KAKEYA CONFIGURATIONS IN LIE GROUPS AND HOMOGENEOUS SPACES

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Abstract. In this paper, we study continuous Kakeya line and needle configurations, of both the oriented and unoriented varieties, in connected Lie groups and some associated homogenous spaces. These are the analogs of Kakeya line (needle) sets (subsets of $\mathbb{R}^n$ where it is possible to turn a line (respectively an interval of unit length) through all directions **continuously, without repeating a "direction"**.) We show under some general assumptions that any such continuous Kakeya line configuration set in a connected Lie group must contain an open neighborhood of the identity, and hence must have positive Haar measure. In connected nilpotent Lie groups G, the only subspace of G that contains such an unoriented line configuration is shown to be G itself. Finally some similar questions in homogeneous spaces are addressed.

These questions were motivated by work of Z. Dvir in the finite field setting.
Keywords: Kakeya needle sets, Lie groups, homogeneous spaces, fiber bundles.
2010 *Mathematics Subject Classification.* Primary: 55M99, 22E25; Secondary: 42B99, 51A15.

Contents

1. Introduction

The purpose of this paper is to study Kakeya like configurations in the setting of connected Lie groups. We will only be looking at continuous configurations and so this paper has little to say about the famous Kakeya conjecture (described below) but instead focuses on Kakeya needle-like configurations in general settings.

Kakeya posed the Kakeya needle question in $\mathbb{R}^2$: What is the smallest area of a subset of $\mathbb{R}^2$ for which it is possible to turn a needle (interval of length 1) around **continuously** while remaining entirely in the subset. Such a subset is called a Kakeya needle set. When the subset is required to be convex, Pal ([Pa]) showed the answer was given by the equilateral triangle of base height 1 (area $\frac{1}{\sqrt{3}}$). Similar minimizers of positive area exist when the subset is required to be star-convex. However Besicovitch ([Be1], [Be2]) showed that in general, there is no lower bound on the area, there exist Kakeya needle subsets of the plane with arbitrarily small positive Lebesgue measure. Cunningham ([Cu]) later refined this work to show there also exist simply-connected Kakeya needle subsets of the unit disk of arbitrarily small positive Lebesgue measure.

Besicovitch also considered more general Kakeya sets which were defined as subsets of $\mathbb{R}^n$ which contain a line segment of length 1 in every given direction of $\mathbb{R}^n$. Here, unlike the case for Kakeya needle sets, the variation of the needle as a function of direction need not be continuous, or even Borel measurable. Besicovitch showed that there are Kakeya sets of Lebesgue measure zero in every $\mathbb{R}^n, n \geq 2$. Regardless these examples have maximal Hausdorff dimension n and the famous Kakeya conjecture asks to show that this is true in general, i.e., that every Kakeya subset of $\mathbb{R}^n$ has Hausdorff (and hence Minkowski) dimension n. This conjecture is still open for $n \geq 3$. This conjecture is related to many questions in Harmonic analysis, for example C. Fefferman ([Fe]) used Besicovitch's construction of Kakeya sets of measure zero to disprove the ball multiplier conjecture for $L^p(\mathbb{R}^n), p \neq 2$. Further interesting applications to analysis are discussed in [Bo], [Ta] and [Wo].

In this paper we will not consider Kakeya sets but only configurations similar to the Kakeya needle sets where the needle is required to move continuously. In fact, we will most often be interested in continuous line configurations where the motion goes through a line in each direction (or parallel class) exactly once. These are sets in which it is possible to move a line continuously such that every direction in the ambient space is achieved exactly once. We consider two variations of these continuous Kakeya configurations: unoriented and oriented.

Definition 1.1 (Continuous unoriented Kakeya configuration)**.** Let $\mathbb{R}P^{n-1}$ denote the space of lines through the origin (projective space) of $\mathbb{R}^n$. We call a continuous function $\sigma\colon \mathbb{R}P^{n-1} \to \mathbb{R}^n$ a *continuous unoriented Kakeya configuration*, and we use $|\sigma|$ to denote the space of lines swept out by the configuration:

$$|\sigma| = \bigcup_{L \in \mathbb{R}P^{n-1}} (\sigma(L) + L).$$

For each line L through the origin, σ picks out a displacement $\sigma(L)$ in a continuous manner and we study the set $|\sigma| = \bigcup_{L\in\mathbb{R}P^{n-1}}(\sigma(L) + L)$. Notice this set contains a line in every direction but is more constrained as the line placement varies continuously and since the configuration is minimal in the sense that for each line through the origin, **exactly one** line parallel to it is used during the motion.

Oriented configurations are defined similarly, but with the space of lines $\mathbb{R}P^{n-1}$ replaced by unit sphere S^{n-1}, which may be identified with the space of directed lines in $\mathbb{R}^n$.

Definition 1.2 (Continuous oriented Kakeya configuration)**.** Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. We call a continuous function $\vec{\sigma}\colon S^{n-1} \to \mathbb{R}^n$ a *continuous oriented Kakeya configuration*, and we use $|\vec{\sigma}|$ to denote the space of lines swept out by the configuration:

$$|\vec{\sigma}| = \bigcup_{L \in S^{n-1}} (\vec{\sigma}(L) + L).$$

Notice for either the oriented or unoriented configurations above, one can consider either the space $|\sigma|$ swept out of the lines of the configuration or the smaller needle space $|\hat{\sigma}| = \bigcup_{L\in\mathbb{R}P^{n-1}}(\sigma(L) + I(L))$ where $I(L)$ is the interval of length 1 centered about the origin in L. The classical Kakeya conjecture is equivalent to showing for any $\sigma : \mathbb{R}P^{n-1} \to \mathbb{R}$ (not necessarily continuous or even Borel measurable), that any Borel set containing $|\hat{\sigma}|$ has Hausdorff dimension n. In this paper we will not address this and consider exclusively only continuous σ.

This study was motivated by a paper of Dvir ([Dv]) which looked at an analogous question over finite fields. Dvir showed that if E is a subset of a finite dimensional vector space over a finite field which contains a line in every direction, then E contains a "positive proportion" of the vector space. Under suitable assumptions, we recover the same picture for continuous Kakeya line configurations in this paper.

The underlying space of the configuration $\sigma : \mathbb{R}P^{n-1} \to \mathbb{R}^n$ is defined to be $|\sigma| = \bigcup_{L \in RP^{n-1}} (\sigma(L) + L)$. A continuous unoriented Kakeya line set is a subset of $\mathbb{R}^n$ that contains the underlying space of some continuous unoriented Kakeya line configuration. Intuitively, it is a set where it is possible to move a line continuously such that during the motion every parallel class of lines is used exactly once.

Using some basic algebraic topology, we show:

Theorem 1.3. *The only continuous unoriented Kakeya line set in $\mathbb{R}^n$ is $\mathbb{R}^n$ itself; that is, for every continuous map $\sigma \colon \mathbb{R}P^{n-1} \to \mathbb{R}^n$, we have*

$$|\sigma| = \bigcup_{L \in \mathbb{RP}^{n-1}} (\sigma(L) + L) = \mathbb{R}^n.$$

Examples are also given showing that this theorem fails if either the continuity assumption is dropped or if more than one line from each parallel class is allowed during the motion (in which case the assignment σ is not a function).

We prove Theorem 1.3 by showing that continuous Kakeya configurations are sections of a natural vector bundle over $\mathbb{R}P^{n-1}$. This allows us to apply tools from algebraic topology, which assert that such sections must have a zero; this implies that the associated Kakeya line set contains the origin. By a slight modification of this argument, we can show that that any specified point must be contained in the Kakeya line set, which proves the theorem.

In the oriented case we show a similar conclusion holds in odd dimensions, while in even dimensions, a continuous oriented Kakeya set may not fill all of Euclidean space.

Theorem 1.4. *A continuous oriented Kakeya set in $\mathbb{R}^d$ satisfies one of the following possibilities:*

(1) *If $d = 2n+1$ is odd, then the only continuous oriented Kakeya line set in $\mathbb{R}^d$ is $\mathbb{R}^d$ itself. That is, for any $n \geq 1$ and for every continuous map $\vec{\sigma} \colon S^{2n} \to \mathbb{R}^{2n+1}$ we have*

$$|\vec{\sigma}| = \bigcup_{x \in S^{2n}} (\vec{\sigma}(x) + L_x) = \mathbb{R}^{2n+1},$$

where L_x is the line passing through the origin and x oriented in direction pointing from the origin to x.

(2) *If $d = 2n$ is even, then for any bounded subset A of $\mathbb{R}^{2n}$ there exists a continuous oriented Kakeya configuration $\vec{\sigma} \colon S^{2n-1} \to \mathbb{R}^{2n}$ such that $A \cap |\vec{\sigma}| = \emptyset$. In particular, for a given $\vec{\sigma} \colon S^{2n-1} \to \mathbb{R}^{2n}$, the space $|\vec{\sigma}|$ need not be all of $\mathbb{R}^{2n}$.*

Again this theorem fails if either $\vec{\sigma}$ is not continuous or if $\vec{\sigma}$ is not a function (that is, more than one line in a given direction is used during the motion).

The proof of Theorem 1.4 (for odd dimensions) is similar to the proof of Theorem 1.3: we show that an oriented Kakeya configuration yields a section of a certain vector bundle over the sphere, and then apply a theorem of algebraic topology to show that this section must have a zero.

Extension to Lie Groups. It is natural to extend these questions to the setting of connected Lie groups G. Indeed, extensions of the Kakeya problem to curves in Euclidean space and geodesics in Riemannian manifolds have been studied extensively in the analytic setting [BG, MiS, S, Wi]. Several of these results are "negative" in that the Kakeya conjecture is false in certain extensions, while we confirm that Kakeya sets in certain Lie groups do have full dimension (however we do so under continuity assumptions, while the aforementioned results do not require continuity). Generalizing our method to geodesics in Riemannian manifolds would require a very controlled parallelization of the tangent bundle in order to achieve a global "direction space", and so we restrict our attention to Lie groups.

Every connected Lie group has a Lie algebra $\mathfrak{g}$, and it is well known (see Warner, chapter 3) that the Lie subalgebras of $\mathfrak{g}$ are in one to one correspondence with the connected subgroups of G. In particular for every line through the origin in $\mathfrak{g}$ there corresponds a unique one-parameter subgroup of G (which will either be an immersed line or a circle group.) through the exponential-log correspondence. The one-parameter subgroups of $\mathbb{R}^n$ are exactly the lines through the origin and these will take the place of lines in a general connected Lie group. The analog of parallel lines, will be left translates gH of a given one parameter subgroup H. (One can also replace left with right through out this paper without changing the results though the translates in general will be different in a non-abelian Lie group).

To extend our results to Lie groups, we will find conditions under which the exponential map is a covering map of the Lie group. Once we know that the exponential map is a covering map, we can use covering space theory to lift Kakeya configurations in the Lie group G to Kakeya configurations in the Lie algebra $\mathfrak{g}$ (definitions are given below). Since $\mathfrak{g}$ is a finite dimensional vector space, we can then apply Theorems 1.4 and 1.3 to show that any Kakeya configuration in $\mathfrak{g}$ must "sweep out" all of $\mathfrak{g}$, which implies that the corresponding Kakeya configuration in G must "sweep out" all of G.

Let $P(\mathfrak{g})$ be the projective space corresponding to the finite dimensional real vector space $\mathfrak{g}$. For a line $L \in \mathfrak{g}$ we let $\exp(L)$ denote the corresponding one-parameter subgroup of G and $g\exp(L)$ for the typical "parallel" left translate. We then define Kakeya configurations in G:

Definition 1.5. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $P(\mathfrak{g})$ be the projective space associated to $\mathfrak{g}$. A continuous unoriented Kakeya line configuration is a continuous map $\sigma : P(\mathfrak{g}) \to G$. The underlying space to this configuration, denoted $|\sigma|$ is defined as $\sigma = \bigcup_{L\in P(\mathfrak{g})}(\sigma(L)\exp(L))$. A continuous unoriented Kakeya line set X is a subspace of G which contains $|\sigma|$ for some continuous unoriented Kakeya line configuration σ. Similar definitions hold for oriented Kakeya line configurations where $P(\mathfrak{g})$ is replaced by $S(\mathfrak{g})$, the sphere of $\mathfrak{g}$ with respect to some positive definite inner product on $\mathfrak{g}$.

We prove, among other things:

Theorem 1.6. *If G is a connected nilpotent Lie group, then the only continuous unoriented Kakeya line set in G is G itself, i.e., $|\sigma| = G$ for any continuous unoriented Kakeya configuration σ in G. A similar theorem holds for oriented Kakeya line sets if in addition, G is assumed to be odd dimensional. This theorem also holds for connected solvable Lie groups of type E.*

Again, the theorem is easily seen to not be true without the continuity assumption.

In a general connected Lie group, one can prove weaker results. One has to assume the Kakeya line configuration is "linear" (liftable via the exponential map) or taut (see sections below) which follows for example if $\sigma : P(\mathfrak{g}) \to G$ has image contained in a suitable open neighborhood of the identity (one which is diffeomorphic to an open ball in $\mathfrak{g}$ under the exponential-log correspondence.) Then one has:

Theorem 1.7. *Let G be a connected Lie group and let $\sigma : P(\mathfrak{g}) \to G$ be a linear continuous unoriented Kakeya line configuration, then $|\sigma|$ contains the identity element. Furthermore any taut unoriented Kakeya line configuration σ has $|\sigma|$ contain an open neighborhood of the identity in G and hence has positive (left) Haar measure. The same statements hold for oriented Kakeya line configurations if the dimension of G is odd.*

Finally the definition of continuous Kakeya line configuration $\sigma : P(\mathfrak{g}) \to G$ behaves well with respect to Lie group covering maps $\pi : G \to M$. Here $\mathfrak{g} = \mathfrak{M}$ canonically and $\pi \circ \sigma$ is then a continuous Kakeya line configuration in M. Furthermore, basic covering space theory shows (see section below) we can lift continuous Kakeya line configurations from M to G under mild hypothesis. Thus for a given finite dimensional, real, Lie algebra $\mathfrak{g}$ it is often sufficient to understand the questions about continuous Kakeya line configurations in the unique simply-connected Lie group with Lie algebra $\mathfrak{g}$.

Given a closed subgroup K (whether normal or not) of a Lie group G, $K\backslash G$ is a smooth manifold, and one has a quotient map $\pi : G \to K\backslash G$ which takes left cosets of one-parameter subgroups of G to a distinguished set of curves in $K\backslash G$. Thus a continuous Kakeya line configuration $\sigma : P(\mathfrak{g}) \to G$ gives rise to a continuous unoriented Kakeya line configuration $\hat{\sigma} = \pi \circ \sigma : P(\mathfrak{g}) \to K\backslash G$. Furthermore, given a continuous map $P(\mathfrak{g}) \to K\backslash G$, one can use fiber bundle theory to decide when it comes from a Kakeya configuration in G. We study such configurations briefly in section 4.

As a final comment in this introductory section, we remark that given a (not necessarily continuous) configuration $\sigma : P(\mathfrak{g}) \to G$ in a connected Lie group, and a choice of positive definite inner product on $\mathfrak{g}$, we can form,

$$|\hat{\sigma}| = \bigcup_{L \in P(\mathfrak{g})} (\sigma(L) exp(I(L)))$$

where $I(L)$ is the part of the line L within distance $\frac{1}{2}$ of the origin. A Borel set containing such a σ is the analog of a Kakeya set in a connected Lie group. To distinguish this from the case where σ is continuous we will refer to these as Kakeya-Besicovitch sets. Note that Hausdorff and Minkowski dimension are metric dependent and one can easily find examples of two metrics that metrize the same topological group with different dimension theory. (The 2-adic integers with the 2-adic metric have Hausdorff dimension 1 while with the metric induced from the real line by the homeomorphism with the Cantor set, it has a fractional Hausdorff dimension.) However for any two positive definite inner products on $\mathfrak{g}$, the corresponding left-invariant Riemannian metrics on G have the same Hausdorff dimensions for any Borel subset. We will use such a canonical version of Hausdorff dimension on a Lie group to state:

Question 1.8 (Lie variant of Kakeya Conjecture). Let G be a connected Lie group and K a Kakeya-Besicovitch subset of G. Does the Hausdorff dimension of K have to equal the dimension of G?

Note the case of simply-connected, abelian Lie groups, i.e., $(\mathbb{R}^n, +)$ is exactly the classical Kakeya conjecture which is known in dimensions 1 and 2 (see [Da]). The Lie group $\{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} | a > 0, b \in \mathbb{R}\}$ is the smallest non-abelian example to consider where an answer might be obtainable from current technology. Though the underlying space of this example is diffeomorphic to $\mathbb{R}^2$, the one-parameter subgroups do not correspond to lines in general under this correspondence.

Also we state a discrete version of this question, let us define a subset E of a finite group G to be a Kakeya set if it contains a left coset of every cyclic subgroup of G. We then ask:

Question 1.9 (Discrete variant of Kakeya Conjecture). Is there a "sharp" explicit formula for a positive constant $c = c(e, g)$, such that for every finite g-generated group G of exponent dividing e, and Kakeya subset $E \subseteq G$, one has $|E| \geq c|G|$.

Note, by the solution of the restricted Burnside conjecture ([Ze], [Ze2]), there are only finitely many g-generated groups whose exponent divides e. It is clear in this case there is an absolute constant $c = c(e, g) > 0$ such that the size of any Kakeya set in these groups is at least $c|G|$. (This is because in any given group, a Kakeya set has to have at least size 1 and so has to be at least $\frac{1}{|G|}$ worth of G. Thus we may take $c(e, g)$ to be the reciprocal of the size of the largest g-generated subgroup of exponent e. This c works but is somewhat silly and definitely not sharp.) Thus the existence of a positive $c = c(e, g)$ in the last question is clear, but a sharp, useful, explicit formula for it would be nice.

In work of Dvir ([Dv]) on the case of G a finite product of cyclic groups of prime order, a relatively sharp formula for c was found as were examples showing that it must depend on g, the number of generators of the group. Sharper formulas for c in this case were obtained in [SS]. In [DH], the case of G a finite product of two cyclic groups of order p^k was worked out and an explicit sharp formula for c found. In this work, examples were found showing c must depend on e, the exponent of G.

Thus the question as framed definitely has a solution and the dependence of the constant on g and e as stated is necessary - explicit sharp formulas for $c = c(e, g) > 0$ are the hopeful goal.

2. Continuous Kakeya Line Configurations in $\mathbb{R}^n$

For convenience, we recall Definition 1.1 and fix notation regarding Kakeya needle sets.

Definition. A continuous unoriented Kakeya line configuration in $\mathbb{R}^n$ is a continuous map $\sigma : \mathbb{R}P^{n-1} \to \mathbb{R}^n$. The underlying space of σ, $|\sigma|$ is defined as

$$|\sigma| = \bigcup_{L \in \mathbb{R}P^{n-1}} (\sigma(L) + L)$$

The associated unoriented Kakeya needle set is $|\hat{\sigma}| = \bigcup_{L \in \mathbb{R}P^{n-1}} (\sigma(L) + I(L))$ where $I(L)$ is the interval of length 1 centered around the origin in L.

For $1 \leq R < \infty$, the R-elongation of the Kakeya needle set is the set obtained when we replace $I(L)$ with an interval of length R centered abound the origin in L in this definition. Similar definitions hold for oriented variants where $\mathbb{R}P^{n-1}$ is replaced by S^{n-1}.

Since projective spaces, spheres and intervals are compact spaces, it is not hard to show that the associated Kakeya needle sets (oriented or unoriented) and their elongations are compact subspaces of $\mathbb{R}^n$ when σ is continuous. Indeed, if $\vec{\sigma}$ is a continuous, oriented Kakeya configuration, then for any $R \geq 0$ the map $f\colon S^{n-1} \times I \to \mathbb{R}^n$ given by $f(u,t) = \vec{\sigma}(u) + Rt \cdot u$ is then the composition of the maps $(u,t) \mapsto (\vec{\sigma}(u), Rtu)$ and $(x,y) \mapsto x + y$, which are continous because scalar multiplication and vector addition are continous. Since the continous image of a compact set is compact, it follows that the R-elongation $f(S^{n-1}, I)$ is compact. The proof that unoriented Kakeya needle sets (and their elongations) are compact then follows by noting that given an unoriented configuration $\sigma : \mathbb{R}P^{n-1} \to \mathbb{R}^n$ we can form an oriented configuration $\vec{\sigma} = \sigma \circ \pi$ by composing with the double cover map $\pi : S^{n-1} \to \mathbb{R}P^{n-1}$. The two configurations are easily seen to have the same underlying needle sets and R-elongations for any $R \geq 0$. It follows by taking a union over positive integers R that $|\sigma|$ itself is σ-compact and hence a Borel set.

Consider the trivial vector bundle ϵ^n given by $\pi\colon \mathbb{R}P^{n-1} \times \mathbb{R}^n \to \mathbb{R}P^{n-1}$. Notice that for any $L \in \mathbb{R}P^{n-1}$, the fiber $\pi^{-1}(L)$ splits as $\mathbb{R}^n = L \oplus L^\perp$, where $L^\perp$ is the orthogonal complement of L with respect to the standard dot product of $\mathbb{R}^n$. The vector bundle over $\mathbb{R}P^{n-1}$ whose fiber over L is L is called the *canonical line bundle* of $\mathbb{R}P^{n-1}$ and is denoted by γ (see Example 4 on page 15 of [MS]). The preceding argument shows that γ is a subbundle of ϵ^n so by Theorem 3.3 of [MS] it follows that $\epsilon^n = \gamma \oplus \gamma^\perp$ where $\oplus$ is Whitney sum of vector bundles (see chapter 3 of [MS] for the definition of a Whitney sum). Alternatively, because $\oplus$ is an *continuous functor*, the decomposition of the fibers of ϵ^n yields the corresponding Whitney sum decomposition (see the exercises on page 31 of [MS]).

We will need the well known fact that any continuous section of $\gamma^\perp$ has a zero.

Lemma 2.1. *Every continuous section of $\gamma^\perp$ has a zero.*

We will sketch the proof of this lemma since it encapsulates much of the machinery behind our proof.

Proof. First we recall some basic facts, see chapter 4 of [MS] for details: any vector bundle ξ over a paracompact, Hausdorff space has a sequence of Stiefel-Whitney classes $w_0, w_1, w_2, \dots$ such that $w_k(\xi) = 0$ when $k > \dim(\xi)$. The class w_i is an element of $H^i(X; \mathbb{F}_2)$ and they fit together to give a total Stiefel-Whitney class in the $\mathbb{F}_2$ cohomology ring:

$$w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \cdots \in \bigoplus_{i=0}^{\infty} H^i(X, \mathbb{F}_2).$$

If $\xi = \xi_1 \oplus \xi_2$ then $w(\xi) = w(\xi_1)w(\xi_2)$ by Cartan's formula. Finally recall $w(\epsilon^k) = 1$ for any trivial bundle ϵ^k and $w(\gamma) = 1 + a$ where a is a generator of $H^1(\mathbb{R}P^{n-1}, \mathbb{F}_2)$.

Since $\epsilon^n = \gamma \oplus \gamma^\perp$ we have by applying w to both sides and using Cartan's formula that

$$1 = w(\gamma)w(\gamma^\perp) = (1 + a)w(\gamma^\perp)$$

from which it follows $w(\gamma^\perp) = \frac{1}{1+a} = 1 + a + a^2 + \cdots + a^{n-1}$ as we are in characteristic two and $H^*(\mathbb{R}P^{n-1}; \mathbb{F}_2) = \mathbb{F}_2[a]/(a^n)$ is a truncated polynomial algebra. In particular $w_{n-1}(\gamma^\perp) = a^{n-1} \neq 0$.

Now suppose that there existed a nowhere vanishing section of $\gamma^\perp$. Then $\gamma^\perp = \mu \oplus \epsilon^1$ where μ is a $(n-2)$-dimensional vector bundle. Thus Cartan's formula would show that $\gamma^\perp$ would have the same Stiefel-Whitney classes as a $(n-2)$-dimensional vector bundle and in particular, $w_{n-1}(\gamma^\perp) = 0$ contradicting what we have established. Thus we conclude $\gamma^\perp$ possesses no nowhere vanishing continuous section. □

Lemma 2.2. *If $\sigma : \mathbb{R}P^{n-1} \to \mathbb{R}^n$ is a continuous unoriented Kakeya line configuration, then $0 \in |\sigma|$.*

Proof. For $L \in \mathbb{R}P^{n-1}$, let $P_L \colon \mathbb{R}^n \to L^\perp$ denote the orthogonal projection onto $L^\perp$. The following claim is the key to our argument:

Claim. *The function $L \mapsto P_L(\sigma(L))$ is a continuous section of $\gamma^\perp$.*

Assuming this claim for now, it follows from lemma 2.1 that $P_{L_0}(\sigma(L_0)) = 0$ for some line L_0 in $\mathbb{R}P^{n-1}$. Since $P_{L_0}(\sigma(L_0)) - \sigma(L_0)$ is parallel to L_0, it follows that $\sigma(L_0) + L_0 = P_{L_0}(\sigma(L_0)) + L_0 = L_0$. As $|\sigma| = \bigcup_{L \in \mathbb{R}P^{n-1}} (\sigma(L) + L)$, we conclude that $|\sigma|$ must contain the line L_0 through the origin and hence must contain the origin 0. This completes the proof of the lemma, assuming the claim.

To prove the claim, to our main task is to show that the map $L \mapsto P_L(\sigma(L))$ is continuous. To do so, we introduce a map $f \colon S^{n-1} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(x, v) = v - (v \cdot x)x.$$

Since f is the restriction to $S^{n-1} \times \mathbb{R}^n$ of a polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, it follows that f is continuous in x and v. Further, for fixed x, $f(x, v)$ is the orthogonal projection of v onto $x^\perp$:

$$f(x, v) \cdot x = (v - (v \cdot x)x) \cdot x = v \cdot x - (v \cdot x)(x \cdot x) = v \cdot x - (v \cdot x)(1) = 0,$$

and hence

$$f(x, f(x, v)) = (v - (v \cdot x)x) - [(v - (v \cdot x)x) \cdot x]x = (v - (v \cdot x)x) - (0)x = v - (v \cdot x)x.$$

Now if $L \in \mathbb{R}P^{n-1}$, we may identify L with $\{\pm x\}$, where x is a unit vector parallel to L. It's easy to see that $f(x, v) = f(-x, v)$, hence f descends to a continuous map $\bar{f} \colon \mathbb{R}P^{n-1} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $\bar{f}(L, v) = f(\pm x, v)$. Since $\bar{f}(L, v) \perp L$, we see that $v \mapsto \bar{f}(L, v)$ is the perpendicular projection from $\mathbb{R}^n$ to $L^\perp$; thus $P_L(v) = \bar{f}(L, v)$. Since $\sigma(L)$ is continuous and $\bar{f}(L, v)$ is continuous in both variables, it follows that $P_L(\sigma(L)) = f(L, \sigma(L))$ is continuous, and hence $L \mapsto P_L(\sigma(L))$ is a continuous section of $\gamma^\perp$, as claimed. □

Theorem 2.3. *Let $\sigma : \mathbb{R}P^{n-1} \to \mathbb{R}^n$ be a continuous unoriented Kakeya line configuration, then $|\sigma| = \mathbb{R}^n$.*

Proof. For any $\hat{x} \in \mathbb{R}^n$, let $T_{-\hat{x}}$ denote the operator of translation by $-\hat{x}$. Note that $T_{-\hat{x}} \circ \sigma$ is another continuous unoriented Kakeya line configuration and $|T_{-\hat{x}} \circ \sigma| = |\sigma| - \hat{x}$. By lemma 2.2, this set must contain the origin. Thus $|\sigma|$ itself must contain $\hat{x}$. Since $\hat{x}$ was arbitrary, $|\sigma| = \mathbb{R}^n$. □

Corollary 2.4. *Let σ be a continuous unoriented Kakeya line configuration, then for some $1 \leq R < \infty$, the R-elongation of the corresponding unoriented Kakeya needle set has positive Lebesgue measure.*

Proof. $|\sigma| = \mathbb{R}^n$ is the nested union of the N-elongations of the corresponding Kakeya needle set, where $N \in \mathbb{Z}_+$. The result follows immediately. □

We now shift gears and consider oriented configurations, whose definition we recall here:

Definition. We call a continuous function $\vec{\sigma}\colon S^{n-1} \to \mathbb{R}^n$ a *continuous oriented Kakeya configuration*, and we use $|\vec{\sigma}|$ to denote the space of lines swept out by the configuration:

$$|\vec{\sigma}| = \bigcup_{L \in S^{n-1}} (\vec{\sigma}(L) + L).$$

Notice that the trivial vector bundle ϵ^n given by $\pi : S^{n-1} \times \mathbb{R}^n \to S^{n-1}$ has each fiber over a point $x \in S^{n-1}$ split into the normal line at x and the tangent plane of the sphere at x. Since the normal bundle of the embedding of S^{n-1} in $\mathbb{R}^n$ is trivial this gives $\epsilon^n = \epsilon^1 \oplus \tau$ where τ is the tangent bundle of the sphere S^{n-1}. We may once again consider the orthogonal projection $p : \mathbb{R}^n \to T_x(S^{n-1})$ and note that the function $x \to p(\vec{\sigma}(x))$ is a continuous section of the tangent bundle of the sphere, i.e., a continuous vector field on the sphere.

In this case the Hairy Ball Theorem says that every continuous vector field on an even dimensional sphere, has a zero (whereas the odd dimensional spheres have non vanishing continuous vector fields). We will not prove this theorem as the proof is found in every first year algebraic topology textbook (see Corollary 21.6 on page 120 of [Mu2], or Theorem 2.28 on page 135 of [Ha]) and is well-known in general. Using this we prove the oriented versions of our theorems:

Theorem 2.5. *Let $\vec{\sigma} : S^{2n} \to \mathbb{R}^{2n+1}$ be a continuous oriented Kakeya line configuration. Then $|\vec{\sigma}| = \mathbb{R}^{2n+1}$.*

This theorem fails in $\mathbb{R}^{2n}$, even for continuous sections. In fact, for any compact subset K of $\mathbb{R}^{2n}$, there is a continuous oriented Kakeya configuration that is disjoint from K.

Proof. Let $p \circ \vec{\sigma}$ denote the corresponding continuous section of the tangent bundle of S^{2n}. By the Hairy Ball Theorem, $p \circ \vec{\sigma}$ has a zero. In other words, there exists a point $x \in S^{2n}$ such that $\vec{\sigma}(x)$ is normal to the sphere and hence lies in the line that x generates. Thus $|\vec{\sigma}| = \bigcup_{y \in S^{2n}} (\vec{\sigma}(y) + L_y)$ has L_x, the line through x and the origin contained in $|\vec{\sigma}|$. Thus $\vec{\sigma}$ contains the origin. From here the same trick with translations used in the unoriented case, shows that $|\vec{\sigma}| = \mathbb{R}^{2n+1}$.

To show that this fails for $\mathbb{R}^{2n}$ i.e., odd spheres, even with continuity, take any nowhere vanishing vector field on S^{2n-1}. For any $C > 0$, one can scale the vector field so that its norm at any point is greater than or equal to C. Then it is not hard to see that $|\vec{\sigma}|$ is a union of lines each with distance $\geq C$ from the origin. Thus $|\vec{\sigma}|$ is disjoint from the open ball of radius C about the origin. Since C is arbitrary, we can find examples of continuous oriented Kakeya line configurations where $|\vec{\sigma}|$ is disjoint from any prescribed bounded set in $\mathbb{R}^{2n}$.

For an explicit example, the reader should consider the collection of (counter clockwise oriented) tangent lines to a standard circle of radius C about the origin in $\mathbb{R}^2$, as in Figure 1.

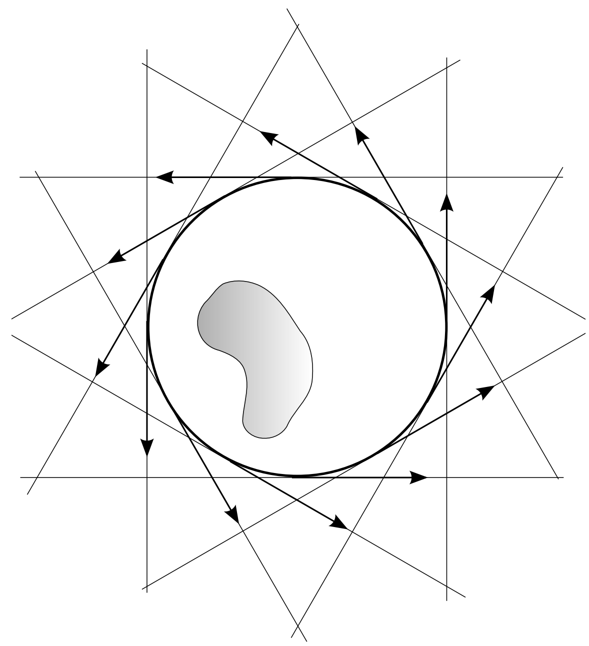

FIGURE 1. An oriented Kakeya needle set excluding a bounded region.

This collection has $|\vec{\sigma}|$ equal to the complement of the open disk of radius C and hence $|\vec{\sigma}| \neq \mathbb{R}^2$. As we have shown, these examples only exist in even Euclidean dimensions and in the case of oriented configurations. This example does not work in the unoriented case as it has two parallel lines in each parallel class and so there is no function $\vec{\sigma}\colon \mathbb{R}P^1 \to \mathbb{R}^2$ corresponding to this motion in the unoriented case. □

It follows from this theorem, that any continuous oriented Kakeya needle set in $\mathbb{R}^{2n+1}$ has an R-elongation with positive Lebesgue measure just as in the unoriented case.

3. Continuous Kakeya Line Configurations in connected Lie groups

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Recall that the Lie subalgebras of $\mathfrak{g}$ are in bijective correspondence with the connected subgroups of G. (See [Wa], chapter 3). The 1-dimensional subspaces of $\mathfrak{g}$ are automatically Lie subalgebras and are in bijective correspondence (exponential-logarithm correspondence) with the 1-parameter subgroups H of G. In the case of $G = (\mathbb{R}^n, +)$, these are exactly the lines through the origin. For any $x \in G$, xH is a left coset of H which will serve as a "parallel" copy of H going through the point x.

Thus in the setting of connected Lie groups, the left cosets of 1-parameter subgroups will play the role that lines did in $\mathbb{R}^n$. $P(\mathfrak{g})$, the projective space of the vector

space $\mathfrak{g}$, then serves as the space of "parallel-classes" i.e., the space of 1-parameter subgroups of G. For $L \in P(\mathfrak{g})$, we will denote the corresponding 1-parameter group by $\exp(L)$. These constructions have many analogies with the $\mathbb{R}^n$-case (and indeed reduce to it when $G = (\mathbb{R}^n, +)$):
(1) Through any point x in G, and $L \in P(\mathfrak{g})$, there is a unique left coset ("parallel line") $x\exp(L)$ which goes through x.
(2) "Parallel lines" do not meet.
(3) For each $L \in \mathfrak{g}$, there is a smooth surjective homomorphism

$$(\mathbb{R}, +) \to \exp(L) \subseteq G.$$

However the reader is warned that even though this map is an immersion, the image can be a circle group (map need not be injective) and the image does not have to be closed in G (for example a dense line in a torus).

Similarly fixing a positive definite inner product on $\mathfrak{g}$, the sphere $S(\mathfrak{g})$ functions as the space of oriented one-parameter subgroups of G, where the orientation is induced by moving outward from 0 to the point $x \in S(\mathfrak{g})$ in the corresponding one-parameter subgroup of G.

We now make definitions that generalize those in $(\mathbb{R}^n, +)$.

Definition 3.1. Let G be a connected Lie group and $\mathfrak{g}$ be its Lie algebra. Fix a positive definite inner product on $\mathfrak{g}$. A continuous unoriented Kakeya line configuration in G is a continuous map $\sigma : P(\mathfrak{g}) \to G$. The underlying space of σ, $|\sigma|$ is defined as

$$|\sigma| = \bigcup_{L \in P(\mathfrak{g})} (\sigma(L) \star \exp(L))$$

where $\star$ denotes multiplication in G.

The associated unoriented Kakeya needle set is $|\hat{\sigma}| = \bigcup_{L \in P(\mathfrak{g})} (\sigma(L) \star \exp(I(L)))$ where $I(L)$ is the interval of length 1 centered around the origin in L. For $1 \leq R < \infty$, the R-elongation of the Kakeya needle set is the set obtained when $I(L)$ is replaced with an interval of length R centered about the origin in L, in the last definition.

We define similar oriented versions of these concepts where a continuous oriented Kakeya line configuration in G is given by a continuous map $\sigma : S(\mathfrak{g}) \to G$, where $S(\mathfrak{g})$ is the unit-sphere in $\mathfrak{g}$.

A difficulty one first encounters in a general Lie group is that the underlying space $|\sigma|$ of a configuration σ lives in G while the direction space $P(\mathfrak{g})$ lives in the Lie algebra $\mathfrak{g}$. To help deal with this we make use of the exponential map $\exp : \mathfrak{g} \to G$ and make the following definition:

Definition 3.2. A continuous unoriented Kakeya line configuration $\sigma : P(\mathfrak{g}) \to G$ is called **linear** if there exists a continuous lift $\mu : P(\mathfrak{g}) \to \mathfrak{g}$ such that $\exp \circ \mu = \sigma$. We make a similar definition in the oriented case where $S(\mathfrak{g})$ replaces $P(\mathfrak{g})$.

The exponential map is a smooth map from $\mathfrak{g}$ to G but it need not be surjective in general even for connected Lie groups. (For example for the connected Lie group $G = SL(2, \mathbb{R})$ the exponential map is not surjective.) Thus not every continuous Kakeya line configuration in G will be linear, for example if the image of σ does not lie in the image of the exponential map, there can be no linearizing lift.

However as the derivative of the exponential map at $0 \in \mathfrak{g}$ is the identity map $\mathfrak{g} \to \mathfrak{g}$, the exponential induces a diffeomorphism from an open neighborhood of 0 in $\mathfrak{g}$ to an open neighborhood of the identity element e in G. We make the following definition:

Definition 3.3. A continuous unoriented Kakeya line configuration $\sigma : P(\mathfrak{g}) \to G$ is called **taut** if the image of σ lies in an open neighborhood of e in G which is diffeomorphic to an open neighborhood of 0 in $\mathfrak{g}$ via (the local inverse of) the exponential map. We make similar definitions in the oriented case. Note such configurations are automatically linear.

Thus every connected Lie group has a lot of taut configurations, though in general the inclusions

$$\text{Taut configurations} \subseteq \text{ Linear configurations} \subseteq \text{ Continuous configurations}$$

are all proper. In $(\mathbb{R}^n, +)$ these 3 notions coincide and we will see that in connected nilpotent Lie groups of dimension > 2, that every continuous configuration is linear.

First we prove analogs of our results in $\mathbb{R}^n$ with restrictions:

Theorem 3.4 (Linear Configurations)**.** *Let G be a connected Lie group and let $\sigma : P(\mathfrak{g}) \to G$ be a linear unoriented configuration. Then $e \in |\sigma|$. A similar result holds for linear oriented configurations as long as the dimension of G is odd.*

Proof. Let $\mu : P(\mathfrak{g}) \to \mathfrak{g}$ denote a continuous lift of σ. By the proof of Theorem 2.3, we find that there must exist a line $L_0 \in P(\mathfrak{g})$ such that $\mu(L_0) \in L_0$. Thus $\sigma(L_0) \in \exp(L_0)$. This in turn implies $\exp(L_0) \subseteq \bigcup_{L \in P(\mathfrak{g})} (\sigma(L) \star \exp(L)) = |\sigma|$. As $e \in \exp(L_0)$, we are done.

The proof of the oriented case proceeds similarly using the proof of Theorem 2.5 which imposes the restriction that the dimension of $\mathfrak{g}$ and hence of G is odd. $\square$

The problem now is that the translation trick that worked in $(\mathbb{R}^n, +)$ does not necessarily work in a general Lie group as left translating (in G) a linear ("liftable") configuration need not yield a configuration which is also linear. One could left translate in the Lie algebra $\mathfrak{g}$ but this does not work well with respect to the exponential correspondence in general. For example

$$e^{\mathbb{A}+t\mathbb{B}} \neq e^{\mathbb{A}} e^{t\mathbb{B}}$$

when the matrices $\mathbb{A}$ and $\mathbb{B}$ don't commute, as one can readily check using the Baker-Campbell-Hausdorff identity.

Thus in a general connected Lie group we will settle for a partial result (though in nilpotent Lie groups we'll see we can do better!):

Theorem 3.5 (Taut Configurations)**.** *Let G be a connected Lie group and let $\sigma : P(\mathfrak{g}) \to G$ be a taut unoriented configuration. Then $|\sigma|$ contains an open neighborhood of the identity and hence has positive Haar measure. A similar result holds for taut oriented configurations as long as the dimension of G is odd.*

Proof. Let σ be a taut configuration. As projective spaces (respectively spheres) are compact, the image of σ is a compact subset of an open neighborhood U of the identity (which is diffeomorphic to an open neighborhood of 0 in $\mathfrak{g}$ under a local inverse of the exponential). As multiplication $M : G \times G \to G$ is continuous, $M^{-1}(U)$ is an open neighborhood of $e \times Image(\sigma)$ in $G \times G$. By the tube lemma (see [Mu]), one has an open neighborhood V of e such that $V \times Image(\sigma) \subseteq M^{-1}(U)$. Let $W = V \cap V^{-1}$ where $V^{-1} = \{x^{-1} | x \in V\}$ then W is an open neighborhood of

e in G such that for $w \in W$, with left translation operator T_w we have $T_w \circ \sigma$ is still a continuous configuration with image in U and hence is still taut.

Hence $|T_w \circ \sigma| = w \star |\sigma|$ contains e by Theorem 3.4. Thus $|\sigma|$ contains w^{-1} for all $w \in W$ and hence $W \subseteq |\sigma|$ which proves the theorem in the unoriented case. The proof of the oriented case is similar and left to the reader.

□

In general, in order to get stronger results in a connected Lie group, we have to look at the exponential map carefully. This is a well-studied subject. If E is the image of the exponential map, it is a basic fact that E generates the connected Lie group G as a group. In fact $E \star E = G$ (see [MoS]). However in general, as already remarked $E \neq G$. It is known that $E = G$ in the case of compact connected Lie groups, connected nilpotent groups and connected solvable Lie groups of type E. (see [MoS]).

Of particular use for us is a result of Dixmier and Saito (see [Di] and [Sa]) which states that for simply-connected solvable groups, the exponential map is surjective iff it is bijective iff it is a global diffeomorphism. These in turn are shown to be equivalent to a condition that in the adjoint representation of the Lie algebra, no nontrivial purely imaginary roots exist (solvable Lie algebras of type E). In particular this applies to the case of nilpotent Lie algebras. Thus in a simply-connected nilpotent Lie group, the exponential map is a diffeomorphism $\mathfrak{g} \to G$ just as in the case $G = (\mathbb{R}^n, +)$. Here there is then no difference between the notions of taut, linear and continuous configurations. Since a connected nilpotent Lie group (or more generally a connected solvable Lie group of type E) is covered by a simply-connected one, and since the exponential maps commute with covering maps, it follows that that $exp : \mathfrak{g} \to G$ is a covering map. Basic covering space theory then says that a continuous Kakeya line configuration $\sigma : P(\mathfrak{g}) \to G$ has a continuous lift, i.e., is linear if and only if σ_*, the induced homomorphism of fundamental groups, is trivial.

The fundamental group $\pi_1(G)$ (we will omit base points as we are discussing path-connected spaces) is abelian as Lie groups are H-spaces and furthermore the fundamental group of a nilpotent connected Lie group (or more generally a connected solvable Lie group of type E) acts freely on its universal cover $\mathbb{R}^n$ which implies it is torsion free—this is Theorem 19.3 on page 103 of [M] (see also exercise 2 on page 36 [Br] for some facts needed for the proof of Theorem 19.3). On the other hand, $\pi_1(P(\mathfrak{g}))$ is isomorphic to the cyclic group of order two when $n > 2$, the infinite cyclic group when $n = 2$ and is trivial when $n \leq 1$. Thus σ_* must be trivial as long as $n \neq 2$ as there is no nontrivial group homomorphism from a torsion group to a torsion-free group.

We have thus proved:

Theorem 3.6. *If G is a connected nilpotent Lie group (or more generally a connected solvable Lie group of type E) of dimension > 2, then every continuous unoriented or oriented Kayeka configuration σ is linear. Furthermore $|\sigma| = G$ in the unoriented case and also in the oriented case when the dimension of G is odd.*

Proof. As explained in the paragraph preceding this theorem, any continuous unoriented configuration is linear (lifts) as long as we are in dimensions > 2. In the case of oriented configurations, the statement is still true as spheres in real vector spaces of dimension > 2 are simply connected. Thus by Theorem 3.4, we conclude

$e \in |\sigma|$ for **any** unoriented configuration or oriented configuration in odd dimensions. Now we can play the translation trick as for any T_x, left translation operator, $T_x \circ \sigma$ is still a configuration and so $|T_x \circ \sigma| = x \star |\sigma|$ contains e for all $x \in G$ and hence $|\sigma|$ contains x^{-1} for all $x \in G$ from which it follows that $|\sigma| = G$. □

Theorem 3.6 does not fully hold in dimension 2 as there exist configurations which do not lift to the Lie algebra, i.e., that are not linear. However it turns out that it is still true that $|\sigma| = G$ for continuous unoriented Kakeya configurations. We now seek to remove the dimension > 2 constraint in this regard for theorem 3.6. The only nontrivial case to prove it for is the case of dimension 2.

In dimension 2, the only connected Lie groups up to isomorphism are $(\mathbb{R}^2, +)$, the 2-torus $S^1 \times S^1$, the cylinder $S^1 \times \mathbb{R}$ and the affine group $\{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} | a > 0, b \in \mathbb{R}\}$ which is non-abelian but solvable. This is because there are only two 2-dimensional Lie algebras, the abelian one, which corresponds to the simply connected group $(\mathbb{R}^2, +)$ and the non-abelian one, which has simply connected Lie group equal to the affine group. As the affine group is centerless, it is the only connected Lie group with a non-abelian 2-dimensional Lie algebra. Since the only discrete subgroups of $(\mathbb{R}^2, +)$ are free-abelian of rank ≤ 2, it only covers cylinders and tori, and so the above list is complete. (Any rank 2 lattice in $\mathbb{R}^2$ is isomorphic to the standard one under an automorphism of $(\mathbb{R}^2, +)$ so all the tori and cylinders are isomorphic to the standard ones.)

The exponential map is a diffeomorphism in the case of $(\mathbb{R}^2, +)$ and the affine group and so Theorem 3.6 holds for them as every continuous unoriented Kakeya configuration is linear.

If $\sigma : RP^1 \to S^1 \times S^1$ is a continuous unoriented Kakeya configuration in the torus, the image of $\sigma_* : \pi_1(RP^1) = \mathbb{Z} \to \pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$ is free abelian of rank ≤ 1. If the image of σ_* is trivial, then the same is true for any translate (they are homotopic) and both the configuration and its translates are linear and one can then readily prove $|\sigma| = G$ using the translation trick. Otherwise the image of σ_* is free abelian of rank 1. The corresponding covering group of $S^1 \times S^1$ is then a cylinder and the configuration and all its translates lift to this cylinder.

Thus to prove that $|\sigma| = G$ also holds for the torus $S^1 \times S^1$ it reduces to proving it for the cylinder $S^1 \times \mathbb{R}$.

Thus to extend the part of Theorem 3.6 which states that $|\sigma| = G$ so that it holds for all connected solvable Lie groups of type E, it remains only to prove it holds in the case $G = S^1 \times \mathbb{R} \cong \mathbb{C}^*$.

Let $\sigma : RP^1 \to \mathbb{C}^*$ be continuous. We can and will view σ as a continuous map $S^1 \to \mathbb{C}^*$ such that $\sigma(u) = \sigma(-u)$, i.e., we will consider the oriented Kakeya configuration which goes through the same collection of lines twice, assigning opposite orientations on each run through. The underlying set of this oriented configuration is the same as its unoriented counterpart. Note $exp : \mathbb{C} \to \mathbb{C}^*$, the complex exponential, is the exponential map for the cylinder Lie group $\mathbb{C}^*$ and it is a covering map with kernel $K = \{2\pi i n | n \in \mathbb{Z}\}$. Covering space theory says we can lift σ to a path $p : [0, 2\pi] \to \mathbb{C}$, where at $p(t)$, is attached a parallel copy L_t of a line of angle t radians with respect to the x-axis, oriented outward from the origin. (As the Lie group $\mathbb{C}^*$ is abelian, the exponential map takes all lines, even those not through the origin, in $\mathbb{C}$ to cosets of one-parameter subgroups of $\mathbb{C}^*$.) To show the underlying set of any continuous unoriented Kakeya configuration in $\mathbb{C}^*$ contains the identity element 1, it then is sufficient to show that $|p| = \bigcup_{t\in[0,2\pi]} L_t$ contains some point of K. If we can show this, then every continuous unoriented Kakeya configuration in the cylinder $\mathbb{C}^*$ will contain the identity element 1 and we can use the translation trick to show $|\sigma| = \mathbb{C}^*$ for all of them. As mentioned before, this will complete the proof that $|\sigma| = G$ for connected Lie groups in dimension 2.

Thus let $p : [0, 2\pi] \to \mathbb{C}$ be a continuous map such that L_t is a line that makes angle t radians with the x-axis, going through the point $p(t)$. We need to show $|p| = \bigcup_{t\in[0,2\pi]} L_t$ contains some point of K. Suppose this were not true, i.e., $|P| \cap K = \emptyset$.

Define $H : [0, 2\pi] \times \mathbb{R} \to \mathbb{C} - K$ via $H(t, s) = p(t) + se^{it}$. Notice for fixed t, and varying s, $H(t, s)$ sweeps out the line L_t. Also notice that $Image(H)$ is exactly $|p|$ and that H is continuous. Now notice that on each line of fixed t, the map H is proper, thus it extends to a continuous map between one-point compactifications of $\mathbb{R}$ and $\mathbb{C}$ and gives continuous $\bar{H} : [0, 2\pi] \times S^1 \to S^2 - K$ where S^2 is the Riemann sphere. Thus $\bar{H}$ gives a base point preserving homotopy of maps $h_t : S^1 \to S^2 - K$ where the base points are the points at infinity. Notice as $h_{\frac{\pi}{2}}$ corresponds to a vertical line which is disjoint from K, it must be a line $x = c$ where $c \neq 0$, on the other hand h_0 corresponds to a horizontal line which is disjoint from K and so it must be a line $y = d$ where $d \neq 2\pi n$ for any integer n. Let a and b be points in K that are below and above d respectively.

We can then view $\bar{H}$ as a continuous base point preserving homotopy $[0, 2\pi] \times S^1 \to S^2 - \{a, b\}$. $S^2 - \{a, b\}$ is homeomorphic to the punctured plane, $\mathbb{R}^2 - \{0\}$, under stereographic projection from the point $a \in S^2$, composed with a translation that ensures the point b maps to 0. Under this modified stereographic projection Ψ, the map $h_{\frac{\pi}{2}}$ clearly projects to a null-homotopic curve $S^1 \to \mathbb{R}^2 - \{0\}$ while the map $h_0 : S^1 \to \mathbb{R}^2 - \{0\}$ projects to a curve with nonzero winding number about the origin. As these are homotopic via $\Psi \circ \bar{H}$, we achieve a contradiction.

Thus no continuous line configuration $p : [0, 2\pi] \to \mathbb{C}$ can have $|p|$ disjoint from K and so no continuous unoriented Kakeya configuration in $\mathbb{C}^*$ can have $|\sigma|$ not contain the identity element 1. Thus every continuous unoriented Kakeya configuration in the cylinder, has underlying space which contains the identity. We are hence done proving $|\sigma| = G$ in dimension 2 as mentioned earlier.

Thus we have proved:

Corollary 3.7. *Let G be a connected nilpotent Lie group (or more generally a connected solvable Lie group of type E) and let σ be a continuous unoriented Kakeya configuration. Then $|\sigma| = G$. The same result holds in the oriented case when the dimension of G is odd.*

4. Continuous Kakeya Line Configurations in homogeneous spaces

Let G be a connected Lie group and K a closed subgroup. Then it is well known that the projection map $\pi : G \to K\backslash G$ gives a fiber bundle with fiber K and that the homogeneous space $K\backslash G$ is a smooth manifold (see chapter 3 of [Wa]). In fact, this fiber bundle is a principal K-bundle. The translates of one-parameter subgroups in G map to a distinguished class of curves in $K\backslash G$. (These can be constant curve if the one-parameter subgroup lies in K for example).

Definition 4.1. A continuous unoriented G-Kakeya configuration in $X = K\backslash G$ is a continuous map $\sigma : P(\mathfrak{g}) \to X$. We define $|\sigma| = \bigcup_{L\in\mathfrak{g}}(\sigma(L) \star \exp(L))$ where $\star$

denotes the right action of G on X. We say the configuration lifts to a configuration in G if there is a continuous unoriented Kakeya configuration in G, $\hat{\sigma} : P(\mathfrak{g}) \to G$ such that $\pi \circ \hat{\sigma} = \sigma$ where $\pi : G \to K\backslash G$ is the quotient map. Notice in this case that $|\sigma| = \pi(|\hat{\sigma}|)$. We make similar definitions for oriented Kakeya configurations where $P(\mathfrak{g})$ is replaced with $S(\mathfrak{g})$ for some choice of positive definite inner product on $\mathfrak{g}$.

Notice that as fiber bundles are Serre fibrations, the question of when a continuous map like σ lifts is a homotopy question, i.e., if σ_1 is homotopic to σ_2 then σ_1 has a continuous lift if and only if σ_2 does. Thus in asking when a G-Kakeya configuration σ in X lifts to one in G, only the unbased homotopy class of σ is relevant.

The following lemma follows immediately from this observation:

Lemma 4.2. *Let $\sigma : P(\mathfrak{g}) \to X = K\backslash G$ be a continuous unoriented G-Kakeya configuration in X whose image set lies in a contractible subspace of X (like a chart). Then σ is homotopic to a constant map and hence has a continuous lift to a continuous unoriented Kakeya configuration in G. Similar statements hold for oriented configurations.*

In general a continuous oriented G-Kakeya configuration $\sigma : S(\mathfrak{g}) \to X = K\backslash G$ determines a unbased homotopy class in $[S^{n-1}, X]$ where n is the dimension of G. As X is path connected, this in turn determines a $\pi_1(X)$-orbit C in $\pi_{n-1}(X)$ under the action of $\pi_1(X)$ on $\pi_*(X)$. The configuration lifts to a continuous oriented Kakeya configuration in G if and only if some element in C is in the image of $\pi_* : \pi_{n-1}(G) \to \pi_{n-1}(X)$ if and only if some element in C is in the kernel of the boundary operator $\partial : \pi_{n-1}(X) \to \pi_{n-2}(K)$. Using these observations and similar variants for projective space, the liftability question can be resolved in most examples.

Once the configuration is lifted to the Lie group, all of our previous results apply. As an example we record the following corollary:

Corollary 4.3. *Let G be a connected solvable Lie group of type E and K a closed subgroup with corresponding homogeneous space $X = K\backslash G$. If σ is an unoriented continuous G-Kakeya configuration in X which lifts to one in G, then $|\sigma| = X$. The same conclusions hold for oriented configurations when the dimension of G is odd.*

Let us look at one simple example to illustrate working with Kakeya configurations in general homogeneous spaces. Let $G = SO(3)$, then G is a compact connected Lie group which is homeomorphic to $\mathbb{R}P^3$ as a space. The Lie algebra $\mathfrak{so}(3)$ is isomorphic to the Lie algebra given by $\mathbb{R}^3$ with the cross-product as bracket. Given a nonzero vector $\hat{v}$ in the Lie algebra, the exponential flow generates the rotation about the axis given by $\hat{v}$ in the counter clockwise direction (right hand rule). Thus the one-parameter subgroup of $SO(3)$ corresponding to any line in $\mathbb{R}^3$ is just the set of rotations about that line.

Now consider the homogeneous space $SO(2)\backslash SO(3)$ which is diffeomorphic to the standard sphere $S^2 \subseteq \mathbb{R}^3$. Under the quotient map, the translates of one-parameter subgroups map to closed geodesics (great circles or constant curves). A continuous oriented $SO(3)$-Kakeya configuration in S^2 is then a map $\sigma : S(\mathbb{R}^3) \to S^2$ which represents a continuously varying family of these curves in S^2 which cover all the "directions" in $SO(3)$. (Thus there can be redundancy in the directions in S^2).

(To avoid such redundancy in general, a choice of identification of the tangent space of the homogeneous space with a subspace of the tangent space of the Lie group G can be chosen and only the corresponding subspace of the projective space of $\mathfrak{g}$ used. However this requires a choice of horizontal lift in the fiber bundle, i.e., a choice of connection for the principal K-bundle. We will not pursue this topic here.)

Since $\pi_2(SO(3)) = 0$, an oriented $SO(3)$-Kakeya configuration in S^2, $\sigma : S(\mathbb{R}^3) = S^2 \to S^2$ lifts to an oriented Kakeya configuration in $SO(3)$ if and only if it has degree 0 as a map, i.e., is homotopic to a constant.

5. Acknowledgements

We would like to thank Michael Gage and Alex Iosevich for many useful discussions, and we would like to thank the anonymous referee for their helpful suggestions.

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