MATH 436 Notes: Chain conditions.

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1 Chain conditions

We next consider sequences of ideals in a ring R.

Definition 1.1 (Chains). *A sequence of (left) (right) ideals* $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ *is called an ascending chain of (left) (right) ideals.*

A sequence of (left) (right) ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$ *is called a descending chain of (left) (right) ideals.*

A chain is said to stabilize if there exists $N \in \mathbb{N}$ *such that* $I_N = I_{N+k}$ *for all* $k \in \mathbb{N}$*.*

Definition 1.2 (Noetherian rings). *A ring* R *is called (left)(right)(two-sided) Noetherian if every ascending chain of (left)(right)(two-sided) ideals in* R *stabilizes.*

Note that if a ring R *is either left or right Noetherian then it is two-sided Noetherian.*

A ring R *is called Noetherian if it is both left* **and** *right Noetherian.*

Definition 1.3 (Artinian rings). *A ring* R *is called (left)(right)(two-sided) Artinian if every ascending chain of (left)(right)(two-sided) ideals in* R *stabilizes.*

Note that if a ring R *is either left or right Artinian then it is two-sided Artinian.*

A ring R *is called Artinian if it is both left* **and** *right Artinian.*

The following is an important characterization of Noetherian rings:

Theorem 1.4 (Noetherian rings). *Let R be a ring. The following are equivalent:*
(1) R is left Noetherian.
(2) Every nonempty set S of left ideals of R has a maximal element I, i.e., there exists $I \in S$ such that $J \in S$, $I \subseteq J$ implies $I = J$.
(3) Every left ideal of R is finitely generated.
Similar theorems hold for right and two-sided Noetherian rings.

Proof. **(1) $\implies$ (2):** Suppose there exists a nonempty set of left ideals S of R without a maximal element. Note for any $I_k \in S$. Since I_k is not maximal in S, we may find $I_{k+1} \in S$ such that $I_k \subset I_{k+1}$ (we will use $\subset$ to denote proper subsets). Thus since S is not empty, we may construct an infinite chain $I_1 \subset I_2 \subset I_3 \subset I_4 \subset \dots$ which does not stabilize, contradicting the fact that R is left Noetherian. Thus every nonempty set of left ideals of R must have a maximal element.
(2) $\implies$ (1): Let $S = \{I_n\}_{n=1}^{\infty}$ be an ascending chain of left ideals. By assumption S has a maximal element say I_N. Since I_N is maximal and $I_N \subseteq I_n$ for all $n \geq N$ we see that $I_N = I_n$ for all $n \geq N$ and so the chain stabilizes. Thus every ascending chain of left ideals stabilizes and so R is left Noetherian.
(3) $\implies$ (1): Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ be an ascending chain of left ideals in R. Then $I = \cup_{n=1}^{\infty} I_n$ is a left ideal of R. By assumption $I = (x_1, \dots, x_k)_L$. For each $1 \leq j \leq k$, $x_j \in I$ so $x_j \in I_{n_j}$ for some positive integer n_j. Thus $\{x_1, \dots, x_k\} \subseteq I_{max(n_1, n_2, \dots, n_k)}$ from which it follows that $I \subseteq I_{max(n_1, n_2, \dots, n_k)}$ and hence $I = I_{max(n_1, n_2, \dots, n_k)}$). Setting $N = max(n_1, n_2, \dots, n_k)$ it then follows easily that $I_N = I_{N+s}$ for $s \in \mathbb{N}$ and so the chain stabilizes. Thus any ascending chain of left ideals in R stabilizes and so R is left Noetherian.
(1) $\implies$ (3): Let I be a left ideal of R. Suppose that I is not finitely generated as a left ideal of R. Then in particular $I \neq (0)$. Hence we may choose nonzero $a_1 \in I$. Suppose we have chosen $a_1, \dots, a_k \in I$. Then since I is not finitely generated as a left ideal, there exists $a_{k+1} \in I - (a_1, \dots, a_k)_L$. Thus using the axiom of countable choice we may choose $a_1, a_2, \dots$ such that the left ideals $(a_1, \dots, a_k)_L = I_k$ form an infinite ascending chain of left ideals which does not stabilize. This contradicts the fact that R is left Noetherian and so we conclude that every left ideal is finitely generated. □

The following is an important corollary:

Corollary 1.5 (Principal Ideal rings are Noetherian). *Let R be a principal ideal ring, then R is a Noetherian ring. Thus every PID is a Noetherian integral domain.*

Proof. In a principal ideal ring R, every left or right ideal is generated by a single element and hence in particular, it is finitely generated. Thus R is a Noetherian ring by the Theorem 1.4. □

It is a deeper theorem that every Artinian ring is Noetherian. (we will see this next semester time permitting.)

The converse is not true in general as the following example shows:

Example 1.6 ($\mathbb{Z}$ is a Noetherian ring which is not Artinian). *We have seen that $\mathbb{Z}$ is a PID and so it is Noetherian by Corollary 1.5. However it is not Artinian. For example, if $n > 1$ is an integer then the descending chain of ideals $(n) \supset (n^2) \supset (n^3) \supset (n^4) \supset \dots$ does not stabilize.*

This example shows that even a PID (where every ideal is generated by a single element) might not be Artinian!