MATH 437: Homework VI.
Due in class on Friday, March 19

1. **Homology of Posets**
Let $(P, \preceq)$ be a POSET (partially ordered set). We define the bar complex $(C_*(P), d)$ for $(P, \preceq)$ as follows:
A n-simplex of $(P, \preceq)$ is a sequence $[\alpha_0 \preceq \alpha_1 \preceq \cdots \preceq \alpha_n]$ where $\alpha_i \in P$ for all i. Note that a n-simplex of $(P, \preceq)$ involves $n+1$ elements of P.

$C_n(P)$ will be the free Abelian group with $\mathbb{Z}$-basis the n-simplices of P. (Thus $C_n = 0$ for $n < 0$.) For $n \geq 1$, we define the homomorphism $d_{n-1} : C_n \to C_{n-1}$ on the basis by

$$d_{n-1}([\alpha_0 \preceq \cdots \preceq \alpha_n]) = \sum_{i=0}^{n} (-1)^i [\alpha_0 \preceq \cdots \preceq \alpha_{i-1} \preceq \hat{\alpha}_i \preceq \alpha_{i+1} \preceq \cdots \preceq \alpha_n]$$

where $\hat{\alpha}_i$ indicates that we have deleted the α_i entry. Note that the subscript for d indicates the subscript of the codomain. For $n < 0$ we define $d_n = 0$.

(a) Show that $(C_*(P), d)$ is a chain complex, i.e., that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$.

Thus we may define $H_n(C_*(P))$ for all $n \in \mathbb{Z}$ and will call this the **homology of the poset** $(P, \preceq)$ and denote it by $H_n((P, \preceq))$ or just $H_n(P)$ if the $\preceq$ is understood.
(b) Let $f : (P_1, \preceq_1) \to (P_2, \preceq_2)$ be a map of POSETs. This means $f : P_1 \to P_2$ is a function with $f(x) \preceq_2 f(y)$ in P_2 whenever $x \preceq_1 y$ in P_1. Define a chain map $f_\sharp : C_*(P_1) \to C_*(P_2)$ that agrees with f on 0-simplices and show that it is indeed a chain map. Thus f induces a well defined map $f_* : H_n(P_1) \to H_n(P_2)$ for all $n \in \mathbb{Z}$.
(c) Suppose a poset $(P, \preceq)$ has a maximum element M, i.e., $\alpha \preceq M$ for all $\alpha \in P$. Show that $\phi_\sharp : C_*(P) \to C_*(P)$ given by $\phi_n = 0$ for $n \neq 0$ and $\phi_0(\alpha) = M$ for all 0-simplices α (note this gives ϕ_0 on a basis of C_0 which determines a homomorphism $\phi_0 : C_0 \to C_0$), is a chain map. (The one case that one has to be careful with is $\phi_0 \circ d_0 = d_0 \circ \phi_1$.)

Now define $T_n : C_n(P) \to C_{n+1}(P)$ by $T_n([\alpha_0 \preceq \cdots \preceq \alpha_n]) = [\alpha_0 \preceq \cdots \preceq \alpha_n \preceq M])$ for $n \geq 0$, $T_n = 0$ for $n < 0$. Show that

$$d_n T_n - T_{n-1} d_{n-1} = \begin{cases} (-1)^{n+1} Id \text{ if } n \geq 1 \\ \phi_0 - Id \text{ if } n = 0 \end{cases}$$

and thus that $dT - Td = \phi_n - (-1)^n Id : C_n(P) \to C_n(P)$ for all $n \in \mathbb{Z}$. Thus we can conclude that $\phi_\sharp \overset{T}{\simeq} (-1)^n Id_\sharp$ where $Id_\sharp : C_*(P) \to C_*(P)$ is the identity chain map.

(d) Use (c) to show that if a poset $(P, \preceq)$ has a maximum element M then $\phi_* = (-1)^n Id : H_n(P) \to H_n(P)$ for all $n \in \mathbb{Z}$ and use this to show that $H_n(P) = 0$ for all $n \neq 0$ and $H_0(P)$ is cyclic generated by the homology class represented by the maximum element M. (Note M is a basis element of $C_0(P)$ which is automatically a 0-cycle as $C_{-1}(P) = 0$. Thus $M + Im(d_0)$ is an element in $H_0(P)$ and we say this element is the homology class represented by M.)

2. **Euler-Poincare Lemma**

Let k be a field and let C_* be a chain complex over k which is **finite** . This means $dim_k(C_n) < \infty$ for all $n \in \mathbb{Z}$ and there exists $M \in \mathbb{N}$ such that $C_n = 0$ for all n with $|n| > M$.

We define the **Euler characteristic** $\chi(C_*) \in \mathbb{Z}$ of a finite complex C_* as

$$\chi(C_*) = \sum_{n \in \mathbb{Z}} (-1)^n dim_k(C_n).$$

(a) Let $Z_n = \{w \in C_n | dw = 0\}$ denote the k-vector space of n-cycles and let $B_n = \{w \in C_n | w = ds\}$ denote the k-vector space of n-boundaries. As usual let $H_n = Z_n / B_n$ be the nth homology of C_* and of course it is also a k-vector space.

Show that we have short exact sequences of the following sort for all $n \in \mathbb{Z}$:

$$0 \to B_n \to Z_n \to H_n \to 0$$

and

$$0 \to Z_n \to C_n \to B_{n-1} \to 0.$$

Conclude that $Z_n \cong B_n \oplus H_n$ and $C_n \cong Z_n \oplus B_{n-1}$ as k-vector spaces for all $n \in \mathbb{Z}$. Use this to show that

$$\sum_{n \in \mathbb{Z}} (-1)^n dim_k(C_n) = \sum_{n \in \mathbb{Z}} (-1)^n dim_k(H_n)$$

and thus the Euler characteristic is also the alternating sum of the dimensions of the homology. This is called the Euler-Poincare lemma.

(b) Given a sequence of finite dimensional k-vector spaces $\{V_n | n \in \mathbb{N}\}$, we

may define the Euler-Poincare power series $P_{V_*}(t) = \sum_{n\in\mathbb{N}} dim_k(V_n)t^n$. If C_* is a finite chain complex over k with $C_n = 0$ for $n < 0$ this gives us a few Euler-Poincare polynomials:
$P_{C_*}(t), P_{B_*}(t), P_{Z_*}(t), P_{H_*}(t)$ corresponding to the sequence of vector spaces coming from the chains, boundaries, cycles and homology of C_* respectively. **Show** that

$$P_{C_*}(t) = (1+t)P_{B_*}(t) + P_{H_*}(t)$$

in $\mathbb{Z}[t]$ and that $\chi(C_*) = P_{C_*}(-1)$.
(c) Let R be a ring, let $m_0(R)$ denote the Abelian monoid of (isomorphism classes of) finitely generated R-modules where the addition $M + N$ is given by the direct sum $M \oplus N$. (Note: Since every finitely generated R-module is isomorphic to a quotient of R^N for some $N \in \mathbb{N}$, this is actually a set!). One can check that this is indeed a well-defined Abelian monoid with identity 0, the zero module. The Grothendieck group of $m_0(R)$ is denoted by $M_0(R)$. Thus the typical element of $M_0(R)$ is of the form of a formal difference $M-N$ where M, N are finitely generated R-modules. Here $M - N = M' - N'$ if and only if $M \oplus N' \oplus S \cong M' \oplus N \oplus S$ for some finitely generated R-module S.

Now suppose $R = kG$ where k is a field, G is a finite group and $char(k)$ does not divide $|G|$. **Show** that a kG-module V is finitely generated if and only if it is finite dimensional as a k-module.

Thus by HW#3, a nonzero finitely generated kG-module V will decompose uniquely as a finite direct sum of simple kG-modules. **Explain** why this implies that the monoid $m_0(kG)$ has cancellation. Conclude that $M - N = M' - N' \in M_0(kG)$ if and only if $M \oplus N' \cong M' \oplus N$. Use this to **show** that $M_0(kG)$ is a free Abelian group on a $\mathbb{Z}$-basis set S of isomorphism classes of simple kG-modules. Since in HW#3, it was showed that kG has at most $|G|$ many distinct simple modules up to isomorphism, it follows that $M_0(kG)$ is a free Abelian group of rank at most $|G|$.
(d) Let k be a field and G be a finite group such that $char(k)$ does not divide $|G|$. Recall that by Maschke's Theorem, this means that kG is a semisimple ring (every module is projective). Let C_* be a chain complex of kG-modules that is finite as a chain complex of k-modules. Show that

$$\sum_{n\in\mathbb{Z}}(-1)^n C_n = \sum_{n\in\mathbb{Z}}(-1)^n H_n$$

in $M_0(kG)$. (Note: One can recover (a) as the case $G = e$.)

3. **Lefschetz numbers.**
Let k be a field and $f : V \to V$ a k-homomorphism. Then note matrices representing f in different basis of V are similar matrices and hence have the same trace by our previous work. Thus $tr(f) \in k$ is well-defined independent of the basis used to represent f as a matrix. Thus to every k-linear map $f : V \to V$ we may define $tr(f) \in k$.
(a) Let C_* be a finite chain complex over k and $f_\sharp : C_* \to C_*$ be a chain map. The **Lefschetz number** of $f_\sharp$ is defined by:

$$L(f_\sharp) = \sum_{n\in\mathbb{Z}} (-1)^n tr(f_n : C_n \to C_n).$$

If $Id_\sharp : C_* \to C_*$ is the identity chain map, show that $L(Id_\sharp) = \chi(C_*)$.
(b) Let V be a finite dimensional k-vector space and let W be a subspace. Suppose $f : V \to V$ is k-linear with $f(W) \subseteq W$. Then show that f induces a well-defined map $V/W \to V/W$ and that

$$tr(f : V \to V) = tr(f : W \to W) + tr(f : V/W \to V/W).$$

(c) **Show** that for any finite chain complex C_* over k and chain map $f_\sharp : C_* \to C_*$ we have

$$L(f_\sharp) = \sum_{n\in\mathbb{Z}} (-1)^n tr(f_n : C_n \to C_n) = \sum_{n\in\mathbb{Z}} (-1)^n tr(f_* : H_n \to H_n)$$

in k. **Explain** why this shows that the Lefschetz number for two chain homotopic chain maps will be the same.
(Note: Lefschetz numbers are the basis for the important Lefschetz fixed point theorem of Algebraic topology.)

4. **Ext computations**
(a) Let R be a PID and let $I = (\alpha)$ be a nonzero proper ideal of R. Show that for any R-module M

$$Ext^n_R(R/I, M) = \begin{cases} ker(\alpha : M \to M) \text{ if } n = 0 \\ M/IM \text{ if } n = 1 \\ 0 \text{ otherwise.} \end{cases}$$

(Hint: First find a projective resolution of R/I of the form $0 \to R \to R \to R/I \to 0$.)
(b) Let $R = \mathbb{R}[x]$ where $\mathbb{R}$ is the field of real numbers. Compute $Ext^n_R(R/(x^6 - 1), R/(x^4 - 1))$ for all $n \in \mathbb{N}$.