Parabolic Mandelbrot Set.

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§1. Moduli Space $\mathcal{M}_2$.

$$f(z) = \frac{p(z)}{q(z)}, \quad p(z), q(z) \in \mathbb{C}[z], \quad (p(z), q(z)) = 1.$$

$$d = \max\{\deg(p(z)), \deg(q(z))\}.$$

$$\mathrm{Rat}_d := \left\{ f(z) = \frac{p(z)}{q(z)} \;\middle|\; \text{degree } d \right\}$$

We want $d=2$. If $p(z) = az^2+bz+c$, $q(z) = dz^2+ez+f$.

$$\rightsquigarrow (a:b:c:d:e:f) \in \mathbb{CP}^5$$

$$\mathrm{Result}(p(z), q(z)) = \det \begin{bmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{bmatrix} \neq 0$$

$$\mathrm{Rat}_2 \cong \{(a:b:c:d:e:f) \in \mathbb{CP}^5 \mid \mathrm{Result}(az^2+bz+c, dz^2+ez+f) \neq 0\}$$

↳ Zariski open set.

We can regard f as a dyn. syst. on $\hat{\mathbb{C}}$ = Riemann sphere.

$PGL(2,\mathbb{C})$ ($\cong \mathrm{Rat}_1$) acts on Rat_2 by conjugation.

$f \in \mathrm{Rat}_2$
$\phi \in PGL(2,\mathbb{C})$

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\ \phi \downarrow & & \downarrow \phi \\ \hat{\mathbb{C}} & \longrightarrow & \hat{\mathbb{C}} \end{array}$$

$\phi \circ f \circ \phi^{-1} =: f^{\phi}$
has degree 2.

We say that f is (holomorphically) conjugate to g if they belong to the same orbit, i.e., $\exists \phi \in PGL(2,\mathbb{C})$ s.t. $g = f^{\phi}$. ($f \sim g$).

$\mathcal{M}_2 := \mathrm{Rat}_2 /$

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⚠ WARNING $PGL(2,\mathbb{C})$ is not free: $\phi(z) = -z$ acts trivially on odd funct.

$\mathcal{M}_2$ could have singularities.

Thm. (Milnor) $\mathcal{M}_2 \cong \mathbb{C}^2$.

Sketch of the sketch. $\quad f(z) = z \iff \dfrac{az^2+bz+c}{dz^2+ez+f} = z$

λ_1, λ_2 and $\lambda_3 \rightsquigarrow$ multipliers of the fixed pts., $f \in \mathrm{Rat}_2$.

$$\prod_{i=1}^{3} (T+\lambda_i) = \sum_{i=0}^{3} \sigma_i(f)\, T^{3-i}.$$

$\sigma_0(f) = 1, \quad \longleftarrow$ gives no inf. about f.

$\sigma_1(f) = \lambda_1 + \lambda_2 + \lambda_3,$

$\sigma_2(f) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3,$

$\sigma_3(f) = \lambda_1\lambda_2\lambda_3.$

Lemma. Let $f \in \mathrm{Rat}_d$, $d \geq 2$, s.t. $\lambda_p \neq 1$, $\forall p \in \mathrm{Fix}(f)$, then
$$\sum_{p \in \mathrm{Fix}(f)} \frac{1}{1-\lambda_p} = 1.$$

Assume $\lambda_1, \lambda_2, \lambda_3 \neq 1$. $\Rightarrow \dfrac{1}{1-\lambda_1} + \dfrac{1}{1-\lambda_2} + \dfrac{1}{1-\lambda_3} = 1$

$\iff (1-\lambda_2)(1-\lambda_3) + (1-\lambda_1)(1-\lambda_3) + (1-\lambda_1)(1-\lambda_2) = (1-\lambda_1)(1-\lambda_2)(1-\lambda_3)$

$\iff 2 \underbrace{- \lambda_1 - \lambda_2 - \lambda_3}_{-\sigma_1(f)} + \underbrace{\lambda_1\lambda_2\lambda_3}_{\sigma_3(f)} = 0$

Extends to all $f \in \mathrm{Rat}_2$.

Lemma (Normal Forms) $f \in \mathrm{Rat}_2$, $\lambda_1, \lambda_2, \lambda_3$ mult. of its fixed pts.
(a) If $\lambda_1 \lambda_2 \neq 1$, then there exists $\phi \in PGL(2,\mathbb{C})$ s.t.

$$f^\phi(z) = \frac{z^2 + \lambda_1 z}{\lambda_2 z + 1}$$

(b) If $\lambda_1 \lambda_2 = 1$, then $\lambda_1 = \lambda_2 = 1$ and $\exists\, \phi \in PGL(2,\mathbb{C})$ s.t.

$$f^\phi(z) = z + \sqrt{1-\lambda_3} + 1/z.$$

$$\sigma : \mathrm{Rat}_2 \longrightarrow \mathbb{C}^2$$
$$f \longmapsto (\sigma_1(f), \sigma_2(f)).$$

If f, g s.t. $\sigma(f) = \sigma(g) \Rightarrow \sigma_i(f) = \sigma_i(g)$, $i = 1, 2$.
$\Rightarrow$ f and g have the same multipliers.
$\Rightarrow f \sim g$

Surjectivity also follows from the lemma.

$\sigma(f^\phi) = \sigma(f)$

$$\rightsquigarrow \quad \tilde{\sigma} : \mathcal{M}_2 \xrightarrow{\sim} \mathbb{C}$$

§2. $\mathrm{Per}_1(1)$.

Def. For each $\eta \in \mathbb{C}$, let $\mathrm{Per}_1(\eta) \subset \mathcal{M}_2$ be the set of conjugacy classes $[f]$ of maps with a fixed pt. with multiplier η.

Want $\mathrm{Per}_1(1)$

$f(z) = \dfrac{az^2+bz+c}{dz^2+ez+f}$ $\quad$ R-H $\Rightarrow$ $2\deg(f) - 2 = \sum_{p \in \hat{\mathbb{C}}} (e_p - 1)$

$\hookrightarrow$ 2 crit. pts. with mult.

$$f'(z) = \frac{(ae-bd)z^2 + (2af-2cd)z + (bf-ce)}{(dz^2+ez+f)^2}$$

$\alpha \neq \infty$ crit. $\Rightarrow (ae-bd)\alpha^2+(2af-2cd)\alpha+(bf-ce)=0$

$\Delta = 4(a^2f^2 - abef + ace^2 - 2acdf + b^2df - bcde + c^2d^2) = 4\,\mathrm{Result}(p,q) \neq 0$

$\alpha = \infty$, take $\phi(z) = 1/z$.

$$f^{\phi}(z) = \frac{1}{f(1/z)} = \frac{d/z^2 + e/z + f}{a/z^2 + b/z + c} = \frac{d+ez+fz^2}{a+bz+cz^2}.$$

$$\mathrm{Result}(d+ez+fz^2,\ a+bz+cz^2) = \det\begin{bmatrix} d & e & f & 0 \\ 0 & d & e & f \\ a & b & c & 0 \\ 0 & a & b & c \end{bmatrix} = \mathrm{Result}(p(z),q(z)) \neq 0$$

$\alpha = \infty$ is also not double.

$\rightsquigarrow f$ has
- at most 3 fixed pts.
- exactly 2 critical pts.

FACT: $\alpha \in \mathrm{Fix}(f)$, $\lambda_\alpha = 1 \iff \alpha$ has multiplicity ≥ 2. $(\alpha \neq \infty)$

$$f(z) - z = f(\alpha) + \lambda_\alpha(z-\alpha) + o(z-\alpha)^2 - z$$

$$= \alpha + \lambda_\alpha(z-\alpha) + o(z-\alpha)^2 - z$$

$$= (z-\alpha)(\lambda_\alpha - 1 + o(z-\alpha)) \quad \checkmark$$

So $f \in \mathrm{Rat}_2$ has a fixed pt. with multip. 1, then

$\rightsquigarrow$ at most 2 fixed pts.

$\longrightarrow$ exactly 2 crit. pts.

Want: $\mathrm{Per}_1(1) = \{[P_A] \mid A \in \mathbb{C}\}$, $P_A(z) = z + 1/z + A$

Take $\phi \in PGL(2,\mathbb{C})$ sending the crit. pts. to ± 1 and the fixed pt. with mult. $= 1$ to ∞.

- (∞ fixed) $\displaystyle\lim_{z\to\infty} \frac{az^2+bz+c}{dz^2+ez+f} = \infty \Rightarrow \boxed{d=0}$

- (± 1 crit.) $f'(z) = \underline{aez^2 + 2afz + (bf-ce)}$

• (± 1 crit.) $f'(z) = \dfrac{aez^2 + 2afz + (bf - ce)}{(ez+f)^2}$

$$\left.\begin{array}{ll} e(a-c) + f(2a+b) = 0 & (1 \text{ crit.}) \\ e(a-c) - f(2a+b) = 0 & (-1 \text{ crit.}) \end{array}\right\} \boxed{\begin{array}{l} f = 0 \\ e \neq 0 \end{array}}$$

$$\rightsquigarrow f(z) = \frac{az^2 + bz + a}{ez}$$

• $(\lambda_\infty = 1)$, $\psi(z) = 1/z$

$$f^\psi(z) = \frac{ez}{a + bz + az^2} \Rightarrow (f^\psi)'(z) = \frac{e(a + bz + az^2) - ez(b + 2az)}{(a + bz + az^2)^2}$$

$$\Rightarrow (f^\psi)'(0) = \frac{ea^2}{a} = \boxed{\frac{e}{a} = 1}$$

$$\Rightarrow f(z) = \frac{az^2 + bz + a}{az} = z + 1/z + \underbrace{(b/a)}_{A}$$

$$\mathrm{Per}_1(1) \subseteq \{ [P_A] \mid A \in \mathbb{C} \}$$

$\supseteq$ ∞ fixed, $\lambda_\infty = 1$.

$f'(z) = 1 - 1/z^2 = 0 \Leftrightarrow z^2 = 1$.

fixed pt. $\alpha_A = -1/A$, $\lambda_{\alpha_A} = 1 - A^2$

$P_A(z) = P_{-A}(-z)$ $\quad$ $P_A \sim P_{-A}$ by $\psi(z) = -z$.

$$\mathbb{C} \xrightarrow[\sim]{\text{biholom.}} \mathrm{Per}_1(1)$$

$$B = 1 - A^2 \longrightarrow [P_A].$$

$$\sigma_1(f) = 2 + (1 - A^2) = 3 - A^2 = 2 + B$$
$$\sigma_2(f) = 1 + 1 - A^2 + 1 - A^2 = 1 + 2B$$

§3. Parabolic Mandelbrot Set.

Def. The Fatou set $\mathcal{F}(f)$ of f is defined to be the set of points in $\hat{\mathbb{C}}$ for which exists nbhd U s.t. the seq of iterates $\{f^n\}$ is normal in U.

Def. The Fatou set $\mathcal{F}(f)$ of f is defined to be the set of points in $\hat{\mathbb{C}}$ for which exists nhood $\mathcal{U}$ s.t. the seq of iterates $\{f^n\}$ is normal in $\mathcal{U}$.
The Julia set $\mathcal{J}(f)$ is its complement $\mathcal{J}(f) = \hat{\mathbb{C}} \setminus \mathcal{F}(f)$.

Rmk: $\mathcal{F}(f)$ open, $\mathcal{J}(f)$ compact.

Thm. $\phi \in PGL(2,\mathbb{C})$, then $\mathcal{F}(f^\phi) = \phi(\mathcal{F}(f))$ and $\mathcal{J}(f^\phi) = \phi(\mathcal{J}(f))$.

Sketch. ϕ conformal $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so $\{f^{n_k}\}$ conv. uniformly on $\mathcal{U}$ to some limit $f_0 \iff$ the seq. $\{(f^\phi)^{n_k}\}$ conv. unif. on $\phi(\mathcal{U})$ to $(f_0)^\phi$.

Thm. $f(\mathcal{F}(f)) = \mathcal{F}(f) = f^{-1}(\mathcal{F}(f))$, $f(\mathcal{J}(f)) = \mathcal{J}(f) = f^{-1}(\mathcal{J}(f))$.

FACT: $\mathcal{J}(f)$ disconnected $\iff$ the crit pts. are in the same Fatou comp. containing a fixed pt. in its closure w/ multiplier in $\mathbb{D} \cup \{1\}$.

Put $M_\lambda := \{ [f] \in \mathrm{Per}_1(\lambda) \mid \mathcal{J}(f) \text{ is connected} \}$ for $\lambda \in \mathbb{D} \cup \{1\}$.

- $\mathrm{Per}_1(0) = \{[z^2+c] \mid c \in \mathbb{C}\} \Rightarrow M_0 =$ Mandelbrot set.
- $|\lambda| < 1$, any rat. map is quasi-conformally conj. to $\mathcal{S}$ (polynomial-like maps) with a fixed pt. with $|\lambda|<1$.
- $\lambda = 1 \longrightarrow M_1 =$ parabolic Mandelbrot set.

Conjecture (Milnor) M_1 homeomorphic to M_0. $\longrightarrow$ Petersen & Roesch.

Milnor. M_1 is connected.