MATH 437: Homework IV.
Due in class on Monday, Feb 23

1. **[List of groups with trivial automorphism group.]**
(a) Let p be a prime and let A be an elementary Abelian p-group, i.e., an Abelian group such that $pa = 0$ for all $a \in A$. Show that A is isomorphic to a direct sum $\oplus_{i \in I} \mathbb{Z}/p\mathbb{Z}$ even in the case that A is not finitely generated. (Hint: $\mathbb{F}_p$ is a field.)
(b) Find all groups G (including potentially infinite or nonAbelian ones) with $Aut(G) = \{Id\}$ i.e., whose automorphism group is trivial. (Hint: Reduce to the elementary Abelian 2-group case.)

2. **[The Eilenberg swindle.]**
Let R be a ring and let P be a projective R-module. Show that there is a free module F such that $P \oplus F \cong F$ as R-modules. (Hint: Take Q such that $Q \oplus P = T$ is a free R-module, then consider the infinite direct sum $\oplus_{n \in \mathbb{N}} T$.)

3. **[Jacobson Density and Artin-Wedderburn Theorems.]**
Let R be a ring and S a simple R-module. Recall that by Schur's Lemma, $D = Hom_R(S, S)$ is a division ring. Note that S is also a D-module under $L \cdot s = L(s)$ for all $L \in D, s \in S$. Since D is a division ring, S will have a D-basis and $dim_D(S)$ is well-defined.
(a) For $r \in R$, show that $\alpha_r : S \to S$ is a D-module homomorphism where α_r is defined as $\alpha_r(s) = rs$. Show that $\alpha : R \to Hom_D(S, S)$ given by $\alpha(r) = \alpha_r$ is a ring homomorphism with kernel $Ann_R(S)$.
Conclude that if $n = dim_D(S) < \infty$ then $R/Ann_R(S)$ is isomorphic to a subring of $Mat_n(D^{op})$.
(b) In general it is interesting to ask how big $Im(\alpha)$ is in $Hom_D(S, S)$. The Jacobson density theorem (See Hungerford or Lang for a proof) answers this question as follows:

Theorem 0.1 (Jacobson density theorem). *R a ring, S a simple R-module. For arbitrary $\theta \in Hom_D(S, S)$ and a finite dimensional D-subspace V of S, there exists $r \in R$ such that $\theta|_V = \alpha_r|_V$.*

Explain how the Jacobson density theorem implies that if $dim_D(S) < \infty$ that $R/Ann_R(S) \cong Mat_n(D^{op})$ as rings.
(c) Use the Jacobson density theorem to show that if R is (left) Artinian then $dim_D(S) < \infty$ and hence $R/Ann_R(S)$ is isomorphic to $Mat_n(D^{op})$ where

$n = dim_D(S)$.
(Hint: If $dim_D(S) = \infty$, take a D-independent set $\{e_n | n \in \mathbb{N}\}$ and consider $I_n = \{r \in R | re_i = 0 \text{ for } 1 \leq i \leq n\}$.)
(d) Using that $Mat_n(D)$ is a simple ring when D is a division ring, show that $Ann_R(S)$ is a maximal 2-sided ideal of R when R is Artinian and S is a simple R-module.
(e) [**Artin-Wedderburn Theorem**]
Show that for an Artinian ring R, the Jacobson radical $J(R) = \cap_{i=1}^{N} Ann_R(S_i)$ is a **finite** intersection of annihilators of simple R-modules S_i. WLOG we may assume $Ann_R(S_i) \neq Ann_R(S_j)$ in this intersection. Use this to show that $R/J(R) \cong \prod_{i=1}^{N} Mat_{n_i}(D_i)$ for some division rings D_i and positive integers n_i.

4. [**Semisimple rings.**]
Recall that a ring R is semisimple if every R-module is projective. This is equivalent to every R-module being injective which is also equivalent to all short exact sequences of R-modules being split.
Thruout this exercise R will be a semisimple ring with $1 \neq 0$.
(a) Let $M \neq 0$ be a Noetherian and Artinian R-module. Show that M decomposes as a finite direct sum of simple R-modules:

$$M \cong S_1 \oplus \cdots \oplus S_n.$$

(Hint: Since M is Artinian, there exists a minimal nonzero R-submodule S_1 of M. Consider the short exact sequence $0 \to S_1 \to M \to M/S_1 \to 0$.)
(b) Let $M \neq 0$ be a Noetherian and Artinian R-module. By (a), M decomposes into a finite direct sum of simple R-modules $M \cong S_1 \oplus \cdots \oplus S_n$. Suppose M also decomposes as $M \cong T_1 \oplus \cdots \oplus T_k$ where the T_i are also simple R-modules. This yields an isomorphism:

$$\theta : S_1 \oplus \cdots \oplus S_n \to T_1 \oplus \cdots \oplus T_k.$$

If S_i is identified as a submodule of the direct sum in the usual way, then show that $\pi_j \circ \theta(S_i) = T_j$ for some index $1 \leq j \leq k$. (Here $\pi_j : T_1 \oplus \cdots \oplus T_k \to T_j$ is the canonical projection.) Conclude that for this index j, $S_i \cong T_j$. Also show that $\pi_m \circ \theta(S_i) = 0$ if $S_i \ncong T_m$.

Thus so far you have shown that the set of isomorphism types of simple modules in the S-decomposition of M is the same as the set of isomorphism types of simple modules in the T-decomposition of M. However at this stage, it is possible that a simple module L occurs with different multiplicities in the two decompositions.

Thus we are reduced to consider an isomorphism:

$$\theta : L_1^{m_1} \oplus \cdots \oplus L_k^{m_k} \to L_1^{n_1} \oplus \cdots \oplus L_k^{n_k}$$

where the L_i are simple R-modules with $L_i \ncong L_j$ if $i \neq j$ and the m_i and n_i are positive integers. Here L^m denotes the m-fold direct sum of L with itself.

Show that θ induces an isomorphism between $L_i^{m_i}$ and $L_i^{n_i}$ for all $1 \leq i \leq k$. In (c) we will use this to show $m_i = n_i$.

(c) Suppose R is a ring and S is a simple R-module M. Show that if $\theta : S^m \to S^n$ is an isomorphism for $1 \leq m \leq n$ then $\lambda : S^n \to S^n$ given by $\lambda(s_1, \ldots, s_n) = \theta(s_1, \ldots, s_m)$ is a surjective R-endomorphism of S^n. Show that S^n is a Noetherian module and hence conclude that $ker(\lambda) = 0$ and use this to show $m = n$.

Thus you have proven that if R is a semisimple ring and $M \neq 0$ is a Noetherian and Artinian R-module then M decomposes **uniquely** into a finite direct product of simple R-modules in the sense that in any other decomposition, the factors are isomorphic after a permutation of factors.

(d) Suppose R is a semisimple Artinian ring. Then by a theorem of Hopkins, every Artinian ring is also Noetherian so R is also a Noetherian ring. Thus $R \cong S_1 \oplus \cdots \oplus S_n$ as left R-modules where the S_i are simple R-modules. Use this to show that $J(R) = 0$ and hence that R is isomorphic to a finite direct product of matrix rings over division rings.

5. **Application of Maschke's Theorem**: Let k be a field and G be a finite group. Let kG denote the group ring over k. We will assume thruout that $char(k)$ does not divide $|G|$. (This is called the non-modular case, i.e., the "nice case".)

Recall that every kG-module M is also a k-vector space. We say that M is a finite dimensional linear representation of G over k if $dim_k(M) < \infty$.

(a) Show that any finite dimensional linear representation M of G over k is both a Noetherian and an Artinian kG-module.

Conclude by Maschke's Theorem that M decomposes uniquely as $M \cong S_1 \oplus \cdots \oplus S_k$ where the S_i are simple kG-modules.

(b) Note kG is itself a finite dimensional linear representation of G. (where kG acts on itself by left multiplication.) kG is called the regular representation of G over k.

Thus $kG \cong S_1 \oplus \cdots \oplus S_k$ where the S_i are simple kG-modules. Show that every simple kG-module S is isomorphic to one of these S_i. (Hint: Find a kG-epimorphism $kG \to S$.) Conclude that there are at most $|G|$ simple kG-modules up to isomorphism.
(c) Let $P = k[x_1, x_2, x_3]$ denote a polynomial algebra in three variables. As a k-vector space, P decomposes into $\oplus_{s=0}^{\infty} P_s$ where P_s is the k-vector space spanned by the homogeneous polynomials of degree s. For example P_2 has k-basis $\{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3\}$. Define a linear Σ_3 action on P via $\sigma(x_i) = x_{\sigma(i)}$ extended to be an algebra map. Thus for example, if $\sigma = (12)$ then $\sigma(x_1^2 + x_2x_3) = x_2^2 + x_1x_3$. One can check that each P_m becomes a $k\Sigma_3$-module which defines a finite dimensional linear representation of Σ_3. Find the decomposition of P_1 into simple $k\Sigma_3$-modules when $k = \mathbb{R}$, the real numbers. To do this consider $\epsilon : P_1 \to k$ given by $\epsilon(c_1x_1+c_2x_2+c_3x_3) = c_1+c_2+c_3$ and show it is a $k\Sigma_3$-hom where k is given the "trivial" $k\Sigma_3$-module structure where all elements in Σ_3 act via the identity. Show that $\{x_2 - x_1, x_3 - x_1\}$ is a k-basis of $K = ker(\epsilon)$ and describe how the elements of Σ_3 act on K by writing down the matrices respresenting them with respect to this basis. Use all this to make the decomposition of P_1 into simple $\mathbb{R}\Sigma_3$-modules.
(d) Find the decomposition of P_2 into simple $\mathbb{R}\Sigma_3$-modules.
(e) The homomorphism $\Sigma_3 \to \pm 1$ defines a linear representation of Σ_3 on $\mathbb{R}$ called the sign representation and denoted $\mathbb{R}_{sign}$. Here $\sigma x = \pm x$ for all $x \in \mathbb{R}$ where the sign is plus for even permutations and minus for odd permutations.
Check that $\mathbb{R}_{sign}$ is a simple $\mathbb{R}\Sigma_3$-module which is not isomorphic to the trivial representation $\mathbb{R}_{triv}$ as $\mathbb{R}\Sigma_3$- modules. Use this to show that P_2 is not isomorphic to the regular representation $\mathbb{R}\Sigma_3$ even though they have the same $\mathbb{R}$-dimension.