow & & \downarrow \pi & & \downarrow \tau \\ \Sigma P^m & \xrightarrow{=} & \Sigma P^m & \xrightarrow{\tau\circ\overline{g}} & B. \end{array}$$

The fact that the inclusion $\Sigma P^{n,m} \to E^{n,m}$ is an equivalence implies that $\overline{f} \vee \overline{g}$ can be extended to a map of fibrations $\Phi^{n,m} : E^{n,m} \to E$. The fact that $\Sigma P^{n,m} \times [0,1] \cup E^{n,m} \times \{0,1\} \to E^{n,m} \times [0,1]$ is an equivalence implies that this extension is unique up to fibre homotopy.

In the above, $\Psi^{n,m} : F^{n,m} \to F$ is the restriction of $\Psi^{n,m}$ and extends $\overline{f}$.

Note the factorization

$$[\iota_n, \iota_m] = (\Omega\iota) \circ [\iota_n, \iota_m]_r : P^{n+m} \to \Omega F^{n,m} \to \Omega E$$

The Hilton-Milnor theorem says that $\Omega F^{n,m} \to \Omega E^{n,m}$ has a retraction and hence the choice of the factorization $[\iota_n, \iota_m]_r$ is unique. Define

Definition 16.4. The relative Samelson product $[f, g]_r \in \pi_{n+m}(\Omega f, Z/p^r Z)$ is

$$[f, g]_r = (\Omega\Psi^{n,m})_*([\iota_n, \iota_m]).$$

We have compatibility with the preceding Samelson product, that is,

$$(\Omega\iota)_*[f, g]_r = [(\Omega\iota)_* f, g].$$

Not always, but when $\Omega F \to \Omega E$ admits a retraction, this property alone is enough to determine $[f, g]_r$ uniquely.

Similarly, given homotopy classes $g : P^m \to \Omega E$ and $f : P^n \to \Omega F$ with adjoints as before, we use the universal 2-variable model $\Sigma P^{n,m} \to P^m$ with the change to the homotopy class $[\iota_m, \iota_n] = [\iota_m, \iota_n]_r$ in $\Omega F^{n,m}$ to define

$$[g, f]_r = (\Omega\Psi^{n,m})_*([\iota_m, \iota_n]_r).$$

Remark 16.5. The relative Samelson products

$$[\ \ ,\ \ ]_r : \pi_n(\Omega F; Z/p^r Z) \times \pi_m(\Omega E; Z/p^r Z) \to \pi_{n+m}(\Omega F; z/p^r Z)$$
$$[\ \ ,\ \ ]_r : \pi_m(\Omega E; Z/p^r Z) \times \pi_n(\Omega F; Z/p^r Z) \to \pi_{n+m}(\Omega F; z/p^r Z)$$

have been constructed to be natural with respect to loop maps of $\Omega F \to \Omega E \to \Omega B$, that is, natural with respect to maps of the fibration sequences $F \to E \to B$.

Lemma 16.6.

$$[g, f]_r = -(-1)^{nm}[f, g]_r.$$

Proof. Since $\Omega F^{n,m}$ is a retract of $\Omega E^{n,m}$, an equation $[\iota_m, \iota_n] = -(-1)^{mn}[\iota_n, \iota_m]$ which is true in $\Omega E^{n,m}$ implies that the corresponding equation $[\iota_m, \iota_n]_r = -(-1)^{nm}[\iota_n, \iota_m]_r$ is true in $\Omega F^{n,m}$ Hence,

$$[g, f]_r = (\Omega\Psi^{n,m})_*([\iota_m, \iota_n]_r) = -(-1)^{nm}(\Omega\Psi^{n,m})_*([\iota_n, \iota_m]_r) = -(-1)^{nm}[f, g]_r.$$

□

Lemma 16.7.

$$\beta([f, g] = [\beta f, g] + (-1)^n[f, \beta g].$$

Proof. The equation $\beta([\iota_n, \iota_m]) = [\beta\iota_n, \iota_m] + (-1)^n[\iota_n, \iota_m]$ is true in $\Omega E^{n,m}$. Therefore, the equation $\beta([\iota_n, \iota_m]_r) = [\beta\iota_n, \iota_m]_r + (-1)^n[\iota_n, \iota_m]_r$ is true in $\Omega F^{n,m}$. Hence,

$$\beta[f, g]_r = \beta(\Omega\Psi^{n,m})_*([\iota_n, \iota_m]_r) =$$
$$(\Omega\Psi^{n,m})_*(\beta[\iota_n, \iota_m]_r) = (\Omega\Psi^{n,m})_*([\beta\iota_n, \iota_m]_r + (-1)^n[\iota_n, \beta\iota_m]_r) =$$
$$= [\beta f, g]_r + (-1)^n[f, \beta g]_r.$$

□

Similarly, the 3-variable models

$$\Sigma P^{n,n} \to \Sigma P^{n,n,m} \to \Sigma P^{m},$$
$$\Sigma P^{n} \to \Sigma P^{n,m,m} \to \Sigma P^{m,m},$$
$$\Sigma P^{n} \to \Sigma P^{n,m,q} \to \Sigma P^{m,q},$$

are used in the proofs of the following identities:

THEOREM 16.8. *If*

$$f \in \pi_n(\Omega F; Z/p^r Z),\ g \in \pi_m(\Omega E; Z/p^r Z),\ h \in \pi_q(\Omega E; Z/p^r Z),$$

then

$$[f + f_1, g]_r = [f, g]_r + [f_1, g]_r,$$
$$[f, g + g_1]_r = [f, g]_r + [f, g_1]_r,$$
$$[f, [g, h]]_r = [[f, g]_r, h]_r + (-1)^{nm}[g, [f, h]_r]_r.$$

17. Samelson products over the loops on an H-space

If $H \subseteq G$ is a normal subgroup of a group with an abelian quotient group G/H, then the commutator map factors as

$$[\quad,\quad] : G \times G \to H \subseteq G.$$

The analog of this for Samelson products is the following:

Suppose that $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ is a fibration sequence. There is a map of differential graded Lie algebras $(\Omega\iota)_* : \pi_*(\Omega F; Z/p^r Z) \to \pi_*(\Omega E; Z/p^r Z), \quad * \geq 1$. If B is an H-space with multiplication μ, then the Lie bracket into $\pi_*(\Omega E; Z/p^r Z)$ compresses into a bilinear pairing

$$[\quad,\quad]_\mu : \pi_*(\Omega E; Z/p^r Z) \otimes \pi_*(\Omega E; Z/p^r Z) \to \pi_*(\Omega F; Z/p^r Z).$$

The compression depends on the multiplication μ. The mere existence of the factorization follows immediately from the fact that ΩB is homotopy commutative and hence all Samelson products vanish in it. The hard part is to show that the Lie algebra identities hold, that is,

THEOREM 17.1. *Suppose that B is an H-space. If p is a prime greater than 3, then, with the exception of anti-commutativity, the map*

$$(\Omega\iota)_* \pi_*(\Omega F; Z/p^r Z) \to \pi_*(\Omega E; Z/p^r Z)$$

is a strong extended differential ideal in the sense of the definition below. The anti-commutativity is replaced by the twisted anti-commutativity,

$$[f, g]_m u = -(-1)^{deg(f)deg(g)}[g, f]_{\mu \circ T}.$$

If the multiplication in B is homotopy commutative, then anti-commutativity is valid without the twisting.

If $p = 3$, all of the above, minus the Jacobi identity and the triple vanishing identity, are satisfied.

REMARK 17.2. Suppose B is a connected H-space which is localized at an odd prime p and such that the rational Pontrjagin ring $H_*(B; Q)$ is graded commutative. If B has only finitely many nonzero homotopy groups, then Zabrodsky [**32**] has shown that the multiplication in B can be altered so that it is homotopy commutative. This result was loosely and incorrectly stated in the author's book [**24**].

Definition 17.3. Let $L' \to L$ be a morphism of graded Lie algebras. We call L' a strong extended ideal of L if there is a bilinear pairing (called a Lie bracket):

$$[\quad,\quad] : L \times L \to L'$$

such that

1) the diagram of Lie brackets commutes

$$\begin{array}{ccc} L' \times L' & \xrightarrow{[\ ,\ ]} & L' \\ \downarrow & & \downarrow \\ L \times L & \xrightarrow{[\ ,\ ]} & L' \\ \downarrow & & \downarrow \\ L \times L & \xrightarrow{[\ ,\ ]} & L \end{array}$$

2) for all x, y, and z in the union of L' and L,

$$[x, y] = -(-1)^{deg(x)\cdot deg(y)}[y, x]$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{deg(x)\cdot deg(y)}[y, [x, z]].$$

A strong extended differential ideal $L' \to L$ is a morphism of differential graded Lie algebras which is a strong extended ideal and such that the differential d is a derivation in the sense that, for all x and y in the union of L' and L,

$$d[x, y] = [dx, y] + (-1)^{deg(x)}[x, dy].$$

If $f,\ g \in \pi_*(\Omega E; Z/p^r Z)$, the subscript μ on $[f, g]_\mu \in \pi_*(\Omega F; Z/p^r Z)$ indicates that the factorization depends on the multiplication in the H-space B. The definition is via a universal model

$$\begin{array}{ccc} \Sigma P^{n,m} & \xrightarrow{\overline{f}\vee\overline{g}} & E \\ \downarrow & & \downarrow \pi \\ \Sigma P^n \times \Sigma P^m & \xrightarrow{\Phi} & B \end{array}$$

where the extension Φ is defined using the H-space multiplication

$$\Sigma P^n \times \Sigma P^m \xrightarrow{\overline{f}\times\overline{g}} B \times B \xrightarrow{\mu} B.$$

Since the universal model in this case does not have a base which is an H-space, more care is needed in the use of the models. Further details can be found in the book [**24**].

Example 17.4. Let x be an even degree element and consider the formula $\mathcal{F}$ valid in any differential graded Lie algebra

$$\mathcal{F} \qquad d(ad^{k-1}(x)(dx)) = \frac{1}{2}\sum_{j=1}^{k-1}(j, k-j)[ad^{j-1}(x)(dx), ad^{k-j-1}(x)(dx)].$$

The formula $\mathcal{F}$ is proved by induction on k. The proof uses the formula for binomial coefficients $(i-1, j) + (i, j-1) = (i, j)$.

Now suppose that $d = \beta$ is the Bockstein and that p is a prime greater than 3. The above formula $\mathcal{F}$ is valid for Samelson products in $\pi_*(\Omega E; Z/p^r Z)$ since this is a differential graded Lie algebra.

If, in addition, $\Omega F \to \Omega E \to \Omega B$ is the loops on a fibration sequence and $\beta x \in \pi_*(\Omega F; Z/p^r Z)$, then the relative Samelson product of the previous section, denoted here with a subscript $[x, y]_r \in \pi_*(\Omega F : Z/p^r Z)$ to distinguish it from the usual internal product in $\pi_*(\Omega E; Z/p^r Z)$, allows us to interpret the formula $\mathcal{F}$ as a valid formula in $\pi_*(\Omega F; Z/p^r Z)$, that is, we write

$$ad(x)(\beta x) = [x, \beta x]_r, \quad ad^{k-1}(x)(\beta x) = [x, ad^{k-2}(x)(\beta x)]_r.$$

The same proof of formula $\mathcal{F}$ works for these relative Samelson products.

Suppose that B is an H-space with multiplication μ but we require only that x, βx lie in $\pi_*(\Omega E; Z/p^r Z)$. We can use the Samelson product of this section to show that the formula $\mathcal{F}$ is valid in $\pi_*(\Omega F; Z/p^r Z)$, that is, we write

$$ad(x)(\beta x) = [x, \beta x]_\mu, \quad ad^{k-1}(x)(\beta x) = [x, ad^{k-2}(x)(\beta x)]_m u = [x, ad^{k-2}(x)(\beta x)]_r.$$

The same proof is valid as before since we never have to use the formula for twisted anti-commutativity. In addition, note that, once the Samelson products have length ≥ 3, the product of this section becomes identical to the relative product of the previous section, for example,

$$[x, [x, \beta x]_\mu]_\mu = [x, [x, \beta x]_m u]_r.$$

In other words, just one use of the product depending on the multiplication μ of B puts the product into the loops on the fibre, and after that, the products are the same as the relative products.

18. Mod p homotopy Bockstein spectral sequences

Let p be a prime. Consider the cofibration sequence

$$S^1 \xrightarrow{p} S^1 \xrightarrow{\overline{\beta}} P^2(Z/pZ) \xrightarrow{\overline{\rho}} S^2 \xrightarrow{p}$$

$$S^2 \xrightarrow{\overline{\beta}} P^3(Z/pZ) \xrightarrow{\overline{\rho}} \dots.$$

If X is any space, this leads to the long exact sequence

$$\pi_1(X) \xleftarrow{p} \pi_1(X) \xleftarrow{\beta} \pi_2(X; Z/pZ) \xleftarrow{\rho} \pi_2(X) \xleftarrow{p} \pi_2(X) \xleftarrow{\beta} \pi_3(X; Z/pZ) \xleftarrow{\rho} \dots.$$

If X is an arbitrary space, we shall alter, shorten, and terminate this sequence by

$$0 \leftarrow \pi_2(X) \otimes Z/pZ \xleftarrow{\rho} \pi_2(X) \xleftarrow{p} \pi_2(X) \xleftarrow{\beta} \pi_3(X, Z/pZ) \xleftarrow{\rho} \dots.$$

This being done, it is a short exact sequence of groups. All the groups except for $\pi_3(X; Z/pZ)$ are abelian. If p is odd, even this is abelian. But, in any case, the image of $\rho : \pi_3(X) \to \pi_3(X; Z/pZ)$ is a central subgroup.

If $X = G$ is a group-like space, we shall alter, extend, and terminate this sequence by

$$0 \leftarrow \pi_1(G) \otimes Z/pZ \xleftarrow{\rho} \pi_1(G) \xleftarrow{p} \pi_1(G) \xleftarrow{\beta} \pi_2(G; Z/pZ) \xleftarrow{\rho} \pi_2(G) \xleftarrow{p}$$

$$\pi_2(G) \xleftarrow{\beta} \pi_3(G; Z/pZ) \xleftarrow{\rho} \dots.$$

Once again, we have a short exact sequence of groups. All the groups except possibly for $\pi_2(G; Z/pZ)$ are abelian and, if p is odd, this is abelian also. And the image of $\rho : \pi_2(G) \to \pi_2(G; Z/pZ)$ is always a central subgroup.

We set $A_* = \pi_*(X)$. In the first case of an arbitrary space X, this is understood to be 0 if $* = 1$. In the second case of a group-like space X, it is understood to be 0 if $* \leq 0$.

We set $E_* = \pi_*(X; Z/pZ)$. In the first case, this is understood to be 0 if $* \leq 1$ and equal to $\pi_2(X) \otimes Z/pZ$ if $* = 2$. In the second case, this is understood to be 0 if $* \leq 0$ and equal to $\pi_1(X) \otimes Z/pZ$ if $* = 1$.

In either of the above situations we get an exact couple C, that is, two graded modules A, E and an exact triangle of homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\iota} & A \\ \partial \nwarrow & & \swarrow j \\ & E & \end{array}$$

where:

$\iota = p : A = \pi_*(X) \to A = \pi_*(X)$ is multiplication by p, hence, has degree 0.

$\partial = \beta : E = \pi_*(X; Z/pZ) \to A = \pi_{*-1}(X)$ is the Bockstein, hence, has degree -1.

$j = \rho : A = \pi_*(X) \to E = \pi_*(X; Z/pZ)$ is the mod p reduction map, hence, has degree 0.

The exact couple is displayed as follows:

$$\begin{array}{ccccccccccc} & & & & \downarrow \iota & & & & \downarrow \iota & & \\ \xrightarrow{j} & E & \xrightarrow{\partial} & & A & \xrightarrow{j} & E & \xrightarrow{\partial} & A & \xrightarrow{\iota} & \\ & & & & \downarrow \iota & & & & \downarrow \iota & & \\ \xrightarrow{j} & E & \xrightarrow{\partial} & & A & \xrightarrow{j} & E & \xrightarrow{\partial} & A & \xrightarrow{\iota} & \\ & & & & \downarrow \iota & & & & \downarrow \iota & & \\ \xrightarrow{j} & E & \xrightarrow{\partial} & & A & \xrightarrow{j} & E & \xrightarrow{\partial} & A & \xrightarrow{\iota} & \\ & & & & \downarrow \iota & & & & \downarrow \iota & & \end{array}$$

In the above, a path which goes two steps to the right, one step down, and repeats this is exact.

The term E has a differential $d : E \to E$ which is defined by $d = j \circ \partial$. We set

$$E' = H(E, d) = Z(E, d)/B(E, d) = ker(d)/im(d).$$

This leads to the derived couple C' of C which consists of the two graded modules:

$$A' = \iota A = im(\iota), \quad E' = H(E, d)$$

together with the maps

$$\iota' : A' \to A'. \quad \iota'(\iota a) = \iota^2 a,$$

$$j' : A' \to E', \quad j'(\iota a) = ja + im(d),$$

$$\partial' : E' \to A', \quad \partial'(e + im(d)) = \partial e.$$

In our case, ι and ι' are both multiplication by p. The map $j = \rho$ is reduction mod p and $j' = \rho \circ p^{-1}$ is the reduction mod p of the class divided by p. The map $\partial = \beta$ is the mod p Bockstein and ∂' is the mod p Bockstein on a coset representative.

The definition of an exact couple and of its derived couple are due to Massey [**16, 17**]. We leave as an exercise:

Lemma 18.1. *1) The maps ι', j', ∂' are well defined, that is, they are independent of choices of representatives and they land in the appropriate groups.*

2) The derived couple C' is exact.

The sequence of successive derived exact couples C, C', C'', C''', ... defines a spectral sequence via

$$E^1 = E,\, E^2 = E',\, E^3 = E'', \dots$$

with differentials $d^r : E^r \to E^r$

$$d^1 = d = j \circ \partial,\, d^2 = d' = j' \circ \partial',\, d^3 = d'' = j'' \circ \partial'',\, d^4 = d''' = j''' \circ \partial''',\, \dots$$

and we have that

$$E^{r+1} = H(E^r, d^r).$$

It is sometimes convenient to define the successive derived couples in one step as in MacLane's book [**14**]:

Let C be an exact couple. Define couples C^r as follows:

$$A^r = im(\iota^{r-1} : A \to A) = \iota^{r-1}A,$$
$$Z^r = \partial^{-1}(im(\iota^{r-1})) \subseteq E,$$
$$B^r = j(ker(\iota^{r-1})) \subseteq E,$$
$$E^r = Z^r/B^r = \partial^{-1}(im(\iota^{r-1})/j(ker(\iota^{r-1}))$$

and maps

$$\iota_r = \iota : A^r \to A^r, \quad \iota_r(\iota^{r-1}a) = \iota^r a,$$
$$j_r = j \circ \iota^{-(r-1)} : A^r \to E^r, \quad j_r(\iota^{r-1}a) = ja + j(im(\iota^{r-1})),$$
$$\partial_r = \partial : E^r \to A^r, \quad \partial_r(e + j(im(\iota^{r-1})) = \partial e.$$

Note that

$$B^1 \subseteq B^2 \subseteq \cdots \subseteq B^r \subseteq B^{r+1} \subseteq \dots$$
$$\subseteq Z^{r+1} \subseteq Z^r \subseteq \cdots \subseteq Z^2 \subseteq Z^1.$$

We leave as an exercise

Lemma 18.2. *The maps ι_r, j_r, ∂_r are well defined.*

Note that $C^1 = C$. We need:

Lemma 18.3. *The couples C^r are all exact and C^{r+1} is the derived couple of C^r.*

Proof. Assume that C^r is exact. It is sufficient to show that C^{r+1} is the derived couple of C^r.

Suppose $\overline{e} = e + B^r$ is a coset in E^r, that is,

$$B^r = j(ker(\iota^{r-1})),$$

and

$$e \in Z^r = \partial^{-1}(im(\iota^{r-1})).$$

The $r-$th differential is described as follows:

$$d^r(\overline{e}) = ja + j(ker(\iota^{r-1})) = \overline{ja}, \quad \text{where} \quad \partial e = \iota^{r-1}a,\ a \in A.$$

First we determine the group of boundaries $im(d^r) \subseteq E^r$:

Claim:

$$im(d^r) = B^{r+1} + B^r \subseteq E.$$

$\iota^r a = \iota\partial e = 0$ and $ja \in j(ker(\iota^r)) = B^{r+1}$. Thus,

$$im(d^r) \subseteq B^{r+1} + B^r.$$

On the other hand, $\iota^r a = 0$ implies that $\iota^{r-1}a = \partial e$ for some e. Hence, $e \in Z^r$, $d^r(\bar{e}) = ja$, and $B^{r+1} + B^r \subseteq im(d^r)$.

Next we determine the group of cycles $ker(d^r) \subseteq E^r$:

Claim:

$$ker(d^r) = Z^{r+1} + B^r.$$

Observe that $d^r\bar{e} = \bar{0}$ if and only if $ja \in B^r$. That is,

$$ja = jb,\ \iota^{r-1}b = 0$$

$$a - b = \iota c,$$

$$\partial e = \iota^{r-1}a = \iota^{r-1}(b + \iota c) = \iota^r c.$$

Thus, $ker(d^r) \subseteq Z^{r+1} + B^r$.

On the other hand, $\partial e = \iota^r a$ implies that $d^r(\bar{e}) = j\iota a + B^r = 0 + B^r$. Hence, $Z^{r+1} + B^r \subseteq ker(d^r)$.

Therefore,

$$H(E^r, d^r) = E^{r+1}.$$

□

Consider the diagrams

$$\begin{array}{ccccc} \pi_*(X; Z/p^rZ) & \xrightarrow{\rho} & \pi_*(X; Z/pZ) & & \\ & & \downarrow\beta & \searrow\beta & \\ \pi_{*-1}(X) & \xrightarrow{p^{r-1}} & \pi_{*-1}(X) & \xrightarrow{\rho} & \pi_{*-1}(X; Z/p^{r-1}Z) \end{array}$$

$$\begin{array}{ccccc} \pi_{*+1}(X; Z/p^{r-1}Z) & \xrightarrow{\beta} & \pi_*(X) & \xrightarrow{p^{r-1}} & \pi_*(X) \\ & \beta\searrow & \downarrow\rho & & \\ & & \pi_*(X; Z/pZ) & & \end{array}$$

It follows that

$$\begin{array}{rcl} Z^r_* & = & \beta^{-1}(im(\pi_{*-1}(X) \xrightarrow{p^{r-1}} \pi_{*-1}(X))) \\ & = & \beta^{-1}(ker(\pi_{*-1}(X) \xrightarrow{\rho} \pi_{*-1}(X; Z/p^{r-1}Z))) \\ & = & ker(\pi_*(X; Z/pZ) \xrightarrow{\beta} \pi_{*-1}(X; Z/p^{r-1}Z)) \\ & = & im(\rho : \pi_*(X; Z/p^rZ) \to \pi_*(X; Z/pZ)) \end{array}$$

that is to say, classes in E^r are represented by mod p classes which are the reductions of mod p^r classes.

$$\begin{array}{rcl} B^r_* & = & \rho(ker(\pi_*(X) \xrightarrow{p^{r-1}} \pi_*(X))) \\ & = & \rho(im(\pi_{*+1}(X; Z/p^{r-1}Z) \xrightarrow{\beta} \pi_*(X))) \\ & = & im(\beta : \pi_{*+1}(X; Z/p^{r-1}Z) \to \pi_*(X; Z/pZ)) \end{array}$$

that is to say, classes represent zero in E^r if they are Bocksteins associated to the short exact coefficient sequence

$$0 \to Z/pZ \to Z/p^rZ \to Z/p^{r-1}Z \to 0.$$

Note that d^r is defined by the relation

$$\begin{array}{llll} E^1 & \supseteq & Z^r & \rightarrow & E^r \\ \downarrow \partial & & & & | \\ A^1 & & & & |d^r \\ \uparrow \iota^{r-1} & & & & \downarrow \\ A^1 & \xrightarrow{j} & Z^r & \rightarrow & E^r. \end{array}$$

The domain of this relation is $Z^r = \beta^{-1}(im\,\iota^{r-1})$ and the indetermancy is $B^r = im\,\beta : \pi_{*+1}(X; Z/p^{r-1}Z) \to \pi_*(X; Z/pZ.)$

This description of d^r yields the following identification of the differential

LEMMA 18.4. *If there is a factorization*

$$f = F \circ \overline{\rho} : P^n(Z/pZ) \to P^n(Z/p^rZ) \to X,$$

then we have a factorization

$$d^r f = F \circ \overline{\beta} \circ \overline{\rho} \circ \overline{\rho} :$$

$$P^{n-1}(Z/pZ) \xrightarrow{\overline{\rho}} P^{n-1}(Z/p^rZ) \xrightarrow{\overline{\rho}} S^{n-1} \xrightarrow{\overline{\beta}} P^n(Z/p^rZ) \xrightarrow{F} X,$$

that is,

$$d^r f = \rho\rho\beta F = \rho\beta F.$$

PROOF. Combine the definition of d^r by the above relation with the diagram

$$\begin{array}{ccccccccc} X & \xleftarrow{f} & P^n(Z/pZ) & \xrightarrow{\overline{\beta}} & S^{n-1} & & & & \\ & \nwarrow F & \downarrow \overline{\rho} & & \downarrow p^{r-1} & & & & \\ & & P^n(Z/p^rZ) & \xleftarrow{\overline{\beta}} & S^{n-1} & \xleftarrow{\overline{\rho}} & P^{n-1}(Z/p^rZ) & \xleftarrow{\overline{\rho}} & P^{n-1}(Z/pZ) \end{array}$$

□

LEMMA 18.5 (Cartan-Eilenberg [**3**]). *Suppose that there is a commutative diagram with the bottom row exact:*

$$\begin{array}{ccccc} & & W & & \\ & \nearrow & \downarrow \gamma & \searrow \epsilon & \\ X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \end{array}$$

Then β induces an isomorphism $im(\gamma)/im(\alpha) \xrightarrow{\cong} im(\epsilon)$.

Hence,

$$\begin{array}{ccccc} & & \pi_*(X; Z/p^rZ) & & \\ & \nearrow & \downarrow \rho & \searrow \overline{p}^{r-1} & \\ \pi_{*+1}(X; Z/p^{r-1}Z) & \xrightarrow{\beta} & \pi_*(X; Z/pZ) & \xrightarrow{\eta} & \pi_*(X; Z/p^rZ) \end{array}$$

where $\overline{p}$ is the fake multiple yields the identification of E^r and the differential d^r

THEOREM 18.6.

$$E^r_* = Z^r/B^r = im\ p^{r-1} : \pi_*(X; Z/p^rZ) \to \pi_*(X; Z/p^rZ)$$

with the differential d^r having the lift to $\underline{d^r} : Z^r \to Z^r$ given by, if

$$f = \rho(F) \in \pi_*(X; Z/pZ), \quad F \in \pi_*X; Z/p^rZ),$$

then

$$\underline{d^r}(f) = \rho\beta F \in \pi_{*-1}(X; Z/pZ).$$

Remark 18.7. The immediate identification of E^r is as the image of the fake multiple, that is,

$$im\,\overline{p}^{r-1} = \eta\circ\rho : \pi_n(X;Z/p^rZ) \to \pi_n(X;Z/pZ) \to \pi_n(X;Z/p^rZ).$$

But we know that, no matter what the prime, the image of the fake multiple is the same as the image of the true multiple in dimensions ≥ 3. When X is a loop space, this is true in dimensions ≥ 2.

In addition, if p is an odd prime, then the fake and true multiples are the same maps in dimensions ≥ 4, and if X is a loop space, they are the same maps in dimensions ≥ 3.

When $X = G$ is a group-like space, in dimension two

$$E^r = im\,\overline{p}^{r-1} : \pi_2(G;Z/p^rZ) \to \pi_2(G;Z/p^rZ)$$

is the image of the fake multiple. In this case, we are unable to reduce this to the true multiple.

Remark 18.8. The above theorem is valid for all spaces X in dimension two with the understanding that $\pi_2(X;Z/p^rZ) = \pi_2(X)\otimes Z/p^rZ$. In this case, we have the factorizations

$$\pi_3(X;Z/p^{r-1}Z) \xrightarrow{\beta} Tor(\pi_2(X),Z/p^{r-1}Z) \subseteq \pi_2(X) \xrightarrow{\rho} \pi_2(X)\otimes Z/p^rZ \xrightarrow{\rho} \pi_2(X)\otimes Z/p^{r-1}Z$$

and we have the diagram with an exact row

$$\begin{array}{ccccc} & \nearrow & \pi_2(X)\otimes Z/p^rZ & \searrow p^{r-1} & \\ & & \downarrow\rho & & \\ Tor(\pi_2(X),Z/p^{r-1}Z) & \to & \pi_2(X)\otimes Z/pZ & \xrightarrow{\eta} & \pi_2(X)\otimes Z/p^rZ \end{array}$$

Thus, in dimension two,

$$Z^r = im\,\rho : \pi_2(X)\otimes Z/p^rZ\otimes\pi_2(X) \to \pi_1(X)\otimes Z/pZ$$

$$B^r = im\,\beta : \pi_3(X;Z/p^{r-1}Z) \to Tor(\pi_2(X),Z/p^{r-1}Z) \to \pi_2(X)\otimes Z/pZ)$$

$$E^r = Z^r/B^r = im\,p^{r-1} : \pi_2(X)\otimes Z/p^rZ \to \pi_2(X)\otimes Z/p^rZ$$

If $X = G$ is a group-like space, we revert to the usual $\pi_2(G;Z/p^rZ) = [P^2(Z/p^rZ),G]_*$ and adopt the convention that $\pi_1(G;Z/p^rZ) = \pi_1(G)\otimes Z/p^rZ$. Then the theorem is valid in all dimensions including dimension one. The obvious variation of the preceding remarks applies.

We now prove a universal coefficient exact sequence for the $r-$term of the Bockstein spectral sequence E^r.

Theorem 18.9. *There is a short exact sequence*

$$0 \to p^{r-1}(\pi_*(X)\otimes Z/p^rZ) \xrightarrow{\rho} E^r_* \xrightarrow{\beta} p^{r-1}Tor(\pi_{*-1}(X),Z/p^rZ) \to 0$$

and the $r-th$ differential β^r is the composition

$$E^r_* \xrightarrow{\beta} p^{r-1}Tor(\pi_{*-1}(X),Z/p^rZ) \to p^{r-1}(\pi_{*-1}(X)\otimes Z/p^rZ) \xrightarrow{\rho} E^r_{*-1}$$

where the middle map is induced by the inclusion $Tor(\pi_{*-1}(X),Z/p^rZ) \subseteq \pi_{*-1}(X)$.

PROOF. Consider the diagram of universal coefficient sequences

$$\begin{array}{ccccccccc}
0 & \to & \pi_*(X)\otimes Z/p^rZ & \to & \pi_*(X;Z/p^rZ) & \to & Tor(\pi_{*-1}(X),Z/p^rZ) & \to & 0\\
 & & \downarrow 1\otimes\rho & & \downarrow \rho & & \downarrow p^{r-1} & & \\
0 & \to & \pi_*(X)\otimes Z/pZ & \to & \pi_*(X;Z/pZ) & \to & Tor(\pi_{*-1}(X),Z/pZ) & \to & 0\\
 & & \downarrow 1\otimes\eta & & \downarrow \eta & & \downarrow include & & \\
0 & \to & \pi_*(X)\otimes Z/p^rZ & \to & \pi_*(X;Z/p^rZ) & \to & Tor(\pi_{*-1}(X),Z/p^rZ) & \to & 0
\end{array}$$

The universal coefficient sequence that we desire says that the images of the 3 columns form a short exact sequence:

$$0 \to im_1 \to im_2 \to im_3 \to 0.$$

The only nontrivial part is the exactness in the middle. This is an easy consequence of the facts that the upper left hand corner map $1 \otimes \rho$ is an epimorphism and that the lower right hand corner map *include* is a monomorphism.

The description of β^r is just the fact that it is induced on the image by the usual Bockstein.

□

REMARK 18.10. The above universal coefficient sequence is split if p is an odd prime or if $p = 2$ and $r \geq 2$ since it is a short exact sequence of vector spaces.

REMARK 18.11. The differentials in the mod p homotopy Bockstein spectral sequence determine the $p-$primary torsion in the integral homotopy groups $\pi_*(X)$ in the following way.

Assume that $\pi_*(X)$ is finitely generated in each degree and has a decomposition into cyclic summands with a set of cyclic generators $\{x_i,\ y_j,\ z_k\}_{i,j,k}$ with $order(x_i) = \infty$, $order(y_j) = p^{r_j}$, and $order(z_k) = q_k$ with q_k relatively prime to p.

Then $E^1_\pi(X)_* = \pi_*(X;Z/pZ)$ contains the following elements which generate it and, if p is odd, are a basis:

1) $\overline{x_i},\ \overline{y_j} \,\varepsilon\, \pi_*(X;Z/pZ)$ such that $x_i \otimes 1 = \overline{x_i}, y_j \otimes 1 = \overline{y_j}$ via the reduction map.

2) and $\sigma(y_j) \,\varepsilon\, \pi_{*+1}(X;Z/pZ)$ such that

$$\beta(\sigma(y_j)) = p^{r_j-1}y_j \ \in\ Tor^Z(\pi_*(X),Z/pZ) \subseteq \pi_*(X).$$

The differentials are as follows:

1) $\beta^s(\overline{x_i}) = \beta^s(\overline{y_j}) = 0$ for all $1 \leq s < \infty$

2) $\beta^s(\sigma(y_j)) = 0$ for all $1 \leq s < r_j$ and $\beta^{r_j}(\sigma(y_j)) = \overline{y_j}$.

3) if p is odd or $r \geq 2$, $E^r_\pi(X)_*$ has a vector space basis $\overline{x_i}, \overline{y_j}, \sigma(y_j)$, with $r_j \geq r$.

4) $E^\infty_\pi(X)_* = E^r_\pi(X)_*$ for r sufficiently large and has a basis $\overline{x_i}$.

19. Samelson products in Bockstein spectral sequences

THEOREM 19.1. *Let $X = \Omega Y$ be a group-like space and p an odd prime. The Samelson product together with the Bockstein differential makes $E^r = E^r(\Omega Y)$ a differential graded Lie algebra (minus the Jacobi identity and the triple vanishing identity if $p = 3$).*

PROOF. Recall that $Z^r = im\,\rho : \pi_*(\Omega Y;Z/p^rZ) \to \pi_*(\Omega Y;Z/pZ)$. We know

Lemma 19.2. *The reduction map commutes with Samelson products and the inflation maps commute modulo a power of p, that is,*

$$\begin{array}{ccc}
\pi_n(\Omega Y; Z/p^r Z) \otimes \pi_m(\Omega Y; Z/p^r Z) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(\Omega Y; Z/p^r Z) \\
\downarrow \rho \otimes \rho & & \downarrow \rho \\
\pi_n(\Omega Y; Z/pZ) \otimes \pi_m(\Omega Y; Z/pZ) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(\Omega Y; Z/pZ) \\
\downarrow \eta \otimes \eta & & \downarrow p^{r-1}\eta \\
\pi_n(\Omega Y; Z/p^r Z) \otimes \pi_m(\Omega Y; Z/p^r Z) & \xrightarrow{[\ ,\ ]} & \pi_{m+n}(\Omega Y; Z/p^r Z)
\end{array}$$

It follows that the Samelson product defines a pairing on the image of ρ,

$$[\ \ ,\ \ ] : Z^r \otimes Z^r \to Z^r, \quad x \otimes y \mapsto [x, y].$$

This pairing satisfies all the Lie identities with the known restrictions when $p = 3$.

Furthermore, this pairing is compatible with the following pairing on the image of $\eta \circ \rho$,

$$[\ \ ,\ \ ] : im\, \eta \circ \rho \otimes\ im\, \eta \circ \rho \to\ im\, \eta \circ \rho, \quad \eta\rho x \otimes \eta\rho y \mapsto \eta\rho[x, y] = p^{r-1}[x, y].$$

This last pairing is well defined since $\eta\rho x = p^{r-1}x = 0$ implies that

$$\eta\rho[x.y] = p^{r-1}[x, y] = [p^{r-1}x, y] = [0, y] = 0.$$

We use the fact that

$$\begin{array}{ccc}
P^{n+m}(Z/p^r Z) & \xrightarrow{\Delta} & P^n(Z/p^r Z) \wedge P^m(Z/p^r Z) \\
\downarrow p^{r-1} & & \downarrow \eta\rho \wedge 1 \\
P^{n+m}(Z/p^r Z) & \xrightarrow{\Delta} & P^n(Z/p^r Z) \wedge P^m(Z/p^r Z)
\end{array}$$

commutes up to composition with Whitehead products. When dimension one is involved, we also use the commutativity of the corresponding diagrams where $P^n(Z/p^r Z)$ and/or $P^m(Z/p^r Z)$ are replaced by spheres.

Hence, there is a well defined Samelson product on

$$E^r = Z^r/B^r =\ im\, \eta \circ \rho : \pi_*(\Omega Y; Z/p^r Z) \to \pi_*(\Omega Y; Z/p^r Z)$$

which is covered by the Samelson product on Z^r.

□

Lemma 19.3. *The derivation formula is valid, that is,*

$$d^r[x, y] = [d^r x, y] + (-1)^{deg\, x}[x.d^r y].$$

Proof. Let $\rho : \pi_*(\Omega Y; Z/p^r Z) \to \pi_*(\Omega Y; Z/pZ)$ be the reduction map, and let $\beta : \pi_*(\Omega Y; Z/p^r Z) \to \pi_{*-1}(\Omega Y; Z/p^r Z)$ be the Bockstein. Suppose $\rho\overline{x} = x, \quad \rho\overline{y} = y$.

Then the differential is represented by

$$d^r[x, y] = \rho\beta[\overline{x}, \overline{y}] = \rho\{[\beta\overline{x}, \overline{y}] + (-1)^{deg\, x}[\overline{x}, \beta\overline{y}]\} =$$

$$[\rho\beta\overline{x}, \rho\overline{y}] + (-1)^{deg\, x}[\rho\overline{x}, \rho\beta\overline{y}] = [d^r x, y] + (-1)^{deg\, x}[x, d^r y].$$

□

REMARK 19.4. Each $Z^r = Z^r(\Omega Y)$ is the sub-algebra of $E^1 = E^1(\Omega Y) = \pi_*(\Omega Y; Z/pZ)$ which is the image of the reduction map $\rho : \pi_*(\Omega Y; Z/p^r Z) \to \pi_*(\Omega Y; Z/pZ)$. The differential $d^r x$ represented by $\rho\beta\overline{x}$ where $\rho\overline{x} = x$.

It follows that, if $\Omega F \to \Omega E \to \Omega B$ is the loops on a fibration sequence, then

$$Z^r(\Omega F) \to Z^r(\Omega E)$$

is an extended ideal with $d^r x$ represented by $\rho\beta\overline{x}$. Hence,

$$E^r(\Omega F) \to E^r(\Omega E)$$

is also an extended ideal with $d^r x$ represented by $\rho\beta\overline{x}$.

If, in addition, B is an H-space, then

$$Z^r(\Omega F) \to Z^r(\Omega E)$$

and

$$E^r(\Omega F) \to E^r(\Omega E)$$

are strong extended ideals with $d^r x$ represented by $\rho\beta\overline{x}$.

In mod 3^r homotopy, the Jacobi identity and the triple vanishing identity are not universally valid, but we do have

THEOREM 19.5. *In the mod 3 homotopy Bockstein spectral sequence $E^r(\Omega Y)$ of a loop space,*

$$[x, [y, z]] = [[x, y], z] + (-1)^{deg\, x\, deg\, y}[y, [x, z]], \quad r \geq 2$$

and

$$[x, [x, x]] = 0, \quad if\ deg\, x\ is\, odd,\ r \geq 3.$$

PROOF. Let

$$\mathcal{J}(x, y, z) = [x, [y, z]] - [[x, y], z] - (-1)^{deg\, x\, deg\, y}[y, [x, z]]$$

be the error in the Jacobi identity. We know that $3\mathcal{J}(x, y, z) = 0$.

In

$$E^r = \ im\, \eta \circ \rho = p^{r-1} : \pi_*(\Omega Y; Z/p^r Z) \to \pi_*(\Omega Y; Z/p^r Z)$$

we have

$$[x, y] = 3^{r-1}[\overline{x}, \overline{y}]]$$

where $3^{r-1}\overline{x} = x,\ 3^{r-1}\overline{y} = y$.

Hence,

$$[x, [y, z]] = 3^{r-1}[\overline{x}, [\overline{y}, \overline{z}]]$$

and

$$\mathcal{J}(x, y, z) = 3^{r-1}\mathcal{J}(\overline{x}, \overline{y}, \overline{z}) = 0, \quad r \geq 2.$$

The Jacobi identity implies that, if x has odd degree,

$$3[x, [x, x]] = 0.$$

Hence, the Jacobi identity modulo an error term of order 3 implies that $3[x, [x, x]]$ has order 3.

Therefore,

$$[x, [x, x]] = 3^{r-1}[\overline{x}, [\overline{x}, \overline{x}]] = 0, \quad r \geq 3.$$

□

EXAMPLE 19.6. Let p be a prime greater than 3. Given $x \in \pi_{2n}(\Omega X; Z/p^r Z)$ with Bockstein $\beta x \in \pi_{2n-1}(\Omega X; Z/p^r Z)$, consider the elements

$$\tau_k(x) = ad^{p^k-1}(x)(\beta x),$$

$$\sigma_k(x) = \frac{1}{2}\sum_{j=1}^{p^k-1} \frac{(j, p^k-1)}{p}[ad^{j-1}(x)(\beta x), ad^{p^k-1}(x)(\beta x)].$$

We know that $\beta\tau_k(x) = p\sigma_k(x)$.

Hence,

$$\tau_k(x) \in \mathit{kernel}\,\beta = \rho\beta : \pi_{2p^k n-1}(\Omega X; Z/p^r Z) \to \pi_{2p^k n-1}(\Omega X; Z/pZ)$$

and there exists

$$\tilde{\tau}_k(x) \in \pi_{2p^k n-1}(\Omega X; Z/p^{r+1} Z)$$

such that $\rho\tilde{\tau}_k(x) = \tau_k(x)$.

We could also say that the element represented by $\tau_k(x)$ in E^r has $d^r\tau_k(x) = 0$ and survives to be represented by $\tilde{\tau}_k(x)$ in E^{r+1}. Since $\tilde{\tau}_k(x)$ actually reduces mod p^r to $\tau_k(x)$ and not just to the same mod p reduction mod boundaries, this is a sharper statement than just saying the element survives to E^{r+1}.

20. Nonexistence of rational Peterson spaces

In this section, we show that there do not exist rational Peterson spaces $P^n(Q)$, that is, there do not exist spaces which have a single nonvanishing integral cohomology group isomorphic to the rational numbers. This result is due to Kan and Whitehead [**12**]. Here is their proof.

First, recall some algebra:

LEMMA 20.1. *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of abelian groups and* D *is any abelian group, then there is a long exact sequence*

$$0 \to Hom(C, D) \to Hom(B, D) \to Hom(A, D) \to$$

$$Ext(C, D) \to Ext(B, D) \to Ext(A, D) \to 0.$$

Hence, if $Ext(A, D) \neq 0$ for a subgroup A of B, then $Ext(B, D) \neq 0$. And, if $Hom(C, D) \neq 0$ for a quotient group C of B, then $Hom(B, D) \neq 0$.

In the same spirit, if $Ext(A, D)$ is uncountable for a subgroup A of B, then $Ext(B, D)$ is uncountable. And, if $Hom(C, D)$ is uncountable for a quotient group C of B, then $Hom(B, D)$ is uncountable.

We also have a dual exactness which can useful in computations

LEMMA 20.2. *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of abelian groups and* D *is any abelian group, then there is a long exact sequence*

$$0 \to Hom(D, A) \to Hom(D, B) \to Hom(D, C) \to$$

$$Ext(D, A) \to Ext(D, B) \to Ext(D, C) \to 0.$$

Among the important characteristics of Hom and Ext are

LEMMA 20.3. *If* A *is a free abelian group or if* B *is a divisible abelian group, then* $Ext(A, B) = 0$.

LEMMA 20.4. *If $A = \lim_{\to} A_n$ is a direct limit, then we have the inverse limit*

$$Hom(A, D) = \lim_{\leftarrow} Hom(A_n, D)$$

and the short exact sequence

$$0 \to \lim_{\leftarrow}{}^1 Hom(A_n, D) \to Ext(A, D) \to \lim_{\leftarrow} Ext(A_n, D) \to 0.$$

Recall the universal coefficient theorem for cohomology

THEOREM 20.5. *For all $k \geq 1$, there is a short exact sequence*

$$0 \to Ext(\overline{H}_{k-1}(X; Z), Z) \to H^k(X; Z) \to Hom(H_k(X; Z), Z) \to 0.$$

Since $Hom(A, Z)$ has no divisible subgroups, we get

COROLLARY 20.6. *If the reduced integral cohomology groups of X are all divisible, then the groups $Hom(H_k(X; Z), Z) = 0$ and $H^k(X; Z) = Ext(\overline{H}_{k-1}(X, Z), Z)$ for all $k \geq 1$.*

We need

THEOREM 20.7. *If A is an abelian group with $Hom(A, Z) = 0$, then A is torsion free and divisible if and only if $Ext(A, Z)$ is torsion free and divisible.*

PROOF. Let d be any nonzero integer and let ${}_dA = \{a \in A \,|\, da = 0\}$ be the $d-$torsion subgroup. Since ${}_dA$ and A/dA are both torsion groups,

$$Hom({}_dA, Z) = Hom(A/dA, Z) = 0.$$

(The second vanishing statement also follows from the fact that A/dA is a quotient group of A.)

The exact sequence

$$0 \to_d A \to A \xrightarrow{d} A \to A/dA \to 0$$

factors into two short exact sequences

$$0 \to_d A \to A \to dA \to 0,$$

$$0 \to dA \to A \to A/dA \to 0.$$

Applying $Ext(\quad, Z)$ to these two exact sequences yields exact sequences

$$0 \leftarrow Ext({}_dA, Z) \leftarrow Ext(A, Z) \leftarrow Ext(dA, Z) \leftarrow 0,$$

$$0 \leftarrow Ext(dA, Z) \leftarrow Ext(A, Z) \leftarrow Ext(A/dA, Z) \leftarrow 0$$

which splice together to give the exact sequence

$$0 \leftarrow Ext({}_dA, Z) \leftarrow Ext(A, Z) \xleftarrow{d} Ext(A, Z) \leftarrow Ext(A/dA, Z) \leftarrow 0.$$

It follows that

$$Ext({}_dA, Z) = Ext(A, Z)/dExt(A, Z), \quad Ext(A/dA, Z) =_d Ext(A, Z).$$

Since every nonzero torsion group D contains a cyclic subgroup, it follows that, for such groups, $Ext(D, Z) = 0$ if and only if $D = 0$.

So then A has no $d-$torsion, that is, $0 =_d A$ if and only if

$$0 = Ext({}_dA, Z) = Ext(A, Z)/dExt(A, Z)$$

if and only if $Ext(A, Z)$ is d divisible.

Likewise, A is d divisible, that is, $0 = A/dA$ if and only if

$$0 = Ext(A/dA, Z) =_d Ext(A, Z)$$

if and only if $Ext(A, Z)$ has no $d-$torsion.

The result follows.

□

Recall that an abelian group is torsion free and divisible if and only if it is a rational vector space.

LEMMA 20.8. *If A is a nonzero rational vector space, then $Ext(A, Z)$ is uncountable.*

PROOF. It is sufficient to show that $Ext(Q, Z)$ is uncountable.

The exact sequence $0 \to Z \to Q \to Q/Z \to 0$ yields the exact sequence

$$0 \to Hom(Q, Q) \to Hom(Q, Q/Z) \to Ext(Q, Z) \to 0.$$

Since $Hom(Q, Q) \cong Q$ is countable, it is sufficient to show that $Hom(Q, Q/Z)$ is uncountable.

Since $Q = \bigcup \frac{1}{n!} Z$,

$$Hom(Q, Q/Z) = \lim_{\leftarrow} Hom(\frac{1}{n!} Z, Q/Z) = \lim_{\leftarrow} Z/n!Z$$

and this is clearly uncountable.

□

THEOREM 20.9. *If X is a connected space for which all of the reduced integral cohomology groups are rational vector spaces, then $H^k(X; Z) = 0$ for all $k \geq 1$. In particular, there are no rational Peterson spaces $P^n(Q)$.*

PROOF. We know that

$$Hom(H_k(X; Z), Z) = 0$$

and

$$H^k(X; Z) = Ext(\overline{H}_{k-1}(X; Z), Z)$$

for all $k \geq 1$.

The groups $Ext(\overline{H}_{k-1}(X; Z), Z)$ are all rational vector spaces and hence, so are the groups $\overline{H}_{k-1}(X; Z)$. If $\overline{H}_{k-1}(X; Z) \neq 0$ for some $k \geq 1$, then the group $Ext(\overline{H}_{k-1}(X; Z), Z) = H^k(X, Z)$ is uncountable. Hence $\overline{H}_{k-1}(X; Z) = 0$ for all $k \geq 1$. Thus, $H^k(X; Z) = 0$ for all $k \geq 1$. □

21. Nonfinitely generated coefficients

For X a pointed topological space and A a finitely generated abelian group and $n \geq 2$,

$$\pi_n(X; A) = [P^n(A), X]_*.$$

We shall show

THEOREM 21.1. *a) Let $n \geq 2$, let $X \in \mathcal{X}$ be a space in the homotopy category of pointed spaces, and let $F \in \mathcal{F}$ be an abelian group in the category of torsion free abelian groups. There is a natural isomorphism of functors of two variables*

$$\pi_n(X) \otimes F \to \pi_n(X; F).$$

b) Let $n \geq 4$, let $X \in \mathcal{X}$ be a space in the homotopy category of pointed spaces, and let $A \in \mathcal{A}$ be an abelian group in the category of category of abelian groups with no 2-torsion. There is a natural short exact sequence of functors of two variables

$$0 \to \pi_n(X) \otimes A \to \pi_n(X; A) \to Tor(\pi_n(X), A) \to 0.$$

REMARK 21.2. It is sufficient to prove the above theorem for finitely generated coefficient groups. Then

$$\pi_n(X; A) = \lim \pi_n(X; B)$$

the direct limit being taken over all the finitely generated subgroups B of A. Since tensor and Tor commute with direct limits and since limits of exact sequences are exact, we get the natural isomorphism in part a) and the natural short exact sequence in b).

Again, since limits of exact sequences are exact, we get the long exact sequences of a fibration in the case of nonfinitely generated coefficients.

PROOF. Let

$$F = \bigoplus_\alpha Z, \quad G = \bigoplus_\beta Z$$

be finitely generated free abelian groups. Since $[P^n(F), P^n(G)]_* = Hom(G, F), \quad n \geq 2$, it follows that

$$\pi_n(X; F) = [P^n(F), X]_* = \bigoplus_\alpha \pi_n(X) = \pi_n(X) \otimes F$$

is a natural isomorphism of functors of two variables. Merely note that an element

$$f = (f_{\alpha\beta}) \in Hom(G, F)$$

is defined by a matrix of integers and

$$f^* = (f^*_{\beta\alpha}) : \bigoplus_\alpha S^n \to \bigoplus_\beta S^n, \quad f^*_{\beta\alpha} = f_{\alpha\beta}$$

is defined by the dual matrix.

Now let $f : A \to B$ be a homomorphism of finitely generated abelian groups with no 2 torsion and let $h = (h_0, h_1)$:

$$\begin{array}{ccccccc} 0 & \to & F_1 & \xrightarrow{d} & F_0 & \xrightarrow{\epsilon} & A & \to 0 \\ & & \downarrow h_1 & & \downarrow h_0 & & \downarrow f & \\ 0 & \to & G_1 & \xrightarrow{d} & G_0 & \xrightarrow{\epsilon} & B & \to 0 \end{array}$$

be a chain map of finitely generated free resolutions covering the homomorphism f.

The choice of the chain map of free resolutions covering f is unique up to a chain homotopy $H : F_0 \to G_1$, that is, any other choice of a chain map $\overline{h} = (\overline{h}_0, \overline{h}_1)$ is of the form

$$\overline{h}_0 = h_0 + d \circ H, \quad \overline{h}_1 = h_1 + H \circ d.$$

Now consider the maps of spaces

$$\begin{array}{ccccccccc} P^{n-1}(G_0) & \xrightarrow{d^*} & P^{n-1}(G_1) & \xrightarrow{\iota} & P^n(B) & \xrightarrow{\epsilon^*} & P^n(G_0) & \xrightarrow{d^*} & P^n(G_1) \\ \downarrow h_0^* & & \downarrow h_1^* & & \downarrow f^* & & \downarrow h_0^* & & \downarrow h_1^* \\ P^{n-1}(F_0) & \xrightarrow{d^*} & P^{n-1}(F_1) & \xrightarrow{\iota} & P^n(B) & \xrightarrow{\epsilon^*} & P^n(F_0) & \xrightarrow{d^*} & P^n(F_1) \end{array}$$

and the maps

$$H^*: P^{n-1}(G_1) \to P^{n-1}(F_0), \quad H^*: P^n(G_1) \to P^n(F_0).$$

The vertical maps of spaces are all unique by strong coefficient functoriality. The maps ι are the standard maps in the cofibration sequences. The remaining horizontal maps and the maps H^* are unique by strong coefficient functoriality.

In the above diagram, from left to right, the first, third, and fourth square commute by strong coefficient functoriality. The second square commutes by the construction of maps of cofibration sequences.

Since $n \geq 4$, all the above maps are suspensions and hence all compositions distribute over sums. It follows that the above squares all commute when the chain map h is replaced by the chain map $\overline{h}$.

Thus, the dual maps are unique, and they form the commutative diagram below, even if we replace h by the chain homotopic map $\overline{h}$:

$$\begin{array}{ccccccccc}
\pi_{n-1}(X;F_0) & \xleftarrow{d_*} & \pi_{n-1}(X;F_1) & \xleftarrow{\iota^*} & \pi_n(X;A) & \xleftarrow{\epsilon_*} & \pi_n(X;G_0) & \xleftarrow{d_*} & \pi_n(X;G_1) \\
\downarrow h_0 & & \downarrow h_1 & & \downarrow f & & \downarrow h_0 & & \downarrow h_1 \\
\pi_{n-1}(X;G_0) & \xleftarrow{d_*} & \pi_{n-1}(X;G_1) & \xleftarrow{\iota^*} & \pi_n(X;B) & \xleftarrow{\epsilon_*} & \pi_n(X;G_0) & \xleftarrow{d_*} & \pi_n(X;G_1)
\end{array}$$

The horizontal arrows are exact and thus give the unique natural maps of short exact sequences, that is, the maps of tensor and tor are the usual maps of tensor and derived functors, independent of the choice of the chain map h:

$$\begin{array}{ccccccccc}
0 & \leftarrow & Tor(\pi_{n-1}(X),A) & \leftarrow & \pi_n(X;A) & \leftarrow & \pi_n(X)\otimes A & \to & 0 \\
 & & \downarrow f_* & & \downarrow f_* & & \downarrow 1\otimes f_* & & \\
0 & \leftarrow & Tor(\pi_{n-1}(X),B) & \leftarrow & \pi_n(X;B) & \leftarrow & \pi_n(X)\otimes B & \to & 0
\end{array}$$

The uniqueness implies that compositions give the maps of exact sequences associated with compositions. Hence the universal coefficients sequence is a functor.

□

EXAMPLE 21.3. Let $R \subseteq Q$ be any subring of the rationals, for example, $R = Q$, $R = Z_{(p)}$, or $R = Z[1/p]$ where p is a prime. Then, for $n \geq 2$, the above theorem permits us to define

$$\pi_n(X;R) = \lim \pi_n(X;Z)$$

where

a) if $R = Q$, the limit is taken with respect to the maps $n : Z \to Z$ corresponding to the subgroups

$$\frac{Z}{(n-1)!} \subseteq \frac{Z}{n!}.$$

b) if $R = Z_{(p)}$, the limit is taken with respect to the maps $q : Z \to Z$ corresponding to the subgroups

$$\frac{Z}{r} \subseteq \frac{Z}{qr}$$

with r and q relatively prime to p.

c) if $R = Z[1/p]$, the limit is taken with respect to the maps $p : Z \to Z$ corresponding to the subgroups

$$\frac{Z}{p^s} \subseteq \frac{Z}{p^{s+1}}.$$

And, in all these cases, the isomorphism $\pi_n(X;R) \cong \pi_n(X) \otimes R$ is valid.

For all of these, we have well known localized Hurewicz theorems [**24**]:

THEOREM 21.4. *: If X simply connected, $n \geq 2$, and $\pi_i(X;R) = 0, \quad i \leq n-1$, then $\pi_i(X;R) \to H_i(X;R)$ is an isomorphism for $i = n$ and an epimorphism for $i = n+1$.*

The first question that arises in an attempt to prove a general Hurewicz theorem is:

For which abelian groups A and G is it true that

$$Tor(G, A) = G \otimes A = 0$$

implies

$$\overline{H}_*(K(G,n), A) = 0?$$

22. Computations with Hilton-Hopf invariants

DEFINITION 22.1. If $f : X \to \Omega Y$ is a map to a loop space, then the multiplicative extension is the composition of loop maps

$$\overline{f} : \Omega\Sigma X \xrightarrow{\Omega\Sigma f} \Omega\Sigma\Omega Y \xrightarrow{\Omega e} \Omega Y$$

where $\Sigma : Y \to \Omega\Sigma Y, \quad y \mapsto <\ \ , y >$ and $e : \Sigma\Omega Y \to Y, \quad < t, \omega > \mapsto \omega(t)$ are the structure maps of the adjoint functors Σ and Ω.

The commutative diagrams

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Sigma(\Sigma)} & \Sigma\Omega\Sigma X \\ & \searrow 1 & \downarrow \Sigma e \\ & & \Sigma X \end{array} \qquad \begin{array}{ccc} \Omega\Sigma\Omega X & \xrightarrow{\Omega e} & \Sigma X \\ \uparrow \Sigma & \nearrow 1 & \\ \Omega X & & \end{array}$$

make it easy to check that

LEMMA 22.2. *$\overline{f}$ extends f in the sense that $\overline{f} \circ \Sigma = f : X \to \Omega Y$ and it is natural with respect to loop maps in the sense that $\overline{(\Omega g) \circ f} = \Omega g \circ \overline{f}$ where $\Omega g : \Omega Y \to \Omega W$.*

Let $\underline{f} : \Sigma X \to Z$ and $\underline{g} : \Sigma Y \to Z$ be maps with respective adjoints $f : X \to \Omega Z$ and $g : Y \to \Omega Z$. Of course, we have the defining compositions

$$f : X \xrightarrow{\Sigma} \Omega\Sigma X \xrightarrow{\Omega \underline{f}} \Omega Z, \quad \underline{f} : \Sigma X \xrightarrow{\Sigma f} \Sigma\Omega Z \xrightarrow{e} Z.$$

Note that the multiplicative extension of f is the loop of the adjoint, that is, $\overline{f} = \Omega \underline{f}$.

DEFINITION 22.3. Recall that the commutator defines the Samelson product $[f, g] : X \wedge Y \to \Omega Z$ and the adjoint is the Whitehead product $[\underline{f}, \underline{g}] : \Sigma(X \wedge Y) \to Z$. In particular, the multiplicative extension of the Samelson product is the loop of the Whitehead product.

We now describe the Hilton-Milnor theorem in one of its two adjoint forms.

Let $\omega_j(\iota_0, \iota_1)$ run over a basis of monomials for the free (ungraded) Lie algebra $L(\iota_0, \iota_1)$. For example, it could begin with

$$\omega_0(\iota_0, \iota_1) = \iota_0, \quad \omega_1(\iota_0, \iota_1) = \iota_1, \quad \omega_2(\iota_0, \iota_1) = [\iota_0, \iota_1]$$

$$\omega_3(\iota_0, \iota_1) = [\iota_0, [\iota_0, \iota_1]], \quad \omega_4(\iota_0, \iota_1) = [\iota_1, [\iota_1, \iota_0]],$$

$$\omega_5(\iota_0, \iota_1) = [\iota_0, [\iota_0, [\iota_0, \iota_1]]], \ldots$$

Let $\iota_0 = \iota_X : X \xrightarrow{\Sigma} \Omega\Sigma X \to \Omega\Sigma(X \vee Y)$ and $\iota_1 = \iota_Y : Y \xrightarrow{\Sigma} \Omega\Sigma Y \to \Omega\Sigma(X \vee Y)$ be the two inclusions.

Then each $\omega_j(\iota_X, \iota_Y) : \omega_j(X, Y) \to \Omega\Sigma(X \vee Y)$ is a Samelson product where $\omega_j(X, Y)$ is an appropriate smash product of X and Y. For example,

$$\omega_0(\iota_X, \iota_Y) = \iota_X : X \to \Omega\Sigma(X \vee Y), \quad \omega_1(\iota_X, \iota_Y) = \iota_Y : Y \to \Omega\Sigma(X \vee Y),$$

$$\omega_2(\iota_X, \iota_Y) = [\iota_X, \iota_Y] : X \wedge Y \to \Omega\Sigma(X \vee Y),$$

$$\omega_3(\iota_X, \iota_Y) = [\iota_X, [\iota_X, \iota_Y]] : X \wedge (X \wedge Y) \to \Omega\Sigma(X \vee Y),$$

$$\omega_4(\iota_X, \iota_Y) = [\iota_Y, [\iota_Y, \iota_X]] : Y \wedge (Y \wedge X) :\to \Omega\Sigma(X \vee Y),$$

$$\omega_5(\iota_X, \iota_Y) = [\iota_X, [\iota_X, [\iota_X, \iota_Y]]] : X \wedge (X \wedge (X \wedge Y))) \to \Omega\Sigma(X \vee Y), , \dots$$

We state without proof the Hilton-Milnor theorem:

THEOREM 22.4 (Hilton-Milnor). *If X and Y are connected spaces, then there is a natural homotopy equivalence*

$$\Psi : \prod_{j \geq 0} \Omega\Sigma(\omega_j(X, Y)) \to \Omega\Sigma(X \vee Y)$$

where the left hand side is the weak product, that is, the direct limit of the finite products, and we get the map Ψ by using the loop multiplication of $\Omega\Sigma(X \vee Y)$ to multiply the multiplicative extensions of the above countable list of Samelson products.

REMARK 22.5. In order to get a map to the loop space defined on the weak product, the most convenient way to multiply maps is to require the $j-$th loop to run in the interval $[1 - 2^j, 1 - 2^{j+1}]$.

Thus, we have a natural homotopy equivalence from the weak product to the loop suspension of the bouquet

$$\Psi : \Omega\Sigma X \times \Omega\Sigma Y \times \Omega\Sigma(X \wedge Y) \times \Omega\Sigma(X \wedge X \wedge Y) \times \Omega\Sigma(Y \wedge Y \wedge X) \times \dots$$
$$\to \Omega\Sigma(X \vee Y).$$

Let $f : Z \to \Omega\Sigma X$ be a map with adjoint $\underline{f} : \Sigma Z \to X$. Let $\mu : \Sigma X \to \Sigma X \vee \Sigma X$ be the comultiplication of the suspension.

DEFINITION 22.6 (Hilton-Hopf invariants). The $k-$th Hilton-Hopf invariant $h_k(f)$ of f is the projection on the $k-$th factor of the composition $\Psi^{-1} \circ \Omega\mu \circ f$, that is,

$$Z \to \Omega\Sigma X \xrightarrow{\Omega\mu} \Omega(\Sigma X \vee \Sigma X) \xrightarrow{\Psi^{-1}} \prod_{j \geq 0} \Omega\Sigma(\omega_j(X, X)) \xrightarrow{\pi_k} \Omega\Sigma\omega_k(X, X).$$

Hence we have the Hilton-Milnor decomposition

THEOREM 22.7 (Hilton-Milnor decomposition).

$$\Omega\mu(f) = \overline{\iota_0} h_0(f) \times \overline{\iota_1} h_1(f) \times \overline{[\iota_0, \iota_1]} h_2(f) \times \overline{[\iota_0, [\iota_0, \iota_1]]} h_3(f) \times \dots$$

The natural projections $\pi_0, \pi_1 : \Sigma X \vee \Sigma X \to \Sigma X$ show that $h_0(f) = h_1(f) = f$.

PROOF. One uses

$$\pi_0 \circ \mu = \pi_1 \circ \mu = 1_{\Sigma X}$$
$$\pi_0 \circ \iota_0 = \pi_1 \circ \pi_1 = 1_{\Sigma X}, \quad \pi_0 \iota_1 = \pi_1 \circ \pi_0 = 0,$$
$$\pi_0 \circ \omega_j(\iota_0, \iota_1) = \pi_1 \circ \omega_j(\iota_0, \iota_1) = 0$$

for $j \geq 2$ since, in these cases, $\omega(\iota_0, \iota_1)$ contains both ι_0 and ι_1.

For example, applying $\Omega\pi_0$ to the equation for $\Omega\mu(f)$ yields

$$f = \Omega\pi_0 \circ \Omega\mu(f) = h_0(f) \times 0 \times 0 \times \cdots = h_0(f)$$

□

Since the only interesting Hilton-Hopf invariants $h_j(f)$ occur when $j \geq 2$, we shall not refer to $h_0(f) = h_1(f) = f$ as Hilton-Hopf invariants.

If Z is a finite dimensional complex, the above sums are finite. In fact, suppose that the dimension of Z is $\leq m - 1$ and that X is $n - 1$ connected. Observe that, if $m \leq 2n$, then $h_j(f) = 0$ for all $j \geq 2$, that is, all Hilton-Hopf invariants are 0. If $m \leq 3n$, then $h_j(f) = 0$ for all $j \geq 3$, We shall be particularly interested in the last case where the only nonzero Hopf invariant is

$$h_2 : [\Sigma Z, X]_* \to [\Sigma Z, \Sigma X \wedge X]_*$$

.

We now describe the Hilton-Milnor theorem in its equivalent adjoint form which uses Whitehead products instead of Samelson products.

We shall abuse notation and use $\iota_0 = \iota_X : \Sigma X \to \Sigma X \vee \Sigma Y$ and $\iota_1 = \iota_Y : \Sigma Y \to \Sigma X \vee \Sigma Y$ to denote the respective adjoints of the previous $\iota_0 = \iota_X, \iota_1 = \iota_Y$.

Also, $\omega_j(\iota_X, \iota_Y) : \Sigma\omega_j(X, Y) \to \Sigma(X \vee Y)$ is used to represent the Whitehead product which is adjoint to the Samelson product.

DEFINITION 22.8. The $k-$th Hilton-Hopf invariant

$$h_k(\underline{f}) : \Sigma Z \to \Sigma\omega_k(X, X)$$

of the adjoint $\underline{f}$ is simply the adjoint of the previous Hilton-Hopf invariant.

Note that the comultiplication $\mu = \iota_0 + \iota_1 : \Sigma X \to \Sigma X \vee \Sigma X$ and the Hilton-Milnor decomposition can be written in the adjoint form where Whitehead products replace Samelson products. Observe that the adjoint of $\Omega\mu(f) = \Omega(\iota_0 + \iota_1) \circ f$ is $(\iota_0 + \iota_1) \circ \underline{f}$ and the adjoint of $\overline{\omega_j(\iota_0, \iota_1)} h_j(f)$ is $\omega_j(\iota_0, \iota_1) h_j(\underline{f})$. Hence,

THEOREM 22.9 (Adjoint form of the Hilton-Milnor decomposition).

$$(\iota_0 + \iota_1) \circ \underline{f} = \iota_0 \circ \underline{f} + \iota_1 \circ \underline{f} + [\iota_0, \iota_1] \circ h_2(\underline{f}) + [\iota_0, [\iota_0, \iota_1]] \circ h_3(\underline{f}) + \ldots$$

in the group $[\Sigma Z, \Sigma X \vee \Sigma X]_*$. If Z is not itself a suspension, the order of the addition may matter since this group may not be abelian .

The space $\Sigma X \vee \Sigma X$ is a univeral example for the Hilton-Milnor decomposition, that is, if g_0, $g_1 : \Sigma X \to W$ are any two maps, this defines a map $g_0 \vee g_1 : \Sigma X \vee \Sigma X \to W$ and we get

COROLLARY 22.10 (Distributivity formula).

$$(g_0 + g_1) \circ \underline{f} = g_0 \circ \underline{f} + g_1 \circ \underline{f} + [g_0, g_1] \circ h_2(\underline{f}) + [g_0, [g_0, g_1]] \circ h_3(\underline{f}) + \ldots$$

in the group $[\Sigma Z, W]_*$. Again, if Z is not itself a suspension, the order of the addition may matter.

The Hilton-Hopf invariants have two kinds of naturality

LEMMA 22.11. *1) (right linearity) If h, k_0, $k_1 : \Sigma T \to \Sigma Z$ are maps, then*

$$h_j(\underline{f} \circ h) = h_j(f) \circ h$$

$$h_j(\underline{f} \circ (k_0 + k_1)) = h_j(\underline{f}) \circ k_0 + h_j(\underline{f}) \circ k_1$$

2) (left linearity with respect to suspensions) If

$$\underline{f} : \Sigma Z \to \Sigma X, \quad \Sigma \ell : \Sigma X \to \Sigma Y$$

are maps, then

$$h_j(\Sigma \ell \circ \underline{f}) = (\Sigma \ell \wedge \cdots \wedge \Sigma \ell) \circ h_j(\underline{f}).$$

The second part of the above lemma is a consequence of the naturality of the Hilton-Milnor theorem for $\Omega\Sigma(A \vee B)$ with respect to maps of A and B.

We now apply the above to the study of compositions $(g_0 + g_1) \circ f$ where $f, g_0, g_1 : P^3(Z/p^rZ) \to P^3(Z/p^rZ)$. The distributivity formula is

$$(g_0 + g_1) \circ f = g_0 \circ f + g_1 \circ f + [g_0, g_1] \circ h_2(f).$$

The Hilton-Hopf invariants $h_j(f)$ are zero for all $j \geq 3$.

Let $\overline{p} : P^3(p^r) \to P^3(p^r)$ be the fake multiple of the identity and let $p : P^3(p^r) \to P^3(p^r)$ be the actual multiple of the identity. Since both induce multiplication by p on integral cohomology, their difference $\overline{p} - p = \alpha$ induces zero on cohomology. Therefore, α factors as

$$\alpha : P^3(p^r) \xrightarrow{q} S^3 \xrightarrow{\overline{\alpha}} S^2 \xrightarrow{\iota} P^3(p^r).$$

We shall call such an α a fake zero.

THEOREM 22.12. *For all $s \geq 1$,*

$$\overline{p}^s = (p + \alpha)^s = p^s + \alpha \circ (p^{s-1}(p^{s-1} + \cdots + p + 1)).$$

We need a sequence of lemmas:

LEMMA 22.13 (composition of fake zeros is zero). *If β and γ are fake zeros, then $\beta \circ \gamma = 0$.*

This is a consequence of the factorization of fake zeros,

$$\beta \circ \gamma : P^3(Z/p^rZ) \to S^2 \xrightarrow{\iota} P^3(Z/p^rZ) \xrightarrow{q} S^3 \to P^3(Z/p^rZ).$$

LEMMA 22.14. *If β is a fake zero and f, $g : P^3(Z/p^rZ) \to P^3(Z/p^rZ)$ are any maps, then $[f, \beta] \circ h_2(g) = 0$. Hence, we have the distributive law*

$$(f + \beta) \circ g = f \circ g + \beta \circ g.$$

PROOF. Since the dimension of $P^3(Z/p^rZ)$ is 3, the Hilton-Hopf invariant factors through the 3-skeleton as

$$h_2(g) = (\iota \wedge \iota) \circ G : P^3(Z/p^rZ) \to \Sigma S^1 \wedge S^1 \to \Sigma P^2(Z/p^rZ) \wedge P^2(Z/p^rZ).$$

Hence

$$[f, \beta] \circ h_2(g) = [f, \beta] \circ (\iota \wedge \iota) \circ G = [f|_{S^2}, \beta|_{S^2}] \circ G = [f|_{S^2}, 0] \circ G = 0$$

since the fake zero β restricts to zero on S^2. □

The preceding two lemmas show that

LEMMA 22.15. *For fake zeros β and γ,*

$$(f+\beta)\circ\gamma = f\circ\gamma.$$

Since the mod p^r Hopf invariant is an isomorphism on fake zeros, Lemma 11.3, that is,

$$\mathcal{H}(k\circ\beta) = k^2\mathcal{H}(\beta) = \mathcal{H}(\beta\circ k^2)$$

implies

LEMMA 22.16. *If β is a fake zero and k is an integer, then $k\circ\beta = \beta\circ k^2$.*

COROLLARY 22.17. *For all $s \geq 1$,*

$$\overline{p}^s = (p+\alpha)^s = (1+\alpha\circ(p^{s-1}+p^{s-2}+\cdots+p+1))\circ p^{s-1}.$$

PROOF. If p is an odd prime, the 3−rd homotopy groups are all abelian. If $p=2$, we can at least say that $\overline{2} = 2+\alpha$ is a central element in the group structure.

In any case, for any positive integer k,

$$\begin{array}{cl}
(p+\alpha)\circ k = (p+\alpha)+\cdots+(p+\alpha) & = \\
p+(p+\alpha)+\alpha+(p+\alpha)+\cdots+(p+\alpha) & = \\
p\circ 2+\alpha\circ 2+(p+\alpha)+\cdots+(p+\alpha) & = \\
p\circ 2+(p+\alpha)+\alpha\circ 2+(p+\alpha)+\cdots+(p+\alpha) & = \\
p\circ 3+\alpha\circ 3+(p+\alpha)+\cdots+(p+\alpha) & = \dots \\
p\circ k+\alpha\circ k &
\end{array}$$

The inductive step to prove the corollary is

$$\begin{array}{cl}
(p+\alpha)\circ(p^s+\alpha\circ(p^{s-1}(p^{s-1}+\cdots+p+1)) & = \\
(p+\alpha)\circ p^s+(p+\alpha)\circ\alpha\circ(p^{s-1}(p^{s-1}+\cdots+p+1)) & = \\
(p^s+\alpha\circ p^s)+p\circ\alpha p^{s-1}(p^{s-1}+\cdots+p+1) & = \\
(p^{s+1}+\alpha\circ p^s)+\alpha\circ p^2p^{s-1}(p^{s-1}+\cdots+p+1) & = \\
p^{s+1}+\alpha\circ p^s(p^s+\cdots+p+1) &
\end{array}$$

□

We claim that the elements α are divisible by p, more precisely:

THEOREM 22.18. *Let p be any prime and $r \geq 2$. For all $j \geq 1$, there exist fake zeros $\delta_j : P^3(Z/p^rZ) \to P^3(Z/p^rZ)$ which induce zero in integral cohomology such that $\alpha = \delta_1\circ p$ and*

$$\overline{p}^j = p^j+\delta_j\circ p^j = (1+\delta_j)\circ p^j.$$

PROOF. We know that both powers p^r and $\overline{p}^r$ are zero in $\pi_3(P^3(Z;Z/p^rZ);Z/p^rZ)$. Hence, $0 = \alpha\circ p^{r-1}(p^{r-1}+\cdots+p+1)$. Since $(p^{r-1}+\cdots+p+1)$ is relatively prime to p and this is an equation in the cyclic group $K^3(Z/p^rZ, Z/p^rZ) = Z/p^rZ$, it follows that $\alpha = \delta\circ p$ in this group.

For all $j \geq 1$,

$$\overline{p}^j = (p+\alpha)^j = p^j+\delta\circ p\circ(p^{j-1}+p^{j-2}+\cdots+p+1))\circ p^{j-1} =$$

$$p^j+\delta\circ(p^{j-1}+p^{j-2}+\cdots+p+1)\circ p^j = p^j+\delta_j\circ p^j = (1+\delta_j)\circ p^j.$$

□

23. Cohomology of some cubic constructions

Let π be the cyclic group of order 3 with a generator σ. We let π act via $\sigma(x \wedge y \wedge z) = y \wedge z \wedge x$ on the smash product $X \wedge X \wedge X$. Let $\omega \in S^1$ be a primitive cube root of unity. Then π acts on the right of S^1 by multiplication, $\alpha * \sigma = \alpha\omega$. Thus π acts on the left of the product $W = S^1 \times (X \wedge X \wedge X, *)$ via $\sigma * (\alpha, x \wedge y \wedge z) = (\alpha\omega^{-1}, y \wedge z \wedge x)$.

Writing $S^1 = [1, \omega] \cup [\omega, \omega^2] \cup [\omega^2, 1]$ gives fundamental domains for the group action on W, that is, we can write

$$W = W_0 \cup W_1 \cup W_2, \quad W_i = [\omega^i, \omega^{i+1}] \times X \wedge X \wedge X, \quad i = 0, 1, 2.$$

Thus each W_i is a fundamental domain for the π space W and the orbit space pair $\Gamma_1(X) = W/\pi = S^1 \times_\pi (X \wedge X \wedge X \wedge X, *)$ can be written as the fundamental domain modulo an identification of the boundary.

$$W/\pi = W_1/\sim, \quad (1, z) \sim (\omega, \sigma z).$$

We define short filtrations of the orbit space pair by

$$G_0 = \{1, \omega, \omega^2\} \times_\pi (X \wedge X \wedge X, *), \quad G_1 = \Gamma_1(X) = S^1 \times_\pi (X \wedge X \wedge X, *).$$

With mod 3 coefficients, these filtration lead to a homology spectral sequence with

$$\begin{gathered} E^1_0 = H_*(G_0) = H_*(\{1, \omega, \omega^2\} \times_\pi (X \wedge X \wedge X, *)) = 1 \otimes H_*(X \wedge X \wedge X, *), \\ E^1_1 = H_*(G_1, G_0) = H_*(S^1 \times_\pi (X \wedge X \wedge X*), \{1, \omega, \omega^2\} \times_\pi (X \wedge X \wedge X, *)) = \\ H_*([1, \omega] \times (X \wedge X \wedge X, *), \{1, \omega\} \times (X \wedge X \wedge X, *)) = \\ e_1 \otimes H_*(X \wedge X \wedge X, *), \end{gathered}$$

with differential

$$d^1(1 \otimes x) = 0, \quad d^1(e_1 \otimes z) = 1 \otimes (\sigma z - z).$$

Since the spectral sequence is confined to two lines, $H_*(E^1, d^1) = E^2 = E^\infty$.

We can now compute the mod 3 homology and cohomology of the orbit space pair $\Gamma_1(P^n(Z/3^rZ)) = S^1 \times_\pi (P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *)$. Recall that we have the following mod 3 homology

$$H_*(P^n(Z/3^rZ)) = < 1, u_{n-1}, v_n > = < 1, u, v > .$$

Let

$$x = v \otimes u \otimes u, \sigma x = u \otimes u \otimes v, \sigma^2 x = (-1)^{n-1} u \otimes v \otimes u$$

and

$$y = u \otimes v \otimes v, \sigma y = v \otimes v \otimes u, \sigma^2 y = (-1)^n v \otimes u \otimes v.$$

Theorem 23.1.

$$H_*(S^1 \times_\pi (P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *)) = H_*(\Gamma_1(P^n(Z/3^rZ))$$

has a basis

$$\begin{gathered} 1 \otimes u \otimes u \otimes u, 1 \otimes v \otimes v \otimes v, 1 \otimes x = 1 \otimes \sigma x = 1 \otimes \sigma^2 x, 1 \otimes y = 1 \otimes \sigma y = 1 \otimes \sigma^2 y, \\ e_1 \otimes u \otimes u \otimes u, e_1 \otimes v \otimes v \otimes v, e_1 \otimes (1 + \sigma + \sigma^2)x, e_1 \otimes (1 + \sigma + \sigma^2)y. \end{gathered}$$

PROOF. The short filtration of the orbit space gives a spectral sequence. with

$$E_0^1 = 1 \otimes H_*(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *),$$
$$E_1^1 = e_1 \otimes H_*(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *)$$

The first differential is $d^1(1 \otimes z) = 0, \quad d^1(e_1 \otimes z) = 1 \otimes (\sigma - 1)z$. As a π module, there is a direct sum decomposition

$$H_*(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *) = < u \otimes u \otimes u > \oplus < v \otimes v \otimes v > \oplus$$
$$< v \otimes u \otimes u, u \otimes u \otimes v, u \otimes v \otimes u > \oplus < u \otimes v \otimes v, v \otimes v \otimes u, v \otimes u \otimes v > .$$

Since π acts trivially on the first two summands and freely on the last two summands, it follows that E^2 is given by the computation of the theorem. Since $E^2 = E^\infty$, the theorem follows. □

We adopt the Milnor sign conventions for the actions of tensors. Thus

$$< \alpha \otimes \beta \otimes \gamma, x \otimes y \otimes z > = (-1)^{\beta x + \gamma x + \gamma y} < \alpha, x > < \beta, y > < \gamma, z > .$$

Permutations act via adjoints, if

$$\sigma(x \otimes y \otimes z) = (-1)^{z(x+y)} z \otimes x \otimes y,$$

then

$$< (\alpha \otimes \beta \otimes \gamma)\sigma, x \otimes y \otimes z > = < \alpha \otimes \beta \otimes \gamma, \sigma(x \otimes y \otimes z) >$$

so that

$$(\alpha \otimes \beta \otimes \gamma)\sigma = \sigma^{-1}(\alpha \otimes \beta \otimes \gamma) = (-1)^{\alpha(\beta+\gamma)} \beta \otimes \gamma \otimes \alpha.$$

A dual spectral sequence to the above give the following cohomology computation. Let $H^*(P^n(Z/3^rZ), *) = < \mu_{n-1}, \nu_n >, \quad a = \mu \otimes \mu \otimes \nu, b = \nu \otimes \nu \otimes \mu$.

THEOREM 23.2.

$$H^*(S^1 \times_\pi (P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *)) = H_*(\Gamma_1(P^n(Z/3^rZ))$$

has a basis

$$1 \otimes \mu \otimes \mu \otimes \mu, 1 \otimes \nu \otimes \nu \otimes \nu, 1 \otimes a(1 + \sigma + \sigma^2), 1 \otimes b(1 + \sigma + \sigma^2),$$
$$w \otimes \mu \otimes \mu \otimes \mu, w \otimes \nu \otimes \nu \otimes \nu, w \otimes b = w \otimes b\sigma = w \otimes b\sigma^2$$

where $deg(w) = 1$.

The proof is left as an exercise.

Let X be a pointed space and suppose $\mu \in H^n(X)$ is a class in mod 3 cohomology. There is a functorial bundle pair

$$\Gamma(X) = E\pi \times_\pi (X \wedge X \wedge X, *) \to B\pi$$

with fibre pair $(X \wedge X \wedge X, *)$. This bundle pair leads to the definition of the Steenrod operations. First

THEOREM 23.3. *There is a canonical class*

$$\xi(\mu) = 1 \otimes \mu \otimes \mu \otimes \mu \in H^{3n}(\Gamma(X))$$

such that the restriction to any fibre pair $(X \wedge X \wedge X, *)$ *is* $\mu \otimes \mu \otimes \mu$.

Proof. The mod 3 cohomology Serre spectral sequence has

$$E_2^{p,q} = H^p(B\pi; H^q(X \wedge X \wedge X, *)).$$

This E_2 term has local coefficients which reduce to the usual tensor product of base and fibre cohomology when the π action on the fibre cohomology is trivial.

Let $X = K = K(\pi, n)$ and let $\iota \in H^n(K(\pi, n))$ be the universal example for mod 3 cohomology, that is, there is a unique homotopy class $f : X \to K$ such that $f^*\iota = \mu$. Since $(K \wedge K \wedge K, *)$ is $3n - 1$ connected with a π equivariant bottom cohomology class $\iota\otimes\iota\otimes\iota$, the class $1\otimes\iota\otimes\iota\otimes\iota \in E_2^{0,3n} = H^0(B\pi)\otimes H^{3n}(K\wedge K\wedge K, *)$ survives to a well defined class $\xi(\iota) \in H^{3n}(\Gamma(K))$.

Setting

$$\xi(\mu) = f^*\xi(\iota)$$

gives the result. □

Recall the definition of the mod 3 Steenrod operations $\mathcal{P}^i$ on a class $\mu \in H^n(X)$. The reduced diagonal $\delta : X \to X \wedge X \wedge X$ is π equivariant and thus defines the equivariant diagonal

$$\overline{\delta} = 1 \times_\pi \delta : B\pi \times (X, *) = E\pi \times_\pi (X, *) \to E\pi \times_\pi (X \wedge X \wedge X, *).$$

The cohomology of the classifying space is

$$H^*(B\pi) = E(w) \otimes P(\beta w), \quad deg(w) = 1, \quad deg(\beta w) = 2.$$

We have

Definition 23.4. The mod 3 Steenrod operations $\mathcal{P}^i$ and their Bocksteins $\beta\mathcal{P}^i$ are defined by

$$\overline{\delta}^*\xi(\mu) = 1 \otimes D_0 + w \otimes D_1 + \beta w \otimes D_2 + w(\beta w) \otimes D_3 + (\beta w)^2 \otimes D_4 + \dots$$

where $D_j \in H^{3n-j}(X, *)$ and there are fixed nonzero scalars ϵ_j such that

$$if \quad j = 2(n - 2i), \quad then \quad D_j = \epsilon_j \mathcal{P}^i\mu$$

$$if \quad j + 2(n - 2i) - 1, \quad then \quad D_j = \epsilon_j \beta\mathcal{P}^i\mu.$$

Otherwise, $D_j = 0$ (For our purposes, the exact value of the fixed nonzero scalars ϵ_j are unimportant.)

Thus the $i-$th mod 3 Steenrod operation $\mathcal{P}^i\mu$ on a class μ of degree n has degree $n + 4i$.

.

With mod 3 coefficients, $H^*(S^{n-1}) =< 1, e >$ with degree $e = n - 1$ and $H*(P^n(Z/3Z)) =< 1, \mu, \nu >$ with degree $\mu = n-1$ and degree $\nu = n$. We compute the mod 3 cohomology of the pairs

$$\Gamma(S^{n-1}) = E\pi \times_\pi (S^{n-1} \wedge S^{n-1} \wedge S^{n-1}, *)$$

and

$$\Gamma(P^n(Z/3Z)) = E\pi \times_\pi (P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z), *).$$

Theorem 23.5. *a)*

$$H^*(\Gamma(S^{n-1})) = E(w) \otimes P(\beta w) \otimes e \otimes e \otimes e.$$

b)

$$H^*(\Gamma(P^n(Z/3Z))) = \begin{cases} E(w) \otimes P(\beta w) \otimes \mu \otimes \mu \otimes \mu \oplus \\ Z/3Z \otimes (1+\sigma+\sigma^2)(\nu \otimes \mu \otimes \mu) \oplus \\ Z/3Z \otimes (1+\sigma+\sigma^2)(\nu \otimes \nu \otimes \mu) \oplus \\ E(w) \otimes P(\beta w) \otimes \mu \otimes \mu \otimes \mu \end{cases}$$

PROOF. a) Since $H^*(S^{n-1} \wedge S^{n-1} \wedge S^{n-1}, *) = < e \otimes e \otimes e >$ is a module with a trivial π action, the cohomology Serre spectral sequence of the fibration pair $\Gamma(S^{n-1}) \to B\pi$ has

$$E_2 = H^*(B\pi) \otimes e \otimes e \otimes e = E(w) \otimes P(\beta w) \otimes e \otimes e \otimes e.$$

Since it is confined to one horizontal line $E_2^{*,3n-3}$, it collapses and $E_2 = E_\infty$ with no extension problems.

b) The cohomology Serre spectral sequence of the fibration pair

$$\Gamma(P^n(Z/3Z)) \to B\pi$$

is a spectral sequence of modules over the cohomology of the base $H^*(B\pi)$ and has E_2 term given with twisted coefficients by

$$\begin{aligned} E_2^{p,q} &= H^p(B\pi; H^q(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *) \\ &= Ext^p(Z/3Z, H^q(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *)) \end{aligned}$$

The resolution to compute

$$H^*(B\pi : M) = Ext_{Z/3Z[\pi]}(Z/3Z, M), M = H^*(X \wedge X \wedge X, *)$$

is

$$Z/3Z[\pi] \xleftarrow{1-\sigma} Z/3Z[\pi] \xleftarrow{1+\sigma+\sigma^2} Z/3Z[\pi] \xleftarrow{1-\sigma} Z/3Z[\pi] \xleftarrow{1-\sigma} \dots$$

We know the structure of $H^*(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *)$ as a π module. It is a trivial π module in the dimensions $3n-3$ and $3n$ generated by the invariant elements $\mu \otimes \mu \otimes \mu$ and $\nu \otimes \nu \otimes \nu$, respectively. And it is a free module in the dimensions $3n-1$ and $3n-2$ with generators $\alpha = \nu \otimes \nu \otimes \mu, \sigma\alpha, \sigma^2\alpha$ and $\beta = \nu \otimes \mu \otimes \mu, \sigma\beta, \sigma^2\beta$, respectively.

Recall that group rings of finite groups over fields are Frobenius algebras and therefore self-injective [**7**]. Hence, the above free modules are injective. It follows that, when $q = 3n-1$ or $q = 3n-2$,

$$\begin{aligned} &Ext^p(Z/3Z, H^q(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *)) = \\ &Hom_\pi(Z/3Z, H^q(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), *)) = \\ &\begin{cases} \text{invariants of} \quad H^q(P^n(Z/3^rZ) \wedge P^n(Z/3^rZ) \wedge P^n(Z/3^rZ), & p = 0 \\ 0, & p \neq 0. \end{cases} \end{aligned}$$

Hence, the nonzero degrees are

$$E_2^{p,q} = \begin{cases} E(w) \otimes P(\beta w) \otimes \mu \otimes \mu \otimes \mu & q = 3n-3 \\ Z/3Z \otimes (1+\sigma+\sigma^2)(\nu \otimes \mu \otimes \mu) & q = 3n-2 \\ Z/3Z \otimes (1+\sigma+\sigma^2)(\nu \otimes \nu \otimes \mu) & q = 3n-1 \\ E(w) \otimes P(\beta w) \otimes \mu \otimes \mu \otimes \mu & q = 3n \end{cases}$$

(E_2 is a module over $H^*(B\pi)$ with w and βw annihilating the middle two $Z/3Z$.)

The map $S^{n-1} \to P^n(Z/3Z)$ induces a map from this spectral sequence in b) to the previous spectral sequence in a). It follows that nothing on the line $q = 3n-3$ can be hit by a differential.

Since the spectral sequence is a module over $H^*(B\pi) = E(w) \otimes P(\beta w)$ with generators $\mu \otimes \mu \otimes \mu, (1+\sigma+\sigma^2)(\nu \otimes \mu \otimes \mu), (1+\sigma+\sigma^2)(\nu \otimes \nu \otimes \mu), \nu \otimes \nu \otimes \nu$ all on the line $p = 0$, it collapses at $E^2 = E^\infty$. There can be no extension problems and the result follows.

□

$E\pi = S^\infty$ is filtered by the π equivariant subspaces

$$S^1 \subset S^3 \subset S^5 \subset \ldots.$$

Hence, if M is a π module, the cohomology $H^*(S^1/\pi; M)$ is computed via the resolution

$$Z/3Z[\pi] \xleftarrow{1-\sigma} Z/3Z[\pi].$$

We use the cohomology Serre spectral sequence to compute the cohomology of

$$\Gamma_1(X) = S^1 \times_\pi (X \wedge X \wedge X, *)$$

in the two cases $X = S^{n-1}$ and $X = P^n(Z/3^rZ)$. We have $E^2 = H^*(S^1/\pi; H^*(X \wedge X \wedge X, *)$ with local coefficients and get:

Theorem 23.6. *a)*

$$H^*(\Gamma_1(S^{n-1})) = <1, w> \otimes e \otimes e \otimes e.$$

b)

$$H^*(\Gamma_1(P^n(Z/3Z))) = \begin{cases} <1, w> \otimes \mu \otimes \mu \otimes \mu \quad \oplus \\ Z/3Z \otimes (1+\sigma+\sigma^2)(\nu \otimes \mu \otimes \mu) \quad \oplus \\ w \otimes \nu \otimes \mu \otimes \mu \quad \oplus \\ Z/3Z \otimes (1+\sigma+\sigma^2)(\nu \otimes \nu \otimes \mu) \quad \oplus \\ w \otimes \nu \otimes \mu \otimes \mu \quad \oplus \\ <1, w> \otimes \mu \otimes \mu \otimes \mu \end{cases}$$

with $(\nu \otimes \mu \otimes \mu) = \sigma(\nu \otimes \mu \otimes \mu) = \sigma^2(\nu \otimes \mu \otimes \mu)$ in line 3 above and $(\nu \otimes \nu \otimes \mu) = \sigma(\nu \otimes \mu \otimes \mu) = \sigma^2(\nu \otimes \mu \otimes \mu)$ in line 5 above.

Remark 23.7. In cohomology, the map $\Gamma_1(P^n(Z/3^rZ) \to \Gamma(P^n(Z/3^rZ)$ sends most of the generators to the identically named generators but it sends $w \otimes \nu \otimes \mu \otimes \mu$, $w \otimes \nu \otimes \nu \otimes \mu$ and anything involving βw to zero.

We now compute the mod 3 Steenrod operation $\mathcal{P}^1(1 \otimes \mu \otimes \mu \otimes \mu)$ in the mod 3 cohomologies of $\Gamma(P^n(Z/3Z))$ and $\Gamma_1(P^n(Z/3Z))$.

Theorem 23.8. *a) In $H^*(\Gamma(P^n(Z/3Z))$,*

$$\mathcal{P}^1(1 \otimes \mu \otimes \mu \otimes \mu) = aw \otimes \nu \otimes \nu \otimes \nu + b(\beta w)^2 \otimes \mu \otimes \mu \otimes \mu.$$

where $a \neq 0$.

b) In $H^*(\Gamma_1(P^n(Z/3Z))$,

$$\mathcal{P}^1(1 \otimes \mu \otimes \mu \otimes \mu) = \pm w \otimes \nu \otimes \nu \otimes \nu.$$

Proof. Note that part a) immediately implies part b).

By the definition of the mod 3 Steenrod operations the equivariant diagonal

$$\overline{\delta} : B\pi \times (P^n(Z/3Z), *) \to E\pi \times_\pi (P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z), *)$$

induces the cohomology map

$$\overline{\delta}^* : 1 \otimes \mu \otimes \mu \otimes \mu = \xi(\mu) \mapsto \epsilon_{2n-2}(\beta w)^{n-1} \otimes \mu + \epsilon_{2n-3} w(\beta w)^{n-2} \otimes \nu.$$

The naturality of the Steenrod operation $\mathcal{P}^1$ and $\mathcal{P}^1 w = 0,\ \mathcal{P}^1(\beta w)^k = k(\beta w)^{k+2}$ gives

$$\overline{\delta}^* : \mathcal{P}^1(1 \otimes \mu \otimes \mu \otimes \mu) \mapsto \epsilon_{2n-2}(n-1)(\beta w)^{n+1} \otimes \mu + \epsilon_{2n-3}(n-2) w(\beta w)^{n} \otimes \nu.$$

Hence, $\mathcal{P}^1(1 \otimes \mu \otimes \mu \otimes \mu) \neq 0$ and the only possibility is that

$$\mathcal{P}^1(1 \otimes \mu \otimes \mu \otimes \mu) = aw \otimes \nu \otimes \nu \otimes \nu + b(\beta w)^2 \otimes \mu \otimes \mu \otimes \mu.$$

If $a = 0$, then the fact that $\overline{\delta}^*$ is a map of $H^*(B\pi)$ modules shows that

$$\overline{\delta}^* : \mathcal{P}^1(1 \otimes \mu \otimes \mu \otimes \mu) \mapsto b\epsilon_{2n-2}(\beta w)^{n+1} \otimes \mu + b\epsilon_{2n-3} w(\beta w)^{n} \otimes \nu.$$

Hence, $\epsilon_{2n-2}(n-1) = b\epsilon_{2n-2}, \epsilon_{2n-3}(n-2) = b\epsilon_{2n-3}$ and $b = n-1 = n-2$ which is impossible. Hence, $a \neq 0$.

□

24. Nonassociativity in smashes of mod 3 Moore spaces

In this section we show that the wedge decomposition of the smash product of mod 3 Moore spaces is not associative. Hence, the Jacobi identity may not hold for mod 3 homotopy. Unfortunately, in the original published version of this paper, I made a mistake. I convinced myself that a modification of the same argument would show the nonassociativity of the smash product of mod 3^r Moore spaces for $r \geq 2$. I falsely concluded that the Jacobi identity might also fail for homotopy modulo 3^r with $r \geq 2$. This would have been a very subtle point since it was known that the Jacobi identity was valid in the mod 3 homotopy Bockstein spectral sequence from the E^2 term onwards. But Brayton Gray pointed out the associativity of wedge decomposition of mod 3^r Moore spaces with $r \geq 2$. We include in the next section his argument showing this and thus that the Jacobi identity is indeed valid for homotopy groups with coefficients mod 3^r with $r \geq 2$.

We begin by studying the failure of associativity for the decomposition of mod 3 Moore spaces. In other words, we prove the nonassociativity of the comultiplication map for a smash product of mod 3 Moore spaces. This is the Spanier-Whitehead dual of a fact discovered by Toda. [**29**] This argument appeared in this author's thesis [**20**] but was never published.

Theorem 24.1. *The maps*

$$\Delta_{n,m} : P^{n+m}(Z/3Z) \to P^n(Z/3Z) \wedge P^n(Z/3Z)$$

are not associative, that is, for some values of $n, m, q \geq 2$, *the diagrams*

$$\begin{array}{ccc} P^{n+m+q}(Z/3Z) & \xrightarrow{\Delta_{n,m+q}} & P^n(Z/3Z) \wedge P^{m+q}(Z/3Z) \\ \downarrow \Delta_{n+m,q} & & \downarrow 1 \wedge \Delta_{m,q} \\ P^{n+m}(Z/3Z) \wedge P^q(Z/3Z) & \xrightarrow{\Delta_{n,m}\wedge 1} & P^n(Z/3Z) \wedge P^m(Z/3Z) \wedge P^q(Z/3Z) \end{array}$$

do not commute up to homotopy.

It suffices to treat the case when $n = m = q$.

We reason by contradiction. Suppose that the above diagrams are all homotopy commutative. Since the comultiplication map is homotopy commutative, it would follow that, if $\Delta = (1 \wedge \Delta_{n.n}) \circ \Delta_{n,2n}$, then

$$\begin{array}{ccc} P^{3n}(Z/3Z) & \xrightarrow{\Delta} & P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z) \\ \downarrow 1 & & \downarrow \sigma \\ P^{3n}(Z/3Z) & \xrightarrow{\Delta} & P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z) \end{array}$$

would homotopy commute for any even permutation σ, for example, for the cyclic permutation $\sigma(x \wedge y \wedge z) = z \wedge x \wedge y$.

This follows from the homotopy commutative diagram

$$\begin{array}{ccccc} P^{3n}(Z/3Z) & \xrightarrow{\Delta} & P^n(Z/3Z) \wedge P^{2n}(Z/3Z) & \xrightarrow{1\wedge\Delta} & P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z) \\ \downarrow 1 & & & & \downarrow 1 \\ P^{3n}(Z/3Z) & \xrightarrow{\Delta} & P^{2n}(Z/3Z) \wedge P^n(Z/3Z) & \xrightarrow{\Delta\wedge 1} & P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z) \\ \downarrow 1 & & \downarrow T & & \downarrow \sigma \\ P^{3n}(Z/3Z) & \xrightarrow{\Delta} & P^n(Z/3Z) \wedge P^{2n}(Z/3Z) & \xrightarrow{1\wedge\Delta} & P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z) \end{array}$$

(Note that the lower right hand square is actually strictly commutative.)

Let $\Delta : P^{3n}(Z/3Z) \to P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z)$ be any map which induces an isomorphism in the top dimension $3n$ of mod 3 homology and therefore also in mod 3 cohomology. Suppose that Δ is homotopy equivariant in the sense that Δ is homotopic to the composition $\sigma\Delta$ with the cyclic permutation σ where $\sigma(x \wedge y \wedge z) = z \wedge x \wedge y$. We shall show that this leads to a contradition. This gives

Corollary 24.2. *The coproduct maps $\Delta : P^{n+m}(Z/3Z) \to P^n(Z/3Z) \wedge P^m(Z/3Z)$ are not all associative. Hence, the Jacobi identity may fail for Samelson products in mod 3 homotopy.*

We proceed to derive the contradiction by constructing a π equivariant map

$$G : S^1 \times (P^{3n}(Z/3Z), *) \to S^1 \times (P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z), *)$$

where, if ω is a primitive cube root of unity, $\pi =< 1, \sigma, \sigma^2 >$ acts on $S^1 \times (P^{3n}(Z/3Z), *)$ via $\sigma(\alpha, x) = (\alpha\omega^{-1}, x)$ and acts on $S^1 \times (P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z), *)$ via $\sigma(\alpha, x \wedge y \wedge z) = (\alpha\omega^{-1}, \sigma(x \wedge y \wedge z))$.

Write $S^1 = [1, \omega] \cup [\omega, \omega^2] \cup [\omega^2, 1]$. Suppose

$$F : [1, \omega] \times P^{3n}(Z/3Z) \to P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z)$$

is a homotopy with $F(1, x) = \Delta(x)$ and $F(\omega, x) = \sigma^{-1}\Delta(x)$.

Extend F to an equivariant map

$$F : S^1 \times P^{3n}(Z/3Z) \to P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z)$$

by requiring $F(\alpha, x) = \sigma F(\alpha\sigma^{-1}, x)$.

Now G is defined by $G(\alpha, x) = (\alpha, F(\alpha, x))$.

Theorem 24.3. *The induced map on orbit spaces*

$$\overline{G} : S^1 \times_\pi (P^{3n}(Z/3Z), *) = S^1/\pi \times (P^{3n}(Z/3Z), *) \to$$

$$\Gamma_1(P^n(Z/3Z)) = S^1 \times_\pi (P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z), *)$$

induces an isomorphism in the top dimension $3n+1$ *of mod 3 cohomology, that is,*

$$\overline{\Delta}^*(w \otimes \nu \otimes \nu \otimes \nu) = w \otimes \nu_{3n}$$

.

PROOF. Since the action of π on $P^{3n}(Z/3Z)$ is trivial, the Künneth theorem shows that $H^*(S^1 \times_\pi (P^{3n}(Z/3Z), *) = <1, w> \otimes <\mu_{3n-1}, \nu_{3n}>$.

The computation now follows from the fact that

$$\overline{\Delta} : S^1 \times_\pi (P^{3n}(Z/3Z), *) \to \Gamma_1(P^n(Z/3Z))$$

is a bundle map which covers the identity on the base spaces $S^1 = S^1/\pi \to S^1/\pi$ and is $F(\alpha, \quad) : P^{3n}(Z/3Z) \to P^n(Z/3Z) \wedge P^n(Z/3Z) \wedge P^n(Z/3Z)$ on the fibre over $[\alpha]$. Of course, all $F(\alpha, \quad)$ are homotopic to Δ and induce a cohomology isomorphism in dimension $3n$. □

25. Associativity in smashes of 3 primary Moore spaces

Let $\Delta : P^{n+m}(Z/3^rZ) \to P^n(Z/3^rZ) \wedge P^m(Z/3^rZ)$ be the coproduct. In this section we prove it is associative if $r \geq 2$ and $n, m \geq 3$. That is, abbreviate $P^n(Z/3^rZ) = P^n(3^r)$ and consider the two maps

$$(\Delta \wedge 1)\Delta : P^{n+m+q}(3^r) \to P^{n+m}(3^r) \wedge P^q(3^r) \to P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r)$$

$$(1 \wedge \Delta)\Delta : P^{n+m+q}(3^r) \to P^n(3^r) \wedge P^{m+q}(3^r) \to P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r).$$

THEOREM 25.1. *If* $m, n, q \geq 3$ *and* $r \geq 2$ *the above maps are homotopic.*

PROOF. Recall the coreduction maps $\overline{\rho} : P^n(3) \to P^n(3^r)$ which are characterized by the commutativity of

$$\begin{array}{ccccccc} S^{n-1} & \xrightarrow{3} & S^{n-1} & \xrightarrow{\iota} & P^n(3) & \xrightarrow{q} & S^n \\ \downarrow = & & \downarrow 3^{r-1} & & \downarrow \overline{\rho} & & \downarrow = \\ S^{n-1} & \xrightarrow{3^r} & S^{n-1} & \xrightarrow{\iota} & P^n(3^r) & \xrightarrow{q} & S^n \end{array}$$

Recall the diagrams

$$\begin{array}{ccc} P^{n+m}(3) & \xrightarrow{\Delta} & P^n(3) \wedge P^m(3) \\ \downarrow \overline{\rho} & & \downarrow \overline{\rho} \wedge \overline{\rho} \\ P^{n+m}(3^r) & \xrightarrow{\Delta} & P^n(3^r) \wedge P^m(3^r) \end{array}$$

These diagrams are homotopy commutative modulo the addition of Whitehead products. Since $m, n \geq 3$, there can be no relavant Whitehead products and they are in fact homotopy commutative. Hence, if $D = (\Delta \wedge 1)\Delta - (1 \wedge \Delta)\Delta$, the diagram

$$\begin{array}{ccc} P^{n+m+q}(3) & \xrightarrow{D} & P^n(3) \wedge P^m(3) \wedge P^q(3) \\ \downarrow \overline{\rho} & & \downarrow \overline{\rho} \wedge \overline{\rho} \wedge \overline{\rho} \\ P^{n+m+q}(3^r) & \xrightarrow{D} & P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r) \end{array}$$

is homotopy commutative.

Now suppose that

$$F : P^{n+m+q}(3^s) \to P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r)$$

is any map which induces zero in mod 3^r cohomology. Recall that $P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r) \simeq P^{n+m+q}(3^r) \vee P^{n+m+q-1}(3^r) \vee P^{n+m+q-1}(3^r) \vee P^{n+m+q-2}(3^r)$.

Lemma 25.2. *a) F factors as*

$$P^{n+m+q}(3^s) \xrightarrow{\overline{F}} P^{n+m+q-2}(3^r) \xrightarrow{\iota} P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r)$$

where ι is the inclusion of the wedge summand.

b) Any such map $\overline{F}$ factors as

$$P^{n+m+q}(3^s) \xrightarrow{q} S^{n+m+q} \xrightarrow{\gamma} S^{n+m+q-3} \xrightarrow{\iota} P^{n+m+q-2}(3^r)$$

where q is the pinch map and ι is the inclusion of the bottom cell.

Proof. a) The Hilton-Milnor theorem shows that the adjoint of F factors through

$$\Omega P^{n+m+q}(3^r) \times \Omega P^{n+m+q-1}(3^r) \times \Omega P^{n+m+q-1}(3^r) \times \Omega P^{n+m+q-2}(3^r).$$

The fact that F is zero in mod 3^r cohomology implies that F factors through $P^{n+m+q-2}(3^r)$ since F cannot map nontrivially to any of the other summands.

b) It is elementary that $\overline{F}$ is null when composed with the pinch map $q : P^{n+m+q-2}(3^r) \to S^{n+m+q_2}$. Hence, it factors through the fibre G of this map. But in the relevant range of dimensions, G is homotopy equivalent to the bottom cell $S^{n+m+q-3} \subset P^{n+m+q-2}(3^r)$. Hence, $\overline{F}$ factors as

$$P^{n+m+q}(3^s) \to S^{n+m+q-3} \xrightarrow{\iota} P^{n+m+q_2}(3^r).$$

Since all maps $S^{n+m+q-1} \to S^{n+m+q-3}$ are null at the prime 3, we get the required factorization through the pinch map.

□

Use the above lemma to factor $D : P^{n+m+q}(3) \to P^n(3) \wedge P^m(3) \wedge P^q(3)$ and consider the homotopy commutative diagram

$$\begin{array}{ccccccc} P^{n+m+q}(3) & \xrightarrow{q} & S^{n+m+q} & \xrightarrow{\gamma} & S^{n-1} \wedge S^{m-1} \wedge S^{q-1} & \xrightarrow{\iota} & P^n(3) \wedge P^m(3) \wedge P^q(3) \\ & & & & \downarrow 3^{r-1} \wedge 3^{r-1} \wedge 3^{r-1} & & \downarrow \overline{\rho} \wedge \overline{\rho} \wedge \overline{\rho} \\ & & & & S^{n-1} \wedge S^{m-1} \wedge S^{q-1} & \xrightarrow{\iota} & P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r) \end{array}$$

No matter what the parity of $n+m+q-3$, it follows that the compositions

$$(\overline{\rho} \wedge \overline{\rho} \wedge \overline{\rho}) \cdot D = D \cdot \overline{\rho} : P^{n+m+q}(3) \to P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r)$$

are null.

Since D factors through a wedge summand as

$$P^{n+m+q}(3^r) \xrightarrow{\overline{D}} P^{n+m+q-2}(3^r) \subset P^n(3^r) \wedge P^m(3^r) \wedge P^q(3^r),$$

it follows that the composition $\overline{D} \cdot \overline{\rho} : P^{n+m+q}(3) \to P^{n+m+q}(3^r) \to P^{n+m+q-2}(3^r)$ is null and $\overline{D}$ factors through the cofibre $\overline{\eta}$, that is,

$$\overline{D} = E \cdot \overline{\eta} : P^{n+m+q}(3^r) \to P^{n+m+q}(3^{r-1}) \to P^{n+m+q-2}(3^r).$$

Since $\overline{\eta}$ is characterized by the commutative diagram

$$\begin{array}{ccccccc} S^{n-1} & \xrightarrow{3^r} & S^{n-1} & \xrightarrow{\iota} & P^n(3^r) & \xrightarrow{q} & S^n \\ \downarrow 3 & & \downarrow = & & \downarrow \overline{\eta} & & \downarrow 3 \\ S^{n-1} & \xrightarrow{3^{r-1}} & S^{n-1} & \xrightarrow{\iota} & P^n(3^{r-1}) & \xrightarrow{q} & S^n \end{array}$$

and since E factors as

$$P^{n+m+q}(3^{r-1}) \xrightarrow{q} S^{n+m+q} \xrightarrow{\gamma} S^{n+m+q-3} \xrightarrow{\iota} P^{n+m+q-2}(3^r)$$

it follows that $\overline{D} = E \cdot \overline{\eta} = \iota \cdot \gamma \cdot 3 \cdot q$ and D are null.

□

It follows that

THEOREM 25.3. *If $r \geq 2$ the Jacobi identity holds for Samelson products of classes of dimensions ≥ 3 in the mod 3^r homotopy groups of a group-like space.*

Since the reduction map $\rho : \pi_*(G; Z/p^r Z) \to \pi_*(G; Z/pZ)$ induces a surjective morphism of Lie structures $\pi_*(G; Z/p^r Z) \to E^r$ onto the mod p homotopy Bockstein spectral sequence of a group-like space G, this gives an alternate proof of

COROLLARY 25.4. *In the mod 3 homotopy Bockstein spectral sequence of a group-like space, the Jacobi identity is valid from E^2 onwards for classes of dimensions ≥ 3.*

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