MATH 437: Homework II.
Due in class on Monday, Feb 2

1. **[Endomorphisms of modules.]**
Let R be any ring. A R-module M is called Noetherian (resp. Artinian) if it has the ascending (resp. descending) chain condition on submodules. This means that any nonempty set of submodules contains a maximal (resp. minimal) element. This condition is equivalent to the condition that any ascending (resp. descending) chain of submodules has to stabilize.
(a) If M is a Noetherian R-module and $f : M \to M$ is an epimorphism then show f is an isomorphism. (Hint: Consider $ker(f^n)$ where f^n is the n-fold composition of f with itself.)
(b) If N is an Artinian R-module and $g : N \to N$ is a monomorphism, then show g is an isomorphism.
(c)**[Schur's Lemma]** If S is a simple R-module then show that the endomorphism ring $End_R(S) = Hom_R(S, S)$ is a division ring.

2. **[The Nil-radical]**
Let A be a commutative ring with $1 \neq 0$ thruout this question.
(a) We define the nil-radical $Nil(A)$ of A as the intersection of all the prime ideals of A. Show that $Nil(A)$ is an ideal of A with $Nil(A) \subseteq J(A)$ where $J(A)$ is the Jacobson radical of A. Moreover show that these two radicals are distinct for the power series ring $k[[x]]$ where k is a field.
(b) An element x in a ring is called nilpotent if $x^n = 0$ for some positive integer n. Show that any nilpotent element $x \in A$ must lie in $Nil(A)$.
(c) If S is a submonoid of $(A, \cdot)$ that does not contain 0 then show that $S^{-1}A$ is not the zero ring and that there exists a prime ideal P of A such that $P \cap S = \emptyset$. (Hint: Recall the prime ideal correspondence between A and $S^{-1}A$.)
(d) Show that $Nil(A) = \{x \in A | x \text{ is nilpotent}\}$. (Hint: If x is not nilpotent, $S = \{x^n | n \in \mathbb{N}\}$ is a multiplicative set not containing zero.)
(e) An ideal I is called nilpotent if $I^n = 0$. Check that in a nilpotent ideal, every element is nilpotent. If A is an Artinian ring show that $J(A)$ is a nilpotent ideal where $J(A)$ is the Jacobson radical of A.
(f) Find an example of an ideal where every element is nilpotent but where the ideal is not a nilpotent ideal. (Hint: Consider an infinite direct product of rings.)
(g) If R is a non-commutative ring show that the sum of two nilpotent elements need not be nilpotent in general.

3. **[Inversion for modules]**
Let R be a commutative ring, S a submonoid of $(R, \cdot)$ and M be a R-module. We define $S^{-1}M = \{[\frac{m}{s}] | m \in M, s \in S\}$ where we consider two fractions to be equal, i.e., $[\frac{m}{s}] = [\frac{m'}{s'}]$ whenever there exists a $t \in S$ such that $ts'm = tsm'$.

The reader may check that $S^{-1}M$ is a well-defined $S^{-1}R$-module under the operations

$$[\frac{m}{s}] + [\frac{m'}{s'}] = [\frac{s'm + sm'}{ss'}]$$

and

$$[\frac{r}{s}][\frac{m}{s'}] = [\frac{rm}{ss'}]$$

for all $m, m' \in M$, $s, s' \in S$, $r \in R$.

Now given $f : A \to B$ a R-module homomorphism, we define $S^{-1}f : S^{-1}A \to S^{-1}B$ via

$$(S^{-1}f)([\frac{m}{s}]) = [\frac{f(m)}{s}]$$

for all $m \in M, s \in S$. One may check that $S^{-1}f$ is a $S^{-1}R$-module homomorphism.

The correspondence S^{-1} is then easily checked to define a covariant functor from the category of R-modules to the category of $S^{-1}R$-modules.

(a) Show that S^{-1} is an exact functor. In other words for every short exact sequence of R-modules

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

show that

$$0 \to S^{-1}A \xrightarrow{S^{-1}f} S^{-1}B \xrightarrow{S^{-1}g} S^{-1}C \to 0$$

is a short exact sequence of $S^{-1}R$-modules.

(b) Now let $R = \mathbb{Z}$ the ring of integers and let $S = \mathbb{Z} - 0$ so that $S^{-1}R = \mathbb{Q}$ the field of rational numbers. Show that if A is a torsion Abelian group (i.e., $\mathbb{Z}$-module), then $S^{-1}A$ is the zero $\mathbb{Q}$-vector space.

(c) Show that if $f : A \to B$ is a homomorphism of $\mathbb{Z}$-modules such that $ker(f)$ and $coker(f) = B/Im(f)$ are torsion Abelian groups, then

$$S^{-1}f : S^{-1}A \to S^{-1}B$$

is an isomorphism of $\mathbb{Q}$-vector spaces.

4. Let R be any ring and let $J = J(R)$ be the Jacobson radical of R. Note that if I is a 2-sided ideal of R and N is a (left) R-module then $IN = \{\sum_{k=1}^{n} i_k n_k | i_k \in I, n_k \in N\}$ is a R-submodule of N.

(a)[**Nakayama's lemma**]

If M is a finitely generated (left) R-module then show that $JM = M$ implies $M = 0$.

(Hint: If $M \neq 0$ then take a minimal size generating set $\{m_1, \dots, m_k\}$ of M (note $k \geq 1$) and try to write m_1 as a R-combination of the others.)

(b) Let R be a commutative Noetherian ring. Show that

$$I = \cap_{n=1}^{\infty} J^n$$

is always an 2-sided ideal with the property that $JI = I$.

(Hint: See Hungerford, Pg 393, Ex 1 and Ex 2).

(c)[**Krull Intersection Theorem for Noetherian rings**]

Let R be a commutative Noetherian ring. Show that

$$\cap_{n=1}^{\infty} J^n = 0.$$