1 Chain conditions

We next consider sequences of ideals in a ring $R$.

**Definition 1.1 (Chains).** A sequence of (left) (right) ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \ldots$ is called an ascending chain of (left) (right) ideals.

A sequence of (left) (right) ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ldots$ is called a descending chain of (left) (right) ideals.

A chain is said to stabilize if there exists $N \in \mathbb{N}$ such that $I_N = I_{N+k}$ for all $k \in \mathbb{N}$.

**Definition 1.2 (Noetherian rings).** A ring $R$ is called (left)(right)(two-sided) Noetherian if every ascending chain of (left)(right)(two-sided) ideals in $R$ stabilizes.

Note that if a ring $R$ is either left or right Noetherian then it is two-sided Noetherian.

A ring $R$ is called Noetherian if it is both left and right Noetherian.

**Definition 1.3 (Artinian rings).** A ring $R$ is called (left)(right)(two-sided) Artinian if every ascending chain of (left)(right)(two-sided) ideals in $R$ stabilizes.

Note that if a ring $R$ is either left or right Artinian then it is two-sided Artinian.

A ring $R$ is called Artinian if it is both left and right Artinian.

The following is an important characterization of Noetherian rings:
Theorem 1.4 (Noetherian rings). Let $R$ be a ring. The following are equivalent:

1. $R$ is left Noetherian.
2. Every nonempty set $S$ of left ideals of $R$ has a maximal element $I$, i.e., there exists $I \in S$ such that $J \in S$, $I \subseteq J$ implies $I = J$.
3. Every left ideal of $R$ is finitely generated.

Similar theorems hold for right and two-sided Noetherian rings.

Proof. (1) $\implies$ (2): Suppose there exists a nonempty set of left ideals $S$ of $R$ without a maximal element. Note for any $I_k \in S$. Since $I_k$ is not maximal in $S$, we may find $I_{k+1} \in S$ such that $I_k \subset I_{k+1}$ (we will use $\subset$ to denote proper subsets). Thus since $S$ is not empty, we may construct an infinite chain $I_1 \subset I_2 \subset I_3 \subset I_4 \subset \ldots$ which does not stabilize, contradicting the fact that $R$ is left Noetherian. Thus every nonempty set of left ideals of $R$ must have a maximal element.

(2) $\implies$ (1): Let $S = \{I_n\}_{n=1}^{\infty}$ be an ascending chain of left ideals. By assumption $S$ has a maximal element say $I_N$. Since $I_N$ is maximal and $I_N \subseteq I_n$ for all $n \geq N$ we see that $I_N = I_n$ for all $n \geq N$ and so the chain stabilizes. Thus every ascending chain of left ideals stabilizes and so $R$ is left Noetherian.

(3) $\implies$ (1): Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be an ascending chain of left ideals in $R$. Then $I = \bigcup_{n=1}^{\infty} I_n$ is a left ideal of $R$. By assumption $I = (x_1, \ldots, x_k)_L$. For each $1 \leq j \leq k$, $x_j \in I$ so $x_j \in I_{n_j}$ for some positive integer $n_j$. Thus $\{x_1, \ldots, x_k\} \subseteq I_{\max(n_1, n_2, \ldots, n_k)}$ from which it follows that $I \subseteq I_{\max(n_1, n_2, \ldots, n_k)}$ and hence $I = I_{\max(n_1, n_2, \ldots, n_k)}$. Setting $N = \max(n_1, n_2, \ldots, n_k)$ it then follows easily that $I_N = I_{N+s}$ for $s \in \mathbb{N}$ and so the chain stabilizes. Thus any ascending chain of left ideals stabilizes and so $R$ is left Noetherian.

(1) $\implies$ (3): Let $I$ be a left ideal of $R$. Suppose that $I$ is not finitely generated as a left ideal of $R$. Then in particular $I \neq (0)$. Hence we may choose nonzero $a_1 \in I$. Suppose we have chosen $a_1, \ldots, a_k \in I$. Then since $I$ is not finitely generated as a left ideal, there exists $a_{k+1} \in I - (a_1, \ldots, a_k)_L$. Thus using the axiom of countable choice we may choose $a_1, a_2, \ldots$ such that the left ideals $(a_1, \ldots, a_k)_L = I_k$ form an infinite ascending chain of left ideals which does not stabilize. This contradicts the fact that $R$ is left Noetherian and so we conclude that every left ideal is finitely generated.

The following is an important corollary:
Corollary 1.5 (Principal Ideal rings are Noetherian). Let $R$ be a principal ideal ring, then $R$ is a Noetherian ring. Thus every PID is a Noetherian integral domain.

Proof. In a principal ideal ring $R$, every left or right ideal is generated by a single element and hence in particular, it is finitely generated. Thus $R$ is a Noetherian ring by the Theorem 1.4.

It is a deeper theorem that every Artinian ring is Noetherian. (we will see this next semester time permitting.)

The converse is not true in general as the following example shows:

Example 1.6 (Z is a Noetherian ring which is not Artinian). We have seen that $\mathbb{Z}$ is a PID and so it is Noetherian by Corollary 1.5. However it is not Artinian. For example, if $n > 1$ is an integer then the descending chain of ideals $(n) \supset (n^2) \supset (n^3) \supset (n^4) \supset \ldots$ does not stabilize.

This example shows that even a PID (where every ideal is generated by a single element) might not be Artinian!