MATH 436: Homework III.
Due in class on Friday, Oct 3

1. Let G be any group. Show that the subgroup of inner automorphisms, $Inn(G)$ is normal in $Aut(G)$.

2. [**Automorphisms of** $\mathbb{Z}/d\mathbb{Z}$]. For $\bar{j} \in \mathbb{Z}/d\mathbb{Z}$, define $f_{\bar{j}} : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ via $f_{\bar{j}}(\overline{n}) = \overline{jn}$ for all $\overline{n} \in \mathbb{Z}/d\mathbb{Z}$. Here $\overline{n}$ denotes the class of the integer n in $\mathbb{Z}/d\mathbb{Z}$.
(a) Check that $f_{\bar{j}}$ is a well-defined endomorphism of $(\mathbb{Z}/d\mathbb{Z}, +)$ and that $\theta : \mathbb{Z}/d\mathbb{Z} \to End((\mathbb{Z}/d\mathbb{Z}, +))$ defined by $\theta(\bar{j}) = f_{\bar{j}}$ induces an isomorphism of the monoid $End((\mathbb{Z}/d\mathbb{Z}, +))$ with $(\mathbb{Z}/d\mathbb{Z}, \cdot)$ i.e., $\mathbb{Z}/d\mathbb{Z}$ under multiplication.
(b) Show that $Aut((\mathbb{Z}/d\mathbb{Z}, +)) = \{f_{\bar{m}} | 1 \leq m < d, gcd(d, m) = 1\}$ and that $Aut((\mathbb{Z}/d\mathbb{Z}, +)) \cong (\mathbb{Z}/d\mathbb{Z})^*$. Thus $|Aut((\mathbb{Z}/d\mathbb{Z}, +))| = \phi(d)$ where ϕ is Euler's Phi function.

3. **Torsion in groups.** In any group G, an element x is called a torsion element if it has finite order, i.e., $x^k = e$ for some integer $k \geq 1$. A group where all elements have finite order is called a torsion group. A group where all non-identity elements have infinite order is called torsion-free. Let $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote respectively the groups of integers, rationals and real numbers under addition.
(a) Show that $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ are torsion-free groups.
(b) Show that $\mathbb{Q}/\mathbb{Z}$ is an infinite torsion group.
(c) Let G be a group and let x, y be torsion elements with $o(x) = m$ and $o(y) = n$. Show that if $xy = yx$ then $o(xy)$ divides $lcm(m, n)$ where lcm stands for least common multiple. If $gcd(m, n) = 1$ show that $o(xy) = mn$. Conclude that in an Abelian group A, the set of torsion elements A_{tor} forms a normal subgroup. Show that A/A_{tor} is torsion-free.
(d) Consider the matrices $\mathbb{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathbb{B} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Show that they are in the special linear group $SL_2(\mathbb{R})$. Find the orders of $\mathbb{A}, \mathbb{B}$ and $\mathbb{AB}$. Do the results in (c) apply here? Do the set of torsion elements in $SL_2(\mathbb{R})$ form a subgroup of $SL_2(\mathbb{R})$?
(e) Let $S^1 = \{z \in \mathbb{C} || z| = 1\}$ be the set of complex numbers of unit length. Show that S^1 is a group under multiplication and construct an isomorphism between it and $(\mathbb{R}/\mathbb{Z}, +)$. S^1 is called the unit circle group.

4. **Order of elements in** Σ_n**.** For $\sigma \in \Sigma_n$ we define the support of σ, denoted $Supp(\sigma)$ by $Supp(\sigma) = \{j | 1 \leq j \leq n, \sigma(j) \neq j\}$. Thus on

$\{1, \ldots, n\} - Supp(\sigma)$, σ acts like the identity map.
(a) Show that if $Supp(\sigma) \cap Supp(\mu) = \emptyset$ then $\sigma \circ \mu = \mu \circ \sigma$ in Σ_n.
(b) $\sigma \in \Sigma_n$ is called a cycle if in the cycle notation it is represented as a single cycle $(a_1, \ldots, a_k)$. For example (12) and (134) are cycles in Σ_4 whereas $(12)(34)$ is not. Show that if $\sigma = (a_1, \ldots, a_k)$ then σ has order k in Σ_n.
(c) Now take $\mu \in \Sigma_n$, we have seen that we may write μ as a product of disjoint cycles. What is the order of $(12)(34)(567) \in \Sigma_7$? What is the order of a general element $\mu \in \Sigma_n$ if μ can be written as a product of disjoint cycles of (not necessarily distinct) lengths $m_1, \ldots, m_s$.

5. **Minimal sets of generators.** Let $\{s_1, \ldots, s_d\}$ be a minimal set of generators for G. ("Minimal" means no smaller set generates G. By convention we say that the emptyset generates the trivial group $\{e\}$.) Consider the set

$$T = \{s_1^{\delta_1} \ldots s_d^{\delta_d} | \delta_j = 0, 1 \text{ for all } 1 \leq j \leq d\}.$$

Show that $|T| = 2^d$ (Be careful here! You need to show there are no repeats in the list of elements in the definition of T.) and conclude that $d \leq log_2(|G|)$. Here log_2 is defined via $2^{log_2(a)} = a$ for all real $a > 0$. Thus any group G can be generated by less than or equal to $log_2(|G|)$ generators.

6. **Conjugation and normalcy in Σ_n.**
(a) Show that Σ_n is generated by the $n-1$ transpositions

$$\{(12), (13), (14), \ldots, (1n)\}.$$

[Hint: Compute $(1i)(1j)(1i)$. Also recall that Lang proves that the set of all transpositions generates Σ_n on Page 13.]
(b) Show that Σ_n is generated by the $n-1$ transpositions

$$\{(12), (23), (34), (45), \ldots, (n-1n)\}.$$

[Hint: Compute $(1, j-1)(j-1, j)(1, j-1)$.]

7. **Some generating sets for Σ_n.**
(a) Show that if $\sigma = (i_1, i_2, \ldots, i_k)$ is a cycle in Σ_n and $\tau \in \Sigma_n$ then $\tau\sigma\tau^{-1}$ is a cycle of the form $(\tau(i_1), \ldots, \tau(i_k))$.
(b) Let $\alpha = (12)(346)$ and $\tau = (354)(21)$ in Σ_6. Explain how to use (a) to calculate $\tau\alpha\tau^{-1}$.
(c) Let $K = \{e, (12)(34), (13)(24), (14)(23)\} \subset \Sigma_4$. Show that K is a normal

subgroup of Σ_4 and that K is Abelian but not cyclic. K is called the Klein four subgroup of Σ_4. Show that $N = \{e, (12)(34)\}$ is a normal subgroup of K but not a normal subgroup of Σ_4. Thus N is normal in K, K is normal in Σ_4 but N is not normal in Σ_4. Hence the relation of being a normal subgroup is not transitive.

8. **Metrics and Volumes for Groups.** Recall a metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that
(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(a) Let G be a group with generating set S. We define

$$d_S(x, y) = min\{k | k \in \mathbb{N}, x s_1^{\pm 1} \dots s_k^{\pm 1} = y, s_j \in S\}.$$

Show that d_S is a metric on G. It is called the Cayley metric on G with respect to the generating set S.
(b) If G is a finitely generated group with two different finite generating sets S and T, show that d_S and d_T are equivalent, i.e., there are real numbers $c, C > 0$ such that $c d_T(x, y) \leq d_S(x, y) \leq C d_T(x, y)$ for all $x, y \in G$. Thus Cayley metrics coming from different finite generation sets for the same finitely generated group are equivalent metrics.
(c) Let G be a group with finite generating set S. We define the volume function $V_S : \mathbb{N} \to \mathbb{N}$ by $V_S(n) = |\{g \in G | d_S(g, e) \leq n\}|$. Thus $V_S(n)$ counts the number of elements whose distance from e does not exceed n in the Cayley metric. Show that $V_S(n) \leq (2|S| + 1)^n$ for all $n \in \mathbb{N}$.
(d) Let $\mathbb{Z} \times \mathbb{Z} = \{(a, b) | a, b \in \mathbb{Z}\}$. This is a subgroup of $\mathbb{R}^2$ under addition. It is easy to check that $S = \{(0, 1), (1, 0)\}$ is a generating set for $\mathbb{Z} \times \mathbb{Z}$. Show that $d_S((m_1, m_2), (n_1, n_2)) = |m_1 - n_1| + |m_2 - n_2|$ and $V_S(n) = n^2 + (n + 1)^2$ in this example.