MATH 436: Homework VI.
Due in class on Monday, Nov 3

1. **[Descending Central Series.]** For any group G we define a descending series inductively by $DCS_0 = G$,

$$DCS_i = [G, DCS_{i-1}] =< [g, \alpha] | g \in G, \alpha \in G_{i-1} >$$

for all integers $i > 0$. Here $[x, y] = xyx^{-1}y^{-1}$ is the usual commutator.
(a) Check that $DCS_1 = G'$ the commutator subgroup of G in general and that $G'' \leq DCS_2 \leq G'$ in general. However show that $DCS_2 \neq G''$ for $G = \Sigma_3$ and so this series is not the same as the derived series for G.
(b) Show that each DCS_i is a fully invariant subgroup of G. Thus conclude that $\hat{DCS}_* \downarrow^G : \cdots \trianglelefteq DCS_1 \trianglelefteq DCS_0 = G$ is a normal series for G. Show in fact that it is a central series for G. It is called the canonical descending central series.
(c) Suppose $\hat{S}_* \downarrow_e^G$ is a central series for G. Show that $DCS_i \leq S_i$ for all i. Conclude that a group G is nilpotent if and only if $\hat{DCS}_* \downarrow_e^G$. (Hint: $[\alpha, g] \in S_{i+1}$ for all $\alpha \in S_i$ in a descending central series $\hat{S}_*$.)

2. **[Exact Sequences]** A sequence of groups and group homomorphisms $G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-2}} G_{n-1} \xrightarrow{f_{n-1}} G_n$ is called exact if $Im(f_i) = Ker(f_{i+1})$ for all $i = 1, \ldots, n-2$. [Let 1 denote the trivial group of order 1 in this exercise.]
(a) A short exact sequence of groups is an exact sequence of the form

$$1 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 1.$$

Here it is understood that the homomorphisms into or out of 1 have to be the trivial homomorphisms that take all elements to the identity element. Check that in a short exact sequence f is a monomorphism, g is an epimorphism and $Im(f) = Ker(g)$.
(b) Given an epimorphism of groups $g : G \to H$, since it is onto, there is a **function** s such that $g \circ s = 1_H$. However in general such a right inverse function for g need not be itself a **homomorphism** of groups. If we can find a homomorphism $s : H \to G$ such that $g \circ s = 1_H$ then we call the epimorphism g a split epimorphism, and s is called a splitting homomorphism. If we cannot, we call g a nonsplit epimorphism. Show that the canonical quotient map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is an example of a nonsplit epimorphism. Show that the induced epimorphism $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is also nonsplit.

(c) A short exact sequence $1 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 1$ is called split if the epimorphism g is a split epimorphism and nonsplit if it is not. Let $K = Im(f) = Ker(g)$ in the above short exact sequence. Show that the short exact sequence is split if and only if there is a subgroup $H \leq G_2$ such that $HK = G_2$, $H \cap K = \{e\}$. Conclude that the short exact sequence is split if and only if $G \cong K \rtimes_\phi H$ for some $\phi : H \to Aut(K)$.
(Note: It turns out there is a gluing process for nonsplit extensions but it is more complicated than the semidirect product - it involves the notion of "factor sets" and is best discussed in books on the "cohomology of groups".)
(d) Assume that $e = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ is a series for G such that each of the epimorphisms $G_i \to G_i/G_{i-1}$ is split, then show that

$$G \cong (((G_1 \rtimes_{\phi_2} G_2/G_1) \rtimes_{\phi_3} G_3/G_2 \dots) \rtimes_{\phi_n} G_n/G_{n-1})$$

is an iterated semidirect product of the factors of the series.

3. [**Normal Abelian groups in a Solvable Group**].
(a) Show that any solvable group has a nontrivial normal Abelian subgroup. (Hint: Consider the end terms of the derived series.)
(b) Show that if K is characteristic in H and $H \trianglelefteq G$ then $K \trianglelefteq G$. Recall $K \leq_{char} H$ means for all $\Psi \in Aut(H)$, $\Psi(K) \subset K$.
(c) If A is a finite Abelian group and p is a prime, show that $A(p) = \{a \in A | o(a) \text{ divides } p\}$ is a characteristic subgroup of A.
(d) For a prime p, an elementary abelian p-group is an Abelian group E such that every element of E has order dividing p. Show that any finite solvable group has a nontrivial normal elementary abelian p-subgroup for some prime p.

4. [**Faithful and Free Actions.**]
An action of a group G on a set X is called faithful if the action homomorphism $\phi : G \to \Sigma(X)$ has trivial kernel, i.e., $g \cdot x = x$ for **all** $x \in X$ implies $g = e$.
On the other hand we call an action of G on a set X free if for every $g \neq e$, $g \cdot x \neq x$ for all $x \in X$.
(a) Show that an action of a group G on a set X is faithful if and only if $\cap_{x \in X} G_x = e$, i.e., the intersection of all the isotropy subgroups is the trivial group. Show that an action of G on a set X is free if $G_x = e$ for all x, i.e., all isotropy subgroups are trivial. Note that free actions are always faithful but not necessarily vice versa.

(b) Check that the standard action of Σ_3 on $X = \{1, 2, 3\}$ is a transitive faithful action which is not free.
(c) Show that if G acts faithfully and transitively on X and G_x is the isotropy subgroup for some point $x \in X$ then G_x contains no normal subgroups of G, in other words, show that $Core_G(G_x) = e$.
(d) If an Abelian group A acts faithfully and transitively on a set X then show that the action is free.

5. **[Primitive Actions.]**
Let G act transitively on a set X with $|X| \geq 2$ throughout this problem. A domain for the action is a set T such that for all $g \in G$, either:
(i) $gT = T$ or
(ii) $gT \cap T = \emptyset$.
(a) Check that any singleton set $\{x_0\}$, the empty set, and the set X itself are examples of domains for the action.
(b) Let $x_0 \in X$. Show that for any subgroup H with $G_{x_0} \leq H \leq G$, the set $T_H = \{h \cdot x_0 | h \in H\}$ is a domain of the action containing the point x_0. Conversely given a domain T containing x_0, show that $H_T = \{g \in G | gT = T\}$ has $G_{x_0} \leq H \leq G$. Thus conclude that there is a bijection between the set of domains for the action containing x_0 and the set of subgroups H with $G_{x_0} \leq H \leq G$.
(c) We say that a transitive action of G on X is primitive if there are no domains T for the action with $2 \leq |T| < |X|$. We say that $M \leq G$ is a maximal subgroup of G if $M \neq G$ and there are no subgroups H with $M < H < G$. Show that for any transitive action of G on X, $|X| \geq 2$, we have the action is primitive if and only if every isotropy subgroup is maximal in G.
(d) Recall we say a group G acts on X k-transitively if given any two sequences of k distinct elements $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ in X, we have a $g \in G$ with $g \cdot x_i = y_i$ for $1 \leq i \leq k$.
Show that if G acts doubly transitively on X, $|X| \geq 2$ then the action is primitive. (Hint: Consider a domain T with $2 \leq |T| < |X|$ and let $x_0, x_1, y_0 \in T$, $y_1 \notin T$.)
(e) Suppose G acts on X, $|X| \geq 2$, primitively and faithfully. Show that any isotropy subgroup H is a maximal subgroup of G which contains no normal subgroup of G. Conclude that $N_G(H) = H$ and that G is not nilpotent.
(f) Check that Σ_n acts n-transitively and faithfully on $X = \{1, 2, \ldots, n\}$ and that A_n acts $n-2$-transitively and faithfully on $X = \{1, 2, \ldots, n\}$ for $n \geq 3$.
(h) Suppose G acts on X primitively and faithfully and that G is a finite

solvable group. Then take E a nontrivial normal elementary abelian p subgroup of G and H an isotropy subgroup for the action. Show that $G = EH$. Use this to show that E acts transitively and hence freely on X and that $E \cap H = \{e\}$. Conclude that $G \cong E \rtimes_\phi H$ and that $|E| = |X|$ is a power of p. Thus you have shown that if a finite group G acts primitively and faithfully on X, where $|X|$ is not a power of a prime, then G is not solvable.
(i) Use (h) to analyze the doubly-transitive faithful action of A_4 on $X = \{1, 2, 3, 4\}$. For example show that H the isotropy group of the point 4 is isomorphic to $A_3 \cong \mathbb{Z}/3\mathbb{Z}$. Since A_4 is solvable (as its order is less than 60 for example) conclude that $A_4 \cong E \rtimes_\phi A_3$ where E is an elementary abelian subgroup of A_4 of order 4. Describe the elements of E.

6. **[Primitive Linear Actions]**
Let p be a prime, recall $\mathbb{F}_p$ denotes the field of p elements. Let $\mathbb{F}_p^2 = \{(a, b)|a, b \in \mathbb{F}_p\}$. Note we may give $\mathbb{F}_p^2$ the additive group structure of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. We will commonly use column vector notation for the elements of $\mathbb{F}_p^2$ and denote them by vectors such as $\hat{u} = \begin{bmatrix} a \\ b \end{bmatrix} = (a, b)$. We call $(0, 0)$ the zero vector and denote it by $\hat{0}$.

$GL_2(\mathbb{F}_p)$ acts on $\mathbb{F}_p^2$ in the usual way by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au + bv \\ cu + dv \end{bmatrix}$$

(a) Check that this is indeed an action and that $\hat{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a fixed point for the action. If X is the set of all nonzero vectors, i.e., $\mathbb{F}_p^2 - \{\hat{0}\}$ then show that $GL_2(\mathbb{F}_p)$ acts transitively and faithfully on X and that the restricted action of $SL_2(\mathbb{F}_p)$ on X is still transitive. (Hint: Consider the orbit of $\hat{e}_1 = (1, 0)$.)
(b) Given a nonzero vector $\begin{bmatrix} u \\ v \end{bmatrix}$, it determines a line thru the origin in $\mathbb{F}_p^2$ via $L = < \begin{bmatrix} u \\ v \end{bmatrix} > = \{\begin{bmatrix} ku \\ kv \end{bmatrix} | k \in \mathbb{F}_p\}$. We say two lines are equal exactly when they are equal as sets. Show that two nonzero vectors $\begin{bmatrix} u \\ v \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ determine the same line if and only if $\begin{bmatrix} u \\ v \end{bmatrix} = k \begin{bmatrix} a \\ b \end{bmatrix}$ for some nonzero $k \in \mathbb{F}_p$. Let $P = \{L | L \text{ a line thru the origin in } \mathbb{F}_p^2\}$. P is called the projective space

for $\mathbb{F}_p^2$. Show that $|P| = p+1$.
(d) Show that the $GL_2(\mathbb{F}_p)$ action on $\mathbb{F}_p^2$ induces a well-defined action on P via $\mathbb{A} < \begin{bmatrix} u \\ v \end{bmatrix} > = < \mathbb{A} \begin{bmatrix} u \\ v \end{bmatrix} >$. Show that the center of $GL_2(\mathbb{F}_p)$ acts trivially under this action and conclude that there is a well-defined action of $PGL_2(\mathbb{F}_p)$ (and similarly $PSL_2(\mathbb{F}_p)$) on the projective space via $\bar{\mathbb{A}} < \begin{bmatrix} u \\ v \end{bmatrix} > = < \mathbb{A} \begin{bmatrix} u \\ v \end{bmatrix} >$ where we have used the convenient convention of denoting the image of a matrix $\mathbb{A}$ in $PGL_2(\mathbb{F}_p)$ by $\bar{\mathbb{A}}$. Thus the projective groups act on the projective space!
(e) Let E_1 be the line $< \begin{bmatrix} 1 \\ 0 \end{bmatrix} >$ and similarly let E_2 be the line $< \begin{bmatrix} 0 \\ 1 \end{bmatrix} >$. Given any two distinct lines L_1, L_2 show that there is $\bar{\mathbb{A}} \in PSL_2(\mathbb{F}_p)$ such that $\bar{\mathbb{A}} E_i = L_i$ for $i = 1, 2$. Conclude that the actions of $PSL_2(\mathbb{F}_p)$ and $PGL_2(\mathbb{F}_p)$ are doubly transitive on P. Check also that they are faithful actions. Show that the isotropy group of the line E_1 is the image $\bar{B}$ of the Borel subgroup $B \leq SL_2(\mathbb{F}_p)$ under the map $SL_2(\mathbb{F}_p) \to PSL_2(\mathbb{F}_p)$. Conclude that $\bar{B}$ is a maximal subgroup of $PSL_2(\mathbb{F}_p)$ of index $p+1$ with trivial core.