MATH 436 Notes: Series.

Jonathan Pakianathan

October 16, 2003

1 Series

We now generalize the process of cutting up a group into smaller parts with the concept of a series:

Definition 1.1 (Series). *Fix a group G and subgroups $H \leq K \leq G$.*

An ascending series connecting H and K, $\hat{G}_* \uparrow_H^K$ is a sequence of groups of the form:

$$H = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = K.$$

Note it is only assumed that G_i is normal in G_{i+1} and not neccesarily in G. If $G_i \trianglelefteq G$ for all i we call $\hat{G}_* \uparrow_H^K$ a normal series.

A descending series connecting K and H, $\hat{G}_* \downarrow_H^K$ is a sequence of groups of the form:

$$H = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = K$$

A normal descending series is a series where $G_i \trianglelefteq G$ for all i.

Notice we may reindex a series by $i \leftrightarrow n - i$ to change from a descending series to an ascending one and vice versa. For a finite series, we will often write $\hat{G}_*|_H^K$ if we do not want to emphasize the indexing. If we refer to $\hat{G}_*$ as a series for G without specifying H and K it is understood that $\hat{G}_* = \hat{G}_*|_e^G$, i.e., that $H = e$ and $K = G$.

We also will consider infinite series on occasion:
An infinite ascending series $\hat{G}_* \uparrow_H$ is one of the form:

$H = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots$.

Similarly an infinite descending series $\hat{G}_* \downarrow^K$ is one of the form:

$\dots \trianglelefteq G_2 \trianglelefteq G_1 \trianglelefteq G_0 = K$.

Definition 1.2 (Factors of a series). *Given an ascending series (finite or infinite) $\hat{G}_* \uparrow_H$, the factors of the series are $\{G_{i+1}/G_i | |G_{i+1}/G_i| > 1\}$. This set is denoted $Factor(\hat{G}_*)$. Note $Factor(\hat{G}_*|_H^K) = \emptyset$ when $H = K$ but otherwise is nonempty.*

The length of the series $\hat{G}_*$ is the cardinality of $Factor(\hat{G}_*)$. If $\hat{G}_*$ is a series for G we loosely think of G as being somehow cut up into pieces consisting of the groups in $Factor(\hat{G}_*)$.

Of course similar definitions hold for descending series, the only difference is that we replace G_{i+1}/G_i with G_i/G_{i+1}.

An important example to keep in mind is the following series associated to a semidirect product.

Example 1.3 (Series for the semidirect product). *Let H, K be groups and $G = K \rtimes_\phi H$ for some gluing map $\phi : H \to Aut(K)$. Then as long as $|K|, |H| > 1$ we have a series of length 2 for G given by:*

$$(e = G_0) \trianglelefteq (K = G_1) \trianglelefteq (G = G_2)$$

with factors $G_1/G_0, G_2/G_1$ isomorphic to K and H respectively. (To show $G/K \cong H$ is left as an exercise to the reader).

Notice how we can think of the group G as made up from the factors K and H in a certain way. There will be a similar idea for series of longer length. However the gluing can be more complicated than that of a semidirect product even for series of length 2 as you will see in the homework.

For example we have seen that $\Sigma_3 \cong \mathbb{Z}/3\mathbb{Z} \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$ and so there is a series for Σ_3 with factors isomorphic to $\{\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$. We will see later in our studies such a series with Abelian factors will indicate that we can solve certain equations associated to the group in a nice way. This motivates both the name and content of the following definition:

Definition 1.4 (Solvable Series and Groups). *A series $\hat{G}_*|_H^K$ is called solvable if all its factors are Abelian groups. A group G is called solvable if it has a solvable series, i.e., a series $\hat{G}_*|_e^G$ with Abelian factors.*

Before we go any further, we shall look at an important example of a solvable series called the derived series.

Example 1.5. *Let G be any group then the derived series $G_*^{Der} \downarrow^G$ is a descending series defined as follows: $G_0 = G$ and we then define inductively* *that $G_{i+1} = G_i' =< [x,y] | x, y \in G_i >$, the commutator subgroup of G_i for all $i \geq 0$. Recall that G_i' is fully invariant and hence in particular normal in G_i. Since the property of being fully invariant is transitive (unlike normality) we have each G_i fully invariant in G.*

Thus the series

$$\cdots \trianglelefteq G_{i+1} \trianglelefteq G_i \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G$$

is in particular a normal series.

Now notice $G_i/G_{i+1} = G_i/G_i' = (G_i)_{ab}$ is the abelianization of G_i and hence is an Abelian group.

In a finite group, we must have $G_n = G_{n+1}$ for some n in the derived series and then it is easy to see from the inductive definition that $G_n = G_m$ for all $m > n$. Thus the derived series will terminate at some subgroup H. This subgroup has the property that $H' = H$. We call such groups with $H = H'$ perfect groups.

So in a finite group, $\hat{G}_*^{Der} \downarrow_H^G$ where H is some perfect subgroup of G. If $H = e$ then $\hat{G}_*^{Der}$ is a solvable series for G and so G is solvable.

The next theorem illustrates that the converse is also true:

Theorem 1.6 (Derived series and Solvability). *Let G be any group and let $\hat{G}_*^{Der} : \ldots G_2 \trianglelefteq G_1 \trianglelefteq G_0$ be the derived series of G. If $\hat{S}_*$ is a solvable series for G, i.e., $e = S_n \trianglelefteq S_{n-1} \cdots \trianglelefteq S_1 \trianglelefteq S_0 = G$ with S_{i-1}/S_i Abelian for all $i = 1, \ldots, n$ then $G_i \leq S_i$ for all i.*

Thus G is solvable if and only if $\hat{G}_*^{Der} \downarrow_e^G$, i.e., $G_n = e$ for some n in the derived series.

Proof. We first prove $G_i \leq S_i$ by induction on i. For $i = 0$, we have $G_0 = S_0 = G$ so this is true in this case. Thus assume $i \geq 0$ and $G_i \leq S_i$ is proven, we seek to show $G_{i+1} \leq S_{i+1}$. Well since $G_i \leq S_i$ clearly $G_{i+1} = G_i' \leq S_i'$. Call this inclusion (*).

On the other hand considering the canonical quotient map $\phi : S_i \to S_i/S_{i+1}$ we have $S_i' \subseteq ker(\phi)$ as S_i/S_{i+1} Abelian. (All commutators are trivial in an Abelian group.) However $ker(\phi) = S_{i+1}$ and so we have $S_i' \leq S_{i+1}$. Using this in (*) we get $G_{i+1} \leq S_{i+1}$ as desired and so by induction the first part of the theorem is proven.

Now we seek to show that G is solvable if and only if $\hat{G}_*^{Der} \downarrow_e^G$.

$\leftarrow$: This is immediate as $\hat{G}_*^{Der}$ has Abelian factors in general so if it is a series for G (i.e., reaches the identity subgroup e), then G has a solvable series and so is solvable.

$\rightarrow$: If G is solvable, then we know there is some solvable series $\hat{S}_*|_e^G$ for G. However (after indexing $\hat{S}_*$ to be decreasing if need be) we have seen $G_i \subseteq S_i$ for all i and so $G_n \subseteq S_n = e$ for large n and so $\hat{G}_*^{Der} \downarrow_e^G$.

This completes the proof. $\square$

By Theorem 1.6, it follows that if G has a solvable series, $\hat{G}_*^{Der}$ will be a solvable series of "quickest descent". On the other hand it is perfectly possible that the derived series has $\hat{G}_*^{Der} \downarrow_H^G$ for some perfect subgroup H other than e. In this case Theorem 1.6 lets us know that G is not solvable, i.e., there is no solvable series at all possible for G.

Notice that any nonAbelian simple group H is perfect as H' is a normal nontrivial (as H is nonAbelian) subgroup of H and hence $H' = H$ by simplicity. The next example will give us an example of a situation where the derived series terminates at a nontrivial perfect subgroup:

Example 1.7 (Σ_n is solvable if and only if $n < 5$.). *For all $n \geq 2$, since Σ_n/A_n is cyclic of order 2, it is Abelian and so $\Sigma_n' \leq A_n$. On the other hand, if $n \geq 5$ then $A_n = A_n' \leq \Sigma_n' \leq A_n$ since A_n is a nonAbelian simple group. Thus $\Sigma_n' = A_n$ for $n \geq 5$. Since $A_n' = A_n$, the derived series for Σ_n looks like: $\ldots A_n \trianglelefteq A_n \trianglelefteq A_n \trianglelefteq \Sigma_n$ and so Σ_n is not solvable for $n \geq 5$.*

On the other Σ_1 and Σ_2 are Abelian and we have seen that Σ_3 has a solvable series with factors $\{\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$ and so these groups are solvable.

For Σ_4 note that we have a series:

$$e \trianglelefteq K \trianglelefteq A_4 \trianglelefteq \Sigma_4$$

where K is the Klein 4-group. The factors are isomorphic to $K, \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively and so this series is a solvable series for Σ_4. Hence Σ_4 is solvable.

2 Central Series and Nilpotent Groups

Definition 2.1. *If G is a group, an (ascending) central series connecting H and K is a normal series of the form $\hat{G}_* \uparrow_H^K$ such that $G_{i+1}/G_i \leq Z(G/G_i)$ for all i.*

Thus for all $g \in G$ and $\alpha \in G_{i+1}$ we have $g\alpha G_i = \alpha g G_i$. As G_i is normal in G by assumption, we may write this as $G_i g\alpha = G_i \alpha g$ and this is the same as saying $[g, \alpha] = g\alpha g^{-1}\alpha^{-1} \in G_i$ whenever $g \in G, \alpha \in G_{i+1}$.

Notice any central series has Abelian factors as G_{i+1}/G_i lies in the center of G/G_i which is Abelian.

Similar definitions and comments hold for descending central series.

Definition 2.2 (Nilpotent Groups). *Let G be a group. If G has a central series $\hat{G}_*|_e^G$ then we call G a nilpotent group. Since such a central series is also a solvable series for G, we see that nilpotent groups are always solvable.*

Basically the notion of a central series is stronger than that of a series with Abelian factors - to see that this is actually a strengthening we consider the following lemma:

Lemma 2.3 (Nontrivial nilpotent groups have nontrivial centers). *Let G be a nilpotent group with $|G| > 1$ then $|Z(G)| > 1$.*

Proof. Since G is nilpotent there is a central series $e = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$. Without loss of generality we may assume $G_1 \neq e$ as if not we can remove the redundant repeated terms.

Then G_1/G_0 central in G/G_0 by definition of a central series and so $G_1 \leq Z(G)$. This shows $Z(G) \neq e$ as desired. □

Example 2.4 (Σ_3 is solvable but not nilpotent). *We have previously seen that Σ_3 is solvable. However as $Z(\Sigma_3) = e$ we see that Σ_3 is not nilpotent by Lemma 2.3.*

Just as there is a canonical series for detecting solvability, there is also a canonical ascending central series:

Definition 2.5 (The Canonical Ascending central series $A\hat{C}C_*$). *We define $A\hat{C}C_* \uparrow_e$ inductively via $A\hat{C}C_0 = e$ and $A\hat{C}C_{i+1} = \phi^{-1}(Z(G/A\hat{C}C_i))$ where $\phi : G \to G/A\hat{C}C_i$ is the canonical quotient map. Since it is the preimage of a normal subgroup under a homomorphism, each $A\hat{C}C_i$ is normal in G.*

By definition, it is also clear that $A\hat{C}C_{i+1}/A\hat{C}C_i = Z(G/A\hat{C}C_i)$ and so this series is an ascending central series.

In a finite group G, it must eventually terminate at a subgroup H, i.e., $H = A\hat{C}C_n$ for all large enough n. This H will have $H \trianglelefteq G$ and G/H with trivial center as the reader can easily verify.

The first fact we prove is that the canonical ascending central series $A\hat{C}C_*$ is enough to determine nilpotency of a group:

Theorem 2.6. *Let G be a group and $e = S_0 \trianglelefteq S_1 \trianglelefteq \cdots \trianglelefteq S_n = G$ be a central series for G. Then $S_i \leq ACC_i$ for all i.*

Thus G is nilpotent if and only if $A\hat{C}C_* \uparrow_e^G$.

Proof. We first prove that $S_i \leq ACC_i$ by induction.

For $i = 0$ we have $S_0 = ACC_0 = e$. Thus assume that $i \geq 0$ and that we have proven $S_i \leq ACC_i$. We wish to show $S_{i+1} \leq ACC_{i+1}$.

Since $\hat{S}_*$ is a central series, for every $g \in G$, $\alpha \in S_{i+1}$ we have $[g, \alpha] \in S_i$ and hence $[g, \alpha] \in ACC_i$. This means $[g, \alpha]ACC_i = ACC_i$ and so $g\alpha ACC_i = \alpha g ACC_i$ for all $g \in G$. Thus $\alpha ACC_i \in Z(G/ACC_i) = ACC_{i+1}/ACC_i$ and so $\alpha \in ACC_{i+1}$. Thus $S_{i+1} \leq ACC_{i+1}$ as we desired to show and so by induction, the first part is proven.

Now we show that G is nilpotent if and only if $A\hat{C}C_* \uparrow_e^G$.

$\leftarrow$: If $A\hat{C}C_* \uparrow_e^G$ then it is a central series for G and hence G is nilpotent.

$\rightarrow$**:** If G is nilpotent, it has some central series $\hat{S}_*$. After reindexing to an increasing series if need be, the first part shows that $S_n \leq ACC_n$ for all n. Thus for large enough n, $G = S_n \leq ACC_n$ and so $ACC_n = G$, i.e., $A\hat{C}C_* \uparrow_e^G$. □

Thus we see that in the case of a nilpotent group, the canonical ascending central series is a central series for G which ascends "most quickly".

Proposition 2.7 (p-groups and Abelian groups are nilpotent). *Any Abelian group A is nilpotent. If p is a prime, any p-group P is nilpotent.*

Proof. For Abelian groups A, $ACC_0 = e$ and $ACC_1 = Z(A) = A$ and so they are nilpotent.

For any finite group $A\hat{C}C_* \uparrow_e^H$ where $H \trianglelefteq G$ and G/H has a trivial center. If G is a p-group, then G/H is a p-group and so it can have a trivial center only if it is trivial. Thus $G = H$ and so $A\hat{C}C_* \uparrow_e^G$ and G is nilpotent. □

We now go thru a series of important lemmas on the way to characterizing all finite nilpotent groups!!

Lemma 2.8 (Finite direct products of nilpotent groups are nilpotent). *If $G = G_1 \times \cdots \times G_n$ is a finite direct product of groups then if each G_i is nilpotent, so is G.*

Proof. It is simple to check that $Z(G) = Z(G_1) \times \dots Z(G_n)$ and that

$$G/Z(G) \cong G_1/Z(G_1) \times \cdots \times G_n/Z(G_n).$$

Using this, it is not hard to show by induction that

$$ACC_k(G) = ACC_k(G_1) \times \cdots \times ACC_k(G_n)$$

where $ACC_k(H)$ stands for the kth term in the canonical ascending central series for H. Since for any $1 \leq i \leq n$, $ACC_k(G_i) = G_i$ for large enough k, it follows that $ACC_k(G) = G$ for large enough k and so G is nilpotent. □

Lemma 2.9. *If G is a nilpotent group and $H < G$ is a proper subgroup then $H < N_G(H)$, i.e., H is a proper subgroup of its normalizer.*

Proof. Since $ACC_0 = e$ and $ACC_n = G$ for large enough n, we have an index s such that $ACC_s \subseteq H$ but $ACC_{s+1} \not\subseteq H$. Thus there is $\alpha \in ACC_{s+1} - H$. Now for all $h \in H$ we have $[\alpha, h] \in ACC_s \subseteq H$ since $A\hat{C}C_*$ is a central series. Thus $[\alpha, h] = \alpha h \alpha^{-1} h^{-1} \in H$ which gives $\alpha h \alpha^{-1} \in H$ for all $h \in H$. This shows that $\alpha \in N_G(H)$. Since $\alpha \notin H$ this shows $H < N_G(H)$. □

Lemma 2.10 (Frattini Argument). *If G is a finite group and p is a prime, with $P \in \mathcal{S}yl_p(G)$ then $N_G(N_G(P)) = N_G(P)$.*

Proof. First note that P is a Sylow p-subgroup for $N_G(P)$. Since $P \trianglelefteq N_G(P)$, it is the unique Sylow p-subgroup for $N_G(P)$. Let $\alpha \in N_G(N_G(P))$ then $\alpha P \alpha^{-1} \leq \alpha N_G(P) \alpha^{-1} = N_G(P)$. Thus $\alpha P \alpha^{-1}$ is a Sylow p subgroup of $N_G(P)$ also and so $\alpha P \alpha^{-1} = P$. Thus $\alpha \in N_G(P)$.

So we have seen $N_G(N_G(P)) \subseteq N_G(P)$. Since in general a group is always contained in its normalizer, we have $N_G(N_G(P)) = N_G(P)$ as desired. □

We are now ready to classify all finite nilpotent groups:

Theorem 2.11 (Characterization of finite nilpotent groups). *Let G be a finite group. Then the following are equivalent:*
(1) G is nilpotent.
(2) $H < G$ implies $H < N_G(H)$.
(3) For every prime p, G has a unique Sylow p subgroup.
(4) G is isomorphic to the direct product of its (nontrivial) Sylow p subgroups.

Proof. (1) $\to$ (2)**:** Follows from Lemma 2.9.
(2) $\to$ (3)**:** Let p be a prime, and $P \in \mathcal{S}yl_p(G)$. Then by Lemma 2.10, $N_G(N_G(P)) = N_G(P)$. By (2) then it follows that $N_G(P) = G$ and so $P \trianglelefteq G$. By Sylow theory, this implies there is a unique Sylow p subgroup.
(3) $\to$ (4)**:** If each Sylow p subgroup is unique for a given prime p, then they are each normal by Sylow theory. Let $P_1, \ldots, P_k$ be a list of the Sylow subgroups of G for the various primes dividing the order of G. Since each $P_i \trianglelefteq G$ then $P_1 \ldots P_i \trianglelefteq G$ for all i. Considering orders of elements we see that $P_i \cap P_j = e$ for $i \neq j$ and more generally $(P_1 \ldots P_i) \cap P_{i+1} = e$. Thus $P_1 \ldots P_{i+1}$ has two normal subgroups $H = P_1 \ldots P_i$ and $K = P_{i+1}$ such that $H \cap K = e$ and $P_1 \ldots P_{i+1} = HK$. Thus $P_1 \ldots P_{i+1}$ is isomorphic to the direct product $(P_1 \ldots P_i) \times P_{i+1}$. Applying this repeatedly to $G = P_1 \ldots P_k$ we conclude that G is isomorphic to the direct product of its nontrivial Sylow-p-subgroups.
(4) $\to$ (1)**:** Assume G is isomorphic to a direct product of its nontrivial Sylow p subgroups. Since p-groups are nilpotent, then finite direct products of p groups are nilpotent and so G will be nilpotent. □

3 Composition Series

In this section we will assume all series $\hat{G}_*|_H^K$ have been cleaned up so that $H = G_0 < G_1 < \cdots < G_n = K$ i.e., that no $G_i = G_{i+1}$. We can always clean up a series by removing redundant terms in a way that it will not change the set of factors of the series. (Recall the factors were the quotients G_{i+1}/G_i which were not trivial.)

Given a series $\hat{G}_* \uparrow_H^K$ we may insert a new term into the series as follows: $G_i \trianglelefteq N \trianglelefteq G_{i+1}$, where let us suppose $G_i \neq G_{i+1}$ as we said before.

Note that the factor G_{i+1}/G_i of the original series is replaced by potentially two new factors G_{i+1}/N and N/G_i. Of course when $N = G_i$ or $N = G_{i+1}$, there really is only one new nontrivial factor which is the same as that of the original series.

Thus we will call an insertion proper if $G_i < N < G_{i+1}$ and $N \trianglelefteq G_{i+1}$. Note that since $G_i \trianglelefteq G_{i+1}$ we have $G_i \trianglelefteq N$ automatically and also note a proper insertion yields a series with strictly more factors than the original. Finally note since $|G_{i+1}/G_i| = |N/G_i||G_{i+1}/N|$, the cardinality of the two new factors are a factorization of the cardinality of the original factor. Furthermore N/G_i is a normal subgroup of the original factor and G_{i+1}/N is a quotient of the original factor by this normal subgroup.

Definition 3.1. *A proper refinement of a series $\hat{G}_*|_H^K$ is a series $\hat{S}_*|_H^K$ which is obtained from $\hat{G}_*$ by a sequence of proper insertions.*

A series is called a composition series if it has no proper refinements.

We will next characterize composition series in a useful way:

Theorem 3.2 (Compositions series have simple factors). *Given two groups $A \trianglelefteq B$, the subgroups N with $A \leq N \trianglelefteq B$ correspond bijectively with the normal subgroups of B/A. Under this bijection the cases $N = A$ and $N = B$ correspond respectively with the subgroups e and B/A of B/A.*

Thus a series is a composition series if and only if each factor of the series is a simple group.

Proof. The second part of the theorem follows directly from the first part given the previous comments so we will prove only the first part.

Let $N(B/A)$ denote the set of normal subgroups of B/A and let $S = \{N | A \leq N \trianglelefteq B\}$. We define $\Theta : N(B/A) \to S$ as follows: Let $\phi : B \to B/A$ be the canonical quotient map. Then $\Theta : N(B/A) \to S$ is given by $\Theta(W) = \phi^{-1}(W)$.

We first check that Θ is a well-defined function. Note $\phi^{-1}(W) \leq B$ as it is the preimage of a subgroup under a homomorphism. It contains A as $A = \phi^{-1}(e) = \Theta(e)$ and $e \in W$. Finally it is normal in B since if $b \in B$ and $a \in \phi^{-1}(W)$ we have $\phi(bab^{-1}) = \phi(b)\phi(a)\phi(b)^{-1} \in W$ as $\phi(a) \in W$ and $W \trianglelefteq B/A$. Thus $bab^{-1} \in \phi^{-1}(W)$ for all $a \in \phi^{-1}(W), b \in B$ showing that $A \leq \phi^{-1}(W) \trianglelefteq B$ as desired. Thus Θ is a well-defined function.

Similarly we define $\Psi : S \to N(B/A)$ by $\Psi(N) = \phi(N) = N/A$. It is easy to show Ψ is a well-defined function too and that Ψ is a two-sided inverse for Θ. Thus Θ is a bijection and we are done.

□

Intuitively a composition series is one whose factors W can no longer be broken up into smaller bits N and W/N using some nontrivial proper normal subgroup N of W.

Let G be a finite group. Starting from any series for example, $e = G_0 \trianglelefteq G_1 = G$ for G, we can perform a sequence of proper insertions until we can no longer do any more. This will eventually happen as at each stage the factors will decompose into groups of strictly smaller cardinality. This shows that we can make a proper refinement of any series for the finite group to arrive at a composition series $\hat{C}_* \uparrow_e^G$ for the group. In particular, we have seen that every finite group has a composition series.

This proves the first part of the Jordan-Holder Theorem which is stated next:

Theorem 3.3 (Jordan-Holder Theorem). *Any finite group G has a composition series $\hat{C}_*|_e^G$.*

Furthermore for any two composition series $\hat{C}_*|_e^G$ and $\hat{D}_*|_e^G$ for G, there is a bijection between the factors of $\hat{C}_*$ and those of $\hat{D}_*$. Thus the set of simple groups together with their multiplicities that occur in a composition series for G is an invariant of G, independent of the particular composition series chosen.

We will not prove the second part of the Jordan-Holder theorem here - for a proof see Lang or Hungerford.

Example 3.4 (Infinite groups do not necessarily have composition series). *Suppose $\hat{C}_* \uparrow_e^{\mathbb{Z}}$ is a composition series for $\mathbb{Z}$. Then $C_0 = e = (0)$ but C_1 is some nontrivial subgroup of $\mathbb{Z}$. However we then know $C_1 = d\mathbb{Z}$ for some $d \geq 1$, and we may set $N = 2d\mathbb{Z}$ to get a proper insertion $C_0 \trianglelefteq N \trianglelefteq C_1$. This contradicts that $\hat{C}_*$ was a composition series for $\mathbb{Z}$. So we conclude that $\mathbb{Z}$ has no (finite) composition series $\hat{C}_*|_e^{\mathbb{Z}}$.*

However we may define for any prime p, a descending series $D_n = p^n\mathbb{Z}$. Then $D_n/D_{n+1} = p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ which is seen to be isomorphic to $\mathbb{Z}/p\mathbb{Z}$ via the isomorphism $\theta : p^n\mathbb{Z} \to \mathbb{Z}$ given by $\theta(p^n k) = k$ for $k \in \mathbb{Z}$.

Thus this series has simple factors all isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Furthermore $\cap_{n=1}^{\infty} D_n = (0) = e$. So this is like an infinite composition series connecting e and $\mathbb{Z}$. However notice that even if we allow these infinite composition series, we no longer have uniqueness since we can do this same process for various primes $p = 2, 3, 5, \ldots$ and get infinitely many distinct such series for $\mathbb{Z}$.

The following is an important property to keep in mind about solvable groups:

Proposition 3.5. *Any finite solvable group G has a composition series $\hat{C}_*|_e^G$ with factors of the type $\{\mathbb{Z}/p\mathbb{Z} | p$ various primes$\}$.*

In fact if $|G| = p_1^{m_1} \ldots p_k^{m_k}$ where $p_1 < \cdots < p_k$ are primes and $m_i \geq 1$ are multiplicites then any composition series for G has m_i factors isomorphic to $\mathbb{Z}/p_i\mathbb{Z}$ for $i = 1, \ldots, k$.

Proof. Since G is solvable, it has a solvable series $\hat{G}_*|_e^G$ with Abelian factors. Since G is finite, we can refine this series to get a composition series $\hat{C}_*|_e^G$. Since we started with Abelian factors and since under each proper insertion, the new factors are subgroups or quotients of the old ones, we see that the final composition series will also have Abelian factors. Since these factors are simple, then they must be of the form $\mathbb{Z}/p\mathbb{Z}$ for various primes p.

Now notice if $e = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$. Then $\prod_{i=0}^{n-1} |G_{i+1}/G_i| = |G_n/G_0| = |G|$ and so the orders of the factors provide a factorization of the order of G. This completes the proof of the proposition. □

This provides the following corollary:

Corollary 3.6. *A finite group G is solvable if and only if it has only Abelian factors in a (any) composition series $\hat{C}_*|_e^G$.*

All groups of order < 60 are solvable.

Proof. If G is solvable, the previous proposition showed that it has Abelian factors in a composition series and hence in any composition series $\hat{C}_*|_e^G$.

On the other hand if G has a composition series $\hat{C}_*|_e^G$ with Abelian factors, then $\hat{C}_*$ is also a solvable series for G and so G is solvable.

Finally we have seen (in Homework) that the smallest nonAbelian simple group has order 60 and so we conclude that the composition series for any group of size < 60 involves only Abelian factors and so any group of size < 60 is solvable.

□

Finally we warn the reader that the simple factors in a composition series do not determine the group. There are in general many groups with the same factors in their composition series. The "gluing" between the factors is crucial and has and will be explored further in the homework.

Example 3.7 (Distinct groups with the same composition factors). *We have seen that Σ_3 has a series with factors $\{\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$ which are hence the composition factors for Σ_3. However $\mathbb{Z}/6\mathbb{Z}$ is Abelian (hence solvable) so it has composition factors $\{\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$ given by a factorization of 6. Thus $\mathbb{Z}/6\mathbb{Z}$ and Σ_3 have the same composition factors but clearly they are not isomorphic groups.*

Recall $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ while $\Sigma_3 \cong \mathbb{Z}/3\mathbb{Z} \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$ for some nontrivial gluing map $\phi : \mathbb{Z}/2\mathbb{Z} \to Aut(\mathbb{Z}/3\mathbb{Z})$. Thus it is the way the factors are put together that created the distinct groups.

For another example $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ are groups which are not isomorphic. (As one has elements of order 4 while the other doesn't). They both are Abelian (hence solvable) so have composition factors $\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$. In this case $\mathbb{Z}/4\mathbb{Z}$ is not even a semidirect product of $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ but represents a "non-split" extension - a concept that will be explored in future homework.

We end with one final example:

Example 3.8 (Cooking up a finite group with any given set of composition factors). *Let $S_1, \ldots, S_N$ be a finite sequence of simple groups (repetitions allowed). Then we may create $G = S_1 \times S_2 \times \cdots \times S_N$ the direct product of these groups. (Recall, the group stucture is given by $(a_1, \ldots, a_N) \star (b_1, \ldots, b_N) = (a_1 b_1, \ldots, a_N b_N)$.)*

Then we may set $G_i = \{(s_1, \ldots, s_N) \in G | s_j = e \text{ for all } j > i\}$. Then it is easy to check that each G_i is a normal subgroup of G and that therefore $e = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$ is a normal series $\hat{G}_* \uparrow_e^G$.

Define $\theta_i : G_i \to S_i$ by $\theta_i((s_1, \ldots, s_i, e, e, \ldots, e)) = s_i$. Then it is easy to check that θ_i is an epimorphism with kernel G_{i-1} and hence by the first isomorphism theorem, $G_i/G_{i-1} \cong S_i$.

Thus $\hat{G}_*|_e^G$ is a composition series for G with factors given by the original sequence of simple groups.

Since this construction works for any given finite sequence of simple groups, we see that every possible such sequence occurs as the composition factors of at least one finite group.