MATH 437: Homework I.
Due in class on Monday, Jan 26

1. **[Commutative Artinian Rings.]** Let A be a commutative ring with $1 \neq 0$. Recall the product of two ideals I and J is denoted IJ and equals the ideal of elements of the form $\sum_{i=1}^{n} \alpha_i \beta_i$ where $\alpha_i \in I$ and $\beta_i \in J$ for all $1 \leq i \leq n$.
(a) We have defined a prime ideal P as an ideal that has the property that whenever $x, y \in A$ has $xy \in P$ then either $x \in P$ or $y \in P$. Show that if I and J are ideals of A such that $IJ \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$.
(b) Suppose the zero ideal of A is a finite product of maximal ideals, i.e.,

$$0 = M_1 M_2 \dots M_n.$$

Show then that every prime ideal of A is actually maximal and in fact that $Spec(A) = MaxSpec(A) = \{M_1, \dots, M_n\}$, i.e., that the only maximal ideals of A are $\{M_1, \dots, M_n\}$.
(c) Fill in the details of the sketch of a proof below that in a commutative Artinian ring A, the zero ideal is a finite product of maximal ideals. From this fact we can conclude that in a commutative Artinian ring, every prime ideal is maximal and there are only a finite number of maximal ideals.
(i) Let $S = \{$ Ideals which are finite products of maximal ideals$\}$. Argue $S \neq \emptyset$ and so $\exists J$ a minimal element of S by the Artinian property. Thus $J = M_1 \dots M_n$ where each M_i is a maximal ideal (not necessarily distinct). Suppose $J \neq 0$ and proceed to get a contradiction in the next steps.
(ii) Argue that if M is a maximal ideal, $JM = J$ and so J is contained in every maximal ideal of A. Argue that $1 - \alpha$ is a unit whenever $\alpha \in J$.
(iii) Argue $J^2 = J$ and that $T = \{$ Ideals I with $IJ \neq 0\}$ is nonempty. By the Artinian property we may find an ideal I which is a minimal ideal such that $IJ \neq 0$. Argue that $IJ = I$ and that I is principal, say $I = (f)$. Conclude that there exists $\alpha \in J$ such that $f\alpha = f$ and use this to show that $I = IJ = 0$ a contradiction to $IJ \neq 0$. This completes the proof.

2. **[Annihilators]**
(a) If M is a left R-module and $m \in M$, we define

$$Ann_R(M, m) = \{r \in R | rm = 0\}.$$

Check that $Ann_R(M, m)$ is a left ideal of R. We call $Ann_R(M, m)$ the annihilator of m.

(b) A cyclic left R-module (sometimes called a principal R-module) is one that is generated by a single element α, i.e., of the form $\{r\alpha | r \in R\}$. Given a left R-module M and $\alpha \in M$ we sometimes denote the cyclic submodule generated by α as $< \alpha >$. Show that $< \alpha >$ is isomorphic to $R/Ann_R(M, \alpha)$ as left R-modules.
(c) Let $V = \mathbb{R}^n$ be given its canonical left $\mathbb{R}$ and $Mat_n(\mathbb{R})$-module structures. Show that V is a cyclic left $Mat_n(\mathbb{R})$-module generated by $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Find $Ann_{Mat_n(\mathbb{R})}(V, e_1)$ and $Ann_{\mathbb{R}}(V, e_1)$.
(d) Given an R-module N we define

$$Ann_R(N) = \{r \in R | rn = 0 \text{ for all } n \in N\}.$$

Show that $Ann_R(N)$ is a two-sided ideal of R. Calculate $Ann_{Mat_n(\mathbb{R})}(\mathbb{R}^n)$ and $Ann_{\mathbb{R}}(\mathbb{R}^n)$.
(e) Let I be a left ideal of R. Show that

$$Ann_R(R/I) = \{r \in R | rs \in I \text{ for all } s \in R\}.$$

Show that $Ann_R(R/I) \subseteq I$ with equality if R is a commutative ring. However if R is noncommutative show that equality need not hold.
(f) We say that an R-module M is a faithful R-module if $Ann_R(M) = 0$. Suppose N is a R-module. Show that N is also a (well-defined) faithful $R/Ann_R(N)$-module via

$$(r + Ann_R(N)) \cdot n = rn$$

for all $r \in R, n \in N$.

3. **[Simple modules and the Jacobson radical]**
(a) Let R be a ring (not necessarily commutative). A left R-module S is called **simple** if $S \neq 0$ and the only R-submodules of S are 0 and S itself. Find all the simple $\mathbb{Z}$-modules and simple $\mathbb{R}$-modules, where $\mathbb{R}$ is the field of real numbers. Is $\mathbb{R}^n$ a simple $Mat_n(\mathbb{R})$-module under its canonical left $Mat_n(\mathbb{R})$-module structure?
(b) Show that any simple left R-module is cyclic and that there is an isomorphism of R-modules

$$S \cong R/Ann_R(S, \alpha)$$

for any nonzero $\alpha \in S$. Show that $Ann_R(S, \alpha)$ is a maximal left ideal of R.
(c) Show that if M is a maximal left ideal of R then R/M is a simple left R-module.
(d) We define the left Jacobson radical $J_L(R)$ as the intersection of all the left maximal ideals of R, i.e.,

$$J_L(R) = \bigcap_{M \text{ a maximal left ideal of } R} M.$$

Show that $J_L(R)$ is also equal to the intersection of the annihilators of all simple left R-modules, i.e.,

$$J_L(R) = \bigcap_{S \text{ a simple left } R-\text{module}} Ann_R(S).$$

Conclude that $J_L(R)$ is actually a two-sided ideal of R.
(d) Show that $\alpha \in J_L(R)$ if and only if $1 - r\alpha$ has a left inverse in R for all $r \in R$.
(e) Let $\alpha \in J_L(R)$ and $r \in R$, then by (d), $1 - r\alpha$ has a left inverse which we may write as $1 - t_r$ for some $t_r \in R$. Show that t_r is in $J_L(R)$ and use this to conclude that $1 - t_r$ and $1 - r\alpha$ are units of R. Use this to show that $J_L(R) = \{\alpha \in R | 1 - r\alpha \text{ is a unit of } R \text{ for all } r \in R\}$. Finally use that you know that $J_L(R)$ is a two-sided ideal to show that

$$J_L(R) = \{\alpha \in R | 1 - r\alpha s \text{ is a unit of } R \text{ for all } r, s \in R\}$$

(f) Using right R-modules instead of left R-modules in the whole discussion above, one can define the right Jacobson radical $J_R(R)$ as the intersection of all the maximal right ideals of R. Analogously one can show that it is equal to the intersection of all the annihilators of simple right R-modules and that $\alpha \in J_R(R)$ if and only if $1 - \alpha r$ has a right inverse in R for all $r \in R$.

Finally one can show as in (e) that

$$J_R(R) = \{\alpha \in R | 1 - s\alpha r \text{ is a unit of } R \text{ for all } r, s \in R\}.$$

Thus $J_L(R) = J_R(R)$!! So, perhaps surprisingly, the intersection of all the maximal left ideals is always equal to the intersection of all the maximal right ideals in any (noncommutative) ring R. Thus we define the Jacobson radical of R, as $J(R) = J_L(R) = J_R(R)$. It is always a two-sided ideal of R and we will see its importance as we progress in our studies.

Find $J(\mathbb{Z})$, $J(\mathbb{Z}/n\mathbb{Z})$, $J(k[x])$, $J(k[[x]])$ and $J(Mat_n(\mathbb{R}))$ where $n \in \mathbb{N}$ and k is a field.
(g) Show that if S is a simple R-module, then S is naturally a simple $R/J(R)$-module.
(h) If A is a commutative Artinian ring, show that $A/J(A)$ is isomorphic to a finite direct product of fields.